Yours affect.
D F Gregory.

THE MATHEMATICAL WRITINGS

OF

DUNCAN FARQUHARSON GREGORY, M.A.,
LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

EDITED BY

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With a Biographical Memoir,

BY

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DEDICATORY LETTER.

Dear Miss Gregory,

The earlier Numbers of the *Cambridge Mathematical Journal,** of which your lamented brother was the first Editor, having been for some years out of print, many of the essays which he contributed to this valuable Periodical have become inaccessible to most mathematical readers.

The intrinsic value of his articles, by which in a great degree a spirit of originality was awakened in the University of Cambridge at an era of peculiar stagnation in mathematical

* *The Cambridge Mathematical Journal,* extending to Four Volumes, was succeeded by a New Series entitled *The Cambridge and Dublin Mathematical Journal,* consisting of Nine Volumes; and of a Third Series, *The Quarterly Journal of Pure and Applied Mathematics,* Six Volumes have already been published.

invention, could not fail to recommend them permanently to those who delight in the history and the philosophy of abstract science. Animated by this conviction, I have, with your permission, gathered together in this Volume his contributions to the various Numbers of the *Mathematical Journal*, together with an Essay on the Foundations of Algebra, presented by him to the Royal Society of Edinburgh.

The photograph, taken from the excellent crayon drawing by Mr. Crawford of Edinburgh, with which you have enriched this volume, will be very interesting to those who have a personal recollection of your brother, and perhaps also to others. As the sketch, which recently served, under your supervision, as the basis of the crayon drawing, was taken when he was only eighteen years of age, the expression naturally strikes me as rather youthful, when I recall the features, so familiar to me, which were impressed on my memory some years later, when I had the happiness of being admitted among the number of his intimate friends.

To you I would beg to dedicate this collection of the writings of your brother of beloved memory, and to remain

Your faithful Servant,

WILLIAM WALTON.

CAMBRIDGE,
October, 1865.

BY THE SAME AUTHOR.

EXAMPLES OF THE PRINCIPLES OF THE DIFFERENTIAL AND INTEGRAL CALCULUS.

Second Edition. Edited by W. WALTON, M.A.

A TREATISE ON THE APPLICATION OF ANALYSIS TO SOLID GEOMETRY.

Second Edition. Edited by W. WALTON, M.A.

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OF

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BIOGRAPHICAL MEMOIR

OF

DUNCAN FARQUHARSON GREGORY.

THE subject of the following memoir died in his thirty-first year. He had, nevertheless, accomplished enough not only to justify high expectations of his future progress in the science to which he had principally devoted himself, but also to entitle his name to a place in some permanent record.

Duncan Farquharson Gregory was born at Edinburgh in April 1813. He was the youngest son of Dr. James Gregory, the distinguished professor of Medicine, and was thus of the same family as the two celebrated mathematicians James and David Gregory. The former of these, his direct ancestor, is familiarly remembered as the inventor of the telescope which bears his name; he lived in an age of great mathematicians, and was not unworthy to be their contemporary.

Of the early years of Mr. Gregory's life but little need be said. The peculiar bent of his mind towards mathematical speculations does not appear to have been perceived during his childhood; but, in the usual course of education, he shewed much facility in the acquisition of knowledge, a remarkably active and inquiring mind, and a very retentive memory. It may, perhaps, be mentioned here, that his father, whom he lost before he was seven years old, used to predict distinction for him; and was so struck with his accurate information and clear memory, that he had pleasure in conversing with him, as with an equal, on subjects of history and geography. In his case, as in many others, ingenuity in little mechanical contrivances seems to have preceded, and indicated the developement of a taste for abstract science.

Two years of his life were passed at the Edinburgh Academy*; when he left it, being considered too young for the University, he went abroad and spent a winter at a private academy in Geneva. Here his talent for mathematics attracted attention; in geometry, as well as in classical learning, he had already made distinguished progress at Edinburgh.

The following winter he attended classes at the University of Edinburgh,† and soon became

* Till he was nine years old he was taught entirely by his mother, whose early care and love he repaid throughout his life by peculiar affection and reverence. From childhood his great amusement was studying geography and astronomy; in his ninth year he made an orrery: his favourite recreation, when out of doors, was gardening. At the conclusion of his ninth year he had a private tutor, for two hours a day, to assist in his instruction. In October, 1825, he went to the Edinburgh Academy.—*Editor's Note.*

† During the time he attended the classes at the University of Edinburgh, he worked much at Chemistry and made many experiments on Polarized Light, a subject the study of which interested him particularly at that time.—*Editor's Note.*

a favourite pupil of Professor Wallace's, under whose tuition he made great advances in the higher parts of mathematics. The Professor formed the highest hopes of Mr. Gregory's future eminence: those who long afterwards saw them together in Cambridge, speak with much interest of the delighted pride he shewed in his pupil's success and increasing reputation.

In 1833, Mr. Gregory's name was entered at Trinity College in the University of Cambridge, and shortly afterwards he went to reside there. He brought with him a very unusual amount of knowledge on almost all scientific subjects: with Chemistry he was particularly well acquainted, so much so that he had been at Cambridge but a few months when it was proposed to him by one of the most distinguished men in the University to act as assistant to the professor of Chemistry; which for some time he did. Indeed, it is impossible to doubt that, had not other pursuits engaged his attention, he might have achieved a great reputation as a chemist. He was one of the founders of the Chemical Society in Cambridge, and occasionally gave lectures in their rooms.

He had also a very considerable knowledge of botany, and indeed of many subjects which he seemed never to have studied systematically: he possessed in a remarkable degree the power of giving a regular form, and, so to speak, a unity to knowledge acquired in fragments.

All these tastes and habits of thought Mr. Gregory cultivated, to a certain extent, during the first years of his residence in Cambridge, of course in subordination to that which was the end principally in view in his becoming a member of the University, namely, the study of mathematics and natural philosophy.

He became a Bachelor of Arts in 1837, having taken high mathematical honours: more, however, might, we may believe, have been effected in this respect, had his activity of mind permitted him to devote himself more exclusively to the prescribed course of study.

From henceforth he felt himself more at liberty to follow original speculations, and, not many months after taking his degree, turned his attention to the general theory of the combination of symbols.

It may be well to say a few words of the history of this part of mathematics.

One of the first results of the differential notation of Leibnitz, was the recognition of the analogy of differentials and powers. For instance, it was readily perceived that

$$\frac{d^{m+n}}{dx^{m+n}}\,y = \frac{d^m}{dx^m}\,\frac{d^n}{dx^n}\,y,$$

or, supposing the y to be *understood*, that

$$\left(\frac{d}{dx}\right)^{m+n} = \left(\frac{d}{dx}\right)^{m}\left(\frac{d}{dx}\right)^{n},$$

just as in ordinary algebra we have, a being any quantity,
$$a^{m+n} = a^m a^n.$$

This, and one or two other remarks of the same kind, were sufficient to establish an analogy between $\frac{d}{dx}$ the symbol of differentiation and the ordinary symbols of algebra. And it was not long afterwards remarked that a corresponding analogy existed between the latter class of symbols and that which is peculiar to the calculus of finite differences. It was inferred from hence that theorems proved to be true of combinations of ordinary symbols of quantity, might be applied by analogy to the differential calculus and to that of finite differences. The meaning and interpretation of such theorems would of course be wholly changed by this kind of transfer from one part of mathematics to another, but their form would remain unchanged. By these considerations many theorems were suggested, of which it was thought almost impossible to obtain direct demonstrations. In this point of view the subject was developed by Lagrange, who left undemonstrated the results to which he was led, intimating, however, that demonstrations were required. Gradually, however, mathematicians came to perceive that the analogy with which they were dealing, involved an essential identity; and thus results, with respect to which, if the expression may be used, it had only been felt that they must be true, were now actually seen to be so. For, if the algebraical theorems by which these results were suggested, were true, *because* the symbols they involve represented quantities, and such operations as may be performed on quantities, then indeed the analogy would be altogether precarious. But if, as is really the case, these theorems are true, in virtue of certain fundamental laws of combination, which hold both for algebraical symbols, and for those peculiar to the higher branches of mathematics, then each algebraical theorem, and its analogue constitute, in fact, only one and the same theorem, except *quoad* their distinctive interpretations, and therefore a demonstration of either is in reality a demonstration of both.*

The abstract character of these considerations is doubtless the reason why so long a time elapsed before their truth was distinctly perceived. They would almost seem to require, in order that they may be readily apprehended, a peculiar faculty—a kind of mental *disinvoltura* which is by no means common.

Mr. Gregory, however, possessed it in a very remarkable degree. He at once perceived the truth and the importance of the principles of which we have been speaking, and proceeded to apply them with singular facility and fearlessness.

It had occurred to two or three distinguished writers that the analogy, as it was called, of

* The values of certain definite integrals are to be looked upon as merely arithmetical results; in such cases we are not at liberty to replace the constants involved in the definite integrals by symbols of operation. In other cases we are at liberty to do so, and this remarkable application of the principles stated in the text, has already led Mr. Boole of Lincoln, with whom it seems to have originated, to several curious conclusions.

powers, differentials, &c., might be made available in the solution of differential equations, and of equations in finite differences.

This idea, however, probably from some degree of doubt as to the legitimacy of the methods which it suggested, had not been fully or clearly developed: it seems to have been chiefly employed as affording a convenient way of expressing solutions already obtained by more familiar considerations.

To this branch of the subject Mr. Gregory directed his attention, and from the general views of the laws of combination of symbols already noticed, deduced in a regular and systematic form, methods of solution of a large and important class of differential equations (linear equations with constant coefficients, whether ordinary or partial) of systems of such equations existing simultaneously, of the corresponding classes of equations in finite and mixed differences; and lastly, of many functional equations. The steady and unwavering apprehension of the fundamental principle which pervades all these applications of it, gives them a value quite independent of that which arises from the facility of the methods of solution which they suggest.

The investigations of which I have endeavoured to illustrate the character and tendency, appeared from time to time in the *Cambridge Mathematical Journal.*

In this periodical publication Mr. Gregory took much interest. He had been active in establishing it, and continued to be its editor, except for a short interval, from the time of its first appearance in the autumn of 1837, until a few months before his death. For this occupation he was for many reasons well qualified; his acquaintance with mathematical literature was very extensive, while his interest in all subjects connected with it was not only very strong, but also singularly free from the least tinge of jealous or personal feeling. That which another had done or was about to do, seemed to give him as much pleasure as if he himself had been the author of it, and this even when it related to some subject which his own researches might seem to have appropriated.

This trait, as the recollections of those who knew him best will bear me witness, was intimately connected with his whole character, which was in truth an illustration of the remark of a French writer, that to be free from envy is the surest indication of a fine nature.

To the *Cambridge Mathematical Journal*, Mr. Gregory contributed many papers beside those which relate to the researches already noticed. In some of these he developed certain particular applications of the principles he had laid down in an Essay on the Foundations of Algebra, presented to the Royal Society of Edinburgh in 1838, and printed in the fourteenth volume of their Transactions. I may particularly mention a paper on the curious question of the logarithms of negative quantities, a question which, it is well known, has often been discussed among mathematicians, and which even now does not appear to be entirely settled.

In 1840, Mr. Gregory was elected Fellow of Trinity College; in the following year he became Master of Arts, and was appointed to the office of moderator, that is, of principal mathematical examiner. His discharge of the duties of this office (which is looked upon as one of the most honourable of those which are accessible to the younger members of the University) was distinguished by great good sense and discretion.

In the close of the year 1841, Mr. Gregory produced his "Collection of Examples of the Processes of the Differential and Integral Calculus;" a work which required, and which manifests much research, and an extensive acquaintance with mathematical writings. He had at first only wished to superintend the publication of a second edition of the work with a similar title, which appeared more than twenty-five years since, and of which Messrs. Herschel, Peacock, and Babbage were the authors. Difficulties, however, arose, which prevented the fulfilment of this wish, and it is not perhaps to be regretted that Mr. Gregory was thus led to undertake a more original design. It is well known that the earlier work exercised a great and beneficial influence on the studies of the University, nor was it in any way unworthy of the reputation of its authors. The original matter contributed by Sir John Herschel is especially valuable. Nevertheless, the progress which mathematical science has since made, rendered it desirable that another work of the same kind should be produced, in which the more recent improvements of the calculus might be embodied.

Since the beginning of the century, the general aspect of mathematics has greatly changed. A different class of problems from that which chiefly engaged the attention of the great writers of the last age has arisen, and the new requirements of natural philosophy have greatly influenced the progress of pure analysis. The mathematical theories of heat, light, electricity, and magnetism, may be fairly regarded as the achievement of the last fifty years. And in this class of researches an idea is prominent, which comparatively occurs but seldom in purely dynamical enquiries. This is the idea of discontinuity. Thus, for instance, in the theory of heat, the conditions relating to the surface of the body whose variations of temperature we are considering, form an essential and peculiar element of the problem; their peculiarity arises from the discontinuity of the transition from the temperature of the body to that of the space in which it is placed. Similarly, in the undulatory theory of light, there is much difficulty in determining the conditions which belong to the bounding surfaces of any portion of ether; and although this difficulty has, in the ordinary applications of the theory, been avoided by the introduction of proximate principles, it cannot be said to have been got rid of.

The power, therefore, of symbolizing discontinuity, if such an expression may be permitted, is essential to the progress of the more recent applications of mathematics to natural philosophy, and it is well known that this power is intimately connected with the theory of definite integrals. Hence the principal importance of this theory, which was altogether passed over in the earlier collection of examples.

Mr. Gregory devoted to it a chapter of his work, and noticed particularly some of the more remarkable applications of definite integrals to the expression of the solutions of partial differential equations. It is not improbable that in another edition he would have developed this subject at somewhat greater length. He had long been an admirer of Fourier's great work on heat, to which this part of mathematics owes so much; and once, while turning over its pages, remarked to the writer,—"All these things seem to me to be a kind of mathematical paradise."

In 1841, the mathematical Professorship at Toronto was offered to Mr. Gregory: this, however, circumstances induced him to decline. Some years previously he had been a candidate for the Mathematical Chair at Edinburgh.

His year of office as moderator ended in

October, 1842. In the University Examination for Mathematical Honours in the following January, he, however, in accordance with the usual routine, took a share, with the title of examiner,—a position little less important, and very nearly as laborious, as that of moderator. Besides these engagements in the University, he had been for two or three years actively employed in lecturing and examining in the College of which he was a Fellow. In the fulfilment of these duties, he shewed an earnest and constant desire for the improvement of his pupils, and his own love of science tended to diffuse a taste for it among the better order of students. He had for some time meditated a work on Finite Differences, and had commenced a treatise on Solid Geometry, which, unhappily, he did not live to complete. In the midst of these various occupations, he felt the earliest approaches of the malady which terminated his life.

The first attack of illness occurred towards the close of 1842. It was succeeded by others, and in the spring of 1843, he left Cambridge never to return again. He had just before taken part in a college examination, and notwithstanding severe suffering, had gone through the irksome labour of examining with patient energy and undiminished interest.

Many months followed of almost constant pain. Whenever an interval of tolerable ease occurred, he continued to interest himself in the pursuits to which he had been so long devoted; he went on with the work on Geometry, and, but a little while before his death, commenced a paper on the analogy of differential equations and those in finite differences. This analogy it is known that he had developed to a great length; unfortunately, only a portion of his views on the subject can now be ascertained.

At length, on the 23rd of February, 1844, after sufferings, on which, notwithstanding the admirable patience with which they were borne, it would be painful to dwell, his illness terminated in death. He had been for a short time aware that the end was at hand, and, with an unclouded mind, he prepared himself calmly and humbly for the great change; receiving and giving comfort and support from the thankful hope that the close of his suffering life here, was to be the beginning of an endless existence of rest and happiness in another world. He retained to the last, when he knew that his own connection with earthly things was soon to cease, the unselfish interest which he had ever felt in the pursuits and happiness of those he loved.

A few words may be allowed about a character where rare and sterling qualities were combined. His upright, sincere, and honourable nature secured to him general respect. By his intimate friends, he was admired for the extent and variety of his information, always communicated readily, but without a thought of display,—for his refinement and delicacy of taste and feeling,—for his conversational powers and playful wit; and he was beloved by them for his generous, amiable disposition, his active and disinterested kindness, and steady affection. And in this manner his high-toned character acquired a moral influence over his contemporaries and juniors, in a degree remarkable in one so early removed.

To this brief history, little more is to be added; for though it is impossible not to indulge in speculations as to all that Mr. Gregory might have done in the cause of science and for his own reputation, had his life been prolonged, yet such speculations are necessarily too vague to find a place here; and even were it not so, it would perhaps be unwise to enter on a subject so full of sources of unavailing regret.

ON THE REAL NATURE OF SYMBOLICAL ALGEBRA.*

THE following attempt to investigate the real nature of Symbolical Algebra, as distinguished from the various branches of analysis which come under its dominion, took its rise from certain general considerations, to which I was led in following out the principle of the separation of symbols of operation from those of quantity. I cannot take it on me to say that these views are entirely new, but at least I am not aware that any one has yet exhibited them in the same form. At the same time, they appear to me to be important, as clearing up in a considerable degree the obscurity which still rests on several parts of the elements of symbolical algebra. Mr. Peacock is, I believe, the only writer in this country who has attempted to write a system of algebra founded on a consideration of general principles, for the subject is not one which has much attraction for the generality of mathematicians. Much of what follows will be found to agree with what he has laid down, as well as with what has been written by the Abbé Buee and Mr. Warren; but as I think that the view I have taken of the subject is more general than that which they have done, I hope that the following pages will be interesting to those who pay attention to such speculations.

* *Transactions of the Royal Society of Edinburgh*, Vol. XIV., p. 208. [Read 7th May, 1838].

The light, then, in which I would consider symbolical algebra, is, that it is the science which treats of the combination of operations defined not by their nature, that is, by what they are or what they do, but by the laws of combination to which they are subject. And as many different kinds of operations may be included in a class defined in the manner I have mentioned, whatever can be proved of the class generally, is necessarily true of all the operations included under it. This, it may be remarked, does not arise from any analogy existing in the nature of the operations, which may be totally dissimilar, but merely from the fact that they are all subject to the same laws of combination. It is true that these laws have been in many cases suggested (as Mr. Peacock has aptly termed it) by the laws of the known operations of number; but the step which is taken from arithmetical to symbolical algebra is, that, leaving out of view the nature of the operations which the symbols we use represent, we suppose the existence of classes of unknown operations subject to the same laws. We are thus able to prove certain relations between the different classes of operations, which, when expressed between the symbols, are called algebraical theorems. And if we can show that any operations in any science are subject to the same laws of combination as these classes, the theorems are true of these as included in the general case: Provided always, that the resulting combinations are all possible in the particular operation under consideration. For it may very well, and does actually happen, that, though each of two operations in a certain branch of science may be possible, the complex operation resulting from their combination is not equally possible. In such a case, the result is inapplicable to that branch of science. Hence we find, that one family of a class of operations may have a more general application than another family of the same class. To make my meaning more precise, I shall proceed to apply the principle I have been endeavouring to explain, by shewing what are the laws appropriate to the different classes of operations we are in the habit of using.

Let us take as usual F and f to represent any operations whatever, the natures of which are unknown, and let us prefix these symbols to any other symbols, on which we wish to indicate that the operation represented by F or f is to be performed.

I. We assume, then, the existence of two classes of operations F and f, connected together by the following laws:

(1) $FF(a) = F(a)$. (2) $ff(a) = F(a)$.

(3) $Ff(a) = f(a)$. (4) $fF(a) = f(a)$.

Now, on looking into the operations employed in arithmetic, we find that there are two which are subject to the laws we have just laid down. These are the operations of addition and subtraction; and as to them the peculiar symbols of $+$ and $-$ have been affixed, it is convenient to retain these as the symbols of the general class of operations we have defined, and we shall therefore use them instead of F and f. As it is useful to have peculiar names attached to each class, I would propose to call this the class of *circulating* or *reproductive* operations, as their nature suggests.

Again, on looking into geometry, we find two operations which are subject to the same laws. The one corresponding to $+$ is the turning of a line, or rather transferring of a point, through a circumference; the other corresponding to $-$ is the transference of a point through a semicircumference. Consequently, whatever we are able to prove of the general symbols $+$ and $-$ from the laws to which they are subject, without considering the nature of the operations they indicate, is equally true of the arithmetical operations of addition and subtraction, and of the geometrical operations I have described. We see clearly from this, that there is no real analogy between the nature of the operations + and − in arithmetic and geometry, as is generally supposed to be the case, for the two operations cannot even be said to be opposed to each other in the latter science, as they are generally said to be. The relation which does exist is due not to any identity of their nature, but to the fact of their being combined by the same laws. Other operations might be found which could be classed under the general head we are considering. Mr. Peacock and the Abbé Buee consider the transference of property to be one of these; but as there is not much interest attached to it in a mathematical point of view, I shall proceed to the consideration of other operations.

II. Let us suppose the existence of operations subject to the following laws:

$$(1)\ f_m(a) . f_n(a) = f_{m+n}(a). \qquad (2)\ f_m f_n(a) = f_{mn}(a).$$

Where f_m, f_n are different species of the same genus of operations, which may be conveniently named index-operations, as, if we define the form of f by making $f_1(a) = a$, and suppose m and n to be integer numbers, we have those operations which are represented in arithmetical algebra by a numerical index. For if m and n be integers, and the operation a^m be used to denote that the operation a has been repeated m times, then, as we know,

$$a^m . a^n = a^{m+n}, \quad (a^m)^n = a^{mn}.$$

We have now to consider whether we can find any other actual operations besides that of repetition which shall be subject to the laws we have laid down. If we suppose that m and n are fractional instead of integer, we easily deduce from our definition that the notation $a^{\frac{p}{q}}$ is equivalent to the arithmetical operation of extracting the q^{th} root of the p^{th} power of a, or generally the finding of an operation, which being repeated q times, will give as a result the operation a^p. Thus we find, as might have been expected, a close analogy existing between the meanings of a^m when m is integer, and when it is fractional. Again, we might ask the meaning of the operation a^{-m}; and we find without difficulty, from the law of combination, that a^{-m} indicates the inverse operation of a^m, whatever the operation a may be. When, instead of supposing m to be a number integer or fractional, we suppose it to indicate any operation whatever, I do not know of any interpretation which can be given to the notation, excepting in the case when it indicates the operation of differentiation, represented by the symbol d. For we know by Taylor's theorem, that

$$\varepsilon^{h\frac{d}{dx}} f(x) = f(x+h),$$

or,

$$a^{\frac{d}{dx}} f(x) = f(x + \log a).$$

In the case of negative indices, we have combined two different classes of operations in one manner, but we may likewise do it in another. What meaning, we may ask, is to be attached to such complex operations as $(+)^m$ or $(-)^m$? When m is an integer number, we see at once that the operation $(+)^m$ is the same as $+$, but $(-)^m$ becomes alternately the same as $+$ and as $-$, according as m is odd or even, whether they be the symbols of arithmetical or geometrical operations. So far there is no difficulty. But if it be fractional, what does $(+)^m$ or $(-)^m$ signify? In arithmetic, the first may be sometimes interpreted, as because $(+)^m = +$ when m is integer, $(+)^{\frac{1}{m}}$ also $= +$, and as $(-)^{2m} = +$, also $(+)^{\frac{1}{2m}} = -$: But the other symbol $(-)^m$ has, when m is a fraction with an even denominator, absolutely no meaning in arithmetic, or at least we do not know at present of any arithmetical operation which is subject to the same laws of combination as it is. On the other hand, geometry readily furnishes us with operations which may be represented by $(+)^{\frac{1}{m}}$ and $(-)^{\frac{1}{m}}$, and which are analogous to the operations represented by + and −. The one is the turning of a line through an angle equal to $\frac{1}{m}$th of four right angles, the other is the turning of a line through an angle equal to $\frac{1}{m}$th of two right angles. Here we see that the geometrical family of operations admits of a more extended application than the arithmetical, exemplifying a general remark we had previously occasion to make. Whether when the index is any other operation, we can attach any meaning to the expression, has not yet been determined. For instance, we cannot tell what is the interpretation of such expressions as $(+)^{\frac{d}{dx}}$ or $(-)^{\frac{d}{dx}}$, or $(+)^{\log}$.

III. I now proceed to a very general class of operations, subject to the following laws:

$$(1)\ f(a)+f(b)=f(a+b).$$

$$(2)\ f_{\prime}f(a) \quad =ff_{\prime}(a).$$

This class includes several of the most important operations which are considered in mathematics; such as the numerical operation usually represented by a, b, &c., indicating that any other operation to which these symbols are prefixed is taken a times, b times, &c.; or as the operation of differentiation indicated by the letter d, and the operation of taking the difference indicated by Δ. We therefore see what an important part this class of functions plays in analysis, since it can be at once divided into three families which are of such extensive use. This renders it advisable to comprehend these functions under a common name. Accordingly, Servois, in a paper which does not seem to have received the attention it deserves, has called them, in respect of the first law of combination, *distributive* functions, and in respect of the second law, *commutative* functions. As these names express sufficiently the nature of the functions we are considering, I shall use them when I wish to speak of the general class of operations I have defined.

It is not necessary to enter at large here, into the demonstration that the symbols of differentiation and difference are subject to the same laws of combination as those of number. But it may not be amiss to say a few words on the effect of considering them in this light. Many theorems in the differential calculus, and that of finite differences, it was found might be conveniently expressed by separating the symbols of operation from those of quantity, and treating the former like ordinary algebraic symbols. Such is Lagrange's elegant theorem, the first expressed in this manner, that

$$\Delta^n u_x = (\varepsilon^{h\frac{d}{dx}} - 1)^n u_x;$$

or the theorem of Leibnitz, with many others. For a long time these were treated as mere analogies, and few seemed willing to trust themselves to a method, the principles of which did not appear to be very sound. Sir John Herschel was the person in this country who made the freest use of the method, chiefly, however, in finite differences. In France, Servois was, I believe, the only mathematician who attempted to explain its principles, though Brisson and Cauchy sometimes employed and extended its application: and it was in pursuing this investigation that he was led to separate functions into distributive and commutative, which he perceived to be the properties which were the foundation of the method of the separation of the symbols, as it is called. This view, which, so far as it goes, coincides with that which it is the object of this paper to develope, at once fixes the principles of the method on a firm and secure basis. For, as these various operations are all subject to common laws of combination, whatever is proved to be true by means only of these laws, is necessarily equally true of all the operations. To this I may add, that when two distributive and commutative operations are such that the one does not act on the other, their combinations will be subject to the same laws as when they are taken separately; but when they are not independent, and one acts on another, this will no longer be true. Hence arises the increased difficulty of solving linear differential equations with variable coefficients; but for more detailed remarks on this, as well as for examples of a more extended use of the method of the separation of symbols than has hitherto been made, I refer to the *Cambridge Mathematical Journal*, Nos. 1, 2, and 3.*

As we found geometrical operations which were subject to the laws of circulating operations, so there is a geometrical operation which is subject to the laws of distributive and permutative operations, and therefore may be represented by the same symbols. This is transference to a distance measured in a straight line. Thus if x represent a point, line, or any geometrical figure, $a(x)$ will represent the transference of this point or line; and it will be seen at once that

$$a(x) + a(y) = a(x+y);$$

or the operation a is distributive. What, then, will the compound operation $b\{a(x)\}$ represent? If x represent a point, $a(x)$, which is the transference of a point to a rectilinear distance, or the tracing out of a straight line, will stand for the result of the operation; and then $b\{a(x)\}$ will be the transferring of a line to a given distance from its original position. In order to effect this, the line must be moved parallel to itself, the effect of which will be the

* Of the articles here referred to, those written by Mr. Gregory are published in this volume.—*Editor's Note.*

tracing out of a parallelogram. The effect will be the same if we suppose a to act on $b(x)$, since in this, as in the other case, the same parallelogram will be traced out: that is to say,

$$a\{b(x)\}=b\{a(x)\}$$

or a and b are commutative operations.

The binomial theorem, the most important in symbolical algebra, is a theorem expressing a relation between distributive and commutative operations, index operations, and circulating operations. It takes cognizance of nothing in these operations except the six laws of combination we have laid down, and, as we shall presently shew, it holds only of functions subject to these laws. It is consequently true of all operations which can be shewn to be commutative and distributive, though apparently, from its proof, only true of the operations of number. The difficulties attending the general proof of this theorem are well known, and much thought has been bestowed on the best mode of avoiding them. The principles I have been endeavouring to exhibit appear to me to shew in a very clear light the correctness of Euler's very beautiful demonstration. Starting with the theorem as proved for integer indices, which he uses as a suggestive form, he assumes the existence of a series of the same form when the index is fractional or negative, which may be represented by $f_m(x)$. He then considers what will be the form of the product $f_m(x)\times f_n(x)$. This form must depend only on the laws of combination to which the different operations in the expression are subject. When x is a distributive and commutative function, and m and n integer numbers, we know that $f_m(x)\times f_n(x)=f_{m+n}(x)$. Now integer numbers are one of the families of the general class of distributive and permutative functions; and if we actually multiplied the expressions $f_m(x)$ and $f_n(x)$ together, we should, even in the case of integers, make use only of the distributive and permutative properties. But these properties hold true also of fractional and negative quantities. Therefore, in their case, the form of the product must be the same as when the indices are integer numbers. Hence $f_m(x) \times f_n(x) = f_{m+n}(x)$ whether m and n be integer or fractional, positive or negative, or generally if m and n be distributive and permutative functions.

The remainder of the proof follows very readily after this step, which is the key-stone of the whole, so that I need not dwell on it longer. I will only say, that this mode of considering the subject shews clearly, that not only must the quantities under the vinculum be distributive and commutative functions, but also the index must be of the same class,—a limitation which I do not remember to have seen any where introduced. Therefore the binomial theorem does not apply to such expressions as $(1+a)^{\log}$ or $(1+a)^{\sin}$; and, though it does apply to $(1+a)^{\frac{d}{dx}}$, since both a and $\frac{d}{dx}$ are distributive and commutative operations, it does not apply to $\{1+f(x)\}^{\frac{d}{dx}}$, as $f(x)$ and $\frac{d}{dx}$ are not relatively commutative.

Closely connected with the binomial theorem is the exponential theorem, and the same remarks will apply equally to both. So that, in order that the relation

$$\varepsilon^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$$

may subsist, it is necessary, and it suffices, that x should be a distributive and commutative function. On this depends the propriety of the abbreviated notation for Taylor's theorem

$$f(x+h) = \varepsilon^{h\frac{d}{dx}} f(x).$$

Properly speaking, however, the symbol ε ought not to be used, as it implies an arithmetical relation, and instead, we ought to employ the more general symbol of $\log^{-1}$. But this depends on the existence of a class of operations on which I may say a few words.

IV. If we define a class of operations by the law

$$f(x)+f(y)=f(xy),$$

we see that, when x and y are numbers, the operation is identical with the arithmetical logarithm. But when x and y are any thing else, the function will have a different meaning. But so long as they are distributive and commutative functions, the general theorems such as

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\&c.$$

being proved solely from laws we have laid down, are true of all symbols subject to those laws. It happens that we are not generally able to assign any known operation to which the series is equivalent when x is any thing but a number, and we therefore say that $\log(1+x)$ is an abbreviated expression for the series $x-\frac{x^2}{2}+\frac{x^3}{3}-\&c.$ But there may be distinct meanings for such expressions as $\log\left(1+\frac{d}{dx}\right)$ or $\log\left(\frac{d}{dx}\right)$, as there are for $\varepsilon^{h\frac{d}{dx}}$, that is $\log^{-1}\left(\frac{d}{dx}\right)$. In the case of another operation, Δ, we know that $\log(1+\Delta)=\frac{d}{dx}$.

V. The last class of operations I shall consider is that involving two operations connected by the conditions

(1) $\quad aF(x+y)=F(x)f(y)+f(x)F(y),$

and (2) $\quad af(x+y)=f(x)f(y)-cF(x)F(y).$

These are laws suggested by the known relation between certain functions of elliptic sectors; and when a and c both become unity, they are the laws of the combinations of ordinary sines and cosines, which may be considered in geometry as certain functions of angles or circular sectors, but in algebra we only know of them as abbreviated expressions for certain complicated relations between the first three classes of operations we have considered. These relations are,

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5}, \text{ \&c.},$$

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4}, \text{ \&c.}$$

The most important theorem proved of this class of functions is that of Demoivre, that

$$\{\cos x + (-)^{\frac{1}{2}} \sin x\}^n = \cos nx + (-)^{\frac{1}{2}} \sin nx.$$

It is easy to see that, in arithmetical algebra, the expression $\cos x + (-)^{\frac{1}{2}} \sin x$ can receive no interpretation, as it involves the operation $(-)^{\frac{1}{2}}$. In geometry, on the contrary, it has a very distinct meaning. For if a represent a line, and $a \cos x$ represent a line bearing a certain relation in magnitude to a, and $a \sin x$ a line bearing another relation in magnitude to a, then $a\{\cos x + (-)^{\frac{1}{2}} \sin x\}$ will imply, that we have to measure a line $a \cos x$, and from the extremity of it we are to measure another line $a \sin x$; but in consequence of the sign of operation $(-)^{\frac{1}{2}}$, this new line is to be measured, not in the same direction as $a \cos x$, but turned through a right angle. As, for instance, if $AB = a \cos x$, and $BC' = a \sin x$, we must not measure it in the prolongation of AB (fig. 1), but turn it round to the position BC; and thus, geometrically, we arrive at the point C. Also, from the relation between $\sin x$ and $\cos x$, we know that the line AC will be equal to a, and thus the expression $a\{\cos x + (-)^{\frac{1}{2}} \sin x\}$ is an operation expressing that the line whose length is a, is turned through an angle x. Hence, the operation indicated by $\cos \frac{2\pi}{n} + (-)^{\frac{1}{2}} \sin \frac{2\pi}{n}$ is the same as that indicated by $(+)^{\frac{1}{n}}$, the difference being, that, in the former, we refer to rectangular, in the latter to polar co-ordinates. Mr. Peacock has made use of the expression $\cos x + (-)^{\frac{1}{2}} \sin x$ to represent direction, while Mr. Warren has employed one which, though disguised under an inconvenient and arbitrary notation, is the same as $(+)^{\frac{1}{n}}$. The connection between these expressions is so intimate, that, being subject to the same laws, they may be used indifferently the one for the other. This has been the case most particularly in the theory of equations. The most general form of the root is usually expressed by $a\{\cos\theta + (-)^{\frac{1}{2}} \sin\theta\}$, while the more correct symbolical form would be $(+)^{\frac{p}{q}} a$, since the expression

$$x^n + P_1 x^{n-1} + P_2 x^{n-2} + \&c. + P_n = 0$$

does not involve any sine or cosine, but may be considered as much a function of $+$ as of x, so that the former symbol may be easily supposed to be involved in the root. Hence, instead of the theorem that every equation must have a root, I would say every equation must have a root of the form $(+)^{\frac{p}{q}} a$, p and q being numbers, and a a distributive and commutative function.

ON THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.*

THE following method of integrating linear differential equations deserves attention, not only as leading readily to the solution of these equations, but also as placing their theory in a clear light, and pointing out the cause of the success of the method usually employed.

M. Brisson appears to have been the first person who applied the principle of the separation of the signs of operation from those of quantity to the solution of differential equations. This he did in two memoirs of the dates of 1821 and 1823, but we have not been fortunate enough to meet with them, (if indeed they have been published), and our knowledge of them is derived from a casual notice in a memoir of Cauchy on the same subject in his *Exercices*, Vol. II., p. 159. This last author seems to have pursued a different course from Brisson; and as it does not appear to be the best for putting the subject in a clear light, we have taken the liberty of deviating very considerably from his method, and in so doing we have probably approached nearer to that of Brisson.

* *Cambridge Mathematical Journal*, Vol. I., p. 22.

If we take the general linear equation with constant coefficients

$$\frac{d^n y}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} + B\frac{d^{n-2}y}{dx^{n-2}} + \ldots + R\frac{dy}{dx} + Sy = X,$$

when X is any function of x, and separate the signs of operation for those of quantity, it becomes

$$\left(\frac{d^n}{dx^n} + A\frac{d^{n-1}}{dx^{n-1}} + B\frac{d^{n-2}}{dx^{n-2}} + \ldots + R\frac{d}{dx} + S\right) y = X.$$

The quantity within the brackets, involving only constants and the signs of operation, may be considered as one operation performed on y, and it may be represented by

$$f\left(\frac{d}{dx}\right) y = X.$$

Here y is given at once explicitly if we are able to perform the inverse operation of $f\left(\frac{d}{dx}\right)$. For if we represent the inverse operation by the usual symbol $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$, and perform that operation on both sides, we get

$$\left\{f\left(\frac{d}{dx}\right)\right\}^{-1} \cdot f\left(\frac{d}{dx}\right) y = \left\{f\left(\frac{d}{dx}\right)\right\}^{-1} X,$$

or,

$$y = \left\{f\left(\frac{d}{dx}\right)\right\}^{-1} X.$$

It is plain that in its general form we cannot easily perform the inverse operation $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$; but if we begin with a simple case we shall be easily led to a means of effecting it.

Let us take the equation

$$y + \frac{dy}{dx} = ax^n,$$

or,

$$\left(1 + \frac{d}{dx}\right) y = ax^n.$$

Now the inverse operation of $\left(1+\frac{d}{dx}\right)$ is $\left(1+\frac{d}{dx}\right)^{-1}$. Therefore

$$y=\left(1+\frac{d}{dx}\right)^{-1} ax^n.$$

But as in integration there must be added an arbitrary constant which vanishes by differentiation, so here we must add a function which will vanish when the operation $\left(1+\frac{d}{dx}\right)$ is performed on it. This complementary function may be found from that condition, but the following more direct method is perhaps preferable. Since the result of the operation $1+\frac{d}{dx}$ on the function is 0, we may put the value of y under the form

$$y=\left(1+\frac{d}{dx}\right)^{-1} ax^n+\left(1+\frac{d}{dx}\right)^{-1} 0.$$

Now if we treat the symbols of operation as if they were symbols of quantity, we have

$$\left(1+\frac{d}{dx}\right)^{-1} 0=\frac{d^{-1}}{dx^{-1}}\left(1+\frac{d^{-1}}{dx^{-1}}\right)^{-1} 0.$$

But $\frac{d^{-1}}{dx^{-1}}$ is the same as $\int dx$. Hence

$$\left(1+\frac{d}{dx}\right)^{-1} 0=\left(1+\frac{d^{-1}}{dx^{-1}}\right)^{-1} C,$$

(C being the arbitrary constant arising from the integration)

$$=\left(1-\frac{d^{-1}}{dx^{-1}}+\frac{d^{-2}}{dx^{-2}}-\ldots\right) C;$$

or, performing the operations indicated,

$$=C\left(1-x+\frac{x^2}{1.2}-\frac{x^3}{1.2.3}+\ldots\right)=C\varepsilon^{-x}.$$

Hence $$y=\left(1+\frac{d}{dx}\right)^{-1} ax^n+C\varepsilon^{-x}.$$

Now expanding the first term

$$y=\left(1-\frac{d}{dx}+\frac{d^2}{dx^2}-\frac{d^3}{dx^3}+\dots\right)ax^n+C\varepsilon^{-x}.$$

Therefore

$$y=a\{x^n-nx^{n-1}+n(n-1)x^{n-2}-\dots\}+C\varepsilon^{-x}.$$

As the operation $\left(1+\frac{d}{dx}\right)^{-1}$ frequently occurs in these equations it is convenient to recollect that we must always add the function $C\varepsilon^{-x}$. And in the same way it would be seen if the operation be $\left(a+\frac{d}{dx}\right)^{-1}$ the complementary function is $C\varepsilon^{-ax}$, and similarly for all Binomial symbols of operation of this kind.

Equations of the first degree, when the coefficients of y and $\frac{dy}{dx}$ are functions of x, are easily reduced to this case by a change of the independent variable. Let us take as an example the equation

$$\frac{dy}{dx}+\frac{ny}{\sqrt{(1+x^2)}}=a,$$

or

$$\left\{1+\frac{\sqrt{(1+x^2)}}{n}\frac{d}{dx}\right\}y=\frac{a\sqrt{(1+x^2)}}{n}.$$

Let $\frac{ndx}{\sqrt{(1+x^2)}}=dt$, therefore $\frac{t}{n}=\log\{x+\sqrt{(1+x^2)}\}$;

whence

$$\sqrt{(1+x^2)}=\tfrac{1}{2}(\varepsilon^{\frac{t}{n}}+\varepsilon^{-\frac{t}{n}}),$$

and the equation becomes

$$\left(1+\frac{d}{dt}\right)y=\frac{a}{2n}(\varepsilon^{\frac{t}{n}}+\varepsilon^{-\frac{t}{n}});$$

therefore

$$y=\left(1+\frac{d}{dt}\right)^{-1}\frac{a}{2n}(\varepsilon^{\frac{t}{n}}+\varepsilon^{-\frac{t}{n}})+c\varepsilon^{-t};$$

or, expanding the first term,

$$y = \frac{a}{2n}\left(1 - \frac{d}{dt} + \frac{d^2}{dt^2} - \&c.\right)\left(\varepsilon^{\frac{t}{n}} + \varepsilon^{-\frac{t}{n}}\right) + c\varepsilon^{-t}$$

$$= \frac{a}{2n}\left(1 - \frac{1}{n} + \frac{1}{n^2} - \&c.\right)\varepsilon^{\frac{t}{n}}$$

$$+ \frac{a}{2n}\left(1 + \frac{1}{n} + \frac{1}{n^2} - \&c.\right)\varepsilon^{-\frac{t}{n}} + c\varepsilon^{-t};$$

therefore $$y = \frac{a}{2(n+1)}\varepsilon^{\frac{t}{n}} + \frac{a}{2(n-1)}\varepsilon^{-\frac{t}{n}} + c\varepsilon^{-t},$$

or substituting for t its value in terms of x,

$$y = \frac{a}{2(n+1)}\{\sqrt{(1+x^2)} + x\} + \frac{a}{2(n-1)}\{\sqrt{(1+x^2)} - x\} + c\{\sqrt{(1+x^2)} + x\}^n.$$

It is needless to multiply examples, as the principle of the method in the case of equations of the first order is sufficiently obvious from those given. But we will proceed to prove a theorem which is very useful, particularly in equations of the higher orders. The theorem is, that

$$\left(\frac{d}{dx} \pm a\right)^n X = \varepsilon^{\mp ax}\left(\frac{d}{dx}\right)^n \varepsilon^{\pm ax} X.$$

For, if we expand the first side, we have

$$\left(\frac{d}{dx} \pm a\right)^n X = \left\{\frac{d^n}{dx^n} \pm na\frac{d^{n-1}}{dx^{n-1}} + \frac{n(n-1)}{1.2}a^2\frac{d^{n-2}}{dx^{n-2}} \pm \&c.\right\} X.$$

Now $$\pm a^p = \varepsilon^{\mp ax}\left(\frac{d}{dx}\right)^p \varepsilon^{\pm ax},$$

so that the second side may be put under the form

$$\varepsilon^{\mp ax}\left\{\frac{d^n}{dx^n} + n\frac{d^{n-1}}{dx^{n-1}}\cdot\frac{d'}{dx} + \frac{n(n-1)}{1.2}\frac{d^{n-2}}{dx^{n-2}}\frac{d'^2}{dx^2} + \&c.\right\}\varepsilon^{\pm ax} X,$$

(where the accented letters refer to $\varepsilon^{\pm ax}$, and the unaccented to X), and this is equivalent to

$$\varepsilon^{\mp ax}\left(\frac{d}{dx} + \frac{d'}{dx}\right)^n \varepsilon^{\pm ax} X = \varepsilon^{\mp ax}\left(\frac{d}{dx}\right)^n \varepsilon^{\pm ax} X,$$

by the Theorem of *Leibnitz*.

When $X = \varepsilon^{mx}$, the proposition takes the form

$$\left(\frac{d}{dx} \pm a\right)^n \varepsilon^{mx} = (m \pm a)^n \varepsilon^{mx}.$$

By this theorem all operations of the nature of $\left(\frac{d}{dx} \pm a\right)^n$ are reduced to differentiation, or, as in the cases to which we have generally to apply it n is negative, to integration.

To return now to the general equation which we represented by

$$f\left(\frac{d}{dx}\right) y = X.$$

The inverse operation of $f\left(\frac{d}{dx}\right)$ cannot easily be performed directly, but we conceive the operation $f\left(\frac{d}{dx}\right)$ to be made up by the combination of n binomial operations of the form of $\left(\frac{d}{dx} - a\right)$; and, by what we have shown before, we can perform the inverse operation for each of these successively, and this will be equivalent to performing the whole inverse operation of $f\left(\frac{d}{dx}\right)$ at once. For, treating the operation $\frac{d}{dx}$ exactly as if it were a function of x of the same form, we can resolve it into factors, so that it becomes

$$\left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right)\left(\frac{d}{dx} - a_3\right) \&c. \left(\frac{d}{dx} - a_n\right),$$

where a_1, a_2, a_3, &c. are the roots of the equation

$$f(z) = 0.$$

Hence the equation $f\left(\frac{d}{dx}\right) y = X$ becomes

$$\left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right)\left(\frac{d}{dx} - a_3\right) \&c. \left(\frac{d}{dx} - a_n\right) y = X.$$

Now, performing the inverse operation of $\left(\frac{d}{dx} - a_1\right)$, we have

$$\left(\frac{d}{dx} - a_2\right)\left(\frac{d}{dx} - a_3\right) \&c. \left(\frac{d}{dx} - a_n\right) y = \left(\frac{d}{dx} - a_1\right)^{-1} X$$
$$= \varepsilon^{a_1 x} \int \varepsilon^{-a_1 x} X dx,$$

by the theorem prefixed, since in this case $n = -1$.

We should properly add a term $\left(\frac{d}{dx}-a_1\right)^{-1}0 = c\varepsilon^{a_1x}$, but as we may suppose the arbitrary constant to be included in the sign of integration, we may leave out this term for the sake of brevity.

Again, performing the inverse operation of $\left(\frac{d}{dx}-a_2\right)$, we have

$$\left(\frac{d}{dx}-a_3\right)\text{\&c.}\left(\frac{d}{dx}-a_n\right)y = \left(\frac{d}{dx}-a\right)^{-1}(\varepsilon^{a_1x}\int\varepsilon^{-a_1x}Xdx)$$

$$=\varepsilon^{a_2x}\int\varepsilon^{(a_1-a_2)x}\left(\int\varepsilon^{-a_1x}Xdx\right)dx.$$

Integrating by parts, this becomes

$$\left(\frac{d}{dx}-a_3\right)\text{\&c.}\left(\frac{d}{dx}-a_n\right)y = \frac{\varepsilon^{a_1x}\left(\int\varepsilon^{-a_1x}Xdx\right)}{a_1-a_2} - \frac{\varepsilon^{a_2x}\left(\int\varepsilon^{-a_2x}Xdx\right.}{a_1-a_2}$$

$$=\frac{\varepsilon^{a_1x}\left(\int\varepsilon^{-a_1x}Xdx\right)}{a_1-a_2} + \frac{\varepsilon^{a_2x}\left(\int\varepsilon^{-a_2x}Xdx\right)}{a_2-a_1}.$$

Performing the inverse operation of $\left(\frac{d}{dx}-a_3\right)$, we have

$$\left(\frac{d}{dx}-a_4\right)\text{\&c.}\left(\frac{d}{dx}-a_n\right)y$$

$$=\left(\frac{d}{dx}-a_3\right)^{-1}\left\{\frac{\varepsilon^{a_1x}\left(\int\varepsilon^{-a_1x}Xdx\right)}{a_1-a_2} + \frac{\varepsilon^{a_2x}\left(\int\varepsilon^{-a_2x}Xdx\right)}{a_2-a_1}\right\}$$

$$=\frac{\varepsilon^{a_3x}\int\varepsilon^{(a_1-a_3)x}\left(\int\varepsilon^{-a_1x}Xdx\right)dx}{a_1-a_2} + \frac{\varepsilon^{a_3x}\int\varepsilon^{(a_2-a_3)x}\left(\int\varepsilon^{-a_2x}Xdx\right)}{a_2-a_1}.$$

And integrating each of the terms separately by parts, we get, as before,

$$\frac{\varepsilon^{a_1x}\left(\int\varepsilon^{-a_1x}Xdx\right)}{(a_1-a_2)(a_1-a_3)} - \frac{\varepsilon^{a_3x}\left(\int\varepsilon^{-a_3x}Xdx\right)}{(a_1-a_2)(a_1-a_3)}$$

$$+\frac{\varepsilon^{a_2x}\left(\int\varepsilon^{-a_2x}Xdx\right)}{(a_2-a_1)(a_2-a_3)} - \frac{\varepsilon^{a_3x}\left(\int\varepsilon^{-a_3x}Xdx\right)}{(a_2-a_1)(a_2-a_3)}$$

$$=\frac{\varepsilon^{a_1x}\left(\int\varepsilon^{-a_1x}Xdx\right)}{(a_1-a_2)(a_1-a_3)} + \frac{\varepsilon^{a_2x}\left(\int\varepsilon^{-a_2x}Xdx\right)}{(a_2-a_1)(a_2-a_3)} + \frac{\varepsilon^{a_3x}\left(\int\varepsilon^{-a_3x}Xdx\right)}{(a_3-a_1)(a_3-a_2)},$$

and so on for every successive factor, so that at last

$$y = \frac{\varepsilon^{a_1 x}\left(\int \varepsilon^{-a_1 x} X dx\right)}{(a_1 - a_2)(a_1 - a_3)\dots(a_1 - a_n)} + \frac{\varepsilon^{a_2 x}\left(\int \varepsilon^{-a_2 x} X dx\right)}{(a_2 - a_1)(a_2 - a_3)\dots(a_2 - a_n)} + \&c.$$

$$+ \frac{\varepsilon^{a_n x}\left(\int \varepsilon^{-a_n x} X dx\right)}{(a_n - a_1)(a_n - a_2)\dots(a_n - a_{n-1})}.$$

We shall leave to the reader the application of the general method to particular cases, and shall proceed to show how some equations of an order higher than the first, may be conveniently solved without operating with each factor separately.

For instance, if we take the example of the equation

$$y + \frac{d^2y}{dx^2} = a \cos mx,$$

or

$$\left(1 + \frac{d^2}{dx^2}\right) y = a \cos mx;$$

therefore $y = \left(1 + \dfrac{d^2}{dx^2}\right)^{-1} a \cos mx + \left(1 + \dfrac{d^2}{dx^2}\right)^{-1} 0.$

$$\begin{aligned}
\text{Now, } \left(1 + \frac{d^2}{dx^2}\right)^{-1} 0 &= \frac{d^{-2}}{dx^{-2}}\left(1 + \frac{d^{-2}}{dx^{-2}}\right) 0 \\
&= \left(1 + \frac{d^{-2}}{dx^{-2}}\right)(c_1 x + c_2) \\
&= \left(1 - \frac{d^{-2}}{dx^{-2}} + \frac{d^{-4}}{dx^{-4}} - \&c.\right)(c_1 x + c_2) \\
&= c_1\left(x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2\dots5} - \&c.\right) \\
&\quad + c_2\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.\right) \\
&= c_1 \sin x + c_2 \cos x.
\end{aligned}$$

Also, $\left(1 + \dfrac{d^2}{dx^2}\right)^{-1} a \cos mx = a\left(1 - \dfrac{d^2}{dx^2} + \dfrac{d^4}{dx^4} - \&c.\right) \cos mx$

$$= a\,(1 + m^2 + m^4 + \&c.) \cos mx = \frac{a}{1 - m^2} \cos mx;$$

whence $y = \dfrac{a}{1 - m^2} \cos mx + c_1 \sin x + c_2 \cos x.$

Again, if we have the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2,$$

where the binomial factors of operation are equal, it may be put under the form

$$\left(\frac{d}{dx} - 2\right)^2 y = x^2,$$

whence $$y = \left(\frac{d}{dx} - 2\right)^{-2} x^2 + \left(\frac{d}{dx} - 2\right)^{-2} 0.$$

Now $$\left(\frac{d}{dx} - 2\right)^{-2} x^2 = \left(2 - \frac{d}{dx}\right)^{-2} x^2$$

$$= \left(2^{-2} + 2.2^{-3}\frac{d}{dx} + 3.2^{-4}\frac{d^2}{dx^2} + \&c.\right) x^2 = \frac{x^2}{2^2} + \frac{4x}{2^3} + \frac{6}{2^4}.$$

Also,

$$\left(\frac{d}{dx} - 2\right)^{-2} 0 = \frac{d^{-2}}{dx^{-2}}\left(1 - 2\frac{d^{-1}}{dx^{-1}}\right)^{-2} 0 = \left(1 - 2\frac{d^{-1}}{dx^{-1}}\right)^{-2}(cx + c_1)$$

$$= \left(1 + 2.2\frac{d^{-1}}{dx^{-1}} + 3.2^2\frac{d^{-2}}{dx^{-2}} + 4.2^3\frac{d^{-3}}{dx^{-}} + \&c.\right)(cx + c_1)$$

$$= c\left(x + 2.\frac{2x^2}{1.2} + 3.\frac{2^2x^3}{1.2.3} + \&c.\right)$$

$$+ c_1\left(1 + 2.2x + 3.2^2\frac{2^2x^2}{1.2} + \&c.\right)$$

$$= cx\left(1 + 2x + \frac{2^2x^2}{1.2} + \frac{2^3x}{1.2.3} + \&c.\right)$$

$$+ c_1\left(1 + 2x + \frac{2^2x^2}{1.2} + \&c.\right) + 2c_1x\left(1 + 2x + \frac{2^2x^2}{1.2} + \&c.\right)$$

$$= (c_1 + c_2x)\,\varepsilon^{2x}, \text{ if } c + 2c_1 = c_2;$$

therefore we have

$$y = \frac{x^2}{2^2} + \frac{4x}{2^3} + \frac{6}{2^4} + (c_1 + c_2x)\,\varepsilon^{2x}.$$

We might have omitted the latter part of this example, as it is easy to show, in the usual manner, what is the form of the complementary function when the two factors are equal, but we perferred the method given, as shewing how we may arrive at the same result directly.

On looking back on the method pursued, it is easy to see the causes of some of the known peculiarities in the usual solution of linear differential equations with constant coefficients. In the first place, their solution is attended with greater facility than that of other differential equations, because in fact y is given *explicitly* at once. In the next place, the exponential function which is assumed for the solution of these equations, is derived from the binomial factors of operation, $\left(a_1 - \frac{d}{dx}\right)$ &c.; and as there are n factors in an equation of the n^{th} order, there will be n exponential functions in the complete solution. Lastly, the equation

$$f\left(\frac{d}{dx}\right) y = X + 0$$

may be derived from the equation

$$f\left(\frac{d}{dx}\right) y = 0,$$

by differentiation only; for in operating with each factor of the form $\left(\frac{d}{dx} - a\right)^{-1}$ on X, we have only to expand according to powers of $\frac{d}{dx}$, and perform the operations indicated, and then add a term which must be the same as the term arising from the corresponding operation in the equation

$$f\left(\frac{d}{dx}\right) y = 0.$$

The application of this method to linear differential equations with variable coefficients is attended with considerable difficulty, and indeed neither Brisson nor Cauchy seem to have made any progress in the solution of these equations. There are, however, some which can be thus integrated, but we shall defer to a future number any observations we have to make on them, as well as the application of the same method to equations of finite and mixed differences, in which it is probably more useful than in differential equations.

But, before leaving the subject, we would say a few words on the legitimacy of the processes employed in this method. In the preceding pages we have spoken of treating the symbols of operation like those of quantity, so that at first sight it would appear as if the principles on which the method is founded, were drawn only from analogy. But a little consideration will show that this is not really the case, and that the reasoning on which we proceed is perfectly strict and logical. We have spoken as if there were a distinction between what are usually called symbols of operation, and those which are called symbols of quantity. But we might with perfect propriety call these last also symbols of operation. For instance, x is the operation designated by (x) performed on unity, x^n is the same operation performed n times in succession on unity, $a+x$ is the operation $(a+x)$ performed on unity, $(a+x)^n$ is the operation $(a+x)$ performed n times in succession on unity. By the phrase "in succession" is to be understood, that the operations are performed, so to speak, successively one on the back of the other; and perhaps it would be better to say, that the operation (x) is repeated n times on unity. And in this x^n is to be distinguished from nx, which represents that n of the operations (x) on unity are taken simultaneously. In the same way as $a\,(1)$ represents the operation (a) performed on (1), $a\,(x)$ would represent the same operation performed on x, and $a^n\,(x)$ would represent the operation repeated n times on (x). These operations are usually written ax, $a^n x$.

If, then, we take this view of what are usually called symbols of quantity, we shall have little difficulty in seeing the correctness of the principle by which other operations, such as we represent by $\left(\frac{d}{dx}\right)$, (Δ), &c., are treated in the same way as a, b, &c. For whatever is proved of the latter symbols, from the known laws of their combination, must be equally true of all other symbols which are subject to the same laws of combination. Now the laws of the combinations of the symbols a, b, &c. are, that

$$a^m . a^n x = a^{m+n} . x \quad \text{.................. (1)},$$

$$a \{b (x)\} = b \{a (x)\} \quad \text{..................(2)},$$

and
$$a (x) + a (y) = a (x + y) \quad \text{................ (3)}.$$

And, if f, f_1, &c. be any other general symbols of operation (f and f_1 being of the same kind) subject to the same laws of combination, so that

$$f^m . f^n (x) = f^{(m+n)} (x) \quad \text{................ (1)},$$

$$f \{f_1 (x)\} = f_1 \{f (x)\} \quad \text{..................(2)},$$

and
$$f (x) + f (y) = f (x + y) \quad \text{..................(3)}.$$

Then, whatever we may have proved of a, b, &c. depending on these three laws, must necessarily be equally true of f, f_1, &c.

Now we know that the symbol (d) is subject to these laws for

$$d^m . d^n (x) = d^{(m+n)} (x)$$

$$\frac{d}{dx} \left\{ \frac{d}{dy} (z) \right\} = \frac{d}{dy} \left\{ \frac{d}{dx} (z) \right\} \quad \text{................ (2)}$$

$$d (x) + d (y) = d (x + y),$$

and the same is true for the symbol Δ.

Hence the binomial theorem (to take a particular case) which has been proved for (a) and (b) is equally true for $\left(\frac{d}{dx}\right)$ and $\left(\frac{d}{dy}\right)$: so that we require no farther proof for the proposition, that when u is a function of two independent variables x and y,

$$d^n (u) = \left(\frac{d}{dx} dx + \frac{d}{dy} dy \right)^n u = \frac{d^n u}{dx^n} dx^n + n \frac{d^{n-1}}{dx^{n-1}} \frac{d}{dy} u dx^{n-1} dy + \ldots$$

But this reasoning will not apply in the case of those functions where the same laws do not hold. For instance, if we take the function log, we have not the condition

$$\log(x) + \log(y) = \log(x+y).$$

But $$\log(x) + \log(y) = \log(xy).$$

Consequently the binomial theorem will not hold for this function, though a binomial theorem might possibly be deduced for it, if the expressions did not become so complicated as to be unmanageable.

We have as yet only considered the combinations of operations of one kind, but in the preceding pages we frequently made use of operations of different kinds together, as in the expression $\left(\frac{d}{dx} - a\right)$. Now so long as each of the operations is subject to the same laws, and that they are independent, that is to say, that the one symbol is not supposed to act on the other, the same deductions will follow as when the operations are of the same kind. Hence we assumed that as the expression

$$x^n + Ax^{n-1} + Bx^{n-2} + \&c. + S$$

can be resolved into the factors

$$(x - a_1)(x - a_2)(x - a_3) \&c.,$$

the expression

$$\frac{d^n}{dx^n} + A\frac{d^{n-1}}{dx^{n-1}} + B\frac{d^{n-2}}{dx^{n-2}} + \&c. + S$$

can be resolved into the factors

$$\left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right)\ldots\left(\frac{d}{dx} - a_n\right),$$

which is the foundation of the method we have explained.

But if we have united together such symbols as $\left(\frac{d}{dx} + x\right)$, the same result will not hold. For though (x) is an operation of the same kind as (a), yet it bears a different relation to $\left(\frac{d}{dx}\right)$, as by the nature of this last operation it affects the operation (x), so that

$$x\left\{\frac{d}{dx}(z)\right\} \text{ is not equal to } \frac{d}{dx}\{x(z)\},$$

or the second law of combination does not hold with regard to these symbols of operation, and, consequently, theorems for other symbols deduced from this law are not true for such symbols as $\left(\frac{d}{dx}\right)$ and (x) together. It is this peculiarity with regard to the combinations of the symbols (x) and $\frac{d}{dx}$ which gives rise to the difficulty in the solution of linear equations with variable coefficients.

Since this article was written, we have learnt that a report by *Cauchy* on *Brisson's* Memoirs, which appears to have been favourable, was rejected by the Academy of Sciences. We know not for what reason.

ON THE SOLUTION OF CERTAIN TRIGONOMETRICAL EQUATIONS.*

THERE are several trigonometrical equations whose roots can be readily obtained by taking into consideration their connection with the general binomial equation whose last term is unity. If for example we have the equation

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos(n-1)\,\theta = 0,$$

we can find its roots by means of the equation

$$x^{2n} - 1 = 0.$$

We know that the roots of this last equation are of the form

$$\pm 1,\ \cos\phi \pm \sqrt{(-1)}\sin\phi,\ \cos 2\phi \pm \sqrt{(-1)}\sin 2\phi \ldots$$
$$\cos(n-1)\,\phi \pm \sqrt{(-1)}\sin(n-1)\,\phi.$$

Now as the equation wants the second term, the sum of its roots must be 0: and as the possible and impossible parts do not affect each other, they must be separately equal to 0; also, as the roots $+1$ and -1 destroy each other, there remains

$$\cos\phi + \cos 2\phi + \cos 3\phi + \ldots \cos(n-1)\,\phi = 0.$$

Comparing this with the original equation, we see that the former will be satisfied by making $\theta = \phi$. But to determine ϕ, we have the equation

$$\{\cos\phi + \sqrt{(-1)}\sin\phi\}^{2n} - 1 = 0,$$

or

$$\cos 2n\phi + \sqrt{(-1)}\sin 2n\phi = 1.$$

* *Cambridge Mathematical Journal*, Vol. I., p. 44.

Whence, as the possible and impossible parts are independent,

$$\cos 2n\phi = 1, \quad \sin 2n\phi = 0;$$

which give
$$2n\phi = 2m\pi,$$

or
$$\phi = \frac{m\pi}{n},$$

where m has any integer value from 0 to n, making in all $n+1$ values of ϕ. But two of these cannot be taken as values of θ, as we excluded the roots $+1$ and -1, which correspond to the values 0 and n of m. So that θ will be found from the equation

$$\theta = \frac{m\pi}{n},$$

where m has any value from 1 to $n-1$, making in all $n-1$ values of θ which answer the given equation. In exactly the same way we might shew how to solve the equation

$$\cos\theta + \cos 3\theta + \cos 5\theta + \ldots + \cos(2n-1)\,\theta = 0.$$

For its roots would be deduced from the equation

$$\{\cos\theta + \sqrt{(-1)}\,\sin\theta\}^{2n} + 1 = 0,$$

or
$$\cos 2n\theta + \sqrt{(-1)}\,\sin 2n\theta = -1;$$

which gives
$$\cos 2n\theta = -1, \quad \sin 2n\theta = 0.$$

Therefore
$$2n\theta = (2m+1)\,\pi,$$

and
$$\theta = \frac{(2m+1)}{2n}\pi;$$

where m has any value from 0 to $n-1$, giving on the whole n values of θ which satisfy the equation.

Similarly, the equations

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = 0,$$

and
$$\cos\theta + \cos 3\theta + \ldots + \cos(2n-1)\,\theta = 1,$$

may be solved by means of the equations

$$x^{2n+1} - 1 = 0,$$

and
$$x^{2n+1} + 1 = 0.$$

For the first we shall have

$$\theta = \frac{2m\pi}{2n+1},$$

where m has n values from 1 to n.

For the second we shall have

$$\theta = \frac{2m+1}{2n+1}\pi,$$

where m has n values from 0 to $n-1$.

The same method may be extended to other equations. For the roots of $x^{2n} - 1 = 0$ being of the form

$$\cos\phi + \sqrt{(-1)}\sin\phi,$$

if we multiply each by $\cos a + \sqrt{(-1)}\sin a$, it becomes

$$\cos(a+\phi) + \sqrt{(-1)}\sin(a+\phi).$$

But the sum of the roots will still remain equal to 0 when multiplied by $\cos a + \sqrt{(-1)}\sin a$, and taking away the terms which destroy each other, there will remain

$$\cos(a+\phi) + \cos(a+2\phi) + \cos(a+3\phi) + \ldots + \cos\{a+(n-1)\phi\} = 0,$$

consequently the equation

$$\cos(a+\theta) + \cos(a+2\theta) + \ldots + \cos\{a+(n-1)\theta\} = 0,$$

will be satisfied by the same values of θ as the first of the given equations; and similarly we might proceed with the others.

EVOLUTE TO THE ELLIPSE.*

THE equation to the evolute of the ellipse may be found very readily by considering it as the locus of the ultimate intersection of consecutive normals.

Let $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be the equation to the ellipse. Then the equation to a normal passing through a point x, y, will be

$$\frac{b^2(\beta - y)}{y} - \frac{a^2(\alpha - x)}{x} = 0,$$

or $$\frac{b^2\beta}{y} - \frac{a^2\alpha}{x} = a^2 - b^2 \ldots\ldots\ldots\ldots\ldots (2),$$

where α and β are the coordinates of the normal itself. To find the locus of the ultimate intersection of the normals, we must differentiate considering α and β as constant, x and y as variable. We then have from equations (1) and (2)

$$\frac{xdx}{a^2} + \frac{ydy}{b^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\frac{b^2\beta dy}{y^2} - \frac{a^2\alpha dx}{x^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$\lambda(3) + (4)$ gives, on equating to 0 the coefficients of each differential,

$$\frac{\lambda x}{a^2} = -\frac{a^2\alpha}{x^2} \cdot \frac{\lambda y}{b^2} = \frac{b^2\beta}{y^2} \cdot$$

* *Cambridge Mathematical Journal*, Vol. I., p. 47.

Multiply the first of these by x, and the second by y, and add.

Then $\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\lambda=\frac{b^2\beta}{y}-\frac{a^2\alpha}{x}=a^2-b^2.$

Substituting

$$\frac{a^2-b^2}{a^2}x=-\frac{a^2\alpha}{x^2}\cdot\frac{a^2-b^2}{b^2}y=\frac{ba^2\beta}{y^2}.$$

Therefore $\frac{x^3}{a^3}=-\frac{a\alpha}{a^2-b^2}\cdot\frac{y^3}{b^3}=\frac{b\beta}{a^2-b^2},$

and these values of $\frac{x}{a}$ and $\frac{y}{b}$ being substituted in the equation to the ellipse, give

$$a^{\frac{2}{3}}\alpha^{\frac{2}{3}}+b^{\frac{2}{3}}\beta^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}.$$

ON THE SOLUTION OF LINEAR EQUATIONS OF FINITE AND MIXED DIFFERENCES.*

In the preceding Number† of the *Mathematical Journal* the principle of the separation of the symbols of operation from those of quantity, was applied to the solution of Differential Equations. We shall here apply the same principle to equations of Finite and Mixed Differences; but, as the method is not very different from that previously delivered, there will be no necessity for dwelling long on it.

The general form of the linear equation of Finite Differences with constant coefficients, is

$$u_{x+n} + Au_{x+n-1} + \&c. + Ru_{x+1} + Su_x = X,$$

where A, B, C, &c. are constants, and X a function of x. By a known relation we can express the quantities u_{x+n}, u_{x+n-1}, &c. in terms of u_x, and its successive differences, so that the equation may be transformed into one which may be put under the form

$$f(\Delta)\, u_x = X;$$

which we might proceed to solve by means of the separation of the symbols. But it will be more convenient to proceed in the following manner. Since we have, generally,

$$u_{x+n} = (1+\Delta)^n u_x,$$

we may consider $1 + \Delta$ as a separate symbol, subject to the same laws as d and Δ, the effect of which, when applied

* *Cambridge Mathematical Journal*, Vol. I., p. 54.

† See p. 14 of this volume.

on u_x, is to convert u_x into u_{x+1}. If we represent, for the sake of shortness, $1+\Delta$ by D, the equation will take the form

$$(D^n + AD^{n-1} + \&c. + RD + S)\, u_x = X,$$

or, as we may write it for convenience,

$$F(D)\, u_x = X.$$

Then, converting $F(D)$ into factors of the form $D-a$, a being one of the roots of the equation $F(z)=0$, we can perform the inverse operation for each factor separately, and so arrive at the value of u_x. To do this more readily, we shall avail ourselves of a theorem similar to that given in page 18. This theorem is, that

$$(D-a)^n X = a^{n+x} \Delta^n (Xa^{-x}).$$

For if we expand the first side, it may be put under the form

$$a^n \left\{ D^n a^{-n} - nD^{n-1} a^{-(n-1)} + \frac{n(n-1)}{1.2} D^{n-2} a^{-(n-2)} - \&c. \right\} X.$$

Now $\qquad D^p a^{-x} = a^{-(x+p)}$, and $a^{-p} = a^x D^p a^{-x}$;

so that the expression may be put under the form

$$a^{n+x} (D^n D'^n - nD^{n-1} D'^{n-1} + \&c.)\, Xa^{-x},$$

where the accented letters refer to a^{-x} and the unaccented to X. This expression is equivalent to

$$a^{n+x} (DD' - 1)^n Xa^{-x}$$

$$= a^{n+x} (\Delta + \Delta' + \Delta\Delta')^n Xa^{-x} = a^{n+x} \Delta^n (Xa^{-x}),$$

by a theorem analogous to that of Leibnitz.

To apply this to the general equation $F(D)\, u_x = X$, suppose $F(D)$ to be resolved into binomial factors, such as $D-a$, so that it becomes

$$(D-a_1)(D-a_2)\ldots(D-a_x)\, u_x = X.$$

Then taking the inverse operation of $D-a$,

$$F'(D)\, u_x = (D-a_1)^{-1} X = a_1^{x-1} \Sigma (Xa_1^{-x}),$$

representing the product of the $n-1$ binomial factors by $F''(D)$, and omitting the arbitrary constant, as included in the sign of integration. Again, performing the inverse operation of $D-a_2$, we have

$$F''(D)\,u_x = a_2^{x-1}\Sigma\{a_1^{x-1}a_2^{-x}\Sigma(Xa_1^{-x})\}$$

$$= \frac{a_2^x}{a_1a_2}\Sigma\left\{\left(\frac{a_1}{a_2}\right)^x\Sigma(Xa_1^{-x})\right\},$$

and integrating this by parts,

$$F''(D)\,u_x = \frac{a_1^{x-1}}{a_1-a_2}\Sigma(Xa_1^{-x}) + \frac{a_2^{x-1}}{a_2-a_1}\Sigma(Xa_2^{-x}).$$

Proceeding in the same manner with the next factor,

$$F'''(D)\,u_x = \frac{a_3^{x-1}}{a_1-a_2}\Sigma\{a_1^{x-1}a_3^{-x}\Sigma(Xa_1^{-x})\}$$

$$+ \frac{a_3^{x-1}}{a_2-a_1}\Sigma\{a_2^{x-1}a_3^{-x}\Sigma(Xa_2^{-x})\}$$

$$= \frac{a_1^{x-1}}{(a_1-a_2)(a_1-a_3)}\Sigma(Xa_1^{-x}) + \frac{a_2^{x-1}}{(a_2-a_1)(a_2-a_3)}\Sigma(Xa_2^{-x})$$

$$+ \frac{a_3^{x-1}}{(a_3-a_1)(a_3-a_2)}\Sigma(Xa_3^{-x}),$$

after integration by parts and reduction. And in the same way we may proceed till all the binomial factors of operation are exhausted. It is obvious from this, how close an analogy this result bears with the corresponding one in Differential Equations.

If there be n equal roots, the theorem given above enables us to arrive directly at the solution. For in this case we should have

$$u_x = (D-a)^{-n}X = a^{x-n}\Sigma^n(Xa^{-x}),$$

or introducing the arbitrary constants, which were before neglected,

$$u_x = a^{x-n}\Sigma^n(Xa^{-x}) + a^{x-n}(C + C_1x + C_2x^2 + \ldots + C_nx^n).$$

Linear equations of the first degree, with variable coefficients, may be easily converted into equations with constant coefficients, by a change analogous to the change of the independent variable in the Differential Calculus. For let

$$u_{x+1} + aP_x u_x = X$$

be the given equation, where P_x is a function of x. We may assume $P_x = \frac{Q_{x+1}}{Q_x}$, so that dividing by Q_{x+1} the equation becomes

$$\frac{u_{x+1}}{Q_{x+1}} + a\frac{u_x}{Q_x} = \frac{X}{Q_x};$$

which may be solved as a linear equation in $\frac{u_x}{Q_x}$, from which, when found, we may determine u_x. The form of Q_{x+1} is evidently

$$P_1 P_2 \ldots P_x.$$

The same method may be sometimes applied to equations of a higher order. If, for instance, we have the equation

$$u_{x+2} + a\phi(x+1)\,u_{x+1} + b\phi(x)\,\phi(x+1)\,u_x = c.$$

Let us represent the continued product $\phi(1).\phi(2)\ldots\phi(x+1)$ by P_{x+1}. Then

$$\phi(x)\,\phi(x+1) = \frac{P_{x+1}}{P_{x-1}} \text{ and } \phi(x+1) = \frac{P_{x+1}}{P_x}.$$

Dividing by P_{x+1} we get

$$\frac{u_{x+2}}{P_{x+1}} + a\frac{u_{x+1}}{P_x} + b\frac{u_x}{P_{x-1}} = \frac{C}{P_{x+1}},$$

which may be solved as an equation in $\frac{u_x}{P_{x-1}}$, and so the value of u_x determined.

It will be seen that the method here laid down of integrating equations of differences, possesses an advantage over the common one of not requiring any assumption of a form of solution; but this advantage is even more displayed in solving equations of mixed differences. In these the two symbols d and Δ are involved at the same time; but as they are the characteristics of independent operations, the same principles which were applied to each separately will hold when they are combined. Without discussing the general equations of mixed differences, we shall proceed to a few of the more interesting examples. Let us have the equation

$$\frac{d}{dx}\Delta y + a\frac{dy}{dx} + b\Delta y + aby = X.$$

By separating the symbols this may be put under the form

$$\left(\frac{d}{dx} + b\right)(\Delta + a)\, y = X.$$

We shall obtain a different result according as we operate first with the one or the other factor; taking then the differential factor first, we have

$$(\Delta + a)\, y = \varepsilon^{-bx}\int X\varepsilon^{bx}dx + C\varepsilon^{-bx}.$$

Now $\qquad \Delta + a = 1 + \Delta - (1-a) = D - (1-a).$

Therefore, performing the inverse operation by the theorem given above, page 34, we obtain

$$y = (1-a)^{x-1}\Sigma\left\{(1-a)^{-x}\varepsilon^{-bx}\int(X\varepsilon^{bx}dx)\right\}$$
$$+ C(1-a)^{x-1}\Sigma\left\{\varepsilon^{-bx}(1-a)^{-x}\right\} + (1-a)^x\phi(\sin 2\pi x, \cos 2\pi x),$$

the last term being the complementary function arising from the operation $\{D-(1-a)\}^{-1}$.

And performing the operation indicated in the second term, we have

$$y = (1-a)^{x-1}\Sigma\left\{(1-a)^{-x}\varepsilon^{-bx}\int(X\varepsilon^{bx}dx)\right\}$$
$$+ C_1\varepsilon^{-bx} + (1-a)^x\phi(\sin 2\pi x, \cos 2\pi x).$$

If we begin with the other factor we have

$$\left(\frac{d}{dx} + b\right) y = (1-a)^{x-1}\Sigma\left\{X(1-a)^{-x}\right\} + (1-a)^x\phi,$$

and $\qquad y = \varepsilon^{-bx}\int dx\left[\varepsilon^{bx}(1-a)^{x-1}\Sigma\left\{X(1-a)^{-x}\right\}\right]$

$$+ \varepsilon^{-bx}\int dx\left\{\varepsilon^{bx}(1-a)^x\phi\right\} + C\varepsilon^{-bx},$$

where ϕ is put for $\phi(\sin 2\pi x, \cos 2\pi x)$.

The next example we shall take is one given by Paoli, where two variables are involved,

$$u_{x+1,y} - \frac{d}{dy} u_{x,y} = P_{x,y},$$

which may be put under the form

$$\left(D - \frac{d}{dy}\right) u_{x,y} = P_{x,y},$$

where the D refers to the x only.

Then, following the same principle as that pursued by Mr. Greatheed in the *Philosophical Magazine* for September, 1837, that is, considering the characteristic $\frac{d}{dy}$ as an independent quantity, we have

$$u_{x,y} = \left(\frac{d}{dy}\right)^{x-1} \Sigma \left(\frac{d}{dy}\right)^{-x} P_{x,y} + \left(\frac{d}{dy}\right)^{x} \phi(y),$$

an arbitrary function of y being substituted for a constant, as the operations are partial. This may evidently be put under the form

$$u_{x,y} = \frac{d^x \left(\Sigma \int^{x+1} P_{x,y} dy^{x+1}\right)}{dy^x} + \frac{d^x \phi(y)}{dy^x}.$$

If $P_{x,y} = 0$, the result is the same as that given by Sir John Herschel, in p. 38 of his *Examples*. He likewise gives the equation

$$u_{x+2,y} - a\left(\frac{d}{dy}\right) u_{x+1,y} + b\left(\frac{d^2}{dy^2}\right) u_{x,y} = 0,$$

which may be put under the form

$$\left(D^2 - aD\frac{d}{dy} + b\frac{d^2}{dy^2}\right) u_{x,y} = 0,$$

where D affects x only. Or, putting it into the form of factors,

$$\left(D - m\frac{d}{dy}\right)\left(D - n\frac{d}{dy}\right) u_{x,y} = 0,$$

where m, n are the roots of the equation

$$z^2 - az + b = 0.$$

Then $$\left(D - n\frac{d}{dy}\right) u_{x,y} = m^{x-1} \cdot \left(\frac{d}{dy}\right)^{x-1} \cdot \phi(y),$$

and $$u_{x,y} = n^{x-1} \cdot \left(\frac{d}{dy}\right)^{x-1} \Sigma \left\{ m^{x-1} . n^{-x} . \left(\frac{d}{dy}\right)^{x-1} . \left(\frac{d}{dy}\right)^{-x} . \phi(y) \right\}$$

$$+ n^{x-1} \left(\frac{d}{dy}\right)^{x-1} \phi_1(y);$$

whence $$u_{x,y} = \frac{m^{x-1} d^{x-1} \phi_2(y)}{dy^{x-1}} + \frac{n^{x-1} d^{x-1} \phi_1(y)}{dy^{x-1}},$$

as the functions of y are quite arbitrary.

In the same manner we might integrate this equation, if the second side were some function of x; and also equations of the form

$$\Delta^2 u_{x,y} - a\Delta \frac{d}{dy} u_{x,y} + b \frac{d^2}{dy^2} u_{x,y} = X,$$

where Δ affects x only. But it is needless to dwell on these, and we shall therefore proceed to the integration of an equation, in which the advantages of this method are very conspicuous. Mr. Airy, on the hypothesis that the distances of the particles of the luminiferous ether are not infinitely small compared with their sphere of action, has given the following equation for determining the disturbance of a particle:

$$\frac{d^2}{dt^2} u_{x,t} = \frac{a^2}{h^2} \Delta^2 u_{x-h,t},$$

where Δ affects x only. This may be put under the form

$$\left(\frac{d^2}{dt^2} - \frac{a^2}{h^2} \frac{\Delta^2}{1+\Delta}\right) u_{x,t} = 0.$$

Following the same method as before, by splitting the operating factor into two, and integrating with each separately, we obtain

$$u_{x,t} = \varepsilon^{\frac{a}{h} t \frac{\Delta}{\sqrt{(1+\Delta)}}} \phi(x) + \varepsilon^{-\frac{a}{h} t \frac{\Delta}{\sqrt{(1+\Delta)}}} \psi(x),$$

where $\phi(x)$ and $\psi(x)$ are arbitrary functions of x. To reduce this to a more intelligible form, we may avail ourselves of Fourier's formula for the transformation of functions. From it we have

$$\pi\phi(x)=\int_{-\infty}^{+\infty}d\alpha\phi(\alpha)\int_0^\infty dp\cos p(x-\alpha).$$

Substituting this form,

$$\pi u_{x,t}=\int_{-\infty}^{+\infty}d\alpha\phi(\alpha)\int_0^\infty dp\varepsilon^{\frac{a}{h}t\frac{\Delta}{\sqrt{(1+\Delta)}}}\cos p(x-\alpha)$$

$$+\int_{-\infty}^{+\infty}d\alpha\psi(\alpha)\int_0^\infty dp\varepsilon^{-\frac{a}{h}t\frac{\Delta}{\sqrt{(1+\Delta)}}}\cos p(x-\alpha).$$

We must now expand the exponential, and operate with each term separately on the cosine. When expanded it becomes

$$\left(1+\frac{a}{h}t\frac{\Delta}{\sqrt{(1+\Delta)}}+\frac{a^2h}{h^2 1.2}t^2\frac{\Delta^2}{1+\Delta}+\&c.\right)\cos p(x-\alpha).$$

Now

$$\frac{\Delta}{\sqrt{(1+\Delta)}}f(x)=\left\{\sqrt{(1+\Delta)}-\frac{1}{\sqrt{(1+\Delta)}}\right\}f(x)=f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right);$$

therefore $\frac{\Delta}{\sqrt{(1+\Delta)}}\cos p(x-\alpha)=-2\sin\frac{ph}{2}\sin p(x-\alpha),$

$$\frac{\Delta^2}{1+\Delta}\cos p(x-\alpha)=-\left(2\sin\frac{ph}{2}\right)^2\cos p(x-\alpha),$$

and so on for every repetition of the operation, so that the series becomes

$$\left\{1-\frac{a^2.h^2}{h^2.1.2}\left(2\sin\frac{ph}{2}\right)^2+\frac{a^4.t^4}{h^4.1.2.3.4}\left(2\sin\frac{ph}{2}\right)^4-\&c.\right\}\times$$

$$\times\cos p(x-\alpha)$$

$$-\left\{\frac{a}{h}\frac{t}{1}\left(2\sin\frac{ph}{2}\right)-\frac{a^3.t^3}{h^3.1.2.3}\left(2\sin\frac{ph}{2}\right)^3+\&c.\right\}\sin p(x-\alpha)$$

$$=\cos\left(\frac{2at}{h}\sin\frac{ph}{2}\right)\cos p(x-\alpha)-\sin\left(\frac{2at}{h}\sin\frac{ph}{2}\right)\sin p(x-\alpha).$$

Similarly for the other term we shall obtain

$$\cos\left(\frac{2at}{h}\sin\frac{ph}{2}\right)\cos p(x-\alpha)+\sin\left(\frac{2at}{h}\sin\frac{ph}{2}\right)\sin p(x-\alpha);$$

and if these expressions be substituted in the expression for $u_{x,t}$, we shall obtain a result free from the symbol Δ: but a slight change in the form of the function will reduce it to a somewhat better form; for if we make

$$f(\alpha)=\psi(\alpha)+\phi(\alpha) \text{ and } F(\alpha)=\psi(\alpha)-\phi(\alpha),$$

the expression becomes

$$\pi u_{x,t}=\int_{-\infty}^{+\infty} d\alpha f(\alpha)\int_0^{\infty} dp \cos\left(\frac{2at}{h}\sin\frac{ph}{2}\right)\cos p(x-\alpha)$$
$$+\int_{-\infty}^{+\infty} d\alpha F(\alpha)\int_0^{\infty} dp \sin\left(\frac{2at}{h}\sin\frac{ph}{2}\right)\sin p(x-\alpha),$$

which is the general solution of the equation. The forms of the arbitrary functions are easily determined from the initial circumstances. For if in the last expression, or in the original integral, we make $t=0$, we find

$$u_{x,0}=f(x),$$

which determines the form of the function f from the initial state of the particles. If we differentiate the expression for $u_{x,t}$ with regard to t, and then make $t=0$, there results

$$\frac{du_{x,0}}{dt}=\frac{a}{h}\left\{F\left(x-\frac{h}{2}\right)-F\left(x+\frac{h}{2}\right)\right\},$$

which determines the form of the function F from the initial velocity.

Mr. Airy has found a particular integral of this equation by assuming as a form of solution

$$u=A\sin\frac{2\pi}{\lambda}(vt-x+\alpha),$$

and then determining v. But his result may be arrived at, merely by assuming the form of the arbitrary function to be that used by Taylor in the solution of the problem of vibrating chords, and which is usually taken as the form of the function in the undulatory theory, and is for that reason well adapted for comparing the results. Let then

$$\phi(x)=\sin\frac{2\pi}{\lambda}(x+\alpha),\quad \psi(x)=0.$$

Effecting the operations indicated in the original integral by the same method as that used in reducing Fourier's function, we obtain

$$u = \sin\frac{2\pi}{\lambda}\left\{x + \frac{a\lambda}{\pi}\sin\left(\frac{\pi h}{\lambda}\right)t + a\right\}.$$

This expression gives us for the velocity of propagation of the wave, which is the coefficient of t,

$$v = \frac{a\lambda}{\pi}\sin\frac{\pi h}{\lambda},$$

which is no longer independent of λ, and therefore the velocity will be different for different values of λ; and as the index of refraction is inversely as the velocity, it will be nearly inversely as λ, and will thus be greater for violet than red light. This result then, as far as it goes, accounts for the phenomena of dispersion.

DEMONSTRATIONS OF SOME PROPERTIES OF THE CONIC SECTIONS.*

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1. THE position of the circle of curvature at any point of a Conic Section, may be readily determined by the following construction. Describe a circle touching the curve at the given point and cutting it in two others, then the chord in the conic section which passes through the given point and is parallel to the line joining the two points of section, is a chord of the circle of curvature at the given point.

Taking the origin at the point in the curve, the equation to the conic section is

$$y^2 + Bxy + Cx^2 + Dy + Ex = 0.$$

If we make the normal the axis of x and the tangent that of y, when $x = 0$ the two values of y must also $= 0$, and therefore $D = 0$, which reduces the equation to

$$y^2 + Bxy + Cx^2 + Ex = 0.$$

The equation to a circle touching the curve at the given point, that is, being also a tangent to the axis of y at that point, is

$$y^2 + x^2 + E'x = 0.$$

At the intersection of the curves we may combine the equations in any manner; subtracting them we have

$$Bxy + (C - 1)\,x^2 + (E - E')\,x = 0,$$

which gives $x = 0,\ By + (C - 1)\,x + E - E' = 0.$

* *Cambridge Mathematical Journal*, Vol. I., p. 61.

The first of these equations gives the origin, the second gives a relation between the coordinates of each of the points of section: and as it is linear it is the equation to the straight line joining them. Now the angle which this line makes with the axis of x is determined by the ratio $\frac{1-C}{B}$; and as this is independent of the circle, the line will remain parallel to one position, as the circle varies in size. But when the circle becomes the circle of curvature, one of the points of intersection coincides with the point of contact, and the line joining the points of intersection, which remains parallel to one position, must pass through the point of contact. Therefore, if from this point a line be drawn parallel to a known position of the line of intersection, it will pass through the point in which the circle of curvature cuts the curve, and will thus be a chord both in the curve itself and in the circle of curvature. Knowing now one chord of the circle of curvature, and the position of its diameter, which coincides with that of the normal, we can determine the circle altogether.

2. The following is another and a very curious method of determining the centre of curvature in the Conic Sections. It was first given by Keill, but seems to have been rather strangely neglected by the subsequent writers on this subject.

From any point P (fig. 2) in the curve draw the normal PN cutting the axis in N; at the point N draw NQ perpendicular to the normal, and meeting the focal chord through P in Q. From Q draw QO perpendicular to the focal chord and meeting the normal in O; then O is the centre of the circle of curvature at the point P. Draw FY from the focus perpendicular to the tangent, and let

$$FP = r, \quad FY = p, \quad PN = N.$$

In the right-angled triangle PQO, we have plainly

$$PO.PN = PQ^2.$$

But from the similar triangles QPN, FPY,

$$PQ.FY = FP.PN;$$

therefore $$PQ^2 = N^2 \frac{r^2}{p^2}, \text{ and } PO = N \frac{r^2}{p^2}.$$

Also in a conic section we have, if R be the radius of curvature,

$$R = \frac{N^3}{m^2} = \text{also } \frac{mr^3}{p^3},$$

where m is half the *latus rectum*.

From which we have $\frac{r^2}{p^2} = \frac{N^2}{m^2}$, and therefore

$$PO = \frac{N^3}{m^2} = R,$$

and O is the centre of the circle of curvature at P.

3. In the *Cambridge Transactions*, Vol. III., Mr. Morton has demonstrated a number of curious properties of the Conic Sections in relation to the generating cone; but he does not seem to have noticed the following one. If a sphere be described round the vertex of a cone as centre, the *latera recta* of all sections of the cone made by planes touching the sphere are equal. Taking the vertex of the cone as the origin, and the axis of the cone as the axis of x, the equation to the cone will be

$$x^2 + y^2 = m^2 z^2.$$

And if we suppose the cutting plane to be perpendicular to the plane of xz, its equation will be

$$z \cos\alpha + x \sin\alpha = r;$$

where r is the radius of the sphere, and α the angle which the perpendicular from the origin on the plane makes with the axis of z. Eliminating z between the two equations, we get

$$x^2(\cos^2\alpha - m^2\sin^2\alpha) + y^2\cos^2\alpha + 2m^2rx\sin\alpha = m^2r^2,$$

which is the equation to the projection of the section on the plane of xy. This equation, which is that to an ellipse, is not referred to its centre, but if we so refer it, it becomes

$$x^2(\cos^2\alpha - m^2\sin^2\alpha)^2 + y^2\cos^2\alpha(\cos^2\alpha - m^2\sin^2\alpha) = m^2r^2.\cos^2\alpha.$$

Now, if a', b' be the axes of the projection,

$$a'^2 = \frac{m^2r^2\cos^2\alpha}{(\cos^2\alpha - m^2\sin^2\alpha)^2}, \quad b'^2 = \frac{m^2r^2}{\cos^2\alpha - m^2\sin^2\alpha}.$$

If a, b be the axes of the section, as the cutting plane is perpendicular to the plane of xz, and makes an angle α with the plane of xy,

$$a' = a\cos\alpha, \quad b' = b.$$

And for the *latus rectum*,

$$\frac{2b^2}{a} = 2mr,$$

which being independent of α, is the same for all sections for which r is the same; that is, for all those which are made by planes touching the sphere.

From this it appears, that the *latus rectum* is equal to the diameter of the sphere multiplied by the tangent of half the vertical angle of the cone.

SOLUTIONS OF SOME PROBLEMS IN TRANSVERSALS.*

THE name of Transversals was given by Carnot to lines considered in their relations of mutual intersection. Many of their properties are very curious, and form interesting problems in Analytical Geometry, though it was not in this way that Carnot considered them. His method was, to proceed step by step from the more simple properties to the more complicated; but it seems better to consider each independently.

1. The following problem, under a slightly different form, was given in one of the Problem Papers for 1836.

If two lines AB, CD intersect in O so that AB is bisected, and if the lines AC, BD meet, when produced, in E, and AD, BC in F, then the line EF is parallel to AB.

Take O as the origin, and AB, CD as the axes of x and y.

Let $OB=a$, $OA=-a$, $OD=b$, $OC=-c$.

The equations to BD and AC are

$$\frac{x}{a}+\frac{y}{b}=1, \quad \frac{x}{a}+\frac{y}{c}=-1.$$

At their intersection the equations may be combined in any manner; therefore, subtracting them,

$$y\left(\frac{1}{b}-\frac{1}{c}\right)=2 \text{ ....(1).}$$

* *Cambridge Mathematical Journal*, Vol. I., p. 87.

The equations to BC and AD are

$$\frac{x}{a} - \frac{y}{c} = 1, \quad \frac{x}{a} - \frac{y}{b} = -1.$$

At their intersection, subtracting them,

$$y\left(\frac{1}{b} - \frac{1}{c}\right) = 2 \text{ .... } (2),$$

which is the same as the equation (1), and therefore is the equation to the line passing through the points of intersection, that is, to EF; and from its form it is evident that it is parallel to the axis of x, that is, to AB.

2. If three lines be drawn from the angles of a triangle A, B, C, (fig. 3), through one point O, and meeting the sides of the triangle in P, Q, R, the sides of the triangle P, Q, R will, when produced, meet those of A, B, C in three points, which are in the same straight line.

Take L, the point of intersection of AB and PQ, as the origin, and these lines as the axes of x and y.

Let $LB = a_1$, $LR = a_2$, $LA = a_3$, $LP = b_1$, $LQ = b_2$.

The equations to BC and QR are

$$\frac{x}{a_1} + \frac{y}{b_1} = 1, \quad \frac{x}{a_2} + \frac{y}{b_2} = 1.$$

At their intersection we have, by subtraction,

$$x\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + y\left(\frac{1}{b_1} - \frac{1}{b_2}\right) = 0 \text{ .... } (1).$$

Again, the equations to AC and PR are

$$\frac{x}{a_3} + \frac{y}{b_2} = 1, \quad \frac{x}{a_2} + \frac{y}{b_1} = 1;$$

and at their intersection we have, by subtraction,

$$x\left(\frac{1}{a_2} - \frac{1}{a_3}\right) + y\left(\frac{1}{b_1} - \frac{1}{b_2}\right) = 0 \text{ .... } (2).$$

Again, combining by addition the equations to BC and AC, we get at their point of intersection C,

$$x\left(\frac{1}{a_1}+\frac{1}{a_3}\right)+y\left(\frac{1}{b_1}+\frac{1}{b_2}\right)=2 \;\ldots\ldots\ldots\ldots(3).$$

Also, the equations to AP and BQ are

$$\frac{x}{a_3}+\frac{y}{b_1}=1,\quad \frac{x}{a_1}+\frac{y}{b_2}=1;$$

and at their intersection we have, by addition,

$$x\left(\frac{1}{a_1}+\frac{1}{a_3}\right)+y\left(\frac{1}{b_1}+\frac{1}{b_2}\right)=2 \;\ldots\ldots\ldots\ldots(4),$$

which is the same as (3), and therefore is the equation to COR, which passes through the two points of intersection.

If in this equation we make $y=0$, we have $x=\dfrac{2a_1a_3}{a_1+a_3}$, but in this case $x=LR=a_2$; therefore

$$a_2=\frac{2a_1a_3}{a_1+a_3},\ \text{or}\ \frac{1}{a_1}-\frac{1}{a_2}=\frac{1}{a_2}-\frac{1}{a_3};$$

therefore equations (1) and (2) coincide, and represent the line passing through M and N, and as from the form of the equation it evidently passes through the origin, the three points L, M, N are in one straight line.

In somewhat a similar manner it might be shown, that if three circles be touched, two and two, by pairs of tangents, the points of intersection of these tangents are in one straight line. But we shall pass on to another problem.

3. If from any point A (fig. 4) in a line of indefinite length equal distances AQ_1, Q_1Q_2, &c. be measured off, and other equal distances AP_1, P_1P_2, &c. in the opposite direction; and if Q_1, Q_2, Q_3, &c. be joined by straight lines with a point O_1, and P_1, P_2, P_3, &c. with a point O_2, O_1 and O_2 being situate in a line parallel to the given line; then the corresponding points of intersection of these lines will lie

in straight lines forming two bundles of lines converging to two points O', O'', situate in O_1O_2.

Take A as the origin, AO_1 as the axis of x, AQ as the axis of y.

Let $AO_1=m$, $O_1O_2=c$, $AQ=a$, $AP=b$.

Let one of the lines O_2P_n meet the axis of x in a point A_n; then by similar triangles it is easy to show that

$$\frac{1}{AA_n}=\frac{1}{m}+\frac{c}{mnb}.$$

The equation to O_1Q_r is

$$\frac{x}{m}+\frac{y}{ra}=1\text{.......................}(1),$$

and the equation to O_2P_s is

$$\frac{x}{m}+\frac{cx}{msb}-\frac{y}{sb}=1\text{ .... }(2).$$

Combining them by subtraction, after multiplying by r and s,

$$(s-r)\frac{x}{m}+\frac{cx}{mb}-y\left(\frac{1}{a}+\frac{1}{b}\right)=s-r\text{ ....}(3),$$

which gives a relation between the coordinates of the point of intersection of the lines. We shall arrive at the same result for all pairs of lines for which $s-r$ is the same; consequently equation (3) will represent a line passing through the point of intersection of each pair. Now, let $x=m$, then

$$y=\frac{ac}{a+b},$$

which, being independent of r and s, will be the same for all the lines represented by (3) for which $s-r$ is different, and therefore they all converge to one point, which it is easy to see lies between O_1 and O_2, as the value of y is less than that of c. If we make $x=0$ in (3), we get

$$y=\frac{r-s}{\frac{1}{a}+\frac{1}{b}}:$$

and as $s-r$ increases by unity, the distance between the points where the different lines diverging from O' cut the axis of y is half the harmonic mean between a and b.

If, instead of subtracting, we add equations (1) and (2), we get

$$(r+s)\frac{x}{m}+\frac{cx}{mb}+y\left(\frac{1}{a}-\frac{1}{b}\right)=r+s,$$

which, as before, may be shown to be the equation to a line passing through the intersections of all lines for which $r+s$ is the same. And if $r+s$ vary, we find, as before, that the different lines converge to a point O'', such that

$$O_1O''=-\frac{ac}{b-a}.$$

If $b>a$, O'' is situate in O_2O_1 produced. If $a=b$, O_1O'' becomes infinite, and the lines are then parallel to AQ. If $b<a$, the point O'' is above O_2. The distance between the points in which the lines diverging from O'' cut the axis of y is $\dfrac{1}{\frac{1}{a}-\frac{1}{b}}$. It is easy also to show, that all the lines measuring from the points of convergence are divided harmonically; the equation to any line, as O_2P_s, is

$$\frac{x}{m}+\frac{cx}{msb}-\frac{y}{sb}=1,$$

when $$y=0,\quad \frac{x}{m}\left(1+\frac{c}{sb}\right)=1;$$

whence $$\frac{1}{O_1A_s}=\frac{1}{m-x}=\frac{1}{m}+\frac{sb}{mc};$$

similarly, $$\frac{1}{O_1A_r}=\frac{1}{m}+\frac{rb}{mc},$$

and so on for the others; and as these are in arithmetic progression, O_1A_s, O_1A_r, &c. are in harmonic progression. Instead of taking O_1A as axis, we might have taken any other line, as O_1Q_s, and the result would be the same.

And in like manner we might proceed, taking O_2 as the origin, as also O', O'', and, as the cases are similar, we should get the same result. Hence the property holds for all the lines in the figure. For other problems in Transversals the reader is referred to Carnot's Memoir, or to a small work by Brianchon on the same subject, where he will find some practical applications of the theory.

MATHEMATICAL NOTE.*

THE area of the parallelogram formed by tangents applied at the extremities of any two conjugate diameters of an ellipse is constant.

Let x, y be the coordinates of the extremity of one diameter, and x', y' those of the other; θ, θ' the angles which they make with x. Then $x = a' \cos\theta$, $y = a' \sin\theta$, $x' = b' \cos\theta'$, $y' = b' \sin\theta'$.

The tangent whose inclination to the axis of x is θ, passes through the point $x'y'$: therefore, by the equation

$$y = \alpha x + (a^2\alpha^2 + b^2)^{\frac{1}{2}},$$

substituting $\tan\theta$ for α, and multiplying by $\cos\theta$,

$$y' \cos\theta - x' \sin\theta = \sqrt{\{(b \cos\theta)^2 + (a \sin\theta)^2\}}.$$

Multiply by a', and substitute for x' and y' their values in b' and θ', therefore

$$a'b' \sin(\theta' - \theta) = \sqrt{\{b^2 (a' \cos\theta)^2 + a^2 (a' \sin\theta)^2\}}$$
$$= \sqrt{(b^2x^2 + a^2y^2)} = ab.$$

* *Cambridge Mathematical Journal*, Vol. I., p. 96.

CIRCULAR SECTIONS IN SURFACES OF THE SECOND ORDER.*

IN determining the circular sections in the Surface of Elasticity, Fresnel has made use of a method which is very readily applicable to surfaces of the second order; and as it has not yet been introduced into any work on Analytical Geometry, it may be useful to insert it here.

Taking first the surfaces which have a centre, let their equation be

$$Px^2 + P'y^2 + P''z^2 = H \text{..............} (1),$$

and let this be cut by a plane

$$z = mx + ny \text{......................} (2),$$

which we suppose to pass through the centre, as all sections made by parallel planes are similar. Let this plane also cut the sphere

$$x^2 + y^2 + z^2 = r^2 \text{....................} (3).$$

Now, as m, n, r are indeterminate, we can so assume the position of the plane and the magnitude of the sphere, that the circular section of the surface (1), if it exist, shall coincide with the section of the sphere; and if these coincide, the equations to their projections on the plane of xy must be identical, which gives us conditions for determining m and n. Substituting for z in (1) and (3) its value from (2), we get

$$(P + P''m^2)\,x^2 + (P' + P''n^2)\,y^2 + 2P''mnxy = H \text{...} (4),$$

$$(1 + m^2)\,x^2 + (1 + n^2)\,y^2 + 2mnxy = r^2 \text{........} (5).$$

* *Cambridge Mathematical Journal*, Vol. I., p. 100.

Comparing each term separately, those involving xy will coincide if either

$$m=0, \quad n=0, \quad \text{or } r^2=\frac{H}{P''}.$$

Taking the first condition and comparing the other terms,

$$\frac{H}{P}=r^2, \text{ and } \frac{P'+P''n^2}{H}=\frac{1+n^2}{r^2},$$

which gives $$P'+P''n^2=P(1+n^2),$$

and $$n=\pm\sqrt{\left(\frac{P-P'}{P''-P}\right)}.$$

If we suppose $n=0$, we find in the same manner

$$m=\pm\sqrt{\left(\frac{P'-P}{P''-P'}\right)}.$$

The third condition leads to no result, and therefore is not to be considered.

In the ellipsoid, P, P', P'' are all positive, and

$$P<P'<P''.$$

This shows that the value of n is impossible, and that of m possible; therefore there are two directions arising from the doubtful sign in which the ellipsoid may be cut in circular sections, determined by the equation to the cutting plane,

$$z=\pm\sqrt{\left(\frac{P'-P}{P''-P'}\right)}\,x.$$

In the hyperboloid of one sheet P'' is negative, and the value of n is possible and m impossible. In the hyperboloid of two sheets P' and P'' are both negative, and $P''<P'$, m is possible and n impossible. It is true, that for a plane passing through the centre the section is impossible, but a plane drawn parallel to this at a sufficient distance from the centre will cut the surface in a circle.

The equation to the surfaces without a centre is

$$p'y^2+pz^2=pp'x.$$

Let this be cut by a plane

$$x=mz+ny,$$

which also cuts the sphere

$$x^2 + y^2 + z^2 = 2rx.$$

The equations to the projections of the sections on the plane of zy are

$$p'y^2 + pz^2 - mpp'z - npp'y = 0,$$

$$(1 + n^2)\, y^2 + (1 + m^2)\, z^2 + 2mnzy - 2mrz - 2mry = 0.$$

In order that these may coincide, the term involving zy must vanish, which will be the case if $m = 0$ or $n = 0$.

If $m = 0$, then $1 + n^2 = \frac{p'}{p}$ and $n = \pm\sqrt{\left(\frac{p' - p}{p}\right)}$.

If $n = 0$, then $1 + m^2 = \frac{p}{p'}$ and $m = \pm\sqrt{\left(\frac{p - p'}{p'}\right)}$.

In the elliptic paraboloid p and p' are both positive, and according as p' is greater or less than p, the first or second is to be taken, the other becoming impossible. In either case there are two series of circular sections corresponding to the positive and negative sign.

In the hyperbolic paraboloid p or p' is negative, so that there are no sections in which it is cut in a circle. This would appear also from the nature of the surface, as it can never be cut by a plane in a closed curve.

The same method may be applied to the oblique cone, so as to determine the sub-contrary sections.

By determining the circular sections in this manner, it is seen at once that any two belonging to different series are situate on the same sphere.

NOTES ON FOURIER'S HEAT.*

THE method employed by Fourier to integrate the partial differential equations which occur in the Theory of Heat, is to assume some simple form of a singular solution, and afterwards to extend it so as to include all the circumstances of the problem. It is in effecting this that he has displayed the great resources of his analysis, and imparted so great an interest to his work by the variety and ingenuity of his methods. Indeed there is a freshness and originality in the writings of Fourier which make them in no ordinary degree arrest the attention of the reader. But however much we may admire the means by which Fourier has overcome the difficulties of the problems he had to deal with, yet it seems more agreeable to the usual style of mathematical investigation to deduce a result by limiting the general solution by means of the conditions of the problem, than by extending a particular case.

That this may be sometimes done with even more readiness than by Fourier's method, will be seen by the following solution of a problem given in page 161 of the *Théorie de la Chaleur*. We may remark, that there is in general no difficulty in the solution of the partial differential equations, but only in the proper determination of the arbitrary functions in the solution, so as to suit the conditions of the problem.

* *Cambridge Mathematical Journal*, Vol. I., p. 104.

If a rectangular plate, bounded by two infinite parallel edges, have one of its extremities kept at a constant temperature 1, while the infinite edges perpendicular to the heated edge are retained at a constant temperature 0, the equation from which the temperature is to be determined is

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where v is the temperature at the point x, y, the origin being at the middle point of the heated edge, the axis of x bisecting the plate, and the axis of y parallel to the heated edge. For the sake of shortness Fourier represents the breadth of the plate by π.

The solution of the equation (1) by the method of the separation of the symbols of operation from those of quantity, is

$$v = \cos\left(y\frac{d}{dx}\right)\phi(x) + \sin\left(y\frac{d}{dx}\right)\psi(x) \quad \ldots\ldots \quad (2),$$

$\phi(x)$ and $\psi(x)$ being arbitrary functions of x. And it may also be put under the form

$$v = F\{x + y\sqrt{(-1)}\} + f\{x - y\sqrt{(-1)}\},$$

where $F(x) = \frac{1}{2}\{\phi(x) + \psi(x)\}$ and $f(x) = \frac{1}{2}\{\phi(x) - \psi(x)\}$.

Now, on looking at the circumstances of the problem, it will be seen that it must be subject to the following conditions:

1st. v must be symmetrical with regard to y and $-y$.

2nd. $v = 0$ when $y = \frac{\pi}{2}$ or $-\frac{\pi}{2}$, whatever x may be.

3rd. $v = 1$ when $x = 0$, whatever y may be.

4th. v must be very small when x is very large.

From the first condition we must have $\psi(x) = 0$, as otherwise the second term would change its sign when $-y$ is put for y. Hence we have only

$$v = \cos\left(y\frac{d}{dx}\right)\phi(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

By the second condition, putting $\frac{\pi}{2}$ for y in equation (3), we have

$$0 = \cos\left(\frac{\pi}{2}\frac{d}{dx}\right)\phi(x) \text{ ....(4).}$$

Now this is in fact a linear differential equation with constant coefficients, and of an infinite order. By the principles laid down in Art. V. of our first Number of the Journal,* we can integrate this equation if we know the roots of the equation $\cos\left(\frac{\pi}{2} z\right) = 0$. Now these are

$$\pm 1, \ \pm 3, \ \pm 5, \ \&c.$$

being in number infinite. Hence the solution of (4) is

$$\phi(x) = \left\{\begin{matrix} C_1\varepsilon^{-x} + C_3\varepsilon^{-3x} + C_5\varepsilon^{-5x} + \&c. \\ + C'_1\varepsilon^{x} + C'_3\varepsilon^{3x} + C'_5\varepsilon^{5x} + \&c. \end{matrix}\right\} \text{ ...... (5),}$$

the number of terms and arbitrary constants being infinite. By the fourth condition it appears that the second line of (5) must disappear, as otherwise v would be very large when x is very large. Hence we must have

$$C'_1 = 0, \ C'_3 = 0, \ C'_5 = 0, \ \&c.$$

and equation (5) is reduced to

$$\phi(x) = C_1\varepsilon^{-x} + C_3\varepsilon^{-3x} + C_5\varepsilon^{-5x} + \&c. \text{ ....(6),}$$

and v becomes

$$v = \cos\left(y\frac{d}{dx}\right)(C_1\varepsilon^{-x} + C_3\varepsilon^{-3x} + C_5\varepsilon^{-5x} + \&c.) \text{ ... (7).}$$

By the third condition $v = 1$ when $x = 0$. If then we expand the symbol of operation in (7), operate on each term separately, and then make $x = 0$, we shall find

$$1 = C_1 \cos y + C_3 \cos 3y + C_5 \cos 5y + \&c. \text{.......(8),}$$

where y is contained between the limits $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

* See page 14 of this volume.

In order to determine the arbitrary constants, we shall follow Fourier's method of definite integrals. If we multiply both sides of (8) by $\cos y\, dy$, and integrate between the limits $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$, all the terms except the first will disappear, as they can each be decomposed into the cosines of even multiples of y, which, on integration, vanish at both limits. Hence we have

$$\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} dy \cos y = C_1 \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} dy \cos^2 y = \frac{C_1}{2} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} dy\, (1 + \cos 2y),$$

whence we find $C_1 = \frac{4}{\pi}$.

In a similar manner we should find

$$C_3 = -\tfrac{1}{3}\frac{4}{\pi}, \quad C_5 = -\tfrac{1}{5}\frac{4}{\pi}, \quad C_7 = -\tfrac{1}{7}\frac{4}{\pi},$$

and so on.

Substituting these values in equation (7), it becomes

$$\frac{\pi}{4}\, v = \cos\left(y\,\frac{d}{dx}\right)(\varepsilon^{-x} - \tfrac{1}{3}\varepsilon^{-3x} + \tfrac{1}{5}\varepsilon^{-5x} - \tfrac{1}{7}\varepsilon^{-7x} + \&c.).$$

Now if we expand the sign of operation, and apply it to such a term as ε^{-nx}, we shall find that it becomes $\cos ny\varepsilon^{-nx}$. Hence the expression for v becomes

$$\frac{\pi}{4}\, v = \varepsilon^{-x} \cos y - \tfrac{1}{3}\varepsilon^{-3x} \cos 3y + \tfrac{1}{5}\varepsilon^{-5x} \cos 5y - \&c.$$

which is one form of the solution which Fourier gives. It may easily be reduced to a more simple form. For if we substitute for the cosines their exponential values, we have

$$\frac{\pi}{2}\, v = \begin{cases} \varepsilon^{-\{x - y\sqrt{(-1)}\}} - \tfrac{1}{3}\varepsilon^{-3\{x - y\sqrt{(-1)}\}} + \tfrac{1}{5}\varepsilon^{-5\{x - y\sqrt{(-1)}\}} - \&c. \\ + \varepsilon^{-\{x + y\sqrt{(-1)}\}} - \tfrac{1}{3}\varepsilon^{-3\{x + y\sqrt{(-1)}\}} + \tfrac{1}{5}\varepsilon^{-5\{x + y\sqrt{(-1)}\}} - \&c. \end{cases}$$

which, by Gregorie's series, becomes

$$\frac{\pi}{2}v = \tan^{-1}\varepsilon^{-\{x-y\sqrt{(-1)}\}} + \tan^{-1}\varepsilon^{-\{x+y\sqrt{(-1)}\}}$$

$$= \tan^{-1}\frac{\varepsilon^{-\{x-y\sqrt{(-1)}\}} + \varepsilon^{-\{x+y\sqrt{(-1)}\}}}{1 - \varepsilon^{-2x}}$$

$$= \tan^{-1}\left(\frac{2\cos y}{\varepsilon^{x} - \varepsilon^{-x}}\right),$$

which is the simplest form that the expression can assume.

ON THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS.*

THE integration of Partial Differential Equations is much facilitated by the principle which we have developed in our preceding Numbers, and it is the more remarkable, that it has been so little applied to these equations, as the first step was taken many years ago. Fourier, in his *Traité de la Chaleur*, published in 1822, has shewn that the series which are obtained in the solution of several partial differential equations may be conveniently expressed by the separation of the symbols of operation from those of quantity. But though he has used this method very frequently, yet he appears to have had some unwillingness to give himself up to it entirely as a guide in his investigations, as if he were not familiar with the principles on which it is founded. His idea apparently was, that the expression which he obtained as solutions might be conveniently expressed by separating the symbols of operation, and not that the symbolical expressions are the proper solutions of the equations, and the series merely the expansion of them. Other French writers seem to have avoided carefully entering at all on the track which Fourier opened: Poisson in particular, in a digression on the subject of partial differential equations in the second volume of his *Mécanique*, does not put in the symbolical form the solution of a very simple equation, which is so given by Fourier. Mr. Greatheed, in a paper

* *Cambridge Mathematical Journal*, Vol. I., p. 123.

published in the number of the *Philosophical Magazine* for September, 1837, was, we believe, the first to call the attention of mathematicians to the utility of this method in the case of partial differential equations, but he had not then reduced it to its greatest degree of simplicity: and his paper is chiefly occupied with a particular class of equations of the first order with variable coefficients, which are not so interesting as many others with constant coefficients. We shall here, therefore, proceed to give several examples of the application of the principles which we laid down in Art. V. of our first Number of the Journal.*

And, first, we may observe generally, that linear partial differential equations between any number of variables with constant coefficients, are to be treated exactly like ordinary differential equations with regard to one of the variables, the symbols of operation of the others being treated as constants. If, for instance, we have the equation

$$a\frac{dz}{dx} + b\frac{dz}{dy} = c.$$

This may be put under the form

$$\left(a\frac{d}{dx} + b\frac{d}{dy}\right) z = c,$$

and therefore $$z = \left(a\frac{d}{dx} + b\frac{d}{dy}\right)^{-1} (c + 0),$$

where we suppose x to be the variable, and $\frac{d}{dy}$ a constant with regard to it. Now the operation $\left(a\frac{d}{dx} + b\frac{d}{dy}\right)^{-1}$ by the theorem given in page 18, is equivalent to

$$a^{-1}\varepsilon^{-\frac{b}{a}x\frac{d}{dy}} \int dx\,\varepsilon^{\frac{b}{a}x\frac{d}{dy}},$$

which being performed, gives

$$z = \left(\frac{d}{dy}\right)^{-1}\frac{c}{b} + \frac{1}{a}\varepsilon^{-\frac{b}{a}x\frac{d}{dy}} f(y),$$

* See page 14 of this volume.

$f(y)$ being an arbitrary function of y, taking the place of the constant in ordinary differential equations.

From this expression we get

$$z=\frac{cy}{b}+f(ay-bx),$$

as by Taylor's theorem

$$\varepsilon^{h\frac{d}{dx}}f(x)=f(x+h).$$

We might equally well have supposed y to be the variable, and $\frac{d}{dx}$ a constant with regard to it.

Again, taking the well-known equation for the motion of waves,

$$\frac{d^2z}{dt^2}-a^2\frac{d^2z}{dx^2}=0,$$

it may be put under the form

$$\left(\frac{d^2}{dt^2}-a^2\frac{d^2}{dx^2}\right)z=0;$$

and integrating it like the ordinary differential equation

$$\left(\frac{d^2}{dt^2}-n^2\right)z=0,$$

we find
$$z=\varepsilon^{at\frac{d}{dx}}\phi(x)+\varepsilon^{-at\frac{d}{dx}}\psi(x),$$

or
$$z=\phi(x+at)+\psi(x-at).$$

The equation $r-2as+a^2t=0$ may be put under the form

$$\left(\frac{d}{dx}-a\frac{d}{dy}\right)^2z=0,$$

the solution of which is, by the Theorem in page 18,

$$z=\varepsilon^{ax\frac{d}{dy}}\int^2dx^2.0$$

$$=\varepsilon^{ax\frac{d}{dy}}\{x\phi(y)+\psi(y)\},$$

$\phi(y)$ and $\psi(y)$ being arbitrary functions of y arising from the integration. Hence, finally, we have

$$z=x\phi(y+ax)+\psi(y+ax).$$

The equation

$$r - a^2t + 2abp + 2a^2bq = 0$$

is equivalent to

$$\left\{\frac{d}{dx} - \left(a\frac{d}{dy} - 2ab\right)\right\}\left(\frac{d}{dx} + a\frac{d}{dy}\right)z = 0.$$

Integrating with regard to the first factor, we have

$$\left(\frac{d}{dx} + a\frac{d}{dy}\right)z = \varepsilon^{x\left(a\frac{d}{dy} - 2ab\right)}\phi(y).$$

Integrating with regard to the remaining factor,

$$z = \varepsilon^{-ax\frac{d}{dy}}\int dx\, \varepsilon^{2ax\left(\frac{d}{dy} - b\right)}\phi(y) + \varepsilon^{-ax\frac{d}{dy}}\psi(y)$$

$$= \varepsilon^{-ax\frac{d}{dy}}\varepsilon^{2ax\left(\frac{d}{dy} - b\right)}\left\{2a\left(\frac{d}{dy} - b\right)\right\}^{-1}\phi(y) + \psi(y - ax)$$

$$= \varepsilon^{ax\frac{d}{dy}}\varepsilon^{-2abx}\varepsilon^{by}\phi_1(y) + \psi(y - ax),$$

by changing the arbitrary function,

$$= \varepsilon^{-2abx}\varepsilon^{b(y+ax)}\phi_1(y + ax) + \psi(y - ax)$$

$$= \varepsilon^{b(y-ax)}\phi_1(y + ax) + \psi(y - ax).$$

Let us take also the equation

$$r - c^2t = xy.$$

Without operating on each factor separately, we may arrive more readily at the result by the same means as those employed in page 28 of the first number of the Journal.* For we have

$$z = \left(\frac{d^2}{dx^2} - c^2\frac{d^2}{dy^2}\right)^{-1}(xy) + \phi(y + ax) + \psi(y - ax)$$

$$= \frac{d^{-2}}{dx^{-2}}\left(1 - c^2\frac{d^2}{dy^2}\frac{d^{-2}}{dx^{-2}}\right)^{-1}(xy) + \phi(y + ax) + \psi(y - ax)$$

$$= \left(1 - c^2\frac{d^2}{dy^2}\frac{d^{-2}}{dx^{-2}}\right)^{-1}\left(\frac{x^3y}{6}\right) + \phi(y + ax) + \psi(y - ax).$$

* See page 21 of this volume.

Therefore

$$z = \frac{x^3 y}{6} + \phi(y + ax) + \psi(y - ax),$$

the arbitrary functions being added as in the second example.

A very simple equation, being one which occurs in the theory of heat, is

$$\frac{dv}{dt} = a\frac{d^2 v}{dx^2},$$

which is the expression for the rectilinear propagation of heat. The solution of this is easily seen to be, if we integrate with regard to t,

$$v = \varepsilon^{at\frac{d^2}{dx^2}} f(x),$$

$f(x)$ being an arbitrary function. If the sign of operation be expanded, we shall obtain a series which is the solution derived by Poisson from the method of indeterminate coefficients. Laplace has deduced from the series an elegant expression for the solution under the form of a definite integral; but it may be more easily deduced from the symbolical solution in the following manner.

Since
$$\int_{-\infty}^{\infty} \varepsilon^{-\omega^2} d\omega = \sqrt{(\pi)},$$

and also
$$\int_{-\infty}^{\infty} \varepsilon^{-(\omega - b)^2} d\omega = \sqrt{(\pi)},$$

we can put the expression for v under the form

$$\sqrt{(\pi)}\, v = \int_{-\infty}^{\infty} d\omega \varepsilon^{at\frac{d^2}{dx^2}} \varepsilon^{-\{\omega - \sqrt{(at)}\frac{d}{dx}\}^2} f(x)$$

$$= \int_{-\infty}^{\infty} d\omega \varepsilon^{-\omega^2} . \varepsilon^{2\omega\sqrt{(at)}\frac{d}{dx}} f(x)$$

$$= \int_{-\infty}^{\infty} d\omega \varepsilon^{-\omega^2} f\{x + 2\omega\sqrt{(at)}\}$$

by Taylor's theorem; and this is Laplace's expression.

This equation $\frac{dv}{dt} = a\frac{d^2v}{dx^2}$ not being homogeneous in the index of the operations, admits of two solutions of very different characters. The one, which we have found by integrating with regard to t, contains only one arbitrary function of x. The other, which may be found by integration with regard to x, must contain two arbitrary functions of t, as the index of operation is of the second degree. If we write the equation in the form

$$\frac{d^2v}{dx^2} - \frac{1}{a}\frac{dv}{dt} = 0,$$

and integrate by the method employed in page 21, we find for the integral

$$v = \left(x + \frac{1}{a}\frac{x^3}{1.2.3}\frac{d}{dt} + \frac{1}{a^2}\frac{x^5}{1.2.3.4.5}\frac{d^2}{dt^2} + \&c.\right)\phi(t)$$

$$+\left(1 + \frac{1}{a}\frac{x^2}{1.2}\frac{d}{dt} + \frac{1}{a^2}\frac{x^4}{1.2.3.4}\frac{d^2}{dt^2} + \&c.\right)\psi(t).$$

It seems at first sight anomalous, that the same equation should have two solutions so different in character: the following is the explanation of the difficulty. Since by Maclaurin's theorem any function of a variable may be expressed by means of its differential coefficients taken with regard to that variable, for the particular value 0 of the variable, we know the function if we can determine its successive differential coefficients. Now from the equation $\frac{dv}{dt} = a\frac{d^2v}{dx^2}$ we can, when we know the value of v when $t=0$, determine the values of all the differential coefficients with regard to t when $t=0$. So that in the resulting expression deduced from Maclaurin's theorem there is only one quantity left undetermined, which is the arbitrary function $f(x)$, introduced in the integration. But from the equation $\frac{d^2v}{dx^2} = \frac{1}{a}\frac{dv}{dt}$, we can only determine from the value of v when $x=0$ the values of the alternate differential coefficients with regard to x. There must consequently be introduced another undetermined quantity, namely, the value of $\frac{dv}{dx}$ when $x=0$; for knowing these two quantities we can determine the values of all the successive differential coefficients with regard to x.

The equation for determining the vibratory motion of an elastic spring is

$$\frac{d^2v}{dt^2}+\frac{d^4v}{dx^4}=0.$$

The solution of which is readily seen to be

$$v=\cos\left(t\frac{d^2}{dx^2}\right)F(x)+\sin\left(t\frac{d^2}{dx^2}\right)f(x).$$

The equation for determining the vibratory motion of elastic plates is

$$\frac{d^2v}{dt^2}+\frac{d^4v}{dx^4}+2\frac{d^4v}{dx^2dy^2}+\frac{d^4v}{dy^4}=0,$$

which may be put under the form

$$\frac{d^2v}{dt^2}+\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)^2v=0;$$

the integral of which is

$$v=\cos t\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)F(x,y)+\sin t\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)f(x,y).$$

In investigating the motion of heat in a ring, we obtain as an equation for determining the temperature at any time and point,

$$\frac{dv}{dt}=k\frac{d^2v}{dx^2}-hv;$$

the solution of which is

$$v=\varepsilon^{\left(k\frac{d^2}{dx^2}-h\right)t}f(x)$$

$$=\varepsilon^{-ht}\varepsilon^{kt\frac{d^2}{dx^2}}f(x),$$

an expression closely connected with one previously given.

In the examples which we have given, the coefficients are constant; but if one of the variables only enters into the coefficients, and we integrate with regard to the other, as the one variable is unaffected by the sign of operation with regard to the other, it may be considered as a constant in the integration.

A good example of this kind of equation is that which expresses the motion of heat in a solid cylinder of infinite length, namely

$$\frac{dv}{dt} = a\left(\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx}\right),$$

the solution of which is

$$v = \varepsilon^{at\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}\right)} f(x),$$

or, as it may be expressed,

$$v = \varepsilon^{at\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right)} f(x).$$

But this will only be possible when the integration is of the first degree, since, as we shewed in the first Number of the Journal,* the symbols of operation become subject to different laws when the variable itself and the sign of differentiation are both involved. And this leads us to the consideration of equations with variable coefficients. As in the case of ordinary differential equations, the solution of this class is attended with great difficulties, so as to become almost impossible for equations of an order higher than the first, it is to be supposed that these difficulties are no way diminished in the case of partial differential equations. Of these, however, when of the first order, Mr. Greatheed has shewn, that a large class may be solved like ordinary differential equations. It is included under the form

$$\frac{dz}{dx} + XY\frac{dz}{dy} = Pz + Q,$$

* See page 14 of this volume.

where X is a function of x only, Y of y only, and P and Q of both x and y. This may be reduced to the form

$$\frac{dz}{dx}+X\frac{dz}{dy}=Pz+Q,$$

by a change of the variable; for if $y'=\int\frac{dy}{Y}$, then $Y\frac{dz}{dy}=\frac{dz}{dy'}$: so that it is only necessary to consider the latter equation. Let it be put under the form

$$\frac{dz}{dx}+\left(X\frac{d}{dy}-P\right)z=Q,$$

and treated as an ordinary linear differential equation between z and x. If we integrate by the method of integrating factors, the factor is $\varepsilon^{\int dx(X\frac{d}{dy}-P)}$, and the equation becomes

$$\frac{d}{dx}\{\varepsilon^{\int dx(X\frac{d}{dy}-P)}.z\}=\varepsilon^{\int dx(X\frac{d}{dy}-P)}.Q;$$

whence

$$z=\varepsilon^{-\int dx(X\frac{d}{dy}-P)}\int dx\,\{\varepsilon^{\int dx(X\frac{d}{dy}-P)}.Q\}+\varepsilon^{-\int dx(X\frac{d}{dy}-P)}.\phi(y),$$

$\phi(y)$ being an arbitrary function of y.

We shall take as an example the equation

$$x\frac{dz}{dx}+y\frac{dz}{dy}=nz.$$

Let $\frac{dy}{y}=dt$, and therefore $y=\varepsilon^t$. Then

$$\frac{dz}{dx}+\frac{1}{x}\frac{dz}{dt}=n\frac{z}{x}.$$

The integral of which, by the preceding formula, is

$$z=\varepsilon^{-\int dx(\frac{1}{x}\frac{d}{dt}-\frac{n}{x})}\phi(t),$$

or $\quad z=\varepsilon^{n\log x}.\varepsilon^{-\log x\frac{d}{dt}}\phi(t)$

$$=x^n\phi(t-\log x)=x^n\phi(\log y-\log x)=x^n\psi\left(\frac{y}{x}\right).$$

The equation $$x\frac{dz}{dx} - y\frac{dz}{dy} = \frac{x^2}{y}$$
may be integrated in the same way. Or if we change both the independent variables, making $\frac{dy}{y} = dt$ and $\frac{dx}{x} = du$, it becomes
$$\left(\frac{d}{du} - \frac{d}{dt}\right) z = \varepsilon^{2u}\varepsilon^{-t},$$
which gives $$z = \varepsilon^{u\frac{d}{dt}} \int du \varepsilon^{-u\frac{d}{dt}} \varepsilon^{2u}\varepsilon^{-t} + \varepsilon^{u\frac{d}{dt}} \phi(t)$$
$$= \varepsilon^{u\frac{d}{dt}} \int du \varepsilon^{3u}\varepsilon^{-t} + \varepsilon^{u\frac{d}{dt}} \phi(t)$$
$$= \frac{\varepsilon^{2u}\varepsilon^{-t}}{3} + \phi(t+u)$$
$$= \frac{x^2 y}{3} + \psi(xy),$$
as $u = \log x$, $t = \log y$.

The equation $$y\frac{dz}{dx} + x\frac{dz}{dy} = z$$
may, by changing $y\,dy$ into $\frac{1}{2}d.y^2$, be transformed into
$$\frac{dz}{dx} + \left(2x\frac{d}{d.y^2} - \frac{1}{y}\right) z = 0.$$

The integral of which is
$$z = \varepsilon^{-\int dx \left(2x\frac{d}{d.y^2} - \frac{1}{y}\right)} \phi(y^2)$$
$$= \varepsilon^{\int \frac{dx}{y}} \varepsilon^{-x^2\frac{d}{d.y^2}} \phi(y^2)$$
$$= \varepsilon^{-x^2\frac{d}{d.y^2}} \left\{\varepsilon^{x^2\frac{d}{d.y^2}} \left(\varepsilon^{\int\frac{dx}{y}}\right) \phi(y^2)\right\}$$
$$= \varepsilon^{-x^2\frac{d}{d.y^2}} \left\{\varepsilon^{\int\frac{dx}{\sqrt{(y^2+x^2)}}} \phi(y^2)\right\}$$
$$= \varepsilon^{-x^2\frac{d}{d.y^2}} \left[\varepsilon^{\log\{x+\sqrt{(x^2+y^2)}\}} . \phi(y^2)\right]$$
$$= \varepsilon^{-x^2\frac{d}{d.y^2}} \left[\{x + \sqrt{(x^2+y^2)}\} \phi(y^2)\right]$$
$$= (x+y)\,\phi(y^2 - x^2).$$

Other equations, which at first sight do not appear to come under this form, may be reduced to it by a proper assumption of a new independent variable. For instance, the equation

$$(x+y)\frac{dz}{dx}+(y-x)\frac{dz}{dy}=z,$$

is converted into

$$u\frac{dz}{dx}-2x\frac{dz}{du}=z,$$

by supposing $y=u-x$. These however are particular cases which come under no general rule.

ON THE RESIDUAL CALCULUS.*

CAUCHY, in several articles in his "*Exercices*," has developed a new Calculus, to which he has given the name of the "*Calcul des Résidus.*" Few persons appear to have followed the author in the use of it; and in our own language I do not recollect having met even with an allusion to it: yet it deserves some attention at the hands of mathematicians, were it only from the deserved celebrity of its author. I propose to give here merely a slight sketch of its principles, with a few examples; and if any one should be induced from this specimen to wish for a more complete knowledge of this Calculus, I must refer him to the Memoirs of its author.

The Residual Calculus bears a certain analogy to the Differential Calculus, for in both the object is to investigate the nature and properties of certain functions which appear in an indeterminate form, but yet have finite values.

Let $f(x)$ be a function of x, which becomes infinite when $x = a$; that is, let a be a root of the equation

$$f(x) = \infty, \text{ or } \frac{1}{f(x)} = 0 \ldots\ldots\ldots\ldots\ldots (1).$$

Then the expression $(x - a)f(x)$ becomes indeterminate when $x = a$, as it takes the form $0 \times \infty$, though it may still have really a finite value. Suppose this to be the case, and

* *Cambridge Mathematical Journal*, Vol. I., p. 145.

that it is a certain function of a which we may call $R(a)$. Then we have

$$(x-a)f(x) = R(x) \text{ when } x = a \text{ .... } (2),$$

which gives

$$f(x) = \frac{R(x)}{x-a} \text{ when } x = a \text{ .... } (3).$$

This function $R(a)$ is called the residue of $f(x)$ with respect to a. The operation of finding the residue is called the extraction of the residue, and is represented by the characteristic symbol $\mathcal{E}$, so that we write

$$R(a) = \mathcal{E}f(x).$$

In equation (3) let us suppose $x = a + h$. Then

$$f(a+h) = \frac{R(a+h)}{h} = \frac{1}{h}\{R(a) + hR'(a) + \&c.\},$$

when

$$x = a, \text{ or } h = 0.$$

Whence we find the residue $R(a)$ of $f(x)$ is the coefficient of $\frac{1}{h}$ in the expansion of $f(a+h)$, where a is a root of the equation $f(x) = \infty$.

We have spoken only of one root of this equation, but there may be several, and for each of these there is a corresponding residue. The sum of the residues corresponding to each root of the equation is called the *integral* residue of $f(x)$. To distinguish the integral from the partial residues, I shall suffix the root to the Residual Symbol in the latter case.* Thus, $\mathcal{E}f(x)$ will represent the integral residue of $f(x)$, $\mathcal{E}_a f(x)$, $\mathcal{E}_b f(x)$, $\mathcal{E}_c f(x)$, &c. will represent the partial residues with respect to a, b, c, &c., these being roots of the equation $f(x) = \infty$.

Let us suppose, now, that instead of the expression

$$(x-a)f(x)$$

having a finite value when $x = a$, that it is the expression

$$(x-a)^m f(x)$$

* This is not Cauchy's notation.

which has a finite value when $x = a$; in which case $f(x) = \infty$ is said to have m equal roots. If, as before, we suppose $R(a)$ to be the finite value, we shall have

$$f(x) = \frac{R(x)}{(x-a)^m} \text{ when } x = a,$$

or putting $x = a + h$, and expanding $R(a+h)$ by Taylor's theorem,

$$f(a+h) = \frac{1}{h^m}\left\{R(a) + \frac{h}{1} R'(a) + \&\text{c.} + \frac{h^{(m-1)} R^{(m-1)}(a)}{1.2\ldots(m-1)} + \&\text{c.}\right\}.$$

The coefficient of $\frac{1}{h}$ in this expansion is

$$\frac{R^{(m-1)}(a)}{1.2\ldots(m-1)},$$

which is the residue of the function $f(x)$ corresponding to $x = a$.

Since $\quad R(x) = h^m f(x) = h^m f(a+h)$,

$$R^{(m-1)}(a) = \frac{d^{m-1}}{dh^{m-1}}\{h^m f(a+h)\} \text{ when } h = 0;$$

and therefore we have

$$\mathcal{E}_a f(x) = \frac{1}{1.2\ldots(m-1)} \frac{d^{m-1}}{dh^{m-1}}\{h^m f(a+h)\} \text{ when } h = 0 \ldots (4).$$

I shall now proceed to shew how the extraction of residues may be simplified when $f(x)$ has particular forms: but previously it is necessary to explain the following notation. When $f(x)$ consists of distinct factors, which are functions of x, the roots of $f(x) = \infty$ may be the roots of any one factor equated to ∞.

When then we wish to indicate that the extraction of residues has reference only to the roots of one of the factors, that factor is enclosed in brackets [ ]. As, for instance, if $f(x) = \phi(x).\psi(x)$, and we wish to represent the extraction of the residues with reference to the roots of $\phi(x) = \infty$, we shall write

$$\mathcal{E}[\phi(x)].\psi(x).$$

If a factor is of the form $\frac{1}{\phi(x)}$, we should properly represent the extraction of the residues with respect to it by

$$\mathcal{E}\left[\frac{1}{\phi(x)}\right].\psi(x).$$

But since $\frac{1}{\phi(x)} = \infty$ gives the same roots as $\phi(x) = 0$, which, therefore, is the equation to which we look, we shall find it convenient to use the notation

$$\mathcal{E}\frac{\psi(x)}{[\phi(x)]}.$$

Hence, if a be a root of $f(x) = \infty$, the residue with regard to it may be expressed by $\mathcal{E}\frac{(x-a)f(x)}{[x-a]}$ instead of $\mathcal{E}_a f(x)$, so that

$$\mathcal{E}\frac{(x-a)f(x)}{[x-a]} = hf(a+h) \text{ when } h = 0,$$

if there be only one root of $f(x) = \infty$; and

$$\mathcal{E}\frac{(x-a)f(x)}{[x-a]} = \frac{1}{1.2\ldots(m-1)}\frac{d^{m-1}}{dh^{m-1}}\{h^m f(a+h)\} \text{ when } h = 0,$$

if there be m roots equal to a.

Now suppose $f(x)$ to be of the form $\frac{\mathrm{f}(x)}{F(x)}$, and that a is a root of $F(x) = 0$; then, as $F(a) = 0$, we have

$$F(a+h) = hF'(a+\theta h),$$

and therefore

$$hf(a+h) = \frac{h\mathrm{f}(a+h)}{hF'(a+\theta h)}.$$

So that when $h = 0$,

$$\mathcal{E}_a\frac{\mathrm{f}(x)}{[F(x)]} = \frac{\mathrm{f}(a)}{F'(a)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

As an example, take $\frac{x^m}{x-a}$, whence

$$\mathcal{E}_a\frac{x^m}{[x-a]} = a^m.$$

Also $$\mathcal{E}_a \frac{x^2 - px + q}{[x^2 - a^2]} = \frac{a^2 - pa + q}{2a},$$

$$\mathcal{E}_a \frac{x^m}{[(x-a)(x-b)]} = \frac{a^m}{a-b}.$$

If the fraction be of the form $\dfrac{\mathrm{f}(x)}{(x-a)(x-b)\dots(x-r)}$, where there are r factors in the denominator, we may extract the residue with respect to each factor, and the sum of all these partial residues will be the integral residue with respect to the denominator. Hence

$$\mathcal{E} \frac{\mathrm{f}(x)}{[(x-a)(x-b)\dots(x-r)]} = \frac{\mathrm{f}(a)}{(a-b)(a-c)\dots(a-r)}$$
$$+ \frac{\mathrm{f}(b)}{(b-a)(b-c)\dots(b-r)} + \&\text{c.} + \frac{\mathrm{f}(r)}{(r-a)(r-b)\dots(r-q)}.$$

When the denominator has m roots, each equal to a,

$$\mathcal{E}_a f(x) = \mathcal{E}_a \frac{\mathrm{f}(x)}{[F(x)]} = \frac{1}{1.2\dots(m-1)} \frac{d^{m-1}}{dh^{m-1}} \{h^m f(a+h)\}$$
when $h = 0$.

But $$f(a+h) = \frac{\mathrm{f}(a+h)}{F(a+h)},$$
and
$$F(a+h) = F(a) + F'(a)\frac{h}{1} + \&\text{c.} + \frac{F^{(m-1)}(a)\,h^{m-1}}{1.2\dots(m-1)} + \frac{F^{(m)}(a)\,h^m}{1.2\dots m} + \&\text{c.}$$

and as these are m equal roots of $F(x) = 0$, we have

$$F(a) = 0, \ F'(a) = 0 \dots F^{(m-1)}(a) = 0.$$

Hence $$F(a+h) = \frac{h^m}{1.2\dots m}\left\{F^{(m)}(a) + F^{(m+1)}(a)\frac{h}{m+1} + \&\text{c.}\right\},$$

or $$F(a+h) = \int^m dh^m \frac{d^m}{da^m} F(a+h);^*$$

* By the notation $\frac{d^m}{da^m} f(a+h)$, I mean to express the value which $\frac{d^m}{dx^m} f(x+h)$ takes when $x = a$.

whence we find

$$\mathcal{E}_a \frac{\mathrm{f}(x)}{[F(x)]} = \frac{1}{1.2\ldots(m-1)} \frac{d^{m-1}}{dh^{m-1}} \left\{ \frac{h^m \mathrm{f}(a+h)}{\int^m dh^m \frac{d^m}{da^m} F(a+h)} \right\}$$

$$\text{when } h = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

When $F(a+h)$ is of the form $(x-a)^m$, this expression is considerably simplified. For

$$\frac{d^m}{dx^m}(x-a)^m = m(m-1)\ldots 2.1,$$

and
$$\int^m dh^m \frac{d^m}{da^m} F(a+h) = h^m;$$

so that

$$\mathcal{E}_a \frac{\mathrm{f}(x)}{(x-a)^m} = \frac{1}{1.2\ldots(m-1)} \frac{d^{m-1}}{dh^{m-1}} \{\mathrm{f}(a+h)\} \text{ when } h = 0 \ldots (7).$$

Thus
$$\mathcal{E}_a \frac{x^n}{(x-a)^4} = \frac{n.(n-1)(n-2)}{1.2.3} a^{n-3}.$$

I shall now proceed to the proof of one of the fundamental theorems of the Residual Calculus. It is, that

$$f(x) - \mathcal{E} \frac{f(z)}{x-z} = \psi(x) \ldots\ldots\ldots\ldots\ldots (8),$$

$\psi(x)$ being a function of x, which takes generally a finite value when

$$x = a_1, = a_2, = a_3, \text{ \&c.},$$

a_1, a_2, a_3, &c., being the roots of the equation $f(x) = \infty$.

We have generally

$$f(x) = \frac{R(x)}{(x-a_1)^m} \text{ when } x = a_1.$$

Expanding $R(x) = R(a_1 + h)$ by Taylor's theorem, and putting $x - a_1$ for h,

$$f(x) = \frac{R(a_1)}{(x-a_1)^m} + \frac{1}{1} \frac{R'(a_1)}{(x-a_1)^{m-1}} + \text{\&c.}$$

$$+ \frac{1}{1.2\ldots(m-1)} \frac{R^{(m-1)}(a_1)}{x-a_1} + \psi(x) \ldots\ldots\ldots (9),$$

where $\psi(x)$ is a function of x, which generally takes for $x=a_1$ the finite value $\dfrac{R^{(m)}(a_1)}{1.2\ldots m}$.

But we have also from the definition of a residue

$$\frac{R^{(m)}(a_1)}{1.2\ldots m}=\mathcal{E}_{a_1}\frac{R(z)}{(z-a_1)^{m+1}}\ldots\ldots\ldots\ldots\ldots(10),$$

and similarly for the others. Hence

$$f(x)=\frac{1}{(x-a_1)^m}\left\{\mathcal{E}_{a_1}\frac{R(z)}{z-a_1}+\mathcal{E}_{a_1}\frac{R(z)}{(z-a_1)^2}(x-a_1)\right.$$
$$\left.+\mathcal{E}_{a_1}\frac{R(z)}{(z-a_1)^3}(x-a)^2+\&c.\right\}$$
$$=\mathcal{E}_{a_1}\frac{R(z)}{(x-a_1)^m(z-a_1)^m}\{(z-a_1)^{m-1}+(z-a_1)^{m-2}(x-a_1)+\&c.\}+\psi(x)$$
$$=\mathcal{E}_{a}\frac{R(z)}{(x-a_1)^m(z-a_1)^m}\left\{\frac{(x-a_1)^m-(z-a_1)^m}{x-z}\right\}+\psi(x)$$
$$=\mathcal{E}_{a_1}\frac{R(z)}{(x-z)(z-a_1)^m}-\frac{1}{(x-a_1)^m}\mathcal{E}_{a_1}\frac{R(z)}{x-z}+\psi(x).$$

Now the second term of this expression is equal to 0, because $\dfrac{R(z)}{x-z}$ does not become infinite for $z=a_1$, and consequently has no corresponding residue. We have therefore

$$f(x)=\mathcal{E}_{a_1}\frac{R(z)}{(x-z)(z-a_1)^m}+\psi(x)=\mathcal{E}_{a_1}\frac{f(z)}{x-z}+\psi(x),$$

since $R(z)=(z-a_1)^m f(z)$.

Hence, in order to deduce from $f(x)$, which becomes infinite for $x=a_1$, a function of x, which shall remain finite under the same circumstances, we must subtract from $f(x)$ the partial residue with respect to a_1 of $\dfrac{f(z)}{x-z}$. Similarly, if we wish to find a function of x which shall not become infinite when $x=a_1$, $=a_2$, &c., we must subtract from $f(x)$

the partial residues with respect to each of these roots. So that if we make

$$f(x) - \mathcal{E}\frac{[f(z)]}{x-z} = \phi(x),$$

$\phi(x)$ will have a finite value when $x = a_1, = a_2, = a_3$, &c.

The notation $\mathcal{E}\frac{[f(z)]}{x-z}$ it must be remembered, means the sum of all the residues with respect to the roots of the equation $f(z) = \infty$.

From this theorem we can deduce a number of important results. If $f(x)$ is a fraction, as $\frac{\mathrm{f}(x)}{F(x)}$, $\phi(x)$ must be a fraction of the same kind, the denominator of which must never become 0, since the fraction $\phi(x)$ can never become infinite.

Therefore the denominator must be independent of x or constant, and $\phi(x)$ an integral function of x. If $F(x)$ be of higher dimensions than $\mathrm{f}(x)$, $f(x)$ will vanish for $x = \infty$, as likewise will the residue, so that we must also have $\phi(x) = 0$. Hence in this case

$$f(x) = \mathcal{E}\frac{[f(z)]}{x-z} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

which gives the means of decomposing a rational fraction unto simple fractions.

As an example, take the fraction $\frac{x^2-7x+1}{(x-1)(x-2)(x-3)} = f(x)$.

$$f(x) = \mathcal{E}\frac{z^2-7z+1}{(x-z)\,[(z-1)(z-2)(z-3)]}$$

$$= \mathcal{E}_1\frac{z^2-7z+1}{(x-z)(z-1)(z-2)(z-3)} + \mathcal{E}_2\frac{z^2-7z+1}{(x-z)(z-1)(z-2)(z-3)}$$

$$+ \mathcal{E}_3\frac{z^2-7z+1}{(x-z)(z-1)(z-2)(z-3)}$$

$$= -\frac{5}{2}\,\frac{1}{x-1} + 9\,\frac{1}{x-2} - \frac{11}{2}\,\frac{1}{x-3}.$$

Again, take $$\frac{1}{(x-1)^2(x+1)}=f(x)$$

$$f(x)=\mathcal{E}_{-1}\frac{1}{(x-z)(z-1)^2(z+1)}+\mathcal{E}_1\frac{1}{(x-z)(z-1)^2(z+1)}$$

$$=\tfrac{1}{4}\frac{1}{x+1}+\frac{d}{dh}\frac{1}{(x-1-h)(2+h)}\text{ when } h=0$$

$$=\tfrac{1}{4}\frac{1}{x+1}+\tfrac{1}{2}\frac{1}{(x-1)^2}-\tfrac{1}{4}\frac{1}{x-1}.$$

When the fraction is of the form $\frac{\mathrm{f}(x)}{(x-a)^m}$, we can easily deduce by this method the series of component simple fractions.

For $$\frac{\mathrm{f}(x)}{(x-a)^m}=\mathcal{E}_a\frac{\mathrm{f}(z)}{(x-z)(z-a)^m}$$

$$=\frac{1}{1.2\ldots(m-1)}\frac{d^{m-1}}{dh^{m-1}}\left(\frac{\mathrm{f}(a+h)}{x-a-h}\right)\text{ when } h=0;$$

and if we actually perform the operations indicated, we get

$$\frac{\mathrm{f}(x)}{(x-a)^m}=\frac{1}{1.2\ldots(m-1)}\left\{\frac{\mathrm{f}^{(m-1)}(a)}{(x-a)}+(m-1)\frac{\mathrm{f}^{(m-2)}(a)}{(x-a)^2}\right.$$

$$+(m-1)(m-2)\frac{\mathrm{f}^{(m-3)}(a)}{(x-a)^2}+\&\text{c.}$$

$$\left.+(m-1)(m-2)\ldots 2.1\frac{\mathrm{f}(a)}{(x-a)^m}\right\}\ \ldots\ldots\ldots\ (12).$$

This also appears from (9) by inverting the order of the terms.

By the application of the theorem (11) we can easily prove Lagrange's formula for interpolation. For if

$$f(x)=\frac{\mathrm{f}(x)}{(x-x_1)(x-x_2)\ldots(x-x_m)},$$

we have

$$\mathcal{E}\frac{\mathrm{f}(z)}{(z-x_1)(z-x_2)\ldots(z-x_m)}\frac{1}{x-z}$$

$$=\mathcal{E}_{x_1}\frac{\mathrm{f}(z)}{(z-x_1)(z-x_2)\ldots(z-x_m)}\frac{1}{x-z}$$

$$+\mathcal{E}_{x_2}\frac{\mathrm{f}(z)}{(z-x_1)(z-x_2)\ldots(z-x_m)}\frac{1}{x-z}+\&\text{c.}$$

$$+\mathcal{E}_{x_m}\frac{\mathrm{f}(z)}{(z-x_1)(z-x_2)\ldots(z-x_m)}\frac{1}{x-z}$$

$$=\frac{\mathrm{f}(x_1)}{(x_1-x_2)\ldots(x_1-x_m)}\frac{1}{x-x_1}+\&\text{c.}$$

$$+\frac{\mathrm{f}(x_m)}{(x_m-x_1)\ldots x_m-x_{m-1}}\frac{1}{x-x_m};$$

whence

$$\mathrm{f}(x)=\frac{(x-x_2)\ldots(x-x_m)}{(x_1-x_2)\ldots(x_1-x_m)}\mathrm{f}(x_1)+\&\text{c.}$$

$$+\frac{(x-x_1)\ldots x-x_{m-1}}{(x_m-x_1)\ldots x_m-x_{m-1}}\mathrm{f}(x_m).$$

Again, taking the equation

$$\mathrm{f}(x)=\mathcal{E}\frac{[f(z)]}{x-z},$$

and multiplying both sides by x, it becomes

$$xf(x)=\mathcal{E}\frac{[f(z)]}{1-\dfrac{z}{x}}.$$

On making $x=\infty$, $f(x)=0$, as the denominator of $f(x)$ is supposed to be of higher dimensions than the numerator; therefore $xf(x)$ may have a finite value. Let this be V, then

$$\mathcal{E}f(z)=V \text{ ....} \quad (13).$$

If $V=0$, which will be the case when the degree of the denominator of $f(x)$ surpasses that of the numerator by more than unity, then we have simply

$$\mathcal{E}f(z)=0 \text{ ....} \quad (14).$$

Let $$f(x) = \frac{x^n}{(x-x_1)(x-x_2)\ldots(x-x_m)},$$
we find that

$$\frac{x_1^{\ n}}{(x_1-x_2)\ldots(x_1-x_m)} + \frac{x_2^{\ n}}{(x_2-x_1)\ldots(x_2-x_m)} + \&c.$$
$$+ \frac{x_m^{\ n}}{(x_m-x_1)\ldots(x_m-x_{m-1})} = 1 \text{ or } = 0,$$

according as $n=m-1$, or as $n<m-1$; since in the former case we have $xf(x)=1$ when $x=\infty$, and in the latter $xf(x)=0$ when $x=\infty$.

We have as yet limited ourselves to the consideration of proper fractions, but the Residual Calculus may be extended to all others. To do this, we must prove the following Theorem:

$$\mathcal{E} f(z) = \mathcal{E} \frac{f\left(\frac{1}{z}\right)}{[z^2]} \ldots\ldots\ldots\ldots\ldots\ldots(15).$$

We know by (8) that

$$\frac{f\left(\frac{1}{u}\right)}{u^2} = \mathcal{E} \frac{f\left(\frac{1}{s}\right)}{[s^2](u-s)} + U,$$

where U is a function of u, which remains finite for $u=0$.

Let there be m roots of $f\left(\frac{1}{s}\right)=\infty$, each equal to 0, and let us expand $\frac{1}{u-s}$; then

$$\frac{f\left(\frac{1}{u}\right)}{u^2} = \frac{1}{u}\mathcal{E}\frac{f\left(\frac{1}{s}\right)}{[s^2]} + \frac{1}{u^2}\mathcal{E}\frac{sf\left(\frac{1}{s}\right)}{[s^2]} + \&c. + \frac{1}{u^{m+2}}\mathcal{E}\frac{s^{m+1}f\left(\frac{1}{s}\right)}{[s^2]} + U.$$

Multiplying by u^2, substituting z for $\frac{1}{u}$, and making $Uu^2=\phi(z)$,

$$f(z) = \frac{1}{z}\mathcal{E}\frac{f\left(\frac{1}{s}\right)}{[s^2]} + \mathcal{E}\frac{sf\left(\frac{1}{s}\right)}{[s^2]} + \&c. + z^m\mathcal{E}\frac{s^{m+1}f\left(\frac{1}{s}\right)}{[s^2]} + \phi(z).$$

Now, extracting the residues with regard to z, and observing that

$$\mathcal{E}\left(\frac{1}{z}\right)=1, \quad \mathcal{E}1=0, \quad \mathcal{E}z=0\ldots\mathcal{E}z^m=0,$$

we find
$$\mathcal{E}f(z)=\mathcal{E}\frac{f\left(\frac{1}{s}\right)}{[s^2]}+\mathcal{E}\phi(z) \quad\ldots\ldots\ldots\ldots (16).$$

Now $\phi(z)=Uu^2$, and U has a finite value for $u=0$, therefore $Uu=z\phi(z)$ must vanish. Hence by (14)

$$\mathcal{E}\phi(z)=z\phi(z)=0.$$

Hence, finally, we have

$$\mathcal{E}f(z)=\mathcal{E}\frac{f\left(\frac{1}{s}\right)}{[s^2]}=\mathcal{E}\frac{f\left(\frac{1}{z}\right)}{[z^2]} \quad\ldots\ldots\ldots\ldots (17).$$

In this last formula substitute $\dfrac{f(z)}{z-x}$ for $f(z)$. As the residue is taken with regard to all the roots of $\dfrac{f(z)}{z-x}=\infty$, and one of these is $x=z$, we may separate that partial residue from the others. Thus we have

$$\mathcal{E}\frac{f(z)}{z-x}=\mathcal{E}_x\frac{f(z)}{z-x}+\mathcal{E}\frac{[f(z)]}{z-x}$$
$$=f(x)+\mathcal{E}\frac{[f(z)]}{z-x};$$

so that, by the last theorem,

$$f(x)+\mathcal{E}\frac{[f(z)]}{z-x}=\mathcal{E}\frac{f\left(\frac{1}{z}\right)}{[z^2]\left(\frac{1}{z}-x\right)}=\mathcal{E}\frac{f\left(\frac{1}{z}\right)}{[z](1-zx)}$$

or
$$f(x)=\mathcal{E}\frac{[f(z)]}{x-z}+\mathcal{E}\frac{f\left(\frac{1}{z}\right)}{[z](1-zx)} \quad\ldots\ldots\ldots (18),$$

which is the equation we ought to substitute for the formula (11), if $f(z)$ becomes infinite when $z=\infty$.

If $f(x)$ is an integral function of x, the residue $\mathcal{E}\,\dfrac{[f(z)]}{x-z}$ vanishes, since $f(z)$ has a finite value for any value of z, so that the expression is reduced to

$$f(x) = \mathcal{E}\,\frac{f\left(\frac{1}{z}\right)}{[z]\,(1-zx)} \quad \text{..................} \quad (19).$$

If $f(x)$ be of the form $\dfrac{\mathrm{f}\,(x)}{F(x)}$, and $F(x)$ be of higher dimensions than $\mathrm{f}\,(x)$, the expression (18) is, as was proved before, reduced to $\mathcal{E}\,\dfrac{[f(z)]}{x-z}$, which enables us to decompose a rational fraction. But if $F(x)$ be of lower dimensions than $\mathrm{f}\,(x)$, the function $\dfrac{\mathrm{f}\,(x)}{F(x)}$ is, by (18), divided into two parts, of which $\mathcal{E}\,\dfrac{[f(z)]}{x-z}$ is the sum of the rational fractions remaining from the division of $\mathrm{f}\,(x)$ by $F(x)$, and $\mathcal{E}\,\dfrac{f\left(\frac{1}{z}\right)}{[z]\,(1-zx)}$ is the quotient arising from that division.

Let us take, as an example, $f(x) = \dfrac{1+x^4}{x\,(1+x^2)}$

$$f(x) = \mathcal{E}\,\frac{(1+z^4)}{[z\,(1+z^2)]\,(x-z)} + \mathcal{E}\,\frac{1+z^4}{[z^2]\,(1+z^2)\,(1-zx)}\,.$$

Now

$$\mathcal{E}\,\frac{1+z^4}{[z\,(1+z^2)]\,(x-z)} = \mathcal{E}_0\,\frac{(1+z^4)}{z\,(1+z^2)\,(x-z)} + \mathcal{E}\,+-^{\frac{1}{2}}1\,\frac{1+z^4}{z\,(1+z^2)\,(x-z)}$$

$$+\,\mathcal{E}\,-\,-^{\frac{1}{2}}1\,\frac{1+z^4}{z\,(1+z^2)\,(x-z)}$$

$$= \frac{1}{x} - \frac{1}{x+-^{\frac{1}{2}}1} - \frac{1}{x--^{\frac{1}{2}}1}\,,$$

and

$$\mathcal{E}\,\frac{1+z^4}{[z^2]\,(1+z^2)\,(1-zx)} = \frac{d}{dh}\,\frac{(1+h^4)}{(1+h^2)\,(1-xh)} \text{ when } h=0, = x.$$

Therefore, finally,

$$\frac{1+x^4}{x(1+x^2)} = x + \frac{1}{x} - \frac{1}{x + -^{\frac{1}{2}}1} - \frac{1}{x - -^{\frac{1}{2}}1}$$

$$= x + \frac{1}{x} - \frac{2x}{1+x^2}.$$

The length to which this article has extended, renders it necessary to postpone to a future Number other illustrations of the use of the Residual Calculus.

DEMONSTRATIONS OF SOME PROPERTIES OF A TRIANGLE.*

THE following properties of a triangle are not perhaps generally known. If we join D, E, F, (fig. 5), the points in which perpendiculars on the sides from the opposite angles intersect the sides, the triangles DEC, DBF, AFE are similar to each other, and to the triangle ABC: and the triangle DEF is the triangle of least perimeter which can be inscribed in the triangle ABC.

We shall first prove that two adjacent sides of the triangle DEF make equal angles with the side of the triangle ABC: that is, that $BDF = EDC$, $DFB = EFA$, $AEF = DEC$.

We have $$BDF = DFC + DCF,$$

as it is the exterior and opposite angle of the triangle DFC. For a similar reason $EDC = DBE + DEB$.

Now, since the three perpendiculars AD, BE, CF pass through the same point G, and in the quadrilateral $BFGD$ the angles at D and F are together equal to two right angles, the four points B, F, G, D are in the circumference of a circle: and the angles DBG and DFG, or DBE and DFC, being angles in the same segment, are equal to one another. In the same way we see that $DCF = DEB$, consequently $BDF = CDE$, and in a similar manner we might prove that $BFD = EFA$ and $FEA = DEC$.

* *Cambridge Mathematical Journal*, Vol. I., p. 157.

We shall next prove that $EDC = BAC$, $DEC = ABC$, and $BFD = BCA$. For we have, as before,

$$EDC = DBE + DEB.$$

Now, since ADB and AEB are right angles, they are angles in a semicircle, so that A, B, D, E are points in the circumference of a circle, and DBE and DAE being angles in the same segment of a circle are equal, and also DAB and DEB for the same reason. But $BAD + DAE = BAE$, so that $EDC = BAC$; and similarly for the others. Comparing the triangles DEC and ABC, we see that the angle at C is common, the angles $CDE = BAC$ and the angle $DEC = ABC$, therefore they are equiangular and similar. In the same way DBF and FAE are similar to ABC, and the three triangles DEC, DBF, and FAE, being similar to ABC are similar to one another.

To prove that DEF is the triangle of least perimeter which can be inscribed in ABC, we must avail ourselves of a known property, that, if from two points without a line we draw two straight lines to a point in the line, their sum will be a minimum when the two lines make equal angles with the given line. Now, in this case, having proved that FD and ED make equal angles with BC, we see that, supposing F and E fixed, the sum of FD and DE is the least of all those which can be drawn from F and E to any point in BC. The same holds true of the other points F and E, so that on the whole the perimeter FED is the least of all those which can be inscribed in ABC.

Various other properties may be demonstrated of this triangle of least perimeter. If we call the angle $BAD = \alpha$, $EBC = \beta$, $FCA = \gamma$, it is easy to see that $FED = 2\alpha$, $EFD = 2\beta$, $FDE = 2\gamma$, and therefore $2(\alpha + \beta + \gamma)$ is equal to two right angles, or $\alpha + \beta + \gamma$ is equal to a right angle. From the construction of the figure it appears that

$$\alpha = \frac{\pi}{2} - B, \quad \beta = \frac{\pi}{2} - C, \quad \gamma = \frac{\pi}{2} - A.$$

Also, since the angles at D, E, F are bisected by the lines intersecting at G, that point is the centre of the circle inscribed in the triangle of least perimeter. If we draw perpendiculars from the vertices of the triangle ABC to the sides of DEF, it will be seen from the similarity of the triangles that they will divide the angles into the same parts as the perpendiculars on the sides of ABC, but in an inverted position. Now, calling the angles as before α, β, γ, AD makes angles α and β with AF, AE respectively, and therefore the perpendicular on FE makes angles α and β with AE, AF respectively. The position of AD may be determined by expressing the relation between α and β, which may be put under the form $\beta = \phi(\alpha)$. Similarly, the position of BE will be determined by $\gamma = \phi_1(\beta)$, and the position of CF by $\alpha = \phi_2(\gamma)$. But since these lines intersect in one point, one of the three equations must be deducible from the other two, as the combination of two at their point of intersection must coincide with each other. Now the perpendiculars on the sides of the triangle of least perimeter being defined by the same equations, of which one is derivable from the other two, must all pass through one point. This may be clearer if we take the actual relations, which will be found to be

$$\alpha = C - B + \beta$$
$$\beta = A - C + \gamma$$
$$\gamma = B - A + \alpha,$$

the latter of which is clearly derivable from the other two, so that they are not independent, and therefore the lines expressed by them must all pass through one point.

The perimeter of the triangle DEF may be easily found. For, from the similarity of the triangles ABC, DEF, we have

$$FD : BD = AC :: AB = b : c, \text{ or } FD = \frac{b}{c} BD,$$

but $BD = c \cos B$, so that $FD = b \cos B$.

Similarly, $DE = c \cos C$, and $FE = a \cos A$; so that if p be the perimeter of DEF,

$$p = a \cos A + b \cos B + c \cos C.$$

We may likewise obtain an expression for the perimeter involving only the sides of the triangle ABC. The area of the triangle DEF

$$= \tfrac{1}{2} FD . DE \sin 2\gamma = \tfrac{1}{2} bc \cos B \cos C \sin 2A.$$

Since $\gamma = \dfrac{\pi}{2} - A$ and $2\gamma = \pi - 2A$.

Now if r be the radius of the circle inscribed in DEF, its area $= \tfrac{1}{2} pr$.

But $r = GD \sin\gamma = GD \cos A = CD \tan\alpha \cos A$

$$= b \cos C \cos A \tan\alpha = b\,\frac{\cos A \cos B \cos C}{\sin B}.$$

Therefore

$$\tfrac{1}{2} bp\,\frac{\cos A \cos B \cos C}{\sin B} = \tfrac{1}{2} bc \cos B \cos C \sin 2A$$

$$= bc \cos A \cos B . \cos C \sin A,$$

or
$$p = 2c \sin A \sin B.$$

Or, substituting for $\sin A$ and $\sin B$, their values in terms of the sides of ABC,

$$p = 2c\,\frac{2}{bc} . \frac{2}{ac}\,\{S(S-a)(S-b)(S-c)\}$$

or
$$p = \frac{8}{abc}\,\{S(S-a)(S-b)(S-c)\}.$$

If R' be the radius of the circle described round ABC, we have, by taking its value in terms of the sides of the triangle,

$$p = \frac{abc}{2R'^2}.$$

Also, if R be the radius of the circle inscribed in ABC, we know that, generally,

$$2RR' = \frac{abc}{a+b+c} = \frac{abc}{P},$$

if P be the perimeter of ABC. Substituting the value of R'^2, from this we have

$$p = 2\frac{P^2R^2}{abc}.$$

And therefore if K be the area of ABC,

$$p = \frac{8K^2}{abc}.$$

The area of DEF is very easily found in terms of the area of ABC, for we found before that it was equal to

$$\tfrac{1}{2}bc \sin 2A \cos B \cos C = bc \sin A \cos A \cos B \cos C$$
$$= 2K \cos A \cos B \cos C = k \text{ suppose.}$$

If a', b', c' be the sides of DEF,

$$a' = a \cos A, \quad b' = b \cos B, \quad c' = c \cos C,$$

and therefore substituting for $\cos A \cos B \cos C$,

$$k = 2K\frac{a'b'c'}{abc}.$$

But we have $\qquad abc = 2PRR' = 4KR'$,

and therefore $\qquad k = \dfrac{a'b'c'}{2R'}$.

It will be seen at once that the areas of the triangles FAE, DBF, DCE, are respectively

$$K \cos^2 A, \quad K \cos^2 B, \quad K \cos^2 C;$$

as the sum of these together with DEF make up ABC, we have, dividing by K,

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\frac{a'b'c'}{abc} = 1.$$

A singular relation exists between the radii of the circles described round ABC and DEF. Let them be R' and r' respectively. Then,

$$a=2R'\sin A,\quad a'=2r'\sin 2\gamma=2r'\sin 2A.$$

Therefore $\quad a'=a\cos A=4r'\sin A\cos A.$

Whence $\quad R'=2r'.$

We found for r the radius of the circle inscribed in DEF,

$$r=\frac{b}{\sin B}\cos A\cos B\cos C,$$

and as $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R';$$

therefore $$r=2R'\cos A\cos B\cos C=R'\frac{k}{K}.$$

Also, since $r=\dfrac{2k}{p}=\dfrac{4K}{p}\cos A\cos B\cos C,$

$$R'=\frac{2K}{p},\text{ and } r'=\frac{K}{p}.$$

Since the sides of any one of the triangles round DEF are equal to the sides of the original triangle multiplied by the cosine of the corresponding angle, all similarly situate lines in the triangles, and therefore the radii of the circumscribing circles, will have the same ratio.

Now the circle described round FAE passes through G, as A, F, E, G, are in the circumference of the same circle: and as AEG, AFG are right angles, AG must be the diameter of the circle, therefore $AG=2R'\cos A$.

Similarly,

$$BG=2R'\cos B,\quad CG=2R'\cos C.$$

We find also,

$$GD=2R'\cos B\cos C,\quad GE=2R'\cos C\cos A,$$

$$GF=2R'\cos A\cos B.$$

To these geometrical properties we may add the following optical one.

If the interior sides of the triangle be reflecting surfaces, and a ray of light be incident on one of the sides in a direction parallel to one of the sides of the inscribed triangle as to FD, it will, after two reflexions from each side, return to its original direction so as to continue the same course *ad infinitum*. This property depends of course on the equality of the angles of incidence and reflection, so that the ray being incident parallel to one of the sides of FDE will continue its course successively parallel to the other sides. Let fd be the incident ray which being incident parallel to FD will be reflected at d parallel to DE, and falling on AC at e is reflected parallel to EF meeting AB in f'. Proceeding from that point in the same manner, it will after reflection again from BC and AC meet AB again in some point f'': our object is to show that f'' coincides with f. For if it does so, it is clear from the course of the ray being parallel to the sides of EFD that it will go over the same track again, and so on.

Since the triangles Bdf, ABC are similar, we have

$$BC.Bd = AB.Bf.$$

For the like reason we have

$$\begin{aligned} BC.Cd &= AC.Ce \\ AB.Af' &= AC.Ae \\ AB.Bf' &= BC.Bd' \\ AC.Ce' &= BC.Cd' \\ AC.Ae' &= AB.Af''. \end{aligned}$$

Adding these together, and observing that we have

$$Bd + Cd = BC, \quad Ce + Ae = AC, \quad Af' + Bf' = AB,$$
$$Ae' + Ce' = AC, \quad Bd' + Cd' = BC,$$

we have

$$BC^2 + AB^2 + AC^2 = AC^2 + BC^2 + AB.Bf + AB.Af'',$$

whence $$AB^2 = AB\,(Bf + Af''),$$

or $$AB = Bf + Af''.$$

Therefore $Af = Af''$ and f and f'' coincide, and the proposition is manifest.

This and several of the other properties of the triangle proved here are due to Professor Wallace of Edinburgh.

ON THE INTEGRATION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS.*

In the present article we shall apply to Simultaneous Linear Differential Equations the principles, the applications of which we have developed in the preceding Numbers of the *Mathematical Journal.* The usual method for solving these equations was first given by D'Alembert, and has received but little improvement since his time, although it is so long and tedious, that some change was highly desirable. The process which we shall give here is at once simple and direct, and shews the advantage of recurring frequently to the principles on which our calculations are founded. The theory of the method is sufficiently simple. Since we have shewn that the symbols of differentiation are subject to the same laws of combination as those of number, they may be always treated in the same manner if the coefficients be all constants, which is the only case we shall consider. We have therefore only to separate the symbol of differentiation from its subject, and then proceed to eliminate one of the variables between the given equations, exactly as if the symbol of differentiation were an ordinary coefficient. Thus the difficulty of elimination becomes reduced to that between ordinary and algebraical equations; and all the facilities for effecting it which have been invented for these in particular cases, may be used for differential

* *Cambridge Mathematical Journal,* Vol. I., p. 173.

equations. When the equations involve various orders of differentials of a variable as well as multiples of it, the symbols of differentiation and of number must be grouped together as one coefficient for the variable. Thus, if we had

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + b\frac{d^2y}{dt^2} + c\frac{dy}{dt} + mx + ny = T,$$

$$\frac{d^2y}{dt^2} + a'\frac{dy}{dt} + b'\frac{d^2x}{dt^2} + c'\frac{dx}{dt} + m'y + n'z = T';$$

they are to be written

$$\left(\frac{d^2}{dt^2} + a\frac{d}{dt} + m\right)x + \left(b\frac{d^2}{dt^2} + c\frac{d}{dt} + n\right)y = T,$$

$$\left(\frac{d^2}{dt^2} + a'\frac{d}{dt} + m'\right)y + \left(b\frac{d^2}{dt^2} + c'\frac{d}{dt} + n'\right)x = T',$$

and then treated as if each symbol of operation were a factor.

We shall proceed to give some examples; and first let us take

$$\frac{dx}{dt} + ay = 0,$$

$$\frac{dy}{dt} + bx = 0.$$

We have to separate $\frac{d}{dt}$ from the variable, and eliminate one of the variables y or x, as would be done if $\frac{d}{dt}$ were an ordinary constant. This will be done if we multiply the first equation by $\frac{d}{dt}$ and the second by a, and subtract, when we obtain

$$\left(\frac{d^2}{dt^2} - ab\right)x = 0.$$

(It is to be observed, that the word "multiply" is used, not because the operation is really multiplication, since that is a numerical operation, but because it bears a close analogy to multiplication, and is represented symbolically in the same manner.)

Having now eliminated y, we may integrate the equation in x at once. The result is

$$x = c_1 \varepsilon^{(ab)^{\frac{1}{2}}t} + c_2 \varepsilon^{-(ab)^{\frac{1}{2}}t}.$$

And from the first equation we get

$$y = c_2 \left(\frac{b}{a}\right)^{\frac{1}{2}} \varepsilon^{-(ab)^{\frac{1}{2}}t} - c_1 \left(\frac{b}{a}\right)^{\frac{1}{2}} \varepsilon^{(ab)^{\frac{1}{2}}t}.$$

As another example, take the equations

$$\frac{dx}{dt} + ax + by = 0,$$

$$\frac{dy}{dt} + a_1 x + b_1 y = 0;$$

which may be put under the form

$$a\left(\frac{d}{dt} + a\right) x + by = 0,$$

$$a_1 x + \left(\frac{d}{dt} + b_1\right) y = 0.$$

Multiply the first by $\frac{d}{dt} + b_1$, and the second by b, and subtract; then

$$\left\{\left(\frac{d}{dt} + a\right)\left(\frac{d}{dt} + b_1\right) - a_1 b\right\} x = 0.$$

The coefficient of x is obviously of the second order. Let it be made up of two factors, so that

$$\left(\frac{d}{dt} + h\right)\left(\frac{d}{dt} + k\right) x = 0,$$

where $\qquad h + k = a + b_1, \quad hk = ab_1 - a_1 b;$

so that they are the roots of the equation

$$z^2 - (a + b_1) z + ab_1 - a_1 b = 0.$$

Integrating in the usual manner,

$$x = c_1\varepsilon^{-ht} + c_2\varepsilon^{-kt};$$

and from the first equation we deduce

$$y = \frac{h-a}{b}\, c\varepsilon^{-ht} + \frac{k-a}{b}\, c_1\varepsilon^{-kt}.$$

It will make no difference in the method of the solution, if there be a function of t on the other side of the equation. As, for instance, we have

$$\frac{dx}{dt} + 4x + 3y = t,$$

$$\frac{dy}{dt} + 2x + 5y = \varepsilon^t,$$

or

$$\left(\frac{d}{dt} + 4\right) x + 3y = t,$$

$$\left(\frac{d}{dt} + 5\right) y + 2x = \varepsilon^t$$

Eliminating y,

$$\left\{\left(\frac{d}{dt} + 4\right)\left(\frac{d}{dt} + 5\right) - 6\right\} x = 1 + 5t - 3\varepsilon^t,$$

or

$$\left(\frac{d}{dt} + 2\right)\left(\frac{d}{dt} + 7\right) x = 1 + 5t - 3\varepsilon^t.$$

Integrating with regard to $\frac{d}{dt} + 7$,

$$\left(\frac{d}{dt} + 2\right) x = \frac{2}{49} + \frac{5}{7}\, t - \frac{3}{8}\,\varepsilon^t + c_1\varepsilon^{-7t},$$

and

$$x = -\frac{31}{196} + \frac{5}{14}\, t - \frac{1}{8}\,\varepsilon^t + c_1\varepsilon^{-7t} + c_2\varepsilon^{-2t},$$

the value of y will be found to be

$$y = \frac{9}{98} - \frac{1}{7}\, t + \frac{5}{24}\,\varepsilon^t - \frac{8}{3}\,\varepsilon^{-7t} - c_2\varepsilon^{-2t}.$$

Take also the equations

$$\frac{dx}{dt} + 5x + y = \varepsilon^t,$$

$$\frac{dy}{dt} + 3y - x = \varepsilon^{2t}.$$

Eliminating y, we obtain

$$\left(\frac{d}{dt} + 4\right)^2 x = 4\varepsilon^t - \varepsilon^{2t};$$

whence $$x = \left(\frac{d}{dt} + 4\right)^{-2} (4\varepsilon^t - \varepsilon^{2t}) + (c_1 t + c_2)\,\varepsilon^{-4t},$$

or $$x = \frac{4}{25}\,\varepsilon^t - \frac{1}{36}\,\varepsilon^{2t} + (c_1 t + c_2)\,\varepsilon^{-4t},$$

and $$y = \frac{1}{25}\,\varepsilon^t + \frac{7}{36}\,\varepsilon^{2t} + (c_1 t + c_2)\,\varepsilon^{-4t}.$$

If there be three simultaneous equations, the same method is to be pursued, but the calculations must be necessarily long, as in ordinary elimination. We may, however, avail ourselves of the method of cross multiplication. If, for instance, we have

$$\frac{dx}{dt} + by + cz = 0,$$

$$\frac{dy}{dt} + ax + c'z = 0,$$

$$\frac{dz}{dt} + a'x + b'y = 0.$$

or $$\frac{d}{dt}\,x + by + cz = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$ax + \frac{d}{dt}\,y + c'z = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$a'x + b'y + \frac{d}{dt}\,z = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

we may eliminate y and z by multiplying

(1) by $\dfrac{d^2}{dt^2} - b'c'$, (2) by $b'c - b\dfrac{d}{dt}$, (3) by $bc' - c\dfrac{d}{dt}$,

and the result will be

$$\left\{\frac{d}{dt}\left(\frac{d^2}{dt^2}-b'c'\right)+a\left(b'c-b\,\frac{d}{dt}\right)+a'\left(bc'-c\,\frac{d}{dt}\right)\right\}x=0,$$

or
$$\left\{\frac{d^3}{dt^3}-(ab+a'c+b'c')\,\frac{d}{dt}+ab'c+a'bc'\right\}x=0,$$

which can be integrated by the usual method. The equations for determining y and z will, of course, be symmetrical with that for x.

There will be three arbitrary constants arising from the integration of this equation; and it would appear, that as there are other two similar equations, from which six more arbitrary constants would arise, there would be nine on the whole. But there are really only three independent arbitrary constants, as we are able to deduce the other two variables y and z from the value of the first, without integration, and consequently the arbitrary constants in their expressions must be derivable from those in the first integral.

The same method of integration may of course be applied to any number of simultaneous differential equations; but the difficulty of elimination rises rapidly with the number of variables, as in the case of ordinary numerical equations. We shall take as an example of four simultaneous equations, those given by Mr. Airy for determining the secular variations of the eccentricity and longitude of the perihelion, *Planetary Theory*, p. 123.

They are of the form

$$0=\frac{d}{dt}\,u+a_1v-a_2v' \quad\text{.................}(1),$$

$$0=a_1u-\frac{d}{dt}\,v-a_2u' \quad\text{.................}(2),$$

$$0=\frac{d}{dt}\,u'+b_1v'-b_2v \quad\text{.................}(3),$$

$$0=b_1u'-\frac{d}{dt}\,v'-b_2u \quad\text{.................}(4).$$

To reduce them to three simultaneous equations, eliminate v by multiplying (1) by $\frac{d}{dt}$ and (2) by a_1, and adding. Then

$$\left(\frac{d^2}{dt^2}+a_1^{\,2}\right)u-a_1a_2u'-a_2\frac{d}{dt}v'=0 \text{ .... (5).}$$

Again, eliminate the same variable between (1) and (3), by multiplying (1) by b_2, and (3) by a_1, and adding, so that

$$b_2\frac{d}{dt}u+a_1\frac{d}{dt}u'+(a_1b_1-a_2b_2)\,v'=0 \text{.........(6),}$$

and the equation (4) is

$$b_2u-b_1u'+\frac{d}{dt}v=0 \text{ ....(7).}$$

Eliminating u' and v' by cross multiplication between (5), (6), and (7), we obtain an equation which reduces itself to

$$\left\{\frac{d^4}{dt^4}+(a_1^{\,2}+b_1^{\,2}+2a_2b_2)\frac{d^2}{dt^2}+a_1b_1-a_2b_2\right\}u=0 \text{ ...(8),}$$

which we shall have no difficulty in integrating.

From its form we see that the operating factor is decomposable into two binomial quadratic factors, so that we may put it under the form

$$\left(\frac{d^2}{dt^2}+k_1^{\,2}\right)\left(\frac{d^2}{dt^2}+k_2^{\,2}\right)u=0,$$

$-k_1^{\,2}$, $-k_2^{\,2}$ being the roots of the equation

$$z^2+(a_1^{\,2}+b_1^{\,2}+2a_2b_2)\,z+a_1b_1-a_2b_2=0 \text{ ....(9).}$$

Integrating with respect to the first factor, as in p. 28 of the *Mathematical Journal*,*

$$\left(\frac{d^2}{dt^2}+k_2^{\,2}\right)u=c_1\cos k_1x+c_2\sin k_1x,$$
$$=c_1\cos(k_1x+\alpha),$$

by changing the form of the constants; whence

$$u=\left(\frac{d^2}{dt^2}+k_2^{\,2}\right)^{-1}\{c_1\cos(k_1x+\alpha_1)\}+c_2\cos(k_2x+\alpha_2).$$

* See page 21 of this volume.

The operating factor will only affect the constant of its subject, and as that is arbitrary, we may write

$$u = c_1 \cos(k_1 x + \alpha_1) + c_2 \cos(k_2 x + \alpha_2).$$

To find the value of v, we may observe, that if we had eliminated v' instead of v at first, we should have, instead of (6) and (7), the similar equations

$$a_2 \frac{d}{dt} u' + b_1 \frac{d}{dt} u + (a_1 b_1 - a_2 b_2) v = 0,$$

$$a_2 u' - a_1 u + \frac{d}{dt} v = 0.$$

Eliminating u' between these by multiplying the second by $\frac{d}{dt}$, and subtracting,

$$\left\{\frac{d^2}{dt^2} - (a_1 b_1 - a_2 b_2)\right\} v = (a_1 + b_1) \frac{du}{dt};$$

or putting for u its value, and effecting the differentiation indicated,

$$\left\{\frac{d^2}{dt^2} - (a_1 b_1 - a_2 b_2)\right\} v$$

$$= -(a_1 + b_1) \{c_1 k_1 \sin(k_1 x + \alpha_1) + c_2 k_2 \sin(k_2 x + \alpha_2)\};$$

whence, putting $a_1 b_1 - a_2 b_2 = A$,

$$v = (a_1 + b_1) \left\{\frac{c_1 k_1}{k_1^2 + A} \sin(k_1 x + \alpha_1) + \frac{c_2 k_2}{k_2^2 + A} \sin(k_2 x + \alpha_2)\right\}.$$

By substituting the actual expressions for k_1 and k_2, this may be simplified. For solving equation (9), we get

$$k_1^2 = \tfrac{1}{2} [a_1^2 + b_1^2 + 2a_2 b_2 - (a_1 + b_1) \{(a_1 - b_1)^2 + 4a_2 b_2\}^{\frac{1}{2}}]:$$

then $k_1^2 + A = \frac{1}{2} (a_1 + b_1) [a_1 + b_1 - \{(a_1 - b_1)^2 + 4a_2 b_2\}^{\frac{1}{2}}]$.

But we have also

$$k_1 = \tfrac{1}{2} [a_1 + b_1 - \{(a_1 - b_1)^2 + 4a_2 b_2\}^{\frac{1}{2}}],$$

and similarly for k_2; so that

$$v = c_1 \sin(k_1 x + \alpha_1) + c_2 \sin(k_2 x + \alpha_2).$$

Knowing u and v, we easily determine u' and v'. For, by equation (2),

$$a_2 u' = a_1 u - \frac{d}{dt} v$$

$$= (a_1 - k_1) c_1 \cos(k_1 x + \alpha_1) + (a_1 - k_2) c_2 \cos(k_2 x + \alpha_2);$$

similarly v' may be found.

In this case we have had an example of the integration of simultaneous differential equations, of an order higher than the first. We shall take another in

$$\frac{d^2 y}{dt^2} - az - by = c,$$

$$\frac{d^2 z}{dt^2} - a'z - b'y = c',$$

being two of the equations for determining the circumstances of the movement of a floating body in a position nearly of equilibrium, the other two being

$$\frac{d^2 \phi}{dt^2} + n\phi - m = 0,$$

$$\frac{d^2 s}{dt^2} - g\frac{d^2 \phi}{dt^2} = 0.$$

Putting the equations under the form

$$\left(\frac{d^2}{dt^2} - b\right) y - az = c,$$

$$\left(\frac{d^2}{dt^2} - a'\right) z - b'y = c',$$

we can eliminate z by multiplying the first by $\frac{d^2}{dt^2} - a'$, and the second by a, and adding, when we get

$$\left(\frac{d^2}{dt^2} - a'\right)\left(\frac{d^2}{dt^2} - b\right) y - ab'y = ac' - ca'.$$

If k_1^2, k_2^2 be the roots of

$$z^2 - (a' + b) z + a'b - ab' = 0,$$

the equation becomes

$$\left(\frac{d^2}{dt^2}-k_1^{\,2}\right)\left(\frac{d^2}{dt^2}-k_2^{\,2}\right)y = ac' - ca';$$

the integral of which is

$$y = \frac{ac'-ca'}{k_1^{\,2}k_2^{\,2}} + c_1\varepsilon^{k_1t} + c_2\varepsilon^{-k_1t} + c_3\varepsilon^{k_2t} + c_4\varepsilon^{-k_2t},$$

$$y = \frac{ac'-ca'}{a'b-ab'} + c_1\varepsilon^{k_1t} + c_2\varepsilon^{-k_1t} + c_3\varepsilon^{k_2t} + c_4\varepsilon^{-k_2t}.$$

The value of z may be found from that of y; and if we know the initial circumstances, we can determine the arbitrary constants.

The same method may be extended to simultaneous partial differential equations, according to the principles developed in the Article 'On the Solution of Partial Differential Equations,' p. 62. Take, for instance, the equations

$$\frac{dz}{dx} + c\frac{du}{dx} + a\frac{dz}{dy} + bz = 0,$$

$$c'\frac{dz}{dx} + \frac{du}{dx} + a\frac{du}{dy} + bu = 0,$$

which may be written

$$\left(\frac{d}{dx} + a\frac{d}{dy} + b\right)z + c\frac{d}{dx}u = 0,$$

$$c'\frac{d}{dx}z + \left(\frac{d}{dx} + a\frac{d}{dy} + b\right)u = 0.$$

To eliminate u, multiply the first by $\left(\frac{d}{dx} + a\frac{d}{dy} + b\right)$, and the second by $c'\frac{d}{dx}$, and subtract. Then

$$\left\{\frac{d^2}{dx^2} + \frac{2\left(a\frac{d}{dy}+b\right)}{1-cc'}\frac{d}{dx} + \frac{\left(a\frac{d}{dy}+b\right)^2}{1-cc'}\right\}z = 0.$$

If we put $1-\sqrt{(cc')}=m_1$, $1+\sqrt{(cc')}=m_2$, this may be resolved into factors,

$$\left\{\frac{d}{dx}+m_1\left(a\frac{d}{dy}+b\right)\right\}\left\{\frac{d}{dx}+m_2\left(a\frac{d}{dy}+b\right)\right\}z=0;$$

the integral of which is

$$z=\varepsilon^{-m_1x\left(a\frac{d}{dy}+b\right)}\phi(y)+\varepsilon^{-m_2x\left(a\frac{d}{dy}+b\right)}\psi(y),$$

or
$$z=\varepsilon^{-m_1bx}\phi(y-am_1x)+\varepsilon^{-m_2bx}\psi(y-am_2x).$$

Mr. Airy, at p. 279, *Note of the Undulatory Theory*, gives as the equations for determining the small disturbances of an elastic medium in three dimensions

$$\frac{d^2u}{dt^2}=a^2\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right),$$

$$\frac{d^2v}{dt^2}=a^2\frac{d}{dy}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right),$$

$$\frac{d^2w}{dt^2}=a^2\frac{d}{dz}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right).$$

We might eliminate v and w, by cross multiplication, between these equations, and so obtain an equation in u which might be integrated, but it will be more convenient to proceed as follows. Let

$$r=a^2\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right):$$

then
$$\frac{d^{-1}}{dx^{-1}}\frac{d^2u}{dt^2}=\frac{d^{-1}}{dy^{-1}}\frac{d^2v}{dt^2}=\frac{d^{-1}}{dz^{-1}}\frac{d^2w}{dt^2}=r.$$

Multiply by $\frac{d^2}{dx^2}$, $\frac{d^2}{dy^2}$, $\frac{d^2}{dz^2}$, and add: then

$$\frac{d^2}{dt^2}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)r,$$

or
$$\left\{\frac{d^2}{dt^2}-a^2\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\right\}r=0;$$

the integral of which is

$$r = \varepsilon^{at\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)^{\frac{1}{2}}} \phi(x, y, z) + \varepsilon^{-at\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)^{\frac{1}{2}}} \psi(x, y, z);$$

knowing r, we can determine u, v, and w, since

$$u = \frac{d^{-2}}{dt^{-2}}\frac{dr}{dx}, \quad v = \frac{d^{-2}}{dt^{-2}}\frac{dr}{dy}, \quad w = \frac{d^{-2}}{dt^{-2}}\frac{dr}{dz}.$$

This solution is due to Mr. Greatheed.

We have now given a sufficient number of examples to enable the student to understand thoroughly the method, and we think that they show clearly the advantages of a process, which, to some persons, might appear to carry out to a startling extent the principles on which it is founded.

GEOMETRICAL THEOREM.*

LET $A_1A_2A_3 \ldots$ be a polygon of n sides inscribed in a circle, α_1, α_2, α_3, &c. the angles which the sides A_1A_2, A_2A_3, &c. subtend at the centre. Then

$$\angle A_1A_2A_3 = \pi - A_1A_5A_3,$$

$$= \pi - \frac{\alpha_1 + \alpha}{2},$$

$$\angle A_3A_4A_5 = \pi - \frac{\alpha_3 + \alpha_4}{2},$$

&c. &c.

$$\angle A_{n-1}A_nA_1 = \pi - \frac{\alpha_{n-1} + \alpha_n}{2}.$$

If n be even, adding all these together, we get

$$\angle A_1A_2A_3 + \angle A_3A_4A_5 + \text{\&c.} + \angle A_{n-1}A_nA_1$$

$$= \frac{n}{2}\pi - \frac{2\pi}{2} = (n-2)\frac{\pi}{2},$$

or the sum of the alternate angles is equal to $n-2$ right angles, a curious extension of Euclid, III. 22.

* *Cambridge Mathematical Journal*, Vol. I., p. 192.

DEMONSTRATIONS OF THEOREMS IN THE DIFFERENTIAL CALCULUS AND CALCULUS OF FINITE DIFFERENCES.*

I PROPOSE in this Article to bring together the more important of the theorems in the Differential Calculus and in the Calculus of Finite Differences, which, depending on one common principle, can be proved by the method of the separation of symbols. These theorems are usually demonstrated by induction in each particular case, which, although a method satisfactory so far as it goes, wants that generality which is desirable in Analytical Demonstrations. As the ordinary Binomial Theorem is the basis on which these theorems are founded, it will not be amiss to say a few words by way of preface regarding the extent of its application, which being said once for all, will prevent useless repetition when we treat of each particular case.

The theorem that

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{1.2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3+\&c.$$

is originally proved when a and b are numbers, and $(a+b)^n$ represents the repetition of the operation n times, implying that n is an integer number. Having the form of the expansion once suggested, it can be shown, by the method

* *Cambridge Mathematical Journal*, Vol. I., p. 212.

of Euler, that the same form is true when n is a fraction or negative number; in which case the left-hand side of the equation acquires different meanings. Moreover, it will be found, on examining Euler's demonstration, that it includes not only these cases, but also all those in which a, b, and n are operations subject to certain laws; for it may be seen, that in the proof no other properties are presumed than that a, b, and n are distributive and commutative functions, and that a^n, b^n are subject to the laws of index functions. These laws are

(1) The commutative, $ab = ba$,

(2) The distributive, $c(a+b) = ca + cb$,

(3) The index law, $a^m.(a^n) = a^{m+n}$.

Now, since it can be shown that the operations both in the Differential Calculus and the Calculus of Finite Differences are subject to these laws, the Binomial Theorem may be at once assumed as true with respect to them, so that it is not necessary to repeat the demonstration of it for each case.* This being premised, I proceed to consider the particular cases of the applications of these theorems.

1. If $u = f(x, y)$ be a function of two independent variables, we find that

$$d(u) = \frac{du}{dx}dx + \frac{du}{dy}dy = \left(\frac{d}{dx}dx + \frac{d}{dy}dy\right)u.$$

by separating the symbols. Now, if we wish to find the n^{th} differential of a function of two variables, we have merely, by the principle of indices, to affix the index n to the sign of operation on both sides, when we get

$$d^n(u) = \left(\frac{d}{dx}dx + \frac{d}{dy}dy\right)^n u.$$

* It is scarcely necessary to add, that those theorems which depend on the binomial, as the polynomial and exponential, are equally extensive, so that they too may be applied to the Differential Calculus and Calculus of Finite Differences.

Now the operation on the second side, being a binomial raised to a power, may, by what has just been said, be expanded by the binomial theorem, so that we have

$$d^n u = \frac{d^n u}{dx^n} dx^n + n \frac{d^{n-1}u}{dx^{n-1}} \frac{du}{dy} + \frac{n(n-1)}{1.2} \frac{d^{n-2}u}{dx^{n-2}} \frac{d^2u}{dy^2} - \&c.$$

This theorem, which can be proved by induction only for positive integer powers of n, that is, for cases of ordinary differentiation, is shown by this method to be true when n is fractional or negative, that is, in the cases of integration and general differentiation.

If we suppose u to be a function of three or more variables, we might by means of the polynomial theorem, expand $d^n.(u)$; but it is not necessary to dwell upon the result, as there is little interest attached to it.

2. I shall next proceed to the elegant theorem of Leibnitz, for finding the n^{th} differential of the product of two functions, a theorem which, when generalized, is most fertile in consequences.

Let u, v be the two functions. Then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

This may be put under the form

$$\frac{d}{dx}(uv) = \left(\frac{d'}{dx} + \frac{d}{dx}\right) uv,$$

if we agree to represent by $\frac{d'}{dx}$ an operation which acts on v, but not on u, and by $\frac{d}{dx}$ an operation which acts on u and not on v. These operations from their nature are distributive, and as they are independent of each other, they must be commutative; hence they come under the circumstances to which the binomial theorem applies. Taking then the n^{th} differential,

$$\left(\frac{d}{dx}\right)^n (uv) = \left(\frac{d'}{dx} + \frac{d}{dx}\right)^n uv$$

$$= \left\{\left(\frac{d'}{dx}\right)^n + n\left(\frac{d'}{dx}\right)^{n-1}\frac{d}{dx} + \frac{n.n-1}{1.2}\left(\frac{d'}{dx}\right)^{n-2}\left(\frac{d}{dx}\right)^2 + \&c.\right\} uv;$$

or applying the operations directly to the quantities which they affect,

$$= u\frac{d^n v}{dx^n} + n\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \frac{n(n-1)}{1.2}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}} + \&c.$$

This theorem is true, like the former, when n is negative or fractional. In the former case, the form is the same as the series at which we arrive by integration by parts, which we thus see to be a particular case of the theorem of Leibnitz.

3. In this expression, when n is negative, let $v = 1$. Then

$$\frac{d^{-n}v}{dx^{-n}} = \frac{x^n}{n\,!}, \quad \frac{d^{-(n+1)}v}{dx^{-(n+1)}} = \frac{x^{n+1}}{(n+1)\,!}, \quad \&c.$$

so that

$$\left(\frac{d}{dx}\right)^{-n} u = \int^n dx^n u = \frac{x^n}{n\,!}u - n\frac{x^{n+1}}{(n+1)\,!}\frac{du}{dx}$$

$$+ \frac{n(n+1)}{1.2}\frac{x^{n+2}}{(n+2)\,!}\frac{d^2u}{dx^2} + \&c.$$

$$= \frac{x^{n-1}}{(n-1)\,!}\left(\frac{x}{n}u - \frac{x^2}{n+1}\frac{du}{dx} + \frac{1}{1.2}\frac{x^3}{n+2}\frac{d^2u}{dx^2} - \&c.\right),$$

which is the general expression for the n^{th} integral of any function.

4. In this last formula, if we make $n = 1$ when it becomes a simple integral, we have

$$\int dxu = xu - \frac{x^2}{1.2}\frac{du}{dx} + \frac{x^3}{1.2.3}\frac{d^2u}{dx^2} - \&c.,$$

the well known series of Bernoulli; which thus appears to be also a particular case of the theorem of Leibnitz when extended to general indices.

5. In this theorem, let us suppose $v = \varepsilon^{ax}$; then as $\frac{dv}{dx} = a\varepsilon^{ax} = av$, we have $\frac{d'}{dx} = a$, and therefore

$$\frac{d^n}{dx^n}(\varepsilon^{ax}u) = \left(a + \frac{d}{dx}\right)^n u\varepsilon^{ax};$$

whence

$$\left(a + \frac{d}{dx}\right)^n u = \varepsilon^{-ax}\left(\frac{d}{dx}\right)^n \varepsilon^{ax}u,$$

which is the theorem given at p. 26 of the *Mathematical Journal*.*

6. In the Calculus of Finite Differences there are more theorems than in the Differential Calculus depending on the expansion of a binomial, in consequence of the relation which subsists between two kinds of operations, that of taking the increment and that of taking the difference. It is not usual to use a separate symbol for the former, but in Art. II. of the Second Number of the *Mathematical Journal*,† I adopted the symbol D to represent this operation, as it simplified greatly the expressions. For the same reason I shall continue to employ it, and I hope that its utility will compensate for any disadvantage which may accrue from using a new notation. Before proceeding, I will say a few words concerning the operation represented by this symbol D. Its definition is, that

$$Df(x) = f(x+1).$$

Now we know by Taylor's theorem that

$$\varepsilon^{h\frac{d}{dx}}f(x) = f(x+h),$$

* See page 18 of this volume. † See page 34 of this volume.

whatever h may be; making $h=1$, we have

$$\varepsilon^{\frac{d}{dx}}=f(x+1).$$

Consequently $$D=\varepsilon^{\frac{d}{dx}};$$

from which we see that $D^h f(x)=f(x+h)$, whatever h may be.

Also, since

$$\Delta f(x)=f(x+1)-f(x)=Df(x)-f(x)=(D-1)f(x),$$

we have $$\Delta=D-1 \text{ and } D=1+\Delta.$$

Consequently, D being a linear compound of commutative and distributive operations, is also a commutative and distributive operation. It is also subject to the laws of index functions, since

$$D^k D^h f(x)=D^k f(x+h)=f(x+h+k)=D^{h+k}f(x).$$

7. This being premised, since we have

$$Du_x=(1+\Delta)\,u_x,$$
$$D^n u_x=(1+\Delta)^n u_x;$$

and by the binomial theorem,

$$D^n u_x=\left\{1+n\Delta+\frac{n(n-1)}{1.2}\Delta^2+\frac{n(n-1)(n-2)}{1.2.3}\Delta^3+\&c.\right\}u_x,$$

or $$u_{x+n}=u_x+n\Delta u_x+\frac{n(n-1)}{1.2}\Delta^2 u_x+\&c.$$

Again, $$D^n u_x=(D^{-1})^{-n}u_x=\{D^{-1}(D-\Delta)\}^{-n}u_x$$
$$=(1-\Delta D^{-1})^{-n}u_x;$$

whence by the binomial theorem

$$D^n u_x=\left(1+n\Delta D^{-1}+\frac{n(n+1)}{1.2}\Delta^2 D^{-2}\right.$$
$$\left.+\frac{n(n+1)(n+2)}{1.2.3}\Delta^3 D^{-3}+\&c.\right)u_x,$$

and therefore

$$u_{x+n} = u_x + n\Delta u_{x-1} + \frac{n(n+1)}{1.2} \Delta u_{x-2} + \&c.$$

8. Besides these there are two other expressions for u_{x+n}, which, though not depending on the binomial theorem, are founded on the same principles.

$$u_{x+n} - u_x = (D^n - 1) u_x$$
$$= \Delta (D-1)^{-1} (D^n - 1) u_x,$$

since $\Delta = D - 1$.

Expanding in the same way as we would expand $\frac{x^n - 1}{x - 1}$, we find

$$u_{x+n} - u_x = \Delta (1 + D + D^2 + \&c. + D^{n-1}) u_x,$$

and therefore

$$u_{x+n} = u_x + \Delta u + \Delta u_{x+1} + \Delta u_{x+2} + \&c. + \Delta u_{x+n-1}.$$

Similarly,

$$u_{x+n} - \Delta^n u_x = (D^n - \Delta^n) u = (D - \Delta)^{-1} (D^n - \Delta^n) u_x,$$

since $D - \Delta = 1$.

Therefore, expanding as we would expand $\frac{x^n - a^n}{x - a}$, we find

$$u_{x+n} - \Delta^n u_x = (D^{n-1} + \Delta D^{n-2} + \Delta^2 D^{n-3} + \&c. + \Delta^{n-2} D + \Delta^{n-1}) u_x,$$

and therefore

$$u_{x+n} = u_{x+n-1} + \Delta u_{x+n-2} + \&c. + \Delta^{n-2} u_{x+1} + \Delta^{n-1} u_x + \Delta^n u_x.$$

9. Corresponding to these theorems for $D^n u_x$, we have theorems for $\Delta^n u_x$.

Since $\quad \Delta u_x = (D-1) u_x, \quad \Delta^n u_x = (D-1)^n u_x,$

and therefore by the binomial theorem

$$\Delta^n u_x = \left(D^n - nD^{n-1} + \frac{n(n-1)}{1.2} D^{n-2} - \&c.\right) u_x$$

$$= u_{x+n} - n u_{x+n-1} + \frac{n(n-1)}{1.2} u_{x+n-2} - \&c.$$

Again, $\Delta^r u_{x+n} = \Delta^r D^r u_{x+n-r} = \Delta^r \{D^{-1}(D-\Delta)\}^{-r} u_{x+n-r}$

$$= \Delta^r (1-\Delta D^{-1})^{-r} u_{x+n-r},$$

and expanding

$$\Delta^r u_{x+n} = \Delta^r \left\{1 + r\Delta D^{-1} + \frac{r(r+1)}{1.2}\Delta^2 D^{-2} + \ldots\right\} u_{x+n-r}$$

$$= \Delta^r u_{x+n-r} + r\Delta^{r+1} u_{x+n-r-1}$$

$$+ \frac{r(r+1)}{1.2}\Delta^{r+2} u_{x+n-r-2} + \ldots.$$

10. Connected with this subject is a formula for the transformation of series, which is useful for the purpose of changing diverging into converging series. The proof of this, which is usually made to depend on the theory of generating functions, can be much more simply derived from the theory I am here developing, and the same may be said of all theorems usually demonstrated by generating functions. Let the given series be

$$S = y_x + y_{x+1} + y_{x+2} + y_{x+3} + \ldots$$

$$= (1 + D + D^2 + D^3 + \ldots) y_x = (1-D)^{-1} y_x;$$

and let it be desired to change this into one depending on

$$ay_x + a_1 y_{x+1} + a_2 y_{x+2} + \ldots + a_n y_{x+n}$$

$$= (a + a_1 D + a_2 D^2 + \ldots + a_n D^n) y_x = \nabla y_x,$$

if we put $\nabla = a + a_1 D + a_2 D^2 + \ldots + a_n D^n$.

Now it is to be observed, that any algebraic combination of the symbols D and Δ with constants will be likewise subject to the same laws of combination as these symbols, and may therefore be treated in the same way.

Hence, making $a + a_1 + a_2 + \ldots + a_n = K$, we shall have

$$S = (1-D)^{-1} y_x = (K-\nabla)^{-1}(K-\nabla)(1-D)^{-1} y_x,$$

since the operations $(K-\nabla)^{-1}$, $K-\nabla$ destroying each other, do not affect the equation. Now

$$(1-D)^{-1}(K-\nabla)=(1-D)^{-1}\{a_1(1-D)+a_2(1-D^2)$$
$$+a_3(1-D^3)+\ldots+a_n(1-D^n)\}$$
$$=a_1+a_2(1+D)+a_3(1+D+D^2)+\ldots$$
$$+a_n(1+D+D^2+\ldots+D^{n-1})$$
$$=a_1+a_2+a_3+\ldots+a_n$$
$$+(a_2+a_3+\ldots+a_n)D$$
$$+(a_3+\ldots+a_n)D^2$$
$$+\ldots+\ldots.$$

And therefore

$$S=(a_1+a\ \ +a_3+\ldots+a_n)(K-\nabla)^{-1}y_x$$
$$+(a\ \ +a_3+\ldots+a_n)(K-\nabla)^{-1}y_{x+1}$$
$$+(a\ \ +\ldots+a_n)(K-\nabla)^{-1}y_{x+2}$$
$$+\ldots+\ldots,$$

and expanding $(K-\nabla)^{-1}$, we find

$$S=(a_1+a_2+a_3+\ldots+a_n)\left(\frac{y_x}{K}+\frac{\nabla y_x}{K^2}+\frac{\nabla^2 y_x}{K^3}+\ldots\right)$$
$$+(a_2+a_3+\ldots+a_n)\left(\frac{y_{x+1}}{K}+\frac{\nabla y_{x+1}}{K^2}+\frac{\nabla^2 y_{x+1}}{K^3}+\ldots\right)$$
$$+(a_3+\ldots+a_n)\left(\frac{y_{x+2}}{K}+\frac{\nabla y_{x+2}}{K^2}+\frac{\nabla^2 y_{x+2}}{K^3}+\ldots\right)+\ldots,$$

which is the required transformation for S.

11. In the particular case where

$$S=ax+a_1x^2+a_2x^3+a_3x^4+\ldots,$$

Euler has employed a very elegant method of transformation, the reason for which appears very clearly, if we follow the method of the separation of symbols.

Let us suppose a, a_1, a_2, a_3, ... to be terms of a series which can be derived one from the other by a certain law, so that $a_1 = Da$, $a_2 = D^2a$, and so on. Then

$$S = (x + x^2D + x^3D^2 + \ldots)\, a$$
$$= x\,(1 - xD)^{-1} a = x\,(1 - x - x\Delta)^{-1} a$$
$$= \frac{x}{1-x}\left(1 - \frac{x}{1-x}\,\Delta\right)^{-1} a$$
$$= \frac{x}{1-x}\left\{1 + \frac{x}{1-x}\,\Delta + \frac{x^2}{(1-x)^2}\,\Delta^2 + \ldots\right\} a$$
$$= \frac{ax}{1-x} + \Delta a\,\frac{x^2}{(1-x)^2} + \Delta^2 a\,\frac{x^3}{(1-x)^3} + \ldots.$$

12. The expression for the total difference of a function of two variables, in terms of the partial differences, is not so simple as its analogue in the Differential Calculus. If we represent the total difference by Δ, and the partial differences with respect to x and y by Δ_x, Δ_y, and the corresponding increments by D_x, D_y, we have, since

$$\Delta_x u_{x,\,y+1} = u_{x+1,\,y+1} - u_{x,\,y+1},$$

and

$$\Delta_y u_{x+1,\,y} = u_{x+1,\,y+1} - u_{x+1,\,y},$$

by adding them together, and subtracting from

$$2\Delta u_{x,y} = 2\,(u_{x+1,\,y+1} - u_{x,y}),$$

$$2\Delta u_{x,y} - \Delta_x u_{x,\,y+1} - \Delta_y u_{y,\,x+1} = u_{x,\,y+1} - u_{x,y} + u_{x+1,\,y} - u_{x,y},$$

whence $2\Delta u_{x,y} = \Delta_y D_x u_{x,y} + \Delta_x D_y u_{x,y} + \Delta_y u_{x,y} + \Delta_x u_{x,y}$,

or $\qquad \Delta u_{x,y} = \frac{1}{2}\,\{\Delta_y\,(1 + D_x) + \Delta_x\,(1 + \Delta_y)\}\; u_{x,y}.$

Now, since all the symbols are relatively commutative, inasmuch as $D.\Delta u_x = \Delta.Du_x$ when they refer to the same variable, and as when referring to different variables they are wholly independent, and therefore commutative; and since all the symbols are also distributive, the binomial theorem may be here applied, and therefore

$$\Delta^n u_{x,y} = \frac{1}{2^n}\{\Delta_y(1+D_x)+\Delta_x(1+D_y)\}^n u_{x,y}$$

$$= \frac{1}{2^n}\Big\{\Delta_y^{\,n}(1+D_x)^n + n\Delta_y^{\,n-1}(1+D_x)^{n-1}\Delta_x(1+D_y)$$

$$+\frac{n(n-1)}{1.2}\Delta_y^{\,n-2}(1+D_x)^{n-2}\Delta_x^{\,2}(1+D_y)^3+\ldots\Big\}u_{x,y}.$$

If each term be expanded, and the operations indicated by D_x, D_y be effected, we shall obtain a result in Δ_x and Δ_y; but it is so complicated, that it is better to keep it in the unexpanded form, as we thus see the law of formation more distinctly.

13. It is also obvious, that as

$$\Delta u_{x,y} = (D-1)\,u_{x,y},$$

when Δ and D are total operations referring to both variables,

$$\Delta^n u_{x,y} = \left\{D^n - nD^{n-1} + \frac{n(n-1)}{1.2}D^{n-2} - \ldots\right\}u_{x,y}$$

$$= u_{x+n,\,y+n} - nu_{x+n-1,\,y+n-1} + \ldots,$$

and similarly

$$D^n u_{x,y} = u_{x,y} + n\Delta u_{x,y} + \frac{n(n-1)}{1.2}\Delta^2 u_{x,y} + \ldots.$$

14. We shall proceed now to the operations on the products of two or more functions of the same variable.

$$\Delta u_x v_x = u_{x+1}v_{x+1} - u_x v_x = (DD_1 - 1)\,u_x v_x,$$

where we suppose D to refer to u_x and D_1 to v_x. Therefore

$$\Delta^n u_x v_x = (DD_1 - 1)^n u_x v_x$$

$$= \left\{D^n D_1^{\,n} - nD^{n-1}D_1^{\,n-1} + \frac{n(n-1)}{1.2}D^{n-2}D_1^{\,n-2} - \ldots\right\}u_x v_x;$$

therefore $\quad \Delta^n u_x v_x = u_{x+n}v_{x+n} - nu_{x+n-1}v_{x+n-1} + \ldots.$

This is true whatever n is. Let it be negative, then

$$\Delta^{-n} u_x v_x = \Sigma^n u_x v_x = u_{x-n}v_{x-n} + nu_{x-n-1}v_{x-n-1} + \ldots.$$

15. The n^{th} difference of the product of two functions may be expanded in another manner. Since

$$\begin{aligned}\Delta u_x v_x &= u_{x+1} v_{x+1} - u_x v_x \\ &= u_x \Delta v_x + v_{x+1} \Delta u_x \\ &= (\Delta + D\Delta_1)\, u_x v_x,\end{aligned}$$

(where Δ, D refer to u_x and Δ_1 to v_x),

$$\begin{aligned}\Delta^n u_x v_x &= (\Delta + D\Delta_1)^n u_x v_x \\ &= \left\{\Delta^n + \Delta^{n-1}\Delta_1 D + \frac{n(n-1)}{1.2}\Delta^{n-2}\Delta_1{}^2 D^2 + \ldots\right\} u_x v_x;\end{aligned}$$

therefore

$$\Delta^n u_x v_x = v_x \Delta^n u_x + n\Delta v_x \Delta^{n-1} u_{x+1} + \frac{n(n-1)}{1.2}\Delta^2 v_x \Delta^{n-2} u_{x+2} + \ldots.$$

When n becomes negative, $\Delta^{-n} u_x v_x = \Sigma^n u_x v_x$, and therefore

$$\Sigma^n (u_x v_x) = v_x \Sigma^n u_x - n\Delta v_x \Sigma^{n+1} u_{x+1} + \frac{n(n+1)}{1.2}\Delta^2 v_x \Sigma^{n+2} u_{x+2} - \ldots,$$

which is the formula for integration by parts; and when $n = 1$,

$$\Sigma (u_x v_x) = v_x \Sigma u_x - \Delta v_x \Sigma^2 u_{x+1} + \Delta^2 v_x \Sigma^3 u_{x+2} - \ldots.$$

16. If we suppose $u_x = x^0 = 1$ in the former expression, we have, since $u_x = u_{x+1} = u_{x+2}$, &c.

$$\Sigma^n u_x = \Sigma^n .1 = \frac{x(x+1)\ldots(x+n-1)}{(n)!},$$

and therefore

$$\Sigma^n (v_x) = \frac{x(x+1)\ldots(x+n-1)}{(n-1)!}\left\{\frac{v_x}{n} - \frac{(x+n)}{1}\frac{\Delta v_x}{n+1} + \frac{(x+n)(x+n+1)}{1.2}\frac{\Delta^2 v_x}{n+2} - \ldots\right\},$$

which, when $n = 1$, becomes

$$\Sigma (v_x) = x v_x - \frac{x(x+1)}{1.2}\Delta v_x + \frac{x(x+1)(x+2)}{1.2.3}\Delta^2 v_x - \ldots,$$

a series which bears a close analogy with that of Bernoulli in the Integral Calculus.

17. Again, since

$$\Delta^n u_x v_x = (DD_1 - 1)^n u_x v_x,$$

if we make $v_x = a^x$, $D_1 a^x = a^{x+1} = a.a^x$, so that $D_1 = a$, and

$$\Delta^n u_x a^x = (Da - 1)^n u_x a^x,$$

and
$$(Da - 1)^n u_x = a^{-x}.\Delta^n (u_x a^x).$$

If for a we put $\frac{1}{a}$, we obtain

$$(D - a)^n u_x = a^{x+n} \Delta^n (u_x a^{-x}),$$

which is the theorem given in Article II. of the second Number of the *Cambridge Mathematical Journal*.*

18. The connexion which exists between the Differential Calculus and the Calculus of Finite Differences, gives rise to various elegant theorems; the first of which is the celebrated theorem of Lagrange, that

$$\Delta^n u_x = (\varepsilon^{\frac{d}{dx}} - 1)^n u_x.$$

For as we have

$$\Delta u_x = (D - 1)\, u_x = (\varepsilon^{\frac{d}{dx}} - 1)\, u_x,$$

raising the symbol of operation to the n^{th} power on each side,

$$\Delta^n u_x = (\varepsilon^{\frac{d}{dx}} - 1)^n u_x.$$

It is usual to make the proof of this theorem a matter of some difficulty, but it follows at once from the theory of the laws of combination of the symbols. It is true whatever n may be, and therefore when n is negative, or

$$\Sigma^n u_x = (\varepsilon^{\frac{d}{dx}} - 1)^{-n} u_x,$$

or for the particular value 1 of n,

$$\Sigma u_x = (\varepsilon^{\frac{d}{dx}} - 1)^{-1} u_x.$$

* See page 34 of this volume.

The second side, when expanded by the numbers of Bernoulli, gives

$$\Sigma u_x = \left\{\left(\frac{d}{dx}\right)^{-1} - \frac{1}{2} + \frac{B_1}{1.2}\frac{d}{dx} - \frac{B_3}{1.2.3.4}\frac{d^3}{dx^3} + \dots\right\} u_x$$

$$= \int u_x dx - \frac{u_x}{2} + \frac{B_1}{1.2}\frac{du_x}{dx} - \frac{B_3}{1.2.3.4}\frac{d^3 u_x}{dx^3} + \dots.$$

19. The theorem for expressing the n^{th} difference in terms of the n^{th} and higher differential coefficients, may be derived very readily without expansion from the fundamental theorem

$$\Delta u_x = (\varepsilon^{\frac{d}{dx}} - 1)\, u_x;$$

for we shall also have

$$\Delta\, (\varepsilon^{h\frac{d}{dx}} u) = \varepsilon^{h\frac{d}{dx}} (\varepsilon^{\frac{d}{dx}} - 1)\, u_x, \text{ when } h = 0.$$

But $\varepsilon^{h\frac{d}{dx}} (\varepsilon^{\frac{d}{dx}} - 1)$ is the difference of $\varepsilon^{h\frac{d}{dx}}$, taken with respect to h, and may be represented by $\Delta_h \varepsilon^{h\frac{d}{dx}}$, where Δ_h implies that the sign of operation affects h only. Hence we have

$$\Delta\, (\varepsilon^{h\frac{d}{dx}} u_x) = \Delta_h\, (\varepsilon^{h\frac{d}{dx}} u_x), \text{ when } h = 0.$$

By this artifice the symbol of operation is transferred from the x to the h. Now, taking the n^{th} difference on both sides,

$$\Delta^n\, (\varepsilon^{h\frac{d}{dx}} u_x) = \Delta_h^{\,n}\, (\varepsilon^{h\frac{d}{dx}} u_x), \text{ when } h = 0.$$

On expanding $\varepsilon^{h\frac{d}{dx}}$ on the second side, and effecting the operation $\Delta_h^{\,n}$, it appears that all the terms will vanish till the $(n+1)^{\text{th}}$, so that replacing h by 0 we have the usual formula

$$\Delta^n u_x = \frac{\Delta^n 0^n}{n\,!}\frac{d^n u}{dx^n} + \frac{\Delta^n 0^{n+1}}{(n+1)\,!}\frac{d^{n+1} u}{dx^{n+1}} + \dots.$$

20. The same method affords an easy proof of a theorem first given by Sir John Herschel in the *Philosophical Transactions*, 1816, for expanding any function of ε^t.

$$f(\varepsilon^t) = f(\varepsilon^t)\,\varepsilon^{xt}, \text{ when } x = 0.$$

And since $$\varepsilon^{\frac{d}{dx}}.\varepsilon^{xt} = \varepsilon^t, \text{ when } x = 0,$$

$$f(\varepsilon^t) = f(\varepsilon^{\frac{d}{dx}})\,\varepsilon^{xt}, \text{ when } x = 0,$$

or expanding ε^{xt}, putting 0 for x and $1+\Delta$ for $\varepsilon^{\frac{d}{dx}}$, we get

$$f(\varepsilon^t) = f(1+\Delta)\,1 + f(1+\Delta)\,0.t + f(1+\Delta)\,\frac{0^2 t^2}{1.2} + \dots,$$

which is the form given by Sir John Herschel.

I cannot mention the name of this mathematician without correcting an error into which I fell in Article V. of the first Number of the *Cambridge Mathematical Journal.** I there stated that, so far as I knew, Brisson was the first person who had applied the method of the separation of symbols to the solution of differential equations. I have since found that Sir John Herschel was really the first person who did so, in a paper published in the *Philosophical Transactions* for 1816, five years before the date of Brisson's Memoir. It is much to be regretted, that neither Sir John Herschel himself, nor any other person, followed up this method, which is calculated to be of so much use in the higher analysis. Perhaps this may have arisen from the theory of the method not having been properly laid down, so that a certain degree of doubt existed as to the correctness of the principle. I trust, however, that the various developments which I have given

* See page 14 of this volume.

in several articles in this Journal, of the principles of the method as well as the proofs of its utility, are sufficient for removing all doubts on this head, and that it will now be regarded as a powerful instrument in the hands of mathematicians.

ON THE IMPOSSIBLE LOGARITHMS OF QUANTITIES.*

In a Paper printed in the fourteenth volume of the *Transactions of the Royal Society of Edinburgh*,† I gave a short sketch of what I conceive to be the true nature of Algebra, considered in its greatest generality; that it is the science of symbols, defined not by their nature, but by the laws of combination to which they are subject. In that paper I limited myself to a statement of the general view, without pretending to follow out all the conclusions to which such views would lead us: such an undertaking would be too extended for the limits of a memoir, and would involve a complete treatise on Algebra. It will not, however, be attempting too much to trace out, in one or two cases, some of the more important elucidations which this theory affords of several disputed and obscure points in Algebra, and therefore in the following pages I shall endeavour to point out the deductions which may be derived from the definition of the operation $+$, given in the paper above alluded to. I there stated, that we must not consider it merely as an affection of other symbols, which we call symbols of quantity, but as a distinct operation possessing certain properties peculiar to itself, and subject, like the more ordinary symbols, to be acted on by any other opera-

* *Cambridge Mathematical Journal*, Vol. I., p. 226.

† See page 1 of this volume.

tions, such as the raising to powers, &c. The definition of the operation represented by this symbol is, that

$$+ + = +,$$

which leads to the equation

$$(+)^r = +,$$

r being any integer. And this peculiarity—that the operation repeated any number of times gives the same result as when only performed once—is the origin of certain analytical anomalies, which do not at first sight appear to be connected.

The first of these is the fact of the existence of a plurality of roots of a quantity, when the corresponding powers have only one value. It seems a fair question, to ask the cause of so great a difference between two operations so analogous in their nature, but it is one which I have not seen anywhere discussed. The distinction is, I conceive, to be traced to the nature of the operation $+$, according to the definition of it which I have given; and much of the obscurity connected with the subject is due to an oversight, by which the existence of this $+$ is wholly overlooked. For it is not a, but $+a$, which has a plurality of roots: and though these quantities are usually reckoned to be the same, this idea is founded on an illegitimate extension of a supposed relation in the science of number. I say *supposed*, because I hold, that even in Arithmetic a and $+a$ are different, and ought not to be confounded—the former being an absolute operation, the other always a relative one, and consequently incapable of existing by itself. But however this may be, there is no doubt that it is entirely illegitimate to suppose that in all cases a and $+a$ are the same, since generally we know not even what their meanings may be. Indeed, in Geometry the distinction is pretty broadly marked, since a represents a line considered with reference to magnitude only, $+a$ with reference both to magnitude and direction.

I therefore maintain, that in general symbolical Algebra we must never consider these quantities as identical; and if at any time we conceive the existence of the $+$, we must take cognizance of its existence throughout all our processes, subjecting it to the operations we may perform on the compound quantity. Now, that in the usual theory of the plurality of roots the existence of $+$ is supposed, though not always expressed, is easily shown from the very first case of plurality of values which occurs. It is argued that, since $a \times a = a^2$ and $-a \times -a = +a^2$ also, we have two values, a and $-a$, for $(a^2)^{\frac{1}{2}}$. But this, it will be seen, depends on the supposition that $+a^2 = a^2$, since in the case of the product $-a \times -a$ the $+$ is exhibited. If, instead of saying $a \times a = a^2$, we were to say that $+a \times +a = +a^2$, we should have undoubtedly $+a^2$ as the result in both cases, and we are therefore entitled to say that $(+a^2)^{\frac{1}{2}}$ has two values, $+a$ and $-a$. The reason for this plurality is now very plain, for

$$(+a^2)^{\frac{1}{2}} = +^{\frac{1}{2}} (a^2)^{\frac{1}{2}} = +^{\frac{1}{2}} a.$$

But from the definition of $+$ it appears that $+^{\frac{1}{2}}$ will be different according as we suppose the $+$ to be equivalent to the operation repeated an even or an odd number of times. In the former case it will be equal to $+$, in the latter to $-$. And generally, if we raise $+a$ to any power m, whether whole or fractional, we have

$$(+a)^m = +^m a^m.$$

Now, as from the definition of $+$ it appears that $+^r = +$, r being any integer, it is indeterminate which power of $+$ it may represent in any case, and therefore we must substitute $+^r$ for $+$, and then, assigning all integer values to r, discover how many values $+^{rm} a^m$ will acquire. So long as m is an integer, rm is an integer, and $+^{rm} a^m$ has only one value; but if m be a fraction of the form $\frac{p}{q}$, $+^{r\frac{p}{q}}$ will acquire different values, according as we assign different values to r.

It will not, however, acquire an infinite number of values, since after r receives the value q, the values will recur in the same order. Hence the number of values of a quantity raised to a fractional power, is equal to the number of digits in the denominator of the index. It is to be observed, that we must never make $r=0$, since that assumption is equivalent to supposing that the operation $+$ is not performed at all, which is contrary to our original supposition. From this we see, that the reason why there is a plurality of values for the roots of a quantity, is to be found in the nature of the operation $+$; and that it is only the compound operation $+a$, which admits of this plurality, a itself having only one value for each root. This view serves to explain an apparent difficulty which is noticed by various writers on Algebra. Since by the rule of signs $- \times -$ gives $+$, we ought to have

$$\surd(-a) \times \surd(-a) = \surd(+a^2) = \pm a;$$

whereas we know that it must be only $-a$.

Now this fallacy arises from the sign of the root not being made to affect the $+$ as well as the a. The process is really this,

$$\surd(-a) \times \surd(-a) = \surd(+a^2) = \surd(+)\ \surd(a^2) = -a;$$

for in this case we know how the $+$ has been derived, namely, from the product $- - = +$ or $-^2 = +$, which of course gives us $+^{\frac{1}{2}} = -$, there being here nothing indeterminate about the $+$.

It was in consequence of sometimes tacitly assuming the existence of $+$, and at another time neglecting it, that the errors in various trigonometrical expressions arose; and it was by the introduction of the factor $\cos 2r\pi + -^{\frac{1}{2}} \sin 2r\pi$ (which is equivalent to $+^r$) that Poinsot established the formulæ in a more correct and general shape. Thus the theorem of Demoivre that

$$(\cos\theta + -^{\frac{1}{2}} \sin\theta)^m = \cos m\theta + -^{\frac{1}{2}} \sin m\theta$$

should be written

$$\{+^r(\cos\theta + -^{\frac{1}{2}}\sin\theta)\}^m = +^{rm}(\cos\theta + -^{\frac{1}{2}}\sin\theta)^m$$
$$= (\cos 2r\pi + -^{\frac{1}{2}}\sin 2r\pi)^m(\cos\theta + -^{\frac{1}{2}}\sin\theta)^m$$
$$= \{\cos(2r\pi + \theta) + -^{\frac{1}{2}}\sin(2r\pi + \theta)\}^m$$
$$= \cos m(2r\pi + \theta) + -^{\frac{1}{2}}\sin m(2r\pi + \theta).$$

It will be seen from what I have said that I suppose the symbol + to play the same part which Professor Peacock ascribes to the symbol 1, when he says that it is the recipient of the affections of a^m; and that what that author considers to be the roots of unity I conceive to be the roots of +.

So far as the correctness of the formulæ is concerned, it makes but little difference which view is taken, if attention be always paid to the existence of this quantity on which the plurality of values depends, whether we denote it by the symbol 1 or +. But in the general Theory of Algebra there is a considerable difference; for 1 being an arithmetical symbol necessarily recalls arithmetical notions; and as the circumstances in which its peculiar nature is evolved occur in general symbolical Algebra, and may be wholly independent of arithmetic, it is of importance to avoid the confusion which must be caused by the introduction into general symbolical Algebra of symbols limited in their signification.

The other point which I propose to elucidate at present, and which is the chief object of this paper, is the plurality of logarithms of quantities, which, although at first sight unconnected with what we have been discussing, will be found to depend also on the existence of a +, which is generally overlooked. This is closely connected also with the discussion concerning the logarithms of negative quantities, which attracted so much attention in the time of Euler, D'Alembert, and John Bernoulli, and the interest of which has been revived of late years by the researches of Vincent, Ohm, and Graves. Euler had apparently set the question at rest by demonstrating the existence of an infinite number of logarithms of a quantity, one only of which is possible; and the formula he gave was that

$$\log a = L(a) + 2r\pi \sqrt{(-1)},$$

representing by $L(a)$ the arithmetical logarithm of a.

Mr. Graves, by a different and very circuitous process, arrives at the result

$$\log(a) = \frac{L(a) + 2r\pi \sqrt{(-1)}}{1 + 2r'\pi \sqrt{(-1)}},$$

the logarithms being taken with respect to the base ε for simplicity.

The correctness of this result is doubted by Professors Peacock and De Morgan, but it is corroborated by the researches of Sir W. Hamilton and Mr. Warren, as well as of M. Ohm. It is therefore both an interesting and an important question to determine which is the correct result, or at least to point out the cause of the differences between them. This I think the system I am advocating is able to do. But it is necessary first to lay down distinctly what is the meaning of the operation denoted by *log;* and this, according to my system, is done by defining its laws of combination. These are

$$\log x + \log y = \log(xy) \quad \ldots\ldots\ldots\ldots (1),$$

$$\log(x^y) = y \log x \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where x and y are distributive and commutative operations,

$$\log a = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

which assumes the species to be that in which the base is a.

The first and third of these laws are the same as those given by Mr. Graves at the suggestion of Sir William Hamilton, but the second he has omitted; I know not whether from oversight, or from considering it to be unnecessary. I have retained it as I conceive it essential for a strict definition of the operation.

This being premised, I proceed to state the position which I lay down, and the truth of which I hope to be able to establish. It is, that the impossible parts of the general logarithms, whether of those given by Euler or by Mr. Graves, are the logarithms of the symbol + which generally is overlooked in the expressions we use; and that the cause of the difference between the two formulæ for logarithms is, that in that of Euler *one* latent + only, and in that of Mr. Graves *two* are exhibited.

This I think is almost apparent from Euler's own process, if we attend to the meaning of the symbols he employs. He substitutes for the number y the expression

$$\{\cos 2r\pi + \sqrt{(-1)}\ \sin 2r\pi\}\ y,$$

r being any integer which he considers to be equivalent to it; and then taking the logarithms with respect to ε, he says that

$$\log y = L\,(y) + \log\{\cos 2r\pi + \sqrt{(-1)}\ \sin 2r\pi\},$$

where $L\,(y)$ represents the arithmetical logarithm of y: and as

$$\cos 2r\pi + \sqrt{(-1)}\ \sin 2r\pi = \varepsilon^{2r\pi\sqrt{(-1)}},$$

we have $$\log y = L\,(y) + 2r\pi\ \sqrt{(-1)}.$$

As r may receive any integer value, this expression has an infinite number of values, one only of which is possible in the case when $r=0$. It will be seen that the correctness of this result depends essentially on the assumption that y and $\{\cos 2r\pi + \sqrt{(-1)}\ \sin 2r\pi\}\ y$ are identical: an assumption which at first it seems very natural to make, since the expression $\cos 2r\pi + \sqrt{(-1)}\ \sin 2r\pi$ is usually considered to be equal to unity. But if we suppose the quantities with which we are dealing to be general quantities, and not numbers merely, a numerical value of $\cos 2r\pi + \sqrt{(-1)}\ \sin 2r\pi$ can have no place in our investigation, and we must seek for its general algebraical meaning. Now, in the paper previously referred to, I have shewn that + and $\cos 2\pi + \sqrt{(-1)}\ \sin 2\pi$

are algebraically equivalent, so that Euler's expression is equivalent to $+^r y$; and, as I remarked before, we cannot assume y and $+y$ to be identical, so that Euler's assumption is not correct. If we do not suppose the existence of $+$ we have only one value for the logarithm of y: if we do suppose its existence, since it is indeterminate what power of $+$ it stands for, we must take all the possible cases, which is easily done by assigning to r all integer values from 0 to ∞. Thus

$$\log(+y) = \log(+^r y) = \log(+^r) + \log y,$$

and as

$$+^r = \{\cos 2\pi + \sqrt{(-1)} \sin 2\pi\}^r = \cos 2r\pi + \sqrt{(-1)} \sin 2r\pi,$$

$$\log(+y) = 2r\pi \sqrt{(-1)} + \log y.$$

It must be observed that, as in the case of the powers of $+y$, we must never suppose $r=0$, since that is the same as supposing y not acted on by $+$, which is contrary to our original supposition.

Let us now consider Mr. Graves's method, starting as he does from the equation

$$y = a^x,$$

where a is the base of the system. If we assume y to stand for $+^r y$, we arrive at the same result as that of Euler. But we may also conceive a to stand for $+^{r'} a$, which is really, though not apparently, what is done by Mr. Graves, and then we obtain a very different result. The equation in this case becomes

$$+^r y = (+^{r'} a)^x,$$

and taking the logarithms with respect to ε for simplicity on both sides, we find

$$\log(+^r) + \log y = x\{\log(+^{r'}) + \log a\}.$$

This gives

$$x = \frac{\log y + \log(+^r)}{\log a + \log(+^{r'})},$$

or, putting for $\log(+^r)$ and $\log(+^{r'})$ their values,

$$x = \frac{\log y + 2r\pi\sqrt{(-1)}}{\log a + 2r'\pi\sqrt{(-1)}},$$

which is Mr. Graves's result. We see that the difference between the methods of Euler and Mr. Graves consists in the nature of the base they assume. It may be remarked however that Euler seems to have had some idea of the view taken by Mr. Graves, as may be seen in his discussion of the Logarithmic Curve, Vol. II., p. 290 of the Latin edition, where he has anticipated the observations of M. Vincent, which nearly coincide in principle with those of Mr. Graves.

Mr. Peacock objects to the system adopted by Mr. Graves, because it involves a circulating function as base: and I am inclined to agree with him. Since the base of the system is now $+^{r'}a$ instead of a; the supposition of a change in the value of r' corresponds to a change in the base, and therefore in the whole system of logarithms, so that the series of values of x corresponding to the different values of r' have as little connexion with each other as if they belonged to systems whose bases were b, c, d, or any other quantities. This of course depends on our believing that it is $(+a)^n$ and not a^n which has a plurality of values, and this I think I have satisfactorily shown. I may observe that if we are to allow a variable base as $+^{r'}a$, we might as well use such quantities as $\sin^{-1}a$, $\cos^{-1}a$ as bases, and reckon the logarithms corresponding to different values to belong to the same system; but this is what I believe no one would admit. The defect of not considering the existence of $+$ will perhaps appear more clearly if we analyse the reasoning by which both M. Vincent and Mr. Graves think that they establish that in certain cases there is a common logarithm for positive and negative numbers. They argue that since $\varepsilon^{\frac{1}{2}}$ or $\sqrt{(\varepsilon)}$ has two values which we may call $+n$ and $-n$, we have therefore

$$+n = \varepsilon^{\frac{1}{2}}, \quad -n = \varepsilon^{\frac{1}{2}},$$

and, from the ordinary definition of logarithms, $\frac{1}{2}$ must be the logarithm both of $+n$ and $-n$. So, indeed, it is, but only when referred to different systems: for, as I maintain, $+n$ and $-n$ are not both values of $\varepsilon^{\frac{1}{2}}$, but one is the value of $(+^2\varepsilon)^{\frac{1}{2}}$ and the other of $(+\varepsilon)^{\frac{1}{2}}$, so that $\frac{1}{2}$ is the logarithm of $+n$ in the system whose base is $+^2\varepsilon$, and of $-n$ in the system whose base is $+\varepsilon$. The same reasoning may be generally extended to such cases as $(+a)^{\frac{1}{n}}$ which admits of n values, and consequently of n quantities, which have a common logarithm $\dfrac{1}{n}$, but in each case referred to a different base. When n is even, one value will be positive and the other negative, all the others being impossible; and the positive and negative values are the only two of which M. Vincent takes notice when discussing the question. It might, perhaps, have weakened his belief in the correctness of the results, if he had come to the conclusion, as he ought to have done, that the same logarithm corresponded to positive, negative, and impossible quantities. These last he seems quite to have overlooked, which may have arisen from his having adopted, with many other mathematicians, the name of *imaginary* quantities. I adhere to the name *impossible* instead of *imaginary*, because the latter involves an idea which I conceive to be very deleterious in analysis. We may be unable to perform an operation though it be by no means an *imaginary* one; and indeed all that we can say of those quantities which have this name affixed to them is, that they are *uninterpretable in arithmetic*. For this reason, if I were permitted to propose a change, I should prefer to call these quantities "operations uninterpretable in arithmetic;" as this involves no theory of their nature, but only expresses what is a fact.

That, according to the system which I adopt, there cannot be a logarithm common to both positive and negative quantities, is easily shown. A positive quantity may be generally expressed by

$$+^{r}a:$$

the logarithm of which is

$$\log a + \log(+^{r}) = \log a + 2r\pi\sqrt{(-1)}.$$

A negative quantity may be expressed by

$$+^{\frac{2r+1}{2}}a:$$

the logarithm of which is

$$\log a + \log(+^{\frac{2r+1}{2}}) = \log a + \frac{2r+1}{2}\pi\sqrt{(-1)}.$$

And these two expressions can never coincide; nor can either ever lose its impossible part, since we are not at liberty to make $r=0$ in the first case, or $=-\frac{1}{2}$ in the second.

It is somewhat remarkable, that Mr. Peacock has been led into the same error as M. Vincent and Mr. Graves, respecting the coincidence in some cases of the logarithms of positive and negative quantities. As the cause of his error has reference to the remark which I have just made, and is not very easy to be detected, I shall point it out more particularly.

He considers $-a^{m}$ to be equivalent to $-1(+a)^{m}$, which gives

$$\begin{aligned}\log-(a)^{m} &= \log(-1) + \log(+a)^{m}\\ &= (2r+2mr'+1)\pi\sqrt{(-1)} + m\log a.\end{aligned}$$

He then supposes $m=\frac{p}{2n}$ where p is prime to n, $r'=-n$, and $r=\frac{p-1}{2}$; and as these values make the multiplier of $\pi\sqrt{(-1)}$ vanish, he concludes that the logarithm of $-(a)^{m}$ coincides with that of a^{m}, since it becomes $m\log a$. Now on this it is to be observed, that since m affects the + in

$(+a)^m$, $-a^m$ is really equal to $-1.+^m.a^m$, or, putting the general values for $-$ and $+$, to

$$+^{\frac{2r+1}{2}}(+^{r'})^m a^m.$$

In this expression, if we make $m=\frac{p}{2n}$, $r'=-n$, $r=\frac{p-1}{2}$, it becomes

$$+^{\frac{p}{2}}.+^{-\frac{p}{2}}.a^{\frac{p}{2n}};$$

and as $+^{\frac{p}{2}}$ and $+^{-\frac{p}{2}}$ are inverse operations, they destroy each other, and we have simply $a^{\frac{p}{2n}}$; the logarithm of which is, as it should be, possible. But these assumptions as to the values of m, r, and r', are plainly not allowable, since they imply, as we have seen, that a^m is not affected by $-$ at all, which is contrary to the original supposition. Hence we perceive that Mr. Peacock's argument for the existence of logarithms common to positive and negative quantities, being based on an unlawful assumption, falls to the ground.

If it be allowable to assume any quantity as base for a system of logarithms, we might, instead of $+^r a$ when r is an integer, take the same quantity, supposing r to be a fraction. We should then have possible quantities corresponding to impossible logarithms, and impossible quantities to possible logarithms; but the subject does not appear to be of sufficient interest to require an extended discussion.

In conclusion, I will recapitulate the conclusions to which I have been led by this mode of considering the symbol $+$.

1. A simple distributive and commutative operation has only one root, but if it be compounded with $+$ it has a plurality of roots depending on the indeterminate nature of $+$.

2. If the base of a system of logarithms and the number be simple distributive and commutative operations, there is only one corresponding logarithm; but if the number of the form be $+^r y$, there is an infinite number of logarithms.

3. If the base of the system be of the form $+^r a$, we are only allowed to assign one value to r, (as otherwise we alter the system,) and then there will be no plurality of logarithms.

4. The logarithms of $+a$ and $-a$ are in all cases different, and neither ever coincide with that of a.

5. The impossible parts of the logarithms, as usually given, are the logarithms of $+$ and of $-$.

MATHEMATICAL NOTE.*

It seems not to be generally known, that the equation

$$1.2.3\ldots n = n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \ldots,$$

which is used in proving Sir John Wilson's theorem respecting prime numbers, can be deduced immediately from the theorems of common Algebra. The following is the method.

By the Binomial Theorem,

$$(\varepsilon^x - 1)^n = \varepsilon^{nx} - \frac{n}{1}\varepsilon^{(n-1)x} + \frac{n(n-1)}{1.2}\varepsilon^{(n-2)x} - \ldots.$$

Substitute for each exponential its expansion according to powers of x, and equate the coefficient of x^n on the two sides. That on the first side, or $\left(x + \frac{x^2}{1.2} + \ldots\right)^n$, is evidently 1; the coefficient of x^n in ε^{nx} is $\frac{n^n}{1.2.3\ldots n}$; in $\varepsilon^{(n-1)x}$, $\frac{(n-1)^n}{1.2.3\ldots n}$; in $\varepsilon^{(n-2)x}$, $\frac{(n-2)^n}{1.2.3\ldots n}$, &c. Hence,

$$1 = \frac{n^n}{1.2.3\ldots n} - \frac{n}{1}\cdot\frac{(n-1)^n}{1.2.3\ldots n} + \frac{n(n-1)}{1.2}\,\frac{(n-2)^n}{1.2.3\ldots n} - \ldots,$$

or $$1.2.3\ldots n = n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \ldots.$$

* *Cambridge Mathematical Journal,* Vol. I., p. 239.

In the same way it is seen that

$$n^m - \frac{n}{1}(n-1)^m + \frac{n(n-1)}{1.2}(n-2)^m - \dots$$

is zero if m and n be any integers, of which n is the greater.

ON THE EXISTENCE OF BRANCHES OF CURVES IN SEVERAL PLANES.*

IN tracing a curve expressed by an equation between two variables, it is customary to make use of negative as well as of positive values of the variables, but to reject those which are usually called impossible or imaginary. This practice was allowable so long as it was supposed that impossible quantities had no meaning in geometry; but if we once admit the possibility of interpreting them in this science, though not in arithmetic, we are bound in strict logic not to neglect them. Accordingly, the Abbé Buée, in his very ingenious paper in the *Philosophical Transactions* for 1806, in which he demonstrated the possibility of interpreting geometrically the symbol usually written $\sqrt{(-1)}$, showed that quantities affected by it corresponded to branches of the curve situate in a plane at right angles to the original plane. Professor Peacock is, I believe, the only author who agrees with Buée in this view of the nature of curves, although it seems difficult to show any reason why it should not be generally allowed. I believe that a name has had great influence in preventing its adoption—the word *imaginary* so frequently applied to the symbol $\sqrt{(-1)}$ appearing to make persons unwilling to believe that it could possibly admit of any interpretation. Yet, after all, the difference between it and the symbol $-$

* *Cambridge Mathematical Journal*, Vol. I., p. 259.

is not so very great, both admitting of easy interpretation in the science of geometry, and neither, if considered independently, in the science of arithmetic, more especially if we consider them, as I have done elsewhere, as fractional powers of the symbol $+$, having peculiar properties depending on the fundamental definition of that symbol.

It appears to me, that if we once admit anything beyond what are called positive values of the variables, that is, pure arithmetical values wholly independent of the symbol $+$, there is no reason why we should confine ourselves to $-$ or $+^{\frac{1}{2}}$, since this is not differently circumstanced from any other power of $+$, analytically considered. I therefore hold, that we must either limit ourselves to the one quadrant formed by the positive axes, or we must be prepared to consider the curve as existing in several planes. Nor need it appear surprising, that by means of an equation between two variables, we are able to take into our view three dimensions, for the symbol $+$ is really equivalent to an angular co-ordinate, and therefore enables us to reach all points of space. In the following pages I propose, therefore, to extend farther than he has done, the principle introduced by Buée, and, not confining myself to values of the variables of the form $+^{\frac{1}{4}}a$, to investigate the forms of curves, when we assume that the variables may be of the general form $+^{\frac{p}{q}}a$.

As a preliminary, we must consider what is the meaning of this expression when substituted in the equation to a curve $y=f(x)$. Since $+^{\frac{p}{q}}$ represents the turning of a line through an angle $\frac{2p\pi}{q}$, the expression $x=+^{\frac{p}{q}}a$ signifies that we are to measure a line whose length is a, along the axis of x, and then to turn the axis through an angle equal to $\frac{2p\pi}{q}$. But we may turn the axis in an infinite number of ways, which at first would make it appear that $x=+^{\frac{p}{q}}a$

would not give a definite point. But it is to be observed, that the axis of x is always to be perpendicular to that of y, so that it is only allowable to move the axis in a plane perpendicular to the plane of xy. If the substitution of $+^{\frac{p}{q}}a$ for x gives a value of y equal to $+^{\frac{r}{s}}b$, this implies that we have to measure a length b along y, and then to turn it through an angle equal to $\frac{2r\pi}{s}$, and the plane passing through the two new axes will be the plane of a branch of the curve formed by assigning all values to a from 0 to ∞, unless, which will not unfrequently happen, the value of a affects the index of $+$ in the expression for y, in which case every element of the curve will lie in a different plane from the contiguous one, and the curve will be one of double curvature. This, perhaps, will appear more clearly, if we illustrate it by an example in the case of the parabola, which is the simplest curve for our purpose. The equation to the parabola being

$$y^2 = mx,$$

if we put $+^{\frac{p}{q}}a$ for x, we find the value of y to be $+^{\frac{p}{2q}}m^{\frac{1}{2}}x^{\frac{1}{2}}$. Hence we see that the axis of y is to be turned round through an angle, which is one half of that through which x is turned round; and for all the values which we may assign to a, if we leave p and q unchanged, the plane of the branch will remain unchanged, and the branch itself will be exactly similar to that in the plane xy, since the *numerical* relation between y and x is the same, whatever values we assign to p and q. By changing these last we obtain different branches in different planes; and as there is no limit to the values we may assign to them, it appears that the equation to the parabola, considered generally, represents a curve of an infinite number of branches, all passing through the origin, and situate in planes, such that the axis of x in any plane makes with the old axis twice the angle which the axis of y in that plane does with the old axis of y. As a particular case, we may take $\frac{p}{q} = \frac{1}{2}$, or make $x = -a$, whence

$$y = +^{\frac{1}{4}} (ma)^{\frac{1}{2}},$$

or the curve lies in a plane at right angles to the old plane.

The existence of this curve on the negative side of the axis of y, serves to explain an apparent anomaly which occurs in a very elementary problem. If we seek the equation to the locus of the intersection of a perpendicular from the focus of a parabola on the tangent, we obtain an equation which divides itself into two—the one representing the axis of y, which is the solution usually taken, the other furnishing only the focus. It seems strange that the focus should in any way be a solution of the problem, since the tangent of the positive branch of the curve never passes through that point. But if we consider the branch of the curve which lies in a plane perpendicular to the plane of xy, we see that all the tangents to that branch must pass through the positive axes of x, and consequently that one, or rather two, must pass through the focus, which thus is the point in which the tangent is met by a perpendicular from that point. Moreover, we find that the values of x and y, which belong to the focus, render the equation to the tangent of the form

$$y = (-)^{\frac{1}{2}} x + \beta,$$

showing that the tangent is in a plane at right angles to its original plane.

The form of the equation to the parabola renders it very easy to determine the value of y corresponding to that of x; but in the other curves of the second degree, though the investigation may not be so simple, we arrive at similar conclusions. Taking the equation to the ellipse referred to the centre,

$$y^2 = m^2 (a^2 - x^2),$$

$\left(\text{putting } \frac{b^2}{a^2} = m^2\right)$, we have for the value of y

$$y = m\,(a^2 - x^2)^{\frac{1}{2}}.$$

So long as $x < a$, whether we reckon x to be positive or negative, the value of y is possible, and the curve exists only in the plane of xy. If we make $x > a$, x being either positive or negative, we find y to be of the form $-^{\frac{1}{2}} mp$, showing that there is a branch of the curve in a plane at right angles to that of xy. Its form, it will be easily seen, is that of a hyperbola, since y increases with x, and becomes infinite when x is so, the relation between them being of the form

$$y = m\,(x^2 + a^2)^{\frac{1}{2}},$$

which is the equation to a hyperbola.

Hence, the vertices of the major axes of the ellipse are the vertices of two hyperbolas in a plane at right angles to the plane of the ellipse, and the ratio of the axes of which is the same as that of the axes of the ellipse. Also, since x and y are symmetrically involved in the equation to the ellipse, there must be a similar result for the extremity of the minor axis, the only difference being, that the axes of the hyperbola will be reversed in position.

If, more generally, we suppose $x = +^{\frac{p}{q}} c$, the result is not so simple, for we have

$$y^2 = m^2\,(a^2 - +^{\frac{2p}{q}} c^2);$$

from which we cannot directly determine the angle through which the axis of y is turned, corresponding to that through which x is supposed to be turned; but if we avail ourselves of the connexion which subsists between powers of $+$ and Demoivre's formula, we are able to determine the value of y.

Let $+^{\frac{p}{q}} = \cos\theta + -^{\frac{1}{2}} \sin\theta$, so that $\theta = \frac{2p\pi}{q}$, then

$$y^2 = m^2\,\{a^2 - (\cos 2\theta + -^{\frac{1}{2}} \sin 2\theta)\,c^2\}$$

$$= m^2\,\{a^2 - c^2 \cos 2\theta - -^{\frac{1}{2}} c^2 \sin 2\theta\};$$

whence

$$y = m\,(a^4 - 2a^2c^2 \cos 2\theta + c^4)^{\frac{1}{2}}\,(\cos\phi - -^{\frac{1}{2}} \sin\phi),$$

where

$$\cos 2\phi = \frac{a^2 - c^2 \cos 2\theta}{(a^4 - 2a^2c^2 \cos 2\theta + c^4)^{\frac{1}{2}}},$$

$$\sin 2\phi = \frac{c^2 \sin 2\theta}{(a^4 - 2a^2c^2 \cos 2\theta + c^4)^{\frac{1}{2}}}.$$

This result differs materially from that in the case of the parabola; for since the value of ϕ depends on c, the circulating function $\cos\phi - -^{\frac{1}{2}} \sin\phi$ depends on c, so that the angle through which the axis of y is to be turned, varies with the length of the abscissa, and the plane of the curve is constantly changing for every value of x, or in other words, the curve is one of double curvature.

When $c = 0$, $y = b$, and the curve passes through the extremities of the axis minor; and since c may be made infinite, the curve is infinite. Also, since x and y are symmetrically involved in the equation, there will be a similar curve passing through the extremities of the axis major. Hence, in addition to the hyperbolas already mentioned, the equation to the ellipse includes an infinite number of curves, with infinite branches passing through the extremities of the axes.

It is not necessary to consider the equation to the hyperbola, since it evidently leads to similar results. Nor is there much interest attached to the discussion of other more complicated curves, which I shall therefore omit with these exceptions—the curves of sines and cosines, and the logarithmic curve. These I shall briefly touch on, as the form of their equations renders their discussion easy, and as there is considerable interest attached to them in a geometrical point of view. If

$$y = a \sin x,$$

then, making $x = +^{\frac{p}{q}} c$, we have

$$y = a \sin(+^{\frac{p}{q}} c);$$

and it remains to be considered what relation this bears to $\sin c$. If we suppose the sector of the circle whose angle is c to turn round the radius from which c is measured, it will be easily seen, that on turning it round through a circumference the angle will return to its original position, and so for any number of revolutions. Therefore, this operation of turning the sector through a circumference is subject to the laws of the symbol $+$, and may therefore be represented by it; and consequently, the operation of turning the sector through the $\frac{p}{q}$th part of a circumference will be properly represented by $+^{\frac{p}{q}}c$. But since the sine of the arc, being in the plane of the sector, is perpendicular to the axis of revolution, it will also be moved through the $\frac{p}{q}$th of a circumference, from which it follows that

$$\sin(+^{\frac{p}{q}}c) = +^{\frac{p}{q}}\sin c.$$

But the cosine being measured along the axis of revolution, experiences no change corresponding to the change in the angle, so that we have

$$\cos(+^{\frac{p}{q}}c) = \cos c.$$

It may be observed, that these propositions are extensions of the two common theorems, that

$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x.$$

Hence, in the case of the curve of sines, we have

$$y = +^{\frac{p}{q}}a \sin c,$$

which shows that the axis of y is to be turned through an angle equal to that through which the axis of x is turned. Hence, if we suppose the plane of xy to turn round an axis in its own plane, passing through the origin, and making equal angles with the two axes of x and y, in every position there will be a curve of sines exactly the same as that in the plane of xy. On the other hand, the equation to the curve of cosines being

$$y = a \cos x,$$

gives for $x = +^{\frac{p}{q}} c$,

$$y = a \cos(+^{\frac{r}{q}} c) = a \cos c,$$

which shows that the axis of y remains fixed; and if we suppose the plane of xy to turn round the axis of y, there will be a curve of cosines, the same as that in xy, corresponding to every position of the plane.

The equation to the logarithmic curve is

$$y = \varepsilon^{nx}.$$

Let now $x = +^{\frac{p}{q}} a = a(\cos\theta + -^{\frac{1}{2}} \sin\theta)$ suppose.

Then $$y = \varepsilon^{na(\cos\theta + -^{\frac{1}{2}}\sin\theta)} = \varepsilon^{na\cos\theta} . \varepsilon^{na\sin\theta -^{\frac{1}{2}}}.$$

Now, since $+^{r} = \varepsilon^{2r\pi -^{\frac{1}{2}}}$, this gives us

$$y = (+)^{\frac{na\sin\theta}{2\pi}} . \varepsilon^{na\cos\theta};$$

thus determining the angle through which the axis of y is to be turned: and as this depends on a, the angle must (as in the case of the ellipse) vary with the length of the abscissa, so that the curve is not situate in one plane, but is a curve of double curvature. The absolute linear value of y, it will be easily seen, is less than that corresponding to the same linear value of x in the plane of xy, since the index of ε is reduced in the ratio of the cosine of θ to unity. It seems scarcely worth while further to discuss the nature of this curve, but having here adopted very different ideas concerning it from those promulgated by M. Vincent in Gergonne's *Annales des Mathématiques*, I think it necessary to state more at length my reasons for differing from that author.

In a paper in the preceding number of this Journal,*

* See p. 124 of this volume.

I developed what I conceive to be the true theory of general logarithms, and I endeavoured to show that the impossible parts are really logarithms of the powers of +, the existence of which is generally overlooked. I pointed out that the formula of Mr. Graves, which agrees with that of M. Vincent, was derived from the supposition, that the base of the system of logarithms was of the form $+^r a$. This gives for the equation to the logarithmic curve,

$$y = (+^r a)^x.$$

Now, M. Vincent assumes, that when x is fractional, y has as many values as there are units in the denominator of x; and when that is even, that two of these are possible—one positive and the other negative. Then he shows that the latter values do not form a continuous curve, but one of a kind which he calls *ponctuées*, whose nature is very peculiar, since, though the points are infinitely near to each other, yet we are able to draw an infinite number of straight lines between any two. This very strange result, so contrary to all our preconceived ideas of the nature of a curve, is sufficient, I think, to make us doubt the correctness of the method: but it is not very easy to point out the error, unless we employ the mode of considering the origin of the plurality of roots, which I have explained in the paper above referred to. According to that system, these various roots arise from our supposing a change to be made in the value of r; and properly speaking there is no plurality of roots, but the nature of the quantity $+^r a$, whose root is taken, is indeterminate. Now here, in the case of the equation to the logarithmic curve, there is no indeterminateness, since r can only have one value. Any change in the value of r is a change in one of the constants of the equation to the curve, and consequently the equation no longer represents the same curve. Each of the points of the "*courbe ponctuée*" of M. Vincent, is really the point

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in which a certain curve meets the plane of xy. They are therefore wholly unconnected with each other, and cannot be reckoned as belonging to the same curve, either in a geometrical or analytical sense. This being granted, we perceive that there is no foundation for M. Vincent's very anomalous conclusion: and we are thus relieved from the necessity of believing in the existence of a species of line, of which we can hardly form a conception, and which has no sort of analogy to support it.

I said, that each point in the "*courbe ponctuée*" of M. Vincent belonged to a separate curve—it may be interesting to consider for a moment its nature. The equation

$$y = (+^r a)^x$$

gives

$$y = +^{rx} a^x.$$

So long as x is an integer, this is the same as the simple equation

$$y = a^x,$$

and gives us the logarithmic curve in the plane of xy; but if we suppose x to be a fraction, it appears that the axis of y is to be turned through rx^{th} part of a circumference. As this angle varies with x, it appears that the curve is one of double curvature, and, as when $x=0$, $y=1$, it intersects the plane of xy at a distance 1 along the axis of y. But it may intersect it again; for if

$$x = \frac{1}{2r}, \quad +^{rx} = +^{\frac{1}{2}} = -,$$

and it cuts the plane of xy on the negative side of y; and if $x = \dfrac{1}{r}$, then $+^{rx} = +$, and it cuts the plane of xy on the positive side of y, meeting the curve traced in the plane of xy. On increasing x we shall obtain a similar curve which cuts the plane of xy, first above and then below the axis of x, the curve meeting the plane of xy whenever $rx = \dfrac{m}{2}$,

m being either odd or even. It is needless to enter into the discussion of the complicated cases when x is of the form $+^{\frac{p}{q}}a$; more particularly as this paper has already exceeded its just limits. And I only will add, that these speculations derive their chief value from their bearing on the General Theory of the Science of Symbols. Practically, little attention will be paid to curves existing out of the plane of reference, since the curves themselves do not come sufficiently under our eye to attract much interest. Perhaps the only way in which the existence of such curves is likely to be brought into notice, is in those cases where they serve to show the possibility of the interpretation of a solution of an equation. One case of this kind I have remarked on in this paper; several have been pointed out by Buée, and many I have no doubt will be added, when the attention of mathematicians is more particularly directed to the subject.

ON THE ELEMENTARY PRINCIPLES OF THE APPLICATION OF ALGEBRAICAL SYMBOLS TO GEOMETRY.*

IN several previous papers in this Journal, I have considered the principles on which certain symbols of operation become subject to the same rules of combination as the symbols of number, which are those usually handled in Algebra. The general theory of this subject I gave in a paper (to which I have elsewhere referred) on the Nature of Symbolical Algebra; in which I endeavoured to exhibit distinctly the principles on which various branches of science may be symbolized—that is to say, on which their study is facilitated by expressing the operations by means of symbols. I use the word *operation* for the purpose of avoiding anything like limitation in the subjects which the symbols may represent, as is too apt to be the case when we employ the word *quantity*, which is generally made to be synonymous with number. Among the sciences whose symbolization I there considered, that of Geometry is the most important; and on that account I wish here to treat of it more at large, especially because it appears to me that the theory of the representation of geometrical quantities by numerical symbols is usually but little attended to, and some obscurity still hangs over the subject. In treating of this matter, I may perhaps

* *Cambridge Mathematical Journal,* Vol. II., p. 1.

appear to some to be raising difficulties where there are none; but I think that a little consideration will show to these persons that the question is not quite so simple as might at first be imagined. Much attention has been bestowed on the theory of the representation of direction by means of the symbols + and −, but the principles on which lines, areas, and solids are represented by numbers has been but little discussed. It is to the latter of these subjects that my remarks will be first directed, and I shall afterwards develop my views of the former.

In the paper I have referred to, I lay down the principle, that an algebraical symbol can only represent an operation in any other science when it is subject to the same laws of combination as that operation. In fact, that as Algebra takes cognizance only of the laws of combination of the symbols, and not of their meaning—in the eye of that science the symbol and the operation are identical. When we turn to the interpretation of our results, we must of course consider the meanings of the symbols—but such interpretation is out of the province of Algebra, and belongs to the science, the operations of which are symbolized. Now, in applying these principles to Geometry, we have first to become acquainted with the operations which require to be symbolized, and then to consider the laws of combination to which they are subject, in order that we may know under which family of algebraical symbols they are to be classed. The ideas with which we are concerned in Geometry are those of magnitude and direction. The former is of three kinds—linear, plane, and solid; and the question is, of what sort of operations these may be considered as the result. Such a one I conceive to be *transference* in *one direction;* for by proper combinations of operations of this description we can represent magnitudes of all kinds. Some persons may think it strange to introduce such an idea as that of transference into so simple a subject as Geometry; but in defence of its adoption, I think it only necessary to plead the simplicity and uniformity of the explanations it affords of the principle of the application of Algebra to Geometry: and I may add, that we are not here considering how Geometry may be treated *geometrically*, but *symbolically;* and we must be content to do so in the way which the subject most readily permits. Besides, for my own part, I think that the idea of transference is quite as simple and elementary as any which occurs in Geometry, and offers itself as readily to the mind of the student. Having fixed on an operation which is to be symbolized, we have also to consider what may be the subject of that operation. The simplest geometrical idea, and that which suits our purpose, is the idea of a point. We may, if we choose, represent this by a symbol, as we represent the fundamental subject-idea in Arithmetic by the symbol 1: but this is not necessary; for, as in Algebra, we have only to consider the combinations of symbols of operation—the subject, being always the same, may be understood, and the symbol for it omitted. Thus it is that we omit in Arithmetic the symbol for unity, which nevertheless requires to be understood at every step as the subject, without which the whole would be unintelligible.

Now, let us assume a to be a symbol representing transference in *one constant* direction through a given space; then, representing the subject-point by the symbol (.), the compound symbol

$$a\,(.)$$

will represent a straight line, as the result of transferring a point through a given space in a constant direction. But as we have agreed to omit the subject-symbol, a line of a given length will be simply represented by the symbol a, which now does not represent the operation, but the result of the operation on the subject.

Again, we may combine this symbol with another symbol for transference in some other given direction, and we may ask the meaning of such a combination as

$$b\{a(.)\},$$

or, omitting the symbol for the subject,

$$b(a).$$

This, it is clear, must signify the transference of a line in one constant direction, that is, the line must move parallel to itself, by which means it will trace out a parallelogram, whose sides are represented by a and b.

In the same way in which we have combined two symbols of transference we may combine three, and ask the meaning of the expression $c\{a(b)\}$. This will, on the same principle, represent the transference of a plane in one constant direction, that is, the transference of a plane parallel to itself, the result of which is a parallelopiped. If we combine more symbols than these, we find no geometrical interpretation for the result. In fact, it may be looked on as an impossible geometrical operation; just as $\sqrt{(-1)}$ is an impossible arithmetical one. For a solid, having equal relations to the three dimensions of space, cannot have any relation with one particular direction, which refers only to one dimension, and direction is essentially involved in the operation we have been considering.

From what has preceded, it appears that we are able, by the combination of the symbol of one kind of operation, to represent the three different geometrical magnitudes—lines, areas, and solids; but, as yet, nothing has been said to point out the *algebraical* nature of these symbols, so that we cannot tell whether or not they coincide algebraically with the symbols for numbers. So far as we have gone, we have not shown how the study of Geometry may be facilitated by having its operations symbolized, as we know not how to treat the symbols, some combinations of which we have been interpreting. But we shall now proceed to show, that these symbols are subject to the two laws of combination which characterize the symbols of number, the ordinary subjects of algebraical operations, viz. the commutative law and the distributive law.

We have found that $b(a)$ represents a parallelogram, the sides of which are a and b; and in the same way $a(b)$ must represent a parallelogram whose sides are also a and b, and which is identical with the former, as the relative inclination of the sides is the same. Hence it follows, that when a and b represent the geometrical operation of transference in a given direction,

$$a(b) = b(a),$$

or the symbols are *commutative.*

Again, with respect to the distributive law: supposing that the symbol $+$ represents the simple arithmetical idea of addition, (the reason for which restriction will be seen afterwards,) $a+b$ will represent a line resulting from the transference of a point in the same direction through distances a and b, and $c(a+b)$ will represent a parallelogram whose sides are c and $a+b$. But $c(a)$ and $c(b)$ will represent respectively parallelograms, whose sides are c, a and c, b, so that $c(a)+c(b)$ will represent the sum of these parallelograms. But, by the first proposition of the second book of Euclid, we know that the sum of these is equal to the first parallelogram. It is true that the proposition in Euclid is proved only for rectangles, but the principle of the demonstration applies to all parallelograms whatsoever. From this it follows, that when c, a, b represent the geometrical idea of transference in a given direction,

$$c(a+b) = c(a) + c(b),$$

or the symbols are distributive.

We are now enabled to see why we can represent geometrical ideas by arithmetical symbols, so as to render geometrical research easier from our previous acquaintance with arithmetical combinations. It is because the symbols in both cases are subject to the same laws of combination, and therefore in the eye of Algebra are identical, at least so far as these laws (which are the algebraical definitions) are concerned. Whatever, therefore, may have been proved in Arithmetic, in dependence solely on these laws, is equally true in Geometry, provided always that we can interpret the result; for there is no reason why we should always be able to interpret a symbolical result either geometrically or arithmetically. And indeed, in Geometry the uninterpretability is soon presented to us in the combination of more than three symbols of transference. From this it appears why areas and solids may be represented by the product of the symbols of lines, or rather by the apparent product: for when a and b are geometrical symbols, we cannot talk of their being multiplied together—but we see that the operation of one on the other bears a close resemblance to the arithmetical operation of multiplication, and from the identity of the laws of combination they may be considered algebraically as the same, though the meanings be wholly different. This question as to the possibility of representing areas and solids by means of the apparent multiplication of the symbols for lines, has always appeared to me to be one of great difficulty in the application of Algebra to Geometry: nor has the difficulty, I think, been properly met in works on the subject. It is not sufficient to say, as is usually done, that if we divide each of the lines into a certain number of units, the number of superficial units in the parallelogram will be equal to the product of the number of units in the two lines: it is also necessary to show how a superficial unit can be represented by the product of two linear units, and this I think cannot be done except on the principle which has here been used.

It is to be observed, that in all which has preceded we have supposed the symbols to represent transference in a constant direction. This limitation is necessary in defining our symbols; for if we were to suppose the direction to vary during the progress of the transference, the same laws would not be found to hold with respect to these symbols as we have seen to hold for the symbols we considered, and we should then be unable to reduce geometrical investigations to processes of arithmetical calculation. We might, certainly, if we chose, use symbols representing different kinds of transference, and we might employ ourselves in investigating their nature and the laws of their combination; but having done so, we should derive no assistance from any previous labours in the science of symbols. It is solely from the previous knowledge which we have of the combinations of arithmetical symbols, that we are enabled to facilitate our researches by the application of Algebra to Geometry, or to any science whatever. And thus it is, that any improvement or discovery in Algebra, however isolated and useless it at first appear, may become ultimately of the utmost importance for the prosecution of other branches of knowledge.

Hitherto we have confined ourselves to the consideration of the means of representing symbolically the geometrical ideas of magnitude; and we have shown how the combination of these symbols to represent areas and solids, bears an analogy to the processes of multiplication in Arithmetic: we shall now proceed to consider the symbolization of direction, and to show that the symbols we adopt bear a striking analogy to a well-known arithmetical symbol.

Direction, in ordinary Plane Geometry, is estimated by means of rectilinear angles, which affords us an easy means of symbolizing this geometrical idea; for by supposing a straight line to revolve round a point situate within it, we can make it generate any given angle. This, therefore, is the operation which we shall express by a symbol, and the laws of which we are to investigate. It is clear, in the first place, that if we take some standard angle as that which is to be the result of the operation symbolized, we may produce multiples or submultiples of that angle by performing the operation a certain number of times, or by performing a certain part of the operation. It is therefore necessary to choose some angle for our standard, and the most convenient for our purpose is that produced by a complete revolution of the line, or revolution through four right angles. Let us assume, then, the symbol Λ to represent the operation of making a line revolve through four right angles, so that, a representing a line in a given direction, $\Lambda(a)$, will represent the same line inclined at an angle equal to four right angles,—that is to say, in a direction coinciding with the original direction. If we repeat the operation, $\Lambda\{\Lambda(a)\}$, or, in accordance with ordinary algebraical notation, $\Lambda^2(a)$, will represent a line inclined to the original at an angle equal to eight right angles, and so on for any number of times that the operation may be performed. As we have introduced integer indices attached to the operation Λ, we may also use fractional indices, and enquire what is the meaning of such an expression as $\Lambda^{\frac{1}{2}}(a)$ or $\Lambda^{\frac{1}{3}}(a)$. In accordance with the algebraical laws for the combination of indices, we easily see that $\Lambda^{\frac{1}{2}}$ must signify an operation which, being performed twice, will give birth to Λ. Such will be the turning of a line through two right angles, or 180°, so that $\Lambda^{\frac{1}{2}}(a)$ will represent a line measured in the opposite direction from the original line. In the same way $\Lambda^{\frac{1}{3}}$ must signify the turning of a line through one third of four right angles, or 120°, as that operation being performed thrice will be equivalent to the turning of a line through four right angles. And generally $\Lambda^{\frac{1}{n}}$ will signify the turning of a line through the n^{th} part of four right angles, or $\dfrac{360°}{n}$. Thus, by the use of the simple algebraical notation of indices, joined to the geometrical operation of turning a line through a given angle, we are able to express the operation of turning a line through any angle whatsoever, and so to express all relations of directions between lines situate in a plane. It is to be observed, that since the operation of turning a line through four right angles, or through any multiple of four right angles, brings it back to its original position, the effect of any number of repetitions of the operation Λ is the same, which may be expressed algebraically by saying that

$$\Lambda^n = \Lambda,$$

n being any integer, which is a law of combination of Λ, and may be considered as its algebraical definition. Now, this is the very law which is known to belong to the arithmetical operation of addition usually represented by $+$, since we have then

$$++=+, \text{ and therefore } +^n=+,$$

n being any integer. Hence it appears, that as the arithmetical operation of addition, and the geometrical operation of turning a line through four right angles, are subject to the same law of combination, they are, so far as that is concerned, algebraically identical, and may be represented by the same symbol. Such, indeed, has long been the case, for the arithmetical symbols for addition and subtraction, along with certain modifications of them, are constantly used to represent geometrical direction. This has given rise to much difficulty and many attempts at explanation; some persons wishing to show that the geometrical operation might be supposed to be derived from the arithmetical, but not finding it very easy to do so in a satisfactory manner—others being inclined to found their views of some points in the arithmetical theory on the basis of the geometrical idea, interpreting the former by the latter. I believe, that the more closely the subject is examined, the more clearly it will be seen, that there is really no resemblance in *kind* between the two operations, but only an identity in the laws of combination; and if this be kept steadily in view, all the difficulties which have been observed in this part of mathematics, and on which so much has been written, will receive a satisfactory explanation. This double meaning of + is the reason of the limitation to the meaning of that symbol assumed in p. 154.

We have only considered the operation Λ or +, as we may now term it, in connection with the symbol for a line, as it was with reference to the direction of a line that its definition was made. But this symbol may also receive interpretation in another case, to which its original definition does not directly refer. It is not *necessary* that it should admit of any other geometrical interpretation, but such is found to be the case when it is applied to areas. The position of a line is determined by the direction in which its length lies; but the position of a plane cannot be determined in like manner by its extension, since that has two dimensions, and direction has only one. But the position of an area may be determined by the direction of the face of the plane, which can be referred to that of any straight line inclined to it at a given angle (such as a right angle), so that we know how one plane is related to another if we know in what direction the face of each is presented. Now, supposing an area to revolve round any line in its own plane, we can make it assume any position we please; and it is easy to see that the operation of turning the area completely round is subject to the same law as that of turning the line, that is to say, that when it is repeated any number of times the result is the same, since the area will always present the same face. Hence it follows, that these two operations may be represented by the same symbol; so that if in any process of Analytical Geometry we find the symbol +, which was originally applied to the symbol for a line, ultimately applied to the symbol for an area, we are able to interpret it. This view of the meaning of +, when applied to the symbol for an area, enables us to offer an explanation of a difficulty in Analytical Geometry.

If x, Ay (fig. 6) be a system of rectangular coordinates, we know, from what has been previously said concerning the representation of the direction of lines, that any abscissa measured along x will be affected with +, and any abscissa measured along x' will be affected with $+^{\frac{1}{2}}$ or $-$; and similarly, any ordinate measured along Ay will be affected with +, and any along Ay' with $+^{\frac{1}{2}}$ or $-$. Therefore the coordinates of a point P will be

$$+x, +y, \quad -x, +y, \quad -x, -y, \quad +x, -y,$$

according as it is in the first, second, third, or fourth quadrant. Now, the rectangle $AxPy$ being represented by the product of the symbols representing its sides, will be

$$+xy, \quad -xy, \quad +xy, \quad -xy,$$

according as it is in the first, second, third, or fourth quadrant. The question then is, what meaning we are to attach to these expressions. It will be seen by a glance at the figure, that if the rectangle AxP_1y turn round the line Ay, or the line Ax through half a circumference, it will occupy the place of $Ax'P_2y$, or $Ay'P_4x$, and therefore these rectangles may be considered as resulting from the turning of the original rectangle round Ay or Ax through half a circumference, so as to present the other face of the plane. Now, we have just shown that the operation of turning a plane through a complete circumference, so as to present the same face as before, may be represented by +, and therefore the operation of turning it through half a circumference may be represented by $-$. Therefore the negative signs attached to expressions for the rectangles in the second and fourth quadrants, are to be interpreted as signifying that these rectangles are equivalent to the original rectangles turned through half a circumference round Ax or Ay, just as the line Ax' would be produced by turning Ax through half a circumference. With respect to the rectangle in the third quadrant, which forms the chief point of difficulty, it can be derived either from that in the second or that in the fourth, by turning them through half a circumference round Ax' or Ay'. And as both of these rectangles present the face opposite to that of the primary rectangle, it is quite consistent with, and indeed follows from the definition of $+$, that the rectangle in the third segment should be represented by $+xy$, since it is derived from the primary rectangle by that rectangle being turned through a circumference, so that it presents the same face in the same direction at it did at first. Or if we suppose that the area in the second or fourth quadrant, instead of continuing to revolve in the same direction as that by revolving in which it was derived from the area in the first quadrant, revolves back so as to undo the operation previously performed, the same result will follow. For the area in the first quadrant being represented by $+xy$, that in the second, being the former turned round Ay through half a circumference, will be represented by $+^{\frac{1}{2}}+xy$: while the area in the third quadrant, being derived from that in the second by its being turned round Ax' through half a circumference in the opposite direction, will be represented by $+^{-\frac{1}{2}}+^{\frac{1}{2}}+xy$, or $+xy$, as in the first quadrant, which ought to be the case, as the same face as before is presented. The same result of course will follow, if we consider the area in the third quadrant as derived from that in the fourth.

These explanations of the meanings of the symbols $+$ and $-$, when applied to areas, are consistent with the original definition, and are closely analogous to their significations when applied to lines, so that I think they must be deemed satisfactory. Should it now be asked whether these principles can be applied to solids, so as to explain the meaning of the symbols $+$ and $-$ prefixed to those of parallelopipeds, I have to answer that they do not; and the reason I conceive to be, as I said before on another subject, that a solid being extended in three dimensions has no relation to one direction, which is essentially only of one dimension. A face or an edge of the solid may be referred to one direction, but the solid itself cannot be so referred. Such expressions as $+abc$ or $-abc$ are, I hold, uninterpretable consistently with the geometrical meaning we attach to the symbols $+$ and $-$. By calling them uninterpretable, I put them in the same class in Geometry as the symbol $\sqrt{(-1)}$ in Arithmetic; we do not at present see any interpretation for them, though there is no reason why farther progress and more extended views in Arithmetic and Geometry should not enable us to understand what is at present beyond our comprehension.

ON THE SYMMETRICAL FORM OF THE EQUATION TO THE PARABOLA.*

WHEN the parabola is referred to a diameter and the tangent at its vertex, although the equation then assumes the simplest form, yet as these lines are not symmetrical with respect to the curve, the equation itself is not symmetrical with respect to the variables. In order, therefore, to get the equation under a symmetrical form, we must refer the curve to lines similarly situated with respect to it: such are two tangents to the parabola. If we take them as axes, and their intersection as origin, the equation to the curve assumes a form which bears a curious analogy to the symmetrical equations of the other conic sections and of the straight line, and is sufficiently remarkable in itself to deserve attention.

The general equation to a curve of the second degree is

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0 \;\ldots\ldots\; (1);$$

the condition that this should represent a parabola is

$$B^2 = 4AC \text{ or } B = \pm 2\sqrt{(AC)} \;\ldots\ldots\ldots\ldots (2),$$

so that (1) is reduced to

$$\{A^{\frac{1}{2}}y \pm C^{\frac{1}{2}}x\}^2 + Dy + Ex + F = 0 \ldots\ldots\ldots\ldots (3).$$

Now, let the parabola be referred to the two tangents AB, AC (fig. 7) as axes, and let $AB = a$, $AC = b$, AB being the axis of x, AC of y. Then, since AB is a tangent at B,

* *Cambridge Mathematical Journal*, Vol. II., p. 14.

if we make $y=0$ in equation (3), the two corresponding values of x must be each equal to a. In this case equation (3) becomes

$$Cx^2+Ex+F=0 \quad\ldots\ldots\ldots\ldots\ldots\quad (4);$$

and the condition for its being a complete square is

$$E^2=4CF \quad\ldots\ldots\ldots\ldots\ldots\quad (5);$$

and as each root is equal to a, we have

$$\frac{F}{C}=a^2, \quad \frac{E}{2C}=-a;$$

therefore $$C=\frac{F}{a^2}, \quad E=-\frac{2F}{a}.$$

In a similar manner we should find that

$$A=\frac{F}{b^2}, \quad D=-\frac{2F}{b},$$

so that equation (3) takes the form

$$\left(\frac{y}{b}\pm\frac{x}{a}\right)^2-2\left(\frac{y}{b}+\frac{x}{a}\right)+1=0 \quad\ldots\ldots\ldots\quad (6).$$

If we take the superior sign in the first term, this equation is equivalent to

$$\left(\frac{y}{b}+\frac{x}{a}-1\right)^2=0,$$

which is the equation to a straight line, or rather to two which coincide; we must therefore take the inferior sign, in order that the equation may represent a parabola. If, now, we add $\frac{4xy}{ab}$ to both sides, the equation becomes

$$\left(\frac{y}{b}+\frac{x}{a}\right)^2-2\left(\frac{y}{b}+\frac{x}{a}\right)+1=\frac{4xy}{ab} \quad\ldots\ldots\ldots\quad (7),$$

the first side of which is a complete square. Extracting, then, the square root on both sides, we have

$$\frac{y}{b}+\frac{x}{a}-1=\pm 2\sqrt{\left(\frac{xy}{ab}\right)};$$

or transposing, $$\frac{y}{b}\mp 2\sqrt{\left(\frac{xy}{ab}\right)}+\frac{x}{a}=1;$$

the first side of which is also a complete square. Extracting the root again, we finally obtain

$$\sqrt{\left(\frac{y}{b}\right)} \pm \sqrt{\left(\frac{x}{a}\right)} = \pm 1 \text{ ....(8)},$$

which is the required symmetrical form.

The form of this equation shows at once, that the curve lies wholly between the positive axes, as neither x nor y can ever become negative. So long as $x < a$ and $y < b$, the positive signs only on both sides must be taken, as the difference between two fractions can never be unity. If $x > a$ and $y > b$, the negative sign only on the left-hand side must be taken, as the sum of two quantities greater than unity can never be equal to unity; and either sign on the right-hand side, according to the relative magnitude of the terms on the left-hand side. If $y > b$ and $x < a$, the negative sign on the first side and the positive on the second are to be taken; and if $x > a$ and $y < b$, the negative sign on both sides. This apparent discontinuity, which renders it necessary to take sometimes one sign and sometimes another, arises from the equation (8) not being the complete form of the equation to the curve. All the cases are included in the expanded form of (9)

$$\frac{y^2}{b^2} - \frac{2xy}{ab} + \frac{x^2}{a^2} - 2\left(\frac{y}{b} + \frac{x}{a}\right) + 1 = 0.$$

If we transpose one term of (8) and square both sides, we have

$$\frac{y}{b} = 1 \pm 2\sqrt{\left(\frac{x}{a}\right)} + \frac{x}{a},$$

or

$$\frac{y}{b} - \frac{x}{a} = 1 \pm 2\sqrt{\left(\frac{x}{a}\right)},$$

so that $\frac{y}{b} - \frac{x}{a} = 1$ is the equation to a diameter passing through C, and similarly

$$\frac{y}{b} - \frac{x}{a} = -1$$

is the equation to a diameter passing through B, and

$$\frac{y}{b} - \frac{x}{a} = 0$$

to one passing through A.

This form of the equation affords an easy proof of a problem in the Senate-House Papers for 1833. The enunciation is as follows: If there are three tangents to a parabola, the triangle formed by their intersection is half of that whose angular points are the points of contact.

Let ARS, BPC (fig. 7) be the triangles; then, taking the equation to the parabola referred to AB, AC as axes, the equation to the tangent is

$$\frac{y}{\sqrt{(by_1)}} + \frac{x}{\sqrt{(ax_1)}} = 1,$$

where $x_1 y_1$ are the coordinates of the point P.

In this equation, making successively $x = 0$, $y = 0$, we find

$$AS = \sqrt{(by_1)}, \quad AR = \sqrt{(ax_1)}.$$

Now

$$\text{area } ASR = \tfrac{1}{2} AR . AS \sin A = \tfrac{1}{2} \sqrt{(abx_1 y_1)} \sin C,$$

and $\quad \text{area } CPB = ACB - NPC - MPB - AMPN.$

Now $\quad ACB = \tfrac{1}{2} ab \sin C,$

$$NPC = \tfrac{1}{2} NC . PN \sin C = \tfrac{1}{2} x_1 (b - y_1) \sin C,$$

$$MPB = \tfrac{1}{2} MB . PM \sin C = \tfrac{1}{2} y_1 (a - x_1) \sin C,$$

and $\quad AMPN = x_1 y_1 \sin C.$

Hence

$$\text{area } CPB = \tfrac{1}{2} \sin C \{ab - x_1 (b - y_1) - y_1 (a - x_1) - 2x_1 y_1\}$$
$$= \tfrac{1}{2} \sin C (ab - bx_1 - ay_1).$$

But, since $x_1 y_1$ are coordinates of a point in the parabola,

$$\sqrt{\left(\frac{x_1}{a}\right)} + \sqrt{\left(\frac{y_1}{b}\right)} = 1,$$

and therefore

$$\frac{x_1}{a} + 2\sqrt{\left(\frac{x_1 y_1}{ab}\right)} + \frac{y_1}{b} = 1;$$

and multiplying by ab, and transposing,

$$2\sqrt{(abx_1y_1)} = ab - bx_1 - ay_1;$$

so that

$$\text{area } CPB = \tfrac{1}{2}\sin C . 2\sqrt{(abx_1y_1)} = \sin C\sqrt{(abx_1y_1)},$$

and therefore $ASR = \frac{1}{2}CPB$.

Since $AR = \sqrt{(ax_1)}$ and $AS = \sqrt{(by_1)}$, we have, making

$$AR = x', \quad AS = y',$$

$$\frac{x'}{a} + \frac{y'}{b} = \frac{\sqrt{(ax_1)}}{a} + \frac{\sqrt{(by_1)}}{b} = \sqrt{\left(\frac{x_1}{a}\right)} + \sqrt{\left(\frac{y_1}{b}\right)} = 1 \;\ldots.(8);$$

and as the equation to BC is

$$\frac{x}{a} + \frac{y}{b} = 1,$$

x' and y' are coordinates of the line BC; so that if from any point Q in BC we draw QS, QR parallel to the axes, the line joining the points where they cut the axes will be a tangent to the parabola. This gives the means of describing a parabola by the ultimate intersection of a line subject to move under a certain condition. For if

$$\frac{x}{m} + \frac{y}{n} = 1$$

be the equation to RS, m and n are subject to the condition

$$\frac{m}{a} + \frac{n}{b} = 1.$$

N THE FAILURE OF FORMULÆ IN THE INVERSE PROCESSES OF THE DIFFERENTIAL CALCULUS.*

If we apply the rule for integrating any power of x to the particular case when the index of the power is -1, we obtain a result having 0 in the denominator, and which is therefore nugatory. This is only one instance of several in which a certain relation of the subject to an inverse operation makes the general formulæ fail; and as these cases give rise to some difficulty, we shall here consider two of the most important of them. The instance to which we have alluded is so well known, that we need do no more than mention it; and for the more general case of failure when the index is of any value, the reader is referred to the *Cambridge Mathematical Journal*, Art. VI., Vol. I., p. 109. The method of arriving at the true value in these cases of failure, is the same as that which we shall pursue in those we are about to consider. The principle is this: since the function in this particular case becomes infinite, we may so assume the arbitrary constant in the complementary function, as to make the formula for the particular value take the indeterminate form $\frac{0}{0}$, the true value of which can easily be determined by the ordinary rules. The assumption made

* *Cambridge Mathematical Journal*, Vol. II., p. 73.

with respect to the arbitrary constant in the complementary function, is to make it negative and infinite, so that the difference of two infinite quantities may be finite. Exactly the same principle holds in the instances we are about to consider.

Suppose we had the differential equation

$$\frac{dy}{dx} - ay = \varepsilon^{ax},$$

we should find, by the usual rule for integrating such equations,

$$y = \frac{\varepsilon^{ax}}{a - a} + C\varepsilon^{ax},$$

the form of which is nugatory. To discover the true value, let us suppose that the multiplier of y is not the same as the multiplier of x, but that the equation is

$$\frac{dy}{dx} - a_1 y = \varepsilon^{ax};$$

the integral of which is

$$y = \frac{\varepsilon^{ax}}{a - a_1} + C\varepsilon^{a_1 x}.$$

Now C being arbitrary, we may conceive it to consist of two parts, so that $C = -\dfrac{1}{a - a_1} + C_1$, which gives

$$y = \frac{\varepsilon^{ax} - \varepsilon^{a_1 x}}{a - a_1} + C_1\varepsilon^{a_1 x}.$$

Now when $a_1 = a$, the first term takes the form $\dfrac{0}{0}$, which is indeterminate; and by the usual method its true value, when $a_1 = a$, is found to be $x\varepsilon^{ax}$, so that

$$y = x\varepsilon^{ax} + C_1\varepsilon^{ax},$$

which is the true solution of the equation.

If the operating factor were of the order r, so that the equation was

$$\left(\frac{d}{dx} - a\right)^r y = \varepsilon^{ax},$$

we should find by the usual rule

$$y = \frac{\varepsilon^{ax}}{(a-a)^r} + (C_0 + C_1 x + \&c. + C_{r-1} x^{r-1})\,\varepsilon^{ax},$$

a nugatory result.

If we suppose the a in the operating factor to be different from the multiplier of x, we should have, by a change of the first arbitrary constant,

$$y = \frac{\varepsilon^{ax} - \varepsilon^{a_1 x}}{(a-a_1)^r} + (C_0' + C_1 x + \&c. + C_{r-1} x^{r-1})\,\varepsilon^{a_1 x}.$$

If we differentiate the numerator and denominator of the first term in order to determine its value when $a_1 = a$, we find

$$y = \frac{x\varepsilon^{ax}}{r(a-a_1)^{r-1}} + (C_0' + C_1 x + \&c. + C_{r-1} x^{r-1})\,\varepsilon^{a_1 x},$$

which is still nugatory when $a = a_1$. We must therefore continue the process, changing the constant in the second term of the complementary function, when we obtain

$$y = \frac{x(\varepsilon^{ax} - \varepsilon^{a_1 x})}{r(a-a_1)^{r-1}} + (C_0' + C_1' x + \&c. + C_{r-1} x^{r-1})\,\varepsilon^{a_1 x},$$

the first term of which we find, as before, to be infinite when we make $a = a_1$. But by continuing the same process as before, we shall at last obtain

$$y = \frac{x^r \varepsilon^{ax}}{r(r-1)\ldots 2.1} + (C_0' + C_1' x + \&c. + C'_{r-1} x^{r-1})\,\varepsilon^{a_1 x},$$

which, when $a_1 = a$, becomes

$$y = \frac{x^r \varepsilon^{ax}}{r(r-1)\ldots 2.1} + (C_0' + C_1' x + \&c. + C'_{r-1} x^{r-1})\,\varepsilon^{ax},$$

being the true solution of the equation.

The other example which we shall here consider is particularly important, as the form of the solution occurs in the second approximation in the Lunar Theory, rendering necessary a change in the form of the equation.

It is met with in the integration of the equation

$$\frac{d^2u}{d\theta^2} + n^2u = \cos m\theta,$$

when $m = n$. For the general solution is

$$u = \frac{\cos m\theta}{n^2 - m^2} + C \cos n\theta + C_1 \sin n\theta,$$

the first term of which is infinite when $m = n$. But if, as before, we change the arbitrary constants in the complementary function, we can put the equation under the form

$$u = \frac{\cos m\theta - \cos n\theta}{n^2 - m^2} + C' \cos n\theta + C_1 \sin n\theta.$$

The value of the first term of this, when $m = n$, determined in the usual way, is $\frac{\theta \sin n\theta}{2n}$; so that

$$u = \frac{\theta \sin n\theta}{2n} + C' \cos n\theta + C_1 \sin n\theta.$$

In the same way, if the original equation were

$$\frac{d^2u}{d\theta^2} + n^2u = \sin n\theta,$$

we should find

$$u = -\frac{\theta \cos n\theta}{2n} + C \cos n\theta + C_1' \sin n\theta.$$

If the original equation were

$$\left(\frac{d^2}{d\theta^2} + n^2\right)^2 u = \cos m\theta,$$

we should have

$$u = \frac{\cos m\theta}{(n^2 - m^2)^2} + \left(\frac{d^2}{d\theta^2} + n^2\right)^{-1} (C_0 \cos n\theta + C_1 \sin n\theta);$$

which, by what we have just found, (observing that the constants are arbitrary,) is equal to

$$u = \frac{\cos m\theta}{(n^2 - m^2)^2} + \theta\,(C_0 \sin n\theta - C_0' \cos n\theta) + C_1' \sin n\theta + C_1' \cos n\theta.$$

The true value of the first term of this, when $m = n$, will be found, by the same process as in the last example, to be

$$-\frac{\theta^2 \cos n\theta}{2.1\,(2n)^2};$$

and generally, if the equation be

$$\left(\frac{d^2}{d\theta^2} + n^2\right)^r u = \cos m\theta,$$

the true value of the first term will be, when $m = n$,

$$\frac{\theta^r \cos\left(n\theta + r\dfrac{\pi}{2}\right)}{r\,(r-1)\ldots 2.1\,(2n)^r}.$$

MATHEMATICAL NOTE.*

To the Editor of the Mathematical Journal.†

SIR,—Your paper on the "Extraneous Solutions of Geometrical Problems," calls my attention to a mistake which I made in an article on the "Existence of Branches of Curves in various Planes." I there state (Vol. I., p. 261)‡ that the extraneous factor of the problem of finding the locus of the intersections of perpendiculars from the focus on the tangents to a parabola, corresponds to the intersection of perpendiculars on the tangents to the branch in the plane perpendicular to xy; whereas it really corresponds to the intersection of those tangents with lines passing through the focus, and making with the axis of x an angle complementary to that of the tangent.

The mistake arose from supposing, that if

$$y = -^{\frac{1}{2}}\alpha x + \beta,$$

in the equation to a tangent in the plane $(+, +^{\frac{1}{4}})$; that to a line perpendicular to it is

$$y = -\frac{1}{-^{\frac{1}{2}}\alpha}(x - m);$$

whereas it ought to be

$$y = - -^{\frac{1}{2}}\frac{1}{\alpha}(x - m);$$

* *Cambridge Mathematical Journal*, Vol. II., p. 91.

† The Editor of this volume was, at the date of this Note, Editor of the Journal.

‡ See p. 142 of this volume.

inasmuch as the $-\frac{1}{2}$ has no reference to the mutual inclination of the lines, which is determined only by α. It is true that the locus of the intersections of the tangent with the perpendiculars on it from the focus does pass through the focus, as the geometrical reasoning I have employed shows; but the extraneoüs factor is not the equation to the locus, which your paper clearly proves. As the nature of the error is, perhaps, not very apparent, I shall feel obliged by your insertion of this correction of it.

I am your obedient servant,

D. F. GREGORY.

ON THE SYMPATHY OF PENDULUMS.*

By the Sympathy of Pendulums is meant the effect on the motions of different pendulums produced by their mutual action, when their points of suspension have any elastic or moveable connexion.

The phenomenon is a striking one, and presents itself in a marked manner to those who are engaged in the art of clock-making. It has been observed by them, that if the pendulums of two clocks, the times of the oscillation of which are different, be so situated that the motion of the one can be in any way communicated to the other—as, for instance, by their centres of suspension being attached to the same beam—the motions of the two pendulums are entirely altered by their mutual action, the periods of both tending to become the same, and the extent of oscillation continually changing. This becomes a serious practical inconvenience, and it is necessary to take precautions to prevent the influence of the one pendulum being communicated to the other. Daniel Bernoulli appears to have been the first who took notice of this phenomenon, at least with any reference to theory; but the case which attracted his attention was far more simple than that to which we have alluded. It was that of the motion of the two scales of a balance, when one has had an oscillatory motion com-

* *Cambridge Mathematical Journal*, Vol. II., p. 120. This article was written conjointly by Mr. Archibald Smith and Mr. Gregory, the physical conceptions being due principally to Mr. Smith and the analysis chiefly to Mr. Gregory.

municated to it. The following is his narration of the phenomenon as he observed it. (*Nova Commen. Petrop.*, Vol. XIX., p. 281).

"Cum aliquando in libra, majori eaque subpigra, alteram "lancem forte fortuna ad latus diducerem, moxque rursus "dimitterem, accidit utique ut protinus hinc inde oscillaret "nec ab initio lanx opposita de loco moveretur: mox autem "et hæc quoque agitari sensimque majores oscillationes for-"mare, dum e contrario lanx prior motum suum oscillatorium "gradatim perderet tandemque fere quiesceret; hoc ipso "momento altera maximum motionis gradum, initiali lancis "sociæ fere æqualem, attingebat: tunc ordine contrario "eædem mutationes repetebantur, usque dum prima lanx "motum suum primitivum integrum resumeret sociaque "quieti ad momentum redderetur; hæc autem oscillationum "communicatio ac reciprocatio diu satis sese manifestabat."

Bernoulli does not seem to have attempted a direct solution of the dynamical problem which this experiment suggested, but contents himself with adducing it as an instance in support of his principle of the coexistence of small oscillations. In the same volume of the *Petersburgh Memoirs*, however, Euler in two papers considers the question theoretically. It is clear that the experiment of Bernoulli admits of being performed in two different ways: the original displacement of the scale may either be in the vertical plane passing through the beam of the balance, or it may be in any other plane. Euler only considers the first case, though another one—that when the original displacement is perpendicular to the vertical plane passing through the beam—is also interesting, and admits of as easy a solution. In his first memoir, Euler supposes that the centre of suspension of the balance is in the same line as the centres of suspension of the two scales; and on investigating the result on this supposition, he finds that the interchange of motion described by Bernoulli could not take place, though the motion of the scale originally put in motion would be different from that which it would have if suspended from a fixed point of support. This result is confirmed by the simple consideration, that when the centre of suspension of the beam is in the line joining the centres of suspension of the scales, these will only receive vertical motions from the motion of the beam; and consequently, if we suppose the second scale to be originally at rest, it will have no horizontal motion communicated to it, so as to cause it to oscillate in a horizontal direction.

In the other memoir Euler considers the case of the centre of suspension being above the line joining the points of suspension, as indeed would be the case in an ordinary balance which is usually suspended by some higher point. This investigation we shall here give, adhering pretty closely to the process adopted by Euler, as it is always both interesting and instructive to see the mode in which the first writers attacked such a problem as this.

Let O (fig. 8) be the point of suspension of the whole balance, G its centre of gravity, AB the beam, P and Q the scales, which are here supposed to be material points. Draw aOb horizontal, $A\alpha$, $B\beta$ vertical. Let $AC = CB = a$, $OC = b$, $OG = c$, $AP = BQ = l$, Mk^2 = moment of inertia of the beam, m the mass of P and of Q supposed to be equal.

Let ϕ be the angle which, at the time t, the beam makes with the horizon.

Let $$PA\alpha = \eta, \quad QB\beta = \theta,$$

$$Op = x, \quad Pp = y, \quad Oq = x', \quad Qq = y'.$$

Then
$$\begin{aligned} x &= a\cos\phi + b\sin\phi + l\sin\eta, \\ y &= b\cos\phi - a\sin\phi + l\cos\eta, \\ x' &= a\cos\phi - b\sin\phi - l\sin\theta, \\ y' &= a\sin\phi + b\cos\phi + l\cos\theta. \end{aligned}$$

Let P, Q, be the tensions of AP and BQ. The equations of their motions are

$$\frac{d^2x}{dt^2} = -\frac{Pg \sin\eta}{m} \text{ .... (1)},$$

$$\frac{d^2y}{dt^2} = g - \frac{Pg \cos\eta}{m} \text{ .... (2)},$$

$$\frac{d^2x'}{dt^2} = \frac{Qg \sin\theta}{m} \text{ .... (3)},$$

$$\frac{d^2y'}{dt^2} = g - \frac{Qg \cos\theta}{m} \text{ ....(4)}.$$

For the motion of the beam we have its own weight at G tending to turn it back to its original position, and the tensions of the strings acting in different directions. The moment of the couple arising from the weight of the beam is

$$Mgc \cos GOA = Mgc \sin\phi.$$

The moment of the couple arising from the tension P is

$$Pg\, Om = Pg\,\{a \cos(\eta - \phi) - b \sin(\eta - \phi)\}.$$

Both these are negative, as tending to bring back the beam to its original position. The moment of the couple arising from Q is

$$Qg\,\{a \cos(\phi - \theta) - b \sin(\phi - \theta)\}.$$

Consequently we have, for the motion of the beam, the equation

$$\frac{d^2\phi}{dt^2} = -\frac{g}{Mk^2}\left[\begin{array}{r} Mc \sin\phi + P\{a \cos(\eta - \phi) - b \sin(\eta - \phi)\} \\ - Q\{a \cos(\phi - \theta) - b \sin(\phi - \theta)\} \end{array}\right] \text{ ....(5)}.$$

These five simultaneous equations, if solved, would serve to determine all the circumstances of the motion, but under their present form the solution is impracticable. To render it possible, we must suppose the displacements to be very small, so that we may put the arc for its sine and unity for the cosine; by this means we find

$$x = a + b\phi + l\eta, \qquad y = b - a\phi + l,$$
$$x' = a - b\phi - l\theta, \qquad y' = a\phi + b + l.$$

Also, the tensions of the strings may be supposed not to be changed, but to remain equal to the weight. Making these substitutions in the five equations of motion, the second and fourth disappear, and there remain

$$b\,\frac{d^2\phi}{dt^2} + l\,\frac{d^2\eta}{dt^2} = -g\eta,$$

$$b\,\frac{d^2\phi}{dt^2} + l\,\frac{d^2\theta}{dt^2} = -g\theta,$$

$$Mk^2\,\frac{d^2\phi}{dt^2} = -g\,\{(Mc + 2mb)\,\phi - mb\,(\eta + \theta)\},$$

and by means of these three simultaneous equations we can easily determine ϕ, η, θ.

Let $\frac{g}{l} = n^2$, $\frac{b}{l} = h$, $\frac{Mc + 2mb}{Mk^2}\,g = p^2$, $-\frac{mbg}{Mk^2} = q$.

Then the equations may be put under the form

$$h\,\frac{d^2\phi}{dt^2} + \left(\frac{d^2}{dt^2} + n^2\right)\eta = 0 \text{ ....(1).}$$

$$h\,\frac{d^2\phi}{dt^2} + \left(\frac{d^2}{dt^2} + n^2\right)\theta = 0 \text{ ....(2).}$$

$$\left(\frac{d^2}{dt^2} + p^2\right)\phi + q\,(\eta + \theta) = 0 \text{ ....(3).}$$

*Add together (1) and (2), which gives

$$2h\,\frac{d^2\phi}{dt^2} + \left(\frac{d^2}{dt^2} + n^2\right)(\eta + \theta) = 0 \text{ .... (4).}$$

Subtract (2) from (1), which gives

$$\left(\frac{d^2}{dt^2} + n^2\right)(\eta - \theta) = 0 \text{ ....(5).}$$

Operate on (3) with $\left(\frac{d^2}{dt^2} + n^2\right)$, multiply (4) by q, and subtract it from the former. Then

$$\left(\frac{d^2}{dt^2} + n^2\right)\left(\frac{d^2}{dt^2} + p^2\right)\phi - 2hq\,\frac{d^2\phi}{dt^2} = 0.$$

* For this method of integrating simultaneous differential equations, see *Cambridge Mathematical Journal*, Vol. I., p. 173; (p. 95 of this volume).

If $-\mu_1^2$, $-\mu_2^2$, be the roots of

$$z^2+(n^2+p^2-2hq)z+n^2p^2=0,$$

this may be put under the form

$$\left(\frac{d^2}{dt^2}+\mu_1^2\right)\left(\frac{d^2}{dt^2}+\mu_2^2\right)\phi=0 \;\ldots\ldots\ldots\ldots(6).$$

The integral of which is

$$\phi=C_1\cos(\mu_1t+\alpha_1)+C_2\cos(\mu_2t+\alpha_2)\;\ldots\ldots\;(A).$$

Substituting this value of ϕ in (3), we find

$$\eta+\theta=\frac{C_1}{q(\mu_1^2-p^2)}\cos(\mu_1t+\alpha_1)+\frac{C_2}{q(\mu_2^2-p^2)}\cos(\mu_2t+\alpha_2).$$

Integrating (5), we have

$$\eta-\theta=C\cos(nt+\alpha);$$

whence

$$2\eta=\frac{C_1}{q(\mu_1^2-p^2)}\cos(\mu_1t+\alpha_1)+\frac{C_2}{q(\mu_2^2-p^2)}\cos(\mu_2t+\alpha_2)$$
$$+C\cos(nt+\alpha)\ldots\ldots(B),$$

$$2\theta=\frac{C_1}{q(\mu_1^2-p^2)}\cos(\mu_1t+\alpha_1)+\frac{C_2}{q(\mu_2^2-p^2)}\cos(\mu_2t+\alpha_2)$$
$$-C\cos(nt+\alpha)\ldots\ldots(C).$$

The equations (A), (B), (C), are the complete solution of the problem, and involve six arbitrary constants, viz. C, C_1, C_2, α, α_1, α_2, which may be determined so as to suit the original circumstances, and according to the nature of these the solution will assume different shapes. To suit the experiment of Bernoulli, we must suppose, when $t=0$, that

$$\phi=0,\qquad \eta=\varepsilon,\qquad \theta=0,$$
$$\frac{d\phi}{dt}=0,\quad \frac{d\eta}{dt}=0,\quad \frac{d\theta}{dt}=0.$$

The last three conditions give us $\alpha=\alpha_1=\alpha_2=0$; and these values substituted in the others reduce them to

$$0=C_1+C_2,\quad \varepsilon=C,$$
$$q\varepsilon=\frac{C_1}{\mu_1^2-p^2}+\frac{C_2}{\mu_2^2-p^2};$$

whence we find

$$C_1 = q\varepsilon \frac{(\mu_1^2 - p^2)(\mu_2^2 - p^2)}{\mu_2^2 - \mu_1^2}, \quad C_2 = q\varepsilon \frac{(\mu_1^2 - p^2)(\mu_2^2 - p^2)}{\mu_1^2 - \mu_2^2}.$$

Substituting these values the equations become

$$\phi = q \frac{\varepsilon (\mu_1^2 - p^2)(\mu_2^2 - p^2)}{\mu_2^2 - \mu_1^2} (\cos \mu_1 t - \cos \mu_2 t),$$

$$2\eta = \frac{\varepsilon}{\mu_2^2 - \mu_1^2} \{(\mu_2^2 - p^2) \cos \mu_1 t - (\mu_1^2 - p^2) \cos \mu_2 t\} + \varepsilon \cos nt,$$

$$2\theta = \frac{\varepsilon}{\mu_2^2 - \mu_1^2} \{(\mu_2^2 - p^2) \cos \mu_1 t - (\mu_1^2 - p^2) \cos \mu_2 t\} - \varepsilon \cos nt.$$

Observing that $\mu_1^2 + \mu_2^2 = n^2 + p^2 - 2hq$, and $\mu_1^2 \mu_2^2 = n^2 p^2$, the values of C_1 and C_2 are reduced to

$$C = \frac{2\varepsilon h p^2 q^2}{\mu_2^2 - \mu_1^2}, \quad C_2 = \frac{2\varepsilon h p^2 q^2}{\mu_1^2 - \mu_2^2};$$

so that we find

$$\phi = \frac{2\varepsilon h p^2 q^2}{\mu_2^2 - \mu_1^2} (\cos \mu_1 t - \cos \mu_2 t),$$

$$2\eta = \frac{2\varepsilon h p^2 q}{\mu_2^2 - \mu_1^2} \left(\frac{\cos \mu_1 t}{\mu_1^2 - p^2} - \frac{\cos \mu_2 t}{\mu_2^2 - p^2} \right) + \varepsilon \cos nt,$$

$$2\theta = \frac{2\varepsilon h p^2 q}{\mu_2^2 - \mu_1^2} \left(\frac{\cos \mu_1 t}{\mu_1^2 - p^2} - \frac{\cos \mu_2 t}{\mu_2^2 - p^2} \right) - \varepsilon \cos nt.$$

It appears from these expressions, that the motion of the beam is compounded of two oscillations of different periods, $\frac{2\pi}{\mu_1}$ and $\frac{2\pi}{\mu_2}$. The relations which these oscillations bear to those which the beam and the scales would separately make, may be easily shown. By means of equations (3) and (4), we see that μ_1^2 and μ_2^2 must satisfy the equation

$$(\mu^2 - n^2)(\mu^2 - p^2) + 2hq\mu^2 = 0.$$

If the mass of the beam be very large, we may suppose p^2 to be less than n^2, and the equation for μ^2 may be put under the forms

$$\mu_1^2 = n^2 - \frac{2hq\mu_1^2}{\mu_1^2 - p^2}, \quad \text{and} \quad \mu_2^2 = p^2 + \frac{2hq\mu_2^2}{n^2 - \mu_2^2},$$

which show that μ_1^2 is less than n^2, and μ_2^2 greater than p^2.

Hence it appears that the period of the one part of the oscillation of the beam is greater than the period of the natural oscillation of the scales, while the period of the other oscillation is less than that of the beam and scales considered as one mass. The motion of the scales consists of these same oscillations, with the addition of one, the period of which is that of a pendulum of the same length as the suspending string.

If we suppose the vibrations of the scales to take place in a plane perpendicular to a vertical plane passing through the beam, the expressions become somewhat simpler. We shall not, in this case, go so much into detail as in the last, but shall at once suppose the displacements to be so small, that the forces of restitution may be considered as proportional to them.

Let AB (fig. 9) be the original position of the beam, PQ its position at the time t; p, q the projections of the positions of scales considered as material points at the same time.

Let $AC=BC=a$, $AP=BQ=z$, $Pp=x$, $Qq=y$, Mk^2 be the moment of inertia of the beam round C, m the mass of each weight, l the length of the string by which each weight is suspended. Then the equations of motion will be

$$\frac{d^2(x+z)}{dt^2}+\frac{g}{l}x=0 \quad\dots\dots\dots\dots\dots\dots (1),$$

$$\frac{d^2(y+z)}{dt^2}+\frac{g}{l}y=0 \quad\dots\dots\dots\dots\dots\dots (2),$$

$$\frac{d^2z}{dt^2}-\frac{g}{l}\frac{ma^2}{Mk^2}(x+y)=0 \quad\dots\dots\dots\dots\dots\dots (3).$$

Subtracting (2) from (1), we have

$$\frac{d^2(x-y)}{dt^2}+\frac{g}{l}(x-y)=0.$$

Add (1) and (2), and subtract (3) multiplied by 2; then

$$\frac{d^2(x+y)}{dt^2}+\frac{g}{l}\left(1+2\frac{ma^2}{Mk^2}\right)(x+y)=0.$$

Let $\frac{g}{l} = n^2$, $\frac{g}{l}\left(1 + 2\,\frac{ma^2}{Mk^2}\right) = n'^2$; then these equations become

$$\frac{d^2(x-y)}{dt^2} + n^2(x-y) = 0,$$

$$\frac{d^2(x+y)}{dt^2} + n'^2(x+y) = 0.$$

Integrating, we have

$$x - y = C\ \cos(nt + \alpha),$$

$$x + y = C_1\ \cos(n't + \alpha_1).$$

If we suppose that at the beginning of the motion

$$x = c, \quad y = 0, \quad \frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0,$$

these equations become

$$x - y = C\ \cos nt,$$

$$x + y = C\ \cos n't;$$

whence $$x = \quad C \cos\frac{n'-n}{2}\,t \cos\frac{n'+n}{2}\,t,$$

$$x = -C \sin\frac{n'-n}{2}\,t \sin\frac{n'+n}{2}\,t.$$

Substituting the value of $x+y$ which has been found in (3), and integrating, we find

$$z = -\frac{g}{l}\,\frac{ma^2}{Mk^2}\,\frac{c}{n'^2}\cos n't + At + B.$$

If at the beginning of the motion we suppose $z = 0$, $\frac{dz}{dt} = 0$, we shall find $A = 0$, $B = \frac{g}{l}\,\frac{ma^2}{Mk^2}\,\frac{c}{n'^2}$, so that

$$z = \frac{g}{l}\,\frac{ma^2}{Mk^2}\,\frac{c}{n'^2}(1 - \cos n't),$$

or $$z = \frac{2g}{l}\,\frac{ma^2}{Mk^2}\,\frac{c}{n'^2}\sin^2\frac{n't}{2}.$$

If we suppose the beam to have an original angular velocity given to it, then A will not be 0, and the expression for z will no longer be simply periodic, but will increase continually with the time. This will also appear from the consideration, that there is no force independent of the oscillations of the scale acting on the beam, so that any originally impressed velocity will not be destroyed, but will continue to carry the beam round with a motion subject to periodic inequalities.

If we suppose the mass of the beam to be very great in comparison with the masses of the weights, so that $\frac{ma^2}{Mk^2}$ is very small, n is very nearly equal to n', $\frac{n'-n}{2}$ is very small, and $\sin\frac{n'-n}{2}t$ and $\cos\frac{n'-n}{2}t$ vary very slowly.

So that we may represent our result as that of two pendulums, whose arcs of vibration are respectively

$$C\cos\frac{n'-n}{2}t \text{ and } C\sin\frac{n'-n}{2}t.$$

These are complementary; they show that the arc of vibration of the first pendulum will gradually diminish, and that of the second increase, till after a time $=\frac{2}{n'-n}\cdot\frac{\pi}{2}$ they have interchanged motions and the converse process is repeated, and the system returns to its original state after a time $=\frac{2\pi}{n'-n}$.

The common time of oscillation is that of a pendulum whose length is

$$g\cdot\frac{4}{(n+n')^2}=l\left(1-\frac{ma^2}{Mk^2}\right) \text{ nearly.}$$

The most general case of the influence of one pendulum on another, when the motions as we have supposed are all in the same horizontal direction and infinitesimal, will be when, calling A and B the points of support, each of these when disturbed performs vibrations in known times: and a disturbance given to A communicates a known motion to B, and *vice versâ*.

To investigate the motion in this case, let u, v be the coordinates of A and B, $u+x$, $v+y$ of the balls suspended to them; then we have the equations

$$\frac{d^2(u+x)}{dt^2}+m^2x=0,$$

$$\frac{d^2(v+y)}{dt^2}+n^2y=0,$$

$$\frac{d^2u}{dt^2}+p^2u-ax-fv=0,$$

$$\frac{d^2v}{dt^2}+q^2v-by-gu=0.$$

A solution of these equations is

$$x=RM\cos\{\sqrt{(\rho)}\,t-r\},\quad u=RP\cos\{\sqrt{(\rho)}\,t-r\},$$
$$y=RN\cos\{\sqrt{(\rho)}\,t-r\},\quad v=RQ\cos\{\sqrt{(\rho)}\,t-r\};$$

and to determine ρ we get the equation

$$\{(\rho-m^2)(\rho-p^2)-a\rho\}\{(\rho-n^2)(\rho-q^2)-b\rho\}$$
$$-fg(\rho-m^2)(\rho-n^2)=0.$$

The four values of ρ determined from this equation are all positive, and therefore the angular functions real. The complete solution is the sum of the particular solutions, therefore

$$x=R_1M_1\cos(\sqrt{\rho_1}t-r_1)+R_2M_2\cos(\sqrt{\rho_2}t-r_2)$$
$$+R_3M_3\cos(\sqrt{\rho_3}t-r_3)+R_4M_4\cos(\sqrt{\rho_4}t-r_4),$$

$$y=R_1N_1\cos(\sqrt{\rho_1}t-r_1)+R_2N_2\cos(\sqrt{\rho_2}t-r_2)$$
$$+R_3N_3\cos(\sqrt{\rho_3}t-r_3)+R_4N_4\cos(\sqrt{\rho_4}t-r_4).$$

If three of the quantities R are equal to zero, the vibrations of the two pendulums are isochronous, and there are therefore four modes of this isochronous vibration. In all cases the pendulums affect each other, so that none of the points oscillates in its natural time. The extreme generality of the equations we have assumed, and consequently of the solution derived from them, prevents us from interpreting our result in a more precise manner. For this purpose it would be necessary to assign some relations between the constants in the equations, but it would lead us too far if we were to attempt any such investigation; and we may add, that any particular case will in general be more easily solved by a direct reference to its own circumstances than by a reduction of the general solution.

ON THE EXPANSION OF COSINES AND SINES OF MULTIPLE ARCS IN ASCENDING POWERS OF THE COSINES AND SINES OF THE SIMPLE ARCS.*

THE method adopted by Lagrange for expressing the cosine and sine of multiple arcs in terms of the powers of the cosines and sines of the simple arcs, depended on the expansion of functions of the form

$$\{x+\sqrt{(x^2-1)}\}^n \text{ and } \{x-\sqrt{(x^2-1)}\}^n.$$

The same method is pursued by Poinsot in his *Recherches sur l'Analyse des Sections Angulaires*, where the complete theory of these circular functions was first given; but he has also indicated another way, which is far less tedious and complicated—that of assuming the form of the series, and determining the coefficients by differentiation. As this may be useful to those who are studying the subject, we shall here briefly fill up the outline which Poinsot has sketched.

1. To expand $\cos n\theta$ in terms of $\cos\theta$ and its powers.

Assume

$$\cos n\theta = a_0 + a_1 \cos\theta + \ldots + a_p (\cos\theta)^p + \ldots + a_{p+2} (\cos\theta)^{p+2} + \ldots.$$

* *Cambridge Mathematical Journal*, Vol. II., p. 129.

Differentiating,

$$n \sin n\theta = \{a_1 + 2a_2 \cos\theta + \ldots + pa_p (\cos\theta)^{p-1} + \ldots$$
$$+ (p+2)\, a_{p+2} (\cos\theta)^{p+1} + \ldots\} \sin\theta.$$

Differentiating again,

$$n^2 \cos n\theta = a_1 \cos\theta + \ldots + pa_p (\cos\theta)^p + \ldots + (p+2)\, a_{p+2} (\cos\theta)^{p+2} + \ldots$$
$$- \{2a_2 + \ldots + p\,(p-1)\, a_p (\cos\theta)^{p-2} + \ldots$$
$$+ (p+1)(p+2)\, a_{p+2} (\cos\theta)^p \ldots\} \sin^2\theta.$$

Putting $1 - \cos^2\theta$ for $\sin^2\theta$, and taking the coefficient of $(\cos\theta)^p$, we find it to be

$$pa_p + p\,(p-1)\, a_p - (p+1)(p+2)\, a_{p+2},$$

and this must be equal to the coefficient of $(\cos\theta)^p$ in the first equation multiplied by n^2. Therefore, we have

$$n^2 a_p = pa_p + p\,(p-1)\, a_p - (p+1)(p+2)\, a_{p+2};$$

whence $$a_{p+2} = -\frac{(n^2 - p^2)}{(p+1)(p+2)}\, a_p.$$

By this means any coefficient is found in terms of that two places below it. Consequently the first and second coefficients are left to be determined by other means. For this purpose let $\theta = (2r+1)\dfrac{\pi}{2}$ in the first equation, r being any integer. Every term on the second side vanishes except the first, and we find

$$a_0 = \cos n\,(2r+1)\,\frac{\pi}{2}.$$

To find a_1, make $\theta = (2r+1)\dfrac{\pi}{2}$ in the second equation, when we obtain

$$a_1 = n\,\frac{\sin n\,(2r+1)\,\dfrac{\pi}{2}}{\sin (2r+1)\,\dfrac{\pi}{2}} = n \cos (n-1)(2r+1)\,\frac{\pi}{2}.$$

Starting from these values, and giving p successively all the integer values from 0 upwards, and separating the terms involving odd powers of $\cos\theta$ from those involving even powers, we find

$$\cos n\theta = \cos n\,(2r+1)\frac{\pi}{2}\left\{1 - \frac{n^2}{1.2}(\cos\theta)^2 + \frac{n^2(n^2-2^2)}{1.2.3.4}(\cos\theta)^4 - \&c.\right\}$$

$$+ n\cos(n-1)(2r+1)\frac{\pi}{2}\left\{\cos\theta - \frac{n^2-1^2}{1.2.3}(\cos\theta)^3 + \&c.\right\}.$$

When n is an even integer, the second line being multiplied by the cosine of an odd multiple of $\frac{\pi}{2}$ vanishes, and the first line only remains; when n is an odd integer, the first line vanishes, and the second line only remains. When n is a fraction, both lines must be retained, except for particular values of r, which cause the factor of one or other series to vanish.

2. If we assume

$$\sin n\theta = a_0 + a_1 \sin\theta + a_2(\sin\theta)^2 + \&c. + a_p(\sin\theta)^p + \&c.,$$

we shall obtain, by the same means as in the previous case, the same equation for determining a_{p+2}, viz.

$$a_{p+2} = -\frac{(n^2-p^2)}{(p+1)(p+2)}\,a_p.$$

To determine the first two coefficients, make $\theta = r\pi$ in the above equation and its differential. We thus obtain

$$a_0 = \sin nr\pi,$$

$$a_1 = n\,\frac{\cos nr\pi}{\cos r\pi} = n\cos(n-1)\,r\pi;$$

so that, dividing the series into two parts containing the odd and the even powers, we have

$$\sin n\theta = \sin nr\pi\left\{1 - \frac{n^2}{1.2}(\sin\theta)^2 + \frac{n^2(n^2-2^2)}{1.2.3.4}(\sin\theta)^4 - \&c.\right\}$$

$$+ n\cos(n-1)\,r\pi\left\{\sin\theta - \frac{n^2-1^2}{1.2.3}(\sin\theta)^3 + \&c.\right\}.$$

When n is an integer the first series always vanishes, and the second is positive or negative according as $(n-1)r$ is even or odd. When n is odd the second series terminates; when n is even it goes on to infinity. When n is a fraction, both series are to be retained.

3. If we assume

$$\cos n\theta = a_0 + a_1 \sin\theta + a_2 (\sin\theta)^2 + \&c. + a_p (\sin\theta)^p - \&c.,$$

we obtain as before for determining a_{p+2}, the equation

$$a_{p+2} = -\frac{(n^2-p^2)}{(p+1)(p+2)} a_p.$$

To determine the first two coefficients, make $\theta = r\pi$ in the equation and its differential. Then

$$a_0 = \cos nr\pi,$$

$$a_1 = -\frac{n \sin nr\pi}{\cos r\pi} = -n \sin(n-1) r\pi;$$

so that

$$\cos n\theta = \cos nr\pi \left\{1 - \frac{n^2}{1.2}(\sin\theta)^2 + \frac{n^2(n^2-2^2)}{1.2.3.4}(\sin\theta)^4 - \&c.\right\}$$

$$- n \sin(n-1) r\pi \left\{\sin\theta - \frac{n^2-1^2}{1.2.3}(\sin\theta)^3 + \&c.\right\}.$$

When n is an integer the second series always disappears, and the first series terminates when n is even, and does not terminate when n is odd. When n is a fraction, both series are retained.

4. If we assume

$$\sin n\theta = a_0 + a_1 \cos\theta + \&c. + a_p (\cos\theta)^p + \&c.,$$

we find as before as the condition for determining the coefficients,

$$a_{p+2} = -\frac{(n^2-p^2)}{(p+1)(p+2)} a_p.$$

Make $\theta = (2r+1)\frac{\pi}{2}$ in the equation and its differential. Then

$$a_0 = \sin n\,(2r+1)\frac{\pi}{2}.$$

$$a_1 = -n\,\frac{\cos n\,(2r+1)\frac{\pi}{2}}{\sin(2r+1)\frac{\pi}{2}} = n\,\sin(n-1)\,(2r+1)\frac{\pi}{2};$$

whence we find

$$\sin n\theta = \sin n\,(2r+1)\frac{\pi}{2}\left\{1 - \frac{n^2}{1.2}(\cos\theta)^2 + n^2\frac{(n^2-2^2)}{1.2.3.4}(\cos\theta)^2 + \&c.\right\}$$

$$+\,n\,\sin(n-1)\,(2r+1)\frac{\pi}{2}\left\{\cos\theta - \frac{n^2-1^2}{1.2.3}(\cos\theta)^3 + \&c.\right\}.$$

When n is an odd integer the first line, when n is even the second line only remains; but when n is fractional both series must be retained. In no case do the series ever terminate.

These four results may be put under a convenient mnemonic form if we make use of a particular notation, by which the laws of these series are assimilated to those of sines and cosines.

Let P_n be such an operation performed on n that

$$P_n^{\,0} = 1, \qquad\qquad P_n^{\,1} = n,$$
$$P_n^{\,2} = (n-0)\,(n+0) = n^2, \qquad P_n^{\,3} = n\,(n-1)(n+1) = n\,(n^2-1^2),$$
$$P_n^{\,4} = n^2\,(n-2)(n+2) = n^2\,(n^2-2^2), \qquad P_n^{\,5} = n\,(n^2-1^2)\,(n^2-3^2),$$
$$\&c. \qquad\qquad \&c.$$

from which the law of formation is evident. Then the series of (1) may be put under the form

$$\cos n\theta = \cos n\,(2r+1)\frac{\pi}{2}\left\{1 - \frac{P_n^{\,2}}{1.2}(\cos\theta)^2 + \frac{P_n^{\,4}}{1.2.3.4}(\cos\theta)^4 - \&c.\right\}$$

$$+\cos(n-1)\,(2r+1)\frac{\pi}{2}\left\{P_n^{\,1}\cos\theta - \frac{P_n^{\,3}}{1.2.3}(\cos\theta)^3 + \&c.\right\};$$

in which it will be seen that the series in the first line follows the law of the cosine of $(P_n \cos\theta)$, and the series in the second line that of the sine of $(P_n \cos\theta)$, so that the expression may be put under the form

$$\cos n\theta = \cos n\,(2r+1)\,\frac{\pi}{2}\,\cos(P_n \cos\theta) + \cos(n-1)\,(2r+1)\,\frac{\pi}{2}\,\sin(P_n \cos\theta)$$

$$= \cos n\,(2r+1)\,\frac{\pi}{2}\,\cos(P_n \cos\theta) \pm \sin n\,(2r+1)\,\frac{\pi}{2}\,\sin(P_n \cos\theta);$$

whence $$\cos n\theta = \cos\left\{n\,(2r+1)\,\frac{\pi}{2} \mp (P_n \cos\theta)\right\} \quad \text{.......(I)}.$$

By using the same notation with respect to (2), we have

$$\sin n\theta = \sin nr\pi \left\{1 - \frac{P_n^{\,2}}{1.2}(\sin\theta)^2 + \frac{P_n^{\,4}}{1.2.3.4}(\sin\theta)^4 - \&c.\right\}$$

$$+ \cos(n-1)\,r\pi \left\{P_n \sin\theta - \frac{P_n^{\,3}}{1.2.3}(\sin\theta)^3 + \&c.\right\}$$

$$= \sin nr\pi\,\cos(P_n \sin\theta) \pm \cos nr\pi\,\sin(P_n \sin\theta);$$

and therefore $$\sin n\theta = \sin\{nr\pi \pm (P_n \sin\theta)\} \quad \text{...........(II)}.$$

Proceeding in the same way, we shall find

$$\cos n\theta = \cos nr\pi\,\cos(P_n \sin\theta) \mp \sin nr\pi\,\sin(P_n \sin\theta),$$

or $$\cos n\theta = \cos\{nr\pi \pm (P_n \sin\theta)\} \quad \text{............ (III)}.$$

Similarly $$\sin n\theta = \sin\left\{n\,(2r+1)\,\frac{\pi}{2} \mp (P_n \cos\theta)\right\} \text{...(IV)}.$$

ON THE THEORY OF MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES.*

ALTHOUGH it is usual to illustrate the Theory of Maxima and Minima of Functions of one Variable by a reference to the properties of curve lines, I do not remember that any similar illustration has been used in treating of Maxima and Minima of Functions of two Variables. As far as the first condition is concerned, the illustrations will be the same in both cases, but the second or Lagrange's condition in functions of two variables has nothing analogous in the case of two variables. The geometrical explanation gives a very distinct idea of its meaning, and on that account I think that a notice of it may be useful to the student.

Let us briefly consider how the condition arises analytically, and afterwards proceed to the geometrical interpretation of the various steps.

If $z=f(x, y)$ be a maximum or minimum, and z_1 be the value of z when $x+h$, $y+k$ are substituted for x and y, z_2 the value when $x-h$, and $y-k$ are substituted; then for a maximum we must have

$$z_1 < z \text{ and } z_2 < z,$$

or $$z_1 - z < 0, \quad z_2 - z < 0;$$

and for a minimum,

$$z_1 - z > 0, \quad z_2 - z > 0;$$

* *Cambridge Mathematical Journal*, Vol. II., p. 138.

Now

$$z_1 - z = \frac{dz}{dx} h + \frac{dz}{dy} k + \frac{1}{1.2} \left(\frac{d^2z}{dx^2} h^2 + 2 \frac{d^2z}{dx\,dy} hk + \frac{d^2z}{dy^2} k^2 \right) + \&c.,$$

and

$$z_2 - z = - \left(\frac{dz}{dx} h + \frac{dz}{dy} k \right) + \frac{1}{1.2} \left(\frac{d^2z}{dx^2} h^2 + 2 \frac{d^2z}{dx\,dy} hk + \frac{d^2z}{dy^2} k^2 \right) - \&c.$$

Now either for a maximum or minimum it appears that $z_1 - z$ and $z_2 - z$ must be of the same sign, and as h and k can be assumed so small that the sign of the whole series after them depends on the sign of the terms involving their first powers, and as these are necessarily of opposite signs in the two series, $z_1 - z$ and $z_2 - z$ cannot be of the same sign, unless the terms involving the first powers of h and k vanish, or

$$\frac{dz}{dx} h + \frac{dz}{dy} k = 0,$$

which, as h and k are independent, involves the two conditions

$$\frac{dz}{dx} = 0, \quad \frac{dz}{dy} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

But $z_1 - z$ and $z_2 - z$ must both of them remain of the same sign, whatever value we assign to h and k, and therefore the first remaining term of the series must remain constantly of the same sign, whatever values we assign to h and k. That is to say, the expression

$$\frac{d^2z}{dx^2} h^2 + 2 \frac{d^2z}{dx\,dy} hk + \frac{d^2z}{dy^2} k^2,$$

must not pass from + to –, or conversely, from changes in the magnitude or sign of h and k. Let $k = mh$, then the expression becomes

$$h^2 \left(\frac{d^2z}{dx^2} + 2 \frac{d^2z}{dx\,dy} m + \frac{d^2z}{dy^2} m^2 \right);$$

and, as h^2 is essentially positive, the sign of

$$\frac{d^2z}{dy^2} m^2 + 2 \frac{d^2z}{dx\,dy} m + \frac{d^2z}{dx^2},$$

must not change. Now this expression can only change sign by passing through 0; and, in order that this may never happen from any change in m, the value of m derived from that expression equated to 0 must be impossible.

The solution of the equation gives, if we put

$$\frac{d^2z}{dx^2} = r, \quad \frac{d^2z}{dx\,dy} = s, \quad \frac{d^2z}{dy^2} = t,$$

$$m = -\frac{s}{t} \pm \frac{\sqrt{(s^2 - rt)}}{t};$$

and, in order that this may be impossible, we must have

$$s^2 < rt,$$

or

$$\left(\frac{d^2z}{dx\,dy}\right)^2 - \left(\frac{d^2z}{dx^2}\right)\left(\frac{d^2z}{dy^2}\right) = 0 \;\ldots\ldots\ldots\ldots (B).$$

This is Lagrange's condition.

Let us now consider the geometrical interpretation of the various steps.

$$z = f(x, y)$$

represents the equation to a surface, and the conditions (A) imply that the tangent plane must be parallel to the plane of xy, since the equation to the tangent plane becomes, under those conditions,

$$z - z_1 = 0.$$

The assumption of $k = mh$ establishing a relation between the increments of x and y, independent of z, corresponds to taking a section of the surface by a plane perpendicular to xy, its trace being inclined to the axis of x at an angle whose tangent is m. For the interpretation of the condition (B), we must have recourse to the expression for the radius of curvature of a normal section.

If ρ be the radius of curvature, we have, under the condition that $\frac{dz}{dx} = 0$, $\frac{dz}{dy} = 0$,

$$\rho = \frac{1 + m^2}{r + 2sm + tm^2}.$$

Now the sign of this depends only on that of the denominator, since the numerator equated to zero gives only impossible values for m, but the denominator is exactly the quantity, the sign of which was considered before. Therefore the condition (B), considered geometrically, implies that the sign of the radius of curvature of a normal section shall not change, that is, that every section at the point under consideration must be either convex or concave, but must never pass from one species of curvature to the other. The reason of this is obvious, as the ordinate which is a maximum for a concave section is a minimum for a convex one, and *vice versâ.* It might perhaps be advantageous to have a name appropriated to those points of surfaces for which the conditions (A) hold, but not the condition (B). Such points, though they do not possess the property of being absolute maxima or minima, are yet for many purposes quite as much worthy of attention, since we have frequently to consider geometrically only the fact that the ordinate is stationary for a short space, or that the tangent plane is then perpendicular to the ordinate. Thus, for instance, in investigating the properties of principal diameters in surfaces of the second order, we have only to consider the first condition, and do not require to pay attention to the second. Perhaps the name of "Stationary Points" would be sufficiently distinctive.

ON THE MOTION OF A PENDULUM WHEN ITS POINT OF SUSPENSION IS DISTURBED.*

IN a former article† we investigated the nature of the mutual action of two pendulums united by any elastic or moveable connexion; we shall here consider more particularly the effect produced on the motion of a simple pendulum by a disturbance of its point of support. As before, we shall suppose the motions to be infinitesimal, in order that the equations may be at all manageable, and also for the sake of simplicity we shall assume the disturbances of the point of suspension to be rectilinear.

I. Let the point of suspension have a horizontal motion parallel to that of the pendulum.

The pendulum being a simple one, let u be the horizontal coordinate of the point of suspension, $u+x$ of the ball; let l be the length of the pendulum, and $n^2=\frac{g}{l}$. Then the equation for the motion of the pendulum is

$$\frac{d^2(u+x)}{dt^2}+n^2x=0 \qquad\qquad (1).$$

Two suppositions may be made regarding the nature of the disturbances of the point of suspension, that is, regarding

* *Cambridge Mathematical Journal*, Vol. II., p. 204. This article was written conjointly by Mr. Archibald Smith and Mr. Gregory.

† See page 175 of this volume.

the motion of u: either it has a vibratory motion independent of the motion of the pendulum or depending on it.

In the first case, when the motion of u is independent of that of the pendulum, u is given simply in terms of t, or

$$u = c\cos(\alpha t + \beta) \quad\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

whence equation (1) becomes

$$\frac{d^2x}{dt^2} + n^2x - c\alpha^2\cos(\alpha t + \beta) = 0 \quad\ldots\ldots\ldots\ldots(3).$$

The solution of this is

$$x = A\cos(nt + B) - \frac{c\alpha^2}{\alpha^2 - n^2}\cos(\alpha t + \beta) \quad\ldots\ldots(4).$$

The motion, therefore, consists of two simple oscillations, which may be considered separately. The one expressed by the term $A\cos(nt+B)$, is independent of the motion of the point of suspension, and depends only on the length of the pendulum and the original circumstances of the motion; the other, expressed by the term $\frac{c\alpha^2}{\alpha^2 - n^2}\cos(\alpha t + \beta)$, depends on the motion of the point of suspension, and is synchronous with it, but both in extent and period is quite independent of the initial circumstances. If we neglect the first term, or the regular motion of the pendulum, we have for the disturbed motion

$$x = -\frac{c\alpha^2}{\alpha^2 - n^2}\cos(\alpha t + \beta),$$

and
$$u + x = -\frac{cn^2}{\alpha^2 - n^2}\cos(\alpha t + \beta).$$

The motion of a point at a distance s from the ball will be

$$= \frac{c}{n^2 - \alpha^2}\left(n^2 - \alpha^2\frac{s}{l}\right)\cos(\alpha t + \beta).$$

That point, therefore, will remain at rest, so far as the disturbed motion is concerned, when

$$\alpha^2 = \frac{l}{s}n^2 = \frac{g}{s},$$

or the distance from the ball of the point at rest, is equal to the length of a simple pendulum vibrating in the same time as the point of support, which might have been anticipated.

The preceding formulæ fail when $\alpha = u$, or when the period of the disturbance is equal to that of the pendulum. In this case the integral of the equation (3) would be of the form

$$x = A \cos(nt + \beta) - \frac{ct}{2n} \sin(nt + \beta),$$

into which the time enters as a multiplier, so that x increases indefinitely with the time, and the motions are therefore no longer infinitesimal, as was at first supposed, and the original equations are therefore no longer applicable. This change of form in the integral is the analytical indication of a very important fact, viz. that a force, however small, may produce a motion of any extent in any body capable of oscillating, provided that the application of the force be made at intervals, the length of which is equal to the period in which the body would oscillate under the action of gravity. Thus it is, that a stone in a sling may be made to revolve completely round with a great angular velocity, merely by the synchronous motion of the hand; and many similar examples of this fact are constantly presented to our observation. We remember to have seen in a steam-vessel a lamp, which was so hung that its time of oscillation very nearly coincided with the stroke of the engine; the consequence of which was, that though the water was quite smooth, the lamp being set in motion by the reiterated strokes of the piston swung in a large arc, as if the vessel were rolling in a heavy sea.

In the second case, when the motion of the point of suspension depends partly on its own elasticity, and partly on the motion of the ball, we shall have for the equation of its motion

$$\frac{d^2u}{dt^2} + k^2u - ax = 0 \text{ .... } (5),$$

which, combined with the equation

$$\frac{d^2u}{dt^2} + \frac{d^2x}{dt^2} + n^2x = 0 \text{ ....(1)},$$

will determine u and x. To do this, multiply (5) by $\frac{d^2}{dt^2}$ and (1) by $\left(\frac{d^2}{dt^2} + k^2\right)$, and subtract; then

$$\left\{\left(\frac{d^2}{dt^2} + k^2\right)\left(\frac{d^2}{dt^2} + n^2\right) + a\frac{d^2}{dt^2}\right\} x = 0,$$

or

$$\left(\frac{d^2}{dt^2} + \rho_1^2\right)\left(\frac{d^2}{dt^2} + \rho_2^2\right) x = 0 \text{ .... (6)},$$

ρ_1^2, ρ_2^2 being the roots of the quadratic equation

$$(\rho^2 - k^2)(\rho^2 - n^2) - a\rho^2 = 0 \text{..............(7)}.$$

Hence, integrating (6), we have for the value of x

$$x = A\cos(\rho_1 t + \alpha) + B\cos(\rho_2 t + \beta) \text{ .... (8)}.$$

To deduce the value of u, subtract (1) from (5); then

$$k^2u = \frac{d^2x}{dt^2} + (n^2 + a)\,x,$$

and therefore

$$u = \frac{A}{k^2}(n^2 + a - \rho_1^2)\cos(\rho_1 t + \alpha) + \frac{B}{k^2}(n^2 + a - \rho_2^2)\cos(\rho_2 t + \beta)$$
$$\text{........................(9)}.$$

In general k is much greater than n, and a is very small. The equation for determining the two values of ρ^2 may then be put under the forms

$$\rho_1^2 = n^2 - \frac{a\rho_1^2}{k^2 - \rho_1^2},$$

$$\rho_2^2 = k^2 + \frac{a\rho_2^2}{\rho_2^2 - n^2}.$$

One therefore of the two values of ρ will be a little less than n, and the other a little greater than k. Hence the signs of the coefficients of the first terms in x and u will be the same, and those of the second terms different.

The vibrations of the ball, and of the point of suspension, will thus consist of two parts, which may either co-exist or exist separately. The one part, the argument of which is $\cos(\rho_1 t + \alpha)$, is a synchronous vibration of the ball, and the point of suspension, a little slower than the independent motion of the ball, and such that the ball and the point of suspension are always on the same side of the perpendicular, passing through the original position. The other part, the argument of which is $\cos(\rho_2 t + \alpha)$, is a synchronous vibration, a little quicker than the independent vibration of the point of support, and such that the ball and the point of suspension are always on opposite sides of the perpendicular.

If instead of a simple pendulum we have a rod or other solid body, let a be the distance of its centre of gravity from the point of suspension, and k the radius of gyration; then if $u + x$ be the co-ordinate of the centre of gravity, and if we suppose the motion of the point of suspension to be independent of that of the pendulum, so that $n = c\cos(\alpha t + \beta)$, we shall have, for determining x, the equation

$$\frac{d^2x}{dt^2} + \frac{ga}{a^2 + k^2} x - \frac{a^2}{a^2 + k^2} c\alpha^2 \cos(\alpha t + \beta) = 0 \ldots (10).$$

Let $\frac{ga}{a^2 + k^2} = n^2$. Then integrating

$$x = A\cos(nt + B) - \frac{n^2\alpha^2 ca}{g(\alpha^2 - n^2)} \cos(\alpha t + \beta) \ldots (11).$$

Neglecting the first term, we get, as the expression for the part of the motion independent of the initial circumstances,

$$x = -\frac{n^2\alpha^2 ca}{g(\alpha^2 - n^2)} \cos(\alpha t + \beta).$$

For a point at a distance s from the point of suspension, the coordinate is $u + \frac{sx}{a}$

$$= \left(1 - \frac{n^2}{g}\frac{\alpha^2 s}{\alpha^2 - n^2}\right) c\cos(\alpha t + \beta).$$

From this it appears that a point, the distance of which from the point of suspension is $\left(a + \frac{k^2}{a}\right)\left(1 - \frac{n^2}{\alpha^2}\right)$ will remain at rest. If $\frac{n}{\alpha} = 0$, that is, if α be very great, or the vibrations of the point of suspension become very rapid, the centre of percussion is the point which will remain at rest, as might have been anticipated.

II. Let the point of suspension have a horizontal motion at right angles to that of the pendulum.

Without going into the calculation in this case, it is easy to see that this disturbance will not affect the motion of the pendulum in its original direction, and that it will give rise to an oscillatory motion at right angles to that of the pendulum, the nature of which will be similar to the disturbance described in the first case. The two independent motions at right angles to each other will be combined into one, which will cause the ball of the pendulum to describe a curvilinear path.

III. Let the point of suspension have a small vertical oscillatory motion while the ball of the pendulum oscillates in one plane.

In this case, the disturbance of the motion of the pendulum is produced by the variation in the tension of the string; therefore, if x represent the horizontal coordinate of the ball, and T the tension of the string, the equation of motion is

$$\frac{d^2x}{dt^2} + T\frac{x}{l} = 0 \text{..................(12)}.$$

Let the vertical motion of the point of suspension be

$$u = c \cos(\alpha t + \beta),$$

being independent of the motion of the pendulum. The ball receives the same motion from the change of tension, and therefore

$$\frac{d^2u}{dt^2} = -c\alpha^2 \cos(\alpha t + \beta) = T - g;$$

and therefore, putting n^2 for $\frac{s}{l}$, equation (12) becomes

$$\frac{d^2x}{dt^2} + n^2x - c\frac{\alpha^2}{l}x\cos(\alpha t + \beta) = 0 \;\ldots\ldots\ldots(13).$$

This equation being no longer linear, cannot be integrated as the preceding equations were, and we must therefore have recourse to an approximate solution. If we suppose c to be small, we may substitute in that term the value of x derived from the supposition of $c = 0$. That gives us

$$x = A\cos(nt + B),$$

and therefore

$$\frac{d^2x}{dt^2} + n^2x = \frac{Ac\alpha^2}{2l}\left[\cos\{(n+\alpha)t + B + \beta\} + \cos\{(n-\alpha)t + B - \beta\}\right].$$

Integrating this equation,

$$x = -\frac{Ac\alpha^2}{2l}\left[\frac{\cos\{(n+\alpha)t + B + \beta\}}{\alpha^2 + 2\alpha n} + \frac{\cos\{(n-\alpha)t + B - \beta\}}{\alpha^2 - 2\alpha n}\right] \;\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

The most important conclusion from this result is, that if $A = 0$, or the ball be originally at rest, it will have no motion communicated to it by the vertical motion of the point of suspension: and that if it has an oscillatory motion in any direction, this will not be permanently altered unless $\alpha = 2n$, or the period of oscillation of the pendulum be double of that of the point of suspension. In this case the integral becomes infinite; and if, as before, we put it into another shape, we find that x increases continually with the time. This accords with experiment, which shows that the arc of vibration of a pendulum may be increased indefinitely by giving the point of suspension a vertical motion of oscillation, the period of which is half of that of the pendulum.

ON THE EVALUATION OF A DEFINITE MULTIPLE INTEGRAL.*

In a memoir read before the Academy of Sciences of Paris, and inserted in the *Comptes Rendus*, Vol. VIII., p. 156, M. Lejeune Dirichlet called the attention of mathematicians to the remarkable multiple integral

$$V = \int dx \int dy \int dz \ldots x^{a-1} y^{b-1} z^{c-1} \text{ ....} (1),$$

which is to be taken between the positive limits of the variables determined by the inequality

$$\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \ldots \overset{=}{<} 1 \text{ ....} (2),$$

the number of variables, and therefore of integrals, being any whatever. The result at which M. Dirichlet arrives is, that

$$V = \frac{\alpha^a \beta^b \gamma^c \ldots}{pqr \ldots} \frac{\Gamma\left(\frac{a}{p}\right) \Gamma\left(\frac{b}{q}\right) \Gamma\left(\frac{c}{r}\right) \ldots}{\Gamma\left(1 + \frac{a}{p} + \frac{b}{q} + \frac{c}{r} \ldots\right)},$$

Γ being the second Eulerian Integral.

The actual calculation is not given in the paper referred to, though the process is indicated; but M. Liouville has investigated the value of the integral by a method different from that employed by M. Dirichlet, and his memoir (*Journal*

* *Cambridge Mathematical Journal*, Vol. II., p. 215.

de Mathématiques, Vol. IV., p. 225) is a very elegant specimen of analysis. The integral itself deserves attention, not only as being a remarkable analytical extension of that property of the first Eulerian Integral by which it is connected with the second, but also because it frequently occurs in the investigation of areas of curves, contents of solids, centres of gravity, and other physical and geometrical problems of a similar kind. From its extensive application to cases which are of such frequent occurrence, this multiple integral ought to receive a prominent place in elementary works on the Integral Calculus, and on that account I here bring it before the English reader. The method by which I propose to evaluate this integral is, I believe, new; and I am anxious to show its application in this case, not only because it exhibits very distinctly the nature of the connexion between this integral and the function Γ, but because it can also be applied with great advantage to the calculation of a number of other definite integrals. In the present paper, however, I shall confine myself to the integral of M. Dirichlet, and a more general one of the same kind which is given by M. Liouville.

In the first place, following the method of M. Liouville, we shall transform the integral so that the limits shall be of the first degree only. This is easily done by assuming

$$\left(\frac{x}{\alpha}\right)^p = x', \quad \left(\frac{y}{\beta}\right)^q = y', \quad \left(\frac{z}{\gamma}\right)^r = z', \text{ \&c.,}$$

from which

$$dx = \frac{\alpha}{p} x'^{\frac{1}{p}-1} dx', \quad dy = \frac{\beta}{q} y'^{\frac{1}{q}-1} dy', \quad dz = \frac{\gamma}{r} z'^{\frac{1}{r}-1} dz', \text{ \&c.}$$

On substituting these values for the variables and their differentials, the integral becomes

$$V = \frac{\alpha^a \beta^b \gamma^c \dots}{pqr\dots} U \dots\dots\dots\dots\dots\dots\dots (3),$$

where U is the definite integral

$$\int dx' \int dy' \int dz' \ldots x'^{\frac{a}{p}-1} y'^{\frac{b}{q}-1} z'^{\frac{c}{r}-1} \ldots\ldots\ldots\ldots (4),$$

the limits of the variables being given by the inequality

$$x' + y' + z' \ldots \overline{\overline{<}}\, 1 \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Now let $\dfrac{a}{p} = l, \quad \dfrac{b}{q} = m, \quad \dfrac{c}{r} = n \ldots,$

and, dropping the accents which are no longer necessary for discrimination, we have to calculate the integral

$$U = \int dx \int dy \int dz \ldots x^{l-1} y^{m-1} z^{n-1} \ldots\ldots\ldots\ldots (6):$$

the limits being given by the inequality

$$x + y + z \ldots \overline{\overline{<}}\, 1.$$

If the variables be only two in number, x and y, the integral is reduced to

$$U = \int dx \int dy\, x^{l-1} y^{m-1} = \frac{1}{m} \int dx\, x^{l-1} (1-x)^m,$$

since the limits of y are 0 and $1 - x$.

The evaluation of this integral, by a method due to Professor Jacobi, may be found in the *Cambridge Mathematical Journal*, Vol. I., p. 94, and it is by an extension of that method that M. Liouville has calculated the general integral under consideration. Instead of employing it we shall proceed in the following manner:

Let $$x + y + z + \ldots = v,$$

or $$x = v - y - z - \ldots.$$

Then, as x varies when $y, z \ldots$ are constant, $dx = dv$, and (10) becomes

$$U = \int dv \int dy \int dz\, y^{m-1} z^{n-1} (v - y - z \ldots)^{l-1} \ldots\ldots\ldots (7).$$

We might now integrate with respect to v, but, for the convenience of our future operations, we shall only indicate the operations. The extreme limits of v are 0 and 1, and we may therefore write

$$U = \int_0^1 dv \int dy \int dz\, y^{m-1} z^{n-1} (v - y - z - \ldots)^{l-1}.$$

Now by the symbolical form of Taylor's Theorem we have

$$f(x+h) = \varepsilon^{h\frac{d}{dx}} f(x).$$

Hence we may put

$$(v - y - z - \ldots)^{l-1} = \varepsilon^{-y\frac{d}{dv}} (v - z - \ldots)^{l-1} \ldots\ldots\ldots (9),$$

and we have then

$$U = \int_0^1 dv \int dz\, z^{n-1} \ldots \int dy\, y^{m-1} \varepsilon^{-y\frac{d}{dv}} (v - z - \ldots)^{l-1} \ldots (10),$$

the limits of y being 0 and $v - z - \ldots$.

Now assume $y \dfrac{d}{dv} = t$, so that $dy = dt \left(\dfrac{d}{dv}\right)^{-1}$,

$$U = \int_0^1 dv \int dz\, z^{n-1} \int dt\, t^{m-1} \varepsilon^{-t} \left(\frac{d}{dv}\right)^{-m} (v - z \ldots)^{l-1} \ldots (11).$$

To find the limits of t we have recourse to the following considerations. Supposing, for simplicity, that there were only two variables, x and y, we have

$$(v - y)^{l-1} = \varepsilon^{-y\frac{d}{dv}} v^{l-1} = \varepsilon^{-t} . v^{l-1}.$$

Now, when $y = 0$ the first side becomes v^{l-1}; and in order that the second side should be reduced to that form, we must have $t = 0$. Again, when $y = v$ the first side becomes zero, l being positive; and in order that the second side may also become zero, we must have $t = \infty$.

Hence to the values

$$\left.\begin{matrix} y = 0 \\ y = v \end{matrix}\right\} \text{ correspond } \left\{\begin{matrix} t = 0 \\ t = \infty. \end{matrix}\right.$$

As in this transformation consists the principle of the method I employ, I shall add a few words in explanation of it. When we assume

$$y\frac{d}{dv}=t \text{ and } dy=dt\left(\frac{d}{dv}\right)^{-1},$$

and therefore $$\varepsilon^{-y\frac{d}{dv}}v^{l-1}=\varepsilon^{-t}v^{l-1},$$

we suppose t to be a variable capable of increase or decrease; or, in other words, a symbol of quantity. But, on the other hand, $y\frac{d}{dv}$ is a symbol of operation to which we cannot apply the terms increase or decrease, and it may therefore seem to be scarcely allowable to assume it to be equal to a quantitative symbol. It is to be observed, however, that our use of the symbolical expression is only for the assumption of the *form* of the new function, and that when we put $y\frac{d}{dv}=t$, we really say nothing more than that t is to be such a function of y that it shall satisfy the equation

$$(v-y)^{l-1}=\varepsilon^{-t}v^{l-1},$$

which of course is an assumption which we are quite at liberty to make. Whether in every case of such a transformation we could actually determine t, is a question with which we have fortunately no concern, as it is to be expected that such a determination would be in most cases very difficult if not impracticable. All that we require to know are the limiting values of t corresponding to those of y; and as these can generally be found by considerations such as we have used, the transformation is for our purposes sufficient. It reduces the function to be integrated to a very convenient form for effecting that operation, and the substitution of the values of the limits offers generally no difficulties. What we have said regarding the limits in the case of two variables, applies equally well to a greater number, except that the limits of y being 0 and $v-z-\ldots$, the limits of t will still be 0 and ∞. Hence, recurring to our integral, equation (11) becomes, on affixing the limits to t,

$$U=\int_0^1 dv \int dz\, z^{n-1}\ldots\int_0^\infty dt\, t^{m-1}\varepsilon^{-t}\left(\frac{d}{dv}\right)^{-m}(v-z-\ldots)^{l-1}.$$

Now $\int_0^\infty dt\, t^{m-1}\varepsilon^{-t}=\Gamma(m)$, and therefore

$$U=\Gamma(m)\int_0^1 dv \int dz\, z^{n-1}\ldots\left(\frac{d}{dv}\right)^{-m}(v-z-\ldots)^{l-1}\ldots(12).$$

Proceeding with z in the same manner as with y, by putting

$$(v-z-\ldots)^{l-1}=\varepsilon^{-z\frac{d}{dv}}v^{l-1},$$

and assuming $$z\frac{d}{dv}=s,$$

we find, as before,

$$U=\Gamma(m)\int_0^1 dv\ldots\int_0^\infty ds\, s^{n-1}\varepsilon^{-s}\left(\frac{d}{dv}\right)^{-(m+n)}v^{l-1},$$

$$=\Gamma(m)\,\Gamma(n)\int_0^1 dv\ldots\left(\frac{d}{dv}\right)^{-(m+n)}v^{l-1}\ldots\ldots\ldots\ldots(13).$$

In this manner we might proceed for any number of variables, but restricting ourselves to three, it only remains to integrate with respect to v between the given limits.

Now as $$\left(\frac{d}{dv}\right)^{-(m+n)}v^{l-1}=\frac{\Gamma(l)}{\Gamma(l+m+n)}v^{l+m+n-1},$$

we find $$U=\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)}\int_0^1 dv\, v^{l+m+n-1}$$

$$=\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{(l+m+n)\,\Gamma(l+m+n)}$$

$$=\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(1+l+m+n)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14);$$

since by the fundamental property of the function Γ

$$(l+m+n)\,\Gamma(l+m+n)=\Gamma(1+l+m+n).$$

Substituting this in (3) and putting for l, m, n, their values $\frac{a}{p}$, $\frac{b}{q}$, $\frac{c}{r}$, we finally obtain

$$V = \frac{\alpha^a \beta^b \gamma^c}{pqr} \frac{\Gamma\left(\frac{a}{p}\right)\Gamma\left(\frac{b}{q}\right)\Gamma\left(\frac{c}{r}\right)}{\Gamma\left(1+\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right)} \text{ .... (15).}$$

It is to be observed, that in effecting the operation

$$\left(\frac{d}{dv}\right)^{-(m+n)} v^{l-1},$$

we must not add any arbitrary constants, since this inverse operation has arisen during the process without causing any constants to disappear, and there are therefore none to be restored. All the arbitrary constants arising from the original integrals are eliminated in taking the limits, and no others are to be introduced, otherwise we should have more arbitrary constants than integrals.

The transformation in (11) and the subsequent investigation of the limits are the parts of this method which appear to be new, or at least not to have been hitherto employed to calculate definite integrals. It is easily seen that the same transformation may be applied to many other definite integrals, but it would occupy too much space to enter on the consideration of them at present. I shall therefore pass on to some examples of the application of the formula which has just been proved.

Ex. 1. To find the area of the evolute to the ellipse.

The expression for the area is

$$V = \iint dx\,dy.$$

x and y being subject to the limiting condition

$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1.$$

Here $a=1$, $b=1$, $p=q=\frac{2}{3}$. Therefore by (15)

$$V=\frac{9}{4}\,\alpha\beta\,\frac{\{\Gamma(\frac{3}{2})\}^2}{\Gamma(4)}.$$

Now $\Gamma(4)=3.2.1$, and $\Gamma(\frac{3}{2})=\frac{1}{2}\Gamma(\frac{1}{2})=\frac{1}{2}\sqrt{(\pi)}$, and therefore $V=\dfrac{3\alpha\beta}{32}$, which is the area of the portion of the curve included between the positive axes, since the variables are supposed never to become negative. The area of the whole curve is $\dfrac{3\alpha\beta}{8}$.

Ex. 2. If we wish to find the co-ordinates of the centre of gravity of the same area, we have to calculate

$$\iint x\,dx\,dy \text{ and } \iint y\,dx\,dy.$$

By the formula (15)

$$\iint x\,dx\,dy=\frac{9}{4}\,\alpha^2\beta\,\frac{\Gamma(3)\,\Gamma(\frac{3}{2})}{\Gamma(1+3+\frac{3}{2})}, \quad \iint y\,dx\,dy=\frac{9}{4}\,\alpha\beta^2\,\frac{\Gamma(3)\,\Gamma(\frac{3}{2})}{\Gamma(1+3+\frac{3}{2})}.$$

Hence, if $\bar{x}$, $\bar{y}$ be the coordinates of the centre of gravity,

$$\bar{x}=\frac{2^8\alpha}{9.7.5}, \quad \bar{y}=\frac{2^8\beta}{9.7.5}.$$

Ex. 3. It is shown, *Cambridge Mathematical Journal*, Vol. II., p. 14,* that the equation to the parabola, when referred to two tangents as axes, is

$$\sqrt{\left(\frac{x}{\alpha}\right)}+\sqrt{\left(\frac{y}{\beta}\right)}=1,$$

α and β being the portions of the tangents intercepted between their intersection and the curve. If θ be the angle between the axes, the area is

$$V=\sin\theta\iint dx\,dy,$$

* See p. 163 of this volume.

the limits of x and y being given by the preceding equation. Here $a=1$, $b=1$, $p=q=\frac{1}{2}$. Therefore

$$V=\sin\theta.4\alpha\beta\frac{\{\Gamma(2)\}^2}{\Gamma(5)}=\frac{\alpha\beta\sin\theta}{6}.$$

From this it appears (referring to the figure in the article alluded to above,) that the triangle ABC is three times the area $ABPC$.

Ex. 4. To find the centre of gravity of the eighth part of the ellipsoid

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1.$$

If $\bar{z}$ be one of the coordinates of the centre of gravity,

$$\bar{z}=\frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

the limits being given by the preceding equation.

In the numerator, comparing it with (15), we have

$$a=b=1,\quad c=2,\quad p=q=r=2.$$

Therefore $$\iiint z\,dx\,dy\,dz=\frac{\alpha\beta\gamma^2}{8}\frac{\{\Gamma(\frac{1}{2})\}^2}{\Gamma(3)}=\pi\frac{\alpha\beta\gamma^2}{16}.$$

In the denominator $a=b=c=1$, $p=q=r=2$. Therefore

$$\iiint dx\,dy\,dz=\frac{\pi\alpha\beta\gamma}{8}\frac{\{\Gamma(\frac{1}{2})\}^3}{\Gamma(1+\frac{3}{2})}=\frac{\pi\alpha\beta\gamma}{6}.$$

Hence $\bar{z}=\frac{3}{8}\gamma$, and similarly for the other coordinates.

M. Liouville has given to the Theorem of M. Dirichlet a very important extension, of which the following is the enunciation. If

$$W=\int dx\int dy\int dz\ldots x^{l-1}y^{m-1}z^{n-1}\ldots f(x+y+z+\ldots)\ldots(16),$$

where the limits of $x, y, z\ldots$ are such as to satisfy the inequality

$$x+y+z+\ldots \mathrel{\substack{=\\<}} h,$$

f being any function whatsoever,

$$W = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\ldots}{\Gamma(l+m+n+\ldots)} \int_0^h dv f(v)\, v^{l+m+n+\ldots-1} \ldots (17).$$

It will be seen, as in the first part of this article, that to the form (16) may be reduced the more general one

$$W' = \int dx \int dy \int dz \ldots x^{a-1} y^{b-1} z^{c-1} f\left\{\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \ldots\right\},$$

where the limiting values are given by the inequality

$$\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \ldots \overline{\underline{<}}\, h;$$

for by a simple transformation of the variables we should find

$$W' = \frac{\alpha^a \beta^b \gamma^c}{pqr} W,$$

and it is therefore only necessary to calculate W. To effect this we proceed in the same manner as before.

Let $$x + y + z + \ldots = v,$$

then $$dx = dv,$$

the extreme limits of v being 0 and h. Then

$$W = \int_0^h dv \int dy \int dz \ldots y^{m-1} z^{n-1} \ldots (v - y - z - \ldots)^{l-1} f(v).$$

Now, as before,

$$(v - y - z - \ldots)^{l-1} f(v) = \varepsilon^{-y\frac{d'}{dv}} (v - z - \ldots)^{l-1} f(v),$$

where $\dfrac{d'}{dv}$ is accentuated to imply that it refers only to the v included in $(v - z - \ldots)^{l-1}$ and not to that under the f. Now putting $y\dfrac{d'}{dv} = t$, the limits of t will be 0 and ∞, and

$$W = \int_0^h dv \int dz \ldots z^{n-1} \ldots \int_0^\infty dt\, t^{m-1} \varepsilon^{-t} \left(\frac{d'}{dv}\right)^{-m} (v - z - \ldots)^{l-1} f(v),$$

$$= \Gamma(m) \int_0^h dv \int dz \ldots z^{n-1} \ldots \left(\frac{d'}{dv}\right)^{-m} (v - z - \ldots)^{l-1} f(v).$$

Next, treating z in the same manner as y, we have

$$W = \Gamma(m)\,\Gamma(n)\int_0^h dv \ldots \left(\frac{d'}{dv}\right)^{-(m+n)} (v - \ldots)^{l-1} f(v),$$

and so on for any number of variables. Restricting ourselves to three, and effecting the operation $\left(\frac{d'}{dv}\right)^{-(m+n)}$, which has reference only to v^{l-1}, we find

$$W = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_0^h dv\, v^{l+m+n-1} f(v).$$

As an example of the application of this formula, let us take the expression

$$W = \int dx \int dy \int dz \; \frac{1}{\sqrt{(1 - x^2 - y^2 - z^2)}},$$

where the variables satisfy the condition

$$x^2 + y^2 + z^2 \leqq 1.$$

To change the variables, put $x^2 = x'$, $y^2 = y'$, $z^2 = z'$; then

$$W = \tfrac{1}{8} \iiint \frac{dx'\,dy'\,dz'}{\sqrt{(x'y'z')}} \; \frac{1}{\sqrt{(1 - x' - y' - z')}},$$

and $$x' + y' + z' \leqq 1.$$

In this case, then, $l = m = n = \frac{1}{2}$; therefore

$$W = \tfrac{1}{8} \frac{\{\Gamma(\frac{1}{2})\}^3}{\Gamma(\frac{3}{2})} \int_0^1 \frac{v^{\frac{1}{2}} dv}{\sqrt{(1 - v)}};$$

putting $v = x^2$, we find

$$\int_0^1 \frac{v^{\frac{1}{2}} dv}{\sqrt{(1 - v)}} = 2\int_0^1 \frac{x^2 dx}{\sqrt{(1 - x^2)}} = \tfrac{1}{2}\pi.$$

Also $\Gamma(\frac{1}{2}) = \sqrt{(\pi)}$, and $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{(\pi)}$; therefore

$$W = \frac{\pi^2}{8}.$$

Again, take $W = \iint dx\,dy \sqrt{\left(\frac{1 - x^2 - y^2}{1 + x^2 + y^2}\right)}$,

when $$x^2 + y^2 \leqq 1.$$

By a change of the variable,

$$W = \tfrac{1}{4} \iint \frac{dx\,dy}{\sqrt{(xy)}} \sqrt{\left(\frac{1-x-y}{1+x+y}\right)}, \text{ and } x+y \overset{=}{<} 1.$$

Here $l = m = \frac{1}{2}$; therefore

$$W = \tfrac{1}{4} \frac{\{\Gamma(\frac{1}{2})\}^2}{\Gamma(1)} \int_0^1 dv \sqrt{\left(\frac{1-v}{1+v}\right)}.$$

Now $\Gamma(1) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{(\pi)}$,

$$\int_0^1 dv \sqrt{\left(\frac{1-v}{1+v}\right)} = \int_0^1 dv \left\{\frac{1}{\sqrt{(1-v^2)}} - \frac{v}{\sqrt{(1-v^2)}}\right\} = \frac{\pi}{2} - 1;$$

therefore
$$W = \frac{\pi}{4}\left(\frac{\pi}{2} - 1\right).$$

Other examples of this formula will be found in a paper by M. E. Catalan, *Journal de Mathématiques*, Vol. IV., p. 323.

SINGULAR POINTS IN SURFACES.*

THE nature of those points in surfaces which are analogous to multiple points in curves, has not, so far as I know, been hitherto discussed in any work on Analytical Geometry. Leroy, in his *Analyse Appliquée*, and in his *Géometrie Descriptive*, does little more than indicate their existence; and Monge and Dupin, although they have given much attention to those singular points called *umbilici*, have not spoken at all of the points which are the object of the present article. The subject is, however, one of some interest, from the light which it throws on the form of certain surfaces; and in physical investigations Sir W. Hamilton has shown by his researches on the form of the Wave Surface, that such points are of considerable importance.

In the first place, it is necessary to define precisely what is meant by a *singular* point in the sense in which I use that phrase. If

$$F(x, y, z) = 0 \dots\dots\dots\dots\dots\dots\dots (1),$$

be the equation to a surface, then those points for which all the differential coefficients of F, with respect to x, y, and z, below a certain order, vanish, I call singular points; and the values of x, y, z, which satisfy these conditions, I shall call the *critical* values. We shall have to consider chiefly points for which the differential coefficients, of the

* *Cambridge Mathematical Journal*, Vol. II., p. 252.

first order only, vanish, so that the characteristic of such points is, that the equations

$$\frac{dF}{dx}=0, \quad \frac{dF}{dy}=0, \quad \frac{dF}{dz}=0 \ldots\ldots\ldots\ldots\ldots(2),$$

are satisfied by simultaneous values of x, y, z, which also satisfy the equation to the surface (1). When the conditions (2) do not hold, we know that the locus of the tangent lines to the surface at a given point is a plane, which is called the tangent plane. When the conditions (2) do hold, the direction cosines of the tangent take the form $\frac{0}{0}$, and are therefore indeterminate, showing that there may be more than one tangent plane at the point: and we shall see that there are in general an infinite number of such planes, forming by their intersections a tangent *cone*.

Let us now investigate the locus of the tangent lines at a singular point. This we shall do by the same method as that employed in the *Cambridge Mathematical Journal*, Vol. I., p. 135; and for the convenience of our future operations, let us put

$$U=\frac{dF}{dx}, \qquad V=\frac{dF}{dy}, \qquad W=\frac{dF}{dz},$$

$$u=\frac{d^2F}{dx^2}, \qquad v=\frac{d^2F}{dy^2}, \qquad w=\frac{d^2F}{dz^2},$$

$$u'=\frac{d^2F}{dy\,dz}, \qquad v'=\frac{d^2F}{dz\,dx}, \qquad w'=\frac{d^2F}{dx\,dy}.$$

Let then the equation to a line passing through the point x, y, z, in the surface, be

$$\frac{x'-x}{l}=\frac{y'-y}{m}=\frac{z'-z}{n}=r \ldots\ldots\ldots\ldots\ldots(a),$$

x', y', z', being the current coordinates of the line. This line will generally be cut by the surface in one or more other points. Let the coordinates of the nearest of these be x_1, y_1, z_1, then from equation (a) we have

$$x_1 = x + lr, \quad y_1 = y + mr, \quad z_1 = z + nr.$$

Substituting these values in the equation to the surface, it becomes

$$F\{x + lr, \; y + mr, \; z + nr\} = 0.$$

Expanding this, considering lr, mr, nr, as the increments of x, y, z, we have

$$0 = F(x, y, z) + r\,(lU + mV + nW)$$
$$+ \frac{r^2}{1.2}\{l^2u + m^2v + n^2w + 2\,(mnu' + nlv' + lmw')\} + \frac{r^3}{1.2.3}\{\quad\}.$$

But from the equation to the surface

$$F(x, y, z,) = 0;$$

and when the point is a singular point,

$$U = 0, \quad V = 0, \quad W = 0;$$

therefore the equation is reduced to

$$0 = \frac{1}{1.2}\{l^2\,(u) + m^2\,(v) + n^2\,(w) + 2\,mn\,(u') + 2nl\,(v') + 2lm\,(w')\}$$
$$+ \frac{r}{1.2.3}\{\quad\}.$$

where the differentials are inclosed in parentheses to indicate the values they receive when the critical values of x, y, z, are substituted in them. When the line becomes a tangent, the points x, y, z, x_1, y_1, z_1, ultimately coincide, and r becomes indefinitely small; in which case the preceding equation becomes ultimately

$$l^2\,(u) + m^2\,(v) + n^2\,(w) + 2\,\{mn\,(u') + nl\,(v') + lm\,(w')\} = 0;$$

whence eliminating l, m, n, by means of equation (a), we find

$$(u)\,(x' - x)^2 + (v)\,(y' - y)^2 + (w)\,(z' - z)^2 + 2\{(u')\,(y' - y)\,(z' - z)$$
$$+ (v')\,(z' - z)\,(x' - x) + (w')\,(x' - x)\,(y' - y)\} = 0 \ldots (3),$$

as the equation to the locus of the tangent lines. This is evidently in general a cone of the second degree, though with certain values of the coefficients, it may represent two planes or a straight line. On transferring the origin to the singular point, the equation (3) may be put under the form

$$(u)\, x^2 + (v)\, y^2 + (w)\, z^2 + 2\{(u')\, yz + (v')\, zx + (w')\, xy\} = 0 \ldots (4).$$

If we suppose that all the second differential coefficients also vanish for the critical values of the variables, we should easily find that the locus of the tangent lines at a singular point is a cone of the third order, and so on in succession to any order. The forms of the equations are easily found by taking the corresponding differential of $F(x, y, z)$, and substituting in it x, y, z, for dx, dy, dz. It is unnecessary to write down these expressions, which are rather long, as I shall scarcely have occasion to use them. Before proceeding to give some applications of the formula which has just been investigated, I shall make one or two remarks on the difference, with respect to singular points, between plane curves and surfaces. In the first place it will be readily seen that the three equations (2) may be satisfied by some relation between the variables, without assigning any particular value to each. If we suppose the relation to be given by the equation

$$\phi\,(x, y, z,) = 0,$$

the intersection of the surface (1), with that represented by the last equation, will give a line which is a *locus* of singular points: that is to say, every point in the line is a singular point. Such, for instance, is the edge of regression of a developable surface, or the rectilinear directrix of a conoid. There is nothing analogous to this in plane curves, since the one variable is always determined in terms of the other by the equation to the curve, so that there is nothing left indeterminate. In the second place, I observed before that the equation (4) may represent a cone of the second order, or two planes, or a straight line, which last may indeed be considered as a cone, the vertical angle of which is zero.

In plane curves, the corresponding equation could represent only two straight lines, or two points, since a homogeneous function of two variables can always be decomposed into possible or impossible factors of the first degree. We shall now proceed to some examples in illustration of the general theory given above.

1. Let the equation to the surface be

$$(x^2+y^2+z^2)^2 = a^2x^2+b^2y^2-c^2z^2.$$

This is the locus of the intersection of the tangent planes to the hyperboloid of one sheet, with perpendiculars on them from the centre. Here

$$U=2x(2r^2-a^2), \quad V=2y(2r^2-b^2), \quad W=2z(2r^2+c^2),$$

$$u=2(2r^2-a^2)+8x^2, \quad v=2(2r^2-b^2)+8y^2, \quad w=2(2r^2+c^2)+8z^2,$$

$$u'=8yz, \quad v'=8zx, \quad w'=8xy.$$

At the origin, or when $x=0$, $y=0$, $z=0$, U, V, W, all vanish; that point is therefore a singular point. Substituting the critical values in the expression for the second differentials, we have

$$(u)=-2a^2, \quad (v)=-2b^2, \quad (w)=2c^2,$$

$$(u')=0, \quad (v')=0, \quad (w')=0.$$

The equation to the tangent cone at the singular point is, therefore,

$$a^2x^2+b^2y^2-c^2z^2=0.$$

This represents an elliptical cone, the axis of z being perpendicular to the directrix.

2. The equation to Fresnel's Wave Surface is

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2$$
$$-b^2(c^2+a^2)y^2-c^2(a^2+b^2)z^2+a^2b^2c^2=0.$$

In this case,

$$U=2x\{a^2(r^2-b^2-c^2)+a^2x^2+b^2y^2+c^2z^2\},$$

$$V=2y\{b^2(r^2-c^2-a^2)+a^2x^2+b^2y^2+c^2z^2\},$$

$$W=2z\{c^2(r^2-a^2-b^2)+a^2x^2+b^2y^2+c^2z^2\}.$$

Now these expressions vanish when

$$y = 0, \quad r^2 = b^2,$$

involving as values of x and z

$$x = \pm c \sqrt{\left(\frac{a^2 - b^2}{a^2 - c^2}\right)}, \quad z = \pm a \sqrt{\left(\frac{b^2 - c^2}{a^2 - c^2}\right)}.$$

There are therefore four singular points corresponding to the four different ways in which the double signs of x and z may be combined. We find also

$$(u) = 8a^2c^2 \frac{a^2 - b^2}{a^2 - c^2}, \quad (v) = -2(a^2 - b^2)(b^2 - c^2), \quad (w) = 8a^2c^2 \frac{b^2 - c^2}{a^2 - c^2},$$

$$(u') = 0, \quad (v') = 4ac \sqrt{\{(a^2 - b^2)(b^2 - c^2)\}} \frac{a^2 + c^2}{a^2 - c^2}, \quad (w') = 0.$$

Substituting these values in equation (4), and putting the result in the simplest form, we find

$$\frac{x^2}{b^2 - c^2} - \frac{a^2 - c^2}{4a^2c^2} y^2 + \frac{z^2}{a^2 - b^2} + \frac{a^2 + c^2}{\sqrt{\{(a^2 - b^2)(b^2 - c^2)\}}} \frac{xz}{ac} = 0,$$

as the equation to the tangent cone at each of the singular points. It is supposed throughout that a, b, c, are in order of magnitude.

3. Let the surface be the locus of the intersection of tangent planes to an elliptical paraboloid, with the perpendiculars on them from the vertex, the equation to it being

$$z(x^2 + y^2 + z^2) + ax^2 + by^2 = 0.$$

Here $U = 2x(z + a)$, $V = 2y(z + b)$, $W = x^2 + y^2 + 3z^2$.

These expressions vanish when $x = 0$, $y = 0$, $z = 0$; which values also satisfy the given equation: the origin is therefore a singular point. We find also

$$(u) = 2a, \quad (v) = 2b, \quad (w) = 0, \quad (u') = 0, \quad (v') = 0, \quad (w') = 0.$$

The equation to the locus of the tangent lines is

$$ax^2 + by^2 = 0.$$

As a and b are both positive, this can be satisfied only by

$$x = 0, \quad y = 0;$$

and it therefore represents the axis of z. It will be seen from the equation that z can never be positive since that would render x and y impossible; the point in question is therefore a cusp. The surface surrounds the negative axis of z which it touches at the origin, so that the form of the surface resembles the shape of the flower of the convolvulus. A correct idea of the surface may be formed by supposing a cissoid to revolve round its axis;—the surface generated is the same as that of the surface here discussed when $a=b$. If a and b be of contrary signs, in which case the surface is formed from the hyperbolic paraboloid, the equation to the locus of the tangent lines is

$$ax^2 - by^2 = 0,$$

which represents two planes perpendicular to the plane of x, y.

4. The equation to the Cono-cuneus of Wallis is

$$a^2y^2 - x^2(c^2 - z^2) = 0.$$

Here $\qquad U = -2x(c^2 - z^2), \quad V = 2a^2y, \quad W = 2x^2z.$

These all vanish when $x=0$, $y=0$, independently of any value of z; so that the axis of z is a locus of singular points or a singular line. We find also

$$\begin{aligned} &(u) = -2(c^2 - z^2), \quad &(v) = 2a^2, \quad &(w) = 0, \\ &(u') = 0, &(v') = 0, \quad &(w') = 0, \end{aligned}$$

so that the equation to the locus of the tangent lines is

$$a^2y'^2 - (c^2 - z^2)\,x'^2 = 0,$$

the current coordinates being accentuated to distinguish them from z, the undetermined coordinate of the point under consideration. So long as $z < c$ this equation represents two planes perpendicular to the plane of xy. z cannot be greater than c, as y would then become impossible.

5. Let the surface to the *héliçoide développable*, the equation to which is

$$x \sin\left\{\frac{2\pi}{h}z - \frac{\sqrt{(x^2+y^2-a^2)}}{a}\right\} + y\cos\left\{\frac{2\pi}{h}z - \frac{\sqrt{(x^2+y^2-a^2)}}{a}\right\} = a.$$

If we assume $\frac{2\pi}{h}z - \frac{\sqrt{(x^2+y^2-a^2)}}{a} = \theta$,

we find
$$U = \sin\theta - \frac{x\,(x\cos\theta - y\sin\theta)}{a\sqrt{(x^2+y^2-a^2)}},$$
$$V = \cos\theta - \frac{y\,(x\cos\theta - y\sin\theta)}{a\sqrt{(x^2+y^2+a^2)}},$$
$$W = \frac{2\pi}{h}\,(x\cos\theta - y\sin\theta).$$

From the equation to the surface it is easily seen that
$$x\cos\theta - y\sin\theta = \sqrt{(x^2+y^2-a^2)}.$$
Hence if we assume
$$x = a\sin\frac{2\pi z}{h}, \quad y = a\cos\frac{2\pi z}{h},$$
the three preceding expressions will vanish, and therefore the line determined by these equations, and the equation to the surface, is a locus of singular points. This line is the intersection of the surface by the cylinder
$$x^2+y^2=a^2,$$
and is evidently the generating helix. Since in the equation to the surface x^2+y^2 can never be less than a^2, it appears that no part of the surface lies within the helix, which is therefore truly an edge of regression.

On proceeding to the second differential coefficients, and substituting in them the critical values of x and y, we find, retaining only the terms which become infinite from involving $\sqrt{(x^2+y^2-a^2)}$ in the denominator,
$$(u) = -2\sin\frac{2\pi z}{h}\cos\frac{2\pi z}{h}, \quad (v) = 2\sin\frac{2\pi z}{h}\cos\frac{2\pi z}{h}, \quad (w) = 0,$$
$$(u') = \frac{2\pi}{h}\cos\frac{2\pi z}{h}, \quad (v') = \frac{2\pi}{h}\sin\frac{2\pi z}{h}, \quad (w') = \sin^2\frac{2\pi z}{h} - \cos^2\frac{2\pi z}{h},$$
so that the equation to the locus of the tangent lines is
$$(y'^2 - x'^2)\sin nz\cos nz - x'y'(\cos^2 nz - \sin^2 nz)$$
$$+ nz'(x'\sin nz + y'\cos nz) = 0,$$

where $n = \frac{2\pi}{h}$, and the accentuated letters are the current coordinates of the tangents, and the unaccentuated z is the undetermined coordinate of the point of contact.

This equation may be decomposed into two factors,

$$x' \sin nz + y' \cos nz = 0,$$

and
$$y' \sin nz - x' \cos nz + nz' = 0,$$

which are the equations to two planes.

6. I shall take a single example where it is necessary to proceed to an order of differentiation higher than the second.

Let the equation to the surface be

$$(x^2 + y^2 + z^2)^3 = a^2 (y^2 z^2 + z^2 x^2 + x^2 y^2).$$

It is easy to see that the origin is a singular point; and on making $x = 0$, $y = 0$, $z = 0$, all the differential coefficients of the second and third orders vanish, while of the fourth all vanish except three, which are

$$\frac{d^4 F}{dx^2 dy^2} = \frac{d^4 F}{dz^2 dx^2} = \frac{d^4 F}{dy^2 dz^2} = -4a^2.$$

The equation to the locus of the tangent lines is therefore

$$y'^2 z'^2 + z'^2 x'^2 + x'^2 y'^2 = 0.$$

This is satisfied in three ways only, either by

$$x = 0, \qquad y = 0,$$

or by
$$x = 0, \qquad z = 0,$$

or by
$$y = 0, \qquad z = 0.$$

It therefore represents the three axes, and shows that the surface at the origin touches these lines, forming round each a figure resembling that produced by the revolution of a circular arc round a tangent.

It is needless to multiply examples, more especially as there are few surfaces of a high order which are sufficiently important to require a minute discussion.

MATHEMATICAL NOTE.*

To show that

$$\left(\frac{d}{dx}\right)^r (\varepsilon^{ax}x^n) = \frac{x^{n-r}}{a^{n-r}} \left(\frac{d}{dx}\right)^n (\varepsilon^{ax}x^r).$$

Expanding the differential on the left-hand side by the theorem of Leibnitz, we have

$$\left(\frac{d}{dx}\right)^r (\varepsilon^{ax}x^n) = \left\{d^r + rd^{r-1}d' + \frac{r(r-1)}{1.2} d^{r-2}d'^2 + \&c.\right\} \left(\frac{\varepsilon^{ax}x^n}{dx^r}\right),$$

where d refers to ε^{ax} and d' to x^n. Hence

$$\left(\frac{d}{dx}\right)^r (\varepsilon^{ax}x^n)$$

$$= \left\{a^r x^n + ra^{r-1}nx^{n-1} + \frac{r(r-1)}{1.2} a^{r-2}n(n-1)x^{n-2} + \&c.\right\} \varepsilon^{ax},$$

$$= \frac{x^{n-r}}{a^{n-r}} \left\{a^n x^r + na^{n-1}rx^{r-1} + \frac{n(n-1)}{1.2} a^{n-2}r(r-1)x^{r-2} + \&c.\right\} \varepsilon^{ax},$$

$$= \frac{x^{n-r}}{a^{n-r}} \left\{d^n + nd^{n-1}d' + \frac{n(n-1)}{1.2} d^{n-2}d'^2 + \&c.\right\} \frac{(\varepsilon^{ax}x^r)}{dx^n},$$

$$= \frac{x^{n-r}}{a^{n-r}} \left(\frac{d}{dx}\right)^n (\varepsilon^{ax}x^r),$$

by the theorem of Leibnitz. This theorem is true when one or both of the quantities n and r are fractional, so that we thus arrive at the curious result, that a differential to a fractional index may be expressed by means of a differential to an integer index of a function of the same form.

* *Cambridge Mathematical Journal*, Vol. II., p. 284.

NOTE ON A CLASS OF FACTORIALS.*

WE owe to Vandermonde the interesting Theorem, that Binomial factorials of any order, in which the successive factors differ by a constant quantity, can be expanded in terms of the simple Monomial factorials according to the law of the expansion of Newton's Binomial Theorem. That is to say, that if we put

$$x(x-1)(x-2)\ldots(x-n+1) = x^{|n},$$

we shall have

$$(x+y)^{|n} = x^{|n} + nx^{|n-1}y + n\frac{(n-1)}{1.2}x^{|n-2}y^{|2} + \&c.$$

This proposition, which may be proved by various methods, is readily seen to depend on the fact that these factorials are subject to the laws of combination in virtue of which the Theorem of Newton, as applied to ordinary algebraical quantities, is true. And perhaps the Theorem of Vandermonde derives its chief value from its being one of the few examples which we have of the extension of Algebraical Theorems to operations not originally included in the demonstration. The other examples which are known, are the Theorems in the Differential Calculus and the Calculus of Finite differences, which are proved by the method of the separation of the symbols.

Between these however and the Theorem of Vandermonde, there is one marked point of distinction: for whereas in the

* *Cambridge Mathematical Journal*, Vol. III., p. 89.

former Theorems the operation which is subject to the index-operation is different from that which forms the staple of ordinary algebra, while the index-operation is always the same, viz. the operation of repetition, in the latter Theorem the base which is subject to the index-operation is or may be the same as that in ordinary algebra, while the index-operation is different. As any Theorem which will add to examples of this kind must contribute to extend our knowledge of the combination of symbols in the direction in which such an extension seems to be most important, I will offer no apology for occupying a page or two, in demonstrating that a Theorem similar to that of Vandermonde is true of a class of factorials different from that of which he has treated. The factorials to which I allude, are those which are met with in expanding the cosine or the sine of a multiple arc according to the powers of the cosine or sine of the arc itself. These factorials, which are of a somewhat remarkable form, have, like ordinary factorials, an analogy with powers, and the proposition of which I speak is an example of this analogy.

On referring to the *Cambridge Mathematical Journal*, Vol. II., p. 129,* the reader will find the following expressions for $\cos n\theta$ and $\sin n\theta$ in terms of $\sin\theta$ when n is an integer,

$$\cos n\theta = \cos n\pi \left\{1 - \frac{n^2}{1.2}v^2 + \frac{n^2(n^2-2^2)}{1.2.3.4}v^4 - \frac{n^2(n^2-2^2)(n^2-4^2)}{1.2.3.4.5.6}v^6 + \&c.\right\},$$

$$\sin n\theta = \cos(n-1)\pi \left\{nv - \frac{n(n^2-1^2)}{1.2.3}v^3 + \frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}v^5 - \&c.\right\},$$

v being written for $\sin\theta$.

Now to exhibit the analogy which the factorials, which are the coefficients of the various terms in these expressions, have with powers, let us represent them by a notation corresponding to that of ordinary factorials, and let us write

$$n = n_{|1}, \; n^2 = n_{|2}, \; n(n^2-1^2) = n_{|3}, \; n^2(n^2-2^2) = n_{|4}, \; \&c.;$$

* See p. 187 of this volume.

and generally

$n_{|r} = n^2 (n^2 - 2^2)(n^2 - 4^2)(n^2 - 6^2) \ldots \{n^2 - (r-2)^2\} \ldots r$ being even,

$n_{|r} = n . (n^2 - 1^2)(n^2 - 3^2) \ldots\ldots\ldots\ldots \{n^2 - (r-2)^2\} \ldots r$ being odd.

This notation being employed, the preceding expressions may be written

$$\cos n\theta = (-)^n \left\{1 - n_{|2} \frac{v^2}{1.2} + n_{|4} \frac{v^4}{1.2.3.4} - n_{|6} \frac{v^6}{1.2.3.4.5.6} + \&c.\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\sin n\theta = (-)^{n-1} \left\{n_{|1} v - n_{|3} \frac{v^3}{1.2.3} + n_{|5} \frac{v^5}{1.2.3.4.5} - \&c.\right\} \ldots\ldots\ldots (2).$$

Now the proposition which we have to demonstrate may be expressed, by means of this notation, in the following manner:

$$(m+n)_{|p} = m_{|p} + p m_{|p-1} n_{|1} + \frac{p(p-1)}{1.2} m_{|p-2} n_{|2} + \&c. + n_{|p}.$$

In the demonstration we must distinguish two cases according as p is even or odd.

1st. Let p be even and $= 2r$; then putting $m+n$ instead of n in the series (1) we have

$$\cos(m+n)\,\theta = (-)^{m+n} \left\{1 - (m+n)_{|2} \frac{v^2}{1.2} + \&c.\right.$$
$$\left. + (-)^r (m+n)_{|2r} \frac{v^{2r}}{1.2\ldots r} - \&c.\right\} \ldots\ldots (3).$$

But by an ordinary formula of Trigonometry we have

$$\cos(m+n)\,\theta = \cos m\theta \cos n\theta - \sin m\theta \sin n\theta.$$

In this formula, if we substitute for the cosines and sines their equivalent series given in (1) and (2), and if we equate the coefficients of v^{2r}, and then multiply both sides of the equation by $1.2\ldots 2r$, we find

$$(m+n)_{|2r} = m_{|2r} + 2r m_{|2r-1} n_{|1} + \frac{2r(2r-1)}{1.2} m_{|2r-2} n_{|2} + \&c. + n_{|2r},$$

which proves the theorem when p is even.

2nd. Let p be odd and $=2r+1$; then, by means of the series (2) and the formula

$$\sin(m+n)\,\theta = \sin m\theta\,\cos n\theta + \cos m\theta\,\sin n\theta,$$

we find on equating the coefficients of v^{2r+1}, and multiplying both sides of the equation by $1.2.3\ldots(2r+1)$,

$$(m+n)_{|2r+1} = m_{|2r+1} + (2r+1)\,m_{|2r}n_{|1} + \frac{(2r+1)\,2r}{1.2}\,m_{|2r-2}n_{|2} + \&c.$$

which proves the theorem when p is odd.

It is easily seen that this result applies equally to factorials of the form

$$n^2(n^2-2^2h^2)(n^2-4^2h^2)\ldots\{n^2-(r-2)^2h^2\},$$

since this last may be written under the form

$$h^r\frac{n^2}{h^2}\left(\frac{n^2}{h^2}-2^2\right)\left(\frac{n^2}{h^2}-4^2\right)\ldots\left\{\frac{n^2}{h^2}-(r-2)^2\right\},$$

which, with the exception of the factor h^r, is the same in form as the factorials which we considered before.

I have not time at present to enter into any further developments of the nature of these factorials, and more particularly into the consideration of their interpretation when the index is negative or fractional: but this is of the less importance, as it is not very likely that expressions of this form will ever be extensively used in analysis; and the demonstration of the preceding theorem is given, not on account of its intrinsic value, but because it illustrates a part of the theory of Algebra which stands most in need of such examples.

DEMONSTRATIONS
OF SOME GEOMETRICAL THEOREMS.*

THE following geometrical theorems may perhaps be interesting to some of the readers of the *Mathematical Journal.* They are founded on the decomposition into quadratic factors of the trinomial $x^{2n} - 2a^n x^n \cos\theta + a^{2n}$, and are therefore intimately related to the well-known theorem of Cotes. We assume the theorem

$$x^{2n} - 2a^n x^n \cos\theta + a^{2n} = \left(x^2 - 2ax\cos\frac{\theta}{n} + a^2\right)\left(x^2 - 2ax\cos\frac{\theta+2\pi}{n} + a^2\right)$$
$$\ldots\left\{x^2 - 2ax\cos\frac{\theta + (n-1)\pi}{n} + a^2\right\} \ldots\ldots\ldots (A).$$

whence also, by making $x = a$, and writing $n\phi$ for $\frac{\theta}{2}$, we have the known theorem

$$\sin n\phi = 2^{n-1}\sin\phi\sin\left(\phi + \frac{\pi}{n}\right)\sin\left(\phi + \frac{2\pi}{n}\right)\ldots\sin\left(\phi + \frac{n-1}{n}\pi\right)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B).$$

If a regular polygon of $2n$ sides circumscribe a circle, and if $p_1 p_2 \cdots p_{2n-1} p_{2n}$ be the perpendiculars drawn on its sides from any point in the circumference of the circle, then

$$p_1 p_2 \cdots p_{2n-1} + p_2 p_n \cdots p_{2n} = \frac{r^n}{2^{n-2}},$$

where r is the radius of the circle.

* *Cambridge Mathematical Journal,* Vol. III., p. 144.

Let the arc between the assumed point and the adjacent point of contact subtend an angle ϕ at the centre; then

$$p_1 = r(1-\cos\phi) = 2r\sin^2\frac{\phi}{2}$$

$$p_2 = r\left\{1-\cos\left(\phi+\frac{2\pi}{2n}\right)\right\} = 2r\sin^2\left(\frac{\phi}{2}+\frac{\pi}{2n}\right),$$

$$p_3 = 2r\sin^2\left(\frac{\phi}{2}+\frac{\pi}{n}\right),$$

$$p_4 = 2r\sin^2\left(\frac{\phi}{2}+\frac{3\pi}{2n}\right), \text{ \&c., \&c.}$$

Hence

$$p_1p_3\cdots p_{2n-1} = 2^n r^n \sin^2\frac{\phi}{2}\sin^2\left(\frac{\phi}{2}+\frac{\pi}{n}\right)\ldots\sin^2\left(\frac{\phi}{2}+\frac{n-1}{n}\pi\right),$$

and by the theorem (B), writing $\frac{1}{2}\phi$ for ϕ, we have

$$p_1p_3\cdots p_{2n-1} = \frac{r^n}{2^{n-2}}\sin^2\frac{n\phi}{2}.$$

In the same way we find

$$p_2p_4\cdots p_{2n} = \frac{r^n}{2^{n-2}}\sin^2 n\left(\frac{\phi}{2}+\frac{\pi}{2n}\right) = \frac{r^n}{2^{n-2}}\cos^2 n\frac{\phi}{2},$$

and therefore $p_1p_3\cdots p_{2n-1} + p_2p_4\cdots p_{2n} = \dfrac{r^n}{2^{n-2}}$ .... (1).

Also, by multiplying the preceding expressions together,

$$p_1p_2\cdots p_{2n-1}p_{2n} = \frac{r^{2n}}{2^{2n-2}}\sin^2 n\phi \text{ ....(2).}$$

If the given point be within the circumference of the circle, and if c be its distance from the centre, and ϕ the angle which c makes with the radius drawn to the adjacent point of contact, we have

$$p_1 = r - c\cos\phi = x^2 - 2xy\cos\phi + y^2,$$

if $$x^2+y^2 = r, \text{ and } 2xy = c.$$

There are corresponding expressions for the other perpendiculars, and therefore

$$p_1p_3\cdots p_{2n-1} = (x^2-2xy\cos\phi+y^2)\ldots\left\{x^2-2xy\cos\left(\phi+\frac{n-1}{n}\pi\right)+y^2\right\}$$

$$= x^{2n} - 2x^n y^n\cos n\phi + y^{2n}, \text{ by } (A).$$

Similarly

$$p_2 p_4 \cdots p_{2n} = x^{2n} - 2x^n y^n \cos(n\phi + \pi) + y^{2n} = x^{2n} + 2x^n y^n \cos n\phi + y^{2n};$$

hence by subtraction,

$$p_2 p_4 \cdots p_{2n} - p_1 p_3 \cdots p_{2n-1} = 4x^n y^n \cos n\phi = \frac{c^n}{2^{n-2}} \cos n\phi \ldots (3).$$

If from the given point within the circumference, we draw perpendiculars $q_1, q_2 \ldots q_{2n}$ on the radii drawn to the points of contact, we have

$$q_1 = c \sin\phi, \quad q_2 = c \sin\left(\phi + \frac{\pi}{n}\right) \ldots q_{2n} = c \sin\left(\phi + \frac{2n-1}{n}\pi\right),$$

and therefore, considering magnitude only,

$$q_1 q_2 \ldots q_{2n} = \frac{c^{2n}}{2^{2n-2}} \sin^2 n\phi \ldots\ldots\ldots\ldots\ldots (4).$$

Mr. Leslie Ellis has shown (*Cambridge Mathematical Journal*, Vol. II., p. 272), that, if $f(\phi)$ be a rational and integral function of $\sin\phi$ and $\cos\phi$, in which the highest power of these quantities is r,

$$f(\phi) + f\left\{\phi + \frac{2\pi}{n}\right\} \ldots + f\left\{\phi + \frac{n-1}{n} 2\pi\right\} = \frac{n}{2\pi} \int_0^{2\pi} fx\,dx,$$

when $n > r$. By similar reasoning we may show that, when $n = r$,

$$f(\phi) + \ldots + f\left\{\phi + \frac{n-1}{n} 2\pi\right\}$$

$$= \frac{n}{2\pi} \int_0^{2\pi} f(x)\,dx + \frac{n}{\pi} \int_0^{2\pi} dx f(x) \cos nx . \cos n\phi,$$

$$+ \frac{n}{\pi} \int_0^{2\pi} dx f(x) \sin nx . \sin n\phi.$$

Now if $p_1 p_2 \ldots p_n$ be the perpendiculars drawn from any point in the circumference of a circle, on a regular circumscribing polygon of n sides, we have

$$p_1^n = r^n (1 - \cos\phi)^n, \quad p_2^n = r^n \left\{1 - \cos\left(\phi + \frac{2\pi}{n}\right)\right\}^n, \text{ \&c.}$$

Therefore

$$\frac{n}{2\pi}\int_0^{2\pi} dx f(x) = r^n \int_0^{2\pi} dx\,(1-\cos x)^n = \frac{(2n-1)(2n-3)\ldots 3.1}{(n-1)(n-2)\ldots 2.1}\,r^n,$$

$$\frac{n}{\pi}\int_0^{2\pi} dx f(x)\cos nx = \frac{n}{\pi}\,r^n \int_0^{2\pi} dx\,(1-\cos x)^n \cos nx = (-)^n \frac{n}{2^{n-1}},$$

and $$\frac{n}{\pi}\int_0^{2\pi} dx\,(1-\cos nx)^n \sin nx = 0.$$

These results are easily arrived at, by observing that

$$(1-\cos x)^n = (-)^n \{(\cos x)^n - n(\cos x)^{n-1} + \&c.\},$$

$$= (-)^n \left\{\frac{\cos nx}{2^{n-2}} - \&c.\right\}.$$

Hence,

$$\Sigma(p^n) = r^n \left\{\frac{(2n-1)(2n-3)\ldots 3.1}{(n-1)(n-2)\ldots 2.1} + (-)^n \frac{n}{2^{n-1}}\cos n\phi\right\};$$

but in the same manner as in (1), we find

$$p_1 p_2 \ldots p_n = \frac{r^n}{2^{n-2}}\sin^2 n\frac{\phi}{2} = \frac{r^n}{2^{n-1}}(1-\cos n\phi).$$

Therefore

$$\Sigma(p^n) + (-)^n p_1 p_2 \ldots p_n = r^n \left\{\frac{(2n-1)(2n-3)\ldots 3.1}{(n-1)(n-2)\ldots 2.1} + (-)^n \frac{n}{2^{n-1}}\right\} \quad \ldots\ldots\ldots\ldots(5).$$

In the same way, if c_1, c_2, &c. be the chords drawn from the given point to the angle of a regular inscribed polygon, we have

$$c_1^2 = 2r^2(1-\cos\phi), \ \&c.,$$

and therefore

$$\Sigma(c^{2n}) + (-)^n (c_1, c_2 \ldots c_n)^2$$

$$= r^n \left\{\frac{2^n(2n-1)(2n-3)\ldots 3.1}{(n-1)(n-2)\ldots 2.1} + (-)^n 2n\right\} \ldots\ldots(6).$$

Theorems of this kind are not confined to the circle: thus, in the parabola, if n tangents be drawn such that the arcs between the points of contact subtend equal angles at the focus, and if $p_1, p_2 \ldots p_n$, be the perpendiculars on them from the focus, we shall have

$$p_1 . p_2 \ldots p_n = 2^{n-1} a^n \operatorname{cosec}\frac{n\theta}{2},$$

a being one-fourth of the parameter, and θ being the angle which the axis of the parabola makes with the radius vector drawn to the adjacent point of contact. For the equation to the parabola being

$$\frac{1}{r}=\frac{1-\cos\theta}{2a}=\frac{1}{a}\sin^2\frac{\theta}{2},$$

we have
$$\frac{1}{p_1^2}=\frac{1}{ar}=\frac{1}{a^2}\sin^2\frac{\theta}{2},$$

or
$$\frac{1}{p_1}=\frac{1}{a}\sin\frac{\theta}{2},$$

and similarly for the others. Hence

$$\frac{1}{p_1.p_2\dots p_n}=\frac{1}{a^n}\sin\frac{\theta}{2}\sin\left\{\frac{\theta}{2}+\frac{\pi}{n}\right\}\dots\sin\left\{\frac{\theta}{2}+\frac{n-1}{n}\pi\right\}$$
$$=\frac{1}{2^{n-1}a^n}\sin\frac{n\theta}{2};$$

and therefore
$$p_1.p_2\dots p_n=2^{n-1}a^n\operatorname{cosec}\frac{n\theta}{2}\ \dots\dots\dots\dots\dots(7).$$

The theorems (1) and (5) were discovered by Professor Wallace, of Edinburgh, about the year 1791, but they have not, we believe, been as yet published.

ON A DIFFICULTY IN THE THEORY OF ALGEBRA.*

I OUGHT perhaps to apologize to the reader for calling his attention to a subject so much discussed as the nature of the symbols + and − when used in symbolical Algebra. But the theory which I wish to develope appears to me to remove some part of the difficulties which, after all which has been written, still adhere to the subject; and I am the more anxious to explain my views, because I have in previous papers held, in common I believe with every other writer, an opinion which a more attentive consideration induces me to think erroneous. It is generally assumed that the symbols + and − signify primarily addition and subtraction, and that any other meanings which we may attach to them must be derived from the fundamental significations. The theory which I have now to maintain is the apparently paradoxical one, that the symbols + and − do not represent the arithmetical operations of addition and subtraction; and that though they were originally intended to bear these meanings, they have become really the representatives of very different operations. This is opposed to our preconceived ideas, but I trust that the following statements will show the truth of the assertion.

When it is said that any symbol does not represent a particular operation, it is necessary to explain what is to be

* *Cambridge Mathematical Journal*, Vol. III., p. 153.

understood by an *Algebraical* symbol, and in what way it represents an operation. In previous papers on the Theory of Algebra, I have maintained the doctrine that a symbol is defined *algebraically* when its laws of combination are given; and that a symbol represents a given operation when the laws of combination of the latter are the same as those of the former. This, or a similar theory of the nature of Algebra, seems to be generally entertained by those who have turned their attention to the subject: but without in any degree leaning on it, we may say that symbols are actually subject to certain laws of combination, though we do not suppose them to be so defined; and that a symbol representing any operation must be subject to the same laws of combination as the operation it represents. These assertions are independent of any theory except this, that there is a general Symbolical Algebra different from Arithmetical Algebra, in which the laws of the combination of the symbols are attended to. When therefore I say that the symbol $+$ does not represent the arithmetical operation of addition, or $-$ that of subtraction, I mean that the laws of combination of the symbols $+$ and $-$ are not those of the operations of addition and subtraction. Now the laws of combination of the symbols $+$ and $-$ are four, viz.

$$++a=+a, \quad +-a=-a, \quad -+a=-a, \quad --a=+a;$$

and whatever may be our theory of the meaning of these symbols, there is no doubt that we always assume them to be subject to these laws, which are indeed the very first steps which the student in algebra makes. But no one who considers the nature of the arithmetical operations can hesitate for a moment to say, that they are not subject to these laws; of which a sufficient proof is this, that addition and subtraction are inverse operations, whereas the second and third of the preceding laws are inconsistent with the idea that $+$ and $-$ are inverse symbols, the character of which is, that the one undoes what the other does; so that if f, ϕ are two symbols representing inverse operations, we have

$$f\phi(a) = a \text{ and } \phi f(a) = a.$$

The same conclusion also follows from the fourth law, which is evidently analogous to the relation between a number and its square, not to that between a number and its reciprocal. A natural objection which may be brought against the view which I am here maintaining is, that we do actually define $+$ and $-$ to be the symbols representing addition and subtraction, and therefore that they must represent these operations. A further examination however will shew that this is not the case. We say that $a+x$ is to represent x added to a, and $a-x$, x subtracted from a: we do not directly assert that $+$ signifies addition, and $-$ subtraction. If we did we should contradict ourselves, when we asserted that $+-a=-a$, or $--a=+a$. The fact is, that we are deceived by writing a *sum* and *difference* in a manner different from that in which we express the performance of any other operation. When we wish to denote that an operation f is performed on a subject a, we usually prefix the symbol of operation to the subject, and write $f(a)$: this however is not necessary, for we might connect them in any other way; and indeed Mr. Murphy prefixes the subject to the symbol of operation, apparently for the purpose of avoiding the prejudices which our ordinary mode of writing is apt to produce. But, on the other hand, when we wish to denote the addition of a number x to a number a, or its subtraction from it, we write

$$a+x \text{ or } a-x,$$

the operations being indicated by writing the symbol of the number added or subtracted *after* the subject, and separated from it by certain symbols; whereas in the ordinary mode of writing we should *prefix* the operating symbol. Now if we merely assert that $a+x$ is to signify the addition of x to a, the symbol + might be understood to *indicate* addition, it being used to distinguish the kind of operation involving x which is performed on a, in the same way as $a \times x$ indicates the multiplication of a by x. But by such an assertion we do not make + an *algebraical* symbol in the sense in which I use the word; nor does it *represent* the operation, though it may indicate it. It is only when we arrive at such conclusions as $a+(x+y)=a+x+y$, involving the law $++a=+a$, that we give to + an algebraical individuality as a symbol subject to certain laws of combination, which, we see at once, are not those belonging to the operation of addition. If on this any one chooses to say that he considers + as indicating (he cannot say *representing*) the operation of addition, and that he does not trouble himself about its laws of combination, there can be no objection to his holding such an opinion except this, that the Algebra in which such a symbol is used is not a general science, but simply Arithmetic. He cannot, consistently with this doctrine, hold that direction in Geometry can be indicated by + and –; or that these symbols can receive any other interpretation than that which was originally assigned to them: conclusions inconsistent with any conceptions which we can form of a *general* Algebra. There is no doubt that we *can* give these symbols such a geometrical interpretation, and it is the possibility of so doing which has occasioned the difficulties of the Theory of Algebra considered as something more general than Arithmetic, and which has led to the more extended views which in recent years have been taken of the subject.

The preceding observations may be illustrated by using new symbols to represent the operations of addition and subtraction, by prefixing them in the ordinary way to the subject, and so investigating their laws of combination. Let us assume the symbol A to be that which represents addition, and B that of subtraction, and let us attach to them as a suffix the quantity which is added or subtracted;

so that $A_x(a)$ and $B_x(a)$ represent the addition of x to a and the subtraction of x from a. One of the most obvious laws of the operation of addition is, that if x and y be two quantities which are added to a, it is indifferent in what order the operations are performed: so that if we first add x to a, and then y to the sum, we obtain the same result as if we first added y to a, and then x to the sum. This law is expressed by the equation

$$A_x A_y(a) = A_y A_x(a);$$

that is, the operations A_x, A_y are commutative.

Again: another law is, that each of these sums is the same as if y were first added to x, and then that sum added to a. Now the sum of y and x is represented by $A_y(x)$, and therefore the addition of this sum to a is represented by $A_{A_y(x)}(a)$; so that the law in question is expressed by the equation

$$A_x A_y(a) = A_{A_y(x)}(a).$$

The same law may be extended to any number of variables x, y, z, so that we have the equation

$$A_x A_y A_z = A_{A_{A_z(y)}(x)}(a), \text{ and so on.}$$

The notation is an inconvenient one: but it is not here introduced for the purpose of supplanting the common one, but merely to show how the laws of combination of the operation of addition may be represented by one symbol only, without the aid of a subsidiary symbol such as $+$.

A third law of the operation of addition is, that it is indifferent whether x be added to a, or a to x: this is expressed by the equation

$$A_x(a) = A_a(x).$$

These three laws of combination are sufficient for our purpose, and show distinctly in a symbolical form how different the laws of the operation of addition are from the laws of combination of the symbol $+$, which therefore cannot represent it.

With regard to subtraction, since it is the operation which is inverse to addition, we have plainly

$$A_x B_x (a) = B_x A_x (a) = a.$$

Also, without going into detail, it is easy to see that

$$B_x A_y (a) = A_{B_x(y)} (a) = B_{B_y(x)} (a);$$

and thus the laws of the operation of subtraction are represented by means of the symbols of addition and subtraction.

By means of this notation we see distinctly how the symbol $+$ appears frequently as a *separative* symbol between two others, in such a way that the order of the symbols cannot be changed. Thus we cannot say $+ax$ instead of $a+x$, though we do say that $a\sqrt{(-1)}\,x$, or $a(-)^{\frac{1}{2}}x$, is equivalent to $(-)^{\frac{1}{2}}ax$: for $a+x$ is equivalent to $A_x(a)$, whereas $a(-)^{\frac{1}{2}}x$ signifies only the successive performance of these operations which are commutative; the symbol $\sqrt{(-1)}$, or $(-)^{\frac{1}{2}}$, not having acquired the double signification which is attached in consequence of their position, to $+$ or $-$. In the same way, though we say that $a+(-x)=a-x$, we do not say that $a+\sqrt{(-1)}\,x$, or $a+[(-)^{\frac{1}{2}}x]=a(-)^{\frac{1}{2}}x$, because these two formulæ are equivalent to $A_{-x}(a)$ and $A_{(-)^{\frac{1}{2}}x}(a)$, and the law that $A_{-x}(a)=B_x(a)$ has no analogue in the case of $(-)^{\frac{1}{2}}$, or other powers of $+$ and $-$.

The distributive law, which is met with so frequently in algebraical operations, and which is usually written

$$f(a+x)=f(a)+f(x),$$

is in this notation expressed by the equation

$$f[A_x(a)] = A_{f(x)}[f(a)].$$

In like manner the index law, which is usually written

$$f^m f^n (a) = f^{m+n} (a),$$

becomes in this notation

$$f^m f^n (a) = f^{A_n(m)} (a).$$

To one who is acquainted with the higher branches of mathematics it is obvious that the operation, which is here denoted by A, is the same in kind as that of which so much use is made in the Calculus of Finite Differences for converting $f(x)$ into $f(x+h)$, and which has been represented by the symbol D^h in the papers on the subject in preceding pages of this volume.

Before concluding I would say a few words on what appears to me to be a prejudice relative to the nature of the symbols $+$ and $-$. These are generally considered to be absolutely distinct from literal symbols, and have in consequence a different name assigned to them, being called "signs of affection." Such a distinction exists in arithmetical, but not in general Algebra. In the former, literal symbols are used to represent numbers or magnitudes, and are capable of receiving interpretation, that is, of having different meanings or values assigned to them, while the signs of affection indicate the performance of certain operations, and are incapable of bearing any other meaning than those which are originally assigned to them. As such, the signs $+$ and $-$ are exactly on a par with $\times$ and $\div$, though the latter, from accidental circumstances, have not become so important as the former. When we write the symbol a in Arithmetical Algebra, we mean that we may substitute for it any number we choose; but when we write $a+b$, we say that b is added to a, we attach to $+$ a definite meaning, and we can give no other interpretation to it, without taking into consideration its laws of combination, which are excluded from Arithmetical Algebra. On the other hand, in Symbolical Algebra, where every symbol represents an operation, it is obvious that we cannot *à priori* speak of any difference in kind between different symbols. In such a science what is a, and what is $+$? To neither are definite meanings attached, as to the latter symbol in Arithmetical Algebra. Our conceptions would be clearer, and our minds more free from prejudice, if we never used in the general science symbols to which definite meanings had been appropriated in the particular science. Inveterate practice has however so wedded us to the use of the symbols + and − that we find it difficult to dispense with them, and still more difficult, in using them, to avoid being misled by ideas drawn from Arithmetic. The symbols + and −, × and ÷, were invented for the purpose of indicating the performance of certain operations on numbers; but as the science advanced, it was found that the symbol × might be conveniently omitted, the operation being indicated merely by the juxtaposition of symbols; so that ax stood for $a \times x$. From this the transition was easy to the conception of a as the symbol of the operation; a change of great importance, as leading to the view that Symbolical Algebra is a Calculus of Operations. But it is merely a matter of accident that the symbol × was that which was expunged: that fate might as well have befallen the symbol +, and then ax would have signified the addition of a to x, and the difficulties which have been experienced regarding + and − would then have been transferred to × and ÷. It is perhaps, to a certain extent, unfortunate that we have in multiplication represented by one letter the symbol of the operation as well as that with respect to which it is performed: the latter ought rather to be attached as an index or suffix. Thus, if we represented the multiplication of x by a by the symbol $P_a(x)$, we should have no difficulty in seeing that it was exactly analogous to addition under the notation $A_a(x)$, as I have in this paper written it.

In the preceding remarks I have proceeded on the supposition that Symbolical Algebra must be considered as a science of operations represented symbolically: this view may not appear to every one necessary; but if the subject be considered in all its generality, it will, I am convinced, be found that there is no other way of explaining the difficulties of Algebra in a uniform and consistent manner.

ON CERTAIN CASES OF GEOMETRICAL MAXIMA AND MINIMA.*

WE occasionally meet in Geometry with certain cases of maxima or minima, for which the ordinary analytical process appears to fail, though from geometrical considerations it is obvious that maxima or minima do exist. The explanation of this failure is not given in works on the Differential Calculus, and some notice of it here may be acceptable to our readers. The difficulty and its explanation will be best seen in an example, and none is better suited for the purpose than a question proposed in one of the papers of the Smith's prize examination for 1842. This was—To explain the cause of the failure of the ordinary method of finding maxima and minima, when applied to the problem of finding the greatest or least perpendicular from the focus on the tangent to an ellipse, the perpendicular being expressed in terms of the radius vector.

The usual expression for the perpendicular in terms of the radius vector is

$$p^2 = \frac{b^2 r}{2a - r};$$

and as p^2 will be a maximum or minimum when p is so, the ordinary rule for finding maxima and minima gives us

$$\frac{d}{dr}(p^2) = \frac{2ab^2}{(2a-r)^2} = 0.$$

* *Cambridge Mathematical Journal*, Vol. III., p. 237.

Now this equation can be satisfied only by $r = \pm \infty$, values which are not admissible in this case; whereas we know from geometry that p is a minimum when $r = a(1-e)$, and a maximum when $r = a(1+e)$.

It would appear then that these two values are not given by the analytical process, and the cause of this exception is to be explained. In the general theory of maxima and minima, it is assumed that the independent variable may receive all possible values; whereas in the present case r is limited to those values which are found by assigning all possible values to θ in the expression

$$r = \frac{a(1-e^2)}{1 - e\cos\theta}:$$

in other words, r is not absolutely independent. Now r being a function of another variable, admits itself of maximum and minimum values; and these are the values for which p is a maximum or minimum. The cause of the failure may therefore be thus exhibited: the equation

$$d(p^2) = \frac{2ab^2dr}{(2a-r)^2} = 0$$

is satisfied by $dr = 0$, that is, by making r a maximum or minimum. Hence generally, if we wish to find the maximum and minimum values of $y = f(x)$, we must consider, not only the values of x which satisfy the equation $\frac{dy}{dx} = 0$, but also the maximum and minimum values of x itself.

In Liouville's *Journal*, Vol. VII., p. 163, there is given a similar case of failure of the analytical process in the problem—To draw the shortest or longest line to a circle from a point without it. If we take the line passing through the centre of the circle, and the given point O as the axis of x, and call a the distance of the point from the centre, c the radius of the circle, x the coordinate of any point P in the circle measured from the centre, we shall have

$$OP^2 = a^2 + c^2 - 2ax, \text{ a maximum or minimum;}$$

from which the usual process would give

$$\frac{d}{dx}(OP^2) = -2a = 0,$$

a nugatory result. In this case x is a maximum or minimum, while OP is a minimum or maximum, and therefore the equation to be satisfied is

$$d.(OP^2) = -2adx = 0,$$

which is satisfied by $dx = 0$.

In this example the difficulty may be avoided by taking our coordinates generally, so that x shall not be a maximum or minimum when OP is so. We shall then have, calling a and b the coordinates of the centre of the circle, the other quantities as before,

$$OP^2 = a^2 + b^2 + c^2 - 2b\sqrt{(c^2 - x^2)} - 2ax = \text{maximum};$$

whence, by the usual process,

$$\frac{bx}{\sqrt{(c^2 - x^2)}} = a,$$

and
$$x = \pm \frac{ac}{\sqrt{(a^2 + b^2)}},$$

giving the two values of x, which will solve the problem.

A very instructive example will be found in the problem—To find those conjugate diameters in an ellipse of which the sum is a maximum or a minimum. If r and r_1 be any two conjugate diameters, a, b the axes, we have

$$r + r_1 = \text{maximum or minimum},$$

$$r^2 + r_1^{\,2} = a^2 + b^2 = c^2 \text{ suppose},$$

so that
$$r + \sqrt{(c^2 - r^2)} = \text{maximum or minimum}.$$

From this we have

$$dr\left\{1 - \frac{r}{\sqrt{(c^2 - r^2)}}\right\} = 0.$$

This equation is satisfied either by

$$r = \sqrt{(c^2 - r^2)}, \quad \text{i.e. by } r = \frac{c}{\sqrt{2}} = r_1,$$

or by
$$dr = 0, \text{ which involves } dr_1 = 0.$$

The former of these results gives the equal conjugate diameters, the sum of which is, as we know, a maximum. The latter result implies that both r and r_1 are maxima or minima, or that they are the principal axes, the sum of which is a minimum. By a different method we might have obtained the minimum instead of the maximum value of $r + r_1$, by the usual process for determining maxima and minima. For since $r^2 + r_1^2 = a^2 + b^2$ and $rr_1 \sin\theta = ab$, θ being the angle between the axes, we have

$$(r + r_1)^2 = a^2 + b^2 + \frac{2ab}{\sin\theta},$$

and hence $$\frac{d}{d\theta}(r + r_1)^2 = -\frac{2ab\cos\theta}{(\sin\theta)^2} = 0.$$

This is satisfied by $\cos\theta = 0$ or $\theta = \frac{1}{2}\pi$, implying that r and r_1 are the principal axes. In this case the maximum value of $r + r_1$ is given by $d\theta = 0$, since the equal conjugate diameters are those which make the greatest angle with each other.

ON THE SOLUTION OF CERTAIN FUNCTIONAL EQUATIONS.*

IN the *Cambridge Mathematical Journal*, Vol. III., p. 92, Mr. Leslie Ellis pointed out what appeared to him to be the essential difference between Functional Equations and those which are usually met with in the various branches of analysis. His idea is, that these classes of equations are distinguished by the *order* in which the operations are performed, so that, whereas in our ordinary equations the known operations are performed on those which are unknown, in functional equations the converse is the case, the unknown operations being performed on those which are known. As this view appears to me to be not only correct, but of very great importance for the proper understanding of the higher departments of analysis, I shall endeavour in the following pages to enforce and illustrate it.

On the preceding theory it is easy to see *why* the solution of functional equations must involve difficulties of a higher order than that of equations of the other class. For if we consider an equation as a series of operations performed on a subject, the operations being known and the subject unknown, the solution of the equation involves the finding of the subject, which may be done theoretically by undoing the operations which have been performed on it; that is, by effecting on the second side the inverse of the

* *Cambridge Mathematical Journal*, Vol. III., p. 239.

known operations on the first side. Thus, if we have the equation

$$\frac{dy}{dx} - ay = 0,$$

or, as we may write it,

$$\left(\frac{d}{dx} - a\right) \phi(x) = 0,$$

the object is to find the form of $\phi(x)$, which is readily done by performing the operation $\left(\frac{d}{dx} - a\right)^{-1}$ on both sides, when we have

$$\phi(x) = \left(\frac{d}{dx} - a\right)^{-1} 0 = C\varepsilon^{ax}.$$

Here the whole difficulty lies in the performing of the inverse operation; and although practically the difficulty of doing so may be very great, yet it is a difficulty wholly different in *kind* from that which we meet with in trying to solve an equation in which the unknown operation is performed on that which is known. We have then no direct means of disengaging the unknown from the known operations, as the inverse of an unknown operation of course cannot be performed, and the known operation, being the subject, cannot be directly separated from the equation. Thus in the equation

$$\phi(ax) - \phi(x) = 0,$$

where the object is to determine the form of ϕ, we cannot as before write

$$\phi(ax - x) = 0,$$

since the form of ϕ is unknown, and we therefore cannot assume it to be subject to the distributive law; neither can we write

$$a\phi(x) - \phi(x) = 0,$$

since we cannot assume that ϕ and a are commutative operations.

The method which is followed for the solution of certain functional equations, is indicated by the process for the solution of linear equations in Finite Differences, which are in fact functional equations of a particular form. Thus the equation

$$u_{x+1} - au_x = 0$$

might be written

$$\phi(x+1) - a\phi(x) = 0,$$

in which the form of ϕ is to be determined.

Here the known operations are the subjects of the unknown, and we cannot directly disengage them; but we are enabled to do so by transforming the equation into one in which the unknown operation ϕ is the subject. For, assuming the operation D to be such that

$$D\phi(x) = \phi(x+1),$$

we are able to investigate the laws of combination of this new symbol and its various properties, so as to make it a known operation. The equation then becomes

$$D\phi(x) - a\phi(x) = 0.$$

Now we can shew that D is a distributive symbol with respect to its subject, and that it is commutative with respect to a; we may therefore write the equation in the form

$$(D - a)\ \phi(x) = 0,$$

whence

$$\phi(x) = (D - a)^{-1}\, 0.$$

For the complete solution, there remains only that we should know the inverse operations of $D - a$ or D, and these are found from the investigation of its direct action. The result, as we know, is

$$\phi(x) = Ca^x.$$

It is useless here to show how this theory may be extended to the solution of general linear equations in finite differences, as that has been sufficiently developed in other places: we shall therefore pass on to show that the same method may be applied to other functional equations.

Let us suppose ω to be any known operation performed on x, so that $\omega(x)$ is a known function of x, and let ϕ be an unknown operation; then the equation

$$\phi(\omega^n x) + a_1\phi(\omega^{n-1}x) + a_2\phi(\omega^{n-2}x) + \&c. + a_n\phi(x) = X,$$

in which $a_1, a_2, \ldots a_n$ are constants, and ϕ is a function to be determined, is a functional equation which bears a close analogy to the general linear equation in finite differences, and which may be solved by a similar process.

Let π be the symbol of an operation which, when performed on $\phi(x)$, converts it into $\phi(\omega x)$, so that

$$\pi\phi(x) = \phi(\omega x).$$

This symbol π possesses various properties in common with the symbol D and others, which are often used. Thus, since

$$\pi\pi\phi(x), \text{ or } \pi^2\phi(x) = \pi\phi(\omega x) = \phi(\omega\omega x) = \phi(\omega^2 x),$$

we see generally that when n is an integer

$$\pi^n\phi(x) = \phi(\omega^n x);$$

from which also it is easy to show that

$$\pi^m\pi^n\phi(x) = \pi^{m+n}\phi(x),$$

or the successive operations of π are subject to the index law. Again, we may consider π as a distributive function for

$$\pi\{f(x) + \phi(x)\} = f(\omega x) + \phi(\omega x) = \pi f(x) + \pi\phi(x).$$

Also, since π acts only on a function which involves x, it is commutative with respect to quantities not involving x; so that a being such a quantity,

$$a\pi = \pi a.$$

These are the laws which are used in applying the method of the separation of symbols to the solution of linear differential equations; and hence the same method may be applied to our functional equation. If we introduce into it the symbol π, it becomes

$$\pi^n\phi(x) + a_1\pi^{n-1}\phi(x) + \&c. + a_n\phi(x) = X,$$

which is no longer in a functional form, since the unknown operation ϕ is the subject of known operations. Separating the symbols of operation, we have

$$(\pi^n + a_1\pi^{n-1} + a_2\pi^{n-2} + \&c. + a_n)\ \phi(x) = X.$$

Now if $r_1, r_2, \ldots r_n$ be the roots of the equation,

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \&c. + a_n = 0,$$

the complex operation performed on $\phi(x)$ may, in consequence of the laws of combination given above, be decomposed into the simpler operations

$$(\pi - r_1)(\pi - r_2)\ldots(\pi - r_n)\ \phi(x) = X,$$

exactly as is done in linear differential equations. And if $N_1, N_2, \ldots N_n$ be the coefficients of the partial fractions arising from the decomposition of

$$\frac{1}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \&c. + a_n} = \frac{1}{(z - r_1)(z - r_2)\ldots(z - r_n)},$$

we have, by effecting the inverse operation of that on the left-hand side of the equation,

$$\phi(x) = N_1(\pi - r_1)^{-1} X + N_2(\pi - r_2)^{-1} X + \ldots + N_n(\pi - r_n)^{-1} X$$
$$+ (\pi - r_1)^{-1} 0 + (\pi - r_2)^{-1} 0 + \ldots + (\pi - r_n)^{-1} 0.$$

The binomial operations in the first line may be expanded in integral powers of π, that is, according to successive performances of the known operation indicated by ω, and the results may therefore be assumed as known. But the operations in the second line must be developed in negative powers of π, implying the performance of inverse operations; the results of these must of course vary according to the nature of π or ω; and it is plain that any one of them is of the same form as that at which we should arrive in solving the equation

$$(\pi - r)\ \phi(x) = 0, \quad \text{or} \quad \pi\phi(x) - r\phi(x) = 0,$$

which is the simple functional equation

$$\phi(\omega x) - r\phi(x) = 0.$$

This may always be done, or supposed to be done, by Laplace's method, in which it is reduced to the solution of two equations of differences: one of these is always a linear equation of the first order, the other depends on the nature of the function represented by ω.

The preceding analysis shows us, that the solution of a certain class of functional equations may be reduced, exactly like linear equations in differentials and finite differences, to the determination of certain inverse operations, in the performance of which alone the difficulty of the solution lies: one or two examples may be of use in illustrating the theory.

Let $\omega(x) = mx$, m being a constant; and let the equation be of the second degree,

$$\phi(m^2x) + a\phi(mx) + b\phi(x) = x^n.$$

If α, β be the roots of $z^2 + az + b = 0$, this may by the preceding theory be put under the form

$$(\pi - \alpha)(\pi - \beta)\,\phi(x) = x^n,$$

where π is such that

$$\pi\phi(x) = \phi(mx).$$

Hence $\phi(x) = (\pi - \alpha)^{-1}(\pi - \beta)^{-1}x^n + (\pi - \alpha)^{-1}0 + (\pi - \beta)^{-1}0.$

The first term of the second side of the equation is easily determined: for since

$$\pi(x^n) = (mx)^n = m^n.x^n,$$

we may replace π by m^n, so that the term becomes

$$(m^n - \alpha)^{-1}(m^n - \beta)^{-1}x^n = \frac{x^n}{m^{2n} + am^n + b}.$$

There remains to determine the inverse operations, which are to be found from the solution of the functional equation

$$\phi_1(mx) - \alpha\phi_1(x) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now, by Laplace's method, assume

$$x = u_z, \quad mx = u_{z+1},$$

so that

$$u_{z+1} - mu_z = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2),$$

an equation for determining u_z, which, being known, enables us to express z in terms of x. Equation (1) may be written as

$$\phi_1(u_{z+1}) - \alpha\phi_1(u_z) = 0,$$

or simply

$$v_{z+1} - \alpha v_z = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The integration of the equations (2) and (3), enables us to solve the given functional equation (3). The solution of (2), on the assumption that the arbitrary function is a constant, is

$$u_z = Cm^z = x;$$

whence, by changing the constant, we have

$$z = \frac{\log(cx)}{\log m}.$$

In like manner the solution of (3) is

$$b_z = C\alpha^z = C\varepsilon^{\frac{\log\alpha}{\log m}\log(cx)} = C(cx)^{\frac{\log\alpha}{\log m}} = \phi_1(x),$$

C being an arbitrary function of $\sin 2\pi z$ and $\cos 2\pi z$.

Similarly for β: hence the solution of the given functional equation is

$$\phi(x) = \frac{x^n}{m^{2n} + am^n + b} + C(cx)^{\frac{\log\alpha}{\log m}} + C'(cx)^{\frac{\log\beta}{\log m}}.$$

Again, let $\omega(x) = x^n$, and the functional equation be

$$\phi(x^{n^2}) + a\phi(x^n) + b\phi(x) = \log x.$$

If we assume $x = u_z$, $x^n = u_{z+1}$, the solution of the preceding equation will depend on that of

$$u_{z+1} = u_z^n, \text{ and of } v_{z+1} - av_z = 0.$$

The integral of the former is

$$u_z = c^{n^z} = x;$$

whence, by a change of constant,

$$z = \frac{1}{\log n}\log\log(x^c).$$

The integral of the latter is

$$v_z = C\alpha^z = C\varepsilon^{\frac{\log\alpha}{\log n}\log(\log x^c)} = C(\log x^c)^{\frac{\log\alpha}{\log n}}.$$

Also, if $\phi(x^n) = \pi\phi(x)$, we have

$$\pi \log(x) = \log(x^n) = n \log x,$$

and therefore

$$(\pi^2 + a\pi + b)^{-1} \log x = \frac{\log x}{n^2 + an + b}.$$

Hence the solution of the proposed equation is

$$\phi(x) = \frac{\log x}{n^2 + an + b} + C(\log x^c)^{\frac{\log \alpha}{\log n}} + C_1 (\log x^c)^{\frac{\log \beta}{\log n}}.$$

If the function $\omega(x)$ be a periodic function of the n^{th} order, so that $\omega^n(x) = x$, $\omega^{n+1}(x) = \omega(x)$, &c., the result of such an operation as

$$(\pi - r)^{-1} f(x),$$

can be always readily determined. For, if we expand the binomial in ascending powers of π, it becomes

$$-\frac{1}{r}\left(1 + \frac{\pi}{r} + \frac{\pi^2}{r^2} + \&c. + \frac{\pi^{n-1}}{r^{n-1}} + \frac{\pi^n}{r^n} + \&c. + \frac{\pi^{2n}}{r^{2n}} + \&c.\right) f(x).$$

But as $\pi^n f(x) = f(\omega^n x) = f(x)$, this is equivalent to

$$-\frac{1}{r}\left(1 + \frac{1}{r^n} + \frac{1}{r^{2n}} + \&c.\right)\left(1 + \frac{\pi}{r} + \frac{\pi^2}{r^2} + \&c. + \frac{\pi^{n-1}}{r^{n-1}}\right) f(x)$$

$$= \frac{1}{1 - r^n}\{\pi^{n-1} + r\pi^{n-2} + \&c. + r^{n-2}\pi + r^{n-1}\} f(x)$$

$$= \frac{1}{1 - r^n}\{f(\omega^{n-1}x) + rf(\omega^{n-2}x) + \&c. + r^{n-2}f(\omega x) + r^{n-1}f(x)\}.$$

As an example, let us assume

$$\omega(x) = \frac{1 + x}{1 - x},$$

which is a periodic function of the fourth order, the successive results being

$$\omega^2(x) = -\frac{1}{x}, \quad \omega^3(x) = \frac{x - 1}{x + 1}, \quad \omega^4(x) = x.$$

Let the functional equation be

$$\phi\left(\frac{1 + x}{1 - x}\right) - a\phi(x) = x.$$

Then if $$x = u_z, \quad \frac{1+x}{1-x} = u_{z+1},$$

$$u_{z+1}u_z - u_{z+1} + u_z + 1 = 0.$$

The solution of this is (Herschel's *Examples*, p. 34)

$$u_z = \tan\left(C + \frac{\pi}{4} z\right) = x;$$

whence $$z = \frac{4}{\pi}(\tan^{-1} x - C).$$

The solution of the equation

$$\phi\left(\frac{1+x}{1-x}\right) - a\phi(x) = 0,$$

is therefore

$$\phi(x) = Ca^z = Ca^{\frac{4}{\pi}(\tan^{-1}x - C)} = C_1 a^{\frac{4}{\pi}\tan^{-1}x},$$

by changing the arbitrary constant. Hence the proposed functional equation gives

$$\phi(x) = (\pi - a)^{-1} x + C_1 a^{\frac{4}{\pi}\tan^{-1}x}$$

$$= \frac{1}{1-a^4}\left(\frac{x-1}{x+1} - \frac{a}{x} + a^2 \frac{1-x}{1+x} + a^3 x\right) + C_1 a^{\frac{4}{\pi}\tan^{-1}x}.$$

Again, let $\omega(x) = \frac{a^2}{x}$, a periodic function of the second order, and let the functional equation be

$$\phi\left(\frac{a^2}{x}\right) + \phi(x) = \varepsilon^{nx}.$$

Then if $x = u_z$, $\frac{a^2}{x} = u_{z+1}$, we have

$$u_{z+1}u_z = a^2;$$

the integral of which is

$$u_z = aC^{(-1)^z} = x;$$

whence $$(-1)^z = \left(\log\frac{x}{a}\right)^c.$$

But the functional equation gives us

$$v_{z+1} + v_z = 0,$$

whence $$v_z = C(-1)^z = C\left(\log\frac{x}{a}\right)^c.$$

Hence $$\phi(x) = (\pi + a)^{-1}\,\varepsilon^{nx} + C\left(\log\frac{x}{a}\right)^c$$

$$= \frac{1}{1-a^2}\left(\varepsilon^{nx} - a\varepsilon^{\frac{na^2}{x}}\right) + C\left(\log\frac{x}{a}\right)^c.$$

In conclusion, I may observe that this article does not pretend to give any new results, as the solution of Functional Equations of the kind here treated of is already known, (see Herschel's *Finite Differences*, p. 547): the object of it is merely to illustrate the theory before spoken of, and to show that a method which has been found useful in two departments of analysis may likewise be applied to simplify the processes of a more difficult branch.

NOTE ON A PROBLEM IN DYNAMICS.*

In the problem of finding the trajectory of a body under the action of a central force varying inversely as the square of the distance, when the circumstances of projection are given, it is usual to employ polar coordinates, either r and θ or p and r. The problem may, however, be solved quite as readily and more elegantly by adhering to rectilinear coordinates. To shew this take the equations of motion,

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{r^3} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{d^2y}{dt^2} = -\frac{\mu y}{r^3} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

whence, as usual, we have

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Multiplying (1) by (3), we have

$$h\frac{d^2x}{dt^2} = -\frac{\mu}{r^3}\left(x^2\frac{dy}{dt} - xy\frac{dx}{dt}\right)$$

$$= -\frac{\mu}{r^2}\left(r\frac{dy}{dt} - y\frac{dr}{dt}\right) = -\mu\frac{d}{dt}\left(\frac{y}{r}\right).$$

Therefore on integration

$$-h\frac{dx}{dt} = \mu\frac{y}{r} + g \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

g being an arbitrary constant.

* *Cambridge Mathematical Journal*, Vol. III., p. 267.

Similarly from (2) and (3), we have

$$h\frac{dy}{dt}=\mu\frac{x}{r}+f \qquad \text{(5)}.$$

Multiply (4) by y, and (5) by x, and add, then

$$\mu r+fx+gy=h^2 \qquad \text{(6)}.$$

From this it appears that r the radius vector is a rational and integral function of the coordinates of its extremity (x and y), and therefore this is the equation to a conic section, the focus of which is at the origin.

On comparing (6) with the general polar equation to the conic section,

$$r=\frac{a(1-e^2)}{1+e\cos(\theta-\alpha)},$$

or

$$r+ex\cos\alpha+ey\sin\alpha=a(1-e^2),$$

we find

$$a(1-e^2)=\frac{h^2}{\mu}, \quad e^2=\frac{f^2+g^2}{\mu^2},$$

$$a=\frac{\mu h^2}{\mu^2-(f^2+g^2)}, \quad \tan\alpha=\frac{g}{f}.$$

To determine the velocity in the path, square and add (4) and (5), then

$$h^2v^2=\mu^2+\frac{2\mu}{r}(fx+gy)+a^2+b^2,$$

But from (6)

$$fx+gy=h^2-\mu r,$$

therefore

$$h^2v^2=h^2\frac{2\mu}{r}+f^2+g^2-\mu^2 \qquad \text{(7)}.$$

Now if V be the velocity of projection at the distance ρ, we have from (7),

$$h^2V^2=h^2\frac{2\mu}{\rho}+f^2+g^2-\mu^2 \qquad \text{(8)},$$

whence, by subtracting, and dividing by h^2,

$$v^2=V^2+2\mu\left(\frac{1}{r}-\frac{1}{\rho}\right) \qquad \text{(9)}.$$

Again from (3), we have

$$\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)\frac{ds}{dt}=h.$$

But if δ be the angle between ρ and the direction of projection,

$$x\frac{dy}{ds}-y\frac{dx}{ds}=\rho\sin\delta$$

at the point of projection: and as then we have $\frac{ds}{dt}=V$,

$$h=V\rho\sin\delta \text{ .... } (10).$$

Then from (8) $\dfrac{f^2+g^2-\mu^2}{h^2}=V^2-\dfrac{2\mu}{\rho}$;

and therefore

$$a=\frac{\mu}{\dfrac{2\mu}{\rho}-V^2} \text{ .... } (11).$$

Also

$$e^2=1-\frac{V^2\rho^2\sin^2\delta}{\mu^2}\left(\frac{2\mu}{\rho}-V^2\right) \text{ ....} (12).$$

As the species of conic section depends on whether e is less, equal to, or greater than unity, it appears that the path is an ellipse, a parabola, or a hyperbola, according as

$$\frac{2\mu}{\rho}>=\text{ or }<V^2,$$

and therefore the species of conic section is independent of the angle of projection.

To find the angle α, we have

$$\tan\alpha=\frac{g}{f},\quad\text{or }\cos\alpha=\frac{f}{\sqrt{(f^2+g^2)}}=\frac{f}{\mu e}.$$

Now supposing the direction of projection to coincide with the axis of x, we have $x=\rho$ when $y=0$, hence (6) gives us

$$f=\frac{h^2}{\rho}-\mu=V^2\rho\sin^2\delta-\mu;$$

and therefore

$$\cos\alpha = \frac{V^2\rho\sin^2\delta - \mu}{\mu e} = \frac{V^2\rho\sin^2\delta - \mu}{\left\{\mu^2 - V^2\rho^2\sin^2\delta\left(\frac{2\mu}{\rho} - V^2\right)\right\}^{\frac{1}{2}}}.$$

If we combine (8) and (11) we get, as an expression for the velocity,

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}.$$

OF ASYMPTOTES TO ALGEBRAIC CURVES.*

THE ordinary method of deducing the equations to asymptotes to plane curves, whether by finding finite values of the intercepts of the tangents for infinite values of the coordinates of the point of contact, or by the more convenient method of expansion in descending powers of one of the variables, are essentially unsymmetrical. Moreover the former is often inapplicable from the difficulty of finding the true value of a fraction of which the numerator and denominator are infinite, and the latter fails when the asymptote is parallel to one of the coordinate axes. The following method, though appropriate to algebraic curves only, is for them of very easy application, and, besides being symmetrical, leads us readily to the demonstrations of various properties of asymptotes to curves.

Let the equation to the curve, cleared of fractions and radicals, be put in the form

$$u = \phi_n(x, y) + \phi_{n-1}(x, y) + \phi_{n-2}(x, y) + \&c. + \phi_0 = 0 \ldots (1),$$

the symbol $\phi_r(x, y)$ denoting generally a *homogeneous* function of r dimensions in x and y; then the equation to the tangent at a point (x, y) may be written

$$x'\frac{du}{dx} + y'\frac{du}{dy} + \phi_{n-1}(x, y) + 2\phi_{n-2}(x, y) + \&c. + n\phi_0 = 0 \ldots (2),$$

x', y' being the current coordinates of the tangent.

* *Cambridge Mathematical Journal*, Vol. IV., p. 42.

The definition of an asymptote is, that it is a line which, passing through a point at a finite distance from the origin, touches the curve at an infinite distance. Now if x, y be the coordinates of the point of contact; x', y' those of a point through which the line passes; l, m the cosines of the angles which the line makes with the axes, we have

$$\frac{x-x'}{l}=\frac{y-y'}{m}=r \quad \text{.................} \quad (3),$$

r being the distance between the point x, y and x', y'. Hence

$$x=x'+lr, \quad y=y'+mr,$$

and substituting these values in (1), it becomes

$$\phi_n(l,m)r^n+\left\{\left(x'\frac{d}{dl}+y'\frac{d}{dm}\right)\phi_n(l,m)+\phi_{n-1}(l,m)\right\}r^{n-1}+\&c.=0 \ldots (4).$$

But if the point x, y is at an infinite distance, r, must be infinite, which involves the condition that the coefficient of the highest power of r shall vanish; consequently we must have

$$\phi_n(l, m)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

as an equation for determining the *direction* of the asymptotes. Again, if we substitute in (2) for x and y their values derived from (3), we have, arranging in powers of r,

$$\left\{\left(x'\frac{d}{dl}+y'\frac{d}{dm}\right)\phi_n(l, m)+\phi_{n-1}(l, m)\right\}r^{n-1}+\&c.=0 \ldots (6).$$

Since the asymptote is by definition a particular case of the tangent, this equation also must give an infinite value for r, which involves the condition

$$\left(x'\frac{d}{dl}+y'\frac{d}{dm}\right)\phi_n(l, m)+\phi_{n-1}(l, m)=0 \ldots\ldots (7).$$

Now this equation is linear in x' and y', which are the coordinates of *any* point through which the asymptote passes, that is of any point in the line; so that this equation is in fact the equation to the asymptote, if we substitute in it the relations between l and m, which satisfy equation (5).

As from the homogeneity of the terms, l and m finally disappear, and as they are involved exactly as x and y are in the equation to the curve, we may express the process for finding the asymptotes to a curve simply as follows. Let the equation to the curve be put in the form

$$u_n + u_{n-1} + u_{n-2} + \&c. + u_0 = 0 \text{ .... } (8),$$

u_r being a homogeneous function in x and y of the r^{th} order, then the equation to the asymptotes will be found by eliminating x and y from

$$x' \frac{du_n}{dx} + y' \frac{du_n}{dy} + u_{n-1} = 0$$

by means of the relation between x and y given by the equation

$$u_n = 0.$$

Since the equation $u_n = 0$ is of the n^{th} degree in x and y, it appears that a curve of the n^{th} degree can have n asymptotes and no more. If there be any impossible factors in $u_n = 0$, there are no corresponding asymptotes; and as impossible factors enter the equation by pairs, a curve of the n^{th} degree must have n or $n-2$ or $n-4$ or &c. asymptotes.

As an example take the curve

$$y^3 - x^3 - ax^2 = 0.$$

In this case $u_n = 0$ becomes $y^3 - x^3 = 0$, in which there is only one possible factor $y - x = 0$. The other equation

$$3y'y^2 - 3x'x^2 - ax^2 = 0$$

becomes, on putting $y = x$,

$$y' - x' = \frac{a}{3},$$

which is the equation to the asymptote.

Again, let the curve be

$$xy^2 - x^3 + 2a^2y = 0;$$

then the equation $u_n = 0$ becomes

$$xy^2 - x^3 = 0,$$

giving three factors

$$x=0, \quad y=+x, \quad y=-x.$$

The first of these substituted in the equation

$$x'(y^2-3x^2)+2y'xy=0,$$

gives

$$x'=0,$$

the second and third give

$$x'-y'=0 \text{ and } x'+y'=0,$$

which are the equations to the three asymptotes, the first being evidently the axis of y, and the others inclined at angles $\pm 45^\circ$ to the axis of x. Let the curve be

$$(x+a)\,y^2=(y+b)\,x^2, \quad \text{or } xy^2-yx^2+ay^2-bx^2=0.$$

The equation $u_n=0$ here becomes

$$xy^2-yx^2=0, \quad \text{or } xy\,(y-x)=0,$$

which has three possible factors

$$x=0, \quad y=0, \quad y-x=0.$$

The general equation to the asymptotes is

$$x'(y^2-2xy)+y'(2xy-x^2)+ay^2-bx^2=0:$$

for $x=0$ this is reduced to

$$x'+a=0;$$

for $y=0$ it becomes

$$y'+b;$$

and for $y-x=0$ it becomes

$$y'-x'+a-b=0,$$

and these are the equations to the three asymptotes. One advantage of this method of finding asymptotes is that a simple inspection of the highest terms of the equation shows at once the number and direction of the asymptotes. The method however fails if the equation $u_n=0$ contain equal possible roots, indicating the existence of parallel asymptotes. For in this case $\frac{du_n}{dx}$ and $\frac{du_n}{dy}$ will vanish along with

u_n, and consequently the equation to the asymptote is nugatory; but a slight extension of the process enables us to overcome the difficulty. It will be seen that the two equations are the coefficients of the n^{th} and $(n-1)^{\text{th}}$ power of r in the expansion (4). In the case of failure from the existence of m equal roots in the equation $u_n = 0$, all that is necessary is to equate to zero the coefficient of the highest power of r which is unaffected by the equality of roots that is the coefficient of the $(n-m)^{\text{th}}$ power of r, and this equation combined with the m equal roots of $u_n = 0$ gives the equations to the parallel asymptotes. If there be two equal roots, and the equation to the curve be put in the form (8), the equation for determining the parallel asymptotes, or the coefficient of r^{n-2} equated to zero, is

$$\frac{1}{2}\left(x'^2 \frac{d^2u_n}{dx^2} + 2x'y' \frac{d^2u_n}{dx\,dy} + y'^2 \frac{d^2u_n}{dy^2}\right) + x' \frac{du_{n-1}}{dx} + y' \frac{du_{n-1}}{dy} + u_{n-2} = 0 \ldots\ldots (9).$$

As an example take the curve

$$x^2y - x^3 - 3bxy + 2b^2y = 0.$$

Here $u_n = 0$ becomes $x^2y - x^3 = 0$ gives

$$x^2 = 0, \quad y - x = 0,$$

$$\frac{d^2u_n}{dx^2} = 2y - 6x = 2y \text{ when } x = 0,$$

$$\frac{d^2u_n}{dx\,dy} = 2x = 0 \text{ when } x = 0, \quad \frac{d^2u_n}{dy^2} = 0;$$

$$\frac{du_{n-1}}{dx} = -3by, \quad \frac{du_{n-1}}{dy} = -3bx = 0 \text{ when } x = 0;$$

consequently equation (9) becomes, on dividing by y,

$$x'^2 - 3bx' + 2b^2 = 0,$$

which is decomposable into the two factors

$$x' - b = 0, \quad x' - 2b = 0,$$

the equations to two asymptotes parallel to the axis of y.

The equation to the other asymptote given by the equation $y - x = 0$ is

$$y' - x' - 3b = 0.$$

In like manner we may shew that the curve

$$x^2 (x^2 + y^2) = a^2 (y - x)^2$$

has two parallel asymptotes, of which the equations are

$$x' + a = 0, \text{ and } x' - a = 0$$

The equation (4) for any values of l and m is an equation for determining the value of r, the portion of the line whose direction-cosines are l and m intercepted between a given point and the curve. It is generally of n dimensions, so that the line generally meets the curve in n points; but when the line is an asymptote, the first two terms disappear and the equation is reduced to $n-2$ dimensions. Consequently an asymptote cannot meet its curve in more than $n-2$ points; and as for all lines parallel to an asymptote the first term of (4) vanishes, lines parallel to an asymptote cannot meet the curve in more than $n-1$ points.

Since, from what precedes, it appears that the equation to an asymptote depends only on the terms involving the highest and second highest powers of the variables, all curves for which these are the same have the same asymptotes, and *vice versâ*. And as among the curves of the n^{th} order is to be included that made up of the n asymptotes themselves, the product of their n linear equations must have the same highest and second highest terms as the equation to the curve; that is, the equation to the curve differs from the product of the equations to the n asymptotes only in terms of the $(n-2)^{\text{th}}$ order: so that if the equations to the asymptotes be the n linear equations

$$u' = 0, \quad u'' = 0 \ldots u^{(n)} = 0,$$

that to the curves to which these are asymptotes may be written

$$u'u'' \ldots u^{(n)} + u_{n-2} + u_{n-3} + \ldots + u_0 = 0.$$

Thus if the curve be of the third order, its equation is

$$u'u''u''' + u_1 + u_0 = 0.$$

When an asymptote meets the curve, which it can do in one point only, this is to be combined with $u' = 0$, $u'' = 0$, $u''' = 0$, any one of which reduces the preceding equation to

$$u_1 + u_0 = 0,$$

which being a linear equation common to the three points in which the curve meets its asymptotes, shews that they lie in one straight line.

In like manner we may shew that the six points in which the curve is cut by lines drawn parallel to the asymptotes all lie in a curve of the second order.

THE END.

William Metcalfe, Printer, Green Street, Cambridge.

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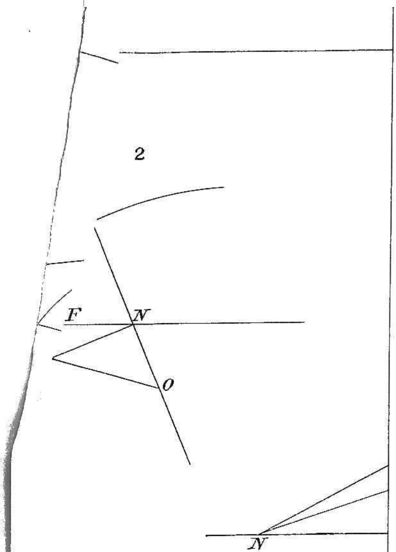
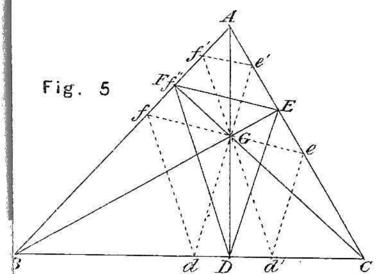


Fig. 5



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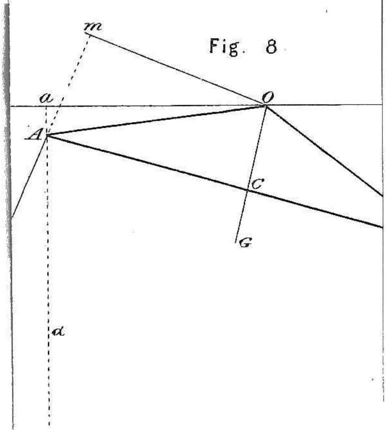

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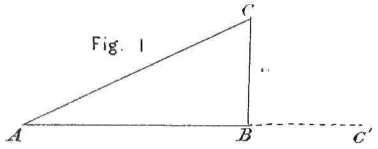

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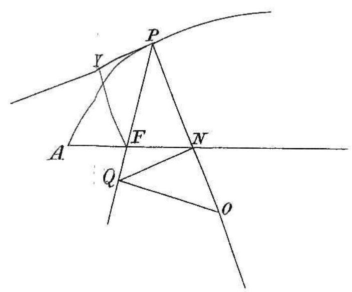

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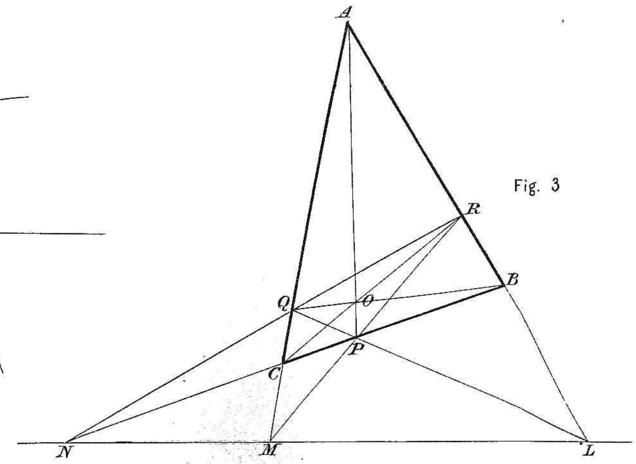


Fig. 4

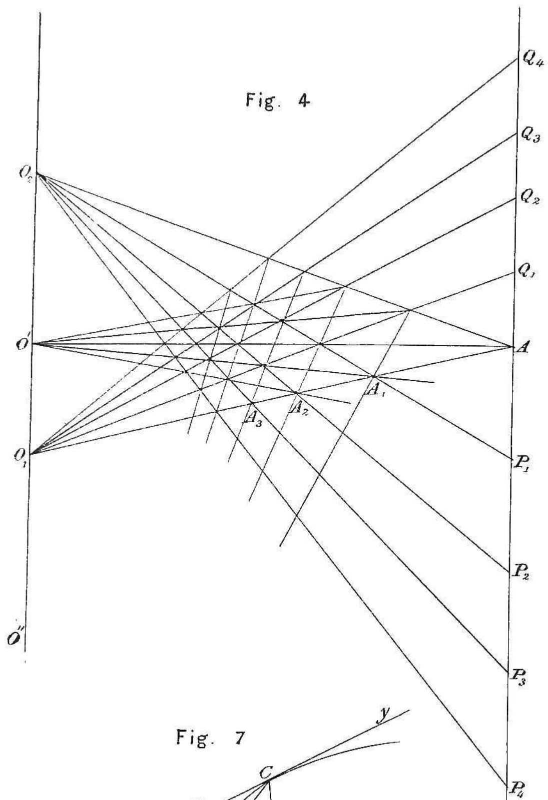
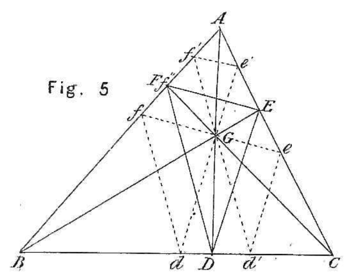


Fig. 5



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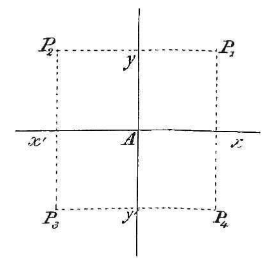

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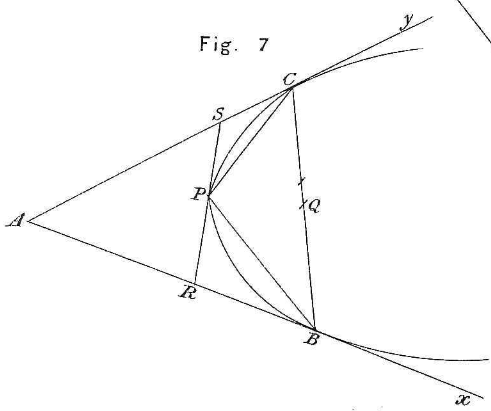


Fig. 8

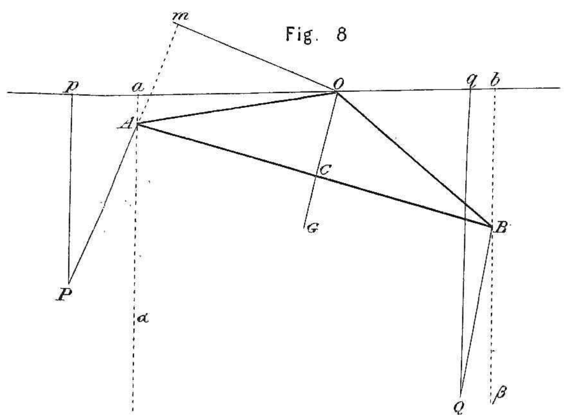
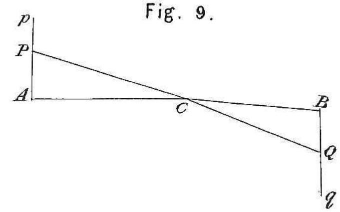


Fig. 9.



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