Bolyai, János,
... The science absolute of space
... 1891.

The loss l'' is therefore truly a loss of kinetic energy.

If we put
$$l''=\frac{k}{2g}V^2,$$
the original equation of energy becomes,
$$wAd=(wW+c)h+(wW+k)h_V+r.$$

In table I, we have
$$l'=l_1'(1+h_V/h)=12+h_V,$$
and within the limits of error of the experiment,
$$l''=100h_V=\frac{50}{g}V^2,$$
for uniform flow, $\therefore$ $k=0.051$.

For flows which are periodic, the law cannot be expected to hold when the values of V are gotten as above, since the discharge takes place in squirts of high velocity, followed by longer periods of rest. The actual velocity of mean flow must therefore be higher than the computed ones, which will bring the series for periodic flow under the law also.

In order to derive reliable conclusions from such experiments a large number of sets should be made upon pipes of different diameters, with a larger range of air flows and values of d and h.

Also different arrangements for introducing the air into the flow pipe should be employed, it is my opinion that introducing it into the side of the pipe, as did Mr. Harris, will do away with periodic flow altogether. I distinguish between periodic flow caused by accumulations of air in the lower end of the pipe, and intermittent flow as caused by the escape of air in the discharge.

I contemplate, if time be allowed, pushing these experiments further, and finally comparing its efficiency with that of the water motor described in the beginning of this paper.

EDITORIAL NOTE.

It has been decided to discontinue the publication of this Journal and its issue ceases with this number, which closes the first volume.

We close the first volume and cease the publication with considerable regret, yet with no small degree of satisfaction, believing as we do, that as a Journal of Elementary Mathematics it has accomplished fairly well the object which it had in view.

Had it done nothing more than to put into English words the papers of Bolyai and Lobatschewsky its life had been well lived. We believe that the time will yet come when the seed thus sown will bear its share of fruit in the advancement of sound geometrical teaching in America.

W. H. E.

SCIENTIÆ BACCALAUREUS.

VOL. I JUNE, 1891. No. 4

THE SCIENCE ABSOLUTE OF SPACE,

INDEPENDENT OF THE TRUTH OR FALSITY OF EUCLID'S AXIOM XI (WHICH NEVER CAN BE ESTABLISHED A PRIORI);

Followed by the geometric quadrature of the circle in the case of the falsity of Axiom XI,

BY

John Bolyai,

CAPTAIN IN THE ENGINEERING CORPS OF THE AUSTRIAN ARMY.

TRANSLATED INTO ENGLISH

BY

GEORGE BRUCE HALSTED,

A. M., Ph. D., Ex-Fellow of Princeton College and Johns Hopkins University, Professor of Mathematics in the University of Texas.

TRANSLATOR'S INTRODUCTION.

Through all its editions up to the last, America's favorite geometry, Wentworth's, taught in all seriousness the following proposition (see 3d edition, 1887, §387, page 224): *To inscribe a regular polygon of any number of sides in a given circle.* But in this, as in some other respects, the book was only more than two thousand years behind the times. Euclid would have smiled at the unconsciousness with which this American Jonah swallowed his impossible whale. Euclid could inscribe regular polygons of 3, 4, 5, 15 sides or numbers obtained by doubling these. Those of 7, 9, 11, 13, 14 sides no man ever could or ever will geometrically inscribe. When on the evening of March 30th, 1796, Gauss showed to his student friend, the Hungarian, Wolfgang Bolyai, the formula which gave the geometric inscription of the regular polygon of 17 sides, it was with the remark that this alone could be his epitaph, if it were not a pity to omit so much that went with it.

Was it this break beyond Euclid's enchanted bounds that started these two young men in that re-sifting of the very foundations of geometry which led to those new conceptions of the whole subject just now, after another hundred years, beginning to be taught in America's foremost universities?

Wolfgang Bolyai was born February 9th, 1775, in that part of Transylvania called Székelyföld. He studied first at Enyed, afterward at Klausenburg, and in 1796, with a son of Baron

Simon Kemény, went first to Jena, then to Göttingen. Here he met Gauss, then in his 19th year, and the two formed a friendship which lasted for life. The letters of Gauss to his friend were sent by Bolyai in 1855 to Professor Sartorius von Walterhausen, then working on his biography of Gauss.

Gauss said that Bolyai was the only man who completely understood his views on the metaphysics of mathematics. Everyone who met him felt that he was a profound thinker and a beautiful character.

Benzenberg said in a letter written to Gauss in 1801 that Bolyai was one of the most extraordinary men he had ever known.

On his return home in 1802 Bolyai was made professor of mathematics in the Reformed College of Maros-Vásárhely.

Here during the 47 years of his active teaching he had for scholars most of the present professors in Transylvania, and nearly all the nobility of the country.

Sylvester has said that mathematics is nearest akin to poetry. Bolyai's first works published were dramas, and translations of English and German poetry into Hungarian.

In 1830 he published an arithmetic. Then came his chief work, to which he constantly refers in his later writings. It is in Latin, two volumes, with title as follows:

Tentamen juventutem studiosam in elementa matheseos purae, elementaris ac sublimioris, methodo intuitivo, evidentiaque huic propria, introducendi. Cum Appendice triplici.

Auctore Professore Matheseos et Physices Chemiaeque publico ordinario.

Tomus primus. Maros Vásárhelyini, 1832.

Tomus secundus. 1833.

The first volume contains:

Preface of two pages: *Lectori salutem.*

A folio table: *Explicatio signorum.*

Index rerum (I—XXXII). *Errata* (XXXIII—LXXIV). *Errores recentius detecti* (LXXIV—XCVIII).

Now comes the body of text (pages 1 502). Then with special paging and a new title page, comes the immortal appendix composed by John Bolyai, son of Wolfgang:

APPENDIX *scientiam spatii absolute veram exhibens: a veritate aut falsitate axiomatis* XI *Euclidei (a priori haud unquam decidenda) independentem; adjecta, ad casum falsitatis, quadratura circuli geometrica. Auctore* JOANNE BOLYAI *de eadem, Geometrarum in Exercitu Caesaris Regio Austriaco Castrensium Capitaneo.* Twenty-six pages of text, two pages of errata.

Finally (pages I XVI), in Hungarian, the names of the subscribers, the nomenclature, and additions to this volume by W. Bolyai. Then 4 plates of figures, the first 3 pertaining to the body of the text, the last to the Appendix.

It is this Appendix which we now give for the first time in English. Milton received but a paltry 5 pounds for his Paradise Lost; but it was at least plus 5, John Bolyai, as we learn from volume second, page 384, of the *Tentamen*, contributed, for the printing of his eternal 26 pages, 104 florins 54 kreuzers.

His father, treating in the body of the work the theory of parallels, says, *à propos* of the systems which are possible when we contradict Euclid's axiom XI, "Appendicis Auctor, rem acumine singulari aggressus, Geometriam pro omni casu absolute veram posuit, quamvis e magna mole, tantum summe necessaria, in Appendice hujus tomi exhibuerit, multis (ut tetraedri resolutione generali, pluribusque aliis disquisitionibus elegantibus) brevitatis studio omissis."

And again: "Nihilominus tamen quaestio suboritur: quid si novum axioma detur, per quod determinetur *u?* Tentamina idcirco, quae olim feceram, breviter exponenda veniunt, ne saltem alius quis operam eodem perdat." He speaks of his son's beautiful treatise with natural admiration: Thus, Vol. I,

p. 502, Nec operae pretium est plura referre; quum res tota ex altiori contemplationis puncto, in ima penetranti oculo, tractetur in Appendice sequente, a quovis fideli veritatis purae alumno digna legi.

And Vol. II, page 380, "Denique aliquid Auctori Appendicis . . . addere fas sit: qúo tamen ignoscat, si quid non acu ejus tetigerim."

This wonderful production of pure genius, this Appendix which makes all preceding space only a special case, only a species under a genus, and so requiring a descriptive adjective, *Euclidean*, this strange Hungarian flower was saved for the world after more than thirty-five years of oblivion, by the rare erudition of Professor Richard Baltzer of Dresden, afterward professor in the University of Giessen. In the second edition of his Elemente der Mathematik in 1867, Dr. Baltzer called attention to this re-making of Geometry, and the name Bolyai was at last given its place in the history of science. Before that, the father Wolfgang Bolyai seems to have been the only person who really appreciated the work of the son John Bolyai. He refers to it in a subsequent work printed in 1846, Uertan elemei kezdöknek, figures for which, we learn, were drawn by his grandson, John's son. Then comes his last work, the only one composed in German, entitled:

Kurzer Grundriss eines Versuchs:

I. *Die Arithmetik*, durch zweckmæssig construirte Begriffe, von eingebildeten und unendlich-kleinen Grössen gereinigt, anschaulich und logisch-streng darzustellen.

II. In der *Geometrie*, die Begriffe der geraden Linie, der Ebene, des Winkels allgemein, der winkellosen Formen, und der Krummen, der verschiedenen Arten der Gleichheit *u. d. gl.* nicht nur scharf zu bestimmen, sondern auch ihr Seyn im Raume zu beweisen; und da die Frage, *ob zwey von der dritten geschnittenen Geraden, wenn die Summe der inneren*

Winkel nicht=2R, sich schneiden oder nicht? niemand auf der Erde ohne ein Axiom (wie Euclid das XI) aufzustellen, beantworten wird; die davon unabhængige Geometrie abzusondern, und eine auf die Ja–Antwort, andere auf das Nein so zu bauen, dass die Formeln der letzten, auf einen Wink auch in der ersten gültig seyen.

Nach einem lateinischen Werke von 1829, M. Vàsàrhely; und eben da selbst gedruckten ungarischen:

Maros-Vàsàrhely, 1851, 88 pages of text.

In this he says, referring to his son's Appendix scientiam spatii ubsolute veram exhibens; "Some copies of the work published here were sent at that time to Vienna, to Berlin, to Goettingen. . . From Goettingen the giant of mathematics, who from his pinnacle embraces in the same view the stars and the abysses, wrote that he was charmed to see executed the work which he had commenced, only to leave it after him in his papers."

On the 9th of March, 1832, Wolfgang Bolyai was made corresponding member in the mathematics section of the Hungarian Academy. As professor he exercised a powerful influence in his country. In his private life he was a type of true originality. He wore roomy black Hungarian pants, a white flannel jacket, high boots, and a broad hat like an old-time planter's. The smoke-stained wall of his antique domicile was adorned by pictures of his friend Gauss, of Schiller, and of Shakespeare, whom he loved to call the child of Nature. His violin was a constant solace. He died the 20th of November, 1856. He ordered that his grave should bear no mark.

His son John died in 1860, seven years before the world began to know of his unique and wonderful work. He was born at Klausenburg, in Transylvania, the 15th of December, 1802.

He studied in one of the institutions founded in Transylvania by the Imperial Academy of Engineering of Vienna, and graduated the 7th of September, 1822, as cadet of engineers. The first of September, 1823, he was made second lieutenant, and the 16th of June, 1833, he was put on the retired list as captain. His profound mathematical ability showed itself physically not only in his handling of the violin, where he was a master, but also of arms, where he was unapproachable. It was this skill which caused his being retired so early from the army, though it saved him from the fate of a kindred spirit, the lamented Galois, killed in a duel when only 19. Bolyai when in garrison with cavalry officers was challenged by 13 of them at once. He accepted all, only stipulating that between each duel he might play a bit on his violin. He was victor thirteen times.

Beyond the *Appendix*, whose translation into English is here given, John Bolyai published nothing; and the thousand pages of manuscript which he left have never been read by a competent mathematician. They are in the library of the Reformed College of Maros-Vàsàrhely. We hear that he had conceived the project of working out a universal language, akin to that which music has, or that of mathematics.

If in this he was only an anticipator of Volapük, we think nothing of it; but it rather seems that he was another Boole, and if so, what discoveries in algorithmic logic might lie hidden in his papers!

In 1853 he must have thought of printing part of his mathematical works, for he left parts of a book with the title:

Principia doctrinae novae quantitatum imaginariarum perfectae uniceque satisfacientis, aliaeque disquisitiones analyticae et analytico-geometricae cardinales gravissimaeque; auctore Johan. Bolyai de eadem, C. R. austriaco castrensium captaneo pensionato.

Vindobonae, vel Maros-Vàsàrhelyini, 1853.

To him who hath shall be given, and it would be natural enough if the world still gives to Gauss, the greatest and best known mathematician of his generation, some of the credit which really belongs to the name of Bolyai. On the completion of his mathematical studies at the university, the Georgia Augusta, Bolyai left Goettingen the 5th of June, 1799.

From Braunschweig, Gauss writes to him in Klausenburg at the end of the year:

"I very much regret that I did not make use of our former proximity to find out *more* of your investigations in regard to the first grounds of geometry; I should certainly thereby have spared myself much vain labor, and would have become more restful than any one such as I can be, so long as, on such a subject, there yet remains so much to be wished for. In my own work thereon I myself have advanced far (though my other wholly heterogeneous employments leave me little time therefor), but *the* way, which I have hit upon, leads not so much to the goal which one wishes, as much more to making doubtful the truth of geometry. I have hit upon much which, with most, would pass for a proof, but which in my eyes proves as good as nothing. For example, if one could prove that a rectilineal triangle is possible whose content may be greater than any given surface, then am I in condition to prove with perfect rigor all geometry. Most would indeed let that pass as an axiom; I not; it might well be possible, that, how far apart soever one took the three vertices of the triangle in space, yet the content was always under a given limit. I have more such theorems, but in none do I find anything satisfying."

From this letter we see that in 1799 Gauss was still trying to prove *a priori* the eternal reality of the Euclidean system, what John Bolyai calls the system Σ. Some time in the next thirty years he comes to Bolyai's conclusion, for in 1829 he writes to Bessel as follows:

"At times in certain free hours, I have meditated again on a theme which, with me, is already nearly 40 years old, I mean the first grounds of geometry. I do not know whether I have spoken to you of my views thereupon. Here also have I much still further consolidated, and my conviction that we cannot found geometry completely *a priori*, has become, if possible, still firmer. Meanwhile, I am still far from attaining to the working out of my *very extended* researches for publication, and perhaps that will never happen in my lifetime, for I dread the outcry of the opposition if I should express my views *fully*."

Later Gauss adds:

"According to my deepest conviction, the science of space has to our science of necessary truths a relation wholly different from the pure science of quantity; there is lacking to our knowledge of the former (space lore) throughout, *that* complete persuasion of its necessity (consequently also of its absolute truth) which is peculiar to the *latter;* we must in humility admit, that, if number is *merely* a product of our mind, space has also a *reality beyond* our mind, of which we cannot fully foreordain the laws *a priori*."

More than twenty years after this, Gauss heard from his own pupil, Riemann, the marvelous dissertation which to Bolyai's spaces, got by denying the axiom of parallels, added as many others got by denying the infinite size of the straight line.

Beltrami showed ("Saggio di interpretazione della geometria non euclidea," Giorm di Matematiche, 1868) that Bolyai's geometry in a plane is equivalent to the Euclidean geometry on a surface of constant negative curvature. Riemann's finite space, of positive curvature, was studied by Felix Klein (1871–2, Math. Annalen IV & VI), and by him named *Elliptic*, while Euclid's he called *Parabolic*, and Bolyai's *Hyperbolic*. I notice that our new Century dictionary confuses Hyperbolic with Elliptic geometry, giving to each the definition of the other.

Cayley carried on the subject to trigonometry in an article entitled, "On the Non-Euclidean Geometry (Mathematische Annalen, v. pp. 630–4, 1872(, which begins as follows: "The theory of the Non-Euclidean Geometry as developed in Dr. Klein's paper "Ueber die Nicht-Euclidische Geometrie" may be illustrated by showing how, in such a system, we actually measure a distance and an angle and by establishing the trigonometry of such a system. I confine myself to the "hyperbolic" case of plane geometry; viz. the absolute is here a real conic, which for simplicity I take to be a circle; and I attend to points *within* the circle.

I use the simple letters a, A, . . to denote (linear or angular) distances measured in the ordinary manner; and the same letters with a superscript stroke, $\bar{a}$, $\bar{A}$, . to denote the same distances measured according to the theory." His result is "that the formulæ are in fact similar to those of spherical trigonometry with only cos h $\bar{a}$, sin h $\bar{a}$, etc., instead of cos a, sin a, etc."

In my first paper on the Bibliography of Hyper-Space and Non-Euclidean Geometry (American Journal of Mathematics, Vol. I, No. 3, pp. 261–276, 1878), I mentioned also Réthy's article: Die Fundamental Gleichungen der nicht-euklidischen Trigonometrie auf elementarem Wege abgeleitet:

Grunert's Archiv, LVIII, 416; also a number of works carrying these ideas on into mechanics.

EXPLANATION OF SIGNS.

The straight ABC means the aggregate of all points situated in the same straight line with A and B.

The sect AB means that piece of the straight AB between the points A and B.

The ray AB means that half of the straight AB which commences at the point A and contains the point B.

The plane ABC means the aggregate of all points situated in the same plane as the three points (not in a straight) A, B, C.

The hemi-plane ABC means that half of the plane ABC which starts from the straight AB and contains the point C.

ABC means the smaller of the pieces into which the plane ABC is parted by the rays BA, BC, or the non-reflex angle of which the sides are the rays BA, BC.

ABCD (the point D being situated within $\angle$ ABC, and the straights BA, CD not intersecting) means the portion of $\angle$ ABC comprised between ray BA, sect BC, ray CD, while BACD designates the portion of the plane ABC comprised between the straights AB and CD.

$\perp$ is the sign of perpendicularity.

$\parallel$ is the sign of parallelism.

$\angle$ means angle.

rt. $\angle$ is right angle.

st. $\angle$ is straight angle.

$\cong$ is the sign of congruence, indicating that two magnitudes are superposable.

AB $\triangleq$ CD means $\angle$ CAB $=$ $\angle$ ACD.

$x \doteq a$ means x converges toward the limit a.

$\triangle$ is triangle.

$\bigcirc r$ means the [circumference of the] circle of radius r.

$\odot r$ means the area of the surface of the circle of radius r.

The Science of Absolute Space.

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1. If the ray AM is not cut by the ray BN, situated in the same plane, but is cut by every other ray BP comprised in the angle ABN, we will call ray BN *parallel* to ray AM, that is to say we will have BN || AM.

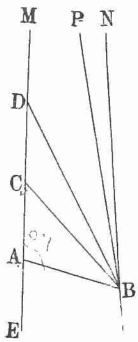

FIG. I.

It is easy to see that *there is one such ray BN, and only one*, passing through any point B (taken outside of the straight AM), and that the sum of the angles BAM, ABN cannot exceed a st. $\angle$. Because, in moving BC around B until BAM + ABC = st. $\angle$, there will be an instant where ray BC will *commence* not to cut ray AM, and it is then that we have BC || AM. It is clear, at the same time, that BN || EM, whatever be the point E taken on the straight AM.

If while the point C goes away to infinity on ray AM, we take always CD = CB, we will have constantly CBD = CDB < NBC. Now NBC $\doteq$ O; therefore also ADB $\doteq$ O.

2. If BN || AM, we will have also CN || AM. Take D any point of MACN. If C is on ray BN, ray BD will cut ray AM, since BN || AM. Therefore ray CD will also cut ray AM. If C is situated on ray BP, take BQ || CD; BQ will fall within the $\angle$ABN (§ 1), and consequently will cut ray AM; therefore ray CD will also cut ray AM. Therefore every ray CD (in ACN) cuts, in each case, the ray AM, without CN itself cutting ray AM. Therefore we have always CN || AM.

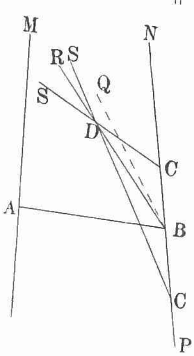

FIG. 2.

3. If BR and CS are each || AM, and C is not situated on the straight BR, then ray BR and ray CS do not intersect. Because if ray BR and ray CS had a common point D, then (§ 2) DR and DS would be each || AM, ray DR (§ 1) would coincide with ray DS, and C would fall on the straight BR, which is contrary to the hypothesis.

4. If MAN>MAB, we will have, for every point B of ray AB, a point C of ray AM, such that BCM=NAM.

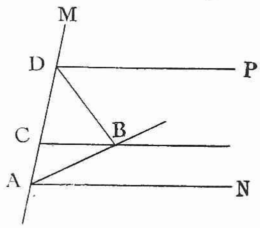

FIG. 3.

For, (§ 1), we may draw BD so that BDM >NAM, and making MDP=MAN, B will be contained in NADP. If therefore we carry NAM along AM, until ray AN arrives on ray DP, ray AN will have necessarily passed through B, and somewhere we have had BCM=NAM.

5. If BN || AM, there is on the straight AM a point F such that FM $\underline{\triangle}$ BN. For we can get (§ 1) BCM> CBN, and if CE=CB, it follows that EC $\underline{\triangle}$ BC, whence BEM< EBN. Move the point P on EC. The angle BPM, for P near E, will commence by being < the corresponding angle PBN, and for P near C, it will finish by being >PBN. Now the angle BPM increases continuously from BEM to BCM, since (§ 4) there exists no angle >BEM and < BCM, to which BPM can not become equal. Likewise PBN decreases continuously from EBN to CBN. There is therefore on EC a point F such that BFM=FBM.

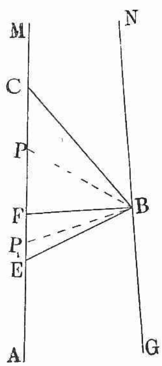

FIG. 4.

6. If BN || AM and E any point of the straight AM, and G any point of the straight BN, then GN || EM and EM || GN.

Because we have (§ 1) BN || EM, whence (§ 2) GN || EM. If now we make (§ 5) ∠BFM=∠FBN, then MFBN ≌ NBFM, and consequently, since BN || FM, we have also FM || BN, and from what precedes EM || GN.

7. If BN and CP are each || AM, and C not on the straight BN, we shall have also BN || CP.

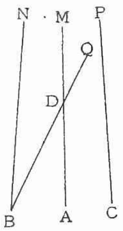

FIG. 5.

The rays BN and CP do not intersect (§ 3). Moreover, AM, BN and CP either are or are not in the same plane, and in the first case, AM either is or is not within BNCP.

1. If AM, BN, CP are in the same plane, and AM falls within BNCP, then every ray BQ drawn within ∠NBC will cut the ray AM somewhere in D, since BN || AM. Moreover, since DM || CP (§ 6), the ray DQ will cut the ray CP, therefore BN || CP.

2. If BN and CP are on the same side of AM, one of them, for example CP, will be contained between the two other straights BN, AM.

Now, every ray BQ within ∠NBA meets the ray AM; consequently it also meets CP. Therefore BN || CP.

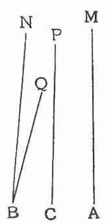

FIG. 6.

3. If the planes MAB, MAC make an angle, then CBN and ABN can have in common nothing but the straight BN, while the straight AM (in ABM) will have nothing in common with the ray BN, and in consequence, also NBC will have nothing in common with the straight AM.

Now every hemi plane BCD, drawn through the ray BD (situated in ∠NBA), will meet the ray AM, since ray BQ meets ray AM (as BN || AM). Therefore in revolving the hemi-plane BCD around BC until this hemi-plane *begins* to leave the ray AM, the hemi-plane BCD will come into coincidence with the hemi-plane BCN. By parity of reasoning this same hemi-plane will come into coincidence with hemi-plane BCP. Therefore BN is in the plane

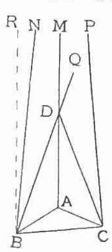

FIG. 7.

BCP. Moreover, if BR $\parallel$ CP, then (AM being also $\parallel$ CP) BR will be by the same reasoning, in the plane BAM, and also (since BR $\parallel$ CP) in the plane BCP. Therefore the straight BR, being common to the two planes MAB, PCB, is identical with the straight BN. Therefore BN $\parallel$ CP.*

If therefore CP $\parallel$ AM, and B exterior to the plane CAM, then the intersection BM of the planes BAM, CAP is $\parallel$ at the same time to AM and CP.

8. If BN $\parallel$ CP and $\angle$CBN$=\angle$BCP, and AM (in NBCP) is $\perp$ the sect BC at its mid point, then BN $\parallel$ AM.

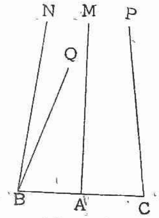

FIG. 8.

For, if ray BN met ray AM, then ray CP would also meet ray AM at the same point (because MABN$\cong$MACP), and this would be common to the rays BN, CP themselves, while on the contrary BN $\parallel$ CP. Moreover every ray BQ interior to $\angle$CBN meets ray CP; therefore also it meets ray AM. Consequently BN $\parallel$ AM.

9. If BN $\parallel$ AM, and MAP $\perp$ MAB, and the dihedral $\angle$DNBA of the planes NBD, NBA (prolonged on that side of MABN where MAP is) is $<$rt. $\angle$, then MAP and NBD intersect.

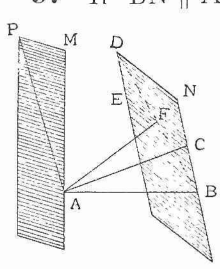

FIG. 9.

Make $\angle$BAM=rt. $\angle$, and AC $\perp$ BN (whether or not C coincides with B), and CE $\perp$ BN (in NBD).

We shall have by hypothesis $\angle$ACE $<$rt. $\angle$, and AF ($\perp$ CE) will fall within $\angle$ACE.

Let ray AP be the intersection of the hemi-planes ABF, AMP (which have the point A common). We shall have (since BAM $\perp$ MAP) $\angle$BAP$=\angle$BAM=rt. $\angle$.

*In placing this third case before the other two, these could be demonstrated with more brevity and elegance, like case 2 of §10. (Author's note.)

If now we move the hemi-plane ABF around the fixed points A and B until it coincides with the hemi-plane ABM, then ray AP will fall on ray AM, and since AC ⊥ BN, and sect AF< sect AC, then sect AF will have its extremity between ray BN and ray AM, and consequently BF will fall within ∠ABN. Now, *in this position*, ray BF will meet ray AP (since BN ∥ AM); therefore ray AP and ray BF intersect also *in the original position*, and the point of meeting is common to the hemi-planes MAP and NBD. Therefore the hemi-planes MAP and NBD intersect. From this we may deduce that the hemi-planes MAP and NBD intersect whenever the sum of the dihedral angles which they make with MAB is <rt. ∠.

10. If BN ∥ AM, and CP ∥ AM, and ∠ABN=∠BAM and ∠ACP=∠CAM, then also BN ∥ CP and ∠BCN=∠BCP.

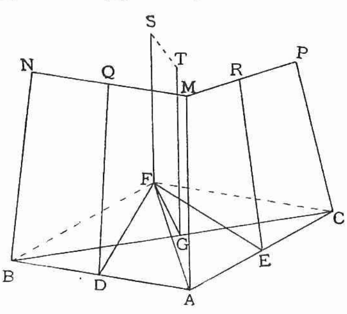

FIG. 10.

For, either the planes MAB, MAC make an angle, or they form one and the same plane.

1. In the first case, draw the hemi-plane QDF ⊥ sect AB at its mid point. Then we will have DQ ⊥ AB and consequently DQ ∥ AM (§ 8). Likewise if hemi-plane ERS is ⊥ sect AC at its mid point, ER ∥ AM.

Consequently (§ 7) DQ ∥ ER.

Hence (§ 9) the hemi-planes QDF and ERS intersect, and have (§ 7) their intersection the ray FS ∥ DQ. Moreover since BN ∥ DQ, we have also FS ∥ BN. Besides for every point F of FS we have the sects FB=FA=FC, and so the ray FS is in the hemi-plane TGF ⊥ sect BC at its mid point. Now since FS ∥ BN we have (§ 7) GT ∥ BN. In the same way GT ∥ CP. But GT ⊥ sect BC at its mid point.

But GT ⊥ sect BC at its mid point. Therefore TGBN ≅ TGCP (§ 1), and BN ∥ CP and ∠CBN=∠BCP.

2. If BM, AM, and CP are in one and the same plane, let FS be exterior to this plane and FS ∥ AM, and ∠AFS=∠FAM. Then from what precedes, FS ∥ BN, FS ∥ CP, ∠BFS=∠FBN, ∠CFS=∠FCP, consequently BN ∥ CP and ∠CBN=∠BCP.

11. Consider the aggregate of the point A and *all* points such that for any one of them B, when BN ∥ AM, also ∠ABN =∠BAM, and designate this aggregate by F; and call L the intersection of F with any plane drawn through the straight AM.

F has a point, and one only, on every straight ∥ AM; and L is divided by AM into two congruent parts.

Call the ray AM *the axis of L.* Evidently, in any one plane passing through the straight AM, there is for the axis ray AM a single line L. Call every line L so defined, the L of ray AM (in the plane, of course, that one considers). By the revolution of L around the straight AM we generate the F of which ray AM is called the axis, and which is, reciprocally, *the F of the axis ray AM.*

12. If B is any point of the L of ray AM, and BN ∥ AM and ∠ABN=∠BAM (§ 11), then the L of ray AM and the L of ray BN *coincide.* For suppose L′ the L of ray BN. Let C be any point of L′, and CP ∥ BN and ∠BCP=∠CBN (§ 11). Since BN ∥ AM and ∠ABN=∠BAM, therefore also CP ∥ AM and ∠ACP=∠CAM (§ 10). Consequently, C will be situated on L. And if C is any point of L, and CP ∥ AM and ∠ACP =∠CAM, then also CP ∥ BN and ∠BCP=∠CBN (§ 10); therefore C is likewise situated on L′ (§ 11). Thus L and L′ are identical, and every ray BN (∥ AM) is a new axis of L, and, if its origin is joined with that of any other axis, they make equal angles with the joining sect.

The same property may be demonstrated in the same manner for the surface F.

13. If BN ∥ AM, and CP ∥ DQ, and ∠BAM+∠ABN= st. ∠, then also ∠DCP+∠CDQ=st. ∠.

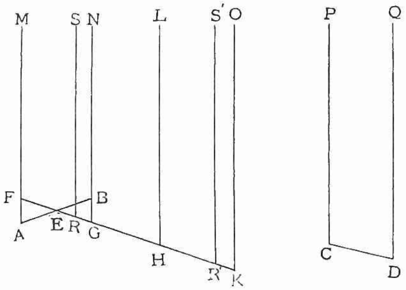

FIG. 11.

Let sect EA=sect EB, and ∠EFM=∠DCP (§ 4). Since ∠BAM + ∠ABN= st. ∠=∠ABN+∠ABG, we have ∠EBG=∠EAF

If therefore we have in addition sect BG=sect AF, then △EBG ≌ △EAF, ∠BEG=∠AEF and G will fall on the ray FE. We have moreover ∠GFM + ∠FGN=st. ∠ (since ∠EGB=∠EFA).

Moreover GN∥FM (§ 6).

Therefore if MFRS ≌ PCDQ, then RS∥GN (§ 7), and R falls within or without the sect FG (unless sect CD=sect FG, in which case the proposition would be evident).

1. In the first case ∠FRS is not>st. ∠—∠RFM=∠FGN, since RS∥FM. But as RS∥GN, ∠FRS is not <∠FGN. Therefore ∠FRS=∠FGN, and ∠RFM+∠FRS=∠GFM+∠FGN=st. ∠. Therefore also ∠DCP+∠CDQ=st. ∠.

2. If R falls without the sect FG, then ∠NGR=∠MFR.

Make MFGN ≌ NGHL ≌ LHKO, and so on until FK=FR or begins to be greater than FR. Then KO∥HL∥FM (§ 7).

If K falls on R then KO falls on RS (§ 1), and consequently ∠RFM+∠FRS=∠KFM+∠FKO=∠KFM+∠FGN=st. ∠ But if R falls within the sect HK, then (as in 1) we have ∠RHL+∠HRS=st. ∠=∠RFM+∠FRS=∠DCP+∠CDQ.

14. If BN∥AM, and CP∥DQ, and ∠BAM+∠ABN< st. ∠, then also ∠DCP+∠CDQ<st. ∠.

Because, if ∠DCP+∠CDQ were not <st. ∠, this sum (§ 1)

would be =st. ∠. Then we should have (§ 13) ∠BAM + ∠ABN=st. ∠, which is contrary to the hypothesis.

15. In consideration of what has been established in §§13 and 14, *we will designate by* Σ *the system of geometry which rests on the hypothesis of the truth of Euclid's axiom* XI, *and by* S *the system founded on the contrary hypothesis.*

All results enunciated without designating expressly whether they belong to the system Σ *or the system* S, *should be considered as enunciated absolutely, that is true whether placed in system* Σ *or system* S.

16. If AM is the axis of a line L, this line L, in the system Σ, will be a straight ⊥ AM.

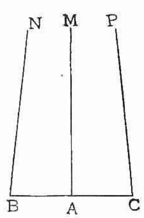

FIG. 12.

Suppose BN an axis at any point B of L; then in Σ, ∠BAM + ∠ABN=st. ∠, therefore ∠BAM =rt. ∠.

And if C is any point of the straight AB, and CP‖AM, then (§ 13) ∠ACP=∠CAM, and consequently C will be on L (§ 11).

But in S, there exists nowhere on L nor on F three points in a straight. For some one of the axes AM, BN, CP, (e. g. AM) falls between the others, and then (§ 14) ∠BAM and ∠CAM are each < rt. ∠.

17. L in S is a line, and F a surface. For (§ 11) every plane drawn perpendicular to the axis ray AM through any point of F, cuts F in [the circumference of] a circle, of which the plane (§ 14) is perpendicular to no other axis BN. If we revolve F about BN, any point of F (§ 12) will remain on F, and the section of F by a plane not ⊥ ray BN will describe a surface. Now, whatever be the points A, B taken on F, F can be so moved *in its trace* that A falls upon the trace of B (§ 12).

Thus F is a uniform surface, a surface which will slide in its own trace.

It follows (§§ 11 and 12) that L is a uniform line, a line which will slide on its trace.*

18. The intersection of F with any plane drawn through a point A of F obliquely to the axis AM, is, in the system S, a circle.

Take A, B, C, three points of this section, and BN, CP, axes. AMBN and AMCP make an angle, otherwise the plane determined by A, B, C, (§ 16), would contain AM, which is contrary to the hypothesis. Therefore the planes ⊥ the sects AB, AC at their mid points intersect (§ 10) in a certain axis ray FS of F, and we have FB=FA=FC.

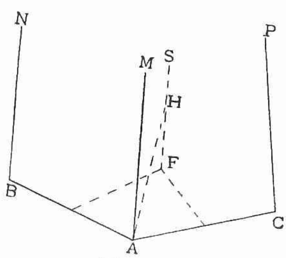

FIG. 13.

Make AH ⊥ FS, and revolve FAH around FS; A will describe a circle of radius HA, passing through B and C, and situated both in F and in the plane ABC; moreover, F and plane ABC have nothing common but the circle (○) HA (§ 16).

It is also evident that in revolving the portion FA of the line L (as radius) in F around A, its extremity will describe the circle with radius HA, ○ HA.

19. The perpendicular BT to the axis BN of L (drawn in the plane of L) is, in the system S, the tangent to the line L.

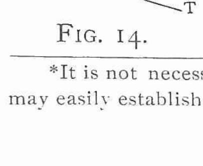

FIG. 14.

For L has in common with ray BT only the point B (§ 14). But if BQ is situated in the plane TBN, then the center of the section made in the F of ray BN by the plane drawn through BQ perpendicular to TBN (§ 18), is evidently on ray BQ; and if sect BQ is a diameter, it is clear that ray BQ will cut in Q the L of ray BN.

*It is not necessary to restrict the demonstration to the system S; we may easily establish that it is true absolutely for S and for Σ.

20. Any two points of F determine a line L (§§ 11 and 18); and since (§§ 16 and 19) L is $\perp$ to all its axes, every $\angle$ of lines L in F is equal to the $\angle$ of the planes drawn through its sides perpendicular to F.

21. Two line-rays, L-ray AP and L-ray BD, in the same surface F, making with a third line L, namely with line AB, interior angles of which the sun is < st. $\angle$, intersect.

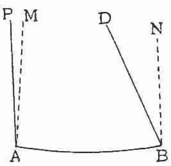

FIG. 15.

We shall designate by line AP, in F, the line L drawn through A and P, and by L-ray AP that half of this line beginning at A, which contains the point P.

Now, if AM, BN are axes of F, the hemi-planes AMP, BND intersect (§ 9), and F will meet their intersection (§§ 7 and 11). Therefore, L-ray AP and L-ray BD intersect.

From this it follows that Euclid's Axiom *XI* and all the consequences deduced from it in geometry and plane trigonometry are true absolutely in F, the lines L playing the role of straights. Consequently the trigonometric functions will be taken here in the same sense as in the system Σ; and the circle traced in F and having for radius a piece of line L equal to r, will have for length $2\pi r$; and area of $\odot$r (in F) $= \pi r^2$ (π designating the length of $\frac{1}{2}\bigcirc 1$ in F, that is to say, the known number 3.1415926+).

22. Let line AB be the L of ray AM, and C a point of ray AM. Suppose the $\angle$CAB (formed by the ray AM and the L-ray AB), translated first along the L-ray AB, then along the L-ray BA, each way to infinity. The path CD of the point C will be the line L of ray CM.

For, calling this latter L′, let D be any point of line CD, let DN be $\parallel$ CM, and B the point of L situated on the straight DN. We shall have BN $\parallel$ AM, and $\angle$ABN$=\angle$BAM, and sect

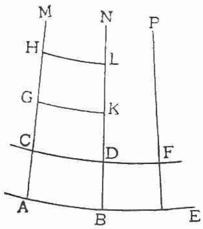

FIG. 16.

AC=sect BD, and consequently DN ∥ CM and ∠CDN=∠DCM; therefore D is on L′. Moreover, if D is on L′ and if DN ∥ CM, and B the point of L on the straight DN, we shall have AM ∥ BN, and ∠BAM=∠ABN, and CM ∥ DN and ∠DCM=∠CDN, whence follows that sect BD=sect AC, and D falls on the path of the point C. Therefore, L′ is identical with the line CD. We shall represent the relation of such a line L′ with L by the notation L′|||L.

23. If the line L represented by CDF is ∥ ABE (§22); if, moreover, AB=BE, and the rays AM, BN, EP are axes, we shall evidently have CD=DF.

If A, B, E are any three points of line AB, and we have AB=n.CD, we shall also have AE=n. CF, and consequently (extending evidently to the case of AB, AE, DC incommensurable), AB:CD=AE:CF The ratio AB:CD is, therefore, *independent of AB, and completely determined by AC.*

We shall designate the value of this ratio AB:CD by the capital letter (as X) corresponding to the small Italic (as x) by which we represent the sect AC.

24. Whatever be x and y, (§ 23), $Y=X^{\frac{y}{x}}$

For, one of the quantities x, y is a multiple of the other (e. g. y is a multiple of x) or it is not.

If $y=n.x$, take x=AC=CG=GH=&c., until we get AH=y.

Moreover, take CD ∥ GK ∥ HL.

We have (§ 23) X=AB:CD=CD:GK=GK:HL, and consequently
$$\frac{AB}{HL}=\left(\frac{AB}{CD}\right)^n,$$

or $Y=X^n=X^{\frac{y}{x}}$.

If x, y are multiples of i, we shall have in accordance with the above, $X = I^m$, $Y = I^n$, and consequently

$$Y = X^{\frac{n}{m}} = X^{\frac{y}{x}}.$$

This conclusion is easily extended to the case where x and y are incommensurable.

If $q = y - x$, then $Q = Y : X$.

In the system Σ, for every value x, we have $X = 1$.

In the system S, on the contrary, $X > 1$, and for any values of AB and ABE there is a line $\|$ AB such that CDF=AB, whence results AMBN $\cong$ AMEP, though the first of these two figures may be any multiple of the second; a singular result, but evidently not showing any absurdity in the system S.

25. *In every rectilineal triangle, the circles with radii equal to its sides are to each other as the sines of the opposite angles.*

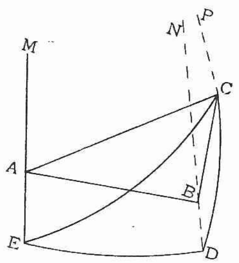

FIG. 17.

Take $\angle$ABC=rt. $\angle$, and AM $\perp$ BAC, and BN and CP $\|$ AM.

We shall have CAB $\perp$ AMBN, and consequently (since CB $\perp$ BA), CB $\perp$ AMBN; therefore, CPBN $\perp$ AMBN. Suppose that the F of ray CP cuts the straights BN, AM respectively in D and E, and the bandes CPBN, CPAM, BNAM along the L-lines CD, CE, DE. Then (§ 20) $\angle$CDE will be equal to the angle of NDC, NDE, and hence=rt. $\angle$; we have in the same way $\angle$CED=$\angle$CAB. Now, (§ 21) in $\triangle$CED formed by the L-lines, (supposing always here the radius=1), we have

$$EC : DC = 1 : \sin DEC = 1 : \sin CAB.$$

We have also (§ 21)

$$EC : DC = \bigcirc.EC : \bigcirc.DC \text{ (in F)} = \bigcirc.AC : \bigcirc.BC \text{ (§ 18)}.$$

Consequently we conclude

$$\bigcirc.AC : \bigcirc.BC = 1 : \sin CAB,$$

whence it follows that the theorem enunciated is established for any triangle.

26. *In any spherical triangle, the sines of the sides are to each other as the sines of the angles opposite.*

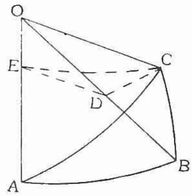

FIG. 18.

Take $\angle$ABC=rt. $\angle$, and CED ⊥ to the radius OA of the sphere. We shall have CED ⊥ AOB, and (BOC being also ⊥ to BOA), CD ⊥ OB. Now, in the triangles CEO, CDO, we have (§ 25)

$\bigcirc$.EC : $\bigcirc$.OC : $\bigcirc$.DC = sin COE : 1 : sin COD = sin AC : 1 : sin BC.

But we have also (§ 25) $\bigcirc$.EC : $\bigcirc$.DC = sin CDE : sin CED. Therefore, sin AC : sin BC = sin CDE : sin CED. But CDE = rt. $\angle$ = CBA, and CED = CAB. Consequently,

sin AC : sin BC = 1 : sin A.

From this follows the whole of spherical trigonometry, which is thus established independently of Euclid's Axiom XI.

27. If AC and BD are ⊥ AB and we translate the $\angle$CAB along the ray AB, we shall have, designating by CD the path described by the point C,

CD : AB = sin u : sin v.

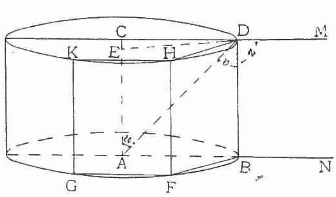

FIG. 19.

Take DE ⊥ CA. In the triangles ADE, ADB, we have (§ 25)

$\bigcirc$.ED : $\bigcirc$.AD : $\bigcirc$.AB = sin u : 1 : sin v.

In revolving BACD around AC, the point B will describe $\bigcirc$.AB, and the point D will describe $\bigcirc$.ED.

Designate here by $\odot$.CD the path of the line CD. Moreover, let there be any polygon BFG . inscribed in $\bigcirc$.AB.

Passing through all the sides BF, FG . planes ⊥ $\odot$.AB we form thus a polygonal figure of the same number of sides in ⊙.CD, and we may demonstrate, as in § 23, that CD : AB=DH : BF=HK : FG=. , and consequently

$$DH+HK+. \quad . : BF+FG+. \ . \ . =CD : AB.$$

If we make each of the sides BF, FG . . approach the limit zero, we have

$$BF+FG+. \ . \ . \doteq ○.AB \quad \text{and}$$
$$DH+HK+ \quad \doteq ○.ED.$$ We have therefore also ○.ED : ○.AB=CD : AB. Now, we already had ○.ED : ○.AB=sin u : sin v. Consequently,

$$CD : AB=\sin u : \sin v.$$

If AC goes away from BD to infinity, then the ratio CD : AB, and consequently also the ratio sin u : sin v remains constant. Now $u \doteq$ rt. ∠ (§ 1), and if DM ∥ BN, $v \doteq z$. Therefore, CD : AB=1 : sin z.

We shall designate this path CD by CD ||| AB.

28. If BN ∥ AM, and ∠ABN=∠BAM, and C a point of ray AM, then putting AC=x (§ 23) we shall have

$$X=\sin u : \sin v.$$

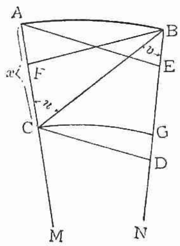

FIG. 20.

For, CD and AE being ⊥ BN, and BF ⊥ AM, we shall have (as in § 27)

$$○.BF : ○.CD=\sin u : \sin v.$$

Now evidently BF=AE. Therefore

$$○.EA : ○.DC=\sin u : \sin v.$$

But in the F-surfaces of AM and CM, which cut AMBN along AB and CG, we have (§ 21)

$$○.EA : ○.DC=AB : CG=X.$$

Therefore also

$$X=\sin u : \sin v.$$

29. If $\angle BAM =$ rt. $\angle$, and sect $AB = y$, and $BN \parallel AM$, we shall have in the system S,

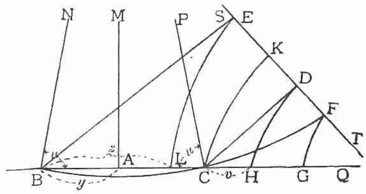

FIG. 21.

$$Y = \text{cotan } \tfrac{1}{2} u.$$

For, if we suppose sect $AB =$ sect AC, and $CP \parallel AM$ (and so $BN \parallel CP$ and $\angle CBN = \angle BCP$), and $\angle PCD = \angle QCD$, then we can draw (§ 19) $DS \perp$ ray CD so that $DS \parallel CP$, and consequently (§ 1) $DT \parallel CQ$. Moreover, if $BE \perp$ ray DS, then (§ 7) $DS \parallel BN$, consequently (§ 6) $BN \parallel ES$, and (since $DT \parallel CG$) $BQ \parallel ET$. Therefore, (§ 1) $\angle EBN = \angle EBQ$. Let BCF be an L-line of BN, and FG, DH, CK, EL, L-lines of FT, DT, CQ, &c. We shall have (§ 22) $HG = DF = DK = HC$; therefore,

$$CG = 2CH = 2v.$$

In the same way $BG = 2BL = 2z$.

Now $BC = BG - GC$; so $y = z - v$, whence (§ 24) $Y = Z : V$

Finally we have (§ 28)

$$Z = 1 : \sin \tfrac{1}{2} u,$$

$$V = 1 : \sin (\text{rt. } \angle - \tfrac{1}{2} u).$$

Therefore, $\quad Y = \text{cotan } \tfrac{1}{2} u.$

30. It is easy to see (after (§ 25) that solution of the problem of Plane Trigonometry, in the system S, requires the expression of the circle in terms of the radius. Now, we are able to obtain this by the rectification of the line L.

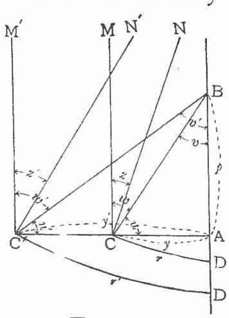

FIG. 22.

Let AB, CM, $C'M'$ be straights $\perp$ ray AC, and B any point of ray AB. We shall have (§ 25)

$$\sin u : \sin v = \bigcirc p : \bigcirc y,$$

$$\sin u' : \sin v' = \bigcirc p : \bigcirc y';$$

Consequently, $\dfrac{\sin u}{\sin v}.\bigcirc y = \dfrac{\sin u'}{\sin v'}.\bigcirc y'$

Now, we have (§ 27) $\sin v : \sin v' = \cos u : \cos u'$.

Therefore, $\dfrac{\sin u}{\cos u}.\bigcirc y = \dfrac{\sin u'}{\cos u'}.\bigcirc y'$.

or $\bigcirc y : \bigcirc y' = \tan u' : \tan u = \tan w : \tan w'$.

Take now CN and C′N′ ∥ AB, and CD, C′D′ L-lines ⊥ AB. We shall have then (§ 21)

$$\bigcirc y : \bigcirc y' = r : r', \text{ whence}$$

$$r : r' = \tan w : \tan w'$$

Make p increase from A to infinity; then $w \doteq z$ and $w' \doteq z'$, whence results also $r : r' = \tan z : \tan z'$.

Designate by i the *constant* ratio

$r : \tan z$ (*independent* of r).

If we suppose $y \doteq 0$, then

$$\frac{r}{y} = \frac{i \tan z}{y} \doteq 1, \text{ and consequently}$$

$\dfrac{y}{\tan z} \doteq i$. From §29, it follows that $\tan z = \frac{1}{2}(Y - Y^{-1})$.

Therefore $\dfrac{2y}{Y - Y^{-1}} \doteq i$,

or (§ 24)

$$\frac{2y . I^{\frac{y}{i}}}{I^{\frac{2y}{z}} - 1} \doteq i.$$

Now, we know that the limit of this expression, for

$y \doteq o$, is $\dfrac{i}{\text{nat. log I}}$. Therefore,

$$\frac{i}{\text{nat. log I}} = i, \text{ and consequently}$$

$$I = e = 2.7182818+,$$

a number which presents itself here in a remarkable manner.

Designating henceforth by i the sect of which the $\mathrm{I}=e$, we shall have

$$r = i \tan z.$$

We have found elsewhere (§ 21) $\bigcirc y = 2\pi r$.

Therefore, $$\bigcirc y = 2\pi i \tan z = \pi i\,(\mathrm{Y}-\mathrm{Y}^{-1}) = \pi i \left(e^{\frac{y}{i}} - e^{-\frac{y}{i}} \right)$$

$$= \frac{\pi y}{\text{nat. log Y}}(\mathrm{Y}-\mathrm{Y}^{-1}) \quad (\S\ 24).$$

31. For the trigonometric solution of all right-angled rectilineal triangles (whence is easily deduced that of all rectilineal triangles whatsoever), in the system S, three equations suffice.

Let c be the hypothenuse, a, b the sides of the right angle, and α, β the angles respectively opposite to a and b. These three equations shall be those which express relations.

I. Between a, c, α;
II. Between a, α, β;
III. Between a, b, c.

From these equations we shall deduce afterward three others by elimination.

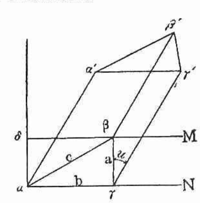

FIG. 23.

I. From §§ 25 and 30 we get

$$1 : \sin \alpha = (\mathrm{C}-\mathrm{C}^{-1}) : (\mathrm{A}-\mathrm{A}^{-1}) =$$

$$= \left(e^{\frac{c}{i}} - e^{-\frac{c}{i}} \right) : \left(e^{\frac{a}{i}} - e^{-\frac{a}{i}} \right),$$ *an equation between c, a, and α.*

II. From §27 we deduce (BM being $\parallel \gamma$n)

$$\cos \alpha : \sin \beta = 1 : \sin u.$$

Now, we have (§ 29)

$1 : \sin u = \frac{1}{2}(\mathrm{A}+\mathrm{A}^{-1})$; therefore $\cos \alpha : \sin \beta = \frac{1}{2}(\mathrm{A}+\mathrm{A}^{-1}) =$

$$\frac{1}{2}\left(e^{\frac{a}{i}} + e^{-\frac{a}{i}} \right),$$ *an equation between α, β, and a.*

III. Take $\alpha\alpha' \perp \beta\alpha\gamma$; $\beta\beta'$ and $\gamma\gamma' \parallel \alpha\alpha'$ (§ 27), and $\beta'\alpha'\gamma' \perp \alpha\alpha'$. We will evidently have (as in § 27)

$$\frac{\beta\beta'}{\gamma\gamma'} = \frac{1}{\sin u} = \tfrac{1}{2}(A + A^{-1}),$$

$$\frac{\gamma\gamma'}{\alpha\alpha'} = \tfrac{1}{2}(B + B^{-1}),$$

$$\frac{\beta\beta'}{\alpha\alpha'} = \tfrac{1}{2}(C + C^{-1}).$$ Consequently

$$\tfrac{1}{2}(C+C^{-1}) = \tfrac{1}{2}(A+A^{-1}).\tfrac{1}{2}(B+B^{-1}),$$ or

$$e^{\frac{c}{i}} + e^{-\frac{c}{i}} = \tfrac{1}{2}\left\{e^{\frac{a}{i}} + e^{-\frac{a}{i}}\right\}\left\{e^{\frac{b}{i}} + e^{-\frac{b}{i}}\right\},$$

an equation between a, b, and c.

If $\gamma\alpha\delta =$ rt. $\angle$, and we have $\beta\delta \perp \alpha\delta$, then we shall get

$$\bigcirc c : \bigcirc a = 1 : \sin \alpha,$$ and

$$\bigcirc c : \bigcirc (d = \beta\delta) = 1 : \cos \alpha.$$

Therefore, designating by $\bigcirc x^2$, for any value of x, the product $\bigcirc x.\bigcirc x$, we shall evidently have

$$\bigcirc a^2 + \bigcirc d^2 = \bigcirc c^2.$$

Now, we have found (§ 27 and § 31, II)

$$\bigcirc d = \bigcirc b.\tfrac{1}{2}(A + A^{-1}),$$ Consequently

$$\left\{e^{\frac{c}{i}} - e^{-\frac{c}{i}}\right\}^2 = \tfrac{1}{4}\left\{e^{\frac{a}{i}} + e^{-\frac{a}{i}}\right\}^2 \cdot \left\{e^{\frac{b}{i}} - e^{-\frac{b}{i}}\right\}^2 + \left\{e^{\frac{a}{i}} - e^{-\frac{a}{i}}\right\}^2,$$

another relation between a, b, and c, the second member of which may be easily put into a form symmetric or invariable.

Finally, from the equations

$$\frac{\cos \alpha}{\sin \beta} = \tfrac{1}{2}(A+A^{-1}),\quad \frac{\cos \beta}{\sin \alpha} = \tfrac{1}{2}(B+B^{-1}),$$ we get (after II)

$$\cot \alpha \cot \beta = \tfrac{1}{2}\left\{e^{\frac{c}{i}} + e^{-\frac{c}{i}}\right\},$$

an equation between α, β, and c.

32. It still remains to show briefly the means of resolving problems in the system S. After having expounded this in regard to the most ordinary examples, we shall see finally what this theory is able to give.

I. Take AB a line in a plane, and $y=f(x)$ its equation in rectangular coordinates. Designate by dz any increment of z, and by dx, dy, du the increments of x, of y, and of the area u, corresponding to this increment dz. Take BH ||| CF; express (§31) $\frac{BH}{dx}$ by means of y, and seek the *limit* of $\frac{dx}{dy}$, when dx tends toward the limit zero (which is always understood when one seeks such limits.

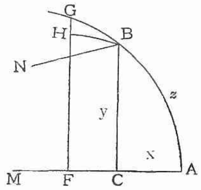

FIG. 24.

We shall then know the limit of $\frac{dy}{BH}$, and so tan HBG; and consequently (since evidently HBC can be neither >, nor <rt. ∠, and so is=rt. ∠), the *tangent* at B of the line BG will be determined by means of y.

II. We can demonstrate that

$$\frac{dz^2}{dy^2+\overline{BH}^2} \doteq 1.$$

Thence we deduce the limit of $\frac{dz}{dx}$, and from it we get, by integration, the expression for z in terms of x.

Given any real curve, we can find its equation in the system S.

For example, to find the equation of a line L. Let ray AM be the axis of the line L; every straight drawn through A, other than the straight AM, meeting L (§ 19), the random ray CB, starting from a point of ray AM, will meet L.

Now, if BN is an axis, we have

$$X=1:\sin \text{CBN } (\S\,28),$$

$$Y=\text{cotan } \tfrac{1}{2} \text{ CBN } (\S\,29), \text{ whence}$$

we get $Y=X+\sqrt{X^2-1}$,

or

$$e^{\frac{y}{i}}=e^{\frac{x}{i}}+\sqrt{e^{\frac{2x}{i}}-1},$$

which is the equation sought.

Hence we get

$$\frac{dy}{dx} \doteq X.(X^2-1)^{-\frac{1}{2}}$$

Now, $\quad \frac{BH}{dx}=1:\sin CBN=X.$ Therefore

$$\frac{dy}{BH} \doteq (X^2-1)^{-\frac{1}{2}}$$

$$1+\frac{dy^2}{BH^2} \doteq X^2.(X^2-1)^{-1},$$

$$\frac{dz^2}{BH^2} \doteq X^2.(X^2-1)^{-1},$$

$$\frac{dz}{BH} \doteq X.(X^2-1)^{-\frac{1}{2}}, \quad \text{and}$$

$$\frac{dz}{dx} \doteq X^2.(X^2-1)^{-\frac{1}{2}}, \quad \text{whence,}$$

integrating, we get (as in §30)

$$z=i(X^2-1)^{\frac{1}{2}}=i \cot CBN.$$

III. Evidently

$$\frac{du}{dx} \doteq \frac{HFCBH.}{dx}$$

If this quantity is not given in y, it is necessary to express it in terms of y, and then we get u from it by integration.

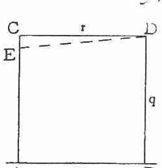

FIG. 25.

Putting $AB=p$, $AC=q$, $CD=r$, and $CABD$ $=S$, we might show (as in II) that

$$\frac{ds}{dq} \doteq r, \text{ a quantity equal to}$$

$$\tfrac{1}{2}p\left(e^{\frac{q}{i}}+e^{-\frac{q}{i}}\right), \text{ whence, integrating,}$$

$$S=\tfrac{1}{2}pi\left(e^{\frac{q}{i}}-e^{-\frac{q}{i}}\right).$$

We might also obtain this result without integration.

For example, if we establish the equation of the circle (after §31, III), of the straight, (from §31, II), of a conic (from what just precedes), we could express also the areas bounded by these lines.

We know that a surface t, ||| a plane figure p (at the distance q) is to p in the ratio of the second powers of homologous lines, that is to say in the ratio of

$$\tfrac{1}{4}\left(e^{\frac{q}{i}}+e^{-\frac{q}{i}}\right)^2 : 1.$$

It is easy to see, moreover, that the calculation of volume, treated in the same manner, requires two integrations (the differential itself being determinable only by integration).

It is necessary first of all to investigate the volume contained between p and t, and the aggregate of all the straights $\perp p$ and joining the boundaries of p and t.

We find for the volume of this solid (whether by means of integration or otherwise)

$$\tfrac{1}{8}\,pi\left(e^{\frac{2q}{i}}-e^{-\frac{2q}{i}}\right)+\tfrac{1}{2}\,pq.$$

The surfaces of bodies may also be calculated in the system S, as well as the *curvatures*, the involutes, the evolutes of any lines, etc.

As to curvature, in the system S, either it will be the curvature of the line L itself, or we may determine it either by the radius of a circle, or by the *distance* of a straight from the curve ||| to this straight; and it is easy to make it evident, after what precedes, that there is not, in a plane any uniform line other than L-lines, circles, and the curves ||| to straights.

IV For the circle we have (as in III) $\frac{d\odot x}{dx} = \bigcirc x$, whence (§ 29), integrating, we get

$$\odot x = \pi i^2\left(e^{\frac{x}{i}}-2+e^{-\frac{x}{i}}\right).$$

V. Take u=CABDC the area comprised between an L-line, AB=r, a ||| to that line, CD=y, and the sects AC=BD=x.

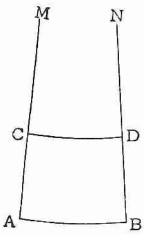

FIG. 26.

We have $\frac{du}{dx} \doteq y$, and (§ 24) $y = re^{-\frac{x}{i}}$, whence integrating $u = ri\left(1 - e^{-\frac{x}{i}}\right)$

If x increases to infinity, then, in the system S, $e^{-\frac{x}{i}} \doteq 0$, and consequently $u \doteq ri$. We shall call this limit the *size* of MABN.

We may see in the same manner that, if p is a figure traced on F, the space comprised between p and the aggregate of axes drawn through the different points of the boundary of p is equal to $\frac{1}{2}pi$.

Let $2u$ be the angle at the center of the spherical calotte z, and p a great circle, and x the arc FC corresponding to the angle u. We shall have (§ 25)

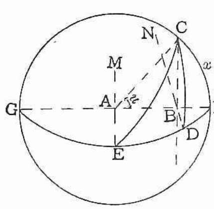

FIG. 27.

$1 : \sin u = p : \bigcirc.\text{BC}$, whence

$\bigcirc.\text{BC} = p \sin u$. We have, besides,

$$x = \frac{pu}{2\pi}, \qquad dx = \frac{pdu}{2\pi}$$

Moreover, $$\frac{dz}{dx} \doteq \bigcirc.\text{BC},$$

$$\frac{dz}{du} \doteq \frac{p^2}{2\pi} \sin u, \text{ and, integrating,}$$

$$z = \frac{\text{ver} \sin u}{2\pi} p^2.$$

Imagine the surface F on which is situated the circle p (passing through the middle F of the calotte). Draw through AF and AC the hemi-planes FEM, CEM, perpendicular to F and cutting F along FEG and CE; and consider the L-line CD (drawn through C perpendicular to FEG), and the L-line CF.

We shall have (§ 20) CEF$=u$, and (§ 21)

$$\frac{fd}{p}=\frac{\text{ver sin } u}{2\pi}, \text{ whence } z=\text{FD}.p.$$

Now (§ 21) $p=\pi.\text{FDG}$; therefore

$z=\pi.\text{FD}.\text{FDG}$. But (§ 21)

$\text{FD}.\text{FDG}=\text{FC}.\text{FC}$; consequently

$z=\pi.\text{FC}.\text{FC}=\odot.\text{FC}$, in F.

Now let $\text{BJ}=\text{CJ}=r$; we shall have (§ 30) $2r=i(\text{Y}-\text{Y}^{-1})$, whence, (§ 21)

$$\odot 2r(\text{in F})=\pi i^2(\text{Y}-\text{Y}^{-1})^2.$$

We also have (*IV*)

$$\odot 2y=\pi i^2(\text{Y}^2-2+\text{Y}^{-2}).$$

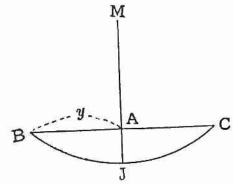

FIG. 28.

Therefore, $\odot 2r$ (in F)$=\odot 2y$, and consequently *the surface* z *of the segment of a sphere is equal to the surface of the circle described with the chord* fc *as radius*.

Therefore the whole sphere has for surface

$$\odot.\text{FG}=\text{FDG}.p=\frac{p^2}{\pi},$$

and the surfaces of spheres are to each other as the second powers of their great circles.

VII. We find in like manner that, in the system S, the volume of the sphere of radius x is equal to

$$\tfrac{1}{2}\pi i^2(\text{X}^2-\text{X}^{-2})-2\pi i^2 x.$$

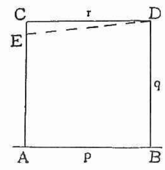

FIG. 29.

The surface generated by the revolution of the line CD around AB is equal to

$$\tfrac{1}{2}\pi i p(\text{Q}^2-\text{Q}^{-2}),$$

and the solid generated by CABDC is equal to

$$\tfrac{1}{4}\pi i p(\text{Q}-\text{Q}^{-1})^2.$$

We suppose, for the sake of brevity, the method by which one may obtain without integration all the results reached from IV thus far.

We can demonstrate that *the limit of every expression containing the letter i* (and consequently founded on the hypothesis that a magnitude i exists), *when i increases to infinity, gives the corresponding expression in the system* Σ (and consequently under the hypothesis that a magnitude i does not exist), provided that we do not meet identical equations.

But we must be very careful not to get the idea that the system itself may be changed at will (for it is entirely determined in itself and by itself); it is only *the hypothesis* which may vary, and which we may change successively, so far as we are not conducted to an absurdity. In *supposing* therefore that, in such an expression, the letter i, in case the system S is that of reality, designates the unique quantity of which the I has e for its value, if we come to recognize that it is the system Σ, which is really actual, we conceive that the limit in question is to be taken in place of the primitive expression. Then it is evident that with this understanding, *all the expressions founded on the hypothesis of the reality of the system S will be true absolutely, even when we are completely ignorant whether or not the system* Σ *is the system of reality.*

So, for example, from the expression obtained in §30 we easily get (either by means of differentiation or otherwise) the known value in the system Σ,

$$\bigcirc x = 2\pi x.$$

From I (§31) we conclude, by suitable transformations,

$$1 : \sin \alpha = c : a;$$

from II we get

$$\frac{\cos \alpha}{\sin \beta} = 1, \text{ and consequently}$$

$$\alpha + \beta = 1.$$

The first equation of III becomes identical, and so it is true in the system Σ, although it there determines nothing. From the second we conclude

$$c^2 = a^2 + b^2.$$

These are the known fundamental equations of plane trigonometry in the system Σ.

Moreover, we find (after §32) in the system Σ, for the area and the volume in III the same value pq.

We have, from IV,

$$\odot x = \pi x^2$$

According to VII, the globe of radius x is

$$= \tfrac{4}{3}\pi x^3, \text{ etc,}$$

The theorems enunciated at the end of VI are evidently *true without conditions.*

33. It still remains to set forth (as we promised in §32) what is the end of this theory.

I. Is it the system Σ or the system S which exists in reality?

That is what we cannot decide.

II. All the results deduced from the falsity of Axiom *XI* (always taking these words in the sense of §32) are *absolutely true*, and in this sense, *depend on no hypothesis.*

There is therefore *a plane trigonometry a priori, in which the system alone really remains unknown;* and where we lack only the *absolute* magnitudes in the expressions, but where a single known case would evidently fix the whole system. On the contrary, spherical trigonometry is established absolutely in §26.

We have, on the surface F, a geometry wholly analogous to the plane geometry of the system Σ.

III. If it were established that it is the system Σ which exists, nothing more would remain to be known on this point.

But if it were *established* that the system Σ *does not exist,* then (§31), being given, for example, in a concrete manner, the sides x, y, and the rectilineal angle which they include, it is clear that it would be impossible in itself and by itself to solve absolutely the triangle, that is to say, to determine *a priori* the other angles and the ratios of the third side to the two given sides, unless one could determine the quantities X, Y. For that, it would be necessary to have in concrete form a certain sect a of which the A was known. Then i would be *the natural unit for length* (as e is the base of *natural* logarithms).

If the existence of this quantity i is supposed to be known, we see how one could construct it, at least with a high degree of approximation, for practical use.

IV. In the sense explained (I and II), we may evidently apply everywhere the modern analytic method (so useful when one employs it within suitable limits).

V. Finally, the reader will not be sorry to see that in case it is the system S, and not the system Σ, which really exists, we can construct a rectilineal figure equivalent to a circle.

34. Through D we may draw DM $\|$ AN in the following manner. From the point D drop DB $\perp$ AN; at any point A of the straight AB erect AC $\perp$ AN (in the plane DBA) and let fall DC $\perp$ AC. We will have (§ 27) ○.CD : ○.AB = 1 : sin z, provided that DM $\|$ BN. Now sin z is not >1; therefore AB is not $>$ DC. Therefore a quadrant described from the center A in BAC, with a radius = DC, will have a point B on ○ in common with ray BD. In the first case, we have evidently z = rt. $\angle$. In the second case we shall have (§ 25)

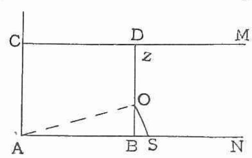

FIG. 30.

$$\bigcirc.\mathrm{AO}(=\mathrm{CD}) : \bigcirc.\mathrm{AB} = 1 : \sin \mathrm{AOB}$$

and consequently z = AOB.

If therefore we take z = AOB, then DM will be $\|$ BN.

35. In the system S we may, as follows, draw a straight $\perp$ to one of the sides of an acute angle and at the same time $\|$ to the other side.

Take AM ⊥ BC, and suppose AB=AC sufficiently small (§19) to make, when we draw BN∥AM (§34) ABN > the given angle.

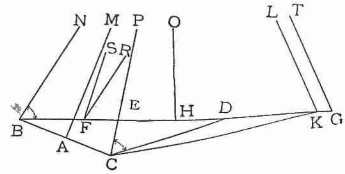

FIG. 31.

Draw also CP∥AM (§34), and take NBG and PCB each equal to the given angle. Rays BG and CD will meet; because if ray BG (falling *by construction* within NBC) cuts ray CP in E, we shall have (since ∠CBN=∠BCP) ∠EBC∠ECB, and so EC< EB. Take EF=EC, EFR=ECD, and FS∥EP, then FS will fall within the angle BFR. Because, since BN∥CP, whence BN∥EP, and BN∥FS, we shall have (§14)

$$\angle FBN + \angle BFS < \text{st. } \angle = FBN + BFR.$$

Therefore, BFS< BFR. Consequently, ray FR cuts ray EP, and so ray CD also cuts ray EG somewhere in D. Take now DG=DC and DGF=DCP=GBN. We shall have (since ∠GCD=∠CGD) ∠GBN=∠BGT and ∠GCP=CGT. Let K (§19) be the point where the line L of BN meets the ray BG and KL the axis of this L-line. We shall have ∠KBN= ∠BKL, and so BKL=BGT=DCP.

Moreover, CKL=KCP. Therefore, evidently K falls on G, and GT∥BN. If now we erect HO ⊥ BG at its mid point, we shall have constructed HO∥BN.

36. Having given the ray CP and the plane MAB, take CB ⊥ the plane MAB, BN (in hemi-plane BCP) ⊥ BC, and CQ∥BN (§34). The meeting of ray CP (if this ray falls within BCQ) with ray BN (in the plane CBN), and consequently with the plane MAB, may be determined, And if we are given

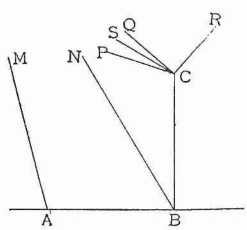

FIG. 32.

the two planes PCQ, MAB, and we have CB ⊥ to plane

MAB, CR ⊥ plane PCQ; and (in plane BCR), BN ⊥ BC, CS ⊥ CR, BN will fall in plane MAB, and CS in plane PCQ; and when we have found the intersection of the straight BN with the straight CS (if there is one), the perpendicular drawn through this intersection, in PCQ, to the straight CS will evidently be the tntersection of plane MAB and plane PCQ.

37. On the straight AM‖BN, there is a point A, such that ∠BAM=∠ABN.

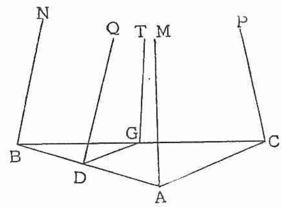

FIG. 33.

If (according to §34) we construct outside of the plane NBM, GT ‖ BN, and make BG ⊥ GT, GC=GB, and CP‖GT; and so draw the hemi-plane TGD that it makes with hemi-plane TGB an angle equal to that which hemi-plane PCA makes with PCB.

Seek then (§36) the intersection DQ of hemi-plane TGD with hemi-plane NBD, and finally draw BA ⊥ DQ.

We shall have, by reason of the similtude of the triangles of L lines traced on the F of BN (§21), DB=DA, and ∠BAM =∠ABN.

We readily conclude from this, that, L-lines being given by their extremities alone, we may obtain in this manner, in F, a fourth proportional, or a mean proportional, and execute, without recourse to Axiom XI, all the geometric constructions made on the plane in the system Σ. Thus, for example, we can geometrically divide a perigon into any special number of equal parts, if we know how to make this special partition in the system Σ.

38. If we construct (§37) for example, NBQ=$\frac{1}{3}$ rt. ∠, and draw (§35), in the system S, AM ⊥ ray BQ and ‖ BN; if we determine, again (§37), ∠BIM=∠IBN, we shall have, putting IA=x (§28), X= $1:\sin\frac{1}{3}$ rt. ∠=2, and x will be constructed *geometrically*.

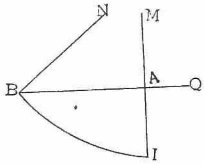

FIG. 34.

We may calculate NBQ so that IA differs from i as little as we choose, which happens for sin NBQ$=\frac{1}{e}$.

39. If in a plane PQ and ST are ||| to the straight MN (§27), and the perpendiculars AB, CD are equal, we shall evidently have $\triangle$DEC $\cong$ $\triangle$BEA; consequently the angles (may be mixtilinear) ECP, EAT would coincide, and we have EC$=$EA. If, besides CF$=$AG, then $\triangle$ACF $\cong$ $\triangle$CAG, and each of them is the half of the *quadrilateral* FAGC.

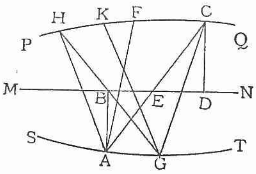

FIG. 35.

If FAGC, HAGK are two of these quadrilaterals, of base AG, contained between PQ and ST, we may demonstrate their equivalence (as in Euclid), as also the equivalence of the triangles AGC, AGH, on a common base AG, and having their vertices on PQ. Moreover, we have ACF$=$CAG, GCQ$=$CGA, and ACF$+$ACG$+$GCQ$=$st. $\angle$ (§32); consequently we also have CAG$+$ACG$+$CGA$=$st. $\angle$. Therefore, in every triangle ACG of this sort, the sum of the angles$=$st. $\angle$. Whether the straight AG coincides with AG (||| MN), or not, the equivalence of the triangles AGC, AGH, *as well in relation to their areas as in relation to the sum of their angles*, is evident.

40. *Equivalent triangles* ABC, ABD, (which we will henceforth suppose rectilineal), *having one side equal, have the sums of their angles equal.*

Draw MN through the mid points of AC and BC, and take (through the point C) PQ ||| MN. The point D will fall on PQ.

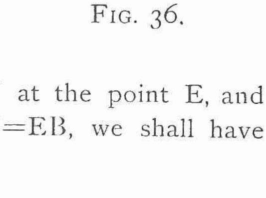

FIG. 36.

For, if ray BD cuts the straight MN at the point E, and and consequently PQ at F making EF$=$EB, we shall have

△ABC=△ABF, and so also △ABD=△ABF, whence it follows that D falls at F.

But if ray BD does not cut the straight MN, let C be the point where the perpendicular from the middle of AB meets PQ, and let GS=HT, so that the straight ST meets the ray BD prolonged in a certain point K (which is possible as we have seen in §4). Take also SR=SA, RO ||| ST, and O the intersection of ray BK with RO. We would then have △ABR=△ABO (§39), and consequently △ABC>△ABD, which would be contrary to the hypothesis.

41. *Equivalent triangles ABC, DEF have the sums of their angles equal.*

Draw MN through the mid points of AC and BC, and PQ through the mid points of DF and FE; and take RS ||| MN, TO ||| PQ.

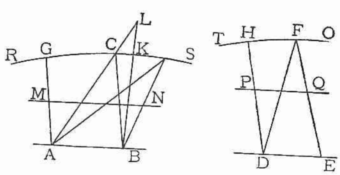

FIG. 37.

The perpendicular AG to RS will equal the perpendicular DH to TO, or will differ from it; for example, DH will be the greater.

In each of these two cases, the ○.DF, described from center A, will have with line-ray GS a common point K, and then (§39) we shall have △ABK=△ABC=△DEF. Now the △ABK (§40) has the same angle-sum as △DEF, and (§39) the same angle-sum as △ABC. Therefore the triangles ABC, DEF have each the same angle-sum.

In the system S the reverse of this theorem is true.

Take ABC, DEF two triangles having the same angle-sum, and △BAL=△DEF. These latter triangles will have, from what precedes, the same angle-sum, and consequently so will △ABC and △ABL.

From this follows evidently

$$BCL+BLC+CBL=\text{st. } \angle.$$

Now (§31) the angle-sum of every triangle, in the system S, is $<$ st. $\angle$

Therefore L falls necessarily on C.

42. Let u be the supplement of the angle-sum of the $\triangle$ABC, and v the supplement of the angle sum of $\triangle$DEF. We shall have $\triangle ABC : \triangle DEF = u : v$.

Let p be the area of each of the equal triangles ACG, GCH, HCB, DFK, KFE, and let $\triangle ABC = m.p$, and $\triangle DEF = n.p$. Designate by s the angle-sum of any one of the triangles equivalent to p. We shall evidently have

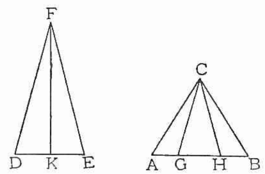

FIG. 38.

$$\text{st.} \angle - u = m.s - (m-1)\text{st.} \angle = \text{st.} \angle - m(\text{st.} \angle - s);$$

and $u = m(\text{st.} \angle - s)$. In the same way $v = n(\text{st.} \angle - s)$.

Therefore $\triangle ABC : \triangle DEF = m : n = u : v$.

The demonstration is easily extended to the case of the incommensurability of the triangles ABC, DEF.

We may demonstrate in the same way that spherical triangles are as their spherical excesses.

If two of the angles of the spherical $\triangle$ are right angles, the third z will be the *excess* in question. Now, designating by p a great circle, this $\triangle$ is evidently

$$= \frac{z}{2\pi} \cdot \frac{p^2}{2\pi} \text{ (§32, VI).}$$

Consequently, any triangle of which the excess is z, is

$$= \frac{zp^2}{4\pi^2}$$

43. Thus, in the system S, the area of a rectilineal $\triangle$ is expressed by means of the sum of its angles.

If AB increases to infinity, then (§42) the relation $\triangle ABC : \text{rt.} \angle - u - v$ will be constant. Now, $\triangle ABC \doteq BACN$ (§32, V) and $\text{rt.} \angle - u - v \doteq z$ (§1).

Therefore, $BACN : z =$

$\triangle ABC : (\text{rt.} \angle - u - v) = BAC'N' : z'$

Moreover, we evidently have (§30)

$BDCN : BD'C'N' = r : r' = \tan z : \tan z'$.

Now, for $y' \doteq 0$, we have

$$\frac{BD'C'N'}{BA'C'N'} \doteq 1, \text{ and also } \frac{\tan z'}{z'} \doteq 1.$$

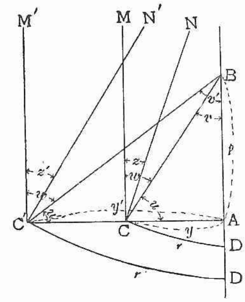

FIG. 39.

Consequently,

$$BDCN : BACN = \tan z : z.$$

But we have found (§32)

$$BDCN = r.i = i^2 \tan z.$$

Therefore,

$$BACN = z.i^2.$$

Designating henceforth, for brevity, every triangle of which the supplement of the angle-sum is z by $\triangle$, we will thus have

$$\triangle = z.i^2.$$

Hence we readily conclude that, if $OR \| AM$ and $RO \| AB$, the *area* contained between the straights OR, ST, BC (which is evidently the absolute limit of the area of rectilineal triangles increasing indefinitely, or the limit of $\triangle$ for $z \doteq \text{st.} \angle$), will be equal to $\pi i^2 = \odot i$ (in F).

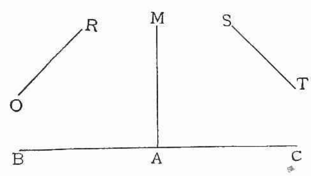

FIG. 40.

Designating this limit by $\square$, we will also have (§30) $\pi r^2 = \tan^2 z . \square = \odot r$ (in F) (§21) $= \odot s$ (§32, VI), representing the chord CD by s.

If now, by means of a perpendicular erected at the mid point of the given radius s of the circle in a plane (or of the radius of form L of the circle in F), we construct (§34) DB∥CN and ∠CDB=∠DCN; by dropping CA ⊥ DB, and erecting CM ⊥ CA, we shall get z; whence (§37), taking arbitrarily a radius of form L for unity, *we shall be able to determine geometrically* $\tan^2 z$, *by means of two uniform lines of the same curvature* (which, their extremities alone being given and their axes constructed, may evidently be treated as straights in seeking their common measure, and are in this respect the equivalent of straights).

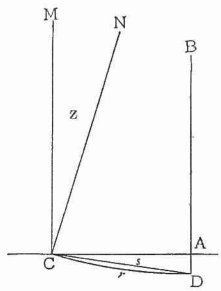

FIG. 41.

We can, moreover, construct as follows a quadrilateral, for example a regular quadrilateral, of area=□.

Take ABC=rt. ∠, BAC=$\frac{1}{2}$ rt. ∠, ACB=$\frac{1}{4}$ rt. ∠, and BC=x.

We can express X (§31, II) by simple square roots, and construct it (§37).

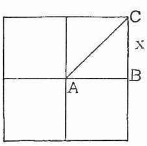

FIG. 42.

Knowing X, we can determine x (§38 or also §§29 and 35). The octuple of △ABC is evidently =□, and thus *a plane circle is geometrically squared by means of a rectilinear figure and of uniform lines of the same species* (that is to say of lines equivalent to straights as to their comparison to each other).

A circle of the surface F is planified in the same manner.

Thus either the Axiom XI *of Euclid is true or we have the geometric quadrature of the circle*, though nothing thus far decides which of the two propositions is real.

Whenever $\tan^2 z$ is either a whole number, or a rational fraction, of which the denominator (after reduction to the simplest form) is either a prime number of the form 2^m+1 (of which $2=2^0+1$ is a particular case), or a product of prime numbers of this form, of which each (with the exception of 2, which alone may enter any number of times) enters *only once* as factor, we can, by the theory of polygons given by Gauss (and for such values alone), construct a rectilineal figure $=\tan^2 z = \odot s$. Because the division of $\square$ (the theorem of §42 extending easily to any polygons) requires evidently the partition of a st. $\angle$, which (as we can demonstrate) is possible geometrically only under the preceding condition.

In all such cases, what precedes conducts easily to the desired end; and every rectilineal figure can be transformed geometrically into a regular polygon of n sides, if n is of the form indicated by Gauss.

It still remains, for the entire completion of our researches, to demonstrate the impossibility of deciding (without having recourse to some hypothesis) whether it is the system Σ, or some one of the systems S (and which one) which really exists. This we reserve for a more favorable occasion.

APPENDIX I.

REMARKS ON THE PRECEDING MEMOIR, BY WOLFGANG BOLYAI.

[From Vol. II of Tentamen, p. 380, et seq.]

I may be permitted to add here certain remarks appertaining to the author of the *Appendix*, who may pardon me if I do not always well express his thought.

The formulas of spherical trigonometry (demonstrated in the preceding memoir independently of Euclid's Axiom XI) *coincide with the formulas of plane trigonometry, when we consider* (to use a provisional method of speaking) *the sides of a spherical triangle as reals, those of a rectilineal triangle as imaginaries;* so that, when it is a question of trigonometric formulas, we may regard the plane as an imaginary sphere, taking for real sphere that in which sin rt. $\angle = 1$.

We demonstrate (§ 30) that there is a certain quantity i (in case of the non-existence of Euclid's axiom), such that the corresponding quantity I is equal to the base e of natural logarithms. In this case, we establish also (§31) the formulas of plane trigonometry, and in such manner (§32, VII) that the formulas are still true for the case of the reality of the axiom in question, taking the limit of the values for $i \doteq \infty$. Thus the Euclidean system is in a certain way the limit of the anti-Euclidean system for $i \doteq \infty$.

Take, in case of the existence of i, the unit $=i$, and extend the definitions of sine and of cosine to imaginary arcs, so that, p designating an arc whether real or imaginary, the expression

$$\frac{e^{p\sqrt{-1}}+e^{-p\sqrt{-1}}}{2}$$ is to be always called

the *cosine* of p, and the expression

$$\frac{e^{p\sqrt{-1}}-e^{-p\sqrt{-1}}}{2\sqrt{-1}}$$ the *sine* of p.

We shall have for q real

$$\frac{e^q-e^{-q}}{2\sqrt{-1}}=\frac{e^{-q\sqrt{-1}\cdot\sqrt{-1}}-e^{q\sqrt{-1}\cdot\sqrt{-1}}}{2\sqrt{-1}}=\sin(-q\sqrt{-1})$$

$$=-\sin(q\sqrt{-1}),$$ and in like manner

$$\frac{e^q+e^{-q}}{2}=\frac{e^{-q\sqrt{-1}\cdot\sqrt{-1}}+e^{q\sqrt{-1}\cdot\sqrt{-1}}}{2}=\cos(-q\sqrt{-1})=\cos(q\sqrt{-1})$$

admitting that, in the imaginary circle as in the real circle, the sines of two arcs equal but of contrary sign are equal and of opposite sign, and that the cosines of two arcs equal but of opposite sign are equal and of the same sign.

We demonstrate, in §25, absolutely, that is to say independently of the axiom in question, that, in every rectilineal triangle *the sines of the angles are to each other as the circles which have for radii the sides opposite to these angles.*

We demonstrate, besides, for the case of the existence of the quantity i, that the circle of radius y is equal to $\pi i\left(e^{\frac{y}{i}}-e^{-\frac{y}{i}}\right)$, which, for $i=1$, becomes

$$\pi(e^y-e^{-y}).$$

Consequently (§31), in a right-angled rectilineal triangle of which the sides of the right angle are a and b, and the hypothenuse c, and of which the angles opposite to the sides a, b, c are α, β, rt. $\angle$, we have (for $i=1$).

From I,

$$1:\sin\alpha=\pi(e^{c}-e^{-c}):\pi(e^{a}-e^{-a});$$

and consequently

$$1:\sin\alpha=\frac{e^{c}-e^{-c}}{2\sqrt{-1}}:\frac{e^{a}-e^{-a}}{2\sqrt{-1}}$$

$$=-\sin(c\sqrt{-1}):-\sin(a\sqrt{-1})=\sin(c\sqrt{-1}):\sin(a\sqrt{-1});$$

From II,

$$\cos\alpha:\sin\beta=\cos(a\sqrt{-1}):1;$$

From III,

$$\cos(c\sqrt{-1})=\cos(a\sqrt{-1}).\cos(b\sqrt{-1}).$$

These formulas, as also all the formulas of plane trigonometry deducible from them, coincide completely with the formulas of spherical trigonometry, to this extent that if, for example, the sides and the angles of a right-angled rectilineal triangle are designated by the same letters as those of a right-angled spherical triangle, the sides of the rectilineal triangle are to be divided by $\sqrt{-1}$ to obtain the formulas relative to the spherical triangle.

So we get, for a spherical triangle,

by I, $\quad 1:\sin\alpha=\sin c:\sin a;$

by II, $\quad 1:\cos a=\sin\beta:\cos\alpha;$

by III, $\quad \cos c=\cos a\cos b.$

As the reader may be stopped by the omission of a demonstration (in §32 at end) it will not be useless to show, for example, how from the equation

$$e^{\frac{c}{i}}+e^{-\frac{c}{i}}=\tfrac{1}{2}\left(e^{\frac{a}{i}}+e^{-\frac{a}{i}}\right)\left(e^{\frac{b}{i}}+e^{-\frac{b}{i}}\right)$$

we deduce the formula

$$c^2=a^2+b^2,$$

or the theorem of Pythagoras for the Euclidean system.

It is probably thus that the author arrived at it, and the other consequences follow in a similar manner.

We have, by the known formula,

$$e^{\frac{k}{i}}=1+\frac{k}{i}+\frac{k^2}{2i^2}+\frac{k^3}{2.3.i^3}+\frac{k^4}{2.3.4.i^4}+\quad\cdot\quad\cdot\quad\cdot\ ,$$

$$e^{-\frac{k}{i}}=1-\frac{k}{i}+\frac{k^2}{2i^2}-\frac{k^3}{2.3.i^3}+\frac{k^4}{2.3.4.i^4}-\quad\cdot\quad,$$ and consequently

$$e^{\frac{k}{i}}+e^{-\frac{k}{i}}=2+\frac{k^2}{i^2}+\frac{k^4}{3.4.i^4}+\ \cdot\ \cdot\ \cdot\ =2+\frac{k^2+u}{i^2},$$

designating by $\frac{u}{i^2}$ the sum of all the terms which follow $\frac{k^2}{i^2}$; and we have $u \doteq 0$, when $i \doteq \infty$. For all the terms which follow $\frac{k^2}{i^2}$, on being divided by i^2, (that is the factor i^2 being taken out of the denominator), will have for first term $\frac{k^4}{3.4i^2}$; and as the ratio of a term to the preceding is throughout $<\frac{k^2}{i^2}$, the sum is less than it would be, if this ratio were $=\frac{k^2}{i^2}$, that is to say less than

$$\frac{k^4}{3.4.i^2}:\left(1-\frac{k^2}{i^2}\right)=\frac{k^4}{3.4.(i^2-k^2)},$$

a quantity which evidently $\doteq 0$ when $i \doteq \infty$.

From the equation

$$e^{\frac{c}{i}}+e^{-\frac{c}{i}}=\tfrac{1}{2}\left(e^{\frac{a+b}{i}}+e^{-\frac{a+b}{i}}+e^{\frac{a-b}{i}}+e^{-\frac{a-b}{i}}\right)$$

there results (calling w, v, λ quantities analogous to u)

$$2+\frac{c^2+w}{i^2}=\tfrac{1}{2}\left(2+\frac{(a+b)^2+v}{i^2}+2+\frac{(a+b)^2+\lambda}{i^2}\right),$$ whence

$$c^2=\frac{a^2+2ab+b^2+a^2-2ab+b^2+v+\lambda-w}{2},$$ a quantity which $\doteq a^2+b^2$.

APPENDIX II.

SOME POINTS IN JOHN BOLYAI'S APPENDIX COMPARED WITH LOBATSCHEWSKY, BY WOLFGANG BOLYAI.

[From *Kurzer Grundriss*, p. 82.]

Lobatschewsky and the author of the *Appendix* each consider two points A, B, of the sphere-limit, and the corresponding axes ray AM, ray BN (§23).

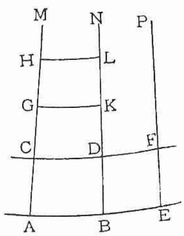

FIG. 43.

They demonstrate that, if α, β, γ designate the arcs of the circle limit AB, CD, HL, separated by segments of the axis AC=1, AH=x, we have

$$\frac{\alpha}{\gamma}=\left(\frac{\alpha}{\beta}\right)^x$$

Lobatschewsky represents the value of $\frac{\gamma}{\alpha}$ by e^{-x}, e having some value >1, dependent on the unit for length that we have chosen, and able to be supposed equal to the Naperian base.

The author of the *Appendix* is led directly to introduce the base of natural logarithms.

If we put $\frac{\alpha}{\beta}=\delta$, and γ, γ' are arcs situated at the distances y, i from α, we shall have

$$\frac{\alpha}{\gamma}=\delta^y=Y,\ \frac{\alpha}{\gamma'}=\delta^i=I,\ \text{whence } Y=I^{\frac{y}{i}}$$

He demonstrates afterward (§29) that, if u is the angle which a straight line makes with the perpendicular to its parallel, we have

$$\mathrm{Y}=\cot \tfrac{1}{2} u.$$

Therefore, if we put $z=\frac{\pi}{2}-u$, we have

$$\mathrm{Y}=\tan(z+\tfrac{1}{2}u)=\frac{\tan z+\tan\frac{1}{2}u}{1-\tan z \tan\frac{1}{2}u},$$

whence we get, having regard to the value of $\tan \frac{1}{2}u=\mathrm{Y}^{-1}$,

$$\tan z=\tfrac{1}{2}(\mathrm{Y}-\mathrm{Y}^{-1})=\tfrac{1}{2}\left\{\mathrm{I}^{\frac{y}{i}}-\mathrm{I}^{-\frac{y}{i}}\right\} \quad (\S 30).$$

If now y is the semi-chord of the arc of circle-limit $2r$, we prove (§30) that $\frac{r}{\tan z}=$constant.

Representing this constant by i, and making y tend toward zero, we have

$$\frac{2r}{2y} \doteq 1, \text{ whence}$$

$$2y \doteq 2\, i \tan z \doteq i\,\frac{\mathrm{I}^{\frac{2y}{i}}-1}{\mathrm{I}^{\frac{y}{i}}}$$

or putting $\frac{2y}{i}=k$, $\mathrm{I}=e^{l}$,

$$kl^{\frac{y}{i}} \doteq e^{kl}-1 \doteq kl(1+\omega),$$

ω being infinitesimal at the same time as k, Therefore, for the limit, $1=l$ and consequently $\mathrm{I}=e$.

The circle traced on the sphere-limit with the arc r of the curve-limit for radius, has for length $2\pi r$. Therefore,

$$\bigcirc y=2\pi r=2\pi\, i \tan z=\pi i(\mathrm{Y}-\mathrm{Y}^{-1}).$$

In the rectilineal $\triangle$ where α, β designate the angles opposite the sides a, b, we have (§25)

$$\sin \alpha : \sin \beta=\bigcirc a : \bigcirc b=\pi i(\mathrm{A}-\mathrm{A}^{-1}) : \pi i(\mathrm{B}-\mathrm{B}^{-1})$$
$$=\sin(a\sqrt{-1}) : \sin(b\sqrt{-1}).$$

Thus in plane trigonometry as in spherical trigonometry, the sines of the angles are to each other as the sines of the opposite sides, only that on the sphere the sides are reals, and in the plane we must consider them as imaginaries, just as if the plane were an imaginary sphere.

We may arrive at this proposition without a preceding determination of the value of I.

If we designate the constant $\frac{r}{\tan z}$ by q, we shall have, as before

$$\bigcirc y = \pi q(\mathrm{Y} - \mathrm{Y}^{-1}),$$

whence we deduce the same proportion as above, taking for i the distance for which the ratio I is equal to e.

If axiom XI is not true, there exists a determinate i, which must be substituted in the formulas.

If, on the contrary, this axiom is true, we must make in the formulas $i = \infty$. Because, in this case, the quantity $\frac{a}{r} = \mathrm{Y}$ is always$=1$, the sphere-limit being a plane, and the axes being parallel in Euclid's sense.

The exponent $\frac{y}{i}$ must therefore be zero, and consequently $i = \infty$.

It is easy to see that Bolyai's formulas of plane trigonometry are in accord with those of Lobatschewsky.

Take for example the formula of §37,

$$\tan \Pi(a) = \sin \mathrm{B} \tan \Pi(p),$$

a being the hypothenuse of a right-angled triangle, p one side of the right angle, and B the angle opposite to this side.

Bolyai's formula of §31, I, gives

$$1 : \sin \mathrm{B} = (\mathrm{A} - \mathrm{A}^{-1}) : (\mathrm{P} - \mathrm{P}^{-1}).$$

Now, putting for brevity, $\frac{1}{2}\Pi(k) = k'$, we have $\tan 2p' : \tan 2a'$
$= (\cot a' - \tan a') : (\cot p' - \tan p') = (\mathrm{A} - \mathrm{A}^{-1}) : (\mathrm{P} - \mathrm{P}^{-1})$
$= 1 : \sin \mathrm{B}.$

APPENDIX III.

LIGHT FROM NON-EUCLIDEAN SPACES ON THE TEACHING OF ELEMENTARY GEOMETRY.

By G. B. Halsted.

The preface to my Elements of Geometry, 1885, says "that around the word 'distance' centers the most abstruse advance in pure science and philosophy."

Recently R. A. Roberts, in his "Modern Mathematics," gives as one of the two main roots from which modern mathematical thought springs, the recognition of the fact that angles and distances in the Euclidean experiential geometry depend upon a certain absolute curve of the second order.

As foreshadowed by Bolyai and Riemann, founded by Cayley, extended and interpreted for hyperbolic, parabolic, elliptic spaces by Klein, and now recast and applied to mechanics by Sir Robert Ball, this projective metrics may in truth be looked upon as the very soul and characteristic of what is highest and most peculiarly modern in all the bewildering range of mathematical achievement.

It permeates like a vital essence, and for questions of method, of teaching, of exposition it is a final criterion. Nearly all mathematicians have already fallen into rank as holding that number is wholly a creation of the human intellect, while on the contrary geometry has an empirical element. Of a number of possible geometries we cannot say *a priori* which shall be that of our actual space, the space in which we move. Of course an advance so important, not only for mathematics but for philosophy, has had some metaphysical opponents, and as long ago as 1878 I mentioned in my Bibligraphy of Hyper-Space and Non-Euclidean Geometry (American Journal of Mathematics Vol. I, 1878, Vol. II, 1879) one of these, Schmitz--Dumont, as a sad paradoxer, and another, J. C. Becker, both of whom would ere this have shared the oblivion of still more antiquated fighters against the light, but that Dr. Schotten, praiseworthy for the very attempt at a comparative planimetry, happens to be himself a believer in the *a priori* founding of geometry, while his American reviewer, Mr. Ziwet, happens to confuse what would be good in a book written for the very necessary preparatory or propaedentric courses in intuitive geometry, with what would be good in a treatise professing to deduce Euclidean geometry from only the necessary assumptions.

He says, "we find that some of the best German text books do not try at all to define what is space, or what is a point, or even what is a straight line." Do any German geometries define space? I never remember to have met one.

In experience, what comes first is a bounded surface, with its boundaries, lines, and their boundaries, points. Are the points whose definitions are omitted anything different or better?

Dr. Schotten regards the two ideas "direction" and "distance" as intuitively given in the mind and as so simple as to not require definition.

As to what Webster's Dictionary says of the meaning of the English word "direction", Professor Cajori has honored me by a quotation on page 383 of his admirable History of Mathematics in the United States, and only today I saw mention of an accident caused while two jockeys were speeding around a track in opposite directions, and also chanced on page 87 of Richardson's Euclid, 1891, to read "The sides of the figure must be produced in the same direction of rotation; . . going round the figure always in the same direction."

No wonder that when Mr. Ziwet had written: "he therefore bases the definition of the straight line on these two ideas," he stops, modifies, and rubs that out as follows, "or rather recommends to elucidate the intuitive idea of the straight line possessed by any well-balanced mind by means of the still simpler ideas of direction" [in a circle] "and distance" [on a curve]. If this is meant for an introductory geometry-for-beginners, all well and good. Elucidate any intuition possessed by the well-balanced baby-mind by anything still simpler which you may happen to think will elucidate it.

But when we come to geometry as a science, as foundation for work like that of Cayley and Ball, I think with Professor Chrystal: "It is essential to be careful with our definition of a *straight line*, for it will be found that virtually the properties of the straight line determine the nature of space.

Our definition shall be that two points *in general* determine a straight line, or that in general a straight line cannot be made to pass through *three* given points."

We presume that Mr. Ziwet glories in that unfortunate expression "a straight line is the shortest distance between two points," still occurring in Wentworth (New Plane Geometry, page 33,) even after he has said, page 5, "the length of the straight line is called the *distance* between two points." If the *length* of the one straight line between two points is the distance between those points how can the straight line itself be the *shortest* distance. If there is only one distance, it is the longest as much as the shortest distance, and if it is the *length* of this shorto-longest distance which is the distance then it is not the straight line itself which is the longo-shortest distance.

But Wentworth also says "Of all lines joining two points the *shortest* is the straight line."

This general comparison involves the measurement of curves, which involves the theory of limits, to say nothing of ratio. The very ascription of length to a curve involves the idea of a limit. And then to introduce this general axiom only to prove a very special case of itself, that two sides of a triangle are together greater than the third, is surely bad logic, bad pedagogy, bad mathematics.

This latter theorem, according to the first of Pascal's rules for demonstrations, should not be proved at all, since every dog knows it. Well and good in our geometry-for-beginners, to which alone Pascal's rules apply; but to this objection, as old as the sophists, Simson long ago answered for the science of geometry, that the number of assumptions ought not to be increased without necessity, or as Dedekind has it: "*Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden.*"

But Mr. Ziwet could correct one of his misapprehensions by looking into Wentworth's book, namely the mistaken idea that American "text books begin with several pages of definitions *to be committed to memory*, followed by a page of axioms *again to be committed to memory*." Wentworth carefully reproduces, whenever he uses them, preceding definitions, axioms, theorems.

It is worth notice that the mistake made in our Century Dictionary, the confusion of hyperbolic with elliptic geometry, is made also on page 186 of Rebiere's enjoyable "Mathématiques et Mathématiciens," 1889, where he says: "De là des *geometries non-euclidiennes* ou la somme des angles d'un triangle n'est plus égale à deux droits: dans celle de Rieman, elle est plus petite que deux droit et dans celle de Lobatschewski, elle est plus grande." Note also that, Frenchman-like, both the proper names are here mis-spelled. May we not fear that here also is a teacher of mathematics who never has read Lobatschewsky's immortal Essay on Parallels? Contrast a distinguished Englishman, Professor Levett, who says: "It is many years since I first made acquaintance with this great work, and I am delighted to see that the good cause of sound geometrical learning has been advanced by the appearance of an English translation. I believe that I am one of the very few schoolmasters who have read the essay with pupils. King Edward's school boys are brought up in the true faith as to the sum of the angles of a triangle."

The brilliant American, Professor W. B. Smith, (Ph. D., Goettingen) has just written: "Nothing could be more unfortunate than the attempt to lay the notion of Direction at the bottom of Geometry."

Was it not this notion which led so good a mathematician as John Casey to give as a demonstration of a triangle's angle-sum the procedure called "a *practical* demonstration" on page 87 of Richardson's Euclid, and there described as "laying a 'straight edge' along one of the sides of the figure, and then turning it round so as to coincide with each side in turn."

This assumes that a straight line may be translated without rotation, which assumption readily comes to view when you try the procedure in two-dimensional double-elliptic geometry, our familiar two-dimensional spherics. It is of the greatest importance for every teacher to know and connect the commonest forms of assumption equivalent to Euclid's Axiom XI. If in a plane two straight lines perpendicular to a third can never meet, are there others, not both perpendicular to any third, which can never meet? Euclid's Axiom XI is the assumption *No*. Playfair's answers *no* more simply, But the very same answer is given by the common assumption of our geometries, usually unnoticed, that a circle may be passed through any three non-collinear points.

This equivalence was first shown by Bolyai, who looks upon this as the simplest form of the assumption. Lobatschewsky's form is, the existence of any finite triangle whose angle-sum is a straight angle; or the existence of a plane rectangle; or that, in triangles, the angle-sum is constant.

One of Legendre's forms was that through every point within an angle a straight line may be drawn which cuts both arms.

But Legendre never saw through this matter because he had not, as we have, the eyes of Bolyai and Lobatschewsky to see with. The same lack of their eyes has caused the author of the charming book "Euclid and His Modern Rivals," to give us one more equivalent form: "In any circle, the inscribed equilateral tetragon is greater than any one of the segments which lie outside it," (A New Theory of Parallels by C. L. Dodgson, 3d. Ed., 1890).