ELASTICITY

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BY

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The Article Heat is entered according to Act of Parliament of Canada, in the year 1880, by WILLIAM THOMSON, Knight, Glasgow, Scotland, in the office of the Minister of Agriculture.
1. Elasticity of matter is that property in virtue of which a body requires force to change its bulk or shape, and requires a continued application of the force to maintain the change, and springs back when the force is removed, and, if left at rest without the force, does not remain at rest except in its previous bulk and shape. The elasticity is said to be perfect when the body always requires the same force to keep it at rest in the same bulk and shape and at the same temperature through whatever variations of bulk, shape, and temperature it be brought. A body is said to possess some degree of elasticity if it requires any force to keep it in any particular bulk or shape. It is convenient to discuss elasticity of bulk and elasticity of shape separately and sometimes jointly.

2. Every body has some degree of elasticity of bulk. If a body possesses any degree of elasticity of shape it is called a solid; if it possesses no degree of elasticity of shape it is called a fluid.

3. All fluids possess elasticity of bulk to perfection. Probably so do all homogeneous solids, such as crystals and glasses. It is not probable that any degree of fluid pressure (or pressure acting equally in all directions) on a piece of common glass, or rock crystal, or of diamond, or on a crystal of bismuth, or of copper, or of lead, or of silver, would make it denser after the pressure is removed, or put it into a condition in which at any particular intermediate pressure it would be denser than it was at that pressure before the application of the extreme pressure. Malleable metals and alloys, on the other hand, may have their densities considerably increased and diminished by hammering and by mere traction. By compression between the dies used in coining, the density of gold may be raised from 19.258 to 19.367, and the density of copper from 8.935 to 8.916, and Mr. McFarlane's experiments quoted below (section 78), show a piece of copper wire decreasing in density from 8.91 to 8.835 after successive simple tractions, by which its length was increased from 287 centimetres to 317 centimetres, while its modulus of rigidity decreased from 443 to 426 million grammes per square centimetre. Later experiments, recently made for this article by the same experimenter, have shown augmentation of density from 8.85 to 8.95, produced by successive tractions which elongated a piece of copper wire from weighing 10.4 grammes per metre to weighing 13.5 grammes per metre, the wire having been first annealed by heating it to redness in sand, and allowing it to cool slowly. Augmentation of density by traction is a somewhat surprising result, but not altogether so when we consider that the wire had been reduced to an abnormally dense condition ("annealing"). The common explanation of these changes of density in metals, which attributes them to porosity, is probably true; by porosity being understood a porous structure with such vast numbers of the ultimate molecules in the portions of the solid substance between pores or interstices that these portions may be called homogeneous in the sense that a crystal or a liquid can be called homogeneous (compare section 40 below).


4. The elasticity of shape of many solids is not perfect; it is not known whether it is perfect for any. It might be expected to be perfect for glass and rock crystal, diamond and other hard, brittle, homogeneous substances; but experiment proves that at all events for glass it is not so, and shows on the contrary a notable degree of imperfection in the torsional elasticity of glass fibres. It might be expected that in copper and soft iron and other plastic metals the elasticity of shape would be very imperfect; experiment shows, on the contrary, that in copper, brass, soft iron, steel, platinum, provided the distortion does not exceed a certain limit in each case, elasticity of shape is remarkably perfect, much more perfect than in glass. It is quite probable that even in the softer metals—zinc, tin, lead, cadmium, potassium, sodium, &c.—the elasticity of shape may be as perfect as in the metals mentioned above, but within narrower limits as to degree of distortion. Accurate experiment is utterly wanting, to discover what is the degree of imperfection, if any, of the elasticity of any metal or alloy, when tested within sufficiently narrow limits of distortion.

5. The "viscosity of metals" described below (sections 21-25) does not demonstrate any imperfectness of elasticity according to the definition of section 1, which is purely statical. The viscosity of solids may (for all we yet know by experiment) depend, as does the viscosity of fluids, upon a resistance varying with the velocity of the change, and vanishing when the velocity of the change is zero, that is to say, when the body is at rest in any configuration; if so, the elasticity of the substance concerned is perfect within the limits of the experiment in question. If, on the other hand (as the discovery of elastic fatigue described below seems to indicate may be to some degree the case), the loss of energy from the vibrations in the experiments described is due to a dependence of the elastic resilient force upon previous conditions of the substance in respect to strain, the "viscosity" would be continuous with a true imperfectness of static elasticity. Here, then, we have a definite question which can be answered by experiment only:—Consider a certain definite stress applied to a solid substance; for example, a certain "couple" twisting a wire or a rod; or a certain weight pulling it out, or compressing it lengthwise; or a certain weight placed on the middle of a beam supported by trestles under its ends. Let it be applied and removed a great many times, and suppose it to be seen that after each application and removal of the stress the body comes to rest in exactly the same configuration as after the previous application or removal of the stress. If now the body be left to itself with the stress removed, and if it be found to remain at rest in the same configuration for minutes, or hours, or days, or years after the removal of the stress, a part of the definition of perfect elasticity is fulfilled. Or, again, if the stress be applied, and kept applied with absolute constancy, and if the body remain permanently in a constant configuration, another item of the definition of perfect elasticity is proved. When any such experiment is made on any metal, unless some of the softer metals (section 4) is to be excepted, there is certainly very little if any change of configuration in the circumstances now supposed. The writer believes, indeed, that nothing of the kind has hitherto been discovered by experiment, provided the stress has been considerably less than that which would break or give a notable permanent twist, or elongation, or bend, to the body, that is to say, provided the action has been kept decidedly within the limits of the body's elasticity as commonly understood (sections 7-20). Mr J. T. Bottomley, with the assistance of a grant of money from the British Association, has commenced making arrangements for secular experiments on the elasticity of metals, in the tower of the university of Glasgow, to answer this question in respect to permanence or non-permanence, through minutes, hours, days, or years, or centuries. If several gold wires are hung side by side, one of them bearing the smallest weight that will keep it approximately straight, another wire $\frac{1}{16}$ of the breaking weight, another wire $\frac{1}{32}$ of the breaking weight, and so on; the one of them bearing $\frac{1}{16}$ of the breaking weight will probably, in the course of a few hours, or days, show very sensible elongation. Will it go on becoming longer and longer till it breaks, or will the time-curve of its elongation be asymptotic? Even with considerably less than $\frac{1}{16}$ of the breaking weight there will probably be a continually augmenting elongation, but with asymptotic time-curve indicating a limit beyond which the elongation never goes, but which it infinitely nearly reaches in an infinite time. It is not probable that a gold wire stretched by $\frac{1}{16}$ of its present breaking weight, or by $\frac{1}{32}$ of its present breaking weight, or even by $\frac{1}{64}$ of its present breaking weight, would break in a thousand or in a million years. The existence of gold ornaments which have been found in ancient tombs and cities, and have preserved their shapes for thousands of years without running down glacier-wise (as does brittle pitch or sealing-wax in the course of a few years in moderately warm climates), seems to prove that for gold (and therefore leaves no doubt also for many other metals) the time-curve is asymptotic, if indeed there is any slow change of shape at all after the application of a moderate stress well within the limits of elasticity. Egyptian and Greek statues, Etruscan vases, Egyptian obelisks, and other stone monuments with their engraved hieroglyphics, flint implements and boulders, and mountains with the geological evidence we have of their antiquity, prove for stones, and pottery, and rocks of various kinds, a permanence for thousands and millions of years of resistance to distorting stress.

6. The complete fulfilment of the definition of perfect elasticity is not proved by mere permanence of the extreme configurations assumed by the substance when a stated amount of the stress is alternately applied and removed. This condition might be fulfilled, and yet the amount of distortion permitted by the definition of perfect elasticity might be different with the same palpable configuration of the body during gradual augmentation and during gradual diminution of the stress. That it is so in fact is proved by the discovery of viscosity referred to below; but it is not yet proved that if, after increasing the stress to a certain definite amount, the body is brought to rest in the same palpable configuration as before, the amounts of stress required to hold it in this configuration are different in the two cases. If they are (section 1) the elasticity is imperfect; if they are not the elasticity is perfect within the limits of the experiment (compare section 39 below).

7. Limits of Elasticity—Elasticity of Shape.—The degree of distortion within which elasticity of shape is found is essentially limited in every solid. Within sufficiently narrow limits of distortion every solid shows elasticity of shape to some degree—some solids to perfection, so far as we know at present. When the distortion is too great, the body either breaks or receives a permanent bend (that is, such a molecular disturbance that it does not return to its original figure when the bending force is removed). If the first notable dereliction from perfection of elasticity is a breakage, the body is called brittle,—if a permanent bend, plastic or malleable or ductile. The metals are generally ductile; some metals and metallic alloys and compounds of metals with small proportions of other substances, are brittle; some of them brittle only in certain states of temper, others it seems essentially brittle. The steel of before the days of Bessemer and Siemens is a remarkable instance. When slowly cooled from a bright
red heat, it is remarkably tough and ductile. When heated to redness and cooled suddenly by being plunged in oil or water or mercury, it becomes exceedingly brittle and hard (glass-hard, as it is called), and to ordinary observation seems capable of taking a permanent bend; (though probably careful observation would prove it not quite so). The definition of steel used to be *approximately pure iron capable of being tempered glass-hard, and again softened to different degrees by different degrees of heat*. Now, the excellent qualities of iron made by Bessemer's and Siemens's processes are called steel, and are reckoned best when incapable of being tempered glass-hard, the possibility of brittleness supervening in the course of any treatment which the metal may meet with in its manufacture being an objection against the use of what was formerly called steel for ship's plates, ribs, stringers, &c., and for many applications of land engineering, even if the material could be had in sufficient abundance.

8. LIMITS OF ELASTICITY (CONTINUED)—ELASTICITY OF BULK.—If we reckon by the amount of pressure, there is probably no limit to the elasticity of bulk in the direction of increase of pressure for any solid or fluid; but whether continued augmentation produces continued diminution of bulk towards zero without limit, or whether for any or every solid or fluid there is a limit towards which it may be reduced in bulk, but smaller than which no degree of pressure, however great, can condense it, is a question which cannot be answered in the present state of science. Would any pressure, however tremendous, give to gold a density greater than 19-6, or to copper a density greater than 9-9, after the pressure is removed (section 3 above)? But whether the body be fluid or a continuous non-porous solid, it probably recovers the same density, however tremendously it may have been pressed, and probably shows perfect elasticity of bulk (section 3 above) through the whole range of positive pressure from zero to infinity, provided the pressure has been equal in all directions like fluid pressure. As for negative pressure, we have no knowledge of what limit, if any, there may be to the amount of force which can be applied to a body pulling its surface out equally in all directions. The question of how to apply the negative pressure is inextricably involved with that of the body's power to resist. The upper part of the mercury of a barometer adhering to the glass above the level corresponding to the atmospheric pressure is a familiar example of what is called negative pressure in liquids. Water and other transparent liquids show similar phenomena, another of which is the warming of water above its boiling point in an open glass or metal vessel varnished with shellac. Attempts to produce great degrees of this so-called negative pressure are baffled by what seems an instability of the equilibrium which supervenes when the negative pressure is too much augmented. It is a very interesting subject for experimental inquiry to find how high mercury or water or any other liquid can be got to stand above the level corresponding to the atmospheric pressure in a tall hermetically sealed tube, and how many degrees a liquid can, with all precautions, be warmed above its boiling point. In each case it seems to be by a minute bubble forming and expanding somewhere at the boundary of the liquid, where it is in contact with the containing vessel, that the possible range of the negative pressure is limited, judging from what we see when we carefully examine a transparent liquid, or the surface of separation between water and glass, in any such experiment. The contrast of the amounts of negative pressure practically obtainable or obtained hitherto in such experiments on liquids (which are at the most those corresponding to the weight of a few metres of the substance), with that obtainable in the case of even the weakest solids, is remarkable; and as for the strongest,
particles may now be regarded as being all completely relaxed. Let next, one end of the bar be fixed, and the other be made to revolve by torsion, till the particles at the circumference of the bar are strained to the utmost extent of which they can admit, without undergoing a permanent change in their mutual connection.

In this condition, equal elements of the cross section of the bar afford resistances proportional to the distances of the elements from the centre of the bar; since the particles are displaced from their positions of relaxation through spaces which are proportional to the distances of the particles from the centre. The couple which the bar now resists, and which is equal to the sum of the couples due to the resistances of all the elements of the section, is what it was in the former case; or, according to the notation already adopted, it is now \( \frac{1}{2} L \).

18. If, after this, all external stress be removed from the bar, it will assume a position of equilibrium, in which the outer particles will have been strained in the direction of tension, and the inner ones in the same direction as that of the twisting, the two sets of opposite couples thus produced among the particles of the bar balancing one another. It is easy to show that the resistance of the outer elements will be greater than those of the inner ones, and that therefore all small elements of the section of the bar afford the same resistance, no matter what their positions from the centre may be, it is easy to prove that the total torsional resistance of the bar is \( \frac{1}{2} L \) what it was in the former case; or, according to the notation already adopted, it is now \( \frac{1}{2} L \).

19. It is not necessary to assume the existence of a definite "elastic limit," or a limit within which, if two particles of a substance be displaced, they will return to their original relative positions when the disturbing force is removed. The opposite conclusion of Professor Hodgkinson seems to have been led by some interesting experimental results, which will be considered in a more advanced part of this paper.

20. To prove this, let the bar be supposed to be divided into an infinite number of cross sections. The resistance of any one of these cross sections to resist a strain tending to make the particles slide past one another, or to resist a shearing strain, as it is commonly called, is represented by a couple amounting to only \( \frac{1}{2} L \). But, as we have seen, it is \( \frac{1}{2} L \) when the wire is twisted forwards. That is, then, the wire in its new state has torsions as much as resistance 25 to resist torsion in one direction as it has to resist it the other.

14. "Principles similar to the foregoing, are applicable in regard to beams subjected to cross strain. As, however, my chief object at present is to point out the existence of such principles, to indicate the mode in which they are to be applied, and to show their great practical importance in the determination of the strength of materials, I need not enter fully into their application in the case of cross strain. The investigation in this case closely resembles that in the case of torsion, but is not equally natural; and upon account of the different ultimate resistances afforded by any material to tension and to compression, and on account of the numerous varieties in the form of section of beams which for different purposes are found advisable to adopt, I shall therefore merely make a few remarks on this subject.

15. "If a bent bar of wrought iron or other ductile material is straightened, its particles will thus be put into such a state that they are not able to resist a strain tending to make the particles slide past one another, or to resist a shearing strain, as it is commonly called. Hence, if we suppose that no change in the form of section of beams which for different purposes are found advisable to adopt, I shall therefore merely make a few remarks on this subject.

16. "What has already been stated is quite sufficient to account for many very discordant and perplexing results which have been arrived at by different experimenters on the strength of materials. It scarcely ever occurs that a material is presented to us, either for experiment or for application to a practical use, in which the particles are free from gross mutual strain. Processes have been pointed out, in which we may produce certain peculiar strains of this kind. Those, or other processes producing somewhat similar strains, are used in the manufacture of almost all materials. Thus, for instance, when malleable iron is hot worked. Its final condition is that of extreme tension and compression, and the internal ones in the form of sections which are occasioned by variations in its molecular arrangement, are much greater even than those which are to be found as occurring in the strength of bars subjected to torsion. The inner ones then contract as they subsequently cool, and thus they, as it were, pull the outer ones together. That is, in the end the outer ones are in a state of compression and the inner ones opposite in the condition.

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cause of an important fact observed by Mr Eaton Hodgkinson in his valuable researches in regard to the strength of cast iron (Report of the British Association for 1837, p. 592). He found, that, contrary to what had been previously supposed, a strain, however small in comparison to that which would occasion rupture, was sufficient to produce a set, or permanent change of form, in the beams on which he experimented. Now this is just what should be expected in accordance with the principles which I have brought forward, for a small change of some of the causes already pointed out, various parts of a beam previously subjected to an external force have been strained to the utmost, when, by the application of such force, however small, they are still further displaced from their position, and must necessarily undergo a permanent alteration in their connexion with one another, an alteration permitted by the ductility of the material; or, in other words, the beam as a whole must take a set.

In connexion with this explanation of the fact observed by Mr Hodgkinson, I do not think we are to conclude with him that "the maxim of loading bodies within the elastic limit has no foundation in nature." It appears to me that the defect of elasticity, which he has shown to occur even with very slight strains, exists only when the strain is applied for the first time; or, in other words, that if a beam has already been subjected to a considerable strain, it may again be subjected to any smaller strain in the same direction without the beam taking a set. It will readily be seen, however, from Mr Hodgkinson's experiments, that the term 'elastic limit,' as commonly employed, is entirely vague, and must tend to lead to erroneous results.

19. The considerations adduced seem to me to show clearly that there really exist two elastic limits for any material, between which the displacements or deflexions, or what may in general be termed the changes of form, must be confined, if we wish to avoid giving rise to the formation of a set, or, in the case of variable strain, if we wish to avoid giving it a continuous succession of sets which would gradually bring about its destruction; that these two elastic limits are usually situated one on the one side and the other on the opposite side of the position assumed by the material when subject to no external strain, though they may be both on the same side of this position of relaxation; and that they may therefore with propriety be called the superior and the inferior limits of the change of form of the material for the particular arrangement which has been given to its particles; that these two limits are not fixed for any given material, but that, if the change of form be continued beyond either limit, two new limits will, by means of an alteration in the arrangement of the particles of the material, be given to it in place of those which it previously possessed; and lastly, that the processes employed in the manufacture of materials are usually such as to place the two limits in close contiguity with one another, thus causing the material to take in the first instance a set from any strain, however slight, while the interval which may afterwards exist between the two limits, and also, as was before stated, the position assumed by each of them are determined by the peculiar strains which are subsequently applied to the material.

20. The introduction of new, though necessary, elements into the consideration of the strength of materials may, on the one hand, seem annoying to the investigations more complicated. On the other hand, their introduction will really have the effect of facilitating difficulties, by removing erroneous views from the subject, and preventing contradictory or incongruous results from being obtained by theory and experiment. In all investigations, in fact, in which we desire to attain or to approach nearly to truth, we must take facts as they actually are, not as we want them to be. For if, for some of the causes already pointed out, various alterations in the arrangement of the particles of the material be given to it in place of those which it previously possessed; and lastly, that the processes employed in the manufacture of materials are usually such as to place the two limits in close contiguity with one another, thus causing the material to take in the first instance a set from any strain, however slight, while the interval which may afterwards exist between the two limits, and also, as was before stated, the position assumed by each of them are determined by the peculiar strains which are subsequently applied to the material.

21. Passing now to homogeneous matter (sec. 35), homogeneously strained (chap. ii. of Math. Theory below), we are met by physical questions of greater interest regarding limits of elasticity. Supposing the solid to be homogenous, and distorted in any particular way to nearly the limit of its elasticity for this kind of distortion, will the limits be widened or narrowed by the superposition of negative or positive pressure equal in all directions producing a dilatation or a condensation? It seems probable that a dilatation would narrow the limits of elasticity, and a condensation widen them. This, however, is a mere guess: experiment alone can answer the question. Take again a somewhat less simple case. A wire is stretched by a weight to nearly its limits of longitudinal elasticity; a couple twisting it is applied to its lower end. Will this cause the weight to run down and give the wire a permanent set, or break it? Probably, yes; but experiment only can decide. The corresponding question with reference to a column loaded with a weight may have the same answer, but not necessarily so. Experiment again is wanting. A wire hanging stretched by a light weight, merely to steady it, is twisted to nearly its limit of torsional elasticity by a couple of given magnitude applied to its lower end; the stretching weight is increased. Will this cause it to yield to the couple and take a permanent set? Probably, yes. [Certainly yes, for steel piano-forte wire experimented on by Mr M'Parlane to answer this question since it was first put in type for the present article.] If so, then the limits of torsional elasticity of a wire having a heavy weight are widened by diminishing or taking off the weight; and no doubt will follow continuously the columns twisted by opposing couples at its two ends will have its limits of torsional elasticity widened by the application of forces to its two ends, pressing them towards one another. Experiment to answer these questions would certainly reward the experimenter with definite and interesting results.

22. Narrowness of Limits of Elasticity—Solids.—The limit of elasticity of metals, stones, crystals, woods, are so narrow that the distance between any two neighbouring points of the substance never alters by more than a small proportion of its own amount without the substance either breaking or experiencing a permanent set, and therefore the angle between two lines meeting in any point of the substance and passing always through the same matter is never altered by more than a small fraction of the radian, before the body either breaks or takes a permanent set. By far the widest limits of elasticity hitherto discovered by experiment, for any substance except cork, india-rubber, jellies, are those of steel piano-forte wire. Take, for example, the piano-forte wire at present in use for deep-sea soundings. It is No. 23 of the Birmingham wire gauge, its density is 7.737, it weighs 0.34 grammes per centimetre, or 6298 kilogrammes per nautical mile of 1852.3 metres, and therefore its sectional area and diameter are 9044 square centimetres and 0.44 centimetres. It bears a weight of 106 kilogrammes, which is equal in weight to about 31 kilometres of its length, and when this weight is alternately hung on and removed the length of the wire varies by 0.5 of its amount. While this elongation takes place there is a lateral shrinking, as we shall see (section 47), of from 1 to 2 of the same amount.

23. Consider now in the unstrained wire two lines through the substance of the wire at right angles to one another in any plane through or parallel to the axis of the wire in directions equally inclined to this line. When the wire is pulled lengthwise the two vertical angles bisected by the length of the wire become acute, and the other two obtuse by a small difference, as illustrated in the diagram (fig. 2),

![Diagram](https://via.placeholder.com/150)

**Fig. 1**
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where the continuous lines represent a portion of the unwirled wire, and the dotted lines the same portion of the wire when pulled. The change in each of the angles would be \( \frac{1}{2} \) of the radian in virtue of the elongation were there no lateral shrinking, and about \( \frac{1}{3} \) of the radian in virtue of the lateral shrinking were there no elongation. The whole change experienced by each of the right angles is therefore actually (section 37) \( \frac{1}{2} + \frac{2}{3} \) or about \( \frac{1}{3} \) of the plane, or \( \theta - \frac{1}{3} \). This is an extreme case. In all other cases of metals, stones, glasses, crystals, the substance either breaks or takes a permanent bend, probably before it experiences any so great angular distortion as a degree; and except in the case of steel we may roughly regard the limits of elasticity as being something between \( \frac{3}{10} \) and \( \frac{2}{3} \) in respect to the linear elongation or contraction, and from \( \frac{1}{3} \) of a degree to half a degree in respect to angular distortion.

24. On the other hand, gelatinous substances, such as india-rubber and elastic jellies, have very wide limits of elasticity. A vulcanised india-rubber band, for instance, is capable of being stretched, again and again, to eight times its length, and returning always to nearly its previous condition when the stress is removed. A shape of transparent jelly presents a beautiful instance of great degree of distortion with seemingly very perfect elasticity. All these instances, india-rubber and jellies, show with great changes of shape but slight changes of bulk. They have, in fact, all, as nearly as experiment has hitherto been able to determine, the same compressibility as water.

25. Cork, another body with very wide limits of elasticity (very imperfect elasticity it is true) is singular, among bodies seemingly homogeneous to the eye, in its remarkably easy compressibility. It is, in fact, the only seemingly homogeneous solid which shows to the unaided eye any sensible change of bulk under any practically applicable forces. A small homogeneous piece torn out of a cork may, by merely pressing it between the fingers, be readily compressed to half its bulk, and a large slab of cork in a Bramah press may be compressed to \( \frac{1}{3} \) of its bulk. An ordinary bottle cork loaded with a small piece of metal presents a very interesting appearance in an Oersted glass compressing vessel; first floating, and when compressed to 20 or 30 atmospheres sinking, and shrivelling in bulk very curiously; then on the pressure being removed, expanding again, but not quite to previous bulk, and floating up or remaining down according to the amount of its load.

The divergencies presented by cork and gelatinous bodies in opposite directions from the regular elasticity of hard solids form an interesting subject, to which we shall return later (section 45).

26. Liquids.—In respect to liquids, there are no limits of elasticity so far as regards the magnitude of the positive pressure applied or conceivably applicable; but in respect to the magnitude of negative pressure, and in respect to the magnitude of the change of bulk, whether by negative or positive pressure, there are probably very decided and not very wide limits. Thus water, though condensed \( \frac{1}{10} \) of its bulk by 2000 atmospheres in Perkins's\(^1\) experiments, is capable of being compressed by much less than \( \frac{1}{3} \) of its volume under a pressure of 7000 atmospheres. How much it or any other liquid is condensed by a pressure of 10,000 atmospheres, or by 20,000 atmospheres, is an interesting subject for experimental investigation.

27. Gases.—In respect to rarefaction, and in respect to proportionate condensation, gases present enormous wider limits of elasticity than any liquids or solids,—in fact no limit in respect to dilatation, and in respect to condensation a definite limit only when the gas is below Andrews's "critical temperature." If the gas be kept at any temperature above that critical temperature, it remains homogeneous, however much it be condensed; and therefore for a fluid above the critical temperature there is, in regard to magnitude of pressure, no superior limit to its elasticity. On the other hand, if a fluid be kept at any constant temperature less than its critical temperature, it remains homogeneous, and presents an increasing pressure until a certain density is reached; when its bulk is further diminished it divides into two parts of less and greater density (the part of less density being called vapour, that of greater density being called liquid, if it is not solid) and presents no further increase of pressure until the vaporous part shrinks to nothing, and the whole becomes liquid (that is to say, homogeneous fluid at the greater of the two densities) or else becomes solid—the question whether the more dense part is liquid or solid depending on the particular temperature below the critical temperature at which the whole substance is kept during the supposed experiment.

28. The thermo-dynamic reasoning of Professor James Thomson, which showed the effect of change of pressure in altering the freezing point of a liquid, leads to analogous considerations regarding the effect of continuous increase or continuous decrease of pressure upon a mass consisting of the same substance partly in the liquid and partly in the solid state at one temperature. The three cases of transition from gas to liquid, from gas to solid, and from liquid to solid, present us with perfectly definite limits of elasticity,—the only perfectly definite limits of elasticity in nature which we have any means of knowing.

29. Viscosity of Fluids and Solids. Closely connected with limits of elasticity, and with imperfectness of elasticity, is viscosity, that is to say, resistance to change of shape depending on the velocity of the change. The full discovery of the viscosity of liquids and gases is due originally to Stokes; and his hypothesis that in fluids the force of resistance is in simple proportion to the velocity of change of shape has been subsequently confirmed by the experimental investigations of Helmholtz, Maxwell, Meyer, Kundt, and Warburg. The definition of a fluid given in section 2 above may, by section 1, be transformed into the following:—A fluid is a body which requires no force to keep in any particular shape, or—A fluid is a body which exercises no permanent resistance to a change of shape. The resistance to a change of shape presented by a fluid, evanescent as it is when the shape is not being changed (or vanishing when the velocity of the change vanishes), is essentially different from that permanent resistance to change of shape, the manifestation of which in solids constitutes elasticity of shape as defined in section 1. Maxwell's admirable kinetic theory of the viscosity of gases points to a full explanation of viscosity, whether of gases, liquids, or solids, in the consideration of configurations and arrangements of relative motions of molecules, permanent in a solid under distorting stress, and temporary in fluids or solids while the shape is being changed, in

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1 Transactions of Royal Society, June 1826, "On the Progressive Compression of Water by high degrees of force, with some trials of its effects on other liquids," by J. Perkins. Communicated by W. H. Wollaston, M.D., V.R.E.
ELASTICITY

virtue of which elastic force in the quiescent solid, and viscous resistance to change of shape in the non-quiescent fluid had been produced.

30. Viscosity of Metals and Fatigue of their Elasticity.—Experimental exercises performed by students in the physical laboratory of the university of Glasgow, during the session 1864–65, brought to light some very remarkable and interesting results, proving a loss of energy in elastic vibrators (sometimes as much as two or three per cent. of energy lost in the course of a single vibration in one direction) incomparably greater than anything that could be due to imperfections in their elasticity (section 1), and showing also a very remarkable fatigue of elasticity, according to which a wire which had been kept vibrating for several hours or days through a certain range came to rest much quicker when left to itself than when set in vibration after it had been at rest for several days and then immediately left to itself. Thus it was found that the rates of subsidence of the vibrations of the several wires experimented on were generally much less rapid on the Monday mornings, when they had been at rest since the previous Friday, than on other days of the week, or than after several series of experiments had been made on a Monday. The following statement (sections 31–34) is extracted from a short article by W. Thomson, in the Proceedings of the Royal Society for May 18, 1865, containing some of the results of these observations.

31. "Viscosity."—By induction from a great variety of observed phenomena, we are compelled to conclude that no change of volume or of shape can be produced in any kind of matter without dissipation of energy. Even in dealing with the absolutely perfect elasticity of volume presented by every fluid, and possibly by some solids, as for instance homogeneous crystals, dissipation of energy is an inevitable result of every change of volume, because of the accompanying change of temperature, and consequent dissipation of heat by conduction or radiation. The same cause gives rise necessarily to some degree of dissipation in connection with every change of shape of an elastic solid. But estimates founded on the thermodynamic theory of elastic solids, which I have given elsewhere, have sufficed to prove that the loss of energy due to this cause is small in comparison with the whole loss of energy observed in many cases of vibration. I have also found, by vibrating a spring alternately in air of ordinary pressure and in the exhausted receiver of an air-pump, that there is an internal resistance to its motions immensely greater than the resistance of the air. The same conclusion is to be drawn from the observation made by Kupffer in his great work on the observation made by Kupffer in his great work on the rapidity of the change, from day to day; the other kept at rest, except when vibrated in an experiment once a day (to test the effect of continued vibration on the viscosity of a metal).

32. "The frictional resistance against change of shape must in every solid be infinitely small when the change of shape is made at an infinitely slow rate, since, if it were finite for an infinitely slow change of shape, there would be infinite rigidity, which we may be sure does not exist in nature. Hence there is in elastic solids a molecular friction which may be properly called viscosity of solids, because, as being an internal resistance to change of shape depending on the rapidity of the change, it must be classed with fluid molecular friction, which by general consent is called viscosity of fluids. But, at the same time, it ought to be remarked that the word viscosity, as used hitherto by the best writers, when solids or heterogeneous semi-solid semi-fluid masses are referred to, has not been distinctly applied to molecular friction, especially not to the molecular friction of a highly elastic solid within its limits of high elasticity, but has rather been employed to designate a property of slow continual yielding through very great, or altogether unlimited, extent of change of shape, under the action of continued stress. It is in this sense that Forbes, for instance, has used the word in stating that "viscous theory of glacial motion," which he demonstrated by his grand observations on glaciers. As, however, he and many other writers after him have used the words plasticity and plastic, both with reference to homogeneous solids (such as wax or pitch even though also brittle, soft metals, &c.) and to heterogeneous semi-solid semi-fluid masses (as mud, moist earth, mortar, glacial ice, &c.), to designate the property common to all those cases of experiencing, under continued stress, either quite continued and unlimited change of shape, or gradually very great change at a diminishing (asymptotic) rate through infinite time, and as the use of the term plasticity implies no more than does viscosity any physical theory or explanation of the property, the word viscosity is without inconvenience left available for the definition I propose.

33. "To investigate the viscosity of metals, I have in the first place taken them in the form of round wires, and have chosen torsional vibrations, after the manner of Coulomb, for observation, as being the easiest way to arrive at definite results. In every case one end of the wire was attached to a rigid vibrator with sufficient firmness (thorough and smooth soldering I find to be always the best plan when the wire is thick enough); and the other to a fixed rigid body, from which the wire hangs, bearing the vibrator at its lower end. I arranged sets of observations to be made for the separate comparison of the following cases:—

(a) The same wire with different vibrators of equal weights to give equal stretching-tractions but different moments of inertia (to test the relation between viscous resistances against motions with different velocities through the same range and under the same stress).

(b) The same wire with different vibrators of equal moments of inertia but unequal weights (to test the effect of different longitudinal tractions on the viscous resistance to torsion under circumstances similar in all other respects).

(c) The same wire and the same vibrator, but different initial ranges in successive experiments (to test an effect unexpectedly discovered, by which the subsidence of vibrations from any amplitude takes place at very different rates according to the immediately previous molecular condition, whether of quiescence or of recurring changes of shape through a wider range).

(d) Two equal and similar wires, with equal and similar vibrators, one of them kept as continually as possible in a state of vibration, from day to day; the other kept at rest, except when vibrated in an experiment once a day (to test the effect of continued vibration on the viscosity of a metal).

34. "Results."—(a) It was found that the loss of energy in

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2 We have no evidence that the precious metals are more elastic than copper, iron, or brass. One of the new bronze pennies given quite as a clear a ring as a two-shilling silver piece tested in the usual manner.
3 Torsional vibrations of a weight hung on a zinc wire subside so rapidly, that it has been found scarcely possible to count more than twenty of them in one case experimented on.
4 Those who believe in the existence of indivisible, infinitely strong and infinitely rigid, very small bodies (finite hard atoms) deny this.
a single vibration through one range was greater the greater
the velocity (within the limits of the experiments); but the
difference between the losses at low and high speeds was
much less than it would have been had the resistance been,
as Stokes has proved it to be, in fluid friction, approximately
as the rapidity of the change of shape. The irregularities in
the results of the experiments which up to this time I
have made seem to prove that much smaller vibrations
(producing less absolute amounts of distortion in the parts
of the wires most stressed) must be observed before any
simple law of relation between molecular friction and
velocity can be discovered.

(b) "When the weight was increased, the viscosity was
always at first much increased; but then day after day
it gradually diminished and became as small in amount as
it had been with the lighter weight. It has not yet been
practicable to continue the experiments long enough in any
case to find the limit to this variation.

(c) "The vibration subsided in aluminium wires much
more rapidly from amplitude 20 to amplitude 10, when the
initial amplitude was 40, than when it was 20. Thus, with
a certain aluminium wire, and vibrator No. 1 (time of
vibration one way 1:757 second), the number of vibrations
counted were in three trials—

| Subsidence from 40 initial amplitude to 20 | 56 64 64
| And from 20 (in course of the same experiments) to 10 | 96 98 98
| The same wire and the same vibrator showed— | 112 vibrations.
| Subsidence from 20 initial amplitude to 10 | (average of four trials)
| Again, the same wire, with vibrator No. 2 (time of vibration one way 1:236), showed in two trials— | 100 vibrations.
| Subsidence from 40 initial amplitude to 20 | 54 62
| And continued from 20 to 10 | 90 90
| Again, same wire and vibrator,— | 108 vibrations.
| From initial amplitude 20 to 10 . . . . 108 (mean of eight trials). This remarkable result suggested the question (d).

(d) "In a wire which was kept vibrating nearly all day,
day by day, after several days very much more mole-
cular friction was found than in another kept quiescent ex-
cept during each experiment. Thus two equal and similar
pieces of copper wire were put up about the 26th of April,
hanging with equal and similar lead weights, the upper and
lower ends of the two wires being similarly fixed by solder-
ing. No. 2 was more frequently vibrated than No. 1 for
a few days at first, but no comparison of viscosities was
made till May 15. Then

No. 1 subsided from 20 initial range to 10 in 97 vibrations.
No. 2 gave the same subsidence in 77 vibrations.

During the greater part of May 16 and 17, No. 2 was kept
vibrating and No. 1 quiescent, and late on May 17 exper-
iments with the following results were made :

<table>
<thead>
<tr>
<th>Time per vibration.</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 1 subsided from 20 to 10 after 99 vibrations in 237 secs.</td>
</tr>
<tr>
<td>&quot;</td>
</tr>
<tr>
<td>&quot;</td>
</tr>
<tr>
<td>No. 2 subsided from 20 to 10 after 68 vibrations in 142 secs.</td>
</tr>
<tr>
<td>&quot;</td>
</tr>
<tr>
<td>&quot;</td>
</tr>
<tr>
<td>again</td>
</tr>
</tbody>
</table>

[Addition, May 27, after the reading of the paper.—] No.
1 has been kept at rest from May 17, while No. 2 has been
kept oscillating more or less every day till yesterday, May
26, when both were oscillated, with the following results :

<table>
<thead>
<tr>
<th>Time per vibration.</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 1 subsided from 20 to 10 after 100 vibrations in 242 secs.</td>
</tr>
<tr>
<td>No. 2</td>
</tr>
</tbody>
</table>

35. The investigation was continued with much smaller
degrees of maximum angular distortion, to discover, if
possible, the law of the molecular friction, the existence of
which was demonstrated by these experiments. Two ques-
tions immediately occurred:—What is the law of sub-
sidence of range in any single series of oscillations, the
vibrator being undisturbed by external force? and (questi-
on (a) of § 33 above) what is the relation between the
law of subsidence in two sets of oscillations having different
periods, with the same elastic body in the same circum-
stances of elastic force, for instance the same or similar
metallic wires with equal weights hung upon them, per-
forming torsional oscillations in different times on account
of the moments of inertia of the suspended masses being
different?

36. So far as the irregularities depending on previous
conditions of the elastic substance allowed any simple law
to be indicated, the experimental answer to the first
question for degrees of angular distortion much smaller
than the palpable limits of elasticity was the Compound
Interest Law, that is to say,—The diminutions of range
per equal intervals of time or per equal numbers of oscilla-
tions bore a constant proportion to the diminishing range;
or, The differences of the logarithms of the ranges were pro-
portional to the intervals of time.

The only approach to an answer to the second question
yet obtained is that the proportionate losses of amplitude in
the different cases are not such as they would be if the
molecular resistance were simply proportional to the velocity
of change of shape in the different cases. If the molecular
friction followed this simple law, the proportionate diminu-
tions of range per period would be directly as the square
roots of the periods, or per equal intervals of time they
would be inversely as the square roots of the periods.
Instead of the proportion being so, the loss was greater
with the longer periods than that calculated according to
the square roots of their respective periods. It was in fact as
it would be if the result were wholly or partially due to imperfect elasticity, or
"elastische Nach-wirkung" — elastic after-working — as
the Germans call it (compare section 6 above). To
form a rough idea of the results, irrespectively of the
ultimate molecular theory (which is to be looked for
in the proper extension of Maxwell's kinetic theory of
viscosity of gases), consider a perfectly elastic vesicular
solid, whether like a sponge with communications be-
tween the vesicles, or with each vesicle separately inclosed
in elastic solid: imagine its pores and interstices filled up
with a viscous fluid, such as oil. Elastic experiments on such a
solid will show perfect circularity of bulk and shape; kinetic
experiments will show losses of energy such as are really
shown by vibrators of india-rubber, jelly, glass, metals, or
other elastic homogeneous solids, but more regular, and
following more closely the compound interest law for single
series and the law of relation to square roots of periods
stated above for sets of oscillations in different periods.
In short, according to Stokes's law of viscosity of fluids,
our supposed vesicular vibrator would follow the law of
subsidence of a simple vibrator experiencing a resistance
simply proportional to the velocity of its motion, while no
such simple law is applicable to the effects of the internal
molecular resistance in a vibrating elastic solid.

37. Hooke's Law,—A law expressed by Hooke with
Latin tenses in the words Ut tensio sic vis is the
foundation of the mathematical theory of the elasticity
of hard solids. By tensio here is meant not force (as is
generally meant by the English word tension), but an
eelongation produced by force. In English, then, Hooke's
law is that elongation (understood of an elastic solid) is
proportional to the force producing it. It is, of course, to
extend continuously from elongation to contraction
in respect to the effect, and from pull to push in respect

3 Of same weight as No. 1, but different moment of inertia.
to the cause; and the experiments on which it is founded prove a perfect continuity from a pulling force to a smaller force in the same direction, and from the less force to zero, and from zero of pulling force to different degrees of push or positive pressure, or negative pull. Experimental proof merely of the 

continuity of the phenomena through zero of force suffices to show that, for infinitely small positive or negative pulls, positive or negative elongation is simply proportional to the positive or negative pull; or, in other words, positive or negative contraction is proportional to the positive or negative pressure producing it. But now must be invoked minutely accurate experimental measurement to find how nearly the law of simple proportionality holds through finite ranges of contraction and elongation. The answer happily for mathematicians and engineers is that Hook's law is fulfilled, as accurately as any experiments hitherto made can tell, for all metals and hard solids each through the whole range within its limits of elasticity; and for woods, cork, india-rubber, jellies, when the elongation is not more than two or three per cent., or the angular distortion not more than a few hundredths of the radian (or not more than about two or three degrees). The same law holds for the condensation of liquids up to the highest pressures under which their compressibility has hitherto been accurately measured. [A decided but small deviation from Hooke's law has been found in steel pianoforte wire under combined influence of torsion and longitudinal pull by Mr M'Farlane in experiments made for the present article after this section was in type. See section 81.]

Boyle's law of the "spring of air" shows that the augmentation of density of a gas is simply proportional to the augmentation of the pressure, through the very wide ranges of pressure through which that law is approximately enough fulfilled. Hence the infinitesimal diminution of volume produced by a given infinitesimal augmentation of pressure varies as the square of the volume, and the proportionate diminution of volume (that is to say, the ratio of the diminution of volume to the volume) is proportional to the volume, or inversely proportional to the density. Andrew's experiments on the compressibility of a fluid, such as carbonic acid, at temperatures slightly above the critical temperature, and of the gas and of liquids at temperatures slightly below the critical temperature, show molecules too small to be directly visible, or measurable but not undiscoverably small,—really, it is to be believed, of dimensions to be accurately determined in future advances of science. Practically the definition of homogeneousness may be applied on a very large scale to masses of building or coarse-grained conglomerate rock, or on a more moderate scale to blocks of common sandstone, or on a very small scale to seemingly homogeneous metals; or on a scale of extreme, undiscovered fineness, to vitreous bodies, continuous crystals, solidified gums, as india-rubber, gum-arabic, &c., and fluids.

39. Isotropic and Anisotropic Substances defined.—The substance of a homogeneous solid is called isotropic when a spherical portion of it, tested by any physical agency, exhibits no difference in quality however it is turned. Or, which amounts to the same, a cubical portion, cut from any position in an isotropic body, exhibits the same qualities relatively to each pair of parallel faces. Or two equal and similar portions cut from any positions in the body, not subject to the condition of parallelism (section 38), are undistinguishable from one another. A substance which is not isotropic, but exhibits differences of quality in different directions, is called anisotropic. The remarks of section 38 relative to homogeneousness in the aggregate, and the supposed ultimately heterogeneous texture of all substances, however seemingly homogeneous, indicate corresponding limitations and non-rigorous practical interpretations of isotropy and anisotropy.

40. Isotropy and Anisotropy of different sets of properties.—The substance of a homogeneous solid may be isotropic in one quality or class of qualities, but anisotropic in others. Or a transparent substance may transmit light at different velocities in different directions through it (that is, be doubly-refracting), and yet a cube of it may (and does in many natural crystals) show no sensible difference in its absorption of white light transmitted across it perpendicularly to any of its three pairs of faces. Or (as a crystal which exhibits dichroism) it may be sensibly anisotropic relatively to the absorption of light, but not sensibly double-refracting, or it may be dichroic and doubly-refracting, and yet it may conduct heat equally in all directions. Still, as a rule, a homogeneous substance which is anisotropic for one quality must be more than infinitesimally anisotropic for every quality which has directional character admitting of a corresponding anisotropy.

41. Moduli of Elasticity.—A modulus of elasticity is the number obtained by dividing the number expressing a stress by the number expressing the strain which it produces. A modulus is called a principal modulus when

though conceivable, is not generally regarded as probable, for any of the real solids or fluids known to us, however seemingly homogeneous. It is held by all naturalists that there is a molecular structure, according to which, in compound bodies such as water, ice, rock-crystal, &c., the constituent substances lie side by side, or arranged in groups of finite dimensions, and even in bodies called simple (that is those not known to be chemically resolvable into other substances) there is no ultimate homogeneousness. In other words, the prevailing belief is that every kind of matter with which we are acquainted has a more or less coarse-grained texture, whether (as great masses of solid brick-work or stone-building, or as natural sandstone or granite rocks) having visible molecules, or (as seemingly homogeneous metals, or continuous crystals, or liquids, or gases) having molecules too small to be directly visible, or measurable but not undiscoverably small,—really, it is to be believed, of dimensions to be accurately determined in future advances of science.

1 Which, however, we know, as proved by Deville and Van Troost, are porous enough at high temperatures to allow very free percolation of gases. Helmholts and Root find percolation of platinum by hydrogen at ordinary temperature (Beri. Sitzungbericht).

2 Thomson and Tait's Natural Philosophy, section 678.

3 Mathematical Theory, below, chap. i.

4 Ibid.
the stress is such that it produces a strain of its own type.

1. An isotropic solid has two principal moduluses—a modulus of compression and a rigidity.

2. A crystal of the cubic class (flour-spar, for instance) has three principal moduluses—one modulus of compression and two rigidities.

3. An anisotropic solid having (what no natural crystal has, but what a drawn wire has) perfect isotropy of physical qualities relative to all lines perpendicular to a certain axis of its substance has three principal moduluses, two determinable from its different compressibilities along and perpendicular to the axis, or from one compressibility and the "Young's modulus" (section 42) of an axial bar of the substance, or determinable from two compressibilities; and one rigidity determinable by measurement of the torsional rigidity of a round axial bar of the substance.

4. A crystal of Iceland spar has four principal moduluses—three like those of case (3), and another rigidity depending on (want of complete circular symmetry, and possession of triple symmetry of form, involving six equal elastic symmetries, round the crystalline axis.

5. A crystal of the rectangular parallelepiped (or "tessaral") class has six distinct principal moduluses which, when the directions of the principal axes are known, are determinable by six single observations—three, of the three (generally unequal) compressibilities along the three axes; and three, of the three rigidities (no doubt generally unequal) relatively to the three simple distortions of the parallelepiped, in any one of which one pair of parallel rectangular faces of the parallelepiped become oblique parallelograms.

6. An anisotropic solid generally has six principal moduluses, which, when a piece of the solid is presented without information, and without any sure indication from its appearance of any particular axis or axes of symmetry of any kind, require just twenty-one independent observations for the determination of the fifteen quantities specifying their types, and the six numerical values of the moduluses themselves.

42. "Young's Modulus," or Modulus of Simple Longitudinal Stress.—Thomas Young called the modulus of elasticity of an elastic solid the amount of the end-pull or end-thrust required to produce any infinitesimal elongation or contraction of a wire, or bar, or column of the substance multiplied by the ratio of its length to the elongation or contraction. In this definition the definite article is clearly misapplied. There are, as we have seen, two moduluses of elasticity for an isotropic solid—one measuring elasticity of bulk, the other measuring elasticity of shape. An interesting and instructive illustration of the confusion of ideas so often rising in physical science from faulty logic is to be found in "An Account of an Experiment on the Elasticity of Ice: By Benjamin Bevan, Esq.," in a letter to Dr Thomas Young, Foreign Sec. R. S., in Young's "Note" upon it, both published in the Transactions of the Royal Society for 1826. Bevan gives an interesting account of a well-designed and well-executed experiment on the flexure of a bar, 3-97 inches thick, 10 inches broad, and 100 inches long, of ice on a pond near Leighton Buzzard (the bar remaining attached by one end to a sheet of ice, but being cut free by a saw along its sides and across its other end), by which he obtained a fairly accurate determination of "the modulus of ice," and says that he repeated the experiment in various ways on ice bars of various dimensions, some remaining attached by one end, others completely detached, and found results agreeing with the first as nearly "as the admeasurement of the thickness could be ascertained." He then proceeds to compare "the modulus of ice" which he had thus found with "the modulus of water," which he quotes from Young's Lectures as deduced from Canton's experiments on the compressibility of water. Young in his "Note" does not point out that the two moduluses were essentially different, and that the modulus of his definition, the modulus determinable from the flexure of a bar, is essentially zero for every fluid. We now call "Young's modulus" the particular modulus of elasticity defined as above by Young, and so avoid all confusion.

43. Modulus of Rigidity.—The "modulus of rigidity" of an isotropic solid is the amount of tangential stress divided by the deformation it produces—the former being measured in units of force per unit of the area to which it is applied in the manner indicated by the annexed diagram (fig. 3), and the latter by the variation of each of the four right angles reckoned in fraction of the radian. By drawing either diagonal of the square in the diagram we see that the distorting stress represented by it gives rise to a normal traction on every surface of the substance perpendicular to the square and parallel to one of its diagonals, and an equal normal pressure on every other diagonal, and that the amount of each of these normal forces per unit of area is equal to the amount per unit area of the tangential forces which the diagram indicates. The corresponding geometrical proposition, also easily proved, is as follows: A strain compounded of a simple extension in one set of parallels, and a simple contraction of equal amount in any other set perpendicular to that, is the same as a simple shear in either of the two sets of planes cutting the two sets of parallels at 45°, and the numerical measuring of this shear or simple distortion is equal to double the amount of the elongation or contraction, each reckoned per unit of length.

Hence we have another definition of "modulus of rigidity" equivalent to the preceding:—The modulus of rigidity of an isotropic substance is the amount of normal traction or pressure per unit of area, divided by twice the amount of elongation in the direction of the traction or of contraction in the direction of the pressure, when a piece of the substance is subjected to a stress producing uniform distortion.  

44. Conditions fulfilled in Elastic Isotropy.—To be elastically isotropic, a spherical or cubical portion of any solid, if subjected to uniform normal pressure (positive or negative) all round, must, in yielding, experience no deformation, and therefore must be equally compressed (or dilated) in all directions. But, further, a cube cut from any position in it, and acted on by tangential or distorting stress in planes parallel to two pairs of its sides, must experience simple deformation, or "shearing" parallel to either pair of these sides, unaccompanied by condensation or dilatation, if the directions of these forces are called the "axes" of the stress. The corresponding directions in the corresponding strain are called the axes of the strain.

1 Mathematical Theory, chap. xvi.
2 The result is given in the Table of Moduluses, sec. 77, below.
3 The directions of these forces are called the "axes" of the stress.
4 Mathematical Theory, chap. vi.
5 This, with several of the following, sections 44-51, is borrowed, with slight change, from the first edition of Thomson and Tait's Natural Philosophy, by permission of the authors.
6 It must be remembered that the changes of figure and volume we are concerned with are so small that the principle of superposition is
and the same in amount for all the three ways in which a stress may be thus applied to any one cube, and for different cubes taken from any different positions in the solid. Hence the elastic quality of a perfectly elastic, homogeneous, isotropic solid is fully defined by two elements—its resistance to distortion and its resistance to compression. The first has been already considered (section 43). The second is measured by the amount of uniform pressure in all directions per unit area of its surface required to produce a stated very small compression. The numerical reckoning of the first is the compressing pressure divided by the diminution of the bulk of a portion of the substance which, when uncompressed, occupies the unit volume. It is sometimes called the "elasticity of bulk," or sometimes the "modulus of bulk-elasticity," sometimes the "resistance to compression." Its reciprocal, or the amount of compression on unit of volume divided by the compressing pressure, or, as we may conveniently say, the compression per unit of volume per unit of compressing pressure, is commonly called the compressibility.

45. Strain produced by a single Longitudinal Stress (subject of Young's Modulus).—Any stress whatever may be made up of simple longitudinal stresses. Hence, to find the relation between any stress and the strain produced by it, we have only to find the strain produced by a single longitudinal stress, which, for an isotropic solid, we may do at once thus:—A simple longitudinal stress $P$ is equivalent to a uniform dilating tension $\frac{1}{k}P$ in all directions, compounded with two distorting stresses, each equal to $\frac{1}{k}P$, and having a common axis in the line of the given longitudinal stress, and their other two axes any two lines at right angles to one another and to it. The diagram (fig. 4), drawn in a plane through one of these latter lines and the former, sufficiently indicates the synthesis,—the only forces not shown being those perpendicular to its plane.

Hence if $n$ denote the rigidity, and $k$ the modulus of compression, or the modulus of bulk-elasticity (being the same as the reciprocal of the compressibility), the effect will be an equal dilatation in all directions, amounting, per unit of volume, to

$$\frac{1}{k}P$$

(compounded with two equal distortions, each amounting to

$$\frac{1}{n}P$$

and having (section 43, footnote) their axes in the directions just stated for the axes of the distorting stresses.

46. The dilatation and two shears thus defined may be conveniently reduced to simple longitudinal strains by following the indications of section 43, thus:—

The two shears together constitute an elongation amounting to $\frac{1}{n}P$ in the direction of the given force $P$, and equal contraction amounting to $\frac{1}{n}P$ in all directions perpendicular to it. And the cubic dilatation $\frac{1}{k}P$ implies a linear dilatation, equal in all directions, amounting to $\frac{1}{9}P$.

On the whole, therefore, we have

\begin{align*}
\text{linear elongation} & = P \left( \frac{1}{3n} + \frac{1}{9k} \right), \\
\text{linear contraction} & = P \left( \frac{1}{9n} + \frac{1}{9k} \right),
\end{align*}

in all directions perpendicular to the applied stress.

47. Hence "Young's Modulus" = $\frac{9nk}{3k + n}$, and when the ends of a column, bar, or wire of isotropic material are acted on by equal and opposite forces, it experiences a lateral lineal contraction equal to $\frac{3k - 2n}{2(3k + n)}$ of the longitudinal dilatation, each reckoned as usual per unit of lineal measure. One specimen of the fallacious mathematics referred to in chap. xvi. of the mathematical theory below is a celebrated conclusion of Navier's and Poisson's that the ratio of lateral contraction to elongation by pull without transverse force is $\frac{1}{3}$. This would require the rigidity to be $\frac{2}{3}$ of the resistance to compression, for all solids; which was first shown to be false by Stokes from many obvious observations, proving enormous discrepancies from it in many well-known bodies, and rendering it most improbable that there is any approach to a constancy of ratio between rigidity and resistance to compression in any class of solids. Thus clear elastic jellies and india-rubber present familiar specimens of isotropic homogeneous solids which, while differing very much from one another in rigidity ("stiffness"), are probably all of very nearly the same compressibility as water, which is about $33 \frac{1}{2}$ per atmosphere. Their resistance to compression, measured by the reciprocal of this, is obviously many hundred times the absolute amount of the rigidity of the stiffest of those substances. A column of any of them, therefore, when pressed together or pulled out, within its limits of elasticity, by balancing forces applied to its ends (or an india-rubber band when pulled out), experiences no sensible change of volume, though very sensible change of length. Hence the proportionate extension or contraction of any transverse diameter must be sensibly equal to half the longitudinal contraction or extension; and such substances may be practically regarded as incompressible elastic solids in interpreting all the phenomena for which they are most remarkable. Stokes gave reasons for believing that metals also have in general greater resistance to compression, in proportion to their rigidities, than according to the fallacious theory, although for them the discrepancy is very much less than for the gelatinous bodies. This probable conclusion was soon experimentally demonstrated by Wertheim, who found the ratio of lateral to longitudinal change of lineal dimensions, in columns acted on solely by longitudinal force, to be about $\frac{1}{3}$ for glass and brass; and by Kirchhoff, who, by a well-devised experimental method, found 0.57 as the value of the ratio for brass, and 0.34 for copper. For copper it is shown to lie between 0.226 and 0.441, by experiments quoted below, measuring the torsional and longitudinal rigidities of copper wires.

48. All these results indicate rigidity less in proportion to the compressibility than according to Navier's and Poisson's theory. And it has been supposed by many naturalists who have seen the necessity of abandoning that theory as inapplicable to ordinary solids that it may be regarded as the proper theory for an ideal perfect solid, and as indicating an amount of rigidity not quite reached in any real substance, but approached to in some of the

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1 Mathematical Theory, chap. viii.


3 "On the Elasticity and Viscosity of Metals" (W. Thomson), Proc. R. S., May 1866.
most rigid of natural solids (as, for instance, iron). But it is scarcely possible to hold a piece of cork in the hand without perceiving the fallacy of this last attempt to maintain a theory which never had any good foundation. By careful measurements on columns of cork of various forms (among them, cylindrical pieces cut in the ordinary way for bottles), before and after compressing them longitudinally in a Bramah's press, we have found that the change of lateral dimensions is insensible both with small longitudinal contractions and with more dilatations, within the limits of elasticity, and with such enormous longitudinal contractions as to $\frac{1}{4}$ or $\frac{1}{3}$ of the original length. It is thus proved decisively that cork is much more rigid, while metals, glass, and gelatinous bodies are all less rigid, in proportion to resistance to compression, than the supposed “perfect solid”; and the practical invalidity of the theory is experimentally demonstrated. By obvious mechanism of jointed bars a solid may be designed which shall swell laterally when pulled, and shrink laterally when compressed, in one direction, and which shall be homogeneous in the same sense (article 40) as crystals and liquids are.

49. Modulus of Simple Longitudinal Strain.—In sections 45, 46, we examined the effect of a simple longitudinal stress in producing elongation in its own direction, and contraction in lines perpendicular to it. With stresses substituted for strains, and strains for stresses, we may apply the same process to investigate the longitudinal and lateral tractions required to produce a simple longitudinal strain (that is, an elongation in one direction, with no change of dimensions perpendicular to it) in a rod or solid of any shape.

Thus a simple longitudinal strain $e$ is equivalent to a cubic dilatation $e$ without change of figure (or linear dilatation $\frac{1}{e}$ equal in all directions), and two distortions consisting each of dilatation $\frac{1}{e}$ in the given direction and contraction $\frac{1}{e}$ in each of two directions perpendicular to it and to one another. To produce the cubic dilatation $e$ alone requires (section 44) a normal traction $ke$ equal in all directions. And, to produce either of the distortions simply, since the measure (section 43) of each is $\frac{1}{e}$, requires a distorting stress equal to $n \times \frac{1}{e}$, which consists of tangential tractions each equal to this amount, positive (or drawing outwards) in the line of the given elongation, and negative (or pressing inwards) in the perpendicular direction. Thus we have in all normal traction $-(k + \frac{1}{e})e$, in the direction of the given strain, and normal traction $(k - \frac{1}{e})e$, in every direction perpendicular to the given strain.

Hence the modulus of simple longitudinal strain is $k + \frac{1}{e}$.

50. Weight-Modulus and Length of Modulus.—Instead of reckoning modules in units of force per unit of area, it is sometimes convenient to express them in terms of the weight of unit bulk of the solid. A modulus thus reckoned, or, as it is called by some writers, the length of the modulus, is of course found by dividing the weight-modulus by the weight of the unit bulk. It is useful in many applications of the theory of elasticity, as, for instance, in this result, which is proved in the elementary dynamics of waves in an elastic solid or fluid (chap. xvii. of the Mathematical Theory, below)—the velocity of transmission of longitudinal vibrations (as of sound) along a bar of cord, or of waves of simple distortion, or of simple longitudinal extension and contraction in a homogeneous isotropic solid, or of sound waves in a fluid, is equal to the velocity acquired by a body in falling from a height equal to half the length of the proper modulus for the case,—that is, the Young's Modulus $\frac{9m}{38 + n}$ for the first case, the modulus of rigidity ($\frac{1}{e}$) for the second, the modulus of simple longitudinal strain ($k + \frac{1}{e}$) for the third, the modulus of compression $k$ for the fourth. Remark that for air the static “length-modulus of compression” is constant, and is the same as what is often technically called the “height of the homogeneous atmosphere.”

51. In reckoning modules there must be a definite understanding as to the unit in terms of which the force is measured, which may be either the kinetic unit or the gravitation unit for a specified locality, that is, the weight in that locality of the unit of mass. Experimenters hitherto have stated their results in terms of the gravitation unit, each for his own locality,—the accuracy hitherto attained being scarcely in any cases sufficient to require corrections for the different intensities of gravity in the different places of observation.

The most useful and generally convenient specification of the modulus of elasticity of a substance is its weight per square centimetre. This has only to be divided by the specific gravity of the substance to give the length of the modulus. British measures, however, being still unhappily sometimes used in practical and even in scientific statements, we too often meet with reckonings of the modulus in pounds per square inch or per square foot, in tons per square inch, or of length of the modulus in feet or in British statute miles.

The reckoning most commonly adopted in British treatises on mechanics and practical statements is pounds per square inch. The modulus thus stated must be divided by the weight of 12 cubic inches of the solid, or by the product of its specific gravity into $\frac{1}{435}$, to find the length of the modulus in feet.

To reduce from pounds per square inch to grammes per square centimetre, multiply by 70.31, or divide by 0.014223. French engineers generally state their results in kilogrammes per square millimetre, and so bring them to more convenient numbers, being $\frac{1}{1000}$ of the incomparable whimsical use of the French metric system is that the mass of the unit volume (1 cubic centimetre) of water at its temperature of maximum density ($3^\circ$ $\deg C.$) is unity (1 gramme) to a sufficient degree of approximation for almost all practical purposes. Thus, according to this system, the density of a body and its specific gravity are one and the same thing; whereas on the British no-system the density is expressed by a number found by multiplying the specific gravity by one number or another, according to the choice of a cubic inch, pint, quart, wine gallon, imperial gallon, cubic foot, cubic yard, or cubic mile that is made for the unit of volume; and the grain, scruple, granmaler's drachm, apothecary's drachm, ounce Troy, ounce avoirdupois, pound Troy, pound avoirdupois, stone (Imperial, Ayrshire, Lanarkshire, Dumbartonshire), stone for hay, stone for corn, quarter (of imperial), hundredweight, or ton that is chosen for unit of mass. It is a remarkable phenomenon, belonging rather to moral and social than to physical science, that a people tending naturally to be regulated by common sense should voluntarily confine themselves, as the British have so long done, to vanity in every action of common business or scientific work related to measurement, from which all the other nations of Europe and America have emancipated themselves. Professor W. H. Miller, of Cambridge, concludes, from a very trustworthy comparison of standards by Kupffer, of St Petersburg, that the weight of a cubic decimetre of water at temperature of maximum density is 1000.013 grammes.

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1 It is to be understood that the vibrations in question are so much spread out through the length of the body that inertia does not sensibly influence the transverse contractions and dilatations which (unless the substance have in this respect the peculiar character presented by cork, section 49) take place along with them.
veniently large numbers expressing moduluses in grammes weight per square centimetre, but it is much better to reckon in millions of grammes per square centimetre.

52. "Resilience" is a very useful word, introduced about forty years ago (when the doctrine of energy was beginning to become practically appreciated) by Lewis Gordon, first professor of engineering in the university of Glasgow, to denote the quantity of work that a spring (or elastic body) can do without breaking or taking a permanent set. In all cases for which Hooke's law of simple proportionality between stress and strain holds, the resilience is obviously equal to the work done by a constant force of half the amount of the extreme forces acting through a space equal to the extreme deflection.

53. When force is reckoned in "gravitation measure," resilience per unit of the spring's mass is simply the height that the spring itself, or an equal weight, could be lifted against gravity by an amount of work equal to that given back by the spring returning from the stressed condition.

54. Let the elastic body be a long homogeneous cylinder or prism with flat ends (a bar as we may call it for brevity), and let the stress for which its resilience is reckoned be produced by normal pressures on its ends. The resilience per unit mass is equal to the greatest height from which the bar can fall with its length vertical, and impinge against a perfectly hard horizontal plane without suffering stress beyond its limits of elasticity. For in this case (as in the case of the direct impact of two equal and similar bars meeting with equal and opposite velocities, discussed in Thomson and Tait's Natural Philosophy, section 303), the kinetic energy of the translational motion preceding the impact is, during the first half of the collision, wholly converted into potential energy of elastic force, which during the second half of the collision is wholly reconverted into kinetic energy of translational motion in the reverse direction. During the whole time of the collision the stopped end of the bar experiences a constant pressure, and at the middle of the collision the whole substance of the bar is for an instant at rest in the same state of compression as it would have permanently if in equilibrium under the influence of that pressure and an equal and opposite pressure on the other end. From the beginning to the middle of the collision the compression advances at a uniform rate through the bar from the stopped end to the free end. Every particle of the bar which the compression has not reached continues moving uniformly with the velocity of the whole before the collision until the compression reaches it, when it instantaneously comes to rest. The part of the bar which at any instant is all that is compressed remains at rest till the corresponding instant in the second half of the collision.

55. From our preceding view of a bar impinging against an ideal perfectly rigid plane, we see at once all that takes place in the real case of any rigorously direct longitudinal collision between two equal and similar elastic bars with flat ends. In this case the whole of the kinetic energy which the bodies had before collision reappears as purely translational kinetic energy after collision. The same would be approximately true of any two bars, provided the times taken by a pulse of longitudinal stress to run through the bars be equal. Thus if the two bars be of the same substance, or of different substances having the same value for Young's modulus, the lengths must be equal, but the diameters may be unequal. Or if the Young's modulus be different in the two bars, their lengths must (Math. Theory, chap. xvii.) be inversely as the square roots of its value. To all such cases the laws of "collision between two perfectly elastic bodies," whether of equal or unequal masses, as given in elementary dynamical treatises, are applicable. But in every other case part of the translational energy which the bodies have before collision is left in the shape of vibrations after collision, and the translational energy after collision is accordingly less than before collision. The losses of energy observed in common elementary dynamical experiments on collision between solid globes of the same substance are partly due to this cause. If they were wholly due to it they would be independent of the substance, when two globes of the same substance are used. They would bear the same proportion to the whole energy in every case of collision between two equal globes, or again, in every case of collision between two globes of any stated proportion of diameters, provided in each case the two which collide are of the same substances; but the proportion of translational energy converted into vibrations would not be the same for two equal globes as for two unequal globes. Hence when differences of proportionate losses of energy are found in experiments on different substances, as in Newton's on globes of glass, iron, or compressed wool, this must be due to imperfect elasticity of the material. It is to be expected that careful experiments upon hard well-polished globes striking one another with such gentle strokes as not to produce even at the point of contact any stress approaching to the limit of elasticity, will be found to give results in which the observed loss of translational energy can be almost wholly accounted for by vibrations remaining in the globes after collision.

56. Examples of Resilience.—Example 1.—In respect to simple longitudinal pull, the extreme resilience of steel pianoforte wire of the gauge and quality referred to in section 22 above (calculated by multiplying the breaking weight into half the elongation produced by it according to the experimental data of section 22) is 6066 metre-grammes (gravitation measure) per ten metres of the wire. Or, whatever the length of the wire, the resilience is equal to the work required to lift its weight through 179 metres.

Example 2.—The torsional resilience of the same wire, twisted in either direction as far as it can be without giving it any notable permanent set, was found to be equal to the work required to lift its weight through 13 metres.

Example 3.—The extreme resilience of a vulcanized india-rubber band weighing 12-5 grammes was found to be equal to the work required to lift its weight through 1200 metres. This was found by stretching it by gradations of weights up to the breaking weight, representing the results by aid of a curve, and measuring its area to find the integral work given back by the spring after being stretched by a weight just short of the breaking weight.

57. Flexure of a Beam or Rod.—In the problem of simple flexure a bar or uniform rod or wire, straight when free from stress, is kept in a circular form by equal opposing couples properly applied to its ends. The parts of the bar on the convex side of the circle must obviously be stretched longitudinally, and those on the concave side contracted longitudinally, by the flexure. It is not obvious, however, what are the conditions affecting the lateral shrinkings and swellings of ideal filaments into which we may imagine the bar divided lengthwise. Earlier writers had assumed without proof that each filament, bent as it is in its actual position in the bar, is elongated or contracted by the same amount as it would be if it were detached, and subjected to the same end pull or end compression with its sides quite free to shrink or
expand, but they had taken no account of the lateral shrinking or swelling which the filament must really experience in the bent bar. The subject first received satisfactory mathematical investigation from St Venant. He proved that the old supposition is substantially correct, with the important practical exception of the flat spring referred to in section 59 below. His theory shows that, in fact, if we imagine the whole rod divided parallel to its length into infinitesimal filaments, each of these shrinks or swells laterally with sensibly the same freedom as if it were separated from the rest of the substance and subjected to end pull or end compression, lengthening or shortening it in a straight line to the same extent as it is really lengthened or shortened in the circular arc which it becomes in the bent rod. He illustrates the distortion of the cross section by which these changes of lateral dimensions are necessarily accompanied in the annexed diagram (fig. 5), in which either the whole normal section of a rectangular, or a rectangular area in the normal section of a beam of any figure, is represented as a strained and unstrained figures, with the central point O common to the two. The flexure is in planes perpendicular to YY, and is concave upwards (or towards X). G, the centre of curvature, being in the direction indicated, but too far to be included in the diagram. The straight sides AC, BD, and all straight lines parallel to them, of the unstrained rectangular area become concentric arcs of circles concave in the opposite direction, whose centre of curvature is represented as being in the case of a broad thin lamina (whether of precisely rectangular section, or of unequal thicknesses in different parts).

51. Flexural Rigidities of a Rod or Beam.—The couple required to give unit curvature in any plane to a rod or beam is called its flexural rigidity for curvature in that plane. When the beam is of circular cross section and of isotropic material, the flexural rigidity is clearly the same, whatever be the plane of flexure through the axis, and the plane of the bending couple coincides with the plane of flexure. It might be expected that in a round bar of anisotropic material, such as a wooden rod with the annual woody layers sensibly plane and parallel to a plane through its axis, would show different flexural rigidities in different planes,—in the case of wood, for example, different according as the flexure is in a parallel or perpendicular to the annual layers. This is not so, however; on the contrary, it is easy to show, by an extension of St Venant's theory, that in the case of the wooden rod the flexural rigidity is equal in all planes through the axis, and that the plane of flexure always agrees with the plane of the bending couple, and to prove generally that the flexure of a bar of anisotropic substance, and composed it may be of longitudinal filaments of heterogeneous materials, is precisely the same as if it were isotropic, and that its flexural rigidities are calculated by the same rule from its Young's modulus, provided that the anisotropy is not such as (section 81) to give rise to alteration of the angle between the length and any diameter perpendicular to the length when weight is hung on the rod, or on any longitudinal filament cut from it. Excluding then all cases in which there is any such oblique anisotropy, we have a very simple theory for the flexure of bars of any substance, whether isotropic or anisotropic, and whether homogeneous or not homogeneous through the cross section.

52. Principal Flexural Rigidities and Principal Planes of Flexure of a Beam.—The flexural rigidity of a rod is generally not equal in different directions, and the plane of flexure does not generally coincide with the plane of the bending couple. Thus a flat ruler is much more easily bent in a plane perpendicular to its breadth than in the plane of its breadth; and if we apply opposing couples to its two ends in any plane through its axis not either perpendicular or parallel to its breadth, it is obvious that the plane in which the flexure takes place will be more inclined to the plane of the breadth than to the plane of the bending couple. Very elementary statical theory, founded on St

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1 Mémoires des Savants Étrangers, 1855, "De la Torsion des Prisms, avec des considérations sur leur Flexion," 10.
2 See Thomson and Tait's Natural Philosophy, vol. 1, § 128.
Venant's conclusions of section 57, shows that, whatever the shape and the distribution of matter in the cross section of the bar, there are two planes at right angles to one another such that if the bar be bent in either of these planes the bending couple will coincide with the plane of flexure. These planes are called principal planes of flexure, and the rigidities of the bar for flexure in these planes are called its principal flexural rigidities. When the principal flexural rigidities are known the flexure of the bar in any plane oblique to the principal planes is readily found by imposing it to be bent in one of the principal planes and simultaneously in the other, and calculating separately the couples required to produce these two component flexures.

The positions of the principal planes of flexure, the relative flexural rigidities, and the law of elongation and contraction in different parts of the cross section, are found according to the following simple rules:

1. Imagine an infinitely thin plane disc of the same shape and size as the cross section loaded with matter in simple proportion to the Young's modulus in different parts of the cross section. Let the quantity of matter per unit area on any point of the disc be equal to the Young's modulus on the corresponding point of the rod when the material is homogeneous; on the other hand, when the material is homogeneous it is more convenient to call the quantity of matter unity per unit area of the disc. Considering different axes in the plane of the disc through its centre of inertia, find the two principal axes of greatest and least moments of inertia, and find the moments of inertia round them.

2. In whatever plane the bar be bent it will experience neither elongation nor contraction in the element which passes perpendicularly through the centre of inertia of the cross section found according to rule (1), nor in the diameter of the cross section perpendicular to the plane of flexure.

3. Thus all the particles which experience neither elongation nor contraction lie in a surface cutting the plane of flexure perpendicularly through the centres of inertia of the cross sections. All the material on the outside of this cylindrical surface is elongated, and all on the inside is contracted, in simple proportion to distance from it: the amount of the elongation or contraction being in fact equal to distance from this neutral surface divided by the radius of its curvature.

4. Hence it is obvious that the portions of the solid on the two sides of any cross section must experience mutual normal forces, pulling them towards one another in the stretched part, and pressing them from one another in the condensed part, and that the amount of this negative or positive normal pressure per unit of area must be equal to the Young's modulus at the place, multiplied into the ratio of its distance from the neutral line of the cross section to the radius of curvature.

The sum of these positive and negative forces over the whole area of the cross section is zero in virtue of condition (2). Their couple resultant has its axis perpendicular to the plane of curvature when this line is either of the principal axes (3) of the cross section; and its moment is clearly equal to the moment of inertia of the material disc (1) divided by the radius of curvature. Hence the principal flexural rigidities are simply equal to the principal moments of inertia of this disc; and the principal flexural planes are the planes through its principal axes and the length of the bar; or taking the quantity of matter per unit area of the disc unity for the ease of a homogeneous bar, we have the rule that the principal rigidities are equal to the product of the Young's modulus into the principal moments of inertia of the cross sectional areas, and the principal planes of flexure are the longitudinal planes through the principal axes of this area.

Torsion.—Of the most beautiful applications of the general equations of internal equilibrium of an elastic solid hitherto made is that of M. de St Venant to "the torsion of prisms." In this work the mathematical methods invented by Fourier for the solution of problems regarding conduction of heat have been most ingeniously and happily applied by St Venant to the problem of torsion. To reproduce St Venant's mathematical investigation here would make this article too long (it occupies 227 quarto pages of the Mémoires des Savants Étrangers); but a statement of some of the chief results is given (sections 63-72), not only on account of their strong scientific interest, but also because they are of great practical value in engineering; and the reader is referred to Thomson and Tait's Natural Philosophy, sections 700-710, for the proofs and for further details regarding results, but much that is valuable and interesting is only to be found in St Venant's original memoir.

64. Torsion Problem stated and Torsional Rigidity defined.—To one end of a long, straight prismatic rod, wire, or solid or hollow cylinder of any form, a given couple is applied in a plane perpendicular to the length, while the other end is held fast: it is required to find the degree of twist produced, and the distribution of strain and stress throughout the prism. The amount of the twist per unit length divided by the moment of the couple is called the torsional rigidity of the rod or prism. This definition is founded simply on the extension of Hooke's law to torsion discovered experimentally by Coulomb, according to which a rod or wire when twisted within limits of torsional elasticity exerts a reactive couple in simple proportion to the angle through which one end is turned relatively to the other. The internal conditions to be satisﬁed in the torsion problem are that the resultant action between the substance on the two sides of any normal section in the couple is equal to the given couple. This problem has not hitherto been attacked for anisotropic solids. Even such a case as that of the round wooden rod (section 61) with annual layers sensibly parallel to a plane through its length, will, when twisted, experience a distribution of strain complicated much by its anisotropy. The following statements of results are confined to rods of isotropic material.

65. Torsion of Circular Cylinder.—For a solid or hollow circular cylinder, the solution (given first, we believe, by Coulomb) obviously is that each circular normal section remains unchanged in its own dimensions, figure, and internal arrangement (so that every straight line of its particles remains a straight line of unchanged length), but is turned round the axis of the cylinder through such an angle as to give a uniform rate of twist equal to the applied couple divided by the product of the moment of inertia of the circular area (whether annular or complete to the centre) into the modulus of rigidity of the substance.

For, if we suppose the distribution of strain thus specified to be actually produced, by whatever application of stress is necessary, we have, in every part of the substance, a simple shear parallel to the normal section, and perpendicular to the radius through it. The elastic reaction against this requires, to balance it (section 43), a simple distorting stress consisting of forces in the normal section, directed as the shear, and others in planes through the axis, and directed parallel to the axis. The amount of the shear is, for parts of the substance at distance r from the axis, equal obviously to nτ r, if τ be the rate of twist reckoned in radians per unit of length of the cylinder. Hence the amount of the tangential force in either set of planes is nτ r per unit of area, if n be the rigidity of the substance. Hence there is no force between parts of the substance lying on the two sides of any element of any circular prismatic inclusion in the bounding cylinder or cylinders; and consequently no force is required on the cylindrical boundary to maintain the supposed state of strain. And the mutual action between the parts of the substance on the two sides of any normal plane section consists of force in this plane, directed perpendicular to the radius through each point, and amounting to nτ r per unit of area. The moment of this distribution of force round the axis of the cylinder is (if dr denote an element of the area) nτ r dr, or the
product of \( \pi \tau \) into the moment of inertia of the area round the perpendicular to its plane through its centre, which is therefore equal to the moment of the couple applied at either end.

66. Prism of any shape constrained to a Simple Twist.—Farther, it is easily proved that if a cylinder or prism of any shape be compelled to take exactly the state of strain above specified (section 65) with the line through the centres of inertia of the normal sections, taken instead of the axis of the cylinder, the mutual action between the parts of it on the two sides of any normal section will be a couple of which the moment will be expressed by the same formula, that is, the product of the rigidity, into the rate of twist, into the moment of inertia of the section round its centre of inertia. But for any other shape of prism than a solid or symmetrical hollow circular cylinder, the supposed state of strain requires, besides the terminal opposed couples, force parallel to the length of the prism, distributed over the prismatic boundary, in proportion to the distance PE along the tangent, from each point of the surface, to the point in which this line is cut by a perpendicular to it from O the centre of inertia of the normal section. To prove this let a normal section of the prism be represented in the annexed diagram (fig. 6). Let PK, representing the shear at any point P, close to the prismatic boundary, be resolved into PN and PT along the normal and tangent respectively. The whole shear PK being equal to \( \pi \tau \) its component PN is equal to \( \pi \tau \sin \omega \) or \( \pi \tau \cdot PE \). The corresponding component of the required stress is \( \pi \tau \cdot PE \), and involves equal forces in the plane of the diagram, and in the plane through TP perpendicular to it, each amounting to \( \pi \tau \cdot PE \) per unit of area.

An application of force equal and opposite to the distribution thus found over the prismatic boundary, would of course alone produce in the prism, otherwise free, a state of strain which, compounded with that supposed above, would give the state of strain actually produced by the sole application of balancing couples to the two ends. The result, it is easily seen, consists of an increased twist, together with a warping of naturally plane normal sections, by infinitesimal displacements perpendicular to themselves, into certain surfaces of anticlastic curvature, with equal opposite curvatures. In bringing forward this theory, St Venant not only pointed out the falsity of the supposition admitted by several previous writers, and used in practice fallaciously by engineers, that Coulomb's law holds for other forms of prism than the solid or hollow circular cylinder, but he discovered fully the nature of the requisite correction, reduced the determination of it to a problem of pure mathematics, worked out the solution for a great variety of important and curious cases, compared the results with observation in a manner satisfactory and interesting to the naturalist, and gave conclusions of great value to the practical engineer.

67. "Hydrokinetic Analogue to Torsion Problem."—We take advantage of the identity of mathematical conditions in St Venant's torsion problem, and a hydrokinetic problem first solved a few years earlier by Stokes, to give the following statement, which will be found very useful in estimating deficiencies in torsional rigidity below the amount calculated from the fallacious extension of Coulomb's law:

"Conceive a liquid of density \( \alpha \) completely filling a closed infinitely light prismatic box of the same shape within as the given elastic prism and of length unity, and let a couple be applied to the box in a plane perpendicular to its length. The effective moment of inertia of the liquid will be equal to the correction by which the torsional rigidity of the elastic prism, calculated by the false extension of Coulomb's law, must be diminished to give the true torsional rigidity."

"Farther, the actual shear of the solid, in any infinitely thin plate of it between two normal sections, will at each point be, when reckoned as a differential sliding (section 43) parallel to their planes, equal to and in the same direction as the velocity of the liquid relatively to the containing box."

68. Solution of Torsion Problem.—To prove these propositions and investigate the mathematical equations of the problem, the process followed in Thomson and Tait's Natural Philosophy, section 706, is first to show that the conditions of sections 63 are verified by a state of strain compounded of (1) a simple twist round the line through the centres of inertia, and (2) a distortion of each normal section by infinitesimal displacements perpendicular to its plane; then find the interior and surface equations to determine this warping; and lastly, calculate the actual moment of the couple to which the mutual action between the matter on the two sides of any normal section is equivalent.

69. St Venant's treatise abounds in beautiful and instructive graphical illustrations of his results, from which the following are selected:

(1.) Elliptic Cylinder.—The plain and dotted curvilinear arcs are (fig. 7) "contour lines" (coupes topographiques) of the section as warped by torsion; that is to say, lines in which it is cut by a series of parallel planes, each perpendicular to the axis. The arrows indicate the direction of rotation in the part of the prism above the plane of the diagram.

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1 Extracted from Thomson and Tait, sections 704, 705.
2 On some cases of Fluid Motion."—Camb. Phil. Trans., 1843.
(2.) Contour lines for St Venant's "étroite à quatre points arrondis."—This diagram (fig. 8) shows the contour lines, in all respects as in case (1), for the case of a prism having for section the figure indicated. The portions of curve outside the continuous closed curve are merely indications of mathematical extensions irrelevant to the physical problem.

(3.) Contour lines of normal section of triangular prism, as warped by torsion, shown as in case (1) (fig. 9).

(4.) Contour lines of normal sections of square prisms as warped by torsion (fig. 10).

(5.) Diagram of St Venant's curvilinear squares for which torsion problem is algebraically solvable.—This diagram (fig. 11) shows the series of lines represented by the equation \( x^2 + y^2 - a(x^2 - a^2y^2 + y^2) = 1 - a \), with the indicated values for \( a \). It is remarkable that the values \( a = 0.5 \) and \( a = 0.5(\sqrt{2} - 1) \) give similar but not equal curvilinear squares (hollow sides and acute angles), one of them turned through half a right angle relatively to the other.

70. Torsional Rigidity less in proportion to sum of principal Flexural Rigidities than according to false extension (section 66) of Coulomb's Law.—Inasmuch as the moment of inertia of a plane area about an axis through its centre of inertia perpendicular to its plane is obviously equal to the sum of its moments of inertia round any two axes through the same point at right angles to one another in its plane, the fallacious extension of Coulomb's law, referred to in section 66, would make the torsional rigidity of a bar of any section equal to the product of the ratio of the modulus of rigidity to the Young's modulus into the sum of its flexural rigidities (section 61) in any two planes at right angles to one another through its length. The true theory, as we have seen (section 67), always gives a torsional rigidity less than this. How great the deficiency may be expected to be in cases in which the figure of the section presents projecting angles, or considerable prominences (which may be imagined from the hydrokinetic analogy given in section 67), has been pointed out by M. de St Venant, with the important practical application, that strengthening ribs, or projections (see, for instance, the second of the annexed diagrams), such as are introduced in engineering to give stiffness to beams, have the reverse of a good effect when torsional rigidity or strength is an object, although they are truly of great value in increasing the flexural rigidity, and giving strength to bear ordinary strains, which are always more or less flexural. With remarkable ingenuity and mathematical skill he has drawn beautiful illustrations of this important practical principle from his algebraic and transcendental solutions.

71. Places of greatest Distortion in Twisted Prisms.—M. de St Venant also calls attention to a conclusion from his solutions which to many may be startling, that in his simpler cases the places of greatest distortion are those points of the boundary which are nearest to the axis of the twisted prism in each case, and the places of least distortion those farthest from it. Thus in the elliptic cylinder the
substance is most strained at the ends of the smaller principal diameter, and least at the ends of the greater. In the equilateral triangular and square prisms there are longitudinal lines of maximum strain through the middles of the sides. In the oblong rectangular prism there are two lines of greater maximum strain through the middles of the broader pair of sides, and two lines of less maximum strain through the middles of the narrow sides. The strain is, as we may judge from the hydrokinetic analogy, excessively small, but not evanescent, in the projecting ribs of a prism of the figure shown in (2) of section 69. It is quite evanescent infinitely near the angle, in the triangular and rectangular prisms, and in each other case, as (5) of section 69, in which there is a finite angle, whether acute or obtuse, projecting outwards. This reminds us of a general remark we have to make, although consideration of space may oblige us to leave it without formal proof.

72. Strain at Projecting Angles, evanescent; at Re-entrant Angles, infinite; Liability to Cracks proceeding from Re-entrant Angles, or any places of too sharp concave curvature.—A solid of any elastic substance, isotropic or anisotropic, bounded by any surfaces presenting projecting edges or angles, or re-entrant angles or edges, however obtuse, cannot experience any finite stress or strain in the unbounded part of a projecting edge (tribhedral, polyhedral, or conical); in the neighbourhood of an angle, an edge, can only experience simple longitudinal stress parallel to the neighbouring part of the edge; and generally experiences infinite stress and strain in the neighbourhood of a re-entrant edge or angle; when influenced by any distribution of force, exclusive of surface tractions infinitely near the edges or angles in question. An important application of the last part of this statement is the practical rule, well known in mechanics, that every re-entering edge or angle ought to be rounded, to prevent risk of rupture, in solid pieces designed to bear stress. An illustration of these principles is afforded by the concluding example of torsion in Thomson and Tait's section 707; in which we have the complete mathematical solution of the torsion problem for prisms of fan-shaped sections, such as the annexed forms (fig. 13).

(1.) A cubical compression of any elastic fluid or solid in an ordinary condition causes an evolution of heat; but, on the contrary, a cubical compression produces cold in any substance, solid or fluid, in such an abnormal state that it would contract if heated while kept under constant pressure. Water below its temperature (3°-9 Cent.) of maximum density is a familiar instance. (See table of section 76.)

(2.) If a wire already twisted be suddenly twisted further, always, however, within its limits of elasticity, cold will be produced; and if it be allowed suddenly to untwist, heat will be evolved from itself (besides heat generated externally by any work allowed to be wasted, which it does in untwisting). It is assumed that the torsional rigidity of the wire is diminished by an elevation of temperature, as the writer of this article had found it to be for copper, iron, platinum, and other metals (compare section 78).

(3.) A spiral spring suddenly drawn out will become lower in temperature, and will rise in temperature when suddenly allowed to draw in. [This result has been experimentally verified by Joule ("Thermodynamic Properties of Solids," Trans. Roy. Soc., 1858) and the amount of the effect found to agree with that calculated, according to the preceding thermodynamic theory, from the amount of the weakening of the spring which he found by experiment.]

(4.) A bar or rod or wire of any substance with or without a weight hung on it, or experiencing any degree of end thrust, to begin with, becomes cooled if suddenly elongated by end pull or by diminution of end thrust, and warmed if suddenly shortened by end thrust or by diminution of end pull; except abnormal cases in which with constant end pull or end thrust elevation of temperature produces shortening; in every such case pull or diminished thrust produces elevation of temperature, thrust or diminished pull lowering of temperature.

(5.) An india-rubber band suddenly drawn out (within its limits of elasticity) becomes warmer; and when allowed to contract, it becomes colder. Any one may easily verify this curious property by placing an india-rubber band in slight contact with the edges of the lips, then suddenly extending it—it becomes very perceptibly warmer; hold it for sometime stretched nearly to breaking, and then suddenly allow it to shrink—it becomes quite startlingly colder, the cooling effect being sensible not merely to the lips but to the fingers holding the band. The first published statement of this curious observation is due to Gough (Memoirs of the Literary and Philosophical Society of Manchester, 2d series, vol. i. p. 288), quoted by Joule in his paper on "Thermodynamic Properties of Solids" (Transactions of Royal Society, 1858). The thermodynamic conclusion from it is that an india-rubber band, stretched by a constant weight of sufficient amount hung on it, must, when heated, pull up the weight, and, when cooled, allow the weight to descend: this Gough, independently of thermodynamic theory, had found to be actually the case. The experiment any one can make with the greatest ease by hanging a few pounds weight on a common india-rubber band, and taking a red-hot coal in a pair of tongs, or a red-hot poker, and moving it up and down close to the band. The way in which the weight rises when the red-hot body is near, and falls when it is removed, is quite startling. Joule experimented on the amount of shrinking per degree of elevation of temperature, with different weights hung on a band of vulcanized india-rubber, and found that they closely agreed with the amounts calculated by Thomson's theory from the heating effects of pull, and cooling effects of ceasing to pull, which he had observed in the same piece of india-rubber.

1 W. Thomson on "Thermo-elastic Properties of Matter," in Quarterly Journal of Mathematics, April 1855 (reprinted in Phil. Mag. 1877, second half year.)
The sudden application of a stress produces a change of temperature agreeing very approximately with theoretical scale, the change of temperature being supposed to be but a very small fraction of it; the geometrical effect (expansion or other strain) produced by an elevation of temperature of one degree when the body is kept under constant stress; the specific heat of the substance per unit mass under constant stress; the density; and the Joule's equivalent (taken as 42400 centimetres). When the law of variation of \( K \) and \( e \) is reckoned ought to be the mean of the stresses kept under constant stress. The constant stress for which the exponent or specific heat in liquids, or in fluids which at less heavy pressures of one atmosphere for air at ordinary pressure; for water or other known fluids \( p \) may, for all we know, amount to twenty atmospheres or one hundred atmospheres without transgressing the limits for which the preceding formula is applicable. When the law of variation of \( K \) and \( e \) with pressure is known, the differential formula is readily integrated to give the integral amount of the change of temperature produced by greater stress than those for which the differential formula is applicable. For air and other permanent gases Boyle's law of compression and Charles's law of thermal expansion supply the requisite data with considerable accuracy up to twenty or thirty atmospheres. The result is expressed by the formula

\[
\frac{t + \theta}{t} = \left( \frac{P + \theta}{P} \right)^{\frac{N}{P}}
\]

where \( k \) denotes the ratio of the thermal capacity, pressure constant, to the thermal capacity, volume constant, of the gas, a number which thermodynamic theory proves to be approximately constant for all temperatures and densities, for any fluid approximately fulfilling Boyle's and Charles's laws; \( P \) and \( t \) the initial pressure and temperature of the gas; \( p \) the sudden addition to the pressure; and, as before, \( \theta \) the elevation of temperature.

For the case of \( p \) a small fraction of \( P \) the formula gives

\[
\theta = (k - 1) \frac{P}{P} t
\]

It is by an integration of this formula that (1) is obtained. For common air the value of \( k \) is very approximately 1.41. Thus if a quantity of air be given at 15° C. (\( = 289^\circ \)) and the ordinary atmospheric pressure, and if it be compressed gradually up to 32 atmospheres, or dilated to \( \frac{1}{32} \) of an atmosphere, and perfectly guarded against gain or loss of heat from or to without, its temperature at several different pressures, chosen for example, will be according to the following table of excesses of temperature above the primitive temperature, calculated by (1).

<table>
<thead>
<tr>
<th>Value of ( P - \theta )</th>
<th>Elevation of temperature produced by compression.</th>
<th>Value of ( P + \theta )</th>
<th>Lowering of temperature produced by dilatation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>95°</td>
<td>4</td>
<td>125</td>
</tr>
<tr>
<td>4</td>
<td>221</td>
<td>8</td>
<td>166</td>
</tr>
<tr>
<td>8</td>
<td>359</td>
<td>16</td>
<td>196</td>
</tr>
<tr>
<td>32</td>
<td>911</td>
<td>32</td>
<td>219</td>
</tr>
</tbody>
</table>

But we have no knowledge of the effect of pressures of several thousand atmospheres in altering the expansibility or specific heat in liquids, or in fluids which at less heavy or at ordinary pressures are "gases."

75. When change of temperature, whether in a solid or a fluid is produced by the application of a stress, the corresponding modulus of elasticity will be greater in virtue of the change of temperature than what may be called the static modulus defined as above, on the understanding that the temperature if changed by the stress is brought back to its primitive degree before the measurement of the stress is performed. The modulus calculated on the supposition that the body, neither losing nor gaining heat during the application of the stress and the measurement of its effect, retains the whole change of temperature due to the stress, will be called for want of a better name the kinetic modulus, because it is this which must (as in Laplace's celebrated correction of Newton's calculation of the velocity of sound) be used in reckoning the elastic forces concerned in waves and vibrations in almost all practical cases. To find the ratio of the kinetic to the static modulus remark that \( e_0 \), according to the notation of section 74, is the diminution of the strain due to the change of temperature \( \theta \). Hence if \( M \) denote the static modulus (section 41), the strain actually produced by it when the body is not allowed either to gain or lose heat is \( \frac{M}{e} - e_0 \), or, with \( \theta \) replaced by its value according to the formula of section 74,
Dividing \( p \) by this expression we find for the kinetic modulus

\[
M' = \frac{1}{\frac{1}{M} - \frac{p}{J_\rho K}}.
\]

Hence

\[
M' = \frac{M}{\frac{1}{M} - \frac{p}{J_\rho K}}.
\]

76. For any substance, fluid or solid, it is easily proved, without thermodynamic theory, that

\[
M' = \frac{M}{\frac{1}{M} - \frac{p}{J_\rho K}} = \frac{M}{\frac{1}{M} - \frac{p}{J_\rho K}}
\]

where \( K \) denotes the thermal capacity of a stated quantity of the substance under constant stress, and \( N \) its thermal capacity under constant strain (or thermal capacity when the body is prevented from change of shape or change of volume). For permanent gases, and generally for fluids approximately fulfilling Boyle's and Charles's laws as said above, \( k \) is proved by thermodynamic theory to be approximately constant. Its value for all gases for which it has been measured differs largely from unity, and probably also differ greatly from for liquids generally (except water near its temperature of constant density). For liquids generally (except water near its temperature of maximum density).

On the other hand, for solids whether the stress considered be uniform compression in all directions or of any other type, the value of \( M' \) or \( K \) differs but very little from unity; and both for solids and liquids it is far from constant at different temperatures (in the case of water it is zero at 3°-9 Cent., and varies as the square of the difference of the temperature from 3°-9 at all events for moderate differences from this critical temperature, whether above or below it). The following tables show the value of \( \frac{M'}{M} \) or \( \frac{K}{N} \), and the value of \( \theta \) by the formula of sec. 74, for different fluid and solid substances at the temperature 15° C. (289° absolute scale). The first table is for compression uniform in all directions; the second, necessarily confined to solids, is for the stress dealt with in "Young's Modulus," that is, normal pressure (positive or negative) on one set of parallel planes, with perfect freedom to expand or contract in all directions in these planes. A wire or rod pulled longitudinally is a practical application of the latter.

### Thermodynamic Table I.


<table>
<thead>
<tr>
<th>Substance</th>
<th>Density (grammes per unit mass)</th>
<th>Thermal Capacity per unit mass ( = K )</th>
<th>Expansibility ( = \theta )</th>
<th>Elevation of Temperature produced by a pressure of one gramme per square centimetre ( = \frac{\theta}{J_\rho K} )</th>
<th>Static Bulk-Modulus ( = \frac{M}{M'} )</th>
<th>Thermodynamic Table II.</th>
<th>Deduced value ( \frac{M}{M'} = \frac{1}{\frac{\theta}{J_\rho K}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>0.005</td>
<td>0.008</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
<td>1.000</td>
</tr>
<tr>
<td>Distilled water</td>
<td>1.000</td>
<td>0.006</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
<td>1.000</td>
</tr>
<tr>
<td>Alcohol</td>
<td>0.785</td>
<td>0.018</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
<td>1.000</td>
</tr>
<tr>
<td>Ether</td>
<td>0.335</td>
<td>0.013</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
<td>1.000</td>
</tr>
<tr>
<td>Mercury</td>
<td>0.706</td>
<td>0.017</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
<td>1.000</td>
</tr>
<tr>
<td>Glass, thin</td>
<td>2.949</td>
<td>0.017</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
<td>1.000</td>
</tr>
<tr>
<td>Brass, drawn</td>
<td>8.471</td>
<td>0.018</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
<td>1.000</td>
</tr>
<tr>
<td>Iron</td>
<td>8.845</td>
<td>0.018</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
<td>1.000</td>
</tr>
<tr>
<td>Copper</td>
<td>8.845</td>
<td>0.018</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
<td>1.000</td>
</tr>
</tbody>
</table>

### Thermodynamic Table II.

Pressures parallel to one direction in a solid—Ratio of Kinetic to Static Young's Modulus. Temperature 15° C. (289° absolute).

<table>
<thead>
<tr>
<th>Substance</th>
<th>Density (grammes per unit mass)</th>
<th>Thermal Capacity per unit mass ( = K )</th>
<th>Expansibility ( = \theta )</th>
<th>Elevation of Temperature produced by a pressure of one gramme per square centimetre ( = \frac{\theta}{J_\rho K} )</th>
<th>Static Young's Modulus ( = \frac{M}{M'} )</th>
<th>Deduced value ( \frac{M}{M'} = \frac{1}{\frac{\theta}{J_\rho K}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zine</td>
<td>7.908</td>
<td>0.002</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
</tr>
<tr>
<td>Tin</td>
<td>7.404</td>
<td>0.004</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
</tr>
<tr>
<td>Silver</td>
<td>10.600</td>
<td>0.005</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
</tr>
<tr>
<td>Copper</td>
<td>8.933</td>
<td>0.004</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
</tr>
<tr>
<td>Lead</td>
<td>11.215</td>
<td>0.004</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
</tr>
<tr>
<td>Glass</td>
<td>2.949</td>
<td>0.017</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
</tr>
<tr>
<td>Iron</td>
<td>7.523</td>
<td>0.008</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
</tr>
<tr>
<td>Platinum</td>
<td>21.279</td>
<td>0.018</td>
<td>0.004</td>
<td>0.002</td>
<td>0.018</td>
<td>1.002</td>
</tr>
</tbody>
</table>

77. Experimental Results.—The following tables show determinations of moduluses of compression, of Young's modulus, and of moduluses of rigidity by various experimenters and various methods. It will be seen that the Young's moduluses obtained by Wertheim by vibrations, longitudinal or transverse, are generally in excess of those which he found by static extension; but the differences are enormously greater than those due to the heating and cooling effects of elongation and contraction (section 76), and are to be certainly reckoned as errors of observation. It is probable that his moduluses determined by static elongation are minutely accurate; the discrepancies of those found by vibrations are probably due to imperfections of the arrangements for carrying out the vibrational method:

### Table of Moduluses of Compressibility.

<table>
<thead>
<tr>
<th>Substance</th>
<th>Moduluses of compressibility in grammes per square centimetre</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distilled water</td>
<td>22.68 x 10^6</td>
</tr>
<tr>
<td>Alcohol</td>
<td>12.4 x 10^6</td>
</tr>
<tr>
<td>Ether</td>
<td>9.5 x 10^6</td>
</tr>
<tr>
<td>Bisulphide of carbon</td>
<td>18.3 x 10^6</td>
</tr>
<tr>
<td>Mercury</td>
<td>5025 x 10^6</td>
</tr>
<tr>
<td>Glass</td>
<td>423 x 10^6</td>
</tr>
<tr>
<td>Steel</td>
<td>1876 x 10^6</td>
</tr>
<tr>
<td>Iron</td>
<td>1485 x 10^6</td>
</tr>
<tr>
<td>Copper</td>
<td>1717 x 10^6</td>
</tr>
<tr>
<td>Brass, different specimens</td>
<td>Mean 1063 x 10^6</td>
</tr>
</tbody>
</table>

### Table of Moduluses of Rigidity.

<table>
<thead>
<tr>
<th>Substance</th>
<th>Moduluses of Rigidity in grammes per square centimetre</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glass, different specimens</td>
<td>Mean 150 x 10^6</td>
</tr>
<tr>
<td>Brass, different specimens</td>
<td>Mean 560 x 10^6</td>
</tr>
</tbody>
</table>

Authority.

- Ann. de Chimie, 1843.
- D. des Chem. de Chim., 1848.
# Elasticity

## Table of Moduluses and Strengths

<table>
<thead>
<tr>
<th>Substance</th>
<th>Density</th>
<th>Young's Modulus</th>
<th>Tensile in Grammes per Square Centimetre</th>
<th>Length Modulus</th>
<th>Modulus of Rupture in Centimètres (or Tensile in Terms of Weight of Unit-Bulk)</th>
<th>Extreme Elastic Inflexion</th>
<th>Resilience per Cable Centimètres in Centimètres</th>
<th>Resilience per Cable Units in Grammes</th>
<th>Authority</th>
<th>Method of Determination</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iron or steel</td>
<td>Abb 910 x 10^9</td>
<td>Abb 9,000,000 ft.</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>Dr. T. Young</td>
<td>Probably to be flexed (Young's Works, vol. ii. p. 139).</td>
</tr>
<tr>
<td>Wood</td>
<td>{ 105 x 10^4 to 290 x 10^4</td>
<td>to 10,000,000 ft.</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Steel</td>
<td>Abb 300 x 10^4</td>
<td>Abb 6,000,000 ft.</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Slate</td>
<td>{ 910 x 10^4 to 1,120 x 10^4</td>
<td>to 21,000,000 ft.</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Ice</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Brass, cast</td>
<td>642 x 10^4</td>
<td>• 004 x 10^4</td>
<td>...</td>
<td>127 x 10^4</td>
<td>0.0198</td>
<td>195</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Zinc, common, drawn</td>
<td>113 x 10^4</td>
<td>112 x 10^4</td>
<td>...</td>
<td>120 x 10^4</td>
<td>0.0106</td>
<td>93</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Tin, cast</td>
<td>55 x 10^4</td>
<td>55 x 10^4</td>
<td>...</td>
<td>91 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Lead, sheet</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Ash</td>
<td>113 x 10^4</td>
<td>113 x 10^4</td>
<td>...</td>
<td>120 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
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<tr>
<td>Beech</td>
<td>68 x 10^4</td>
<td>68 x 10^4</td>
<td>...</td>
<td>80 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
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<td>Birch</td>
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<td>110 x 10^4</td>
<td>...</td>
<td>100 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
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<td>Cedar of Lebanon</td>
<td>31 x 10^4</td>
<td>31 x 10^4</td>
<td>...</td>
<td>32 x 10^4</td>
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<td>94</td>
<td>...</td>
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<tr>
<td>Fir, red pine</td>
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<td>118 x 10^4</td>
<td>...</td>
<td>91 x 10^4</td>
<td>0.0653</td>
<td>94</td>
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<tr>
<td>Spruce</td>
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<td>115 x 10^4</td>
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<td>0.0653</td>
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<tr>
<td>Fir, white</td>
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<td>79 x 10^4</td>
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<td>80 x 10^4</td>
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<td>88 x 10^4</td>
<td>...</td>
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<td>0.0653</td>
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<td>...</td>
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<td></td>
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<td>Oak, European</td>
<td>105 x 10^4</td>
<td>105 x 10^4</td>
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<td>105 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
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<td>Sycamore</td>
<td>70 x 10^4</td>
<td>70 x 10^4</td>
<td>...</td>
<td>70 x 10^4</td>
<td>0.0653</td>
<td>94</td>
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<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
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<tr>
<td>Teak, Indian</td>
<td>169 x 10^4</td>
<td>169 x 10^4</td>
<td>...</td>
<td>169 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
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<tr>
<td>Lost, cast</td>
<td>11.25</td>
<td>11.25</td>
<td>16 x 10^4 cms.</td>
<td>22 x 10^4</td>
<td>0.0653</td>
<td>94</td>
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<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Tin, cast</td>
<td>78.04</td>
<td>78.04</td>
<td>56 x 10^4 cms.</td>
<td>41 x 10^4</td>
<td>0.0653</td>
<td>94</td>
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<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
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<tr>
<td>Cadmium, drawn</td>
<td>865</td>
<td>865</td>
<td>63 x 10^4</td>
<td>63 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Gold, drawn</td>
<td>86.4</td>
<td>86.4</td>
<td>44 x 10^4</td>
<td>44 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Silver, drawn</td>
<td>10.89</td>
<td>10.89</td>
<td>71 x 10^4</td>
<td>71 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Zinc, common, drawn</td>
<td>7.068</td>
<td>7.068</td>
<td>134 x 10^4</td>
<td>134 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Palladium</td>
<td>11.35</td>
<td>11.35</td>
<td>104 x 10^4</td>
<td>104 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Copper, drawn</td>
<td>8.23</td>
<td>8.23</td>
<td>159 x 10^4</td>
<td>159 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Copper, annealed</td>
<td>8.23</td>
<td>8.23</td>
<td>159 x 10^4</td>
<td>159 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Platinum wire, fine</td>
<td>21.76</td>
<td>21.76</td>
<td>75 x 10^4</td>
<td>75 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Steel, cast, drawn</td>
<td>1861 x 10^4</td>
<td>1861 x 10^4</td>
<td>266 x 10^4</td>
<td>266 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Steel wire, English, drawn</td>
<td>86.5 x 10^4</td>
<td>86.5 x 10^4</td>
<td>244 x 10^4</td>
<td>244 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Steel wire, common, tempered blue</td>
<td>86.5 x 10^4</td>
<td>86.5 x 10^4</td>
<td>244 x 10^4</td>
<td>244 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>English steel, Plansee</td>
<td>8.9</td>
<td>8.9</td>
<td>110 x 10^4</td>
<td>110 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
<tr>
<td>Copper wire</td>
<td>8.9</td>
<td>8.9</td>
<td>110 x 10^4</td>
<td>110 x 10^4</td>
<td>0.0653</td>
<td>94</td>
<td>...</td>
<td>...</td>
<td>Bennett, Rankine's Rules and Tables.</td>
<td></td>
</tr>
</tbody>
</table>
To determine rigidities by torsional vibrations, taking advantage of the elasticity of solids, what changes are produced in the molecular condition, has occupied the attention, no doubt, of every "naturalist" who has studied the subject, and valuable contributions to its answer by experiment had been given by Wertheim and other investigators, but solely with reference to Young's modulus. In 1865 an investigation of the effect on the torsional rigidity of wires of different metals, produced by stretching them longitudinally beyond their limits of elasticity, was commenced in the physical laboratory of the university of Glasgow in its old buildings in 1865. The following description of experiments and table of results is extracted from the paper by W. Thomson "On the Elasticity and Viscosity of Metals," already quoted (section 30), with reference to viscosity and fatigue of elasticity.

"To determine rigidities by torsional vibrations, taking advantage of an obvious but most valuable suggestion made to me by Dr. Joule, I used as vibrator in each case a thin cylinder of sheet iron, true inside and outside (of which the radius of gravitation must be, to a very close degree of approximation, the arithmetic mean of the radius of the outer and inner cylindrical surfaces) supported by a thin flat rectangular bar, of which the square of the radius of gravitation is one-third of the square of the diameter from the centre to the corner. The wire to be tested passed perpendicularly through a hole in the middle of the bar, and was there firmly soldered. The cylinder was tied to the middle of the bar by a light silk thread so as to hang with its axis vertical. Each wire, after having been suspended and stretched with just force enough to make it as nearly straight as was necessary for accuracy, was vibrated. Then it was stretched by hand (applied to the cross bar soldered to its lower end) and vibrated again, and stretched again, and so on till it broke." The experiments were performed with great care and accuracy by Mr. Donald McFarlane. "The results, as shown in the accompanying table, were most surprising."

The highest and lowest rigidities found for copper in the table are as follows:

<table>
<thead>
<tr>
<th>Substance</th>
<th>Length of Wire (in centimetres)</th>
<th>Volume in cubic centimetres</th>
<th>Density</th>
<th>Moment of Inertia of Wire</th>
<th>Time of Vibration for one complete revolution</th>
<th>Rigidity in grammes weight per square centimetre</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum 1</td>
<td>60-3</td>
<td>11843</td>
<td>2704</td>
<td>31771</td>
<td>1.14</td>
<td>241 x 10^6</td>
</tr>
<tr>
<td>Zinc 2</td>
<td>304-9</td>
<td>2351</td>
<td>105</td>
<td>31896</td>
<td>4.31</td>
<td>369 x 10^6</td>
</tr>
<tr>
<td>Brass 3</td>
<td>237-7</td>
<td>783</td>
<td>473-8</td>
<td>5.456</td>
<td>364 x 10^6</td>
<td></td>
</tr>
<tr>
<td>Copper 4</td>
<td>231-9</td>
<td>1703</td>
<td>8388</td>
<td>5.96</td>
<td>360 x 10^6</td>
<td></td>
</tr>
<tr>
<td>Copper 5</td>
<td>248-5</td>
<td>1510</td>
<td>891</td>
<td>382</td>
<td>438 x 10^6</td>
<td></td>
</tr>
<tr>
<td>Copper 6</td>
<td>234-4</td>
<td>1384</td>
<td>864</td>
<td>3171</td>
<td>5.015</td>
<td>453 x 10^6</td>
</tr>
<tr>
<td>Copper 7</td>
<td>143-7</td>
<td>968</td>
<td>874</td>
<td>2975</td>
<td>6.21</td>
<td>462 x 10^6</td>
</tr>
<tr>
<td>Copper 8</td>
<td>296-8</td>
<td>4275</td>
<td>2012</td>
<td>373</td>
<td>456 x 10^6</td>
<td></td>
</tr>
<tr>
<td>Copper 9</td>
<td>291</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>375</td>
<td>456 x 10^6</td>
</tr>
<tr>
<td>Copper 10</td>
<td>293</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>417</td>
<td>456 x 10^6</td>
</tr>
<tr>
<td>Copper 11</td>
<td>298</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>450</td>
<td>456 x 10^6</td>
</tr>
<tr>
<td>Copper 12</td>
<td>300-0</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>453</td>
<td>456 x 10^6</td>
</tr>
<tr>
<td>Copper 13</td>
<td>303-4</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>404</td>
<td>456 x 10^6</td>
</tr>
<tr>
<td>Copper 14</td>
<td>307-1</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>483</td>
<td>456 x 10^6</td>
</tr>
<tr>
<td>Copper 15</td>
<td>317-4</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>490</td>
<td>456 x 10^6</td>
</tr>
<tr>
<td>Copper 16</td>
<td>315-6</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>490</td>
<td>456 x 10^6</td>
</tr>
<tr>
<td>Copper 17</td>
<td>296-5</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>490</td>
<td>456 x 10^6</td>
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<tr>
<td>Copper 18</td>
<td>291-9</td>
<td>297</td>
<td>874</td>
<td>303</td>
<td>456 x 10^6</td>
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<tr>
<td>Copper 19</td>
<td>252-3</td>
<td>1760</td>
<td>891</td>
<td>345</td>
<td>456 x 10^6</td>
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</tr>
<tr>
<td>Copper 20</td>
<td>229-2</td>
<td>897</td>
<td>872</td>
<td>345</td>
<td>456 x 10^6</td>
<td></td>
</tr>
<tr>
<td>Copper 21</td>
<td>252-8</td>
<td>297</td>
<td>891</td>
<td>345</td>
<td>456 x 10^6</td>
<td></td>
</tr>
<tr>
<td>Copper 22</td>
<td>252-8</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>460</td>
<td>464 x 10^6</td>
</tr>
<tr>
<td>Copper 23</td>
<td>270-4</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>460</td>
<td>464 x 10^6</td>
</tr>
<tr>
<td>Copper 24</td>
<td>279-7</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>460</td>
<td>464 x 10^6</td>
</tr>
<tr>
<td>Copper 25</td>
<td>297-9</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>459</td>
<td>464 x 10^6</td>
</tr>
<tr>
<td>Copper 26</td>
<td>308-8</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>460</td>
<td>464 x 10^6</td>
</tr>
<tr>
<td>Copper 27</td>
<td>327-5</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>460</td>
<td>464 x 10^6</td>
</tr>
<tr>
<td>Copper 28</td>
<td>296-5</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>460</td>
<td>464 x 10^6</td>
</tr>
<tr>
<td>Copper 29</td>
<td>297-9</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>460</td>
<td>464 x 10^6</td>
</tr>
<tr>
<td>Copper 30</td>
<td>308-8</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>460</td>
<td>464 x 10^6</td>
</tr>
<tr>
<td>Copper 31</td>
<td>327-5</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>460</td>
<td>464 x 10^6</td>
</tr>
<tr>
<td>Copper 32</td>
<td>286-0</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>456</td>
<td>468 x 10^6</td>
</tr>
<tr>
<td>Copper 33</td>
<td>297-9</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>456</td>
<td>468 x 10^6</td>
</tr>
<tr>
<td>Copper 34</td>
<td>308-8</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>456</td>
<td>468 x 10^6</td>
</tr>
<tr>
<td>Copper 35</td>
<td>327-5</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>456</td>
<td>468 x 10^6</td>
</tr>
<tr>
<td>Copper 36</td>
<td>1640</td>
<td>1357</td>
<td>7357</td>
<td>8455</td>
<td>760</td>
<td>100 x 10^6</td>
</tr>
<tr>
<td>Platinum</td>
<td>39-4</td>
<td>1744</td>
<td>2085</td>
<td>2092</td>
<td>4.05</td>
<td>622 x 10^6</td>
</tr>
<tr>
<td>Gold</td>
<td>16-9</td>
<td>1825</td>
<td>195</td>
<td>1695</td>
<td>2.05</td>
<td>622 x 10^6</td>
</tr>
<tr>
<td>Silver</td>
<td>75-7</td>
<td>1128</td>
<td>1021</td>
<td>10967</td>
<td>2.77</td>
<td>776 x 10^6</td>
</tr>
</tbody>
</table>

Remarks.

1 Only forty vibrations from initial arc of convenient amplitude could be counted. Had been stretched considerably before this experiment.
2 A piece of the preceding stretched.
3 A smaller copper wire, stretched by successive tractions.
4 Another piece of the long (2435 centim.) wire; stretched by successive tractions.
5 A fine copper wire, softened by being heated to redness and plunged in water. A length of 200 centimetres cut from this, suspended, and elongated by successive tractions; a second length of 200 centimetres cut from the same, and similarly treated.
6 One piece successively elongated by simple tractions till it broke.

Note: 1 It is exactly the square root of the mean of their squares.

2 For example, see paper "On Electrodynamic Qualities of Metals," Philosophical Transactions, 1856, by W. Thomson.
79. The following tables show the effects of differences of temperature on the Young's Modulus, rigidity-modulus, and modulus of compressibility of various substances:

<table>
<thead>
<tr>
<th>Substance</th>
<th>Density</th>
<th>Young's Modulus in million gms. per square centimetre</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>15°</td>
</tr>
<tr>
<td>Lead</td>
<td>11.282</td>
<td>178</td>
</tr>
<tr>
<td>Gold</td>
<td>18.085</td>
<td>558</td>
</tr>
<tr>
<td>Silver</td>
<td>10.304</td>
<td>715</td>
</tr>
<tr>
<td>Palladium</td>
<td>11.295</td>
<td>979</td>
</tr>
<tr>
<td>Copper</td>
<td>8.288</td>
<td>1099</td>
</tr>
<tr>
<td>Platinum</td>
<td>21.352</td>
<td>1562</td>
</tr>
<tr>
<td>Steel, drawn, English</td>
<td>7.522</td>
<td>1738</td>
</tr>
<tr>
<td>Cast steel</td>
<td>7.919</td>
<td>1956</td>
</tr>
<tr>
<td>Iron, Berry</td>
<td>7.757</td>
<td>2079</td>
</tr>
</tbody>
</table>

The above results are from Wertheim's "Mémoires" on Elasticity, Ann. de Chim. et Phys., tom xii. (1844).

The change in the rigidity-modulus produced by change of temperature was investigated by Kohlrausch. He found that it is expressed by the formula \( n = n_0 (1 - \alpha T) \), where \( n_0 \) denotes the value of the rigidity-modulus at \( 0 \)° C., \( n \) its value at temperature \( T \), and \( \alpha, \beta \) coefficients the values of which for iron, copper, and brass are as follows:

<table>
<thead>
<tr>
<th>Substance</th>
<th>( \alpha )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iron</td>
<td>0.000447</td>
<td>0.00000052</td>
</tr>
<tr>
<td>Copper</td>
<td>0.000520</td>
<td>0.00000028</td>
</tr>
<tr>
<td>Brass</td>
<td>0.000428</td>
<td>0.00000038</td>
</tr>
</tbody>
</table>

Modulus of Compressibility of Water, Alcohol, and Ether at Different Temperatures.

<table>
<thead>
<tr>
<th>Temp. Cent.</th>
<th>Water</th>
<th>Alcohol</th>
<th>Ether</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>20·6 x 10^-6</td>
<td>12·4 x 10^-6</td>
<td>9·5 x 10^-6</td>
</tr>
<tr>
<td>15</td>
<td>20·2 x 10^-6</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>41</td>
<td>20·7 x 10^-6</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>108</td>
<td>21·5 x 10^-6</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>134</td>
<td>21·6 x 10^-6</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>140</td>
<td>...</td>
<td>11·4 x 10^-6</td>
<td>...</td>
</tr>
<tr>
<td>150</td>
<td>...</td>
<td>...</td>
<td>8·07 x 10^-6</td>
</tr>
<tr>
<td>180</td>
<td>22·4 x 10^-6</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>250</td>
<td>23·6 x 10^-6</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>54</td>
<td>23·3 x 10^-6</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>48</td>
<td>23·3 x 10^-6</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>53</td>
<td>23·5 x 10^-6</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

80. Tempering soft iron by long-continued stress.

Preliminary experiments by Mr J. T. Bottomley towards the investigation promised in section 5 above have discovered a very remarkable property of soft iron wire, viz., its ultimate tensile strength. Eight different specimens, tested by the gradual application of more and more weight within ten minutes of time in each case until the wire broke, bore from 43½ to 46 lb (average 45·2) just before breaking, with elongations of from 17 per cent to 22 per cent. Another specimen left within the wire, with a weight of 39 lb hung upon it, was twisted to the extent of 95 turns, it became gradually elongated to the extent of 150½ of the length of the wire; when farther twisted it began to shorten still, when 25 turns had been given (in all 120 turns), the weight had risen from its lowest position through nearly 30½ of the length of the wire, so that the previous elongation had been diminished by about 3 of its amount.

Experiments were also made by Mr Andrew Gray and Mr Thomas Gray for the purpose of determining the effects of various amounts of permanent twist in altering the rigidity-modulus and the Young's modulus of wires of copper, iron, and steel. A copper wire, of 3-15 metres in length and 1-64 centimetre diameter, No. 17 B.W.G., which had a rigidity-modulus of 442 million gms per square centimetre to begin with, was found to have 420 after 10 turns, showing a diminution in the modulus of 4 of its own amount. The diminution went on rapidly until 100 turns of permanent twist had been given, when the modulus was as low as 531. The diminution of the modulus continued with further twist, but very slowly, up to 1225 turns, when the modulus was found to be 371, showing a diminution to the extent of 4 of its original value! There was little further change until 1400 turns had been given, when the modulus began to increase. At 1525 turns its value was 573, and at 1825 it was 377. Twenty turns more broke the wire before the torsional elasticity had been again determined.

A piece of iron wire of nearly the same length, about three metres, but of smaller diameter (0·87 centimetre), showed continued diminution of torsional rigidity as far as the modulus seems to be a minimum near the temperature of maximum density.
Def. A stress is an equilibrating application of force to a body.

MATHEMATICAL THEORY OF ELASTICITY.  

The steel pianoforte wire also showed a diminution of torsional rigidity with permanent twist, and (as did the copper wire) showed first a diminution and then a slight augmentation. The amount of the diminution in the steel wire was enormously greater than the surprisingly great amount which had been discovered in the copper wire, and the ultimate augmentation was considerably greater in the steel than what it had been in the copper before rupture.

Thus after 473 turns of permanent twist the torsional modulus had diminished from 751 million grammes per square centimetre to 414. 55 more turns of permanent twist augmented the rigidity from 414 to 430, and when farther twisted the wire broke before another observation had been made. The vibrator used in these experiments was a cylinder of lead weighing 56 lb, which was kept hanging on the wire while it was being twisted, and in fact during the whole of about 100 hours from the beginning of the experiment till the wire broke, except on two occasions for a few minutes, while the top fastening which had given way was being resoldered. The period of vibration was augmented from 39-575 seconds to 51-9 seconds by the twist.

The wire took the twist very irregularly, some parts not beginning to show much signs of permanent twist till near the end of the experiment. In two specimens of copper wire of the same length and gauge as those described above, the Young's modulus was found to be increased 10 per cent. by 100 turns of permanent twist.

Five metres of the steel pianoforte wire, bearing a weight of 39 lb, was in one of Mr M'Farlane's experiments twisted 120 turns, and then allowed to untwist, and 38½ turns came out, leaving the wire in equilibrium with 81½ turns of permanent twist. Its Young's modulus was then found not to differ as much as ½ per cent. from the value it had before the wire was twisted.

CHAPTER II.—Homogeneous Stresses and Homogeneous Strains.

Def. A stress is said to be homogeneous throughout a body when equal and similar portions of the body, with corresponding lines parallel, experience equal and parallel pressures or tensions on corresponding elements of the surfaces.

Cor. When a body is subjected to any homogeneous stress, the mutual tension or pressure between the parts of it on two sides of any plane amount to the same per unit of surface as that between the parts on the two sides of any parallel plane; and the former tension or pressure is parallel to the latter.

A strain is said to be homogeneous throughout a body, or the body is said to be homogeneous strain, when equal and similar portions, with corresponding lines parallel, experience equal and similar alterations of dimensions.

Cor. All the particles of the body in parallel planes remain in parallel planes, when the body is homogeneous strain in any way.

Examples.—A long uniform rod, if pulled, or a pillar loaded with a weight, will experience a uniform strain, except near its ends. There will be a sensible diminution of the rigidity of the strain, because of the elastic reaction of the body, or of an elastic sphere, of other circumstances preventing the ends from expanding latitudinally to the same extent as the middle of the body.

A piece of cloth held in a plane, and distorted so that a warp and woof, instead of being perpendicular to one another, becomes two sets of parallels cutting one another obliquely, experiences a homogeneous strain. The strain is heterogeneous as to intensity, from the axis to the surface of a cylindrical wire under tension, and heterogeneous as to direction in different positions in a circle round the axis.

CHAPTER III.—On the Distribution of Force in a Stress.

Theorem.—In every homogeneous stress there is a system of three rectangular planes, each of which is perpendicular to the direction of the mutual force between the parts of the body on its two sides.

For let $F(X), P(Y), P(Z)$ denote the components, parallel to $X, Y, Z$, any three rectangular lines of reference, of the force experienced per unit of surface at any portion of the solid bounded by a plane parallel to $(Y, Z);$ $Q(X), Q(Y), Q(Z)$, the corresponding components of the force experienced by any surface of the solid parallel to $(Z, X);$ and $R(X), R(Y), R(Z)$, those of the force at a surface parallel to $(X, Y)$. Now by considering the equilibrium of a cube of the solid with faces parallel to the planes of reference (fig. 15), we see that the couple of forces $Q(Z)$ on its two faces perpendicular to $Y$ is balanced by the couple of forces $R(Y)$ on the faces perpendicular to $Z$. Hence we must have

$$Q(Z) = R(Y).$$

Similarly it is seen that

$$R(X) = P(Z);$$

and

$$P(Y) = Q(X).$$

For the sake of brevity, these pairs of equal quantities (being tangential forces respectively perpendicular to $X, Y, Z$) may be denoted by $(T(X), T(Y), T(Z));$ and the forces actually experienced by the sides $A, B, C$, through a point $O$ parallel to the planes of lines of reference, and a third plane $K$ cutting these at angles $\alpha, \beta, \gamma$ respectively; so that as regards the areas of the different sides we shall have

$$A = K \cos \alpha; B = K \cos \beta; C = K \cos \gamma.$$

The forces actually experienced by the sides $A, B, C$ have nothing to balance them except the force actually experienced by $K$. Hence those three forces must have a single resultant, and the force on $K$ must be equal and opposite to it. If, therefore, the force on $K$ be denoted by $F$, and its direction cosines by $l, m, n$, we have

$$F = T(X) + T(Y) + T(Z);$$

$$P(X) = T(Y) + T(Z);$$

$$Q(Y) = T(Z) + T(X);$$

and, by the relations between the cases stated above, we deduce

$$T = F \cos \alpha + T(Y) \cos \beta + T(Z) \cos \gamma;$$

$$Q = T(Z) \cos \alpha + T(X) \cos \beta + T(Y) \cos \gamma;$$

$$P = T(Y) \cos \alpha + T(Z) \cos \beta + T(X) \cos \gamma.$$

Hence the problem of finding $(\alpha, \beta, \gamma)$, so that the force $F$ is parallel to $K$, will be solved by substituting $\cos \alpha, \cos \beta, \cos \gamma$ for $l, m, n$ in these equations. By the elimination of $\cos \alpha, \cos \beta, \cos \gamma$ from the three equations thus obtained, we have the well-known cubic determinant equation, of which the roots, necessarily real, lead, when no two of them are equal, to one and only one system of three rectangular axes having the stated property.
Chapter IV.—On the Distribution of Displacement in a Strain.

Def. The three lines thus proved to exist for every possible homogeneous stress are called its axes. The planes of their pairs are called its normal planes; the mutual forces between parts of the body separated by these planes, or the forces on portions of the bounding surfaces parallel to them, are called the principal tensions.

Cor. 1. The Principal Tensions of the stress are the roots of the determinant cubic referred to in the demonstration.

Cor. 2. If a stress be specified by the notation \( P(X) \), &c., an expression for its principal planes is obtained as follows:

\[
\rho_{XX} + \rho_{YY} + \rho_{ZZ} = 1,
\]

where \( \rho_{XX}, \rho_{YY}, \rho_{ZZ} \) are the principal tensions, and \( \rho \) any infinitely small quantity, represents the stress, in the following manner:

From any point \( P \) in the surface of the ellipsoid draw a line in the tangent plane half-way to the point where this plane is cut by a perpendicular to it through the centre; and from the end of the first line draw a new surface to meet radius sphere of unit radius concentric with the ellipsoid. The tension at this point of the surface of the solid is in the line from it to the point \( P \); and its amount per unit of surface is equal to the length of the small line drawn through \( P \) by \( \rho \).

Cor. 3. Any stress is fully specified by six quantities, viz., its three principal tensions \( (F, G, H) \), and three angles \( (\theta, \phi, \psi) \) or three numerical quantities equivalent to the nine direction cosines specifying its axes.

Chapter IV.—On the Distribution of Displacement in a Strain.

Prop. In every homogeneous strain any part of the solid bounded by an ellipsoid remains bounded by an ellipsoid.

For all particles of the solid in a plane remain in a plane, and two particle distortions remain parallel. Consequently every system of conjugate diametral planes of an ellipsoid of the solid retain the property of conjugate diametral planes with reference to the altered curve surface containing the same particles. This altered surface is therefore an ellipsoid.

Prop. There is a single system (and only a single system, except in the cases of symmetry) of three rectangular planes for every homogeneous strain, which remain at right angles to one another in the altered point of the surface of the solid.

Def. These three planes are called the normal planes of the strain, or simply the strain-normals. Their lines of intersection are called the strain-normals of the strain. The three angles of the surface are perpendicular to the strain-normals.

Remark. The preceding propositions and definitions are not limited to infinitely small strains, but are applicable to whatever extent the body may be strained.

Prop. If a body, while experiencing an infinitely small strain, be held with one point fixed and the normal planes of the strain parallel to three fixed rectangular planes through the point \( O \), a sphere of the solid of unit radius having this point for its centre becomes, when strained, an ellipsoid, whose equation, referred to the strain-normals through \( O \), is

\[
(1-2\alpha X^2 + (1-2\beta Y^2 + (1-2\gamma Z^2) = 1,
\]

if \( \alpha, \beta, \gamma \) denote the elongations of the solid per unit of length in the directions respectively perpendicular to these planes; and the position, on the surface of this ellipsoid, attained by any particular point of the solid, is such that if a line be drawn in the tangent plane, half-way to the point of intersection of this plane with the plane through the centre, a radial line drawn through its extremity cuts the primitive spherical surface in the primitive position of that point.

Cor. 1. For every stress, there is a certain infinitely small strain, and contrary to every infinitely small strain, there is a certain stress, so related that if, while the strain is being acquired, the centre and the strain-normals through it are unmoved, the absolute displacements of particles belonging to a spherical surface of the solid, of the same radius and intensity (accounting as to units for the representation of force by lines) and in direction, the force (reckoned as to intensity, in amount per unit of area) experienced by the enclosed sphere of the solid, at the different parts of its surface subjected to the stress.

Cor. 2. Any strain is fully specified by six quantities, viz., its three principal elongations, and three angles \( (\theta, \phi, \psi) \) or nine direction cosines, equivalent to three independent quantities specifying its axes.

Chapter V.—Conditions of Perfect Concurrence between Stresses and Strains.

Def. 1. Two stresses are said to be coincident in direction, or to be perfectly concurrent, when they only differ in absolute magnitude. The same relative designations are applied to two strains differing from one another only in absolute magnitude.

Def. 2. When a homogeneous stress is such that the normal component of the mutual force between the parts of the body on the two sides of any plane whatever through it is proportional to the augmentation of distance between the same plane and another parallel to it and initial at a certain distance, the stress experienced by the same body, the stress and the strain are said to be perfectly concurrent; also to be coincident in direction. The body is said to be yielding directly to a stress applied to it, when it is acquiring a strain directly related to it. In the same circumstances, the stress is said to be working directly on the body, or to be acting in the same direction as the strain.

Cor. 1. Perfectly concurrent stresses and strains are of the same type.

Cor. 2. If a strain is of the same type as the stress, its reverse will be said to be negatively of the same type, or to be directly opposed to the strain. A body is said to be working against a stress applied to it, when it is acquiring a stress directly opposed to the stress; and in these circumstances, the matter round the body is said to be yielding directly to the reactive stress of the body upon it.

Chapter VI.—Orthogonal Stresses and Strains.

Def. A stress is said to act right across a strain, or to act orthogonally to a strain, or to be orthogonal to a strain, if work is neither done upon nor by the body in virtue of the action of the stress upon it while it is acquiring the strain.

Def. 2. Two stresses are said to be orthogonal when either coincides in direction with a stress orthogonal to the other.

Def. 3. Two strains are said to be orthogonal when either coincides in direction with a stress orthogonal to the other.

Examples.—(1) Uniform cubic compression, and any strain involving no simple shear, or volume, are orthogonal to each other.

(2) Simple extension or contraction in parallel lines unaccompanied by any transverse distortion or contraction in directions at right angles to those parallel lines, is orthogonal to any similar strain in lines at right angles to those parallels.

(3) Any single longitudinal strain is orthogonal to a "simple tangential strain" in which the sliding is parallel to the direction right to the line.

(4) Two infinitely small simple tangential strains in the same plane, or with the directions of sliding mutually inclined at an angle of 45°.

(5) An infinitely small simple tangential strain is orthogonal to every infinitely small strain, in a plane either parallel to its plane of sliding or perpendicular to its line of sliding.

Chapter VII.—Composition and Resolution of Stresses and of Strains.

Any number of simultaneously applied homogeneous stresses are equivalent to a single homogeneous stress which is called their resultant. Any number of superimposed homogeneous strains are equivalent to a single homogeneous resultant strain. Infinitely small strains may be independently superimposed; and in what follows it will be uniformly understood that the strains spoken of are infinitely small, unless the contrary is stated.

Examples.—(1) A strain consisting simply of elongation in one set of parallel lines, and a strain consisting of contraction in a direction right to it, are orthogonal to each other.

(2) Any two simple distortions in one plane may be reduced to a single simple distortion in that plane.

(3) A strain consisting simply of elongation in a plane, and a strain consisting of contraction in a direction right to it, are orthogonal to each other.

(4) Three equal longitudinal elongations or contractions in three directions

\[
\text{Def. 2. A stress and an infinitely small strain related in the manner defined in Cor. 1, are said to be of the same type. The ellipsoid by means of which the distribution of force over the surface of a sphere of unit radius is represented in one case, and by means of which the displacements of particles from the spherical surface are shown in the other, may be called the geometrical type of either.}
\]

Cor. Any stress- or strain-type is fully specified by five quantities, viz., two ratios between its principal strains or elongations and three quantities specifying its angular position of its axes.

Chapter V.—Conditions of Perfect Concurrence between Stresses and Strains.

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Cor. 1. Perfectly concurrent stresses and strains are of the same type.

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(3) Any simple longitudinal strain is orthogonal to a "simple tangential strain" in which the sliding is parallel to the direction right to the line.

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(2) Any two simple distortions in one plane may be reduced to a single simple distortion in that plane.

(3) A strain consisting simply of elongation in a plane, and a strain consisting of contraction in a direction right to it, are orthogonal to each other.

(4) Three equal longitudinal elongations or contractions in three directions.
at right angles to one another are equivalent to a single dilatation or compression in all directions. The single stress equivalent to three equal tensions or compressions at right angles to one another is a negative or positive pressure equal in all directions.

(1) A simple stress or an infinitely small strain may be defined by the ellipsoid

\[(a-A)X^2+(a-B)Y^2+(a-C)Z^2+EYZ+FXY=1,
\]

and another stress or an infinitely small strain by the ellipsoid

\[(a-A)X^2+(a-B)Y^2+(a-C)Z^2+EYZ+FXY=1,
\]

where \(A, B, C, E, F, \) &c, are all infinitely small, their resultant stress or strain is that represented by the ellipsoid

\[(a-A)X^2+(a-B)Y^2+(a-C)Z^2+EYZ+FXY=1,
\]

and another stress or an infinitely small strain by the ellipsoid

\[(a-A)X^2+(a-B)Y^2+(a-C)Z^2+EYZ+FXY=1,
\]

where \(A, B, C, E, F, \) &c, are all infinitely small, their resultant stress or strain is that represented by the ellipsoid

\[(a-A)X^2+(a-B)Y^2+(a-C)Z^2+EYZ+FXY=1,
\]

Chapter IX.—Orthogonal Types of Reference.

**Def.** Strains of any types are said to be to one another in the same ratio as stresses of the same types respectively, when any particular plane of the solid acquires, relatively to another plane parallel to it, motions in virtue of those strains which are to one another in the same ratio as the normal components of the forces between the parts of the solid on the two sides of either plane due to the respective stresses.

**Cor.** If \(\xi, \eta, \zeta, \) &c denote the orthogonal components of a certain strain, and if \(P, Q, R, S, T, U\) denote components of the same type respectively, of a stress applied to a body while acquiring that strain, the work done upon it per unit of its volume will be

\[
F_{\xi}+G_{\eta}+H_{\zeta}+I_{\xi}+J_{\eta}+K_{\zeta}.
\]

**Cor.** The condition that two stresses or strains specified by \((\xi, \eta, \zeta, \) &c) and \((\xi', \eta', \zeta', \) &c) in terms of a normal system of types of reference, may be orthogonal to one another is

\[
\xi\xi'+\eta\eta'+\zeta\zeta'=0.
\]

Now we have, to express the proportionality of the stresses and strains,

\[
\frac{F}{G} = \frac{P}{Q} = \frac{R}{S} = \frac{T}{U} = \frac{V}{W}.
\]

Each member must be equal to

\[
F'+G'+H'+I'+J'+K' = 0
\]

and also equal to

\[
F_{\xi}+G_{\eta}+H_{\zeta}+I_{\xi}+J_{\eta}+K_{\zeta}.
\]

\[
F_{\xi}+G_{\eta}+H_{\zeta}+I_{\xi}+J_{\eta}+K_{\zeta}.
\]

\[
F_{\xi}+G_{\eta}+H_{\zeta}+I_{\xi}+J_{\eta}+K_{\zeta}.
\]

\[
F_{\xi}+G_{\eta}+H_{\zeta}+I_{\xi}+J_{\eta}+K_{\zeta}.
\]

\[
F_{\xi}+G_{\eta}+H_{\zeta}+I_{\xi}+J_{\eta}+K_{\zeta}.
\]

**Cor.** Any stress or strain may be numerically specified in terms of the specifying elements of a strain or stress, or in terms of the specifying elements of a stress or strain, as the case may be. The specifying elements of a strain may also be called its coordinates, with reference to the chosen types.

Examples.—(1) Six strains in each of which one of the six edges of a tetrahedron of the solid is elongated while the others remain unchanged, may be used as types of reference for the specification of any kind of stress or strain. The ellipsoid representing any one of these six types will have its two circular sections parallel to the faces of the tetrahedron which do not contain the specified side.

(2) Six strains consisting, one of them, of an infinitely small alteration of length of one of the edges, or of one of the three angles between the faces of a parallelepiped of the solid, while the other five angles and edges remain unchanged, may be taken as types of reference, for the specification of infinite stresses or strains. In some cases, as for instance in expressing the probable elastic properties of a crystal of Iceland spar, it might possibly be convenient to use an oblique parallelepiped for such a system of types of reference; but more frequently it will be convenient to adopt a system of types related to the deformations of a cube of the solid.

**Chapter IX.—Orthogonal Types of Reference.**

**Def.** A normal system of types of reference is one in which the strains or stresses of the different types are all six mutually orthogonal in a natural condition of their system of types of reference; and may also be called an orthogonal system. The elements specifying, with reference to such a system, any stress or strain, will be called orthogonal components or orthogonal coordinates.

Examples.—(1) The six types described in Example (2) of Chapter VIII. are clearly orthogonal, if the parallelepiped referred to is rectangular. Those of these are simple longitudinal extensions, parallel to the three axes of rectangular coordinates, and the other three, corresponding to the faces, their axes bisecting the angles between the edges. They constitute the system of reference usually employed by writers on the theory of elasticity.

(2) The six strains in which a spherical portion of the solid is changed into ellipsoids having the following equations:

\[
(X^2+Y^2+Z^2=1,
\]

\[
(X^2+Y^2+Z^2=1,
\]

\[
(X^2+Y^2+Z^2=1,
\]

\[
(X^2+Y^2+Z^2=1,
\]

\[
(X^2+Y^2+Z^2=1,
\]

\[
(X^2+Y^2+Z^2=1,
\]

are of the same kind as those considered in the preceding example and therefore constitute a normal system of types of reference. The resultant of the strains specified, according to those equations, by the elements \(A, B, C, D, E, F, \) is a strain in which the sphere becomes an ellipsoid whose equation see above, Chapter VII. Ex. (5) is

\[
(1-A)X^2+(1-B)Y^2+(1-C)Z^2=EYZ+FXY+GYZ.
\]

(3) A compression equal in all directions (1), three simple stresses having the same magnitude to a common point of reference, and their axes bisecting the angles between the lines of intersection of these planes (II), (III), (IV), and simple longitudinal extensions of equal magnitude to the three axes (V), and the distortion (VI), consist from the same elements which are orthogonal to the last, affect a system of six mutually orthogonal types which will be used as types of reference below in expressing the elasticity of cubical isotropic solids. (Compare Chapter X).

**Example 7 below**

---

1 This example, as well as (7) of Chapter IX, or XI, and the example of Chapter XII, are intended to prepare for the application of the theory of elasticity to cubic and cubically and spherically isotropic bodies, in Part II.

2 Chapter XV. 

3 The "axes of a simple distortion" are the lines of its two component longitudinal strains.
component strains, according to the orthogonal system of types described in Examples (1) and (2) of Chap. IX., the resultant strain equivalent to them will be one such that a sphere of radius 1 in the solid becomes an ellipsoid whose equation is

\[(1 - 2r)X^2 + (1 - 2s)Y^2 + (1 - 2t)Z^2 = 2V(2ZY + YX + ZX)\]

and its magnitude will be

\[\sqrt{2r} + \sqrt{2s} + \sqrt{2t} \geq 1\].

(7) The specifications, according to the system of reference used in the preceding Example, of the unit strains of the six orthogonal types defined in Example (3) of Chap. IX. are respectively as follows:

| (I) | \[x = \sqrt{3}r, y = \sqrt{3}s, z = \sqrt{3}t\] |
| (II) | \[x = 0, y = 1, z = 0\] |
| (III) | \[x = 0, y = 0, z = 1\] |
| (IV) | \[x = 0, y = 1, z = 0\] |
| (V) | \[x = r, y = y', z = t\] |
| (VI) | \[x = 1, y = y', z = 0\] |

where \(r, m, n, k, p, v\) denote quantities fulfilling the following conditions:

\[r + m + n = 0,\]

\[r'^2 + m'^2 + n'^2 = 1,\]

\[r + m' + n' = 0.\]

(8) If

\[(1 - 2r)X^2 + (1 - 2s)Y^2 + (1 - 2t)Z^2 = 2V(2ZY + YX + ZX)\]

be the equation of the ellipsoid representing a certain stress, the amount of work done by this stress, if applied, to a body while acquiring the strain represented by the equation of the preceding example, will be

\[x = \sqrt{3}r, y = \sqrt{3}s, z = \sqrt{3}t, x' = x, y' = y, z' = z.\]

Cor. Hence, if variables \(x, y, z\) be transformed to any other set \((x', y', z')\) fulfilling the condition of being the coordinates of the same point, referred to another system of rectangular axes, the coefficients \(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, \pi, \rho, \omega\), in two homogeneous quadratic functions of three variables, corresponding to the transformed \(x', y', z'\), will be so related that

\[a_1' + a_2' + a_3' = a_1 + a_2 + a_3,\]

\[b_1' + b_2' + b_3' = b_1 + b_2 + b_3,\]

\[c_1' + c_2' + c_3' = c_1 + c_2 + c_3,\]

or the function \(a_1' + a_2' + a_3' + b_1' + b_2' + b_3' + c_1' + c_2' + c_3'\) is an "invariant" for linear transformations fulfilling the conditions of transformation from one to another set of rectangular axes. Since \(x + y + z\) and \(x' + y' + z'\) are clearly invariants also, it follows that

\[a_1x_1 + a_2x_2 + a_3x_3 + b_1y_1 + b_2y_2 + b_3y_3 + c_1z_1 + c_2z_2 + c_3z_3\]

is an invariant function of the coefficients of the two quadratics

\[a_1x^2 + a_2y^2 + a_3z^2 + b_1yz + b_2zx + b_3xy + c_1xz + c_2zy + c_3yx,\]

and

\[a_1'y^2 + a_2'y^2 + a_3'y^2 + b_1'y_1y_2 + b_2'y_2z_3 + b_3'y_3x_1 + c_1'y_3z_2 + c_2'y_2x_1 + c_3'y_1z_2,\]

which it is easily proved to be by direct transformation.

This is the simplest form of the algebraic theorem of invariance with which we are concerned.

**Chapter XI.—On Imperfect Concurrences of two Stress or Strain Types.**

Def. The concurrence of any stresses or strains of two stated types is the proportion which the work done when a body of unit volume experiences a stress of either type, while acquiring a strain of the other, bears to the product of the numbers measuring the stress and strain respectively.

Cor. 1. In orthogonal resolution of a stress or strain, its component of any stated type is equal to its own amount multiplied by its concurrence with that type; or the stress or strain of a stated type which, along with another or others of stated type, have a given stress or strain for their resultant, is equal to the amount of the given stress or strain reduced in the ratio of its concurrence with that stated type.

Cor. 2. The concurrence of two coincident stresses or strains is unity; or a perfect concurrence is numerically equal to unity.

Cor. 3. The concurrence of two orthogonal stresses and strains is zero.

Cor. 4. The concurrence of two directly opposite stresses or strains is -1.

Cor. 5. If \(x, y, z, \xi, \eta, \zeta\) are orthogonal components of any strain or stress \(r, s, t\), its concurrence with the types of reference are respectively

\[r = (\xi^2 + \eta^2 + \zeta^2)^{1/2}, \]

\[s = (\xi^2 + \eta^2 + \zeta^2)^{1/2}, \]

\[t = (\xi^2 + \eta^2 + \zeta^2)^{1/2} \]

where \(r, m, n\) are by quantities fulfilling the conditions

\[r + m + n = 0,\]

\[r + m' + n' = 0,\]

\[r + m' + n' = 0,\]

\[x + y + z = 0.\]

**PART II.—On the Dynamical Relations between Stresses and Strains Experienced by an Elastic Solid.**

**Chapter XIII.—Interpretation of the Differential Equation of Energy.**

In a paper on the Thermo-elastic Properties of Matter, published in the first number of the Quarterly Mathematical Journal, April 1855, and republished in the Philosophical Magazine, 1877, second half year, it was proved, from general principles in the theory of the Transformation of Energy, that the amount of work \(w\) required to reduce an elastic solid, kept at a constant temperature, to a specification \(c, \epsilon, \tau, \pi, \sigma\) in terms of the system of reference used in Examples (6) and (7). Chapter IX., is as follows:

\[w = -\frac{1}{2}(\cos \theta - \sin \theta),\]

where \(\theta\) is the angle between the axis of elongation in the given distortion and in the first component type.

The mutual concurrence of a simple longitudinal strain and a simple distortion is

\[\frac{\sqrt{2}}{2}(\cos \theta + \sin \theta),\]

The mutual concurrence of a simple retortional strain and a uniform dilatation is

\[\frac{1}{2}(\cos \theta - \sin \theta)\]
the elasticity under any of the strains it has experienced. Thus for homogeneous, solidly homogeneous, strained, it appears that \( w \) is a function of six independent variables \( x, y, z, \xi, \eta, \zeta \), by which the condition of the solid as to strain is specified. Hence to strain the body to the infinitely small extent expressed by the variation from \( (x, y, z, \xi, \eta, \zeta) \) to \( (x + dx, y + dy, z + dz, \xi + d\xi, \eta + d\eta, \zeta + d\zeta) \), the work required to be done upon it is

\[
\frac{dx}{dx} \cdot \frac{dx}{da} + \frac{dy}{dy} \cdot \frac{dy}{db} + \frac{dz}{dz} \cdot \frac{dz}{dc} + \frac{d\xi}{d\xi} \cdot \frac{d\xi}{d\alpha} + \frac{d\eta}{d\eta} \cdot \frac{d\eta}{d\beta} + \frac{d\zeta}{d\zeta} \cdot \frac{d\zeta}{d\gamma}.
\]

The stress which must be applied to its surface to keep the body in equilibrium in the state \( (x, y, z, \xi, \eta, \zeta) \) must therefore be such that if it would do this amount of work if the body, under its action, were to acquire the arbitrary strain \( dx, dy, dz, d\xi, d\eta, d\zeta \); that is, it must be the resultant of six stresses:—one orthogonal to the five strains \( dy, dz, d\xi, d\eta, d\zeta \) and of such a magnitude as to do the work \( \frac{dy}{dy} \cdot \frac{dy}{db} \) when the body acquires the strain \( dz \); a second orthogonal to \( dz, ds, d\eta, d\zeta, ds' \) and of such a magnitude as to do the work \( \frac{ds}{ds} \cdot \frac{ds}{db} \) when the body acquires the strain \( dy \); and so on.

If \( a, b, c, f, g, h \) denote the respective concurrences of these six stresses, with the types of reference used in the specification \( (x, y, z, \xi, \eta, \zeta) \) of the strains, the amounts of the six stresses which fulfil those conditions will (Chapter XIV) be given by the equations

\[
\begin{align*}
S = & \frac{1}{a} \frac{da}{dx} + \frac{1}{b} \frac{db}{dy} + \frac{1}{c} \frac{dc}{dz} + \frac{1}{f} \frac{df}{d\xi} + \frac{1}{g} \frac{dg}{d\eta} + \frac{1}{h} \frac{dh}{d\zeta}, \\
S = & \frac{1}{a} \frac{da}{dx} + \frac{1}{b} \frac{db}{dy} + \frac{1}{c} \frac{dc}{dz} + \frac{1}{f} \frac{df}{d\xi} + \frac{1}{g} \frac{dg}{d\eta} + \frac{1}{h} \frac{dh}{d\zeta}, \\
S = & \frac{1}{a} \frac{da}{dx} + \frac{1}{b} \frac{db}{dy} + \frac{1}{c} \frac{dc}{dz} + \frac{1}{f} \frac{df}{d\xi} + \frac{1}{g} \frac{dg}{d\eta} + \frac{1}{h} \frac{dh}{d\zeta}.
\end{align*}
\]

and the types of these component stresses are determined by being orthogonal to the five of the six strain-types, wanting the first, and the second, &c., respectively.

Cor. If the types of reference used in expressing the strain of the body constitute an orthogonal system, the types of the component stresses will coincide with them, and each of the concurrences will be unity. Hence the equations of equilibrium of an elastic solid referred to six orthogonal types are simply

\[
\begin{align*}
P = & \frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} + \frac{df}{d\xi} + \frac{dg}{d\eta} + \frac{dh}{d\zeta}, \\
Q = & \frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} + \frac{df}{d\xi} + \frac{dg}{d\eta} + \frac{dh}{d\zeta}, \\
R = & \frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} + \frac{df}{d\xi} + \frac{dg}{d\eta} + \frac{dh}{d\zeta}.
\end{align*}
\]

CHAPTER XIV.—Reduction of the Potential Function, and of the Equations of Equilibrium, of an Elastic Solid to their simplest Forms.

If the condition of the body from which the work denoted by \( w \) is reckoned be that of equilibrium under no stress from without, and if \( x, y, z, \xi, \eta, \zeta \) be chosen for each of these conditions, we shall have, by Maclaurin's theorem (section 37 above) that, within the elastic limits, the stresses are isotropic among them; and (3) that there is no equality between any of the six roots of the determinant in general, when there are twenty-one independent coefficients in the given quadratic.

Prop. Hence a simple system of six mutually orthogonal types may be determined for any homogeneous elastic solid, so that its potential energy when homogeneously strained in any way is expressed by the sum of the products of the squares of the components of the strain, according to those types, respectively multiplied by six determinate coefficients.

Def. The six strain-types thus determined are called the Six Principal Strain-types of the body.

The concurrences of the stress-components used in interpreting the potential equation of elasticity with the types of the strain-coordinates in terms of which the potential of elasticity is expressed, being perfect when these constitute an orthogonal system, each of the quantities denoted above by \( a, b, c, f, g, h \) is unity, if the elastic reaction is balanced by a stress orthogonal to the five others of the same kind.

Principal Types, and \( P, Q, R, S, T, U \) the components according to the same types, of the stress required to hold the body in equilibrium when in the condition of having those strains. The amount of work that must be spent upon it per unit of its volume, to bring it to this state from an unstrained condition, is given by the equation

\[
\begin{align*}
P = & Ax, \quad Q = By, \quad R = Cz, \\
S = & \frac{1}{2}(Ax^2 + By^2 + Cz^2 + \frac{1}{2}P^2 + \frac{1}{2}Q^2 + \frac{1}{2}R^2),
\end{align*}
\]

where \( x, y, z, \xi, \eta, \zeta \) denote strains belonging to the six Principal Types, and \( P, Q, R, S, T, U \) the components according to the same types, of the stress required to hold the body in equilibrium when in the condition of having these strains. The amount of work that must be spent upon it per unit of its volume, to bring it to this state from an unstrained condition, is given by the equation

\[
\begin{align*}
P = & Ax, \quad Q = By, \quad R = Cz, \\
S = & \frac{1}{2}(Ax^2 + By^2 + Cz^2 + \frac{1}{2}P^2 + \frac{1}{2}Q^2 + \frac{1}{2}R^2),
\end{align*}
\]

Def. The coefficients \( A, B, C, F, G, H \) are called the Six Principal Elasticities of the body.

The equations of equilibrium express the following propositions:

Prop. If a body be strained according to any one of its six Principal Types, the stress required to hold it so is directly concurrent with the strain.

Examples.—(1) If a solid be cubically isotropic in its elastic properties, as crystals of the cubic class probably are, any portion of it will, when subjected to a positive or negative pressure or any rectangular diastasis in any ways, to a uniform condensation or dilation in all directions. Hence a uniform condensation or dilation in all directions is a principal strain, and since the three corresponding principal elasticities are equal to one another, any strain whatever compounded of these three is a principal strain. Lastly, a plane distortion whose axes coincide with any two edges of the cube, being clearly a principal distortion, and the principal elasticities corresponding to the three directions of this kind being equal to one another, any distortion compounded of them is also a principal distortion.

Hence the system of orthogonal types treated of in Examples (3) Chap. IX., and (3) Chap. X., or any system in which, for (1), (2), and (4), any three orthogonal strains compounded of them are substituted, constitutes a system of six Principal Strains in a solid cubically isotropic. There are only three distinct Principal Elasticities for such a body, and these are—(A) its modulus of compressibility, (B) its rigidity against diagonal distortion in any of its principal planes (principal elasticities), and (C) its rigidity against rectangular distortion of a cube of symmetry (two equal elasticities).

(2) In a perfectly isotropic solid, the rigidity against all distortions is equal. Hence the rigidity (B) against diagonal distortion must be equal to the rigidity (C) against rectangular distortion, in a cube; and it is easily seen that if this condition is fulfilled for one set of three rectangular planes for which the body is isotropic, the isotropy must be complete. The conditions of perfect or spherical isotropy are therefore expressed in terms of the conditions referred to in the preceding example, with the further condition \( B = C \).

Principal Types in all directions, and any system whatever of four orthogonal distortions, constitute a system of six Principal Strains in a spherically isotropic solid. Its Principal Elasticities are simply its Modulus of Compressibility and its rigidity.

Prop. Unless some of the six Principal Elasticities be equal to one another, the stress required to keep the body strained otherwise than according to one or other of six distinct types is oblique to the strain.

Prop. The stress required to maintain a given amount of strain is a maximum or a maximum-minimum, or a minimum, if it is of one of the six Principal Types.

If \( A \) be the greatest, and the least of the six quantities \( A, B, C, F, G, H \), the principal type to which the first corresponds is the type of strain requiring a greater stress to maintain it than any
other strain of equal amount, and the principal type to which the last corresponds is that of a strain which is maintained by a lesser stress than any other strain of equal amount in the same body. The stress corresponding to the four other principal strain-types have each the maximum-magnitude property in a determinate way.

Prop. If a body be strained in the direction of which the concurrence with the principal strain-types are \( i, m, n, \lambda, \mu, \rho \), and to an amount equal to \( \tau \), the stress required to maintain it in this state will be equal to \( \sigma \), where

\[
\Omega = A\sigma^2 + B\sigma + C, \quad \text{with} \quad A > 0, B > 0, C > 0,
\]

and will be of a type of which the concurrence with the principal types are respectively

\[
A_{j} \quad B_{i} \quad C_{j} \quad \text{for} \quad j = 1, 2, 3
\]

Prop. A homogeneous elastic solid, crystalline or non-crystalline, subject to magnetic force or free from magnetic force, has neither any right-handed or left-handed, nor any dipolar, properties dependent on elastic forces simply proportional to strains.

Cor. The elastic forces concerned in the luminiferous vibrations of a solid or fluid medium possessing the right- or left-handed property, whether axial or rotatory, such as quartz crystal, or tartaric acid, or solution of sugar, either depend on the heterogeneity or on the magnitude of the strains experienced.

Hence as they do not depend on the magnitude of the strain, they do depend on its heterogeneity through the portion of a medium containing a wave.

Cor. There cannot possibly be any characteristic of elastic forces simply proportional to the strains in a homogeneous body, corresponding to certain peculiarities of crystalline form which have been observed—for instance corresponding to the plagiedral faces discovered by Sir John Herschel to indicate the optical character, whether right-handed or left-handed, in different specimens of quartz crystal, or corresponding to the distinguishing characteristics of the crystals of the right-handed and left-handed tartaric acids obtained by M. Pasteur from racemic acid, or corresponding to the dipolar characteristics of form said to have been discovered in electric crystals.

CHAPTER XVI.—Application of Conclusions to Natural Crystals.

It is easy to demonstrate that a body, homogeneous when regarded on a large scale, may be constructed to have twenty-one arbitrarily prescribed values for the coefficients in the expression for its potential energy in terms of any prescribed system of strain coordinates. This proposition was first enunciated in the paper on the Thermo-elastic Properties of Solids, published April 1855, in the Quarterly Mathematical Journal alluded to above. We may infer the following.

Prop. A solid may be constructed to have arbitrarily prescribed values for its six Principal Elasticities and an arbitrary orthogonal system of six strain-types, specified by fifteen independent elements, for its principal strains: for instance, five arbitrarily chosen systems of three rectangular axes, for the normal axes of five of the principal Types; those of the sixth consequently in general distinct from all the others, and determinable; and the two sets of the two ratios between the three stresses or strains of each type, also determine. The fifteen equations expressing (Chap. VI.) the mutual orthogonality of the six principal Types determine the twelve ratios for the six types, and the three quantities specifying the axes of the sixth type in the particular case here suggested; or generally the fifteen equations determine fifteen out of the thirty quantities (viz., twelve ratios and eighteen angular coordinates) specifying six Principal Types.

Cor. There is no reason for believing that natural crystals do not exist for which there are six unequal Principal Elasticities, and six distinct sets of three normal axes constituting six distinct sets of three principal rectangular axes of elasticity.

It is easy to give arbitrary illustrative examples regarding Principal Elasticities: also, to investigate the principal strain-types and the equations of elastic force referred to them or to other natural types, for a body possessing the kind of symmetry as to elastic forces that is possessed by a crystal of Iceland spar, or for a crystal of the "tesseral class," or for the included "cubical classes." Such illustrations and developments, though proper for a student's textbook of the subject, are unnecessary here.

For applications of the Mathematical Theory of Elasticity to the question of the earth's rigidity and elasticity as a whole, and to the problem of elastic solids in general, which are beyond the scope of the present article, the reader is referred to Thomson and Tait's "Natural Philosophy," §§ 558, 740, 822, 849, and Appendix C.

CHAPTER XVII.—Plane Waves in a Homogeneous Anisotropic Solid.

A plane wave in a homogeneous elastic solid is a motion in which every line of particles in a plane parallel to one fixed plane ex-
ELASTICITY

Hence, instead of three different waves with different velocities, we have just two,—a wave (like that of sound in air or other elastic fluid) in which the motions are perpendicular to the wave front, and the other (like the waves of light in an isotropic medium) in which the motions are parallel to the wave front.

Waves in an Incompressible Solid (Isotropic).—If the solid be incompressible, we have \( A = 0 \), and \( u \) must be zero. Hence

\[ w = B_0^2 + C_0^2 + 2D_0^2 \]

and by a determinantal quadratic, instead of cubic, we find two "wave-velocities and two wave-modes, in each of which the motion is parallel to the wave front. In the case of isotropy the two wave velocities are equal.

It is to be noticed that \( M_1, M_2, M_3 \) in the preceding investigation are not generally true "principal moduluses," but special moduluses corresponding to the particular plane chosen for the wave front. In the particular case of isotropy, however, the equal moduluses \( M_1, M_2 \) of (11) are principal moduluses, being each equal to the modulus of rigidity, but \( M_3 \) is a mixed modulus of compressibility and rigidity—not a principal modulus. In the case of incompressibility, the two moduluses found from the determinantal quadratic by the process indicated above are not principal moduluses generally, because the distortions by the differential motions of planes of particles parallel to the wave front must generally give rise to tangential stresses orthogonal to them, which do not influence the wave motion. (W. TH.)
HEAT is a property of matter which first became known to us by one of six very distinct senses.

§ 1. Sense of Heat.—The sense of touch, as commonly meant, has two distinct objects—force and heat. If a person stretches out his hand till it meets anything solid, or holds it out while something solid is placed upon it, he experiences a sensation of force. He perceives resistance to the previous motion of his hand in one case, in the other case the necessity of resisting to prevent his hand from being forced downwards; the immediate object of this perception in each case is force. But there is another very distinct sensation, that of heat or cold, which he may or may not perceive in either of those cases, and which he may also perceive, still by what is commonly called the sense of touch, in other cases even when no sense of force is also experienced. Thus, in the first case, if the solid be a fixed can of warm water, or of iced water, the person perceives the change of the freezing point of a liquid thermometer through imperfect elasticity of the glass, the diminution of change of the freezing point by use of a fluid of high expansion, the error of thermometer due to capillarity, the objections to use of volatile liquids as thermometric substances, air as thermometric fluid—Regnault’s experiments.

1 The sense of smoothness and roughness to which physiologists have sometimes given the special name “tactile sense” is as clearly a sense of force as is what they call the muscular sense. The sense of roughness is a sense of force at places of application distributed over the skin of the finger, while in the muscular sense of force the place of application on a larger scale is determined by the position of the hand perceiving the force. The internal mechanism of tissue and nerves in one case and that of muscles in the other, through which the perception of places of application of force is obtained, are no doubt different, but the thing perceived is essentially the same—force—the complete discrimination of which involves magnitude of the force, its place of application, and its direction.

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The earliest scientific thoughts respecting these sensations of heat and cold must have led to the true conclusion that there is some property of external matter on which the sensations depend, and a little advance into the natural philosophy of the subject has suggested and proved that this property is also possessed by the living body, and that the sensation of heat or cold in the hand, in the instances
HEAT

§ 2. Latent Heat.—There are exceptional cases in which temperature does not vary in a mass of matter when heat is communicated to it from, or taken from it, external matter. For instance, when the body is ice at the melting point, heat communicated to it does not raise its temperature; or if the body be water at the freezing point with ever so small a piece of ice in it (see Matter, Properties of), heat taken from it does not cause its temperature to fall; or if the whole mass considered be ice and water well mixed, heat may be either communicated to it or taken from it without altering its temperature; or if the body be water at the boiling point in the open air, heat very slowly communicated to it in however great quantities does not raise its temperature sensibly, but causes it to disappear by evaporation from its surface; or if the body be steam in a cylinder with a little water in the bottom and with a frictionless piston above it for roof (fig. 1), under atmospheric pressure, heat taken from it very slowly does not cool it until the whole steam has become condensed into water, and heat communicated to it very slowly does not warm it until the whole water has become evaporated into steam; or if the body be ice (or frozen water), in place of the liquid water of the last case, and if the pressure on the upper side of the piston instead of atmospheric pressure be of about 1033 grammes per square centimetre (14-7 lb per square inch), be anything less than \( \frac{1}{4} \) th of a gramme per square centimetre, the same statement will still apply with "ice" substituted for water. Black's celebrated doctrine of latent heat is merely the declaration of a class of phenomena of which the preceding illustrations sufficiently indicate the character. Modern mysticism has been much exercised in respect to the terms sensible heat and latent heat, whether in decrying them, or in continuing to use them, but without investigating historically of the clear meaning of these old doctrine. It has become of late years somewhat the fashion to decry the designation of latent heat, because it had been very often stated in language involving the assumption of the materiality of heat.¹ Now that we know heat to be a mode of motion, and not a material substance, the old "impressive, clear, and wrong" statements regarding latent heat, evolution and absorption of heat by compression, specific heats of bodies and quantities of heat possessed by them, are summarily discarded. But they have not yet been generally enough followed by equally clear and concise statements of what we now know to be the truth. A combination of impressions surviving from the old erroneous notions regarding the nature of heat with imperfectly developed apprehension of the new theory has somewhat liberally perplexed the modern student of thermodynamics with questions unanswerable by theory or experiment, and propositions which escape the merit of being false by having no assignable meaning. There is no occasion to give up either "sensible heat" or "latent heat," and there is a positive need to retain the term latent heat, because if it were given up a term would be needed to replace it, and it seems impossible to invent a better. Heat given to a substance and warming it is said to be sensible in the substance. Heat given to a substance and not warming it is said to become latent. These designations express with perfect clearness the relation of certain material phenomena to our sensory perception of them. Thus when heat given to a quantity of water warms it, the heat becomes sensible to a hand held in the water. When a basin of warm water and a basin of water and ice are placed side by side, a hand dipped first in one and then in the other perceives the heat. If now the warm water be poured into the basin of ice and water, and stirred for a few seconds of time (unless there is enough of warm water to melt all the ice), the hand perceives no warmth; on the contrary, it perceives that the temperature is the same as it was in the basin of ice and water at the beginning. Thus the heat which was sensible in the basin of warm water has ceased to be sensible in the water that was in that basin, and has not become sensible in the other. It is therefore well said to have become latent.

CALORIMETRY.

§ 3. Calorimetry by Latent Heat.—The doctrine of latent heat leads us very smoothly to a most important measurement in thermal science, the measurement of quantities of this wonderful property of matter which we call heat, without our knowing anything of the nature of heat, whether it be a subtle elastic fluid, or a state of motion, or possibly some modification of matter related to action of force. Without, in the first place, admitting into our minds any definite idea as to the nature of heat, we may agree to measure quantities of heat by quantities of ice melted into water without change of temperature. Thus if a kilogramme of ice is melted by a large quantity of water at a lukewarm temperature, or by a comparatively small quantity of very hot water, the same quantity of heat has certainly gone from the warm water to the ice in each case, supposing that the result in each case is the ice and warm water left all in a state of ice-cold water. The measurement of quantities of heat, whether by the melting of ice, or by any other means, received the name of "Calorimetry," when the essence of heat was supposed to be a fluid, and this fluid called caloric. The name calorimetry is still by general consent retained to designate measurement of quantities of heat, as distinguished from thermometry, or of a substance called the matter of heat; and, as I think Sir Isaac Newton's opinion that heat consists in the internal motion of the particles of bodies much the most probable, I chose to use the expression heat is generated' (Phil. Trans., 1783, quoted by Forbes). We shall not now be in danger of any error if we use latent heat as an expression meaning neither more nor less than this: —

¹ A hundred years ago those deeper philosophers who in their judgments anticipated, or tended to anticipate, what we now know to be the true theory of the nature of heat, had indeed good grounds to be jealous of even the phrase latent heat. Maxwell says—"It is worthy of remark that Cavendish, though one of the greatest chemical discoverers of his time, would not accept the phrase latent heat. He prefers to speak of the generation of heat when steam is condensed, a phrase inconsistent with the notion that heat is matter, and objects to Black's term as relating to an hypothesis depending on the supposition that the heat of bodies is owing to their containing more or less
the measurement of temperature (§ 10–67). As long as the truth or falsity of the materialistic hypothesis seemed an open question, the word caloric was held to imply the materiality of heat. Thus Davy, after discussing some of the fundamental dogmas of the "Calorists," as he called them, and describing his own experiments, which proved beyond all doubt the falsity of their fundamental hypothesis that heat is matter, varied the statement of his conclusion by saying, "or caloric does not exist." While accepting Davy's conclusion, however, we need not accept this way of stating it; and as most of our best modern writers still use the word caloriometer, and as French writers have, in comparatively recent times, introduced the word "calorie" to designate a unit quantity of heat, it is decidedly convenient still to retain the name caloric to denote definitely the measurable essence of heat. This is convenient scientifically as tending to give precision to language and ideas respecting the two classes of measurement, caloriometry and thermometry; and it has the advantage of leaving the more popular word heat available for that somewhat lax general usage, from which we cannot altogether displace it, in which it may sometimes mean high temperature, as when we speak of great heat, or summer heat, or blood heat, sometimes a measurable quantity of heat, as in the term latent heat, and sometimes a branch of study or science dealing with the transference of heat by conduction and radiation, as in the title of Fourier's great work Théorie analytique de la Chaleur, or the whole province of science concerned with heat, including caloriometry and thermometry, and conduction and radiation of heat, and generation of heat, and dynamical relations of heat, as in English titles of separate books such as Dixon's, Balfour Stewart's, and Maxwell's, or of chapters or divisions of larger treatises, such as even the present article.

§ 4. Calorimetry by Melting of Ice.—Calorimetry was first practised by means of the melting of ice as explained above, and the first thermal unit, or unit quantity of heat, or "calorie," although not then called caloric, was the quantity of heat required to melt unit weight of ice. This, for example, is the unit on which Fourier founds his reckoning illustratively when he explains the fundamental principles of his theory of the conduction of heat. Ice seems to have been first used for calorimetry by Wilcke, a Swede. For the systematic application of this method for the measurement of quantities of heat in various physical inquiries Laplace and Lavoisier constructed an instrument to which the name of caloriometer was applied, and described it in the memoirs of the French Academy of Sciences for 1780.1 Though in the hands of Laplace and Lavoisier it gave good results, it had a great inconvenience, which with less careful and less scientific experimenters might lead to great inaccuracies, on account of the water adhering by capillary attraction to the broken ice, instead of draining away from it completely and showing exactly how much ice had been melted. To avoid this evil Sir John Herschel suggested that, instead of draining away the water from the ice, the water and ice should all be kept together, and the whole bulk measured. The diminution of bulk of the whole thus gives an accurate measurement of the quantity of ice melted, because ice melting into water completely occupies just 91.675 per cent. of its own volume. This suggestion is admirably carried out by Dunsen2 in his ice-caloriometer, an instrument possessing also other novel features of remarkable beauty and scientific interest. It is particularly valuable for the measurement of small quantities of heat. Its inventor, for example, by means of it succeeded in making satisfactory determinations of the specific heats of some of those rarer metals, such as indium, of which only a few grammes have been obtained.

§ 5. Calorimetry by the Evaporation of Water.—By another application of Black's doctrine of latent heat, the evaporation of water may be used for calorimetry with great advantage in many scientific investigations. It is used generally in engineering practice, particularly for testing the heating power of different qualities of coal and the economy of various forms of furnaces. The thermal unit, which presents itself naturally in this system, is the quantity of heat required to evaporate unit weight of water when the pressure of the atmosphere as measured by the barometer is of some conventional standard amount, such as that called one atmosphere, or one atmo, being that for which the barometer, with its mercury column at zero centigrade (or the temperature at which ice melts), stands at 76 centimetres in the latitude of Paris, 48° 50', or at

\[76 \times \left(1 + \frac{0.00531 \sin 28° 50'}{1 + 0.00531 \sin 28° 50'}\right)\]

in any latitude. This thermal unit (see Thermodynamics) is, according to Regnault's observations, equal to 6.84 times the ice-caloriometric unit.

§ 6. Thermometric Calorimetry.—The most prevalent mode of caloriometry in scientific investigation has been hitherto, however, neither that by the melting of ice, nor that by evaporation of water, nor indeed anything founded on the doctrine of latent heat at all. It has been founded on the elevation of temperature produced in water by the communication to it of the heat to be measured; and, for the sake of distinction from caloriometry by latent heat or otherwise, it may be called thermometric calorimetry. We can only consider it now in anticipation, as we have not yet reached the foundation of any thermometric scale; but even now we can see that, if in any way we fix upon any two particular determinate temperatures, the quantity of water warmed from the lower to the higher of them by the heat to be measured is a perfectly definite measure for the quantity of this heat. The two temperatures chosen for thermometric calorimetry are those marked 0° and 1° on the centigrade scale. The first of these we can understand at present, being the temperature at which ice melts under ordinary atmospheric pressure. The second is fully defined in §§ 55, 37, 51, 67. The quantity of heat required to raise unit mass of water (1 kilogramme, or 1 gramme, or 1 milligramme, or 1 lb, as the case may be) from zero to 1° C. is called the thermal unit centigrade, and sometimes, especially by French writers, the "caloric." The quantity of heat required to raise special units of water (1 kilogramme, or 1 gramme, or 1 milligramme, or 1 lb, as the case may be) from zero to other temperatures is called the thermal unit centigrade at those temperatures.

§ 7. Comparison of Calorimetric Units.—Observations by Prevost and Desains, and by Regnault, on the latent heat of fusion of ice, show it to be 79.25 thermal units centigrade, a result differing but little from Black's original determination, which made it 142 thermal units Fahr.—this being equal to 78.9 thermal units centigrade. Thus if one kilogramme of ice be put into 791 grammes of water at 1° C., and left till the whole is melted (the process may be accelerated by not too violent stirring § 9), the result will be 804 grammes of water at 0° C.

Regnault's experiments on the latent heat of steam show that the quantity of heat required to convert steam unit

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1 The instrument itself is preserved in the Conservatoire des Arts et Métiers in Paris. It is described and explained in Maxwell's Theory of Heat, chap. iii.

mass of water at the boiling temperature, under standard atmospheric pressure (§ 5), is 536 1/2 thermal units centigrade. This number, which is no doubt very accurate, differs but little from Watt's final result, 960 thermal units Fehr. (equal to 533 3/4 thermal units centigrade), obtained by him, in a repetition in 1781 of experiments which he had commenced in 1765 at the invitation of Black, whose pupil he was.

§ 8. Preliminary regarding the Nature of Heat.—Dynamical Calorimetry.—From the dawn of science till the close of last century two rival hypotheses had been entertained regarding the nature of heat, each with more or less of plausibility, but neither on any sure experimental basis: one that heat consisted of a subtle elastic fluid permeating through the pores or interstices among the particles of matter, like water in a sponge; the other that it was an intestine commotion among the particles or molecules of matter. In the year 1799 Davy, in his first published work entitled An Essay on Heat, Light, and Combinations of Light, 1 conclusively overthrew the former of these hypotheses, and gave good reason for accepting as true the latter, by his celebrated experiment of converting ice into water by rubbing two pieces of ice together, without communicating any heat from surrounding matter. A few years earlier Rumford had been led to the same conclusion, and had given very convincing evidence of it in his observation of the great amount of heat produced in the process of boring cannon in the military arsenal at Munich, and the experimental investigation on the excitation of heat by friction 2 with which he followed up that observation. He had not, however, given a perfect logical demonstration of his conclusion, nor even quite a complete experimental basis on which it could be established with absolute certainty. All that the materialistic doctrine, was baffled by this sophism. It is remarkable that Davy's logic, too, was at fault, and on just the same point as Rumford's, but with even more transparently logical fallaciousness, because his argument is put in a more definitely logical form.

Let heat be considered as matter, and let it be granted that the temperature of bodies cannot be increased unless their capacities are diminished from some cause, or heat added to them from some bodies in contact"! !

"Experiment II. I procured two parallelepipedons of ice, 4 of the temperature of 20°, 6 inches long, 2 3/4 inches broad, and 1/4 of an inch thick; they were fastened by wires to two bars of iron. By a peculiar mechanism their surfaces were placed in contact and kept in a continued and violent friction for some minutes. They were then entirely converted into water, which was collected and its temperature ascertained to be 35° after remaining in an atmosphere of a lower temperature for some minutes. The fusion took place only at the plane of contact of the two pieces of ice, and no bodies were in the form of ice. From this experiment it is evident that ice by friction is converted into water, and, according to the supposition, its capacity is diminished; but it is a well-known fact that the capacity of water for heat is much greater than that of ice, and ice must have an absolute quantity of heat added to it before it can be converted into water. Friction consequently does not diminish the capacities of bodies for heat."—Davy's Essay on Heat, Light, and Combinations of Light, pp. 10-12.

[Delete from "and, according to the supposition," to "greater than that of ice" inclusive; and delete the same as inconsistent conclusion in the last line. The residue constitutes an unanswerable demonstration of Davy's negative proposition that heat is not matter.]

§ 9. Joule's Dynamical Equivalent of Heat.—It is remarkable that, while Davy's experiment alone sufficed to overthrow the hypothesis that heat is matter, and Rumford's, with the addition of just a little consideration of its relations to possibilities or probabilities of inevitable alternatives, did the same, fifty years passed before the scientific world became convinced to their conclusion,—a remarkable instance of the tremendous efficiency of bad logic in confounding public opinion and obstructing true philosophic thought.

1 Published in 1799 in Contributions to Physical and Medical Knowledge, principally from the West of England, collected by Thomas Beddoes, M.D., and reprinted in Dr Davy's edition of his brother's collected works, 2 vol. London, 1836.

2 "An Enquiry concerning the Source of Heat which is excited by Friction" (Phil. Trans., abridged, vol. xviii. p. 286).

3 Compare quotation from Cavendish, footnote, § 2 above.

4 "The result of the experiment is the same if wax, tallow, resin, or any substance fusible at a low temperature be used; even iron may be fused by collision, as is evident by the first experiment."
Joule's great experiments from 1840 to 1849, creating new provinces in the thermodynamics of electricity, and magnetism, and electro-chemistry, recalled attention to Davy and Rumford's doctrine regarding the nature of heat, and supplied several fresh proofs, each like Davy's absolutely incontestably cogent, that heat is not a material substance, and each advancing with exact dynamical measurements on the way pointed out by Rumford in his measurement of the quantity of heat generated in a certain time by the action of two horses not urged to overwork themselves. The full conversion of the scientific world to the kinetic theory of heat took place about the middle of this century, and was no doubt an immediate consequence of Joule's work, although Rumford and Davy's demonstrative experiments, and the ingenious and penetrating speculations of Mohr, and Séguin, and Mayer, and the experimental thermodynamic measurements of Colding, all no doubt contributed to the result. Each of the several subjects of thermodynamic measurement undertaken by Joule gave him a means of estimating the quantity of work required to generate a certain quantity of heat; but after several years of trials he was led to prefer to all others the direct method of simply stirring a quantity of water by a paddle, and measuring the quantity of heat produced by a measured quantity of work; and this method he has accordingly used in all his experiments for the purpose of determining the "dynamical equivalent of heat" from the year 1845 to the present time. By this he found his final result of 1849, which was 772 Manchester foot-pounds for the quantity of work required to warm by 1° Fahr. at any temperature between 55° and 61° Fahr., 1 lb of water weighed in vacuum. In 1870 he commenced work for a fresh determination of the dynamical equivalent of heat at the request of the British Association, and the result was communicated to the Royal Society about the end of 1877, with the following preface:

"The committee of the British Association on standards of electrical resistance having judged it desirable that a fresh determination of the mechanical equivalent of heat should be made by observing the thermal effects due to the transmission of electrical currents through resistances measured by the unit they had issued, I undertook experiments with that view, resulting in a larger figure (782.5, Brit. Assoc. Report, Dundee, 1867, p. 582) than that which I had obtained by the friction of fluids (772.6, Phil. Trans., 1850, p. 82). The only way to account for this discrepancy was to admit the existence of error in either my thermal experiments or in the unit of resistance. A committee consisting of Sir William Thomson, Professor P. G. Tait, Professor Clerk Maxwell, Professor B. Stewart, and myself, were appointed at the meeting of the British Association in 1870, and with the funds thus placed at my disposal I was charged with the present investigation, with the purpose of giving greater accuracy to the results of the direct method.

The result of this final investigation of Joule's is 772.43 Manchester foot-pounds for the quantity of heat required to warm from 60° to 61° Fahr. a pound of water weighed in vacuum, which is about 3 per cent. greater than the final result of 1849 expressed in the same terms. According to Regnault's measurements of the thermal capacity of water at different temperatures from 0° to 230° C, it must be about 0.8° per cent. greater at 60° Fahr. than at 32°. According to this, Joule's thermodynamic result would be 771.81 Manchester foot-pounds, for the work required to warm a pound of water from 32° to 33° Fahr., or 1389.26 to warm a pound of water from 0° to 1° C. Reducing 1389.26 feet to metres, we have 423.437 metres. At Paris the force of gravity is about 156.50 per cent, less than in Manchester. Hence for about the middle of France and the southern latitudes of Germany, Joule's result, according to the ordinary reckoning of French and German engineers, may be stated as 423.5 kilogramme-metres for the amount of work required to warm 1 kilogramme of water from 0° to 1° C. The force of gravity at Manchester is 581.94 dynamical centimetre-seconds per second. Multiplying 423.437 by this, we find accordingly 4,553,000 centimetre-dynes, or "ergs," for the amount of work in C.G.S. measure required to warm 1 gramme of water from 0° to 1° C.

THERMODYNAMICS.

§ 10. Preliminary for Thermometry.—Sense of Heat (resumed from § 1).—The sense of heat and cold is not simply dependent on the temperature of the body touched. If a person takes a piece of iron, or a stone, or a piece of wood, or a ball of worsted, or a quantity of finely carded cotton-wool, or of eider down, in his hand, or touches an iron column, or a stone wall, or a wooden beam, or a mass of wool or of down, he will perceive the iron cold, the stone cold, but less cold than the iron, the wood but slightly cold—much less cold—much less cold than the stone, the wood or down decidedly warm.

We now know that if all the bodies before being touched were near one another in similar exposure, they must have been at the same temperature, and from the iron and stone being felt cold we know that this mean temperature is lower than the temperature of the hand. Each of the bodies touched must at the first instant have taken some heat from the hand, and therefore, if the perception were quick enough, all at the very instant of being touched would have seemed cold to the sense. The iron by its high thermal conductivity (§ 76, 78, 80) keeps drawing off heat from the hand and lowering its temperature, till after many seconds of time an approximately permanent temperature is reached, which may be consider-ably lower than the temperature of the hand before contact, but somewhat higher than the previous temperature of the iron, because of the internal furnace (see THERMODYNAMICS) existing that in the iron. In the similar result, but in less time and with less ultimate lowering of temperature of the hand, takes place when stone is touched. When wood is touched it comparatively small conductivity (§ 76) allows its surface to be warmed again after the first few seconds, sometimes to a higher temperature of the hand before contact, but somewhat higher than the previous temperature of the iron, because of the internal furnace (see THERMODYNAMICS) existing that in the iron. If we say that the temperature of the hand was considerably lower than the iron before contact, and that the iron was warmed to a higher temperature than the hand, we should be mistaken, for the temperature of the hand is 32° Fahr. (100° Centigrade), and of the iron is 104° Fahr. (40° Centigrade), and the former is higher than the latter. The iron is much more conducting than wood, which is much more conducting than stone, and the heat of the hand content is transferred from the hand to the more conducting iron, but not so much to the still more conducting wood, and not at all to the stone. We now know that if all the bodies before being touched ware near one another in similar exposure, they must have been at the same temperature, and from the iron and stone being felt cold we know that this mean temperature is lower than the temperature of the hand. Each of the bodies touched must at the first instant have taken some heat from the hand, and therefore, if the perception were quick enough, all at the very instant of being touched would have seemed cold to the sense. The iron by its high thermal conductivity (§ 76, 78, 80) keeps drawing off heat from the hand and lowering its temperature, till after many seconds of time an approximately permanent temperature is reached, which may be considerably lower than the temperature of the hand before contact, but somewhat higher than the previous temperature of the iron, because of the internal furnace (see THERMODYNAMICS) existing that in the iron. If we say that the temperature of the hand was considerably lower than the iron before contact, and that the iron was warmed to a higher temperature than the hand, we should be mistaken, for the temperature of the hand is 32° Fahr. (100° Centigrade), and of the iron is 104° Fahr. (40° Centigrade), and the former is higher than the latter. The iron is much more conducting than wood, which is much more conducting than stone, and the heat of the hand content is transferred from the hand to the more conducting iron, but not so much to the still more conducting wood, and not at all to the stone.
perature than that of the hand before contact; and thus, if the sensation could be perfectly remembered, it would be perceived that the wood was first felt to be cold, and afterwards to be warm. This latter warmth is rendered

be perceived that the wood was first felt to be cold, and

if the sensation could be perfectly remembered, it would

be perfectly abundant supplies of iced water and of

peratures than that of the hand before contact. • and thus,

removing it; a sense of cold is immediately perceived in

some time in the air, it be placed on a varnished wooden

consequence of the exposure of the hand to the air. The

sits down, and holds his feet in the air, the air seems to

some time with it pressed to the floor, the foot be suddenly

be warm by contrast. The same sensations are perceived

walks with bare feet on a wooden floor, a continued sense of

is very startling to find a body which we know to be ice-

is attained by the hand than it had before contact.

in a small fraction of a second of time a higher temperature

ferences of temperature than that, and, what is still more re-

attendants, and persons occupied with hot liquors in various

manufactures, such as dyeing, can detect much smaller dif­

water will, as we have found by experiment, detect a differ­

point. It is described here, not only because it is very

commonest thermometers in popular use being in fact

scientific investigation. In many cases the hand is a more

convenient and easy test than a common mercury thermo­

method would give more accurate results than many (§ 68)

common thermometers sold by instrument makers for ordi­

norality a somewhat delicate thermal test when properly used.

Even an unskilled hand alternately dipped into two basins of

water, as we have found by experiment, detect a differ­

temperature of less than a quarter of a degree centigrade; and there can be no doubt that bath and hospital

attendants, and persons occupied with hot liquors in various

manufactures, such as dyeing, can detect much smaller differ­

ences of temperature than that, and, what is still more re­

markable, can remember permanently sensations of absolute

temperature sufficiently to tell within less than a degree

centigrade that the temperature of a bath, or a poultice, or dyeing liquor is “blood heat,” or “fever heat,” or some other

definite temperature to which they have been accustomed.

§ 12. Thermometry by Sense of Heat—with arbitrary

Centigrade Scale deduced from Mixtures of Hot and Cold

Water.—Without knowing anything of the nature of heat

we might found a complete system of thermometry on the

mixing of hot and cold water with no other thermoscope

(§ 13) than our sense of heat, if we had but two definite

constant temperatures of reference. These in practical

thermometry are supplied by the melting-point of ice and the

temperature of steam from water boiling in air at a definite

pressure (the “atm” or standard atmospheres, § 5). Thus,

suppose perfectly abundant supplies of cold water and of

water at the boiling temperature to be available, and suppose

it to be desired to measure the temperature of a river, or

lake, or sea. Take measured quantities of the boiling and

of the ice-cold water, and mix them by trial until, tested by

the hand, the mixture is found to have the same temperature

as that of the mass of water of which the temperature is to

be determined. Suppose, for example, the mixture giving

the required temperature to consist of 86-6 parts by weight

of ice-cold water, and 13-4 parts by weight of boiling water; An

the required temperature is 13-4 on a perfectly definite arbitrary

scale of thermometry in which the temperature of ice-cold centi-

grade water is called zero, that of boiling water 100, and other

temperatures are reckoned according to the law of propor-

tion of mixtures of water in the manner indicated by the on-

example, and defined generally in § 81 below. For tem-

peratures within the range of sensibility of the hand this

method would give more accurate results than many

§ 86)

of water to be constant.
pended on difference of temperature between the two places, and showing zero continuously when the temperatures of the two places are varied, provided they are kept exactly equal. Every kind of differential thermoscope, and of continuous intrinsic thermoscope, must be founded on some property of matter, continuously varying with the temperature, as density of a fluid under constant pressure, pressure of a fluid in constant volume, volume of the liquid part of a whole mass of liquid and solid kept in constant volume, steam-pressure of a solid or liquid, shape or density of an elastic solid under constant stress, viscosity of an elastic solid in a constant state of strain, viscosity of a fluid, electric current in a circuit of two metals with their junctions at unequal temperatures, electric resistance of a conductor, magnetic moment of a steel or loadstone magnet.

Examples:—(1) Leslie’s differential air thermometer ; (2) steam-pressure differential thermometers (§§ 39–44 below) ; (3) Joule’s hydraulic and pneumatic differential thermoscopes (Memoirs Chem. Soc., vol. iii. p. 201; Proc. Lond. and Phil. Soc., Manchester, vol. iii. p. 73; Ibid., vol. vii. p. 35) ; (4) viscosity differential thermoscope (Proceedings R. S. E., April 5, 1880) ; (5) thermo-electric differential thermometer ; (6) Siemens electric resistance differential thermometer ; (7) thermo-magnetic differential thermometer (see Proceedings R. S. E. for April 5, 1880).

§ 14. Intrinsic Thermoscopes.—An intrinsic thermoscope is an instrument capable of indicating one definite temperature or several definite temperatures, or all temperatures within the range of the instrument, whatever it may be—the temperature or temperatures indicated being intrinsically determined by the constitution of the instrument and indicated by some recognizable feature of the instrument which changes discontinuously or continuously, as the case may be, and which is always the same when the instrument is brought back again and again to the same temperature, whatever changes it may have experienced in the intervals. Discontinuous intrinsic thermoscopes show only a limited number of temperatures. A continuous intrinsic thermoscope shows any temperature whatever throughout the range of efficiency of the instrument, ideally any temperature whatever, though in practice every thermoscope is limited, some with both inferior and superior limit, as the mercury thermometer by the freezing of mercury at about −39° C, and the bursting of the vessel by the melting of the solder when the temperature reaches a certain limit ; also an exceedingly useful guard against overheating in the flue of a stove, by which a stopper is allowed to fall by the melting of the solder when the temperature reaches a certain limit ; also an exceedingly useful guard against overheating in the flue of a stove, by which a stopper is allowed to fall by the melting of the solder when the temperature reaches a certain limit ; also an exceedingly useful guard against overheating in the flue of a stove, by which a stopper is allowed to fall by the melting of the solder when the temperature reaches a certain limit ; also an exceedingly useful guard against overheating in the flue of a stove, by which a stopper is allowed to fall by the melting of the solder when the temperature reaches a certain limit ; also an exceedingly useful guard against overheating in the flue of a stove, by which a stopper is allowed to fall by the melting of the solder when the temperature reaches a certain limit ; also an exceedingly useful guard against overheating in the flue of a stove, by which a stopper is allowed to fall by the melting of the solder when the temperature reaches a certain limit.

The accuracy of an intrinsic thermometer, whether discontinuous or continuous, depends upon permanence of quality of the material and of the mechanical constitution of the instrument, according to which the recognized feature shall always be very accurately the same for the same temperature. The sensibility or delicacy of a continuous intrinsic thermometer depends upon the recognizability of change in its indicating feature with very small change of temperature.

§ 17. The property of matter chosen as the foundation of almost all ordinary continuous intrinsic thermoscopes in common use is interdependence of the density, the temperature, and the pressure of a fluid. The only other thermoscopes which can be said to be in common use at all are “metallic thermometers” (see Thermometer; Matter, Properties of) ; these depend upon the change of shape of a rigid elastic solid under a stated stress, or on the change of shape of a compound solid, composed of two elastic solids of different substances melted or soldered together. For the present we confine our attention to the former and much larger class of instruments. The general type of all those instruments, except the steam-pressure thermometer (§§ 39–46 below), is a glass tube, measuring the bulk of a fluid. To give the requisite practical sensibility to the measurement, the glass, except for the ease of the constant-pressure gas thermometer (§§ 64–67 below) and of the steam-pressure thermometer, is made of a shape which may be generally described as a bottle with a long narrow neck. The body of the bottle, which may either be spherical or of an elongated form, is called the bulb, and the neck is called the tube or stem (stem we shall most frequently call it, to obviate ambiguities without circumlocutions). The thermometric fluid may be all liquid, as mercury, or oil, or alcohol, or other, or glycerine and water ; or it may be all gas, as common air, or hydrogen, or carbonic acid ; or it may be partly liquid and partly steam (steam being a name which we shall invariably use to designate the less dense portion defined of a fluid substance at one temperature and pressure throughout, and in equilibrium, with two parts of different densities). This last case is different from the two preceding, in respect to the character of the thermometric indication : the whole volume of the thermometric substance may be changed from that of all liquid to that of all steam without changing the temperature or the pressure, and the pressure cannot be changed without changing the...

1 This is the principle of the ordinary mercury or spirit thermometer.

2 For definition of steam see § 17 below.
Manometric thermometry.

iment, provided the substance is kept in the double condition of part liquid and part steam; in other words, in this case the pressure depends upon the temperature alone and is independent of the volume. In the steam-pressure thermometer, therefore, there is no need of measuring of volume of the thermometric substance, and the vessel containing it is not in the shape of bulb and stem; but the instrument consists essentially of a means of measuring the pressure of the thermometric substance, with a test that it is really in the twofold condition of part liquid and part steam, whether by seeing it through a glass containing-vessel, or by a proper hydraulic appliance for ascertaining that the pressure is not altered by rarefaction or condensation when the temperature is kept constant. Realized thermometers of this species, quite convenient for many practical purposes, with steam of sulphuric acid, of water, and of mercury, to serve for different ranges of temperature, from below $-30^\circ$ C. to above $+520^\circ$ C., are described in §§ 33-44 below.

§ 18. In respect to general convenience for large varieties of uses, whether for scientific investigation, or for the arts, or for ordinary life, liquid thermometers are generally and with good reason preferred; but the general preference of either mercury or spirits of wine for the liquid, which is so much the rule, is not (§ 20 below) so clearly reasonable. For ordinary uses in which the thermometer has to be moved about and placed in various positions, gas thermometers are much less convenient, because they require essentially an accurate measurement of pressure, and generally for this purpose a column of liquid. But when the thermometer is to be kept always in one position, as for instance when it is devoted to testing the temperature of the air indoors or out of doors, Amonton's air or gas thermometer is really as convenient and as easily read as any liquid thermometer can be: but even it, simple as it is, involves a triple division of the hermetically sealed space, with three different conditions of occupation, one part occupied by the thermometric substance, another by the pressure-measuring vapourless liquid, and the third vacuous: and it is by so much the less simple than the liquid thermometer that in the liquid thermometer the enclosed space is divided into only two parts, one occupied by the thermometric liquid, and the other by its steam, with

1 An instrument closely resembling that shown in the drawing (fig. 2), but with common air instead of hydrogen, was made for the writer of this article, by Casella, about fifteen years ago, and has been used for illustrations in the natural philosophy class in Glasgow university ever since. If it is probably an exceedingly accurate air thermometer. When it was set up in the new lecture-room after the migration to its present locality in 1870, the tube above the manometric liquid column was cleared of air. To do this the instrument must be held in such a sloping position, with the closed end of the tube down, as to allow the bubble of air always found in it to rise and burst in the bulb. If now the instrument is placed in its upright position, the liquid refuses to leave the top of the tube, and it would remain filling the tube (probably for ever) if left in that position. No violence of knocking which has been ventured to try to bring it down has succeeded. To bring it down a bubble of air must be introduced. The bubble must be very small, so that the pressure of the air which fills it may become insensible when this air expands into the space of the tube left above the manometric column after it descends to its proper thermometric position.

2 Special experiments made for this article in September 1879 showed that in the nine years during which the instrument had remained undisturbed in the lecture-room a very sensible quantity of air (enough to produce a temperature difference of about $30^\circ$ C., too low had been introduced from the tube through the sulphuric acid into the tube above the liquid column. This air was eliminated, and the instrument rest set up for use, an operation completed in a minute at any time it need be. Some careful experiments were then made by Mr Macfarlane to ascertain if the pressure of vapour or gas from the sulphuric acid, in the tube, was sensible, with a happily decisive result in the negative. The bulb was kept at a very constant temperature below $0^\circ$ C. for a few months, and the whole of the tube above it were heated to about $100^\circ$ C. by steam blown through a glass jacket-tube, fitted round it for the purpose. The height of the manometric column varied, sometimes to as much as $\frac{1}{2}$ of a degree C. in its freezing point or in its indication at any other absolutely definite temperature, within some inches. If, as from $-20^\circ$ to $-10^\circ$ C. to $+30^\circ$ or $+40^\circ$ C. But the abrupt and irregular changes, produced by exposing the thermometer to temperatures much above or much below some such limited range as that, constitute a very serious difficulty in the way of accurate thermometry by the mercury-in-glass thermometer.
Although the greatest care has been bestowed by Eegnault, Joule, and all other accurate thermometric experimenters to avoid error from this cause, we have still but little definite information as to its natural history. In thermometers of different qualities of glass, different shapes of bulb, and constructed differently in respect to processes of glass-blowing, boiling the mercury, and sealing the stem. We do not even know whether the excess of the atmospheric pressure outside the bulb over the pressure due to mercury and Torricellian vacuum inside is influential sensibly, or to any considerable degree, in producing the gradual initial shrinkage. If it were so we might expect that the effect of heating the thermometer up to 100° or more at any time would be rather to produce an accelerated shrinkage for the time than what it is found to be, which is a return towards the original larger volume, followed by gradual shrinkage from day to day and week to week afterwards. A careful comparison between two thermometers constructed similarly in all respects, except sealing one of them with Torricellian vacuum and the other with air above the mercury, would be an important contribution to knowledge of this subject, interesting, not only in respect to thermometry, but also to that very fundamental question of physical science, the imperfect elasticity of solids (see Elasticity, § 4).

§ 20. The error of a thermometer due to irregular shrinkages and enlargements of the bulb is clearly the less the greater is the expansion of the thermometric fluid with the given change of temperature. By the investigation of § 30 we can calculate exactly how much the error is for any stated amount of abnormal change of bulk in the bulb. But it is enough at present to remark that for different liquids in the same or in similar bulbs the errors are very nearly in the same proportion as the specific expansions of the liquids. Now (Table III. below) in being warmed from 0° to 1°C. alcohol expands 6 times as much as mercury, methyl butyrate 7 times, and sulphuric ether 8½ times. Hence if irregular changes of bulk of the bulb leave, as they probably do in practice, an uncertainty of $\frac{1}{4}$ths of a degree in respect to absolute temperature by the best possible mercury-in-glass thermometers used freely at all temperatures from the lowest up to 100°C, the uncertainty from this cause will be reduced to $\frac{1}{8}$th of a degree by using alcohol, or $\frac{1}{12}$th by using methyl butyrate instead of mercury; it may therefore, in a glass thermometer of alcohol or of the butyrate, be considered as practically annulled (§ 19) after a few weeks or months have passed, and the first main shrinkage is over.

§ 21. An alcohol-in-glass thermometer is easily made strong enough to bear a temperature of 100°C, as this gives by the pressure of the vapour an internal bursting pressure of not quite an atmosphere and a quarter in excess of the atmospheric pressure outside. The boiling point of methyl butyrate (Table III. below) is 102°C; a thermometer of it may therefore be used for temperatures considerably above 100°C, but how much above we cannot admit to be so. It is that when the temperature is rapidly sink-

1 Phil. Trans. for 1860, p. 325.

§ 22. Liquids which wet the glass have another great advantage over mercury in their smaller capillary attraction (see Capillary Action, vol. v. p. 56; and Matter, Properties of) and in the constancy of their 180° angle of contact with the glass, instead of the much greater absolute intensity of capillary attraction in the mercury, and its very variable angle of contact, averaging about 45° when the mercury is rising, and about as much as 90° when it is falling. On account of these variations the bulb of the mercury thermometer is subjected to abrupt variations of pressure when the mercury is rising or falling. The greatest and least pressures due to this cause are experienced when the angle of contact is respectively least and greatest, and differ by the pressure due to a vertical column of mercury equal in height to the difference of depressions of mercury in a capillary tube of the same bore as the thermometer stem when the angle of contact is changing from one to the other of the supposed extreme values. Hence the mercury in a thermometer rises and falls by jerks very noticeable in a delicate thermometer when looked at with a lens of moderate magnifying power, or even with the naked eye. Dr Joule informs us that this defect is much greater in some thermometers than in others, that he believes it is greatly owing to the tube being left unssealed for too long a time after the introduction of the mercury (by which it is to be presumed something of a film of oxide of mercury is left on the glass to reappear on the surface of the mercury when it sinks as it cools after the sealing of the end). In Joule's own thermometers not the smallest indication has ever been detected of what he calls "this untoward phenomenon, which is calculated to drive an observer mad, if he discovers it towards the close of a series of careful experiments." Their admirable quality in this respect is no doubt due to the great care taken by the maker, Mr Dancer, under Joule's own instructions, to have the mercury and the interior of the bulb and tube thoroughly clean, and to guard it from contact with any "matter in its wrong place" until completion of the sealing. But no amount of care could possibly produce a mercury thermometer of moderate dimensions moving otherwise than by jerks of ever so many divisions, if its stem were of fine enough bore to give anything approaching to two or three hundred divisions to the centigrade degree.

§ 23. One chief objection to the use of alcohol or other volatile liquid for the thermometric substance in ordinary glass thermometers is the liability to distillation of some of the liquid into the stem and head reservoir, unless the glass above the level of the liquid be kept at least as warm as the liquid. On this account a spirit thermometer is not suitable for being plunged into a space warmer than the surrounding atmosphere with the stem simply left to take the temperature to which it comes in the circumstances. But for elaborate experiments using ordinary thermometric purposes, there is little difficulty in arranging to keep the port of the stem which is above the liquid surface somewhat warmer than at the liquid surface, and this suffices absolutely to prevent the evil of distillation. The only other objection of any grave validity2

2 There is one other objection which, though often stated as very grave against the thermometric use of any other liquid than mercury, we do not admit to be so. It is that when the temperature is rapidly sink-
against the use of highly expansive liquids instead of mercury is the difficulty of allowing for the expansion of the liquid in the stem, if it is not at the same temperature as the bulb. With the same difference of temperatures in different parts of the instrument, the error on this account is clearly in simple proportion to the expansibility of the liquid; and therefore, the residual error due to want of perfect accuracy in the data, for the allowance will, generally speaking, be greater with the more expansive than with the less expansive liquid. But in every case in which the bulb and stem can all conveniently be kept at one temperature, a thermometer having for its thermometric substance some highly expansive oil or alcohol or ether, or other so-called organic liquid of perfectly permanent chemical constitution, cannot but be much more accurate and sensitive than the mercury thermometer, which has hitherto been used almost exclusively in thermometric work of the highest rank. We shall see (§§ 62 and 64–65) that the ultimate standard for thermometry, according to the absolute thermodynamic scale (§ 34), is practically attained by the use of hydrogen or nitrogen gas as the thermometric substance, but that for ordinary use a gas thermometer can scarcely be made as convenient as one in which the thermometric substance is a liquid. For practical thermometry of the most accurate kind it seems that the best plan will be to use as ordinary working standard thermometers highly sensitive thermometers constructed of some chosen "organic" liquid, and graduated according to the absolute thermodynamic scale, by aid of the thermodynamically corrected air thermometer (§ 62) used as ultimate standard of reference. The great convenience of the mercury thermometer in respect to freedom from liability to distillation and smallness of error on account of temperature differences between the bulb and stem renders it the most convenient for a large variety of scientific and practical purposes in which the most minute accuracy or the most extreme sensibility is not required.

§ 24. Without any thermodynamic reason for preferring air to mercury as thermometric fluid, Regnault preferred it for two very good reasons. (1) Its expansion is 20 times that of mercury and 160 times the cubical expansion of glass, and therefore with air the error due to irregularity in the expansion of the glass is 20 times smaller than with mercury, and small enough to produce no practical deflection from absolute accuracy in thermometry, as he found by elaborate and varied trials. So far as this is concerned, some highly expansive organic liquid would appear as well as air for thermometric fluid, and would have the advantage of giving a thermometer much more easily used. (2) For an ultimate standard of reference air has the advantage over organic liquids generally, that different samples of it taken at different times, or in different parts of the world, and purified of water and carbonic acid by well-known and easily practised processes, are sufficiently uniform to give thermometric results between which the accordance is practically perfect, provided the thermometric plan according to which the different samples are used is the same, or as approximately the same as is easily secured in practice. Two plans for the thermometric use of air naturally present themselves:—(1) augmentation of volume of air kept in constant pressure; and (II.) augmentation of pressure of air kept in constant volume. Regnault tried both plans, but found that he could only arrange his apparatus to give good results by the second, and on it therefore he founded what he called his "normal air" thermometer. For the sake of perfect definiteness he chose, as the density of the air in his normal thermometer, the density which air has when at the temperature of melting ice and under the pressure of one atmosphere. He adopted the centigrade scale in respect to the marking of the freezing and boiling points by 0° and 100°; and the principle which he assumed for the reckoning of other temperatures was to call equal those differences of temperature for which differences of pressure of the air in his normal thermometer are equal. Thus he was led to a definition of temperature expressed by the following formula:

\[ t = \frac{100 \, P - \Pi}{P_{100} - \Pi}, \quad \ldots \ldots \quad (1) \]

where \( \Pi \) denotes the pressure one atmosphere, and \( P \) and \( P_{100} \) the pressures of the air of the normal thermometer at the temperatures denoted by \( t \) and by 100 respectively, the latter being the temperature of steam issuing from water boiling under the pressure of one atmosphere. By the most accurate observations which he could make, Regnault found for his "normal air" \( P_{100} = 1.2665 \times 11 \). Hence his thermometric formula becomes

\[ t = \frac{100 \, P - \Pi}{P_{100} - \Pi} = \frac{272.85 \, P - 1}{11 \, P} \quad \ldots \ldots \quad (2) \]

§ 25. Regnault compared with his normal air thermometers on the same plan of constant volume, but with air at other than the normal density of 1 atmosphere, and with other gases than air; also air and gas thermometers on the plan of constant apparent volume as measured in a glass bulb and stem; also a thermometer founded simply on the dilatation of mercury; and also thermometers of mercury in different kinds of glass, each divided on the glass stem with divisions corresponding to exactly equal volumes of the bore; also overflowing thermometers (thermomètres à déversement), in which a bulb with a short piece of fine stem was perfectly filled with mercury at 0° and the quantity of mercury expelled by the high temperature to be measured was weighed, instead of being thermometrically measured by divisions of a long stem as in the ordinary thermometer.

The whole of this thermometric investigation is full of scientific interest, and abounds with results of great practical value in respect even of the minutest details of Regnault's work. It will be found fully described in the first of his three volumes, entitled Relation des Experiences Machines à Vapeur, which were published at Paris in 1847, 1862, and 1870. Here we can but state some of the most important of the general conclusions:

(1) The air thermometers with pressure at 0° of from 44 to 149 centimetres of mercury agreed perfectly with the normal air thermometer calculated according to the same formula (2), and nearly the same numerical coefficient 272.85. A slightly larger value 272.98 (or 0036635) gave the best agreement for the 44 c.m. pressure, and the somewhat smaller value 272.7 (or 003667) for the pressure 149 c.m.

(2) The hydrogen gas thermometer, with pressure one atmosphere at 0°, and with its indications calculated according to
formula (2) but with a different numerical coefficient,\(^1\) agreed perfectly with the normal air thermometer from 0\(^\circ\) to 325\(^\circ\).

(3) The carbonic acid gas thermometer with pressure 46 c.m. at 0\(^\circ\), and its indications calculated with the coefficient 271·59 (003632\(^{-1}\)), agreed perfectly with the normal air thermometer from 0\(^\circ\) to 305\(^\circ\).

(4) The carbonic acid gas thermometer with pressure, at 0\(^\circ\), 74 centimetres (or nearly 1 atmo), calculated with the coefficient 270·64 (003693\(^{-1}\)) to make it agree with the normal air thermometer at 100\(^\circ\), gave numbers somewhat too large for all temperatures from 200\(^\circ\) to 323\(^\circ\).

The difference seemed to rise to a maximum at about 180\(^\circ\), when it was about \(\frac{1}{2}\), and to diminish so as to be only about \(\frac{1}{2}\) at the highest temperatures of the comparison. Two sulphuric acid gas thermometers, with pressures 59 c.m. and 75 c.m. at 0\(^\circ\), calculated with coefficients 263·6 (003794\(^{-1}\)) and 261·4 (003825\(^{-1}\)) respectively to make them agree at 100\(^\circ\) with the normal air thermometer, each gave numbers too small for the higher temperatures by differences gradually increasing up to \(2\frac{1}{2}\) at 350\(^\circ\) in the case of Choisi le Roi crystal, a hard glass without lead, and to as much as \(3\frac{1}{2}\) in the case of ordinary glass.

In connexion with these observations Regnault remarks that the greatest cause of uncertainty in his air thermometer is the allowance for expansion of the glass. It was only by most carefully made special experiments \(^2\) on each individual bulb and tube that he succeeded in obtaining the great accuracy which we find in his results, according to which the probable error, whether by his normal air thermometer, or by other air or gas thermometers of those stated above to agree with it perfectly, was not more than from \(-1\) to \(-1\frac{1}{2}\) of a degree for any temperature up to 350\(^\circ\).

(6) The mercury-in-glass thermometers which Regnault generally used for comparison with his normal air thermometer were overflowing thermometers, because he found that with such he could more easily obtain the very minute accuracy at which he aimed than with the ordinary volumetric thermometers; but the former thermometer was adapted to give exactly the same result as would have been obtained by the ordinary thermometer with divisions on the stem corresponding to equal volumes of the bore. It must be remembered, however, that this perfect agreement between the volumetric and overflowing thermometers would not be found unless the expansion of the bulb and tube were uniform and isotropic throughout.

(7) The general results of Regnault's comparisons of mercury thermometers with his normal air thermometer were given by himself in a diagram of curves from which the accompanying is copied on a reduced scale (fig. 3). It shows that at a temperature of 320\(^\circ\) the independent mercury thermometer stands 329·8\(^\circ\), the thermometer of mercury in Choisi le Roi crystal at 327·25\(^\circ\), and the thermometer of mercury in ordinary glass at 321·8\(^\circ\); and that the independent mercury thermometer and the mercury in Choisi le Roi crystal stand 10\(^\circ\) higher than the normal air thermometer at the temperatures by it of 323\(^\circ\) and 345\(^\circ\) respectively.

§ 26. The curve for the independent mercury thermometer is merely Regnault's graphic representation of his experiments on the absolute expansion of mercury (Relation des Expériences, vol. i. p. 328). It shows that the addition of bulk given to the same mass of mercury by raising its temperature by elevation of temperature is for the same difference of temperatures as indicated by his normal air thermometer regularly greater and greater the higher the temperature.

§ 27. It is interesting to see by the diagram that at the high temperatures all the mercury thermometers keep nearer to the air thermometer than does the independent mercury thermometer, and that the mercury in ordinary soft glass keeps much nearer to the air thermometer than does the mercury in the hard Choisi le Roi glass. We infer that, still reckoning temperature by the air thermometer, we have regular augmentation of expansion at the high temperatures in all the different glasses, each greater than the augmentation of expansion of mercury, and that this augmentation is greater in the soft ordinary glass than in the hard Choisi le Roi glass, being in the ordinary glass great enough to overcompensate in the resulting thermometric indication the augmenting expansion of the mercury from 100\(^\circ\) to 245\(^\circ\); while above 245\(^\circ\) in the ordinary glass thermometer, and at all temperatures above 100\(^\circ\) in the Choisi le Roi thermometer, the compensation is only partial. Between 0\(^\circ\) and 100\(^\circ\) the independent mercury thermometer stands regularly lower than the air thermometer by as great a difference as \(-35^\circ\) at 50\(^\circ\), where it is a maximum. The curves for the mercury-in-glass thermometers are not shown between 0\(^\circ\) and 100\(^\circ\), but it is clear from the diagram that the Choisi le Roi thermometer must, like the independent mercury thermometer, stand lower than the air thermometer, but by a smaller difference, probably only about \(2^\circ\) at 50\(^\circ\); and the ordinary glass thermometer higher than the air thermometer from 0\(^\circ\) to 100\(^\circ\) by a difference which may be \(2^\circ\) or \(3^\circ\) at 50\(^\circ\). This last inference from the diagram is confirmed by Regnault's table of results facing page 227 of his first volume.

§ 28. In the best modern thermometers the graduations are actually engraved on the glass, but in most popular thermometers, and in many for scientific investigation, they are on an attached scale, Fig. 4, of wood, or ivory, or brass, or paper. Some of the best popular thermometers are the German bath thermometers, in which the graduation is on a paper scale guarded by being enclosed in a wide glass tube hermetically sealed round the stem and over the bulb of the glass which contains the mercury, in the manner shown in fig. 4.
The graduation is clearer and more easily read in this kind of thermometer than in any other. The complete protection of the paper; and as we shall keep the damage afforded by its hermetically sealed glass envelope gives a perennially enduring quality to this form of thermometer,\(^1\) such as is possessed by no others except those graduated, on the glass; and the lightness of the paper renders its proper attachment to the inner stem, by gum or otherwise, thoroughly trustworthy, when once well done by the maker of the instrument. For scientific purposes the paper scale was too cheap, and common, and good, to satisfy the ideas of those instrument makers who in Germany and France substituted the heavy graduated slab of opal glass for the paper, while still adhering to the bath thermometer pattern in hermetically enclosing this scale in an outer containing glass tube—very unnecessarily, as the glass scale, unlike the paper scale, does not require any such protection.

This is now, however, a thing of the past. At the present time all high-class scientific thermometers are graduated on the glass of the stem without any attached scale of other material. Except in respect to ease of reading the indications this simplest form is, both for popular and for scientific purposes, superior even to the German bath thermometer with hermetically sealed paper scale; and this will be the form intended when we speak of a mercury thermometer, or a spirit thermometer, or a liquid thermometer, without any special qualifications.

§ 30. To define exactly the indications of a thermometer founded on the expansion of a fluid, let the volume of the bore of the stem between two consecutive divisions be called for brevity a degree measure. The degree measure is habitually made as nearly possible equal throughout the scale by the manufacturer, and as we shall see (§ 65), it ought to be so in an air thermometer to give indications agreeing with the absolute thermodynamic scale nearly enough for the most accurate practical thermometry. But in practical spirit-thermometers the divisions are made to correspond as nearly as may be to degrees of a standard mercury or air thermometer, and the degree measures are therefore (Table II. below) larger and larger from the lower to the upper end of the scale. For the purpose, however, of comparing the thermometric performance of different liquids, we shall suppose the degree measure to be of equal volume throughout the scale in each case.

Let \( N \) be the number of degree measures contained in the volume of the bulb and stem up to the point marked zero on the scale; and let \( D_i \) denote the volume, at any temperature \( t \), of the degree measure reckoned in absolute units of volume. The volume of the bulb and stem up to zero will be \( N D_0 \). On the supposition of perfect isotropy and freedom from strain in the glass, \( N \) will be independent of the temperature and \( D_i/D_0 \) will be the ratio of the volume of any portion of the glass at temperature \( t \) to its volume at the temperature called zero. If \( D_j \) denote the volume of the degree measure when the glass is at this zero temperature. Let now \( L_0 \) and \( L_t \) denote the volumes of the whole liquid in a thermometer at the two temperatures \( t \) and 0; we have \( L_t = N D_j \). And if \( s \) be the number of scale divisions marking the place of the liquid surface in the thermometer tube, we have \( L_t = (N + s)D_0 \). Hence

\[
L_t/L_0 = (1 + s/N)D_j/D_0.
\]

Hence, if \( E_i \) denote augmentation of bulk of the liquid, and \( E_j \) augmentation of bulk of each degree-division of the stem, when temperature is raised from 0 to \( t \), each reckoned in terms of the bulk at zero temperature, we have

\[
s = N \left( \frac{1 + E_j}{1 + E_i} - 1 \right) - N E_j - E_i.
\]

This is the formula for the ordinary liquid thermometer. It is also applicable to the constant pressure air thermometer, in which, with proper instrumental means to keep the pressure constant, the air expands or contracts with elevation or depression of temperature, and its volume is measured in a properly shaped glass measuring vessel. We may arbitrarily determine to take \( s \) as the numeric for the temperature which is indicated by any one particular thermometer of this kind, for instance, a methyl butyrate thermometer, or an alcohol thermometer, or a mercury or an air thermometer. But if \( s = t \) for any one individual thermometer, it cannot be exactly so for any other. In the first advances towards accurate thermometry it was taken so for the mercury-in-glass thermometer, and by general consent it was continued so until it was found (§ 25) that different mercury-in-glass thermometers cannot be made so near, each to each, in accuracy, differ largely in their reckonings of temperature.

§ 31. Numerical Thermometry.—In § 12 above, a perfectly definite and very simple basis for numerical thermometry was described, not as having been adopted in practice, but defined as an illustration of a very general principle upon which all reckoning of temperature may be done in numbers. The specific principle is this. Two definite temperatures depending on properties of some particular substance or substances are first fixed upon and marked by two arbitrary numbers,—as, for instance, the temperature of melting ice marked zero, and the temperature of steam issuing from boiling water, under atmospheric pressure of exactly one atmosphere, marked 100. Then, if an intermediate temperature \( t \) is obtained by taking \( t \) parts of water at 100° and \( (100 - t) \) parts at 0° and mixing them together. As said in § 12 this method is limited to temperatures at which liquid water can be obtained, and therefore practically it is only applicable between the melting point of ice and the boiling point of water, under ordinary atmospheric pressure.

§ 32. Any other liquid of permanent chemical constitu-
Heat

Equally definitions of numerical reckoning of temperature have been given in §§ 12, 24, 25, 30, 31, 32, several distinct definitions of numerical reckoning of temperature are to be called equal are defined specially, and this is the essence of the thermometric scale in each case (the marking of 0° and 100° for the "freezing" and "boiling" points being common to all as a matter of practical usage and not an essential of the thermometric principle in any case). Thus in §§ 12, 31, and 32 differences of temperature are called equal which are produced by the communication of equal quantities of heat to a given quantity of another substance, or to the thermometric substance itself, for example, or mercury; in other words (§ 68 below), this thermometric system is chosen so as to make the specific heat of a particular thermometric substance the same for all temperatures. Again in § 24 differences of temperature are called equal for which the difference of pressure are equal in air of the particular density which air has if its pressure is one atmo when its temperature is "freezing." This is Regnault's "normal thermometry." In § 25 (1), (2), (3), (4), other reckonings of temperature differing essentially from this, though, as Regnault's experiments proved, by very small differences, are given simply by the substitution of air of other than Regnault's normal density, and of other gases than air, for the air of Regnault's normal thermometer. In § 25 (5) a thermometry founded on a complex coefficient of change of pressure and volume of a gas and change of volume of some one particular glass vessel is defined and compared with Regnault's normal thermometry; and in § 25 (6) and (7) the same is done for the ordinary mercury-in-glass thermometer, which depends on a coefficient of glass and mercury leading to the reckoning of temperature defined in § 30. Again in § 26 and § 24 (I.) is indicated a system of thermometry founded on the absolute dilatation of some fluid, such as mercury or alcohol or butyrate of oxide of methyl or other permanent liquid or air, at some constant pressure, such as one atmo, with equal differences of temperature defined as those which give equal dilatations of the particular substance chosen as the thermometric fluid.

§ 34. Each of all these different definitions of temperature is founded on some particular property of a particular substance. A thermometer graduated to fulfill one of the definitions for one particular substance would not agree with another thermometer graduated according to the same definition for another substance, or according to some of the other definitions. A much more satisfactory foundation for thermometry is afforded by thermodynamic science, which (see Thermodynamics) gives us a definition of temperature depending on certain thermodynamic properties of matter in such a manner that if a thermometer is graduated according to it from observation of one class of thermal effects in one particular substance, it will agree with a thermometer graduated according to the same thermodynamic law from the same class of effects in any other substance, or from the same or from some other class of effects in another substance. Thus we have what is called the absolute thermodynamic scale. This scale is now in modern thermal science the ultimate scale of reference for all thermometers of whatever kind (§ 67). It is defined in §§ 35 and 37 after the following preliminary. A piece of matter which we shall call the "thermometric body" or "thermometric substance" must be given, and at each instant it must be throughout at one temperature, whatever operations we perform upon it. For simplicity we shall suppose it to be of one substance throughout. It may be all solid, or it may be partly solid and the remainder gaseous (as the contents of a wholly frozen cryophorus 1 or any other form of closed vessel full of ice and vapour of water, but with no air); or it may at one particular temperature in the course of its use be partly solid and partly liquid and partly gaseous (as the contents of a partially frozen cryophorus); or it may be partially liquid and partially gaseous (as the contents of an unfrozen cryophorus or of a "philosopher's hammer"); 2 or it may be all liquid; or it may be all gas; or it may be all fluid at a temperature above the Andrews "critical temperature." 3 If it be all solid it may be under any homogeneous stress (Elasticity, Mathematical Theory, part I., chap. I.) but in any case we suppose for simplicity homogeneously divided glass and mercury leading to the reckoning of temperature in a-glass thermometer, which depends on a coefficiency of a particular property other than Eegnault's normal density, and of other gases founded Eegnault's experiments proved, by but very small difference. In particular § 25 (5) a thermometry founded on a complex coefficiency of a par- than air, for the air of Eegnault's normal thermometer. In § 25 (6) a thermometry founded on air of Eegnault's normal thermometer. In § 25 (7) a thermometry founded on a complex coefficiency of a par- than air, for the air of Eegnault's normal thermometer. In § 25 (8) a thermometry founded on a complex coefficiency of a par- than air, for the air of Eegnault's normal thermometer. In § 25 (9) a thermometry founded on a complex coefficiency of a par- than air, for the air of Eegnault's normal thermometer.

§ 35. (1) Alter the bulk or shape of the thermometric First definition by ratio of two temperatures:
(2) Keeping it now at this higher temperature, alter bulk or shape further, and generate the heat which the substance takes to keep its temperature constant, by stirring water, or a portion of the substance itself, if it is partly fluid, and measure the quantity of work spent in this stirring.
(3) Bring it back towards its original bulk and shape till it becomes cooled to its original temperature.
(4) Keeping it at this temperature, reduce it to its original bulk and shape, carrying off, by a large quantity of water, the heat which it must part with to prevent it from becoming solidified. Find by a special experiment how much work must be done to give an equal amount of heat to an equal amount of water by stirring. Then the ratio of the first measured quantity of work to the second is the ratio of the higher temperature to the lower on the absolute thermodynamic scale.

§ 36. The following is equivalent to § 35, and is more convenient for analytical use. It is derived from § 35 by supposing the first and third operations to be so small that the ratio defined as the ratio of the two temperatures is infinitely nearly unity, and conversely § 35—our first form of definition of absolute temperature—may be derived from the second, which is to be now given, by passing through a finite range of temperature by successive infinitesimal steps, and applying the second definition to each step.

§ 37. Let the thermometric body be infinitely small warmed by stirring a portion or the whole of itself if it be partially or wholly fluid, or by stirring a quantity of fluid in space around it if it be all solid; and during the process let the stress upon the body be kept unchanged. The body expanding or contracting or changing its shape with the heat, as the case may be, does work upon the surrounding material by which its stress is maintained.

1 See articles Liquid, Matter (Properties of), and Steam.
2 Ibid.
3 Ibid.
Second definition of absolute temperature — by ratio of an infinitesimal difference of temperatures to whole temperature.

Find the ratio of the amount of work thus done to the amount of work spent in the stirring. For brevity we shall call this the work-ratio. Again, let the stress be infinitely increased, the thermometric body being now for the time enclosed in an impermeable envelope so that it may neither gain nor lose caloric. It will rise (or fall)\(^1\) in temperature in virtue of the augmentation of stress.

The ratio of this infinitesimal elevation of temperature to the whole absolute temperature is equal to the work-ratio multiplied into the ratio of the infinitesimal augmentation of stress to the whole stress.

§ 38. To show how our definition of absolute temperature is to be applied in practice take the following examples.

Example 1. — Any case in which the thermometric substance is part in one condition and the remainder in another of different density, as part solid and part vapour, or part solid and part liquid, or part liquid and part steam. In this last case, as explained above (§ 34), we suppose the stress to be uniform pressure in all directions.

Let \( p \) be its amount, and let \( \dot{t} \) be the absolute temperature corresponding to this pressure. Let \( \sigma \) be the ratio of the density of the rarer to that of the denser portion, \( \rho \) the density of the rarer portion, and \( J \dot{x} \) the quantity of work required to generate the heat taken to convert unit mass of the substance from the lower to the higher condition per unit volume of the latter is \( p(1 - \sigma) \), and the amount of work required to generate the heat taken in doing so is \( J p \) \( \dot{t} \).

Hence the work-ratio of our second definition is

\[
\frac{p(1 - \sigma)}{J p \dot{t}} = \ldots \ldots (1).
\]

Let now the pressure be increased by an infinitely small quantity \( dp \), and the substance being still in the two conditions but of uniform temperature throughout, let \( dt \) be the corresponding rise in temperature. We have by the definition (§ 37)

\[
\frac{dt}{\dot{t}} = \frac{p(1 - \sigma) dp}{\dot{t} \rho J p}.
\]

Hence

\[
\frac{1}{\dot{t}} \frac{dt}{\dot{t}} = \frac{p(1 - \sigma)}{J p \dot{t}} \ldots \ldots (2).
\]

Hence by integration

\[
\log t = \int_{p_0}^{p} \frac{(1 - \sigma) dp}{J p \dot{t}} \int_{t_0}^{t} \ldots \ldots (3).
\]

Water-steam thermometer.

§ 39. Fig. 6 represents a thermometer constructed to show absolute temperature on the plan of example 1, § 38, realized for the case of water and vapour of water as thermometric substance. The containing vessel consists of a tube with cylindrical bulb like an ordinary thermometer; but, unlike an ordinary thermometer, the tube is bent in the manner shown in the drawing. The tube may be of from 1 to 2 or 3 millims. bore, and the cylindrical part of the bulb of about ten times as much. The length of the cylindrical part of the bulb may be rather more than \( \overline{\overline{T}} \) of the length of the straight part of the tube. The contents, water and vapour of water, are to be put in and the glass hermetically sealed to enclose them, with the utmost precautions to obtain pure water as thoroughly freed from air as possible, after better than the best manner of instrument makers in making cryophoruses and water hammers. The quantity of water left in at the sealing must be enough to fill the cylindrical part of the bulb and the horizontal branch of the tube. When in use the straight part of the tube must be vertical with its closed end up, and the part of it occupied by the manometric water-column must be kept at a nearly enough definite temperature by a surrounding glass-jacket of iced water. This glass-jacket is wide enough to allow little lumps of ice to be dropped into it from its upper end, which is open. By aid of an india-rubber tube connected with its lower end, and a little movable cistern, as shown in the drawing, the level of the water in the jacket is kept from a few inches above to a quarter of an inch below that of the interior manometric column. Thus, by dropping in lumps of ice so as always to keep some unmelted ice floating in the water of the jacket, it is easy to keep the temperature of the top of the manometric water-column exactly at the freezing temperature. As we shall see presently, the manometric water below its free surface may be at any temperature from freezing to 10° C. above freezing without more than \( \frac{3}{4} \) per cent. of hydrostatic error. The temperature in the vapour-space above the liquid column may be either freezing or anything higher. It ought not to be lower than freezing, because, if it were so, vapour would condense as hoar frost on the glass, and evaporation from the top of the liquid column would either cryophorously (see Liquid and Thermodynamics) freeze the liquid there, or would cool it below the freezing point.

§ 40. The chief object of keeping the top of the manometric column exactly at the freezing point is to render perfectly definite and constant the steam pressure in the space above it.

A second object of considerable importance when the bore of the tube is so small as one millimetre is to give constancy to the capillary tension of the surface of the water. The elevation by capillary attraction of ice-cold water in a tube of one millimetre bore is about 7 millims. The constancy of temperature provided by the surrounding iced water will be more than sufficient to prevent any perceptible error due to inequality of this effect. To avoid error from capillary attraction the bore of the tube ought to be very uniform, if it is so small as one millimetre. If it be three millimetres or more, a very rough approach to uniformity would suffice.

A third object of the iced-water jacket, and one of much more importance than the second, is to give accuracy to the hydrostatic measurement by keeping the density of the water throughout the long vertical branch definite and constant. But the density of water at the freezing point is only \( \frac{3}{4} \) per cent. less than its maximum density, and is the same as the density at 8° C. and therefore within \( \frac{1}{4} \) per cent. an admissible error on our thermometric pressure, the density will be nearly enough constant with any temperature from 0° to 10° C. throughout the column. But on account of the first object mentioned above the very top of the water column must be kept with exceeding exactness at the freezing temperature.

§ 41. In this instrument the "thermometric substance" (§ 34) is the water and vapour of water in the bulb, or more properly speaking the portions of water and vapour

\(^1\)In the case of fall the elevation of temperature is to be regarded as negative; and in this case the "work-ratio" is negative also.
of water infinitely near their separating interface. The rest of the water is merely a means of measuring hydrostatically the fluid pressure at the interface. When the temperature is so high as to make the pressure too great to be conveniently measured by a water column, the hydrostatic measurement may be done, as shown in the annexed drawing (fig. 7), by a mercury column in a glass tube, surrounded by a glass water jacket not shown in the drawing, to keep it very accurately at some definite temperature so that the density of the mercury may be accurately known.

The simple form of steam thermometer represented with figured dimensions in fig. 6 will be very convenient for practical use for temperatures from freezing to 60°. Through this range the pressure of water-steam, reckoned in terms of the balancing column of water of maximum density, increases (Table V.) from 6 1/2 to 202.3 centimetres; and for this therefore a tube of a little more than 2 metres will suffice. From 60° to 140° the pressure of steam now reckoned in terms of the length of a balancing column of mercury at 0° increases from 14.88 to 271.8 centimetres; and for this a tube of 280 centimetres may be provided. For higher temperatures a longer column, or several columns, as in the multiple manometer, or an accurate air pressure-gauge, or some other means, such as a very accurate instrument constructed on the principle of Bourdon’s metallic pressure-gauge, may be employed, so as to allow us still to use water and vapour of water as thermometric substance.

§ 42. At 230° C, the superior limit of Regnault’s high-pressure steam experiments, the pressure is 27.93 atmos, but there is no need for limiting our steam thermometer to this temperature and pressure. Suitable means can easily be found for measuring with all needful accuracy much higher pressures than 27 atmos. But at so high a temperature as 140°, vapour of mercury measured by a water column, as shown in the diagram (fig. 8), becomes available for purposes for which one millimetre to the degree is a sufficient sensibility. The mercury-steam-pressure thermometer, with pressure measured by water-column, of dimensions shown in the drawing, serves from 140° to 280° C, and will have very ample sensibility through the upper half of its scale. At 280° its sensibility will be about 4 1/2 centimetres to the degree! For temperatures above 280° sufficient sensibility for most purposes is obtained by substituting mercury for water in that simplest form of steam thermometer shown in fig. 6, in which the pressure of the steam is measured by a column of the liquid itself kept at a definite temperature. When the liquid is mercury there is no virtue in the particular temperature 0° C., and a stream of water as nearly as may be of atmospheric temperature will be the easiest as well as the most accurate way of keeping the mercury at a definite temperature. As the pressure of mercury steam is at all ordinary atmospheric temperatures quite imperceptible to the hydrostatic test when mercury itself is the balancing liquid, that which was the chief reason for fixing the temperature at the interface between liquid and vapour at the top of the pressure-measuring column when the balancing liquid was water (§ 40) has no weight in the present case; but, on the other hand, a much more precise definiteness than the ten degrees latitude allowed in the former case for the temperature of the main length of the manometric column is now necessary. In fact, a change of temperature of 2-2° in mercury at any atmospheric temperature produces about the same proportionate change of density as is produced in water by a change of temperature from 0° to 10°, that is to say, about 1/39 per cent.; but there is no difficulty in keeping, by means of a water jacket, the mercury column constant to some definite temperature within a vastly smaller margin of error than 2-2°, especially if we choose for the definite temperature something near the atmospheric temperature at the time, or the temperature of whatever abundant water supply may be available. If the vertical tube for the pressure-measuring mercury column be 830 centimetres long, the simple mercury-steam thermometer may be used up to 520° C, the highest temperature reached by Regnault in his experiments (Table V. below) on mercury-steam. By using an iron bulb and tube for the part of the thermometer exposed to the high temperature, and for the lower part of the measuring column to within a few metres of its top, with glass for the upper part to allow the mercury to be seen, a mercury-steam-pressure thermometer can with great ease be made which shall be applicable for temperatures giving pressures up to as many atmospheres as can be measured by the vertical height available. The apparatus may of course be simplified by dispensing with the Torricellian vacuum at the upper end of the tube, and opening the tube to the atmosphere, when the steam-pressure to be measured is so great that a rough and easy barometer observation gives with sufficient accuracy the air-pressure at the top of the measuring column. The easiest, and not necessarily in practice the least accurate, way of measuring very high pressures of mercury-steam will be by enclosing some air above the cool pressure-measuring column of mercury, and so making it into a compressed-air pressure-gauge, it being understood that the law of compression of the air under the pressures for which it is to be used
in the gauge is known by accurate independent experiments such as those of Regnault on the compressibility of air and other gases.

§ 43. The water-steam thermometer may be used, but somewhat precariously, for temperatures below the freezing point, because water, especially when enclosed and protected as the portion of it in the bulb of our thermometer is, may be cooled many degrees below its freezing point without becoming frozen; but, not to speak of the uncertainty or instability of this peculiar condition of water, the instrument would be unsatisfactory on account of insufficient thermometric sensibility for temperatures more than two or three degrees below the freezing point. Hence, to make a steam thermometer for such temperatures some other substance than water should be taken, and none seems better adapted for the purpose than sulphuric acid, which, in the apparatus represented with figured dimensions in the accompanying diagram (Fig. 9), makes an admirably convenient and sensitive thermometer for temperatures from -20° to something far below -30°, as we see from the results of Regnault's measurements (Table VI. below).

§ 44. To sum up, we have in §§ 39 . . 43 a complete series of steam-pressure thermometers, of sulphuric acid, of water, and of mercury, adapted to give absolutely definite and highly sensitive thermometric indications throughout the wide range from something much below -30° to considerably above 520° of the centigrade scale. The graduation of the scale of these thermometers to show absolute temperature is to be made by calculation from formula (3) of § 38, when the requisite experimental data, that is to say, the values of σ and κ for different values of P throughout the range for which each substance is to be used as thermometric fluid are available. Hitherto these requisites have not been given by direct experiment for any one of the three substances with sufficient accuracy for our thermometric purpose through any range whatever. But water, naturally, is the one for which the nearest approach to the requisite information has been obtained. For it Regnault's experiments have given, no doubt with great accuracy, the values of p*(the steam pressure) and of κ (the latent heat of steam per unit mass) for all temperatures reckoned by his normal air thermometer, which we now regard merely as an arbitrary scale of temperature, through the range from -30° to +230°. If he, or any other experimenter, had been with such equal accuracy through the same range the values of p (the density of steam) and σ (the ratio of the density of steam to the density of water in contact with it), for temperatures reckoned on the same arbitrary scale, we should have all the data from experiment required for the graduation of our water-steam thermometer to absolute thermodynamic scale. For it is to be remarked that all reckoning of temperature is eliminated from the second member of formula (3), and that in our use of it Regnault's normal thermometer has merely been referred to for the values of ρκ and of 1 - σ, which correspond to stated values of p. The arbitrary constant of integration, ε, is truly arbitrary. It will be convenient to give it such a value that the difference of values of t between the freezing point of water and the temperature for which p is equal to one atmo shall be 100, as this makes it agree with the centigrade scale in respect to the difference between the numbers measuring the temperatures which on the centigrade scale are marked -18° and 100°. We shall see (§ 56 below) that indirectly by means of experiments on hydrogen gas this assignation of the arbitrary constant of integration would give 273 for the absolute temperature 0° C., and 373 for that of 100° C. Meantime, as said above, we have not the complete data from direct experiments even on water-steam for graduating the water-steam thermometer; but on the other hand we have, from experiments on air and on hydrogen and other gases, data which allow us to graduate indirectly any continuous intrinsic thermoscope (§ 19 above) according to the absolute scale; and we shall see that by thus indirectly graduating the water-steam thermometer, we learn the density of steam at different temperatures more accurately than it has hitherto been made known by any direct experiments on water-steam itself.

§ 45. Merely viewed as a continuous intrinsic thermoscope, the steam thermometer, in one or other of the forms described above to suit different parts of the entire range from the lowest temperatures to temperatures somewhat above 520°, is no doubt superior in the conditions for accuracy specified in § 16 to every other thermoscope of any of the different kinds hitherto in use; and it may be trusted more surely for accuracy than any other as a thermometric standard when once it has been graduated according to the absolute scale, whether by practical experiments on steam, or indirectly by experiments on air or other gases. In fact, the use of steam-pressure measured in definite units of pressure, as a thermoscopic effect, in the steam thermometer is simply a continuous extension to every temperature of the principle already practically adopted for fixing the temperature which is called 100° on the centigrade scale; and it stands on precisely the same theoretical footing as an air thermometer, or a mercury-in-glass thermometer, or an alcohol thermometer, or a methyl butyrate thermometer, in respect to the graduation of its scale according to absolute temperature. Any one intrinsic thermoscope may be so graduated ideally by thermodynamic experiments on the substance itself without the aid of any other thermometer or any other thermometric substance; but the steam-pressure thermometer has the great practical advantage over all others, especially the air thermometer, that these experiments are easily realizable with great accuracy instead of being, though ideally possible, hardly to be considered possible as a practical means of attaining to thermodynamic thermometry. In fact, for water-steam it is only the most easily obtained of experimental data, the measurement of the density of the steam at different pressures, that has not already been actually obtained by direct experiment. Whether or not, when this lacuna has been filled up by direct experiments, the data from water-steam alone may yield more accurate thermodynamic thermometry than we have at present from the hydrogen or nitrogen gas thermometer (§§ 64 . . 69 below), we are unable at present to judge. But when once we have the means, directly from itself, or indirectly from comparison with hydrogen or nitrogen or air thermometers, of graduating once for all a sulphuric acid steam thermometer, water-steam thermometer, or mercury-steam thermometer, that is to say, when once we have a table of the absolute thermodynamic temperatures corresponding to the different steam pressures of the substances sulphuric acid, water, and mercury, we have a much more accurate and more easily reproducible standard than either the air or gas thermometer of any form, or the mercury thermometer, or any liquid thermometer can give.
In fact, the series of steam thermometers for the whole range from the lowest temperatures can be reproduced with the greatest ease in any part of the world by a person commencing with no other material than a piece of sulphur and air to burn it in, some pure water, and some pure mercury, and with no other apparatus than can be made by a moderately skilled glass-blower, and with no other standard of physical measurement of any kind than an accurate linear measure. He may assume the force of gravity to be that calculated for his latitude, with the ordinary rough allowance for his elevation above the sea, and his omission to measure with higher accuracy the actual force of gravity in his locality can lead him into no thermometric error which is not incomparably less than the inevitable errors in the reproduction and use of the air thermometer, or of mercury or other liquid thermometers. In temperatures above the highest for which mercury-steam pressure is not too great to be practically available, nothing hitherto invented but Deville's air thermometer with a hard porcelain bulb suited to resist the high temperature is available for accurate thermometry.

§ 46. We have given the steam thermometer as our first example of thermodynamic thermometer because intelligence in thermodynamics has been hitherto much retarded, and the student unnecessarily perplexed, and a mere quicksand has been given as a foundation for thermometry, by building from the beginning on an ideal substance called perfect gas, with none of its properties realized rigorously by any real substance, and with some of them (see Matter, Liquid, Steam) unknown, and utterly unassignable, even by guess. But after having been moved by this reason to give the steam-pressure thermometer as our first theoretical example, we have been led into the preceding carefully detailed examination of its practical qualities, and we have thus become convinced that though hitherto used in scientific investigations only for fixing the "boiling point," and (through an inevitable natural selection) by practical engineers for knowing the temperatures of their boilers by the pressures indicated by the Bourdon gauge, it is destined to be of great service both in the strictest scientific thermometry and as a practical thermometer for a great variety of useful applications.

§ 47. Example 2 (including example 1).—Any case in which the stress is uniform pressure in all directions. Let p and v denote the pressure and volume. The condition of the substance (single, double, or triple) as the case may be is determinate when p and v are given, and it will therefore be spoken of shortly as the condition (p, v). Let e be the energy which must be communicated to the substance to bring it from any conveniently defined state above its motivity, to any condition whatever (p, v). Remark that e is a function of the two independent variables p, v, to be found by experiment, and that the finding of it by a perfectly determined practical problem, which can be carried out without the aid of any thermoscope, and without any consideration whatever relating to temperature. We shall see in fact that accurate practical solutions of it for many different substances have been obtained by experiment (see Thermodynamics). The absolute temperature t is also a function of p and v to be also determined by experiment, according to the equivalent definitions of §§ 35 and 37. Let heat be communicated to the substance so as to cause its volume to increase by dv, the pressure being kept constant. The energy of the body will be augmented by

\[ \frac{de}{dv} \cdot dv. \]

At the same time the body in expanding and pressing out the matter around it does work to the extent of

\[ p \cdot dv \]  

Hence the whole work required to generate the heat given to it amounts to

\[ \left( \frac{de}{dv} + p \right) dv \]  

Hence the ratio of (4) to (5), or

\[ \frac{p}{\frac{de}{dv} + p} \]  

is the "work-ratio" of § 37. Hence by the definition

\[ \frac{Dp}{t} \frac{dp}{p} \frac{p}{\frac{de}{dv} + p} = \frac{dp}{p} \]  

where Dt denotes the change of temperature produced by aug. Operando the pressure by dp, and at the same time preventing the loss of substance from either giving heat to or taking heat from the sur-called rounding matter. To express this last condition analytically, let adiabatic be the augmentation of volume (negative, of course, if dp be compressive) which it implies. The work done on the substance by sion by the pressure from without is \(-p \cdot dv\), and the energy of the substance Rankine is augmented by just this amount, because of the condition to be expressed. Hence

\[ \frac{de}{dv} + \frac{de}{dp} \frac{dp}{dv} = -p \cdot dv \]  

whence

\[ \frac{dv}{de} - \frac{de}{dp} \frac{dp}{de} = \frac{dv}{de} \]  

But

\[ \frac{Dp}{t} \frac{dp}{p} \frac{p}{\frac{de}{dv} + p} = \frac{dp}{p} \]  

and so we have

\[ \frac{dp}{p} = \frac{de}{dv} \frac{dv}{de} \frac{dp}{de} \]  

Eliminating Dt/dp from this by (7) we find

\[ t = p + \frac{p \cdot dv}{de} = p + \frac{de}{dv} dp \]  

§ 48. This is a linear partial differential equation of the first order for the determination of t, supposing, as we do for the present, that e is a known function of t and v. The following graphical illustration of the well-known analytical process for finding the complete solutions of such equations shows exactly how much towards determination of temperature can be done with no other data from experiment than the values of \( \frac{de}{dv} \) and \( \frac{de}{dp} \) as functions of p and v, and what additional information is required to fully determine t.

First remark that (11) is the condition that \( \frac{1}{t} \) be a factor of the rendering \( p + \frac{de}{dv} dp + \frac{de}{dp} dy \) a complete differential of a function by Rankine, of two independent variables p and v. Let \( e \) be this function,—

\[ \frac{dp}{de} \frac{1}{(p + \frac{de}{dv})}, \quad \frac{de}{dp} \frac{1}{t \cdot t} \]  

or

\[ t = \frac{p + \frac{de}{dv}}{de} \]  

Then every solution of the differential equation

\[ \frac{dp}{dv} = -\frac{p \cdot dv}{de} \]  

renders \( e \) constant; and conversely, every series of values of p and \( \frac{de}{dv} \) which renders \( e \) constant constitutes a solution of (13). Now this adiabatic differential equation may be solved graphically by taking p and v as rectangular coordinates of a point in a plane, and drawing the Rankine, whole series of curves which satisfy it as follows. Commence with isentropic any point and calculate for its values of p and v the value of the is at its second member of (13). Draw through this point an infinitesimal Willard line in the direction of the tangent to the curve given by the value Gibbs, so found for \( \frac{de}{dv} \). With the altered values of p and v corresponding

\[ \text{Complete diagram of adiabatic batics by Clausius and sub-sequent writers.} \]  

\[ \text{This function is of great importance in practical thermodynamics:} \]

\[ \text{multiplied by} \ t_0, \ \text{it is equal to the excess of the energy of the substance} \]

\[ \text{above its motivity.} \]  

Proceedings R.S.E., 1879) is the amount of work obtainable by letting the substance pass from the state (p, v) in which it is given to the zero condition \( (p_0, v_0) \), without either taking in heat from or giving out heat to matter at any other temperature than \( t_0 \). See Thermodynamics.
to the other end of this infinitesimal line, calculate a fresh value of \( \frac{dp}{dv} \) and continue the curve in the slightly altered direction thus found, and so on. Take another point anywhere infinitesimally near this curve but not in, and draw by a similar process the curve through it satisfying the equation. Take a third point infinitely near this second curve, and draw through it a third curve satisfying (13), and so on till the whole area of values of \( p, v \), possible for the substance in question, is filled with a series of curves one of which passes through, or infinitely nearly through, every point of the area. Assign arbitrarily a particular value of \( \phi \) to each of these curves; then graphically find \( \frac{dp}{dv} \) and \( \frac{dp}{dv} \) for any or every value of \( p \) and \( v \). Then either of the two second forms of equation (12) gives us a value of \( t \) for any values whatever of \( p \) and \( v \).

Completely solution of differential equation for \( t \) with one arbitrary function.

For the arbitrary function of the analytical solution determined by experiment.

Now trace the curve \( p = f(v) \) on our diagram. It must generally cut every one of the previously drawn determinate series of curves. Hence, if \( p \) be given for some one particular temperature, \( t_0 \) is the only additional thing required for the determination of \( t \) for every value of \( p \) and \( v \). For let \( p - f(v) \) be the relation between \( p \) and \( v \) for some one particular temperature, \( t_0 \). If by this we eliminate \( p \) from (12) we find

\[
\frac{dp}{dv} \left( t - t_0 \right) = \frac{dv}{dt} \left( f(v) + \frac{de}{dv} \right) \quad \ldots \ldots \ldots \ldots \ldots (14);
\]

where \( \frac{de}{dv} \) is when \( p - f(v) \) becomes a known function of \( v \) alone.

Hence by integration we find

\[
\phi = \int \left( F(v) + C \right) \quad \ldots \ldots \ldots \ldots (15),
\]

where \( F \) denotes a known function and \( C \) an arbitrary constant.

Now trace the curve \( p = f(v) \) on our diagram. It must generally cut every one of the previously drawn determinate series of curves. Hence, if \( p \) be given for some one particular temperature, \( t_0 \), gives determinately for every value of \( p \) and \( v \). Either of equations (12) then gives \( t \) determinately as a function of \( p \) and \( v \) with only the value \( t_0 \) arbitrary.

The information from experiment, regarding the properties of the thermometric substance, on which this determination is made, consists, founded, consists of a knowledge of the relation between \( p \) and \( v \) for any one temperature, and of the value of \( e - e_0 \) for all values of \( p \) and \( v \) (\( e_0 \) denoting the unknown value of \( e \) for some particular values \( p_0, v_0 \)). Although, theoretically, this information is attainable by purely dynamical operations and measurements, with no other thermal guidance than that afforded by a single-temperature-thermostscope (§15), the whole of it has not in fact been explicitly obtained for any one substance. But less than the whole of it suffices to make a perfect absolute thermometer of any given substance.

Now we examine into the restricted dynamic and thermoscopic investigations upon any particular substance, which will suffice to allow us to make of it a standard absolute thermometer of one or other of these species.

Dynamical and thermoscopic investigation required to graduate, according to the absolute scale, a constant-pressure thermometer of any particular fluid. Let a large quantity of fluid be given, and let proper mechanical means be taken to cause it to flow slowly and uniformly through a pipe, in one short length of which there is a fixed porous plug. If, as is the case with common air, nitrogen, oxygen, carbonic acid, and no doubt many other gases, the fluid leaves the plug cooler than it enters it, let there be a paddle in the stream flowing from the plug, and let this paddle be turned so as to stir the fluid and cause the temperature, when the rapids are fairly past and the eddies due to the stirring subsided, to be the same as in the stream flowing towards the plug. When, as in the case of hydrogen and of all ordinary liquids, the fluid flows away from the plug warmer than it entered it, let a uniform stream of water be kept flowing in a separate channel outside the tube round a portion of it in which the internal fluid is from the plug, and by this means let the temperature of the internal fluid be brought to equality with that which it had on entering the plug. By a separate thermodynamic experiment find how much work would have to be spent in stirring the external stream of water by a paddle to warm it as much as it is warmed by conduction from the internal fluid across the separating tube. Returning now to the internal fluid flowing towards and from the plug, let \( p + e \) be the pressure in the steady stream approaching the disturbed region, and \( p \) the pressure in the steady stream flowing from the disturbed region; and let \( e \) be the quantity of work done by the paddle per unit of mass of the fluid passing by, reckoned positive in the first case, that is, the amount of work is, in which the paddle compensates a cooling effect experienced by the fluid passing through the porous plug. In the second case \(-e\) is (in this case a positive quantity) must denote the work done by the paddle upon the supposed external stream of water in the separate thermodynamic experiment. It is to be reckoned per unit mass of the internal fluid, irrespectively of the rate of flow of the external water. Let \( t \) denote the temperature of the fluid according to the thermodynamic scale, and let \( e \) denote the infinitely small change of temperature which it must experience to produce an infinitesimal expansion from volume \( v \) to volume \( v + 8v \) under constant pressure.

We have

\[
\frac{e}{v} = \frac{1}{t} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (16).
\]

Hence equation (15), with two arbitrarily assigned constants \( t_0 \) and \( C \), gives determinately for every value of \( p \) and \( v \). Either of equations (12) then gives \( t \) determinately as a function of \( p \) and \( v \) with only the value \( t_0 \) arbitrary.

Therefore this determination is made, consists of a knowledge of the relation between \( p \) and \( v \) for any one temperature, and of the value of \( e - e_0 \) for all values of \( p \) and \( v \) (\( e_0 \) denoting the unknown value of \( e \) for some particular values \( p_0, v_0 \)). Although, theoretically, this information is attainable by purely dynamical operations and measurements, with no other thermal guidance than that afforded by a single-temperature-thermostscope (§15), the whole of it has not in fact been explicitly obtained for any one substance. But less than the whole of it suffices to make a perfect absolute thermometer of any given substance.

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§ 52. Now if for any particular fluid at some one given pressure \( p \), with infinitesimal excess \( \delta p \) above this pressure for the higher pressure in the thermodynamic experiment, we find neither heating nor cooling effect in passing through the porous plug, the paddle has nothing to do; that is, \( \delta w = 0 \). If, with always the same pressure \( p \), but with different values of \( v \), that is to say, with the fluid given at different temperatures, but with pressures nearly infinitely nearly the same, we always find the same result, \( \delta w = 0 \), it follows from (16) that for this particular fluid at the particular pressure of the experiment, and for all the temperatures of the experiment, we have

\[
\frac{v}{t} \frac{dv}{d\vartheta} = 1 \quad \ldots \quad (22).
\]

Hence by integration

\[
t = -\vartheta v \quad \ldots \quad (23),
\]

Hence we infer that with this fluid for thermometric substance, with the particular pressure of the experiment, and throughout the range of temperatures for which experiment has given us \( \delta w = 0 \), absolute temperature is shown on a scale graduated and numbered in simple proportion to the whole volume of the fluid.

§ 53. If the thermodynamic test repeated for the same fluid at different pressures gives still the same result, we have, for all pressures and temperatures within the range for which the supposed result \( \delta w = 0 \) has been found by the experiment,

\[
t = f(p) v \quad \ldots \quad (24);
\]

where \( f(p) \) denotes a quantity which depends only on the pressure of the fluid and is independent of its density.

§ 54. Joule and Thomson's experiments on the thermal effects of fluids in motion\(^1\) showed that for pressures of from one to five or six atmos hydrogen gas, common air, nitrogen, oxygen, and carbonic acid, all somewhat approximately fulfil the condition of passing through the porous plug without change of temperature.—Hydrogen much more approximately, carbonic acid much less approximately, than any of the others. Hence we infer that absolute temperature is somewhat approximately proportional to the volume of the fluid, if any one of these gases be used as the thermometric fluid in a constant-pressure thermometer. We shall presently see that the requisite correction of this statement for the case of hydrogen is so small as to be almost within the limits of accuracy of the most accurate thermometric usage.

§ 55. In the case of common air, nitrogen, oxygen, and carbonic acid, the experiments showed a slight cooling effect upon the fluid in passing through the porous plug; in the case of hydrogen, a much smaller heating effect. According to the rigorous dynamical form of our statement of § 51, we have no right to measure these heating and cooling effects on any scale of temperature, as we have not yet formed a thermometric scale. And it is interesting to remark that in point of fact the thermodynamic experiment described in that section involves the use of a differential thermoscope (§ 13) and not of any intrinsic thermoscope at all; and in respect to this requisite it may be contrasted with the thermodynamic investigation of § 49 previously, which involved the use, not of any continuous thermoscope, but only of a single-temperature intrinsic thermoscope (§ 14). Now, instead of reckoning on any thermometric scale the cooling effect or the heating effect of passage through the plug, we have to measure the quantity of work \( \delta w \) required to annul it, in the case of the majority of gases; and in the case of hydrogen, instead of reckoning on any thermometric scale the heating effect, we have to measure \( -\delta w \) as explained in § 51. The experiments as actually made by Joule and Thomson simply gave the cooling effects and heating effects shown by mercury thermometers in the tranquil stream towards and from the plug; but the very thermometers that were used had been used by Joule in his original experiments determining the dynamical equivalent of heat, and again in his later experiments by which for the first time the specific heat of air at constant pressure was measured with sufficient accuracy for our present purpose. Hence by putting together different experiments which had actually been made with those thermometers of Joule's, the operation of measuring \( \delta w \), at all events for the case of air, was virtually completed. Thus according to our present view the mercury thermometers are merely used as a step in aid of the measurement of \( \delta w \), and their scales may be utterly arbitrary, provided we know the quantity of work required to raise unit mass of the fluids concerned through the particular differences of temperature actually shown by the thermometers in the Joules and Thomson experiment. The best way of doing this of course is to take advantage of the best measurements, that is to say Regnault's, of the thermal capacity of air at constant pressure, and then to calculate according to Joule's own measurement the dynamical equivalent of the heat required to warm water through one degree of his own thermometers.

§ 56. Let \( K \) be the thermal capacity, pressure constant, of the fluid experimented on, \( J \) the dynamical equivalent of the thermal unit, and \( S \) the cooling effect (reckoned negative when the effect is rise of temperature), as measured by Joule's thermometers. We have

\[
\delta w = JK \delta t \quad \ldots \quad (25),
\]

\[
v \frac{dv}{d\vartheta} = 1 + \frac{JK}{\delta p} \delta t \quad \ldots \quad (26),
\]

The experiment showed \( \delta t \) to be simply proportional to \( \delta p \) not merely for an infinitesimal difference of pressures but for pressures up to 5 or 6 atmos. For the case of hydrogen\(^2\) the heating effect observed amounted, per 100 inches of mercury, to 100 of a degree centigrade at temperatures of 4° or 5° centigrade, and to 150 of a degree centigrade at temperatures of from 89° to 93° centigrade. The investigation was not carried out in sufficient detail to give any law of variation of this effect with temperature, and it was not even absolutely proved to be greater for the higher than for the lower temperature. In the circumstances we may take the mean of the results for the higher and lower temperatures, say '13 per 100 inches of mercury, or 039 per atm. Hence if \( \Pi \) denotes the force per unit of area in the pressure called 'one atm,' we have

\[
\frac{\delta t}{\delta p} = -0.039KJ/\Pi \quad \ldots \quad (27),
\]

\[
\frac{dv}{dt} = 1 - \frac{0.039JK}{\Pi v} \quad \ldots \quad (28),
\]

which gives by integration

\[
t = -0.039JVK/\Pi \quad \ldots \quad (29),
\]

The arbitrary constant \( C \) depends on the unit adopted for temperature. Let this be such that the difference of temperature between freezing and boiling is 100 (which will make our arbitrary scale agree with the ordinary centigrade scale in respect to the difference between these two temperatures). Denote now by \( \delta t \) the absolute temperature corresponding to 0° C. The absolute temperature corresponding to 100° C. will be \( \delta t = 100 \). Denote also by \( v \) and \( v_{100} \) for the same two temperatures, the bulks of unit mass of hydrogen at any constant pressure within the plug; and from Joule and Thomson's experiments, say, from one to five or six atmos. Then by dividing the value of each member of (28) for 0° C. by the difference of its values for 0° and 100°, we find

\[
\frac{\delta t}{100} = \frac{v - v_{100}}{v_0} \quad \ldots \quad (30),
\]

\[
\frac{\delta t}{100} = \frac{100v_0 - 0.039JKv_0}{v_0} \quad \ldots \quad (31),
\]

\[
\frac{\delta t}{100} = \frac{-0.039JKv_0}{v_0} \quad \ldots \quad (32),
\]

\[
\frac{\delta t}{100} = \frac{100v_0 - 0.039JKv_0}{v_0} \quad \ldots \quad (33).
\]

\(^1\) Transactions Royal Society, June 1853, June 1854, June 1860, and June 1862.

\(^2\) Joule and Thomson, Transactions Royal Society, June 1860.
where \( E \) denotes the expansion of hydrogen, pressure constant, from 0° to 100° C. in terms of its volume at 0°; that is to say,

\[
E = \frac{\text{volume at 0°}}{\text{volume at } 100°} = 0.894 \quad \ldots \quad (31).
\]

Let \( V_o \) denote what the volume would be at 0° C, if the pressure were \( p \) instead of the actual pressure \( p \). We have

\[
V_o = \frac{100}{E} \left( 1 - \frac{\Delta T}{100} \right) \quad \ldots \quad (32).
\]

Regnault finds (Experiences, vol. ii. p. 122) that the value of \( K \) for hydrogen agrees within \( \pm 1/4 \) per cent. with its value for common air; and for common air he finds \( K = 283.8 \). Thus with 423 a for the value of \( J \) in metres (§ 9 above) we find \( JK = 100\times79.9 \) metres. And Regnault's observations on the density of air give for \( IV_o \) (or the height of the homogeneous atmosphere at 0° C.) 7990 metres. Hence for common air, and therefore also for hydrogen, \( JK/IV_o = 0.0126 \); and thus (32) becomes

\[
\frac{\text{volume at 0°}}{\text{volume at } 100°} \left( 1 - \frac{\Delta T}{100} \right) = 0.894 \quad \ldots \quad (38),
\]

with \( c = -0.0064 \) for hydrogen. For this gas expanding under constant pressure of one atmo Regnault found (Experiences, vol. i. p. 80) \( E = -36613 \), which gives \( \Delta T = 273.13 \). Hence (38), with \( V_o - V_d \) gives

\[
\frac{\text{volume at 0°}}{\text{volume at } 100°} \left( 1 - \frac{\Delta T}{100} \right) = 0.894 \quad \ldots \quad (34);
\]

that is to say:

§ 57. We conclude from Regnault's observations on the expansion of hydrogen from 0° to 100° C. under a constant pressure of one atmo, and from the small heating effect discovered in Joule and Thomson's experiments on the forcing of hydrogen through a porous plug, that the absolute temperature of melting ice is 273°C, if the unit or degree of absolute temperature is so chosen as to make the difference one hundred between the temperatures of melting ice and of water with steam at one atmo of pressure.

§ 58. An almost identical number for that most important physical constant, the absolute temperature of melting ice, is obtained from observations on common air, and a not very different number from observations on carbonic acid, the only two gases besides hydrogen for which Regnault (Experiences, vol. i. p. 90) measured the expansion under constant pressure, and for which Joule and Thomson made their experiment on the thermal effect of passage through a porous plug. For each of these two gases the thermal effect observed was a lowering of temperature, and was found to vary at different temperatures very nearly in the inverse proportion of the square of the temperature C. by mercury thermometer, with 273 added. Hence nearly enough for use in the small term of the denominator of (26) we have, for air and carbonic acid,

\[
\frac{\Delta T}{\Delta S} = \frac{273}{t} = \frac{273}{t} \quad \ldots \quad (65),
\]

where \( t \) denotes as before absolute temperature, and \( A \) the amount of the cooling effect per atmo of difference of pressures, on the two sides of the plug, at the temperature of melting ice. The values of \( A \) found for common air and carbonic acid are 273 and 1.588. Regnault (Experiences, vol. ii. p. 126) finds \( JK/IV_o \) greater for carbonic acid than for common air in the ratio of 1.39 to 1 on the average of temperatures from 0° to 210°. But he found also that the specific heat of carbonic acid varies greatly with the temperature; and, taking the mean of the values which he finds for it at 0° and 100°, p. 130, as the proper mean for our present purpose, we find for \( JK/IV_o \) a value 1.29 times its value for common air. From these experimental results we find by the mathematical process below (§ 61) still the same approximate formula (33), but with \( c = +0.0028 \) for common air and \( c = +0.163 \) for carbonic acid. At constant pressure of one atmo Regnault's measurements gave \( E = -36706 \) for common air, and \( E = -3710 \) for carbonic acid; and dividing 100 by these decimals we find respectively 272.44 and 269.5. The corrections on these numbers by formula (33) to give the absolute temperature of freezing are accordingly +70 and +4.4, and the resulting estimate of the required absolute temperature are 273.14 and 273.9. Bringing together the results in the three cases, we see them conveniently in the following table:

<table>
<thead>
<tr>
<th>Name of Gas</th>
<th>Expansion at one atm according to Regnault.</th>
<th>Proper mean cooling-effect of forcing hydrogen through a porous plug per atmo according to Joule and Thomson.</th>
<th>Uncorrected estimate of absolute temperature of melting ice</th>
<th>Correction of absolute temperature from cooling-effect.</th>
<th>Resulting estimate of absolute temperature of melting ice.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>36613</td>
<td>+0°-039</td>
<td>272-13</td>
<td>-0°-13</td>
<td>272-00</td>
</tr>
<tr>
<td>Air</td>
<td>36706</td>
<td>+0°-208</td>
<td>272-44</td>
<td>+0°-70</td>
<td>272-14</td>
</tr>
<tr>
<td>Carbonic acid</td>
<td>37100</td>
<td>+1°-100</td>
<td>273-9</td>
<td>+0°-4</td>
<td>273-9</td>
</tr>
</tbody>
</table>

1 Investigated in § 61 below.

The close agreement of the results from hydrogen and common air is very satisfactory, and it is interesting to see it brought about with so large a correction calculated from the Joule and Thomson effect. It is also interesting to see the sevenfold larger correction of nearly five times bringing so nearly the same results from the one cent. larger expansion of carbonic acid. The \( \pm 1 \) per cent. discrepancy which remains between the results from carbonic acid and from hydrogen is not satisfactory, and requires explanation, particularly when we remark that, of five measurements by Regnault (Experiences, vol. i. p. 84) of the expansion of carbonic acid under constant pressure of one atmo, all lie within 1° 3 per cent. of the mean number 3710 which he has given, and we have taken, as his result.

Notwithstanding that the Joule and Thomson correction is so much greater for common air than for hydrogen, the result from common air is probably the most trustworthy of the three, because both Regnault's experiments and Joule and Thomson's were probably more accurate for air than for either of the other two gases. The true result to one place of decimals may therefore be considered as most probably being 273.1°, but the probability that it is nearer 273-0° than 273-1° is scarcely enough to make it worth while to use in any ordinary thermodynamic calculation any other number than 273°, which is exactly that found from hydrogen.

§ 59. The real meaning of our result 273-1° for the absolute temperature of melting ice, expressed without any choice of degrees or units for temperature, is that the ratio of the temperature at which vapour of water has a pressure of one atmo to the temperature at which ice melts is 373-1/273-1. Still another way of saying the same thing, this time eliminating all numerical reckoning of temperature, is (see Thermodynamics) as follows:

For every hundred units of heat converted into determinable work by a perfect thermodynamic engine, 373.1 are necessary to take from the source, and 273-1 rejected to the refrigerator, if the temperature of the source be that at which steam of water has a pressure of one atmo, and the temperature of the refrigerator source that at which ice melts.

§ 60. Integration of differential equation (28) gives

\[
\frac{\Delta T}{\Delta S} = \frac{JK}{2p} \quad \ldots \quad (28).
\]

For each of the five gases experimented on, namely, common air, oxygen, nitrogen, carbonic acid, hydrogen, the experiment showed that, for all pressures up to five or six atmos., \( \Delta S/\Delta p \) was sensibly independent of the pressure, but that it varied very considerably with the temperature. Hence, if we put \( \Delta S/\Delta p = -\Delta T/\Delta S \), which will
thus denote the cooling effect per atmo of differential pressure, is a
function of the temperature, and is independent of the whole pressure.
With this notation (36) becomes

$$\frac{de}{d\theta} = -\frac{JK}{\Pi} \theta .$$

This is a linear equation in $z$ with a second member, if we put $z = -\log t$. Integrating it, and replacing $t$, we find, as the complete integral,

$$v = t - \frac{V_n}{t} \ln \left( \frac{t^2}{t_0^2} \right) + \frac{JK}{\Pi} \ln \left( \frac{t^0}{t} \right) \cdots \cdots (37).$$

§ 61. Expansions of different gases, pressure constant, calculated from the Joule and Thomson experiment. We have from equation (37)

$$v = \frac{V_n}{t} - \frac{t}{t_0} \ln \left( \frac{t^0}{t} \right) + \frac{JK}{\Pi} \ln \left( \frac{t^0}{t} \right) \cdots \cdots (38).$$

For each of the gases experimented on, except hydrogen, $\theta$ was taken as the proper mean cooling effect per atmo in the Joule experiment; and for hydrogen or any other gas, if there is any other, in which the experiment shows a heating effect). This "proper mean" may be taken as the arithmetic mean of the cooling effects at 0° and 100° C., unless otherwise stated.

Hence, for these gases at pressures from 0 to 5 or 6 atmos, (38) becomes

$$v = \frac{V_n}{t} - \frac{t}{t_0} \left( 1 + \frac{JK}{\Pi} \ln \frac{t^0}{t} \right) \cdots \cdots (39).$$

This shows that the "proper mean cooling effect" ($M$ in the table of § 58) is

$$M = \frac{1}{1 + \frac{JK}{\Pi} \ln \frac{t^0}{t}} \left( 1 - \frac{t^0}{t} \right)^2 \cdots \cdots (40),$$

$M$ denoting the proper mean cooling effect per atmo in the Joule and Thomson experiment (as computed to be equal to the mean for the temperatures $t$ and $t'$, with absolute agreement between our theoretical results and the only direct measurements by Regnault. More of direct measurement, to allow a more extensive comparison, is very desirable.

These formulas must be exceedingly near the truth for all pressures from 0 to 6 atmos, because within this range the thermal effects in the Joule and Thomson experiment were very approximately in simple proportion to the differences of pressure on the two sides of the plug. The following table of results calculated from (43) for several pressures of from 0 to 6 atmos is interesting as showing such different expansions for the different cases, determined by thermodynamic theory from Regnault's measurements of specific heats and Joule and Thomson's of their particular thermal effect, with absolutely no direct measurement of expansion except the one for common air at one atmo, shown as the third entry of column 5 in the table. The other five entries of column 5 show a fair amount of agreement between our theoretical results and the only direct measurements by Regnault. More of direct measurement, to allow a more extensive comparison, is very desirable.

<table>
<thead>
<tr>
<th>Name of Gas</th>
<th>Expansion, pressure constant, from 0° to 100° C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>-3662 (1 + 0-00049 $V_n/V_0$)</td>
</tr>
<tr>
<td>Common Air</td>
<td>-3662 (1 + 0-0025 $V_n/V_0$)</td>
</tr>
<tr>
<td>Oxygen</td>
<td>-3662 (1 + 0-0032 $V_n/V_0$)</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>-3662 (1 + 0-0031 $V_n/V_0$)</td>
</tr>
<tr>
<td>Carbonic Acid</td>
<td>-3662 (1 + 0-0166 $V_n/V_0$)</td>
</tr>
</tbody>
</table>

§ 64. We are now quite prepared to make a practical working thermometer directly adapted to show temperature on the absolute thermodynamic scale through the whole range of temperature, from the lowest attainable by any means to the highest for which glass remains solid. It is to be remarked that our investigation of § 51, and all the deduced formulas and relative calculations, are absolutely independent of the approximate fulfilment of Boyle's law by the gases to which we have applied them, and are equally applicable without any approach to fulfilment of Boyle's law; also that the only experimental data on which we are founded our special numerical conclusions of §§ 59 to 63 are Regnault's measurements of specific heats under constant pressures, and Joule and Thomson's measurements of the thermal effect of forcing through a porous plug. From these experimental data alone we see by formula (38) of § 61 how to graduate a constant-pressure gas thermometer so that it shall show temperature on the absolute thermodynamic scale. Hence, notwithstanding the difficulty (§ 24 above) which Regnault found in the thermometric use of air or other gases on the system of constant pressure, and his practical preference for the constant-volume
air thermometer, it becomes of the highest importance to construct a practical constant-pressure gas thermometer. This we believe may be done by avoiding the objectionable expedient adopted by Pouchet and Regnault of allowing a portion (when high temperatures are to be measured) of the whole gas to be pressed into a cool volumetric chamber out of the thermometric chamber proper by the expansion of the portion which remains in; and instead fulfilling the condition, stated, but pronounced practically impossible, by Regnault (Experiences, vol. i. pp. 168, 169), that the thermometric gas "shall, like the mercury of a mercury thermometer, be allowed to expand freely at constant pressure in a calibrated reservoir maintained throughout at one temperature." We have accordingly designed a constant-pressure gas thermometer to fulfill this condition. It is represented in the accompanying drawing (Fig. 10), and described in the following section.

§ 65. The vessel containing the thermometric fluid, which in this case is to be either hydrogen or nitrogen,1 consists in the main of a glass bulb and tube placed vertically with bulb up and mouth down; but there is to be a secondary tube of much finer bore opening into the bulb or into the main tube near its top, as may be found most convenient in any particular case. The main tube, which, to distinguish it from the secondary tube, will be called the volumetric tube, is to be of large bore, not less than 2 or 3 centimetres, and is to be ground internally to a truly cylindric form. To allow this to be done it must be made of thick, well-annealed glass like that of the French glass-barrelled air-pumps. The secondary tube, which will be called the manometric capillary, is to be of round bore, not very fine, such as to be of a few thousandths of a millimetre diameter. Its lower end is to be connected with a mercury manometer to show if the pressure of the thermometric air is either greater or less than the definite pressure to which it is to be brought every time a thermometric measurement is made by the instrument. The change of volume required to do this for every change of temperature is made and measured by means of a micrometer screw2 lifting or lowering a long solid glass piston, fitting easily in the glass tube, and caulked air-tight by mercury between its lower end and an iron sole-plate by which the mouth of the volumetric tube is closed. To prevent this mercury caulking, when the piston is raised and lowered, mercury is allowed to flow in and out through a hole in the iron sole-plate by a thin pipe, connected with two mercury cisterns at two different levels by branches each provided with a stopcock. When the piston is being lowered the stopcock of the branch leading to the lower cistern is closed, and the other is opened enough to allow the mercury to flow up after the piston and press gently on its lower side, without entering more than infinitesimally into the space between it and the surrounding glass tube (the condition of the upper bounding surface of the mercury in this respect being easily seen by the observer looking at it through the glass tube). When the piston is being lowered the stopcock of the branch leading from the upper cistern is closed, and the one in the branch leading to the lower cistern is opened enough to let the mercury go down before the piston, instead of being forced to any sensible distance into the space between it and the surrounding tube, but not enough to allow it to part company with the lower surface of the piston. The manometer is simply a mercury barometer of the form commonly called a siphon barometer, with its lower end not open to the air but connected to the lower end of the manometric capillary. This connexion is made below the level of the mercury in the following manner. The lower end of the capillary widens into a small glass ball or stout tube of glass of about 2 centimetres bore and 2 centimetres depth, with its lip ground flat like the receiver of an air-pump. The lip or upper edge of the open cistern of the barometer (that is to say, the cistern which would be open to the atmosphere were it used as an ordinary barometer) is also ground flat, and the two lips are pressed together with a greased leather washer between them to obviate risk of breaking the glass, and to facilitate the making of the joint mercury tight. To keep this joint perennially good, and to make quite sure that no air shall ever leak in, in case of the interior pressure being at any time less than the external barometric pressure or being arranged to be so always, it is preserved and caulked by an external mercury-jacket not shown in the drawing. The mercury in the thus constituted lower reservoir of the manometer is above the level of the leather joint, and the space in the upper part of the reservoir over the surface of the mercury, up to a distance into the capillary above, is occupied by a fixed oil or some other practically vapourless liquid. This oil or other liquid is introduced for the purpose of guarding against error in the reckoning of the whole bulk of the thermometric gas, on account of slight irregular changes in the capillary depression of the border of the mercury surface in the reservoir.

§ 66. In the most accurate use of the instrument, the glass and mercury and oil of the manometer are all kept at one definite temperature according to some convenient and

1 Common air is inadmissible because even at ordinary temperatures its oxygen attacks mercury. The film of oxide thus formed would be very inconvenient at the surface of the mercury caulking, round the base of the piston, and on the inner surface of the glass tube, to which it would adhere. Besides sooner or later the whole quantity of oxygen is to save the oil or other liquid which is interposed between it and the mercury of the manometer from being thickened or otherwise altered by oxidation.

2 This screw is to be so well fitted in the iron sole-plate as to be sufficiently mercury-tight without the aid of any soft material, under such moderate pressure as the greatest it will experience when the pressure chosen for the thermometric gas is not more than a few centimetres above the atmospheric pressure. When the same form of apparatus is used for investigation of the expansion of gases under high pressures, a greased leather washer may be used on the upper side of the sole in the screw-hole plate, to prevent mercury from escaping round the screw. It is to be remarked that in no case will a little oil get out of the mercury round the screw while it is being turned, introduce any error at all into the thermometric result; because the correctness of the measurement of the volume of the gas depends only on the mercury being brought up into contact with the bottom of the piston, and not more than just perceptibly up between the piston and volumetric tube surrounding it.
perfectly trustworthy intrinsic thermoscope (§§ 15 and 16 above), by means of thermal appliances not represented in the drawing but easily imagined. This condition being fulfilled, the one desired pressure of the thermometric gas is attained with exceedingly minute accuracy by working the micrometer screw up or down until the oil is brought precisely to a mark upon the manometric capillary.

In fact, if the glass and mercury and oil are all kept rigorously at one constant temperature, the only access for error is through irregular variations in the capillary depressions in the borders of the mercury surfaces. With so large a diameter as the 2 centimetres chosen in the figured dimensions of the drawing, the error from this cause can hardly amount to \( \frac{1}{57} \) per cent. of the whole pressure, supposing this to be one atmo or thereabouts.

For ordinary uses of our constant-pressure gas thermometer, where the most minute accuracy is not needed, the rule will still be to bring the oil to a fixed mark on the manometric capillary; and no precaution in respect to temperature will be necessary except to secure that it is approximately uniform throughout the mercury and containing glass. The oil is raised to the level of the mercury in the lower reservoir, and the quantity of oil is so small that, whatever its temperature may be, the bringing of its free surface to a fixed mark on the capillary secures that the mercury surface below the oil in the lower reservoir is very nearly at one constant point relatively to the glass, much more nearly so than it could be made by direct observation of the mercury surface, at all events without optical magnifying power. Now if the mercury surface be at a constant point of the glass, it is easily proved that the difference of pressures between the two mercury surfaces will be constant, notwithstanding considerable variations of the common temperature of the mercury and glass, provided a certain easy condition is fulfilled, through which the effect of the expansion of the glass is compensated by the expansion of the mercury.

This condition is that the whole volume of the mercury shall bear to the volume in the cylindrical vertical tube from the upper surface to the level of the lower surface the ratio \( \frac{\lambda - \sigma}{\lambda - \sigma} \), where \( \lambda \) denotes the cubic expansion of the mercury, and \( \sigma \) the cubic expansion of the solid for the same elevation of temperature, it being supposed for simplicity of statement that the tube is truly cylindrical from the upper surface to the level of the lower surface, and that the sectional area of the tube is the same at the two mercury surfaces. The cubic expansion of mercury is approximately seven times the cubic expansion of glass. Hence

\[
\left( \frac{\lambda - \sigma}{\lambda - \sigma} \right) = \left( \frac{\lambda - \sigma}{\lambda - \sigma} \right) \div 11.1.11.
\]

Hence the whole volume of the mercury is to be about 1.1.11 times the volume from its upper surface to the level of the lower surface; that is to say, the volume from the lower surface in the bend to the same level in the vertical branch is to be \( \frac{1}{3} \) of the volume in the vertical tube above this surface. A special experiment on each tube is easily made to find the quantity of mercury that must be put in to cause the pressure to be absolutely constant when the surface in the lower reservoir is kept at a fixed point relatively to the glass, and when the temperature is varied through such moderate differences of temperature as are to be found in the use of the instrument at different times and seasons.

A sheet-iron containing water or oil or fusible metal, with external thermal appliances of gas or charcoal furnace, or low-pressure or high-pressure steam heater, and with proper internal stirrer or stirrers, is fitted round the bulb and manometric tube to produce uniformly throughout the mass of the thermometric gas the temperature to be measured. This part of the apparatus, which will be called for brevity the heater, must not extend so far down the manometric tube that when raised to its highest tempera-

ture it can warm the caulking mercury to as high a temperature as 40° C., because at somewhat higher temperatures than this the pressure of vapour of mercury begins to be perceptible (see Table V. below), and would vitiate the thermometric use of the pure hydrogen or nitrogen of our thermometer. To secure sufficient coolness it will probably be advisable to have an open glass jacket of cold water (not shown in the drawing) round the volumetric tube, 2 or 3 centimetres below the bottom of the heater, and reaching to about half a centimetre above the highest position of the bottom of the piston.

§ 67. It seems probable that the constant-pressure hydrogen or nitrogen gas thermometer which we have now described may give even more accurate thermometry than Regnault's constant-volume air thermometers (§ 24 above), and it seems certain that it will be much more easily used in practice.

We have only to remark here further that, if Boyle's law were rigorously fulfilled, thermometry by the two methods would be identical, provided the scale in each case be graduated or calculated so as to make the numerical readings of the thermometer agree at two points,—for example, 0° C. and 100° C. The very close agreement which Regnault found among his different gas thermometers and his air thermometers with air of different densities (§ 25 above), and the close approach to rigorous fulfilment of Boyle's law which he and other experimenters have ascertained to be presented by air and other gases used in his thermometers, through the ranges of density, pressure, and temperature at which they were used in these thermometers, renders it certain that in reality the difference between Regnault's normal air thermometer and thermometry by our hydrogen gas constant-pressure thermometer must be exceedingly small. It is therefore satisfactory to know that for all practical purposes absolute temperature is to be obtained with very great accuracy from Regnault's thermometric system by simply adding 273 to his numbers for temperature on the centigrade scale. It is probable that at the temperatures of 250° or 300° C. (or 523 or 573 absolute) the greatest deviation of temperature thus reckoned from correct absolute temperature is not more than half a degree.

§ 68. The thermometric scale being now thoroughly established in theory and practice (§§ 33–69 and §§ 18–30), we are prepared to define, without any ambiguity, the expressions thermal capacity and specific heat with reference to matter at any temperature and in any physical condition.

Definition 1.—The thermal capacity of a body (whether it be a portion of matter homogeneous throughout or of homogeneous substance in two different conditions as liquid and steam, or solid and vapour of solid, or a piece of apparatus consisting of different parts as glass and metals, and containing as the case may be liquids or gases,—subject only to the condition that the whole matter considered is at one temperature) is the quantity of heat required to raise its temperature by one degree on the absolute thermodynamic scale.

Definition 2.—The specific heat of a substance is the thermal capacity of a stated quantity of it. This stated quantity is generally understood to be the unit of mass, unless some other definite quantity is explicitly designated, as for instance the quantity of the substance which occupies unit of volume at some definite pressure and temperature, for instance, one atmo and temperature 273 absolute. It is of no consequence what unit of mass is chosen, provided it be the same as that which is used in defining the thermal unit; but, unless the contrary be explicitly stated, we always understand one gramme as the unit of mass and the thermal unit as the quantity of heat...
required to raise one gramme of water from 273 to 274 (compare § 6 above).

§ 69. There is scarcely any subject upon which more skilled labour in scientific laboratories, chemical and physical, has been spent than the measurement of specific heats, whether of solids, liquids, or gases. An ample and well-arranged table of results is to be found in Clarke's Constants of Nature, a compilation of numerical results of scientific experiments made in all parts of the world by various observers and experimenters, a most valuable aid to scientific knowledge given to the world as No. 255 of the Smithsonian Miscellaneous Collections. It is most interesting as showing how very differently different substances behave in respect to constancy or variation of specific heat with temperature. Thus it shows that, according to the results of all the experimenters, the specific heats of all the substances experimented on, whether simple or compound, are very nearly constant at all events for ranges between -10° and 200° or 300° C., except the three elementary substances, boron, carbon, silicon. The specific heats of these three have been found by F. Weber to vary greatly with temperature. Thus for diamond he finds the specific heat to be 1° at 0° C. and 27 at 206° C., or nearly threefold of the amount at 0°; at -50° C. the specific heat is 0.63; and at +985° it is 4.69 or about seven and a half times the specific heat at -50° (a curious practical commentary, we may remark in passing, on the doctrine of the calorists on specific heat referred to in § 8 above). The specific heats of carbon in its other forms of graphite and charcoal through wide ranges of temperature according to the same observer, F. Weber, are particularly interesting and significant. The approximate equality of the product of specific heat into the atomic weight for the simple metals is another important point, so is the tendency of constancy and uniformity in the corresponding product for other substances, whether simple or compound. If we were to define a metal as a substance for which through the range of temperature from 0° to 250° C. the product of the specific heat into the atomic weight is not less than 5.86 and not greater than 6.93, we should include every substance commonly called a metal, and no substance not commonly called a metal, except phosphorus, and solid sulphur lately fused.

Some important results of Regnault's regarding the specific heats of gases under constant pressure have been already quoted in §§ 56, 58, and 62 above. Further information from experiments aided by thermodynamic theory, regarding specific heats of gases and vapours under constant pressure and of gases in constant volume, will be found under the heading THERMODYNAMICS. To this also, and to the articles Matter (Properties of), Liquid, and Steam, the reader is referred for information respecting latent heats of liquefaction of solids and of evaporation of solids and liquids, also respecting the thermal capacity of a portion of homogeneous substance in two different states, such as the water-liquid and water-steam of an ordinary crystal of phosphorus or philosopher's hammer, or of the sulphurous acid liquid and steam of a sulphurous acid steam thermometer (§ 43 above).

TRANSPERSION OF HEAT.

§ 70. When two contiguous portions of matter are at different temperatures, heat is transferred from the warmer to the colder. This process is called conduction of heat.

When two bodies at different temperatures are separated by a transparent medium, such as air, or water, or glass, or ice, heat passes from the warmer to the colder irrespectively of the temperature of the intervening medium, except in so far as its transparency may in some slight degree be affected by the temperature. Thus the colder of the two bodies becomes actually heated above the temperature of the intervening medium if the warmer be kept above this temperature, and if heat is not otherwise drawn off from the colder body in greater quantity than the heat entering it from the warmer. This process of transference from one body to another body at a distance through an intervening medium is called radiation of heat. The condition of the intervening matter in virtue of which heat is thus transferred is called light; and radiant heat is light if we could but see it with the eye, and not merely discern with the mind, as we do, that it is perfectly continuous in quality with the species of radiant heat which we see with the material eye through its affecting the retina with the sense of light. Thus a white hot poker in a room perfectly darkened from all other lights is seen as a brilliant white light gradually becoming reddish and less bright, until it absolutely fades from vision in a dull red glow. Long after it has ceased to be visible to the eye, the fact that heat is being transferred from it to colder bodies all round it, or above it or below it, is proved by our sense of heat in a hand or face held near it on any side or above it. By considering the whole phenomenon of the white hot mass, without much of experimental investigation, we judge that there is perfect continuity through the whole process, in the first part of which the radiant heat is visible and in the second part invisible to the human eye: and thorough experimental investigation confirms this conclusion. Thus radiant heat is brought under the undulatory theory of light, which in its turn becomes annexed to heat as a magnificent outlying province of the kinetic theory of heat.

§ 71. In this article we confine ourselves to a practical evaluation of rate of gain or loss of heat across the surface of an isolated solid placed in a medium such as air, and regard all in a solid substance so as is the temperature of the isolated body at any time, as the rate of gain of heat in the first part of which the radiant heat is visible and in the second part invisible to the human eye: and thorough experimental investigation confirms this conclusion. Thus radiant heat is brought under the undulatory theory of light, which in its turn becomes annexed to heat as a magnificent outlying province of the kinetic theory of heat.
ture of 14° C, while the other had the varying temperature of the centre of the ball. Two sets of experiments were made. In one the ball had a bright surface, in the other it was coated with soot from the flame of a lamp, and in both the air was kept moist by a saucer of water placed in the interior of the tinplate enclosure. The results are given in terms of the number of units of heat lost per second, per square centimetre of surface of the copper, per degree of difference between the temperatures of the two junctions.

§ 72. Returning to the conduction of heat, we have first to say that the theory of it was discovered by Fourier and given to the world through the French Academy in his Théorie Analytique de la Chaleur,1 with solutions of problems naturally arising from it, of which it is difficult to say whether their uniquely original quality, or their transcendently intense mathematical interest, or their perennially important instructiveness for physical science, is most to be praised. Here we can but give the very slightest sketch of the elementary law of conduction in an isotropic substance, the mathematical expression for it in terms of orthogonal plane or curved coordinates, and a few of the elementary solutions in Fourier's theory.

§ 73. Consider a slab of homogeneous solid bounded by two parallel planes. Let the substance be kept at two different temperatures over these parallel planes by suitable sources of heat and cold. For example, let one side be kept cold by a stream of cold water, or by a large quantity of ice and water in contact with it, and the other kept warm by a large quantity of warm water or by steam blown against it. Whatever particular plans of heater and refrigerator be adopted, care must be taken that the temperature is kept uniform over the whole, or over a sufficiently large area of each side of the slab, to render the isothermal surfaces sensibly parallel planes through the whole of the slab intercepted between the two calorimetric areas, and that the temperature at each side is prevented from varying with time. It will be found that heat must continually be applied at one side and removed from the other, to keep the circumstances in the constant condition thus defined. When this constant condition of surface temperature is maintained long enough, the temperature at every point of the slab settles towards a constant limiting value; and when this limiting value has been sensibly reached by every point of the slab, the temperature throughout remains sensibly uniform, so long as the surface temperature is kept constant. In this condition of affairs the temperature varies continuously from one side of the slab to the other; and it is constant throughout each interior plane parallel to the sides; in other words, the isothermal surfaces are parallel planes. Let V and V' be the temperatures in two of these isothermals and a the distance between them. The quotient \( \frac{V - V'}{a} \) is the average rate of variation of temperature per unit of length between these two isothermals. Let Q be the quantity of heat taken in per unit of time at a certain area A on one side, and emitted at the corresponding area of the other side of the slab, measured by proper calorimetric appliances to these areas, which we shall call the calorimetric areas of the apparatus. It will generally be found that the value of the quotient \( \frac{V - V'}{a} \) is not the same for consecutive isothermal surfaces. For metals it is ascertained by experiment that it increases continuously from the cold side to the hot side of the slab. In other words, as we shall see presently, the thermal conductivity of the substance is not generally the same at different temperatures, and for metals it is smaller the higher the temperature.

§ 74. Circumstances being as described in § 73, the thermal conductivity of the substance between the isothermals \( \nu \) and \( \nu' \) is the value of

\[
\frac{Qs}{A(\nu - \nu')} \quad \ldots \ldots \ldots \ldots (1).
\]

It must be remembered that the temperatures \( \nu, \nu' \) used in this definition are temperatures of the substance itself. Some experimenters have given largely erroneous results through assuming that the temperatures of the two sides of the slab were equal to those of the calorimetric fluids, such as warm water or steam on one side, and cold water or cold air on the other side. To obtain correct results, the actual temperatures at two points in the conducting body itself must be ascertained by aid of suitable thermometers, or thermometers and differential thermoscopes, applied in such a way as not sensibly to disturb the isothermal surfaces. This, so far as we know, has not been done by any experimenter hitherto in attempting to measure thermal conductivity directly by the method indicated in the definition; and therefore if any results obtained by this method hitherto are trustworthy, it is only in a few cases,—cases in which, unless the substance experimented upon has been of such small conducting power, and the stirring of the calorimetric fluids on its two sides so energetic, that we can feel sure that the observed or assumed temperatures of these fluids, or of the portions of them of which the temperatures have been measured by thermometers, have not differed sensibly from the temperatures of the slab at its surfaces in contact with them.

§ 75. What utter confusion has permeated scientific literature, from experiments on thermal conductivity instituted by non-fulfilment of this condition, is illustrated by results quoted in Everett's Units and Physical Constants (London, 1879), among which we find '19 for the conductivity of copper according to Péclet, and 1.1 according to Angström (which we now know to be correct). When we look to Péclet's and Angström's own papers the confusion becomes aggravated. Péclet, in his Mémoire sur la détermination des coefficients de conductibilité des métaux par la chaleur,2 quotes old experiments of Clémont, and others more recent of Thomas and Laurent, regarding which he gives certain details. Taking his information no doubt from Péclet's paper, Angström gives a statement3 for the conductivity of copper, according to experimenters who had preceded him, which, with the decimal point shifted two places to the left to reduce to C. G. S., is as follows:—

Clémont,.................................................. 0.0281
Thomas and Laurent,.................................. 0.12
Péclet,......................................................... 1.78

But Angström did not notice that Péclet had stated the thickness of the plate experimented on by Clémont to be between 2 and 3 millimetres. Péclet himself in his next sentence seems to have forgotten this when he compares the figure -23 which he had calculated from Clement's results, without taking account of the thickness of the plate, with -22 which he calculates from Thomas and Laurent's experiments on copper, without stating any thickness for the tube of copper on which (instead of a flat plate) they had experimented. Thus we have no data for finding what their results really were in either Péclet's or Angström's paper; but Péclet seems to show enough regarding it to let us now feel perfectly sure that it is only a question of

1 Annates de Chimie et de Physique, Paris, 1841.
2 Angström's own statement the unit quantity of heat is that required to raise 1000 grams of water 1°. The conduction is reckoned per square metre of the copper plate per second of time, and the unit chosen for the rate of variation of temperature across the plate is 1° per millimetre. To reduce his numbers to the C. G. S. system we must therefore multiply by \( 10^2 \times 10^{-4} \times 10^{-1} = 10^{-5} \).
whether it is tens or hundreds of times too small. Omitting it from the preceding statement, completing the correction by multiplying the 0.057 by 2.2 (assuming the thickness of the plate to be 0.22 millimetres, as Péclet says it was between 2 and 3) to give Clément's result, and appending Ångström's result, which we now know to be correct, we have the following statement for thermal conductivity of copper in C. G. S. units:—

\[ 0.057 \times 2.2 = 0.126 \]

\[ \text{Péclet,}^1 \]

\[ 1.1 \times 1.5 = 1.65 \]

\[ \text{Ångström.} \]

§ 76. The comparison of these results is highly instructive. Clément's result is two hundred times too small, and Péclet's five times too small. Clément experimented by exposing one side of a plate of copper of a square metre surface and about two and a half millimetres thick to steam at 100° C, and the other side to water at 28° C. It was assumed that the difference of temperatures between the two sides was 72°. The difference really was about 36 of a degree, as we know from the quantity of heat actually conducted through it in Clément's experiment, indicated by the amount of steam which he found to be condensed into water. In fact, the amount of steam condensed did not differ sensibly from what it would have been if the copper plate had been infinitely thin, or if its substance of infinite thermal conductivity. It is important in engineering, and in many of the arts and manufactures involving thermal processes, and particularly in that one of them with the greatest everyday value to the human race than all the others put together, cookery, to know that for copper or iron boilers, or steam-tubs, or ovens, or frying-pans, the thermal conductivity of heat from radiant burning coal or charcoal, or red or white hot fireclay or other solids, and from hot air in contact with them, on one side, to hot water or steam or oil or melted fat on the other side, or hot liquid or steam on one side and cool air on the other side, is for practical purposes sensibly the same as if the thermal conductivity of the metal were infinite, or its resistance to the transmission of heat nothing. The explanation is obvious to us now with the definite and sure knowledge regarding thermal conductivities of different substances and of matter in different conditions, solid, liquid, and gaseous, gained within the last twenty years. Ångström, Forbes, F. Neumann, and Taill have given, each one of them with thoroughly sufficient experimental evidence to leave no room to doubt the substantial accuracy of his results, absolute values for the thermal conductivities of copper and iron. Clausius and Maxwell have given us thermal conductivities of air and other gases, from their splendid development of the kinetic theory, which are undoubtedly trustworthy as somewhat close approximations to the true values, and which it is quite possible are more accurate than we can hope to see obtained from direct measurements of the conduction of heat through gases. J. T. Bottomley has given a trustworthy and somewhat closely accurate direct measurement of the thermal conductivity of water. From the results of these experimenters' works, reduced to uniform C. G. S. reckoning in our tables of thermal conductivities (Table VIII.), we see that the thermal conductivity of iron is 80 times, and that of copper 500 times that of water. The thermal conductivity of iron is 3500 times, and that of copper is 20,000 times that of air. Hence, although the water or air at the very interface of its contact with the metal is essentially at the same temperature as the metal, there must be great differences of temperature in very thin layers of the fluid close to the interface when there is large flux of heat through the metal, and the temperature of the fluid as measured by any practicable thermometer, or inferred from knowledge of the average temperature of the whole fluid, or from the temperatures of entering and leaving currents of liquid, may differ by scores of degrees from the actual temperature of the solid at the interface. It is remarkable that Péclet, while perceiving that Clément's result was largely erroneous, and moreover, improving the mode of experimenting by introducing a rotating mechanical stirrer to change very rapidly the fluid in contact with the solid, only multiplied Clément's conductivity by 30 instead of by 200, which would have been necessary to annul the error. Notwithstanding his failure to obtain accurate results for metals, we have ventured to include his results for wood, and solids of lower conductivity than wood, in our table, because we perceive that he was alive to the necessity for very energetic stirring of the liquid, and the mechanical means which he adopted for it, though utterly insufficient for the case of even the least conductive of the metals, were probably not so for wood and solids of lower conductivity than wood; and because it is not probable that the complication of heat generated by the stirring (which Ångström suggests as an objection to Péclet's method) was in any case sufficient to produce a sensible influence upon the experimental results.

§ 77. The first correct determinations of thermal conductivities were given by Forbes in his paper on the temperature of the earth, in the Transactions of the Royal Society of Edinburgh for 1846, as calculated from his observations of underground temperature at three localities in the neighbourhood of Edinburgh—the trap rock of Calton Hill, the sand of the Experimental Garden, and the sandstone of Craigleith Quarry—by an imperfect approximate method indicated by Poisson. A more complete analytical treatment of the observational results, analysed harmoniously and interpreted by application of Fourier's formulæ (equation (19) of Math. App.) to each term separately by W. Thomson, gave results (quoted in Table VIII. below) for the conductivities, which differed but little from Forbes's approximate determinations.

§ 78. It has always seemed to us that the best mode of Estimating and comparing the thermal capacities of different substances and of matter in different conditions, solid, liquid, and gaseous, gained within the last twenty years. Ångström, Forbes, F. Neumann, and Taill have given, each one of them with thoroughly sufficient experimental evidence to leave no room to doubt the substantial accuracy of his results, absolute values for the thermal conductivities of copper and iron. Clausius and Maxwell have given us thermal conductivities of air and other gases, from their splendid development of the kinetic theory, which are undoubtedly trustworthy as somewhat close approximations to the true values, and which it is quite possible are more accurate than we can hope to see obtained from direct measurements of the conduction of heat through gases. J. T. Bottomley has given a trustworthy and somewhat closely accurate direct measurement of the thermal conductivity of water. From the results of these experimenters' works, reduced to uniform C. G. S. reckoning in our tables of thermal conductivities (Table VIII.), we see that the thermal conductivity of iron is 80 times, and that of copper 500 times that of water. The thermal conductivity of iron is 3500 times, and that of copper is 20,000 times that of air. Hence, although the water or air at the very interface of its contact with the metal is essentially at the same temperature as the metal, there must be great differences of temperature in very thin layers of the fluid close to the interface when there is large flux of heat through the metal, and the temperature of the fluid as measured by any practicable thermometer, or inferred from knowledge of the average temperature of the whole fluid, or from the temperatures of entering and leaving currents of fluid, may differ by scores of degrees from the actual temperature of the solid at the interface. It is remarkable that Péclet, while perceiving that Clément's result was largely erroneous, and moreover, improving the mode of experimenting by introducing a rotating mechanical stirrer to change very rapidly the fluid in contact with the solid, only multiplied Clément's conductivity by 30 instead of by 200, which would have been necessary to annul the error. Notwithstanding his failure to obtain accurate results for metals, we have ventured to include his results for wood, and solids of lower conductivity than wood, in our table, because we perceive that he was alive to the necessity for very energetic stirring of the liquid, and the mechanical means which he adopted for it, though utterly insufficient for the case of even the least conductive of the metals, were probably not so for wood and solids of lower conductivity than wood; and because it is not probable that the complication of heat generated by the stirring (which Ångström suggests as an objection to Péclet's method) was in any case sufficient to produce a sensible influence upon the experimental results.

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and, because the emission is supplied by conduction from within,
\[
\mathbf{\nabla} \cdot \mathbf{K} \left( \frac{\partial \mathbf{v}}{\partial t} \right).
\]

Hence
\[
\mathbf{v}' = \mathbf{v} + \mathbf{v}_0 .
\]

The value of \( \mathbf{v} \) for a blackened globe hung within a hollow, with blackened bounding surface, is about \( \frac{2}{5} \pi \) according to Macfarlane's experiments (Table VII. below), and considerably less for surfaces with any degree of polish. We may therefore take \( \frac{2}{5} \pi \mathbf{v} \) as a maximum value for \( \mathbf{v} \). The values of \( \mathbf{K} \) for copper and iron at ordinary temperatures are, in C. G. S., approximately \(-95\) and \(-18\). Hence, if \( r = 5 \text{ cm} \) (or the diameter of the bar 10 cm., which is more than it is likely to be in any laboratory experiments), we find
\[
\mathbf{v}' - \mathbf{v} - \mathbf{v}_0 .
\]

Hence the error will be practically nothing if we take \( \mathbf{v}' = \mathbf{v} \). Thus, and if we suppose \( \mathbf{k} \) to be independent of temperature, (1) becomes
\[
\frac{\partial \mathbf{v}}{\partial \mathbf{t}} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \mathbf{v} \frac{\partial \mathbf{v}}{\partial \mathbf{x}}
\]

or
\[
\frac{\partial \mathbf{v}}{\partial \mathbf{t}} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \mathbf{v} \frac{\partial \mathbf{v}}{\partial \mathbf{x}}
\]

which is Fourier's equation for the conduction of heat along a bar or the circumference of a thin ring. Its solution to express simple harmonic variations of temperature produced in an infinitely long bar by properly varied thermal appliances at one end is
\[
v = e^{-\alpha x} \cos\left(\pi t - \frac{\pi x}{A} + \epsilon \right)
\]

\( a, \mathbf{R}, \epsilon \) being arbitrary constants,—the "speed," the semi-range, and the epoch for \( x = 0 \); and \( f, g \) constants given by the formulas
\[
\begin{align*}
\int_0^\infty 
\frac{\partial \mathbf{x}}{\partial \mathbf{t}} 
&= \frac{1}{2} (n^2 + h^2) \frac{1}{\sqrt{h}} 
\int_0^\infty 
\frac{\partial \mathbf{x}^2}{\partial \mathbf{t}} 
&= \frac{1}{2} (n^2 + h^2) \frac{1}{\sqrt{h}}
\end{align*}
\]

For iron and copper the values of \( c \) are respectively \( 95 \) and \( 945 \). Hence, with the previously used values of \( \mathbf{k} \) for these metals, and with \( 1/4000 \) for \( c \), we find \( k = 1/1 \) for copper and \( k = 2 \) for iron; and for either, \( h = 1/1700 \) nearly enough. Suppose, for example, \( r = 2 \text{ cm} \), this makes \( h = 1/1600 \), and suppose the period to be 32 m. (the greatest of those chosen by Angström), this makes \( n = 2 \pi (60 \times 32) \), or roughly \( n = 1/310 \) and \( h/n = 1/17 \). Now when \( h/n = 1 \) is small, we have approximately
\[
\begin{align*}
&= \frac{1}{2} (n^2 + h^2) \frac{1}{\sqrt{h}} 
&= \frac{1}{2} (n^2 + h^2) \frac{1}{\sqrt{h}}
\end{align*}
\]

by which we see that the propagation of the variations of temperature is but little affected by the lateral surface emissivity. Little as this effect is, it is very perfectly eliminated by the relation
\[
f = \frac{g^2}{g_0}
\]

which we find from (5).

It is convenient to remark that \( g \) is the rate of diminution of the Napierian logarithm of the range, and \( f \) the rate of retardation of the epoch (reckoned in radians) per centimetre of the bar. Were there no lateral emissivity these would be equal, and the diffusivity might be calculated from each separately. This was done by William Thomson in his analysis of the Edinburgh underground temperature observations. But in the propagation of periodic variation of temperature along a bar (as of electric potential along the conductor of a submarine cable) lateral emissivity (or imperfect insulation) augments the rate of diminution of the logarithm of the amplitude, and diminishes the rate of retardation of the phase, leaving the product of the two rates unaffected, and allowing the diffusivity to be calculated from it by (6). This was carried out for copper and iron by Angström in Sweden, and the results communicated to the Royal Swedish Academy in January 1861. German and English editions of his paper have been published in Poggendorff's Annalen for 1853, and the Phil. Mag. for 1865 (first half year). The details of the apparatus and of the actual experiments, in which Angström had the assistance of Thalen, are sufficiently described in this paper, and in a subsequent paper (Pogg. Annalen for 1863, p. 428), to allow us to feel perfect confidence in the very approximate accuracy of the results. Hence we have included them in our Table.

§ 79. The question, Does thermal conductivity vary with temperature? was experimentally investigated by Forbes about thirty years ago; and in a first provisional statement of results communicated to the British Association at Belfast in 1852 it was stated that the thermal conductivity of iron is less at high temperatures than at low. Forbes's investigation was conducted by elaborate method of experimenting, in which the static temperature of a long bar of metal is observed after the example of the earlier experiments of Despretz, with a most important additional experiment and measurement by means of which the static result is reduced to give conductivity in absolute measure, and not merely as in Despretz's experiments to give comparisons between the conductivities of different metals. In 1861 and 1865 Forbes published results calculated from his experiments, including the first determination of thermal conductivity of a metal (iron) in absolute measure, and a confirmation of his old result that the conductivity of iron diminishes with rise of temperature. Forbes's bars have been inherited and further utilized, and bars of copper, lead, and other metals have been made and experimented upon according to the same method, by his successor in the university of Edinburgh, Professor Tait. The investigation was conducted partly with a view to test whether the electric conductivities and the thermal conductivities of different metals, more or less approximately pure, and of metallic alloys, are in the same order, and, further, if their thermal conductivities are approximately in the same proportion as their electric conductivities. The following results quoted from his paper on "Thermal and Electric Conductivity" (Transactions R.S.E., 1878) are valuable as an important instalment, but expressly only an instalment, towards the answering of this interesting question:

"Taking the inferior copper ('Copper C') as unit both for thermal and for electric conductivities, we find the following table of conductivities at ordinary temperatures, with the rough results as to specific gravity and specific heat referred to in § 15 above:

<table>
<thead>
<tr>
<th>Material</th>
<th>Thermal Conductivity</th>
<th>Electric Conductivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper, Crown</td>
<td>1.41</td>
<td>1.72</td>
</tr>
<tr>
<td>Copper, Crown</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Forbes's iron</td>
<td>0.29</td>
<td>0.26</td>
</tr>
<tr>
<td>Lead</td>
<td>0.12</td>
<td>0.14</td>
</tr>
<tr>
<td>German silver</td>
<td>0.14</td>
<td>0.17</td>
</tr>
</tbody>
</table>

1 The first paper is marred unhappily by two or three algebraic and arithmetical errors. One algebraic error is very disturbing to a careful reader, and might even to a hasty judgment seem to throw doubt on the validity of the experimental use which is made of the formulae. There is, however, no real foundation for any such doubt. The following details of the correction suffice to put the matter right. For the general term as printed in Angström's paper read
\[
\sum_{n}^\infty \mathbf{e}^{-2\pi^2 n^2} \sin \left( \frac{2\pi l}{T} \mathbf{x} - 2\pi^2 n^2 \right)
\]

with the following values for \( g_0, g_1, g_2, g_3 \):

\[
\begin{align*}
g_0 &= \sqrt{\frac{\pi^2}{K_R^2 + H^2}} \frac{H}{4K_R^2 + 2K} \\
g_1 &= \sqrt{\frac{\pi^2}{K_R^2 + H^2}} \frac{H}{4K_R^2 + 2K} \\
g_2 &= \sqrt{\frac{\pi^2}{K_R^2 + H^2}} \frac{H}{4K_R^2 + 2K} \\
g_3 &= \sqrt{\frac{\pi^2}{K_R^2 + H^2}} \frac{H}{4K_R^2 + 2K}
\end{align*}
\]

instead of these formulæ without the \( \gamma \), as Angström gives them.
The agreement of these numbers is by no means so close as is generally stated; but this is no longer remarkable, for it is well known that the electric conductivity of all pure metals varies very much with the temperature, while we have seen that as regards thermal conductivity there is but slight change with either copper or iron, and that there is a large change with iron. This is confirmed with some results of my own on the electric conductivity of iron at high temperatures (Proc. R. S. E., 1872-73, p. 83), and the results of the repetition of these experiments by a party of my laboratory students. (Proc. R. S. E., 1872-76, p. 620.)

§ 80. The absolute values of Tait's results for the five metals of the preceding list are given in C.G.S. units in our general table. As to change of conductivity with temperature, we have a discrepancy between the results of Angström's and Tait's experiments. Tait finds but little difference in the conductivity of copper through the wide range of temperatures from 0° to 300° C, and that difference an augmentation instead of a diminution at the high temperatures as shown in the following results (where t is measured from 0° C):

Copper
Crown 0.076 (1 - 0.00004#)
C 0.054 (1 - 0.000055#)
Iron 0.015 (1 - 0.000146#)

On the other hand, Angström finds for copper from experiments described in his second paper referred to above, at mean temperatures of from 26°-38° to 71°-8° C, results which reduced to C.G.S. are as follows:

Copper
0.982 (1 - 0.001519#)
Iron
1.99 (1 - 0.00287#)

From the admirable method of experimenting, and the care with which experiments were carried out by himself and Thalen, it is impossible for us to doubt the validity and somewhat closely approximate numerical truth of the result. On account of the discrepancy from Tait's results it is desirable that Angström's method should be carried out for copper through a much wider range of temperatures. This can be done with great ease from the lowest temperature obtainable by freezing mixtures to temperatures up to the melting point of copper, so excellently plastic is Angström's method. Our proposed extension of it is to be carried out by proper thermal appliances to the end of the bar which Angström left to itself,—appliances by which in one series of experiments it may be kept constant at 50° or -60° C, in others left to itself to take nearly the atmospheric temperature, in others kept at high temperatures limited only by the melting temperature of copper, if the experimenter desires to go so far. We would also suggest that the thermo-electric method first introduced by Wiedemann and Franz in their experiments on the static temperature of bars or wires heated at one end and allowed to lose heat by convection and radiation from their sides, (which was rejected, not, we think, judiciously, by Angström), might be used with advantage instead of the mercury thermometers inserted in holes in the bar in Angström's apparatus; or that, if thermometers are to be used, air thermometers in which the bulb of the thermometer is itself a very small hole in the bar experimented on, and the tube a fine-bore glass tube fitted to this hole, would be more preferable to the mercury thermometers hitherto employed in, we believe, all experiments except those of Wiedemann and Franz, on the conduction of heat along metallic bars.

§ 81. Fourier's ninth chapter is entitled "De la Diffusion de la Chaleur." The idea embodied in this title is the spreading of heat in a solid tending to ultimate equalization of temperature throughout it, instead of the transference of heat from one body to another by conduction through the substance considered. Though Fourier makes the special subject of his chapter on "Diffusion" the conduction of heat through an infinite solid, we may conveniently regard as coming under the several designations "Diffusion of Heat" every case of thermal conduction in which the heat conducted across any part of the solid has the effect of warming contiguous parts on one side of it, or of leaving contiguous parts on the other side cooler,—in other words, every case in which the temperature of the body through which the conduction of heat takes place is varying with time, as distinguished from what Fourier calls "Uniform Motion of Heat," or the class of cases in which the temperature at every point of the body is constant. The experiments of Pèlet, Desprets, Forbes and Tait, Wiedemann and Franz, were founded on the uniform conduction of heat across slabs or along boxes, and their determinations of relative and absolute conductivities were made by comparing or by measuring absolutely quantities of heat that were conducted out of the body tested. On the other hand, it is the diffusion of heat that is used in the determinations of thermal conductivity in absolute measure by Forbes and William Thomson from the periodic variations of underground temperature; in those of Angström, from his experiments on the spreading of periodic variations of temperature through bars of iron and copper, and a series of valuable experiments a year or two later by F. Neumann, applying the same general method to bars of brass, zinc, German silver, and iron; in experiments by F. Neumann on substances of lower conductivity (coal, cast sulphur, ice, snow, frozen earth, gritstones) formed into cubes or globes of 5 or 6 inches diameter, and heated uniformly, and then left to cool in an atmosphere of lower temperature, and from time to time during the cooling explored by thermo-electric junctions imbedded in them to show their internal distribution; in similar experiments on the cooling of globes of 14 cm. diameter of porphyritic trachyte by Ayrton and Perry in Japan; and in Kirchhoff and Hansemann's recent experiments, to find the thermal conductivity of iron by the not well-chosen method of suddenly cooling one side of a cube of iron of 14 cm., and observing the temperatures by aid of thermo-electric junctions in several points of the line perpendicular to this side, its middle.

§ 82. When the effect of heat conducted across any part of the body in heating the substance on one side or leaving the substance on the other side cooler is to be reckoned, it is convenient to measure the thermal conductivity in terms, not of the ordinary general gramme water-unit of heat, but of a special unit, the quantity required to raise unit bulk of the substance in 1°. In other words, if \( k \) be the conductivity in terms of any thermal unit, and \( e \) the thermal capacity of unit bulk of the substance, it is \( k \), not merely \( k \), that expresses the quantity of the substance on which the phenomenon chiefly depends. We therefore propose to give to \( k \) the name of thermal diffusivity (or simply diffusivity when heat is understood to be the subject), while still using the word thermal conductivity to denote the conducting power as defined in § 73, without restriction as to the thermal unit employed. It is interesting and important to remark that "diffusivity" is essentially to be reckoned in units of area per unit of time, that its "dimensions" are L²T⁻¹ (see Dimensions). Its regular C.G.S. reckoning is therefore in square centimetres per second. In the article DIFFUSION the relation between diffusion of heat and diffusion of matter is explained. We have added diffusion of electricity through a submarine cable, which has been shown to follow the same law as the thermal conductivity. Wiedemann's (late Poggendorff's) Annalen, 1859, No. 1. Wiedemann and Franz, on the conduction of heat along metallic bars. Diffusion of heat.
"linear" diffusion of heat, as Fourier calls the diffusion of heat when the isothermal surfaces are parallel planes. The curves of the following diagram and Tables A, B, and C show in a practically useful way the result in the course of the times noted, of from fractions of a second to thousands of millions of years, of linear diffusion of two different qualities in an infinite line from an initial condition in which there is sudden transition from one quality to the other, in the thoroughly practical cases specified in the accompanying explanations.

**Fig. 11.—Diagram of Diffusion.**

Curve No. 1 shows temperature, or quantity of substance in solution, or potential in the conductor of a submarine cable through which electricity is diffusing. Curve No. 2 shows rate per unit of distance of variation of the temperature, or of the quantity of substance in solution. Vertical ordinates are actual distances through the medium. Horizontal ordinates represent temperature, or quantity of diffusing substance in No. 1 curve, and rate of variation of temperature or of diffusing substance or of electric potential in No. 2 curve.

### Diffusions (Secular).—Table B.

<table>
<thead>
<tr>
<th>Substance</th>
<th>Time in Years from the commencement of the Diffusion until the Condition represented by the Curves on the Scale of 300 Kilometres, or 1,000,000 times the Actual Scale, is reached.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbonic acid through air</td>
<td>2,320,000 years</td>
</tr>
<tr>
<td>Heat through hydrogen</td>
<td>283,000</td>
</tr>
<tr>
<td>copper</td>
<td>283,000</td>
</tr>
<tr>
<td>iron</td>
<td>1,733,000</td>
</tr>
<tr>
<td>air</td>
<td>1,852,000</td>
</tr>
<tr>
<td>underground strata</td>
<td>2,000,000</td>
</tr>
<tr>
<td>wood</td>
<td>244,000,000</td>
</tr>
<tr>
<td>Common salt through water</td>
<td>1,370,000,000</td>
</tr>
</tbody>
</table>

1 Instructive as to the proportion of carbonic acid in air at different heights, proving its approximate uniformity due to convection, not to diffusion.

### § 83. The following tables contain useful information regarding various thermal properties of matter. Every known property of a piece of matter, except its gravity and its inertia, varies with variation of temperature. For further information respecting the effect of variation of temperature in causing changes in properties of matter reference is made to the article Elasticity, Electricity, Magnetism, Light, Matter (Properties of), &c.

#### Table I.—Linear Expansions of Solids.

<table>
<thead>
<tr>
<th>Name of Cable</th>
<th>Substances</th>
<th>Mean Expansion per Degree C. through Range stated</th>
<th>Range</th>
<th>Authority</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silver</td>
<td>0.0002120</td>
<td>0° to 100°</td>
<td></td>
<td>Muschenbruck</td>
</tr>
<tr>
<td></td>
<td>1.916</td>
<td>0° to 100°</td>
<td></td>
<td>Kupffer</td>
</tr>
<tr>
<td></td>
<td>1.939</td>
<td>0° to 100°</td>
<td></td>
<td>Matthiessen</td>
</tr>
<tr>
<td>Thallium</td>
<td>3.021</td>
<td>40°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td>Sulphur, Sicily</td>
<td>6.413</td>
<td>40°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td>Selenium, cast</td>
<td>3.580</td>
<td>40°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td>Tellurium, cast</td>
<td>1.675</td>
<td>40°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td>Lead</td>
<td>2.799</td>
<td>6° to 100°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td></td>
<td>2.994</td>
<td>6° to 100°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td>Iron</td>
<td>1.115</td>
<td>6° to 100°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td></td>
<td>1.190</td>
<td>6° to 100°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td>Steel, annealed</td>
<td>1.220</td>
<td>6° to 100°</td>
<td></td>
<td>Muschenbruck</td>
</tr>
<tr>
<td>French cast, tempered</td>
<td>1.222</td>
<td>40°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td>French cast, annealed</td>
<td>1.101</td>
<td>40°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td>English cast, annealed</td>
<td>1.066</td>
<td>40°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td>Steel, soft</td>
<td>1.62</td>
<td>6° to 100°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td>Cast iron</td>
<td>1.112</td>
<td>6° to 100°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td>Cobalt, red by H. compressed</td>
<td>1.336</td>
<td>40°</td>
<td></td>
<td>Fizeau</td>
</tr>
<tr>
<td>Nickel, red by H. compressed</td>
<td>1.379</td>
<td>40°</td>
<td></td>
<td>Fizeau</td>
</tr>
</tbody>
</table>

1 Abridged from Clarke's Constants of Nature.

2 Where only one number is given for the range in this and the following table, the corresponding statement is to be understood as applying through a small range on either side of the number stated.
<table>
<thead>
<tr>
<th>Name</th>
<th>Mean Expansion per degree</th>
<th>Range</th>
<th>Authority</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper</td>
<td>0.00001278</td>
<td>0° to 100°</td>
<td>Borda</td>
</tr>
<tr>
<td>native</td>
<td>1969</td>
<td>40°</td>
<td>Mattheissen</td>
</tr>
<tr>
<td>commercial</td>
<td>1978</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>Ruthenium, semi-fused</td>
<td>0.0625</td>
<td>40°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Rhodium, semi-fused</td>
<td>0.0509</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>Palladium, forged</td>
<td>1116</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>Platinum</td>
<td>0.0600</td>
<td>0° to 100°</td>
<td>Calvert, Johnson, Lowe, Mattheissen</td>
</tr>
<tr>
<td>Iridium</td>
<td>0.0700</td>
<td>40°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Osmium, semi-fused</td>
<td>0.0507</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Zinc</td>
<td>2200</td>
<td>0° to 100°</td>
<td>Calvert, Johnson, Lowe, Mattheissen</td>
</tr>
<tr>
<td>distilled</td>
<td>2210</td>
<td>40°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Cadmium</td>
<td>339</td>
<td>0° to 100°</td>
<td>Calvert, Johnson, Lowe, Mattheissen</td>
</tr>
<tr>
<td>distilled</td>
<td>3406</td>
<td>40°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Magnesium, cast</td>
<td>3594</td>
<td>40°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Indium</td>
<td>4170</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>Aresine, sublimed</td>
<td>0.0559</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>Antimony</td>
<td>0.0990</td>
<td>0° to 100°</td>
<td>Calvert, Johnson, Lowe, Mattheissen</td>
</tr>
<tr>
<td>along axis</td>
<td>1958</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>normal to axis</td>
<td>1962</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>mean value</td>
<td>1943</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>Bismuth</td>
<td>333</td>
<td>0° to 100°</td>
<td>Calvert, Johnson, Lowe, Mattheissen</td>
</tr>
<tr>
<td>along axis</td>
<td>1612</td>
<td>40°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>mean value</td>
<td>1605</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>Gold, annealed</td>
<td>1469</td>
<td>0° to 100°</td>
<td>Muschenbruck, Calvert, Johnson, Lowe, Mattheissen</td>
</tr>
<tr>
<td>cast</td>
<td>1476</td>
<td>0° to 100°</td>
<td></td>
</tr>
<tr>
<td>Carbon, diamond</td>
<td>0.0000</td>
<td>-28° to 8°</td>
<td></td>
</tr>
<tr>
<td>graphite</td>
<td>0.0132</td>
<td>50°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>gas carbon</td>
<td>0.0132</td>
<td>50°</td>
<td></td>
</tr>
<tr>
<td>anthracite</td>
<td>2078</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>Tin</td>
<td>2949</td>
<td>0° to 100°</td>
<td>Muschenbruck, Calvert, Johnson, Lowe, Mattheissen</td>
</tr>
<tr>
<td></td>
<td>273</td>
<td>0° to 100°</td>
<td></td>
</tr>
<tr>
<td>Alumium, commercial</td>
<td>222</td>
<td>0° to 100°</td>
<td>Calvert, Johnson, Lowe, Mattheissen</td>
</tr>
<tr>
<td>Folin span, CaF₂</td>
<td>10594</td>
<td>0° to 100°</td>
<td>Pfaff</td>
</tr>
<tr>
<td>Silver iodide, AgI, cylinder, precipitated and compressed</td>
<td>0.000000229</td>
<td>40°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Lanthanum, CeO₃</td>
<td>0.000000229</td>
<td>40°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Mercury iodide, HgI₂</td>
<td>2287</td>
<td>40°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Lead iodide, PbI₂</td>
<td>3539</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>Cadmium iodide, CdI₂</td>
<td>2016</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>Ice, H₂O₃</td>
<td>0.00241</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hematite, Fe₂O₃ along axis</td>
<td>0.000000229</td>
<td>40°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Hematite, Fe₂O₃ normal to axis</td>
<td>0.0886</td>
<td>40°</td>
<td></td>
</tr>
<tr>
<td>Magnetic oxide of iron, Fe₂O₃</td>
<td>0.0540</td>
<td>0° to 100°</td>
<td>Pfaff</td>
</tr>
</tbody>
</table>

**HEAT**

<table>
<thead>
<tr>
<th>Name</th>
<th>Mean Expansion per degree</th>
<th>Range</th>
<th>Authority</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper oxide, CuO</td>
<td>-0.00000008</td>
<td>0° to 100°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Zinc oxide, ZnO</td>
<td>0.0256</td>
<td>0° to 100°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Quartz, SiO₂ along axis</td>
<td>0.0141</td>
<td>0° to 100°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Quartz, SiO₂ along axis</td>
<td>0.0160</td>
<td>0° to 100°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Pyrite, Fe₃S₄</td>
<td>0.0094</td>
<td>0° to 100°</td>
<td>Pfaff</td>
</tr>
<tr>
<td>Beryl, length axis</td>
<td>0.01294</td>
<td>0° to 100°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Emerald, along axis</td>
<td>+0.00000011</td>
<td>0° to 100°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Topaz, lesser horiz. axis</td>
<td>0.00325</td>
<td>0° to 100°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Topaz, greater horiz. axis</td>
<td>0.00322</td>
<td>0° to 100°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Topaz, vertical</td>
<td>0.00423</td>
<td>0° to 100°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Tourmaline, length axis</td>
<td>0.00229</td>
<td>0° to 100°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Tourmaline, horiz. axis</td>
<td>0.00722</td>
<td>0° to 100°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Garnet</td>
<td>0.00478</td>
<td>0° to 100°</td>
<td>Fizeau</td>
</tr>
<tr>
<td>Glass tube</td>
<td>0.0003333</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Rod</td>
<td>0.000339</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Plate</td>
<td>0.001237</td>
<td>20° to 30°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>plate</td>
<td>0.001214</td>
<td>20° to 30°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Crown</td>
<td>0.007572</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Crown</td>
<td>0.008792</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Crowne</td>
<td>0.001271</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Tubing</td>
<td>0.004298</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Tube</td>
<td>0.004298</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Soft Thuringt.</td>
<td>0.001214</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Wedgewood ware</td>
<td>0.008313</td>
<td>15° to 60°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Bayaux porcelain</td>
<td>0.008889</td>
<td>15° to 60°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Lead and tin-solder</td>
<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Lead and antimony</td>
<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Type metal</td>
<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Zinc and tin-solder</td>
<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Copper and tin-solder</td>
<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Copper and tin-solder</td>
<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Silver tapmetal</td>
<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Silver tapmetal</td>
<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Silver tapmetal</td>
<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
<td>Silver tapmetal</td>
<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
<tr>
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<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
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<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
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<td>Silver tapmetal</td>
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<td>Smeaton</td>
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<td>Smeaton</td>
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<td>Smeaton</td>
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<td>Smeaton</td>
</tr>
<tr>
<td>Silver tapmetal</td>
<td>0.000357</td>
<td>0° to 100°</td>
<td>Smeaton</td>
</tr>
</tbody>
</table>
### Table II.—Cubic Expansions of Solids and Liquids.

<table>
<thead>
<tr>
<th>Name of Liquid</th>
<th>Atomic Constitution</th>
<th>Temperature of Boiling Point</th>
<th>Expansion per Degree C.</th>
<th>Range.</th>
<th>Authority.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bromine</td>
<td>(C₆H₅)₂O</td>
<td>35°-5</td>
<td>-0.00755</td>
<td>0° to 100°</td>
<td>Pierre</td>
</tr>
<tr>
<td>Ethylene alcohol</td>
<td>C₆H₅(O)</td>
<td>75°-5</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Methylic alcohol</td>
<td>C₆H₅(OH)</td>
<td>65°-5</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Ammonic alcohol</td>
<td>C₆H₅(OH)</td>
<td>131°-8</td>
<td>-0.00755</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Chloric acid</td>
<td>(C₆H₅)₂Cl</td>
<td>11°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Iodic acid</td>
<td>C₆H₅I</td>
<td>30°-5</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Bromic acid</td>
<td>C₆H₅Br</td>
<td>30°-5</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Iodic acid</td>
<td>C₆H₅IO₂</td>
<td>50°-5</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Acetic acid</td>
<td>C₆H₅CO₂H</td>
<td>74°</td>
<td>-0.00755</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Butyric acid</td>
<td>C₆H₅CO₂H₂</td>
<td>119°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Butyric acid</td>
<td>C₆H₅CO₂H₂</td>
<td>59°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Butyric acid</td>
<td>C₆H₅CO₂H₂</td>
<td>102°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Acetic acid</td>
<td>C₆H₅CO₂H</td>
<td>74°</td>
<td>-0.00755</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Butyric acid</td>
<td>C₆H₅CO₂H₂</td>
<td>119°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Butyric acid</td>
<td>C₆H₅CO₂H₂</td>
<td>59°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Butyric acid</td>
<td>C₆H₅CO₂H₂</td>
<td>102°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Terchloride of phosphorus</td>
<td>P₂Br₃</td>
<td>175°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Terchloride of phosphorus</td>
<td>PCl₃</td>
<td>78°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Terchloride of arsenic</td>
<td>AsCl₃</td>
<td>133°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Bichloride of tin</td>
<td>SnCl₂</td>
<td>115°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Tetrachloride of titanium</td>
<td>TiCl₄</td>
<td>76°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>&quot;Terchloride of silicon&quot;</td>
<td>Si₂Br₃</td>
<td>153°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>&quot;Bichloride of ethylene&quot;</td>
<td>C₂Cl₄</td>
<td>84°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>&quot;Bichloride of ethylene&quot;</td>
<td>C₂Cl₄</td>
<td>84°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>&quot;Bichloride of ethylene&quot;</td>
<td>C₂Cl₄</td>
<td>84°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>&quot;Bromide of ethylene&quot;</td>
<td>C₂Br₄</td>
<td>132°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Bromine</td>
<td>Br</td>
<td>63°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Bismuthide of carbon</td>
<td>CS₂</td>
<td>47°</td>
<td>-0.00750</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
</tbody>
</table>

### Table III.—Expansions of Liquids.

Pierre, having measured the expansions of many different liquids, embodies the results in the empirical formula: 

$$\delta = \alpha + \beta t + \gamma t^2$$

where \( \delta \) represents the expansion of unit volume from \( 0° \) to \( t° \) cent., and \( \alpha, \beta, \gamma \) have the values given in the Table for the different liquids specified. The substances and formulas in quotation marks are quoted from Dixon's Heat, Dublin, 1849.

<table>
<thead>
<tr>
<th>Name of Liquid</th>
<th>Atomic Constitution</th>
<th>Temperature of Boiling Point</th>
<th>Expansion per Degree C.</th>
<th>Range.</th>
<th>Authority.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Antimony</td>
<td>-</td>
<td>-</td>
<td>-0.00003</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Bismuthine</td>
<td>-</td>
<td>-</td>
<td>-0.00001</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Gold</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Diamond</td>
<td>-</td>
<td>-</td>
<td>-0.00005</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Water, H₂O</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Ice</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Sulphur dioxide</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Carbon dioxide, liquid, CO₂</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Copper oxide, CuO</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Zinc, ZnO</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Quartz, SiO₂</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Beryl, BeO</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Topaz</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Tourmaline</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Garnet</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Amethyst</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Iolite</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Zirconite</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Glass, white tube</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Globe, globe</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Green tube, globe</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Crystal tube, globe</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Globe, globe</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
<tr>
<td>Crystal tube, globe</td>
<td>-</td>
<td>-</td>
<td>-0.00004</td>
<td>0° to 100°</td>
<td>Kopp.</td>
</tr>
</tbody>
</table>

* Abridged from Clarke's Constants of Nature.
### Table IV.—Density of Water.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>1.00000</td>
<td>10°</td>
<td>1.00010</td>
<td>20°</td>
<td>1.00020</td>
</tr>
<tr>
<td>1</td>
<td>1.00001</td>
<td>11</td>
<td>1.00011</td>
<td>21</td>
<td>1.00021</td>
</tr>
<tr>
<td>2</td>
<td>1.00002</td>
<td>12</td>
<td>1.00012</td>
<td>22</td>
<td>1.00022</td>
</tr>
<tr>
<td>3</td>
<td>1.00003</td>
<td>13</td>
<td>1.00013</td>
<td>23</td>
<td>1.00023</td>
</tr>
<tr>
<td>4</td>
<td>1.00004</td>
<td>14</td>
<td>1.00014</td>
<td>24</td>
<td>1.00024</td>
</tr>
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<td>1.00015</td>
<td>25</td>
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<td>1.00006</td>
<td>16</td>
<td>1.00016</td>
<td>26</td>
<td>1.00026</td>
</tr>
<tr>
<td>7</td>
<td>1.00007</td>
<td>17</td>
<td>1.00017</td>
<td>27</td>
<td>1.00027</td>
</tr>
<tr>
<td>8</td>
<td>1.00008</td>
<td>18</td>
<td>1.00018</td>
<td>28</td>
<td>1.00028</td>
</tr>
<tr>
<td>9</td>
<td>1.00009</td>
<td>19</td>
<td>1.00019</td>
<td>29</td>
<td>1.00029</td>
</tr>
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<td>10</td>
<td>1.00010</td>
<td>20</td>
<td>1.00020</td>
<td>30</td>
<td>1.00030</td>
</tr>
</tbody>
</table>

### Table V.—Steam Pressures (in Centimeters of Mercury).

<table>
<thead>
<tr>
<th>Water</th>
<th>Mercury</th>
<th>Sulphur</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp.</td>
<td>Pressure</td>
<td>Temp.</td>
</tr>
<tr>
<td>-32°</td>
<td>0.034</td>
<td>0°</td>
</tr>
<tr>
<td>-30</td>
<td>0.040</td>
<td>10°</td>
</tr>
<tr>
<td>-20</td>
<td>0.049</td>
<td>20°</td>
</tr>
<tr>
<td>-10</td>
<td>0.060</td>
<td>30°</td>
</tr>
<tr>
<td>0</td>
<td>0.070</td>
<td>40°</td>
</tr>
<tr>
<td>10</td>
<td>0.090</td>
<td>50°</td>
</tr>
<tr>
<td>20</td>
<td>0.110</td>
<td>60°</td>
</tr>
<tr>
<td>30</td>
<td>0.130</td>
<td>70°</td>
</tr>
<tr>
<td>40</td>
<td>0.150</td>
<td>80°</td>
</tr>
<tr>
<td>50</td>
<td>0.170</td>
<td>90°</td>
</tr>
</tbody>
</table>

### Table VI.—Steam Pressures.

<table>
<thead>
<tr>
<th>Temperature</th>
<th>Pressure</th>
<th>Temperature</th>
<th>Pressure</th>
<th>Pressure</th>
<th>Temperature</th>
<th>Pressure</th>
<th>Pressure</th>
<th>Pressure</th>
<th>Pressure</th>
<th>Temperature</th>
<th>Pressure</th>
<th>Pressure</th>
<th>Pressure</th>
<th>Pressure</th>
<th>Pressure</th>
<th>Pressure</th>
<th>Pressure</th>
<th>Pressure</th>
<th>Pressure</th>
</tr>
</thead>
</table>
Table VII.—Emittance for Heat of Polished and Blackened Copper Surfaces.

<table>
<thead>
<tr>
<th>Difference of Temperature</th>
<th>Amount of Heat lost per second per square centimetre of surface per degree of difference of temperatures</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Polished Surface</td>
<td>Blackened Surface</td>
</tr>
<tr>
<td>5°</td>
<td>000178</td>
<td>000252</td>
</tr>
<tr>
<td>10</td>
<td>000188</td>
<td>000266</td>
</tr>
<tr>
<td>20</td>
<td>000198</td>
<td>000279</td>
</tr>
<tr>
<td>30</td>
<td>000207</td>
<td>000289</td>
</tr>
<tr>
<td>40</td>
<td>000212</td>
<td>000306</td>
</tr>
<tr>
<td>50</td>
<td>000217</td>
<td>000318</td>
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<tr>
<td>60</td>
<td>000225</td>
<td>000328</td>
</tr>
<tr>
<td>70</td>
<td>000229</td>
<td>000339</td>
</tr>
<tr>
<td>80</td>
<td>000232</td>
<td>000349</td>
</tr>
<tr>
<td>90</td>
<td>000236</td>
<td>000358</td>
</tr>
<tr>
<td>100</td>
<td>000239</td>
<td>000362</td>
</tr>
</tbody>
</table>

Table VIII.—Thermal Conductivities.

<table>
<thead>
<tr>
<th>Substances</th>
<th>Conductivity, k</th>
<th>Authority</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper</td>
<td>0.0011</td>
<td></td>
</tr>
<tr>
<td>Iron</td>
<td>0.0016</td>
<td></td>
</tr>
<tr>
<td>Oxygen</td>
<td>0.000049</td>
<td>Clausius and Maxwell, according to kinetic theory.</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>0.000049</td>
<td></td>
</tr>
<tr>
<td>Carbonic oxide</td>
<td>0.000049</td>
<td></td>
</tr>
<tr>
<td>Carbonic acid</td>
<td>0.000033</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Hydrogen</td>
<td>0.000034</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Stone, porphyritic trachyte</td>
<td>-0.0059</td>
<td></td>
</tr>
<tr>
<td>Underground strata</td>
<td>0.0085</td>
<td>Forbes and Wm. Thomson.</td>
</tr>
<tr>
<td>Sandstone of Craigleith</td>
<td>0.01068</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Quarry</td>
<td>0.00815</td>
<td></td>
</tr>
<tr>
<td>Traprock of Calton Hill</td>
<td>0.00202</td>
<td>J. T. Bottomley. (With, in Everett's Units and Physical Constants.)</td>
</tr>
<tr>
<td>Sand of Experimental Garden</td>
<td>0.00202</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Water</td>
<td>0.002</td>
<td>J. T. Bottomley. (With, in Everett's Units and Physical Constants.)</td>
</tr>
<tr>
<td>Fir, across fibres</td>
<td>0.00026</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Walnut, across fibres</td>
<td>0.00023</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Oak, across fibres</td>
<td>0.00029</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Cork</td>
<td>0.00029</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Hemped cloth, new</td>
<td>0.000144</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Writing paper, white</td>
<td>0.000119</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Grey paper, unsized</td>
<td>0.000089</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Calico (new), of all densities</td>
<td>0.000139</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Wool (carded, of all densities)</td>
<td>0.000123</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Finely carded cotton wool</td>
<td>0.000111</td>
<td>Do. do.</td>
</tr>
<tr>
<td>Elder down</td>
<td>0.000108</td>
<td>Do. do.</td>
</tr>
</tbody>
</table>

Table IX.—Diffusivities (Thermal, Material, and Electric).

<table>
<thead>
<tr>
<th>Substance</th>
<th>Thermal Conductivity, k</th>
<th>Thermal Capacity of Unit Bulk, c.</th>
<th>Diffusivity, k.c.</th>
<th>Authority</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper</td>
<td>0.0011</td>
<td>0.0011</td>
<td>0.0011</td>
<td></td>
</tr>
<tr>
<td>Iron</td>
<td>0.0016</td>
<td>0.0016</td>
<td>0.0016</td>
<td></td>
</tr>
<tr>
<td>Air</td>
<td>0.000049</td>
<td>0.000049</td>
<td>0.000049</td>
<td></td>
</tr>
<tr>
<td>Oxygen</td>
<td>0.000049</td>
<td>0.000049</td>
<td>0.000049</td>
<td></td>
</tr>
<tr>
<td>Nitrogen</td>
<td>0.000049</td>
<td>0.000049</td>
<td>0.000049</td>
<td></td>
</tr>
<tr>
<td>Carbonic oxide</td>
<td>0.000049</td>
<td>0.000049</td>
<td>0.000049</td>
<td></td>
</tr>
<tr>
<td>Carbonic acid</td>
<td>0.000033</td>
<td>0.000033</td>
<td>0.000033</td>
<td></td>
</tr>
<tr>
<td>Hydrogen</td>
<td>0.000034</td>
<td>0.000034</td>
<td>0.000034</td>
<td></td>
</tr>
<tr>
<td>Stone, porphyritic trachyte</td>
<td>-0.0059</td>
<td>0.0059</td>
<td>0.0059</td>
<td></td>
</tr>
<tr>
<td>Underground strata</td>
<td>0.0085</td>
<td>0.0085</td>
<td>0.0085</td>
<td></td>
</tr>
<tr>
<td>Sandstone of Craigleith</td>
<td>0.01068</td>
<td>0.01068</td>
<td>0.01068</td>
<td></td>
</tr>
<tr>
<td>Quarry</td>
<td>0.00815</td>
<td>0.00815</td>
<td>0.00815</td>
<td></td>
</tr>
<tr>
<td>Traprock of Calton Hill</td>
<td>0.00202</td>
<td>0.00202</td>
<td>0.00202</td>
<td></td>
</tr>
<tr>
<td>Sand of Experimental Garden</td>
<td>0.00202</td>
<td>0.00202</td>
<td>0.00202</td>
<td></td>
</tr>
<tr>
<td>Water</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td></td>
</tr>
</tbody>
</table>

Table X.—Relative Diffusivities of different Substances through Water.

<table>
<thead>
<tr>
<th>Substance</th>
<th>Diffusivity, k.c.</th>
<th>Authority</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrophilic acid</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Chloride of sodium</td>
<td>2.23</td>
<td></td>
</tr>
<tr>
<td>Sugar</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Alumina</td>
<td>4.9</td>
<td></td>
</tr>
<tr>
<td>Caramel</td>
<td>3.8</td>
<td></td>
</tr>
</tbody>
</table>

Note: The above table has been accidentally printed on page 217 of the original document.
HEAT

Each of the following expressions I. . . . XVII. for \( \psi \) satisfies Fourier (9) or its equivalent (8), as the reader will readily verify for himself. Solutions corresponding to the peculiar character of the particular solution is specially noted in each case.

I. Instantaneous simple point-source; a quantity \( \theta \) of heat suddenly generated at the point \((0, 0, 0)\) at time \( t=0 \), and left to diffuse through an infinite homogeneous solid.

EQUATION (2) OF HEAT

Equation (2) was first given, proved as above, as the direct expression of Fourier's fundamental law of conduction, by W. Thomson (Gam.

If \( \varepsilon \) be constant, and we put \( \varepsilon/\alpha = k \), this becomes

\[
\frac{\partial v}{\partial t} = \frac{1}{c^2} \left( \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \phi^2} \right)
\]

(4).

If \( \kappa \) be constant, and we put \( \kappa/\alpha = \kappa \), this becomes

\[
\frac{\partial v}{\partial t} = \kappa \left( \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v}{\partial \theta} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \phi^2} \right)
\]

(5).

or

\[
\frac{\partial v}{\partial t} = \kappa \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)
\]

(8).

It is this restricted form which, with the further restriction that \( \varepsilon \) be constant, is most generally recognized as Fourier's equation of conduction, and is for it, with these restrictions, that his brilliant solutions were given. These solutions are available for practical use by limiting the range of temperature within which any one solution is continuously applied to a range of temperature within which the values of \( \varepsilon \) and \( \varepsilon \) are each nearly enough constant. We may expect 10° or 20° C. on each side of the mean temperature to be practically not too wide a range for any case, judging from copper and iron (§ 90), the only substance for which hitherto we have any information as to variations of both \( \varepsilon \) and \( \varepsilon \) with temperature.
Remark that (18) is obtainable from (15) by putting \( Q/4\pi a^2 = \sigma \), and \( a = \infty \); or directly from (10) by integration over the plane.

IX. "Linear Motion of Heat"; time-periodic plane-source; rate per unit of area, per unit of time, at time \( t \), a \( \sin 2\pi t \): 
\[
\iint_{0 \leq x, y < \infty} \frac{\partial \phi}{\partial t} - \phi = \sigma \delta(x - ct) dxdy.
\]

Verify that \( \sigma \) satisfies the equation 
\[
\iint_{0 \leq x, y < \infty} \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial \phi}{\partial x} = \iint_{0 \leq x, y < \infty} \frac{\partial \phi}{\partial x}.
\]

\( \sigma \) being any integer.

X. "Linear Motion of Heat"; space-periodic simple harmonic solid source, with plane isothermal surfaces. Initial distribution, \( v = V \sin \omega \), when \( t = 0 \). Solution for any value of \( t \):
\[
(20).
\]

Modifying the integral within the brackets to make it appear as an analytical expression belonging to the general theory of images, for the case of a single infinite row of images, and equating the result to the right-hand member, we see that
\[
(21).
\]

\( \xi \) being any integer.

XI. "Linear Motion of Heat"; space-periodic arbitrary solid source, isothermally plane. Initial distribution, \( v = f(x) \), when \( t = 0 \), \( f \) denoting an arbitrary periodic function, period \( t \); so that \( f(x + t) = f(x) \), \( f \) being any integer. Two solutions (A), (B).

(A) derived synthetically from (19): 
\[
v = \frac{1}{2\pi \sqrt{\nu t}} \int_{-\infty}^{\infty} d\xi f(\xi) \sin \frac{\pi x}{\nu t} \
\]

where \( S = \sum_{n=-\infty}^{\infty} e^{-i \pi n x} / \nu t \).

(B) derived analytically and synthetically from (20).

Find \( A_0, A_1, A_2, ... \), \( B_0, B_1, B_2, ... \), by the harmonic analysis, to satisfy the condition 
\[
\int_{-\infty}^{\infty} \cos \frac{\pi x}{\nu t} d\xi = \cos \frac{\pi x}{\nu t}.
\]

The No. 2 diffusion curve of \( \xi = \sigma \) is the representation of the first term (\( t = 0 \)) of this formula.

The comparison of these two solutions is very interesting physically, and useful arithmetically. To facilitate the comparison, put
\[
(23).
\]

the two solutions become
\[
(24).
\]

The equation between the second and third member, virtually due originally to Fourier, is also an interesting formula of Jacob, \( \text{"Analyse des Equations aux Dérivées Partielles..."} \), as being so long pointed out by Cauchy. \( f(x) \) is a series which converges for every value of \( t \) however small or however great; the first, \( (27) \), the more rapidly the less is \( t \); the second, \( (28) \), the more rapidly the greater is \( t \). For the case of \( t = 1/2\pi \nu t \), \( v = 0 \) (that is, \( q = 1 \) and \( p = 0 \)), the two series become identical. For the more comprehensive case of \( p = 0 \), \( q \) unrestricted, the comparison gives the following very curious arithmetical theorem—
\[
(29).
\]

When \( t = \infty \) the first solution (27) converges with so great suddenness that three terms suffice for most practical purposes; when \( t = 1/2\pi \nu t \) (or \( q = 1 \)) the second solution (28) converges with so great suddenness that one term (after the constant first term) suffices for most practical purposes. Thus by using the solution (27) for all values of \( v \) from zero to something less than \( 1/2\pi \nu t \), and (28) for all values greater than that instant at which (27) is used, we have an exceedingly rapid convergence and easy calculation to find \( v \) for any values of \( x \) and \( t \). These formulas, thus used, have been of great practical value in calculating what is now known as the arrival curve of signals through a submarine cable, and in designing instruments to record it automatically and allow its telegraphic meaning to be read, or without recording it to allow its meaning to be read by watching the motions of a spot of light.

XVI. "Harmonic solutions. Any distribution of heat, whether in an infinite or in a bounded solid, which keeps its type unchanged in subiding towards uniformity, when left without positive or negative sources, except such, essentially negative, as are required to fulfil a proper boundary condition, is called a harmonic distribution, the provided temperature does not increase to infinity in any direction. The boundary condition, if the solid is bounded, is essentially that the rate of emission from the surface at every point of it varies in simple proportion to the temperature, and at such a rate per 1° of temperature at each part of the surface as the boundary conditions requires. X. and XIII. are examples. The general condition for a harmonic solution is
\[
(30).
\]

\( \varrho \) is a function of \( (x, y, z) \).

\( \text{Note by Cayley on an article by W. Thomson, entitled } \text{"On the Calculation of Transcendents of the form } f(x, y) \text{".} \)
From these we may find $h$ and $h'$, so as to let the harmonic character of the solution be fulfilled in the subidence. Or if $h$ and $h'$ be given, we have in (41) two equations which determine the two unknown quantities $A/B$ and $\lambda$. Eliminating $A/B$, we thus find a single transcendental equation for $\lambda$, which is proved to have no imaginary or negative roots, and an infinite number of real positive roots, each $=e^{\lambda(i+1)/r^2}$. In the case of $i=0$, or temperature independent of $\phi$ and $\theta$, (40) gives

$$u - A \cos \theta \sqrt{\frac{p}{\kappa} \sin \phi},$$

For this case the transcendental equation for determining values for $\mu$ is very simple, and its roots are calculated numerically with great ease. With the further restriction of no central hollow, we must have $A=0$, so that $u/r$ may be finite when $r=0$. This case was fully investigated by Fourier, and very beautifully worked out in his fifth chapter. The more general problem of a solid sphere, with any given initial distribution of temperature, without the restriction of temperature independent of $\theta$ and $\phi$, was solved first we believe by Poisson in the 11th chapter of his Théorie Mathématique de la Chaleur, in terms of the formulae (36), (38), (40) above.

XVII. The equation of the transference of heat in terms of columnar coordinates, (7) above, affords naturally another beautiful case of harmonic solution. Assume

$$v = e^{-\mu t} A \phi(r) + B \psi(r) S_i,$$

where $S_i$ denotes a spherical surface-harmonic of order $i$, and $\phi(r)$, $\psi(r)$, two particular solutions of the equation

$$\frac{d^2u}{dr^2} + \left[ \frac{\rho}{\kappa} - \frac{\mu(i+1)}{r^2} \right] u = 0,$$

Then (36) and (35) give finally

$$v = e^{-\mu t} A \phi(r) + B \psi(r) S_i . . . . . . . (41).$$

This solution is in its generality applicable to an infinite solid occupying all space except a hollow round the origin. The solid may of course be bounded externally also by a finite closed surface. If there be no hollow, $A/B$ must fulfill the condition that $[\lambda \phi(r)+B \psi(r)]r^2$ is finite when $r=0$. If there are two boundaries, concentric spherical surfaces, with their common centre at the origin of coordinates, the boundary condition obviously requires uniform emissivity over each, but not necessarily equal for the two. If the two emissivities be denoted by $h$ and $h'$, and the radii of the surfaces by $a$ (outer) and $a'$ (inner), the boundary conditions are

$$\frac{dv}{dr} = -h v, \text{ when } r = a,$$

and

$$\frac{dv}{dr} = -h' v, \text{ when } r = a',$$

where $U = A \phi(r) + B \psi(r) S_i$.

The treatment of this equation and its integral (obviously derivable by $\delta$ differentiations from $u_0$, which is a Bessel's function) for the full solution of the thermal problem is most interesting, and very instructive and suggestive in respect to pure analysis. It was splendidly worked out for the case of $m=0$ and $i=0$ by Fourier in his 6th chapter, "The Motion of Heat in a Solid Cylinder," truly a masterpiece of art. When it was printed in 1821, and published after having with the rest of Fourier's work been buried alive for fourteen years in the archives of the French Academy, and when Bessel found it in so thorough an investigation and so strikingly beautiful an application of the "Besselsche Function," we can imagine the ordinary feeling towards these "qui ante nos nostra dixerunt" reversed into the pleasure of genuine admiration.
ERRATA TO "HEAT."

Page 58, § 82, line 9 from beginning of §, for kc read k/c.
" " 12 " " " kc " k/c.
" " 19 " " " L*D " L*T.

Page 59, Diffusions.—Table A.

For Heat through copper .......... "92 read Heat through copper .......... 93
" " underground strata 10.00 " " underground strata 100.0
" Common salt through water ...... 4320 " " Common salt through water ...... 87150

Page 59, Diffusions (Secular).—Table B.

Carbonic acid through air .......... for 2,229,000 years, read 220,000 years.
Heat through hydrogen .......... " 283,000 " " 28,000 ",
" " copper .......... " 293,000 " " 29,000 ",
" " iron .......... " 1,783,000 " " 174,000 ",
" " air .......... " 1,982,000 " " 198,000 ",
" " underground strata .. " 3,000,000 " " 317,000 ",
" " wood .......... 244,000,000 " " 24,700,000 ",
Common salt through water ...... 1,378,000,000 " " 2,789,000,000 ",

Page 60, table, col. 2, for "Platiniridium one-tenth iron," read "Platiniridium one-tenth iridium."

Page 63, Table VIII.—Thermal Conductivities,

For Copper .......... 0.91, read Copper .......... 0.96, Ångström.
" Iron .......... 0.16, " Iron .......... 0.20, "

Page 63, Table IX.—Diffusivities (Thermal, Material, and Electric).

For Copper .......... 0.91 0.845 1.077, read Copper .......... 0.96 0.845 1.12, Ångström.
Iron .......... 0.16 0.875 0.155, " Iron .......... 0.20 0.873 0.22, Ångström and Tait.