ELEMENTS OF QUATERNIONS.
ADVERTISEMENT TO THE FIRST EDITION.

In my late father's Will no instructions were left as to the publication of his Writings, nor specially as to that of the "Elements of Quaternions," which, but for his late fatal illness, would have been before now, in all their completeness, in the hands of the Public.

My brother, the Rev. A. H. Hamilton, who was named Executor, being too much engaged in his clerical duties to undertake the publication, deputed this task to me.

It was then for me to consider how I could best fulfil my triple duty in this matter—First, and chiefly, to the dead; secondly, to the present public; and thirdly, to succeeding generations. I came to the conclusion that my duty was to publish the work as I found it, adding merely proof-sheets, partially corrected by my late father, and from which I removed a few typographical errors, and editing only in the literal sense of giving forth.

Shortly before my father's death, I had several conversations with him on the subject of the "Elements." In these he spoke of anticipated applications of Quaternions to Electricity, and to all questions in which the idea of Polarity is involved—applications which he never in his own lifetime expected to be able fully to develop, bows to be reserved for the hands of another Ulysses. He also discussed a good deal the nature of his own forthcoming Preface; and I may intimate that, after dealing with its more important topics, he intended to advert to the great labour which the writing of the "Elements" had cost him—labour both mental and mechanical; as, besides a mass of subsidiary and unprinted calculations, he wrote out all the manuscript, and corrected the proof-sheets, without assistance.

And here I must gratefully acknowledge the generous act of the Board of Trinity College, Dublin, in relieving us of the remaining pecuniary liability, and thus incurring the main expense, of the publication of this volume. The announcement of their intention to do so, gratifying as it was, surprised me the less, when I remembered that they had, after the publication of my father's former book, "Lectures on Quaternions," defrayed its entire cost; an extension of their liberality beyond what
was recorded by him at the end of his Preface to the "Lectures," which doubtless he would have acknowledged, had he lived to complete the Preface of the "Elements."

He intended also, I know, to express his sense of the care bestowed upon the typographical correctness of this volume by Mr. M. H. Gill of the University Press, and upon the delineation of the figures by the Engraver, Mr. Oldham.

I annex the commencement of a Preface, left in manuscript by my father, and which he might possibly have modified or rewritten. Believing that I have thus best fulfilled my part as trustee of the unpublished "Elements," I now place them in the hands of the scientific public.

WILLIAM EDWIN HAMILTON.

January 1st, 1866.
PREFACE TO THE FIRST EDITION.

[1.] The volume now submitted to the public is founded on the same principles as the "Lectures,"(1) which were published on the same subject about ten years ago: but the plan adopted is entirely new, and the present work can in no sense be considered as a second edition of that former one. The Table of Contents, by collecting into one view the headings of the various Chapters and Sections, may suffice to give, to readers already acquainted with the subject, a notion of the course pursued: but it seems proper to offer here a few introductory remarks, especially as regards the method of exposition, which it has been thought convenient on this occasion to adopt.

[2.] The present treatise is divided into Three Books, each designed to develop one guiding conception or view, and to illustrate it by a sufficient but not excessive number of examples or applications. The First Book relates to the Conception of a Vector, considered as a directed right line, in space of three dimensions. The Second Book introduces a First Conception of a Quaternion, considered as the Quotient of two such Vectors. And the Third Book treats of Products and Powers of Vectors, regarded as constituting a Second Principal Form of the Conception of Quaternions in Geometry.

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* This fragment, by the Author, was found in one of his manuscript books by the Editor. [W. E. Hamilton.]
PREFACE TO THE SECOND EDITION.

Sir William Rowan Hamilton died on the 2nd of September, 1865, leaving his great work on Quaternions unfinished. He intended to have added some account of the operator* \( \nabla \), an Index, and an Appendix containing notes on Anharmonic Coordinates, on the Barycentric Calculus, and on proofs of his geometrical theorems stated in Nichol’s Cyclopædia. At the time of his death, with the exception of a fragment of the preface, and a small portion of the table of contents, all the manuscript he had prepared was in type. As he rarely commenced writing before his thoughts were fully matured, he has left no outline of the additions contemplated.

In this edition, printed by direction of the Board of Trinity College, Dublin, the original text has been faithfully preserved, except in a few places where trifling errors have been corrected. I have added notes, distinguished in every case by square brackets, wherever I thought they were wanted. I have rendered the work more convenient by increasing the number of cross-references, by including in the page-headings the numbers of the articles (for the original references are generally given to articles and not to pages), by dividing the work into two volumes, and by the addition of an index. The table of contents has been amplified by a brief analysis of each article, designed as far as possible to assist the reader in following and in recapitulating the arguments in the text. Hamilton indicated “a minimum course of study, amounting to rather less than 200 pages (or parts of pages),” suitable for a first perusal, and he intended to have prepared a table containing references to this course. Such a table will be found at the end of the table of contents, but for the convenience of students of Physics, and of those desirous of obtaining a working knowledge of Hamilton’s powerful engine of research, I have amplified it somewhat, duly noting, however, the minimum course.

* In the second volume I hope to devote an appendix to this important subject.

Hamilton’s Elements of Quaternions.
I infer from the fragment of the author’s preface that he proposed to sketch an outline of the method of exposition, of an elementary character and adapted to those readers to whom the subject is new. To those readers chiefly I address the following remarks:

According to the plan of this work, whenever a new conception or notation is introduced, a series of illustrative examples immediately follows. Most of these involve no real difficulty, but occasionally a long and difficult investigation occurs even in the early parts of the book. Intricate investigations, which are merely illustrative, are everywhere omitted from the selected course.

The First Book deals with Vectors, considered without reference to angles or to rotations. In a word, it is concerned with the application of the signs +, -, and = to the algebra of vectors. The sign – is first introduced, and the sign + follows from the formula of relation \((b - a) + a = b\). Sections 3 and 4 (pp. 7-11) are occupied with a series of propositions concerning the commutative and associative laws of the addition of vectors, and the multiplication of vectors by scalars, or algebraical coefficients. Propositions such as these often appear to a student to be mere truisms, and unfortunately it is not easy to find elementary examples to convince him of the contrary. The addition of vector-arcs, he will find on p. 156, is not commutative, though it is associative.† With the exception of a few passages noted in the table of a selected course, there is nothing in chaps. II. and III. essential to a good knowledge of the subject. They contain, however, an account of an extremely elegant theory of anharmonic coordinates, independent of any non-projective property, and intricate and powerful investigations of geometric nets and of systems of barycentres.

The Second Book treats of Quaternions considered as quotients of vectors, and as involving angular relations. It opens with a first conception of a quaternion as a quotient of two vectors, and thus the division of vectors is introduced before that of multiplication, just as in the First Book subtraction precedes addition. If \(q = \beta : a\) is the quotient of two vectors, \(\beta\) and \(a\), it is natural to define the product \(q \cdot a\) by the relation \(q \cdot a = \beta\). It is soon found, if any vector \(\gamma\) is selected in the plane of \(a\) and \(\beta\), that the product \(q \cdot \gamma\) is a vector in the same plane whose length bears to that of \(\gamma\) the same ratio as the length of \(\beta\) to that of \(a\), and which makes the same angle with \(\gamma\) that \(\beta\)

* In fact the commutative law of addition depends on a property of a parallelogram, and therefore ultimately on the validity of Euclid’s fifth postulate. It does not hold except for Euclidean space.
makes with \(a\). Thus, from the first conception of a quaternion as a quantity expressing the relative length and direction of two given vectors, we have come to consider a quaternion as an operator on a special set of vectors, viz. those in its own plane. Observe that, so far, we have not arrived at the conception of the product of two vectors, nor of the product of a quaternion and an arbitrary vector. We have only reached the limited conception of the product \(q \cdot \gamma\) of a quaternion \(q\) and a vector \(\gamma\) in its plane, and while an interpretation is assigned to \(q \cdot \gamma\), as yet the product \(\gamma \cdot q\) is unknown.

After reviewing a class of quaternions derived by fixed laws from a given quaternion, a special class of quaternions, called versors or radial quotients, is considered in detail. The product of a pair of versors is found (p. 147) to depend on the order in which they are multiplied, that is \(qq'\) is not generally equal to \(q'q\), or the commutative law of algebraic multiplication is not true for versors, nor \(\textit{ad fortiori}\) for quaternions.

The multiplication of a special set of versors of a restricted kind occupies section 10, chap. I.; and on p. 160 the famous formula

\[i^2 = j^2 = k^2 = ijk = -1\]  

is deduced, in which \(i, j,\) and \(k\) are right versors* in three mutually perpendicular planes. This section contains the first example of a product of more than two versors, and it is shown that the multiplication of these specially related right versors is associative. Warned by the failure of the commutative law, it is necessary to determine if the remaining laws of algebra are valid in quaternions. In algebra, if we first form the product \(bc\) and then multiply by \(a\), we have the same result as if we multiplied \(c\) by the product \(ab\), and this associative law is expressed in symbols by the equation \(a \cdot bc = ab \cdot c\). This is also true for quaternions, and it may be regarded as the chief feature which distinguishes quaternions from other systems of vector analysis. For example, Grassmann’s multiplication is sometimes associative, but sometimes it is not. It is necessary to prove, moreover, that quaternion multiplication is distributive, or that \(a (b + c) = ab + ac\). This is not true if \(b\) and \(c\) are vector arcs, even when \(a\) is a number as shown on p. 156. Some of Hamilton’s early investigations led him to a non-distributive system of multiplication in 1830.†

Next a quaternion is decomposed in two ways:—(1) in section 11, into the product of its tensor and its versor; (2) in section 12, into the sum of its

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* A right versor turns a vector in its plane through a right angle.
† Preface to Lectures on Quaternions, paragraph [41]. Schefler has reproduced this system.
scalar and its right or vector part. This right or vector part, it is ultimately shown, may be identified with a vector; at present it is regarded as a right quaternion, or a quotient of two perpendicular vectors. By the first of these decompositions, "the multiplication of any two quaternions is reduced to the arithmetical operation of multiplying their tensors, and the geometrical operation of multiplying their versors"; and by the second the addition of quaternions is reduced to the algebraical addition of their scalar parts, and the geometrical addition of their vector parts. Thus it is proved (Arts. 206, 207) that the addition of the vector parts is reducible to the addition of vectors, and, as the addition both of scalars and of vectors is commutative and associative, so likewise is the addition of quaternions.

The multiplication of right quaternions, or of the vector parts of quaternions, is proved in Art. 211 to be distributive; and, as any quaternion is the sum of a scalar and a vector part, it is also proved that the general multiplication of quaternions is distributive. A long series of examples follows, some of which are not easy, including Hamilton's well-known construction of the ellipsoid.

Section 14 is entitled "On the reduction of the general Quaternion to the Standard Quadrinomial Form \(q = w + ix + jy + kz\); with a First Proof of the Associative Principle of the Multiplication of Quaternions." This proof depends on the general Distributive Property lately proved, and on the Associative Property of the particular set of versors \(i, j, k\) (Art. 161); but in chap. III. various proofs are given which are independent of these properties. The first proof is sufficient for all practical purposes.

The laws of combination of quaternions are now established. Addition (and subtraction) is associative and commutative; multiplication (and division) is associative and distributive, but not commutative.

Passing over the second and third chapters in this Second Book, which are chiefly complementary to the development of the theory, we find in chap. I., Book III., three lines of argument traced out in justification of the identification of the vector part of a quaternion with a vector. In fact a restriction is imposed, or a simplification is introduced, and this restriction or simplification is shown to be consistent with the results already obtained.* In much the same way as a couple or an angular

* Compare the note to p. 175, in which Hamilton remarks: "We have thus a new point of agreement, or of connexion, between right quaternions and their index-vectors, tending to justify the ultimate assumption (not yet made), of equality between the former and the latter."
velocity is sometimes represented by a right line, a right quaternion and a vector of appropriate length, perpendicular to the plane of the quaternion, are now represented by the same symbol.*

The scope of the remainder of this volume is, I think, sufficiently indicated in the table of contents. The foregoing sketch of the development of the calculus of Quaternions necessarily presents but a meagre view of the nature of this work; however, my object has been to carry out, as far as I could, the intention of its illustrious author expressed in the fragment of his preface.

CHARLES JASPER JOLY.

THE OBSERVATORY, DUNSINK,
December, 1898.

* With but slight change, much of Books I. and II. might have been extended to space of $n$-dimensions. In Book III. advantage is taken of the peculiar simplicity of space of those dimensions in which but one direction is perpendicular to a given plane, and a legitimate reduction of the number of symbols is consequently made.
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BOOK I.

ON VECTORS, CONSIDERED WITHOUT REFERENCE TO ANGLES, OR TO ROTATIONS, ....... 3-103

CHAPTER* I.

FUNDAMENTAL PRINCIPLES RESPECTING VECTORS.

Section † 1.—On the Conception of a Vector; and on Equality of Vectors, . 3-4

[Art. 1, Definition of a vector: Geometrical subtraction of points; Introduction of the sign —, p. 3. — Art. 2, Equality of vectors; Introduction of sign =, p. 3. — Art. 3, The equation \( b - c = b - a \), considered as an equidifference of points, admits of inversion and alternation, p. 4.]

Section 2.—On Differences and Sums of Vectors, taken two by two, . 5-7

[Art. 4, Definition of the geometrical subtraction of vectors, p. 5. — Art. 5, Introduction of sign +, regarded as a converse to —, by the formula \((b - a) + a = b\), p. 5. — Art. 6, The sum of any two co-initial sides of a parallelogram \(ABCD\) is the co-initial diagonal, or \(AB + AC = AD = AC + AB\), p. 6. — Art. 7, Addition of two Vectors is a commutative operation, or \(a + b = b + a\),§ p. 6.]

Section 3.—On Sums of Three or more Vectors, . 7-8

[Art. 8, Definition of the sum of any number of vectors, p. 7. — Art. 9, The addition of vectors is an associative and commutative operation, p. 7. — Art. 10, Closed polygon, p. 8. — Art. 11, Projection of a closed polygon, p. 8.]

* This Chapter may be referred to, as I. i.; the next as I. ii.; the first Chapter of the Second Book, as II. i.; and similarly for the rest.
† This Section may be referred to, as I. i. 1; the next, as I. i. 2; the sixth Section of the second Chapter of the Third Book, as III. ii. 6; and so on. [Article 180 is referred to as (180), and the third sub-article of (180) as (180 (3.)).]
‡ This is, in words, \(b - a\) is added to \(a\) and their sum is \(b\), but not \(a\) is added to \(b - a\) and their sum is \(b\). See (6) and (7).]
§ In (180 (3.)) it is shown that the addition of vector arcs is not commutative.
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Section 4.—On Coefficients of Vectors, 9-11

[Art. 12, 1a + ma = (1 + m)a if m is a positive or negative integer, p. 9.—
Art. 13, The multiplication of vectors by positive or negative integers is a doubly
distributive operation,* p. 9.—Art. 14, This is also true when the coefficients are in-
commensurable, yα ± xa = (y ± x)α; y(αα) = yαα; x(β ± α) = xβ ± xa, p. 10.
—Art. 15, β = xa and α have the same or opposite directions according as x > or < 0,
p. 10.—Art. 16, Division of a vector by a parallel vector, p. 10.—Art. 17, Intro-
duction of the term scalar, p. 11.]

This short First Chapter should be read with care by a beginner; any misconception
of the meaning of the word "Vector" being fatal to progress in the Quaternions. The
Chapter contains explanations also of the connected, but not all equally important, words
or phrases, "revector," "provector," "transvector," "actual and null vectors," "op-
posite and successive vectors," "origin and term of a vector," "equal and unequal
vectors," "addition and subtraction of vectors," "multiples and fractions of vectors,
&c.; with the notation b – α, for the Vector (or directed right line) αβ: and a deduction
of the result, essential but not peculiar† to quaternions, that (what is here called) the
vector-sum, of the two co-initial sides of a parallelogram, is the intermediate and co-initial
diagonal. The term "Scalar" is also introduced, in connexion with coefficients of vectors.

CHAPTER II.

APPLICATIONS TO POINTS AND LINES IN A GIVEN PLANE.

Section 1.—On Linear Equations connecting two Co-initial Vectors, 12-13

[Art. 18, Origin and change of origin, p. 12.—Art. 19, Equation of a line oα;
β = xa, p. 12.—Art. 20, Or αa + βb = 0, p. 12.]

Section 2.—On Linear Equations between three Co-initial Vectors, 13-19

[Art. 21, αa + βb is a vector coplanar with α and β, p. 13.—Art. 22, αα + ββ
+ γγ = 0 is the condition that α, α, β, and γ, or α, β, and γ, should be coplanar. Vector
equation of a plane, p. 14.—Art. 23, αα + ββ + γγ = 0 is the condition of collinearity
of α, β, and γ, p. 14.—Art. 24, α : β : γ = oc : oc : αa, when α + β + γ = 0, p. 14.—
Art. 25, Equation of a line. Anharmonic of four collinear points, p. 15.—Art. 26,
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first construction, α' = oα · bo, b' = ob · ca, c' = oo · ab, ‡ p. 17.—Art. 28, Ratios
of segments of sides of the triangle ABC, made by α', β', γ'. Hence, α : β : γ =
oa : ob : oc, oc : oc : αb, ‡ p. 18.—Art. 29, Origin at mean point of triangle, p. 18.—Art. 30,
Projection of triangle and mean point, p. 19.]

After reading these two first Sections of the second Chapter, and perhaps the three
first Articles (31-33, pages 20-22) of the following Section, a student to whom the
subject is new may find it convenient to pass at once, in his first perusal, to the third

[* m (β ± a) = mβ ± ma is only true if α + β = β + α. See (180 (3.)).]
† Compare the second Note to page 206.
‡ oa · bo denotes the point of intersection of the lines OA and BC, DE · ABC the point of intersec-
tion of the line DE with the plane ABC.]
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Chapter of the present Book: and to read only the two first Articles (62, 63, pages 44-45) of the first Section of that Chapter, respecting Vectors in Space, before proceeding to the Second Book (pages 107, &c.), which treats of Quaternions as Quotients of Vectors.

SECTION 3.—On Plane Geometrical Nets, .... 20-23

[Art. 31, Properties of points of second construction, \( a' = b' c' \cdot b c; \quad a'' = c' a' \cdot c a; \quad c'' = a' b' \cdot a b \), p. 20.—Art. 32, Equation of six segments made by transversal, p. 21.
- Art. 33, Points of second construction, \( a'' = a b; \quad b'' = a c; \quad c'' = c a \), p. 21.—Art. 34, Construction of plane net, p. 22.]

SECTION 4.—On Anharmonic Co-ordinates and Equations of Points and Lines in one Plane, .... 23-29

[Art. 35, Anharmonics of groups and pencils, p. 23.—Art. 36, Anharmonic co-ordinates, \( \rho = \frac{x a x + y b y + z c z}{x a + y b + z c} \), p. 24.—Art. 37, Anharmonic equation and co-ordinates of a right line, p. 26.—Art. 38, Examples. Line at infinity, p. 27.—Arts. 39, 40, Independent geometrical interpretation of the coordinates of a line, p. 27.—Art. 41, Concurrent and parallel lines, p. 29.]

SECTION 5.—On Plane Geometrical Nets, resumed, .... 29-32

[Art. 42, Points and lines of a net are rational, p. 29.—Art. 43, Converse. Indefinite approximation to irrational points and lines, p. 30.—Art. 44, Anharmonics of nets are rational. Nets are homographic figures, p. 30.—Art. 45, Construction of a net from one of its quadrilaterals, p. 31.]

SECTION 6.—On Anharmonic Equations and Vector Expressions, for Curves in a given Plane, .... 32-43

[Art. 46, Conic touching sides of triangle at points of first construction, p. 32.—Art. 47, An ellipse or circle when \( o \) is interior to triangle, p. 33.—Art. 48, Case of \( o \) exterior to triangle, p. 33.—Art. 49, Geometrical criterion of species, p. 34.—Art. 50, Determination of asymptotes of hyperbola and axis of parabola, p. 34.—Art. 51, Anharmonic symbol for tangent, p. 35.—Art. 52, Vector to centre of conic, p. 35.—Art. 53, Circumscribed conic having double contact with the inscribed conic, p. 36.—Art. 54, Vector expression for a cubic with a conjugate point, p. 37.—Art. 55, Anharmonic generation of general cubic, p. 37.—Art. 56, Tangential Equations, p. 38.—Art. 57, Examples of conics, p. 39.—Art. 58, Equations which are neither rational nor integral, p. 40.—Art. 59, Tangential equation deduced from \( x^3 + y^3 + z^3 = 0 \), p. 41.—Art. 60, General Method, p. 42.—Art. 61, Application to cubic, p. 42.]

Among other results of this Chapter, a theorem is given in page 38, which seems to offer a new geometrical generation of (plane or spherical) curves of the third order. The anharmonic co-ordinates and equations employed, for the plane and for space, were suggested to the writer by some of his own vector forms; but their geometrical interpretations are assigned. The geometrical nets were first discussed by Professor Möbius, in his Barycentric Calculus, but they are treated in the present work by an entirely new analysis: and, at least for space, their theory has been thereby much extended in the Chapter to which we next proceed.

HAMILTON'S ELEMENTS OF QUATERNIONS.
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CHAPTER III.
APPLICATIONS OF VECTORS TO SPACE.

SECTION 1.—On Linear Equations between Vectors not Complanar, Pages 44–50

[Art. 62, $s\alpha + b\beta + c\gamma + d\delta = 0$, if $\alpha, \beta, \gamma, \delta$ are any four vectors, p. 44.—Art. 63, $a + b + c + d = 0$, if $\alpha$ and $\beta$ are coplanar, p. 45.—Art. 64, Intersections of lines in a plane, p. 45.—Art. 65, For any five vectors $s\alpha + b\beta + c\gamma + d\delta + e\epsilon = 0$, and $a + b + c + d + e = 0$, p. 47.—Art. 66, Intersection of a line and a plane, p. 47.—Art. 67, Ratios of coefficients in (65) expressed as ratios of volumes of pyramids, p. 48.—Art. 68, Convention respecting signs of volumes, p. 48.—Art. 69, $I \cdot \alpha \cdot \beta \cdot \gamma \cdot \delta \cdot \epsilon = 0$; II. $\alpha \cdot \beta \cdot \gamma - \beta \cdot \alpha \cdot \gamma + \gamma \cdot \alpha \cdot \beta = 0$; III. $\alpha \cdot \beta \cdot \gamma \cdot \delta \cdot \epsilon = 0$; where in I., $\alpha, \beta, \gamma, \delta, \epsilon$ are any three collinear points; in II., $\alpha, \beta, \gamma, \delta, \epsilon$ are any four coplanar points; and in III., $\alpha, \beta, \gamma, \delta, \epsilon, \rho$ are any five points of space, p. 49.]

It has already been recommended to the student to read the first two Articles of this Section, even in his first perusal of the Volume; and then to pass to the Second Book.

SECTION 2.—On Quinary Symbols for Points and Planes in Space, Pages 50–55

[Art. 70, Indefinite expression for a vector $p = \frac{s\alpha + b\beta + c\gamma + d\delta + e\epsilon}{s\alpha + b\beta + c\gamma + d\delta + e\epsilon}$, p. 50.—Art. 71, Quinary Symbol for a point; Congruence, p. 51.—Art. 72, $(Q') = (Q)$ if $(Q) = t \cdot (Q') + u \cdot (U)$, where $(U) = (11111)$, p. 51.—Art. 73, Condition of collinearity of three points, p. 52.—Art. 74, Condition of coplanarity of four points, p. 52.—Art. 75, Quinary equation and symbol of a plane, p. 53.—Art. 76, Intersection of a line and a plane, p. 54.—Art. 77, Points, lines, and planes rationally related to five given points, p. 54.—Art. 78, Synthetical points, p. 55.]

SECTION 3.—On Anharmonic Co-ordinates in Space, Pages 55–60

[Art. 79, Quaternary or anharmonic symbol of a point in space, p. 55.—Art. 80, Anharmonic symbol and equation of a plane, p. 56.—Art. 81, Trace of a plane on a given plane, p. 56.—Art. 82, Formulae of collineation and coplanarity, p. 56.—Art. 83, Ratios of co-ordinates expressed as anharmonics of pencils of planes, p. 57.—Art. 84, Anharmonics of groups of points and of pencils of planes, p. 58.—Art. 85, Quotients of two homogeneous and linear functions of the co-ordinates of a point expressed as the anharmonic of a pencil of planes, p. 59.—Art. 86, Reciprocal theorem and example, p. 59.—Art. 87, Centre and plane of homology of pyramid, p. 60.]

SECTION 4.—On Geometrical Nets in Space, Pages 61–81

[Art. 88, Net derived from five points, p. 61.—Art. 89, First construction, p. 61.—Art. 90, Second construction, p. 62.—Art. 91, Quinary symbols of constructed lines and planes, p. 63.—Art. 92, Eight types of points $p_5$. Anharmonic properties, p. 63.—Art. 93, Verification that the enumeration of the points $p_5$ is complete, p. 73.—Art. 94, Arrangement of the points on lines and in planes, p. 76.—Art. 95, Extension of results of (44) and (45) to nets in space, p. 79.]
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SECTION 5.—On Barycentres of Systems of Points; and on Simple and Complex Means of Vectors, 81-87

[Art. 96, Simple mean of coinitial vectors; Mean point; Gauche quadrilateral (1), (2); Lines and planes determined by mean points of partial systems (3)-(6), p. 81. —Art 97, Complex means; Barycentres, p. 84.]

SECTION 6.—On Anharmonic Equations, and Vector Expressions, of Surfaces and Curves in Space, 87-96

[Art. 98, Local and tangential equations of surfaces; Ruled quadric (1)-(10); twisted curve (11), (12), p. 87.—Art. 99, Vector expression for a curve \( p = \phi(t) \); for a surface \( p = \phi(t, u) \); Line and plane (1); Ellipse and ellipsoid (2); quadric cone (3), (4); Cubic cone (5); Ruled quadric (6)-(10), p. 94.]

SECTION 7.—On Differentials of Vectors, 96-103

[Art. 100, Definition of the differential of a vector; Hodograph (5); Developable surface (8); Tangents to surfaces (9), p. 96.]

An application of finite differences, to a question connected with barycentres, occurs in p. 84. The anharmonic generation of a ruled hyperboloid (or paraboloid) is employed to illustrate anharmonic equations; and (among other examples) certain cones, of the second and third orders, have their vector equations assigned. In the last Section, a definition of differentials (of vectors and scalars) is proposed, which is afterwards extended to differentials of quaternions, and which is independent of developments and of infinitesimals, but involves the conception of limits. Vectors of Velocity and Acceleration are mentioned; and a hint of Hodographs is given.

BOOK II.

ON QUATERNIONS, CONSIDERED AS QUOTIENTS OF VECTORS,
AND AS INVOLVING ANGULAR RELATIONS, 107-249

CHAPTER I.

FUNDAMENTAL PRINCIPLES RESPECTING QUOTIENTS OF VECTORS.

Very little, if any, of this Chapter II. I., should be omitted, even in a first perusal, since it contains the most essential conceptions and notations of the Calculus of Quaternions, at least so far as quotients of vectors are concerned, with numerous geometrical illustrations. Still there are a few investigations respecting circumscribed cones, imaginary intersections, and ellipsoids, in the thirteenth Section, which a student may pass over, and which will be indicated in the proper place in this Table.

SECTION 1.—Introductory Remarks; First Principles adopted from Algebra, 107-110

[Arts. 101-2, Comparison between Books I. and II.; General principles adopted by definition, p. 107.—Art. 103, I. Division must correspond to converse act of multiplication, or \( \frac{b}{a} = \beta \); \( qa = q \), p. 108.—Art. 104, II. Unequal vectors divided by equal vectors give unequal quotients, p. 109.—Art. 105, III. Quotients equal to
the same are equal to one another, p. 109.—Art. 106, IV. $\frac{\gamma}{a} \pm \frac{\beta}{a} = \frac{\gamma \pm \beta}{a}$ and $\frac{\gamma \cdot \beta}{a} = \frac{\gamma}{a}$; Definition of addition, subtraction and division of quotients with a common

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In the five foregoing Sections it is shown, among other things, that the plane of a quaternion is generally an essential element of its constitution, so that dieplanar quaternions are unequal; but that the square of every right radial (or right versor) is equal to negative unity, whatever its plane may be. The Symbol \( \sqrt{-1} \) admits then of a real interpretation, in this as in several other systems; but when thus treated as real, it is in the present Calculus too vague to be useful: on which account it is found convenient to retain the old signification of that symbol, as denoting the (uninterpreted) Imaginary of Algebra, or what may here be called the scalar imaginary, in investigations respecting non-real intersections, or non-real contacts, in geometry.
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The student ought to make himself familiar with these laws, which are all included in the Fundamental Formula,

\[ i^2 = j^2 = k^2 = ijk = -1. \] (A)

In fact, a Quaternion may be symbolically defined to be a Quadrinomial Expression of the form,

\[ q = w + ix + jy + kz, \] (B)

in which \( w, x, y, z \) are four scalars, or ordinary algebraic quantities, while \( i, j, k \) are three new symbols, obeying the laws contained in the formula (A), and therefore not subject to all the usual rules of algebra: since we have, for instance,

\[ ij = + k, \text{ but } ji = -k; \text{ and } ijk^2 = - (ijk)^3. \]

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as far as the formula VIII. in p. 294, should be read, as a preparation for the Third
Book, to which the Student may next proceed.
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ON DIPLANAR QUATERNIONS, OR QUOTIENTS OF VECTORS IN SPACE: AND especially on the Associative Principle of Multiplication of such Quaternions.

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BOOK III.

ON QUATERNIONS, CONSIDERED AS PRODUCTS OR POWERS OF VECTORS; AND ON SOME APPLICATIONS OF QUATERNIONS, 321 to the end.

CHAPTER I.

ON THE INTERPRETATION OF A PRODUCT OF VECTORS OR POWER OF A VECTOR, AS A QUATERNION.

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SECTION 3.—On a Second Method of arriving at the same Interpretation, of a Binary Product of Vectors, 329-330

This second interpretation treats the product $\beta \cdot a$, as equal to the quotient $\beta : a^{-1}$; where $a^{-1}$ (or $Ba$) is the previously defined Reciprocal (II. ii. 7) of the vector $a$, namely a second vector, which has an inverse length, and an opposite direction. Multiplication of Vectors is thus proved to be (like that of Quaternions) a Distributive, but not generally a Commutative Operation. The Square of a Vector is shown to be always a Negative Scalar, namely the negative of the square of the tensor of that vector, or of the number which expresses its length; and some geometrical applications of this fertile principle, to spheres, &c., are given. The Index of the Right Part of a Product of Two Coinitial Vectors, $oa$, $ob$, is proved to be a right line, perpendicular to the Plane of the Triangle $oab$, and representing by its length the Double Area of that triangle; while the Rotation round this Index, from the Multiplier to the Multiplicand, is positive. This right part, or vector part, $VaB$, of the product vanishes, when the factors are parallel (to one common line); and the scalar part, $SaB$, when they are rectangular.

SECTION 4.—On the Symbolical Identification of a Right Quaternion with its own Index: and on the Construction of a Product of Two Rectangular Lines, by a Third Line, rectangular to both, 331-334

SECTION 5.—On some Simplifications of Notation, or of Expression, resulting from this Identification; and on the Conception of an Unit-Line as a Right Versor, 334-337

In this second interpretation, which is found to agree in all its results with the first, but is better adapted to an extension of the theory, as in the following Sections, to ternary products of vectors, a product of two vectors is treated as the product of the two right quaternions, of which those vectors are the indices (II. i. 5). It is shown that, on the same plan, the Sum of a Scalar and a Vector is a Quaternion.
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This interpretation is affected by the substitution, as in recent Sections, of Right Quaternions for Vectors, without change of order of the factors. Multiplication of Vectors, like that of Quaternions, is thus proved to be an Associative Operation. A vector, generally, is reduced to the Standard Trinomial Form,

\[ \rho = ix + jy + kz; \]  

in which *i, j, k* are the peculiar symbols already considered (II. i. 10), but are regarded now as denoting Three Rectangular Vector-units, while the three scalars *x, y, z* are simply rectangular co-ordinates; from the known theory of which last, illustrations of results are derived. The Scalar of the Product of Three coinitial Vectors, *oa, ob, oc*, is found to represent, with a sign depending on the direction of a rotation, the Volume of the Parallelepiped under these three lines; so that it vanishes when they are complanar.

Constructions are given also for products of successive sides of triangles, and other closed polygons, inscribed in circles, or in spheres; for example, a characteristic property of the circle is contained in the theorem, that the product of the four successive sides of an inscribed quadrilateral is a scalar: and an equally characteristic (but less obvious) property of the sphere is included in this other theorem, that the product of the five successive sides of an inscribed gauche pentagon is equal to a tangential vector, drawn from the point at which the pentagon begins (or ends). Some general Formulae of Transformation of Vector Expressions are given, with which a student ought to render himself very familiar, as they are of continual occurrence in the practice of this Calculus; especially the four formulae (pp. 337, 339):

\[ V \gamma V \beta a = aS\beta \gamma - \beta S\gamma a; \]  
\[ V \gamma V \beta a = aS\beta \gamma - \beta S\gamma a + \gamma S\alpha \beta; \]  
\[ \rho S\alpha \beta \gamma = aS\beta \rho + \beta S\gamma \rho + \gamma S\alpha \rho; \]  
\[ \rho S\alpha \beta \gamma = V\beta \gamma S\alpha \rho + V\gamma S\beta \rho + V\alpha S\beta \gamma; \]

in which \( \alpha, \beta, \gamma, \rho \) are any four vectors, while S and V are signs of the operations of taking separately the scalar and vector parts of a quaternion. On the whole, this Section (III. i. 6) must be considered to be (as regards the present exposition) an important one; and if it have been read with care, after a perusal of the portions previously indicated, no difficulty will be experienced in passing to any subsequent applications of Quaternions, in the present or any other work.

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Section 8.—On an Equivalent Interpretation of the Fourth Proportional to Three Diplanar Vectors, deduced from the Principles of the Second Book, 379-393

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<td>Art. 303, Consequently, adopting the formula of 301, if ( v ) is a right quaternion, ( v^{-1}w = u ), p. 383.</td>
<td>Art. 308, A power of a vector is a quaternion, p. 396.-Art. 309, and a quaternion may be regarded as a power of a vector. Proof of the equation ( 2a 2b 2x 2y 2z = -1 ), p. 399.-Art. 310, which includes the whole doctrine of Spherical Triangles. Spherical sum of angles, p. 404.-Art. 311, And arcanal addition of sides, p. 407.-Art. 312, Solution of the equation of 309, p. 408.-Art. 313, Extension to spherical polygons, p. 414.-Art. 314, Geometrical loci and, p. 417.-Art. 315, Transformations connected with the powers of vectors, p. 420.</td>
</tr>
<tr>
<td>Art. 304, and as a further consequence ( (\beta : \alpha) \gamma = 5 + ew ), ( u ) being now the same for all systems of mutually rectangular lines. Spherical parallelograms, p. 385.-Art. 305, Series of spherical parallelograms, p. 387.</td>
<td>It may be well to read Art. 307, especially for the Exponential Connexions which it establishes, between Quaternions and Spherical Trigonometry, or rather Polygonometry, by a species of extension of Moivre's theorem, from the plane to space, or to the sphere. For example, there is given (in p. 417) an equation of six terms, which holds good for every spherical pentagon, and is deduced in this way from an extended exponential formula. The calculations in the sub-articiles to Art. 312 (pp. 409-414) may however be passed over; and perhaps Art. 315, with its sub-articiles (p. 420). But Art. 314, and its sub-articiles, pp. 417-419, should be read, on account of the exponential form of which they contain, of equations of the circle, ellipse, logarithmic spirals (circular and elliptic), helix, and screw surface.</td>
</tr>
<tr>
<td>Art. 306, Construction of the series, p. 390.</td>
<td>Section 10.—On the Interpretation of a Power of a Vector as a Quaternion, 396-420</td>
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</tbody>
</table>

Section 9.—On the Third Method of interpreting a Product or Function of Vectors as a Quaternion; and on the Consistency of the Results of the Interpretation so obtained, with those which have been deduced from the two preceding Methods of the present Book, . . . . . . 394-396

Section 11.—On Powers and Logarithms of Diplanar Quaternions; with some Additional Formulse, . . . . . . . . . . 421-429

It may suffice to read Art. 316, and its first eleven sub-articiles, pp. 421-423. In this
Section, the adopted Logarithm, \( lq \), of a quaternion \( q \), is the simplest root, \( q' \), of the transcendental equation,

\[
1 + q' + \frac{q'^2}{2} + \frac{q'^3}{2 \times 3} + \text{&c.} = q;
\]

and its expression is found to be,

\[
lq = \Omega q + \vartheta q \cdot \Upsilon q \tag{II}
\]

in which \( \Omega \) and \( \Upsilon \) are the signs of tensor and versor, while \( \vartheta q \) is the angle of \( q \), supposed usually to be between 0 and \( \pi \). Such logarithms are found to be often useful in this Calculus, although they do not generally possess the elementary property, that the sum of the logarithms of two quaternions is equal to the logarithm of their product: this apparent paradox, or at least deviation from ordinary algebraic rules, arising necessarily from the corresponding property of quaternion multiplication, which has been already seen to be not generally a commutative operation (\( q'q'' \) not \( = q''q' \), unless \( q' \) and \( q'' \) be complanar). And here, perhaps, a student might consider his first perusal of this work as closed.*

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CHAPTER II.

ON DIFFERENTIALS AND DEVELOPMENTS OF FUNCTIONS OF QUATERNIONS; AND ON SOME APPLICATIONS OF QUATERNIONS TO GEOMETRICAL AND PHYSICAL QUESTIONS.

It has been already said, that this Chapter may be omitted in a first perusal of the work.

SECTION 1.—ON THE DEFINITION OF SIMULTANEOUS DIFFERENTIALS.


SECTION 2.—ELEMENTARY ILLUSTRATIONS OF THE DEFINITION, FROM ALGEBRA AND GEOMETRY.

[Art. 322, Illustration from Algebra, p. 432.—Art. 323, And from geometry, p. 435.]

In the view here adopted (comp. I. iii. 7), differentials are not necessarily, nor even generally, small. But it is shown at a later stage (Art. 401), that the principles of this Calculus allow us, whenever any advantage may be thereby gained, to treat differentials as infinitesimals; and so to abridge calculation, at least in many applications.

* If he should choose to proceed to the Differential Calculus of Quaternions in the next Chapter (III. ii.), and to the Geometrical and other Applications in the third Chapter (III. iii.) of the present Book, it might be useful to read at this stage the last Section (I. iii. 7) of the First Book, which treats of Differentials of Vectors (pp. 96—102); and perhaps the omitted parts of the Section II. i. 13, namely Articles 213—220, with their sub-articles (pp. 220—242), which relate, among other things, to a Construction of the Ellipsoid, suggested by the present Calculus. But the writer will now abstain from making any further suggestions of this kind, after having indicated as above what appeared to him a minimum course of study, amounting to rather less than 200 pages (or parts of pages) of this Volume, which will be recapitulated for the convenience of the student at the end of the present Table.
TABLE OF CONTENTS.

Section 3.—On some general Consequences of the Definition, 438-451


Partial differentials and derivatives are introduced; and differentials of functions.

Section 4.—Examples of Quaternion Differentiation, 451-464


One of the most important rules is, to differentiate the factors of a quaternion product, in situ; thus (by p. 446),

$$d \cdot qq' = dq \cdot q' + q \cdot dq'.$$

The formula (p. 439),

$$d \cdot q^{-1} = -q^{-1}dq \cdot q^{-1},$$

for the differential of the reciprocal of a quaternion (or vector), is also very often useful; and so are the equations (p. 456),

$$\frac{dTq}{Tq} = S \frac{dq}{q}; \quad \frac{dUq}{Uq} = V \frac{dq}{q};$$

(K)

and (p. 454),

$$d \cdot a\alpha = \frac{\pi}{2} a^{t+1} d\alpha;$$

(L)

$q$ being any quaternion, and $\alpha$ any constant vector-unit, while $t$ is a variable scalar. It is important to remember (comp. III. i. 11), that we have not in quaternions the usual equation,

$$dIq = \frac{dq}{q};$$

unless $q$ and $dq$ be complanar; and therefore that we have not generally,

$$dIp = \frac{dp}{p},$$

if $p$ be a variable vector; although we have, in this Calculus, the scarcely less simple equation, which is useful in questions respecting orbital motion,

$$\frac{d\alpha}{\alpha} = \frac{dp}{p}$$

(M)

if $\alpha$ be a constant vector, and if the planes of $\alpha$ and $p$ be given (or constant).
Section 5.—On Successive Differentials and Developments, of Functions of Quaternions, 465-484


In this Section principles are established (pp. 469-473), respecting quaternion functions which vanish together; and a form of development (pp. 473-475) is assigned, analogous to Taylor's Series, and like it capable of being concisely expressed by the symbolical equation, $1 + \Delta = e^\Delta$ (p. 480). As an example of partial and successive differentiation, the expression (pp. 480-481),

$$\rho = r\Delta j^j k^j k^j,$$

which may represent any vector, is operated on; and an application is made, by means of definite integration (pp. 482, 483), to deduce the known area and volume of a sphere, or of portions thereof; together with the theorem, that the vector sum of the directed elements of a spheric segment is zero: each element of surface being represented by an inward normal, proportional to the elementary area, and corresponding in hydrostatics to the pressure of a fluid on that element.

Section 6.—On the Differentiation of Implicit Functions of Quaternions; and on the General Inversion of a Linear Function, of a Vector or a Quaternion; with some connected Investigations, 484-568


In this Section it is shown, among other things, that a Linear and Vector Symbol, $\phi$, of Operation on a Vector, $p$, satisfies (p. 494) a Symbolic and Cubic Equation, of the form,

$$0 = m - m'\phi + m''\phi^2 - \phi^3;$$

whence

$$m\phi^{-1} = m' - m''\phi + \phi^2 = \psi,$$

= another symbol of linear operation, which it is shown how to deduce otherwise.

* At a later stage (Art. 375), a new Enunciation of Taylor's Theorem is given, with a new proof, but still in a form adapted to quaternions.
BOOK I.

ON VECTORS, CONSIDERED WITHOUT REFERENCE TO ANGLES OR TO ROTATIONS.

Hamilton's Elements of Quaternions.
CHAPTER I.

FUNDAMENTAL PRINCIPLES RESPECTING VECTORS.

SECTION 1.

On the Conception of a Vector; and on Equality of Vectors.

Art. 1.—A right line ab, considered as having not only length, but also direction, is said to be a Vector. Its initial point a is said to be its origin; and its final point b is said to be its term. A vector ab is conceived to be (or to construct) the difference of its two extreme points; or, more fully, to be the result of the subtraction of its own origin from its own term; and, in conformity with this conception, it is also denoted by the symbol b - a: a notation which will be found to be extensively useful, on account of the analogies which it serves to express between geometrical and algebraical operations. When the extreme points a and b are distinct, the vector ab or b - a is said to be an actual (or an effective) vector; but when (as a limit) those two points are conceived to coincide, the vector aa or a - a, which then results, is said to be null. Opposite vectors, such as ab and ba, or b - a and a - b, are sometimes called vector and revector. Successive vectors, such as ab and bc, or b - a and c - b, are occasionally said to be vector and provector: the line ac, or c - a, which is drawn from the origin a of the first to the term c of the second, being then said to be the transvector. At a later stage, we shall have to consider vector-arcs and vector-angles; but at present, our only vectors are (as above) right lines.

2. Two vectors are said to be equal to each other, or the equation ab = cd, or b - a = d - c, is said to hold good, when (and only when) the origin
and term of the one can be brought to coincide respectively with the corresponding points of the other, by transports (or by translations) without rotation. It follows that all null vectors are equal, and may therefore be denoted by a common symbol, such as that used for zero; so that we may write,
\[ A - A = B - B = \&c. = 0; \]
but that two actual vectors, \( \text{ab} \) and \( \text{cd} \), are not (in the present full sense) equal to each other, unless they have not merely equal lengths, but also similar directions. If then they do not happen to be parts of one common line, they must be opposite sides of a parallelogram, \( \text{ABDC} \); the two lines \( \text{AD} \), \( \text{BC} \) becoming thus the two diagonals of such a figure, and consequently bisecting each other, in some point \( E \).

Conversely, if the two equations,
\[ D - E = E - A, \quad \text{and} \quad C - E = E - B, \]
are satisfied, so that the two lines \( \text{AD} \) and \( \text{BC} \) are commedial, or have a common middle point \( E \), then even if they be parts of one right line, the equation \( D - C = B - A \) is satisfied. Two radii, \( \text{ab} \), \( \text{ac} \), of any one circle (or sphere), can never be equal vectors; because their directions differ.

3. An equation between vectors, considered as an equidifference of points, admits of inversion and alternation; or in symbols, if
\[ D - C = B - A, \]
then
\[ C - D = A - B, \quad \text{and} \quad D - B = C - A. \]

Two vectors, \( \text{cd} \) and \( \text{ef} \), which are equal to the same third vector, \( \text{ab} \), are also equal to each other; and these three equal vectors are, in general, the three parallel edges of a prism.
SECTION 2.

On Differences and Sums of Vectors taken two by two.

4. In order to be able to write, as in algebra,

\[(c' - a') - (b - a) = c - b, \text{ if } c' - a' = c - A,\]

we next define, that when a first vector \(ab\) is subtracted from a second vector \(ac\) which is co-initial with it, or from a third vector \(a'c'\) which is equal to that second vector, the remainder is that fourth vector \(bc\), which is drawn from the term \(b\) of the first to the term \(c\) of the second vector: so that if a vector be subtracted from a transvector (Art. 1), the remainder is the revector corresponding. It is evident that this geometrical subtraction of vectors answers to a decomposition of vections (or of motions); and that, by such a decomposition of a null vection into two opposite vections, we have the formula,

\[0 - (b - a) = (a - a) - (b - a) = a - b;\]

so that, if an actual vector \(ab\) be subtracted from a null vector \(aa\), the remainder is the revector \(ba\). If then we agree to abridge, generally, an expression of the form \(0 - a\) to the shorter form, \(-a\), we may write briefly, \(-ab = ba\); \(a\) and \(-a\) being thus symbols of opposite vectors, while \(a\) and \(-(-a)\) are, for the same reason, symbols of one common vector: so that we may write, as in algebra, the identity,

\[-(-a) = a.\]

5. Aiming still at agreement with algebra, and adopting on that account the formula of relation between the two signs, + and -, \n
\[(b - a) + a = b,\]

in which we shall say as usual that \(b - a\) is added to \(a\), and that their sum is \(b\), while relatively to it they may be jointly called summands, we shall have the two following consequences:—

I. If a vector, \(ab\) or \(b - a\), be added to its own origin \(a\), the sum is its term \(b\) (Art. 1); and

II. If a revector \(bc\) be added to a vector \(ab\), the sum is the transvector \(ac\); or in symbols,

I. \((b - a) + a = b;\) and II. \((c - b) + (b - a) = c - a.\)

In fact, the first equation is an immediate consequence of the general formula.
which, as above, connects the signs + and −, when combined with the conception (Art. 1) of a vector as a difference of two points; and the second is a result of the same formula, combined with the definition of the geometrical subtraction of one such vector from another, which was assigned in Art. 4, and according to which we have (as in algebra) for any three points \( A, B, C \), the identity,

\[
(c - A) - (B - A) = C - B.
\]

It is clear that this geometrical addition of successive vectors corresponds (comp. Art. 4) to a composition of successive vectors, or motions; and that the sum of two opposite vectors (or of vector and revector) is a null line; so that

\[
BA + AB = 0, \text{ or } (A - B) + (B - A) = 0.
\]

It follows also that the sums of equal pairs of successive vectors are equal; or more fully that

if \( b' - a' = b - a \), and \( c' - b' = c - b \), then \( c' - a' = c - a \);

the two triangles, \( ABC \) and \( A'B'C' \), being in general the two opposite faces of a prism (comp. Art. 3).

6. Again, in order to have, as in algebra,

\[
(c' - b') + (b - a) = c - a, \text{ if } c' - b' = c - b,
\]

we shall define that if there be two successive vectors, \( AB, BC \), and if a third vector \( b'c' \) be equal to the second, but not successive to the first, the sum obtained by adding the third to the first is that fourth vector, \( AC \), which is drawn from the origin \( A \) of the first to the term \( C \) of the second. It follows that the sum of any two co-initial sides, \( AB, AC \), of any parallelogram \( ABDC \), is the intermediate and co-initial diagonal \( AD \); or, in symbols,

\[
(c - A) + (B - A) = D - A, \text{ if } D - C = B - A;
\]

because we have then (by 3) \( C - A = D - B \).

7. The sum of any two given vectors has thus a value which is independent of their order; or, in symbols, \( a + \beta = \beta + a \). If equal vectors be added to equal vectors, the sums are equal vectors, even if the summands be not given as successive (comp. 5); and if a null vector be added to an actual vector, the sum is that actual vector; or, in symbols, \( 0 + a = a \). If then we agree to abridge generally (comp. 4) the expression \( 0 + a \) to \( + a \), and if \( a \) still denote a
vector, then + a, and (+ a), &c., are other symbols for the same vector; and we have, as in algebra, the identities,

\[-(-a) = +a, \quad (+a) + (-a) = 0, \quad \text{&c.}\]

SECTION 3.

**On Sums of three or more Vectors.**

8. The sum of three given vectors, \(a, \beta, \gamma\), is next defined to be that fourth vector,

\[\delta = \gamma + (\beta + a), \quad \text{or briefly,} \quad \delta = \gamma + \beta + a,\]

which is obtained by adding the third to the sum of the first and second; and in like manner the sum of any number of vectors is formed by adding the last to the sum of all that precede it: also, for any four vectors, \(a, \beta, \gamma, \delta\), the sum \(\delta + (\gamma + \beta + a)\) is denoted simply by \(\delta + \gamma + \beta + a\), without parentheses, and so on for any number of summands.

9. The sum of any number of successive vectors, \(AB, BC, CD\), is thus the line \(AD\), which is drawn from the origin \(A\) of the first, to the term \(D\) of the last; and because, when there are three such vectors, we can draw (as in fig. 9) the two diagonals \(AC, BD\) of the (plane or gauche) quadrilateral \(ABCD\), and may then at pleasure regard \(AD\), either as the sum of \(AB, BD\), or as the sum of \(AC, CD\), we are allowed to establish the following general formula of association, for the case of any three summand lines, \(a, \beta, \gamma\):

\[(\gamma + \beta) + a = \gamma + (\beta + a) = \gamma + \beta + a;\]

by combining which with the formula of commutation (Art. 7), namely, with the equation,

\[a + \beta = \beta + a,\]

which had been previously established for the case of any two such summands, it is easy to conclude that the Addition of Vectors is always both an Associative and a Commutative Operation. In other words, the sum of any number of given vectors has a value which is independent of their order, and of the mode of grouping them; so that if the lengths and directions of the summands be preserved, the length and direction of the sum will also remain
unchanged: except that this last direction may be regarded as indeterminate, when the length of the sum-line happens to vanish, as in the case which we are about to consider.

10. When any $n$ summand-lines, $ab$, $bc$, $ca$, or $ab$, $bc$, $cd$, $da$, &c., arranged in any one order, are the $n$ successive sides of a triangle $abc$, or of a quadrilateral $abcd$, or of any other closed polygon, their sum is a null line, $aa$; and conversely, when the sum of any given system of $n$ vectors is thus equal to zero, they may be made (in any order, by transports without rotation) the $n$ successive sides of a closed polygon (plane or gauche). Hence, if there be given any such polygon ($p$), suppose a pentagon $abcde$, it is possible to construct another closed polygon ($p'$), such as $a'b'c'd'e'$, with an arbitrary initial point $a'$, but with the same number of sides, $a'b'$, $e'a'$, which new sides shall be equal (as vectors) to the old sides $ab$, $ea$, taken in any arbitrary order. For example, if we draw four successive vectors, as follows,

$$A'B' = CD, \quad B'C' = AB, \quad C'D' = EA, \quad D'E' = BC,$$

and then complete the new pentagon by drawing the line $e'a'$, this closing side of the second figure ($p'$) will be equal to the remaining side $de$ of the first figure ($p$).

11. Since a closed figure $abc \ldots$ is still a closed one, when all its points are projected on any assumed plane, by any system of parallel ordinates (although the area of the projected figure $a'b'c' \ldots$ may happen to vanish), it follows that if the sum of any number of given vectors $a, \beta, \gamma, \ldots$ be zero, and if we project them all on any one plane by parallel lines drawn from their extremities, the sum of the projected vectors $a', \beta', \gamma', \ldots$ will likewise be null; so that these latter vectors, like the former, can be so placed as to become the successive sides of a closed polygon, even if they be not already such. (In fig. 11, $a''b''c''$ is considered as such a polygon, namely, as a triangle with evanescent area; and we have the equation,

$$A''B'' + B''C'' + C'A'' = 0,$$

as well as $A'B' + B'C' + C'A' = 0$, and $AB + BC + CA = 0$.)
SECTION 4.

**On Coefficients of Vectors.**

12. The simple or single vector, $a$, is also denoted by $1a$, or by $1 \cdot a$, or by $(+1) a$; and in like manner, the double vector, $a + a$, is denoted by $2a$, or $2 \cdot a$, or $(+2) a$, &c.; the rule being, that for any algebraical integer, $m$, regarded as a coefficient by which the vector $a$ is multiplied, we have always,

$$1a + ma = (1 + m) a;$$

the symbol $1 + m$ being here interpreted as in algebra. Thus, $0a = 0$, the zero on the one side denoting a null coefficient, and the zero on the other side denoting a null vector; because by the rule,

$$1a + 0a = (1 + 0) a = 1a = a,$$

and 

$$0a = a - a = 0.$$

Again, because $(1) a + (-1) a = (1 - 1) a = 0a = 0$, we have $(-1) a = 0 - a = -a = -(1a)$; in like manner, since $(1) a + (-2) a = (1 - 2) a = (-1) a = -a$, we infer that $(-2) a = -a - a = -(2a)$; and generally $(-m) a = -(ma)$, whatever whole number $m$ may be: so that we may, without danger of confusion, omit the parentheses in these last symbols, and write simply, $-1a$, $-2a$, $-ma$.

13. It follows that whatever two whole numbers (positive or negative, or null) may be represented by $m$ and $n$, and whatever two vectors may be denoted by $a$ and $\beta$, we have always, as in algebra, the formulæ,

$$na ± ma = (n ± m) a, \quad n (ma) = (nm) a = nma,$$

and (compare fig. 12),

$$m (\beta + a) = m\beta ± ma;$$

so that the multiplication of vectors by coefficients is a doubly distributive operation,
at least if the multipliers be whole numbers; a restriction which, however, will soon be removed.

14. If $ma = \beta$, the coefficient $m$ being still whole, the vector $\beta$ is said to be a multiple of $a$; and conversely (at least if the integer $m$ be different from zero), the vector $a$ is said to be a sub-multiple of $\beta$. A multiple of a sub-multiple of a vector is said to be a fraction of that vector; thus, if $\beta = ma$, and $\gamma = na$, then $\gamma$ is a fraction of $\beta$, which is denoted as follows, $\gamma = \frac{n}{m} \beta$; also $\beta$ is said to be multiplied by the fractional coefficient $\frac{n}{m}$, and $\gamma$ is said to be the product of this multiplication. It follows that if $x$ and $y$ be any two fractions, (positive or negative or null, whole numbers being included), and if $a$ and $\beta$ be any two vectors, then

$$ya \pm xa = (y \pm x)a, \quad y(xa) = (yx)a = yxa, \quad x(\beta \pm a) = x\beta \pm xa;$$

results which include those of Art. 13, and may be extended to the case where $x$ and $y$ are incommensurable coefficients, considered as limits of fractional ones.

15. For any actual vector $a$, and for any coefficient $x$, of any of the foregoing kinds, the product $xa$, interpreted as above, represents always a vector $\beta$, which has the same direction as the multiplicand-line $a$, if $x > 0$, but has the opposite direction if $x < 0$, becoming null if $x = 0$. Conversely, if $a$ and $\beta$ be any two actual vectors, with directions either similar or opposite, in each of which two cases we shall say that they are parallel vectors, and shall write $\beta \parallel a$ (because both are then parallel, in the usual sense of the word, to one common line), we can always find, or conceive as found, a coefficient $x \geq 0$, which shall satisfy the equation $\beta = xa$; or, as we shall also write it, $\beta = ax$; and the positive or negative number $x$, so found, will bear to $+1$ the same ratio, as that which the length of the line $\beta$ bears to the length of $a$.

16. Hence it is natural to say that this coefficient $x$ is the quotient which results, from the division of the vector $\beta$, by the parallel vector $a$; and to write, accordingly,

$$x = \beta \div a, \quad x = \beta : a, \quad or \quad x = \frac{\beta}{a};$$

so that we shall have, identically, as in algebra, at least if the divisor-line $a$ be an actual vector, and if the dividend-line $\beta$ be parallel thereto, the equations,

$$(\beta : a) \cdot a = \frac{\beta}{a} a = \beta, \quad and \quad xa : a = \frac{xa}{a} = x;$$

which will afterwards be extended, by definition, to the case of non-parallel
vectors. We may write also, under the same conditions, \( a = \frac{\beta}{x} \), and may say that the vector \( a \) is the quotient of the division of the other vector \( \beta \) by the number \( x \); so that we shall have these other identities,

\[
\frac{\beta}{x} \cdot x = (ax) = \beta, \quad \text{and} \quad \frac{ax}{x} = a.
\]

17. The positive or negative quotient, \( x = \frac{\beta}{a} \), which is thus obtained by the division of one of two parallel vectors by another, including zero as a limit, may also be called a Scalar; because it can always be found, and in a certain sense constructed, by the comparison of positions upon one common scale (or axis); or can be put under the form,

\[
x = \frac{c-a}{b-a} = \frac{ac}{ab},
\]

where the three points, \( A, B, C \), are collinear (as in the figure annexed). Such scalars are, therefore, simply the Reals (or real quantities) of Algebra; but, in combination with the not less real Vectors above considered, they form one of the main elements of the System, or Calculus, to which the present work relates. In fact it will be shown, at a later stage, that there is an important sense in which we can conceive a scalar to be added to a vector; and that the sum so obtained, or the combination "Scalar plus Vector" is a Quaternion.
CHAPTER II.

APPLICATIONS TO POINTS AND LINES IN A GIVEN PLANE.

SECTION 1.

On Linear Equations connecting two Co-initial Vectors.

18. When several vectors, $oa$, $ob$, . . are all drawn from one common point $o$, that point is said to be the Origin of the System; and each particular vector, such as $oa$, is said to be the vector of its own term, $A$. In the present and future sections we shall always suppose, if the contrary be not expressed, that all the vectors $a$, $b$, . . which we may have occasion to consider, are thus drawn from one common origin. But if it be desired to change that origin $o$, without changing the term-points $a$, . . we shall only have to subtract, from each of their old vectors $a$, . . one common vector $w$, namely, the old vector $oo'$ of the new origin $o'$; since the remainders, $a - w$, $b - w$, . . will be the new vectors $a'$, $b'$, . . of the old points $A$, $B$, . . For example, we shall have

$$a' = o'a = a - o' = (a - o) - (o' - o) = aa - oo' = a - w.$$  

19. If two vectors $a$, $b$, or $oa$, $ob$, be thus drawn from a given origin $o$, and if their directions be either similar or opposite, so that the three points, $o$, $A$, $B$, are situated on one right line (as in the figure annexed), then (by 16, 17) their quotient $\frac{b}{a}$ is some positive or negative scalar, such as $x$; and conversely, the equation $b = ax$, interpreted with this reference to an origin, expresses the condition of collinearity, of the points $o$, $A$, $B$; the particular values $x = 0$, $x = 1$, corresponding to the particular positions, $o$ and $A$, of the variable point $B$, whereof the indefinite right line $oa$ is the locus.

20. The linear equation, connecting the two vectors $a$ and $b$, acquires a more symmetric form, when we write it thus:

$$aa + b\beta = 0;$$

where $a$ and $b$ are two scalars, of which however only the ratio is important.
The condition of coincidence, of the two points A and B, answering above to
\( x = 1 \), is now \( \frac{a}{b} = 1 \); or, more symmetrically,
\[ a + b = 0. \]
Accordingly, when \( a = -b \), the linear equation becomes
\[ b(\beta - a) = 0, \quad \text{or} \quad \beta - a = 0, \]
since we do not suppose that both the coefficients vanish; and the equation
\( \beta = a \), or \( OB = OA \), requires that the point B should coincide with the point A: a
case which may also be conveniently expressed by the formula,
\[ B = A; \]
coincident points being thus treated (in notation at least) as equal. In general,
the linear equation gives,
\[ a \cdot OA + b \cdot OB = 0, \quad \text{and therefore} \quad a : b = BO : OA. \]

SECTION 2.

On Linear Equations between three Co-initial Vectors.

21. If two (actual and co-initial) vectors, \( a, \beta \), be not connected by any
equation of the form \( aa + b\beta = 0 \), with any
two scalar coefficients \( a \) and \( b \) whatever, their
directions can neither be similar nor opposite
to each other; they therefore determine a
plane \( AOB \), in which the (now actual) vector,
represented by the sum \( aa + b\beta \), is situated.
For if, for the sake of symmetry, we denote
this sum by the symbol \( -\gamma \), where \( c \) is some
third scalar, and \( \gamma = OC \) is some third vector,
so that the three co-initial vectors, \( a, \beta, \gamma \), are connected by the linear equation,
\[ aa + b\beta + c\gamma = 0; \]
and if we make
\[ OA' = \frac{-aa}{c} \quad \text{and} \quad OB' = \frac{-b\beta}{c}; \]
then the two auxiliary points, \( A' \) and \( B' \), will be situated (by 19) on the two
indefinite right lines, \( OA, OB \), respectively: and we shall have the equation,
\[ OC = OA' + OB', \]
so that the figure \( A'OB'c \) is (by 6) a parallelogram, and consequently plane.
22. Conversely, if \( c \) be any point in the plane \( \alpha \beta \), we can draw from it the ordinates, \( \alpha \alpha' \) and \( \beta \beta' \), to the lines \( \alpha \alpha \) and \( \beta \beta \), and can determine the ratios of the three scalars, \( \alpha, \beta, c \), so as to satisfy the two equations,

\[
\frac{\alpha}{c} = -\frac{\alpha \alpha'}{\alpha \alpha'}, \quad \frac{\beta}{c} = -\frac{\beta \beta'}{\beta \beta'};
\]

after which we shall have the recent expressions for \( \alpha \alpha', \beta \beta' \), with the relation \( \gamma c = \alpha \alpha' + \beta \beta' \) as before; and shall thus be brought back to the linear equation \( a\alpha + b\beta + c\gamma = 0 \), which equation may therefore be said to express the condition of complanarity of the four points, \( o, \alpha, \beta, c \). And if we write it under the form,

\[
xa + y\beta + z\gamma = 0,
\]

and consider the vectors \( \alpha \) and \( \beta \) as given, but \( \gamma \) as a variable vector, while \( x, y, z \) are variable scalars, the locus of the variable point \( c \) will then be the given plane, \( o\alpha\beta \).

23. It may happen that the point \( c \) is situated on the right line \( \alpha \beta \), which is here considered as a given one. In that case (comp. Art. 17, fig. 13), the quotient \( \frac{\alpha c}{\alpha \beta} \) must be equal to some scalar, suppose \( t \); so that we shall have an equation of the form,

\[
\frac{\gamma - \alpha}{\beta - \alpha} = t, \quad \text{or} \quad \gamma = a + t(\beta - \alpha), \quad \text{or} \quad (1 - t)\alpha + t\beta - \gamma = 0;
\]

by comparing which last form with the linear equation of Art. 21, we see that the condition of collinearity of the three points \( \alpha, \beta, c \), in the given plane \( o\alpha\beta \), is expressed by the formula,

\[
a + b + c = 0.
\]

This condition may also be thus written,

\[
1 = \frac{-a}{c} + \frac{-b}{c}, \quad \text{or} \quad \frac{\alpha \alpha'}{\alpha \alpha} + \frac{\beta \beta'}{\beta \beta} = 1;
\]

and under this last form it expresses a geometrical relation, which is otherwise known to exist.

24. When we have thus the two equations,

\[
a\alpha + b\beta + c\gamma = 0, \quad \text{and} \quad a + b + c = 0,
\]

so that the three co-initial vectors \( \alpha, \beta, \gamma \) terminate on one right line, and may on that account be said to be termino-collinear, if we eliminate,
successively and separately, each of the three scalars $a$, $b$, $c$, we are conducted to these three other equations, expressing certain \textit{ratios of segments}:

\[ b(\beta - a) + c(\gamma - a) = 0, \quad c(\gamma - \beta) + a(a - \beta) = 0, \]
\[ a(a - \gamma) + b(\beta - \gamma) = 0; \]

or

\[ 0 = b \cdot AB + c \cdot AC = c \cdot BC + a \cdot BA = a \cdot CA + b \cdot CB. \]

Hence follows this \textit{proportion}, between \textit{coefficients} and \textit{segments},

\[ a : b : c = BC : CA : AB. \]

We might also have observed that the proposed equations give,

\[ a = \frac{b\beta + c\gamma}{b + c}, \quad \beta = \frac{c\gamma + \alpha\alpha}{c + \alpha}, \quad \gamma = \frac{\alpha\alpha + b\beta}{a + b}; \]

whence

\[ \frac{AC}{AB} = \frac{\gamma - a}{\beta - a} = \frac{b}{a + b} = -\frac{b}{c}, \text{ &c.} \]

25. If we still treat $a$ and $\beta$ as \textit{given}, but regard $\gamma$ and $\frac{y}{x}$ as \textit{variable}, the equation

\[ \gamma = \frac{xa + y\beta}{x + y} \]

will express that the \textit{variable point} $c$ is situated \textit{somewhere} on the \textit{indefinite right line} $AB$, or that it has this \textit{line} for its \textit{locus} : while it divides the \textit{finite line} $AB$ into \textit{segments}, of which the \textit{variable quotient} is,

\[ \frac{AC}{CB} = \frac{y}{x}. \]

Let $c'$ be another \textit{point} on the \textit{same line}, and let its \textit{vector} be $b + y'\beta$,

\[ \gamma' = \frac{x'a + y'\beta}{x' + y'}; \]

then, in like manner, we shall have this \textit{other ratio of segments},

\[ \frac{AC'}{C'B} = \frac{y'}{x'}. \]

If, then, we agree to employ, \textit{generally}, \textit{for any group of four collinear points}, the \textit{notation},

\[ (ABCD) = \frac{AB}{BC} \cdot \frac{CD}{DA} = \frac{AB}{BC} : \frac{AD}{DC}; \]

so that this \textit{symbol},

\[ (ABCD), \]

\[ (ABCD), \]

may be said to denote the \textit{anharmonic function}, or \textit{anharmonic quotient}, or
simply the anharmonic of the group, \( A, B, C, D \): we shall have, in the present case, the equation,

\[
\frac{AC'BC'}{CB} = \frac{\frac{AC}{CB}}{\frac{C'B}{C'B}} = \frac{yx'}{xy'}.
\]

26. When the anharmonic quotient becomes equal to negative unity the group becomes (as is well known) harmonic. If then we have the two equations,

\[
\gamma = \frac{x\alpha + y\beta}{x + y}, \quad \gamma' = \frac{x\alpha - y\beta}{x - y},
\]

the two points \( c \) and \( c' \) are harmonically conjugate to each other, with respect to the two given points, \( A \) and \( B \); and when they vary together, in consequence of the variation of the value of \( \frac{y}{x} \), they form (in a well-known sense), on the indefinite right line \( AB \), divisions in involution; the double points (or foci) of this involution, namely, the points of which each is its own conjugate, being the points \( A \) and \( B \) themselves. As a verification, if we denote by \( \mu \) the vector of the middle point \( M \) of the given interval \( AB \), so that \( \beta - \mu = \frac{1}{2} (\alpha + \beta) \), we easily find that

\[
\frac{\gamma - \mu}{\beta - \mu} = \frac{y - x}{y + x} = \frac{\beta - \mu}{\gamma' - \mu}, \quad \text{or} \quad \frac{MC}{MB} = \frac{MB}{MC'};
\]

so that the rectangle under the distances \( MC, MC' \), of the two variable but conjugate points, \( c, c' \), from the centre \( M \) of the involution, is equal to the constant square of half the interval between the two double points, \( A, B \). More generally, if we write

\[
\gamma = \frac{x\alpha + y\beta}{x + y}, \quad \gamma' = \frac{l_0x + my\beta}{lx + my},
\]

where the anharmonic quotient \( \frac{l}{m} = \frac{yx'}{xy} \) is any constant scalar, then in another known and modern* phraseology, the points \( c \) and \( c' \) will form, on the indefinite line \( AB \), two homographic divisions, of which \( A \) and \( B \) are still the double points. More generally still, if we establish the two equations

\[
\gamma = \frac{x\alpha + y\beta}{x + y}, \quad \gamma' = \frac{l_0x + my\beta'}{lx + my},
\]

where \( \frac{l}{m} \) is constant, but \( \frac{y}{x} \) variable, while \( \alpha' = 0\alpha', \beta' = 0\beta', \\) and \( \gamma' = 0\gamma' \), the two given lines, \( AB \) and \( AB' \), are then homographically divided, by the two variable points \( c \) and \( c' \), not now supposed to move along one common line.

* See the Géométrie Supérieure of M. Chasles, p. 107. (Paris, 1852.)
27. When the linear equation \( aa + b\beta + c\gamma = 0 \) subsists, without the relation \( a + b + c = 0 \) between its coefficients, then the three co-initial vectors \( a, \beta, \gamma \) are still complanar, but they no longer terminate on one right line; their term-points, \( A, B, C \) being now the corners of a triangle.

In this more general case, we may propose to find the vectors \( a', \beta', \gamma' \) of the three points,

\[
A' = OA \cdot BC, \quad B' = OB \cdot CA, \quad C' = OC \cdot AB;
\]

that is to say, of the points in which the lines drawn from the origin \( O \) to the three corners of the triangle intersect the three respectively opposite sides. The three collineations \( OA', B', C' \), &c.,

give (by 19) three expressions of the forms,

\[
a' = xa, \quad \beta' = y\beta, \quad \gamma' = z\gamma,
\]

where \( x, y, z \) are three scalars, which it is required to determine by means of the three other collineations, \( \lambda'BC, \) &c., with the help of relations derived from the principle of Art 23. Substituting therefore for \( a \) its value \( xa' \), in the given linear equation, and equating to zero the sum of the coefficients of the new linear equation which results, namely,

\[
x^2aa' + b\beta + c\gamma = 0;
\]

and eliminating similarly \( \beta, \gamma \), each in its turn, from the original equation; we find the values,

\[
x = \frac{-a}{b + c}, \quad y = \frac{-b}{c + a}, \quad z = \frac{-c}{a + b};
\]

whence the sought vectors are expressed in either of the two following ways:

I. . . \( a' = \frac{-aa}{b + c}, \quad \beta' = \frac{-b\beta}{c + a}, \quad \gamma' = \frac{-c\gamma}{a + b}; \)

or

II. . . \( a' = \frac{b\beta + c\gamma}{b + c}, \quad \beta' = \frac{c\gamma + aa}{c + a}, \quad \gamma' = \frac{aa + b\beta}{a + b}. \)

In fact we see, by one of these expressions for \( a' \), that \( \lambda' \) is on the line \( OA \); and by the other expression for the same vector \( a' \), that the same point \( \lambda' \) is on the line \( BC \). As another verification, we may observe that the last expressions for \( a', \beta', \gamma' \), coincide with those which were found in Art. 24, for \( a, \beta, \gamma \) themselves, on the particular supposition that the three points \( A, B, C \) were collinear.
28. We may next propose to determine the ratios of the segments of the sides of the triangle ABC, made by the points A', B', C'. For this purpose, we may write the last equations for a', b', c' under the form,

\[ 0 = b(a' - \beta) - c(\gamma - a') = c(\beta' - \gamma) - a(a - \beta') = a(\gamma' - a) - b(\beta - \gamma'); \]

and we see that they then give the required ratios, as follows:

\[
\frac{BA'}{AC'} = \frac{c}{b}, \quad \frac{CB'}{BA'} = \frac{a}{c}, \quad \frac{AC'}{CB'} = \frac{b}{a};
\]

whence we obtain at once the known equation of six segments,

\[
\frac{BA'}{AC'} \cdot \frac{CB'}{BA'} \cdot \frac{AC'}{CB'} = 1,
\]

as the condition of concurrence of the three right lines AA', BB', CC', in a common point, such as O. It is easy also to infer, from the same ratios of segments, the following proportion of coefficients and areas,

\[
a : b : c = OBC : OCA : OAB,
\]

in which we must, in general, attend to algebraic signs; a triangle being conceived to pass (through zero) from positive to negative, or vice versa, as compared with any given triangle in its own plane, when (in the course of any continuous change) its vertex crosses its base. It may be observed that with this convention (which is, in fact, a necessary one, for the establishment of general formulæ) we have, for any three points, the equation

\[
ABC + BAC = 0,
\]

exactly as we had (in Art. 5) for any two points, the equation

\[
AB + BA = 0.
\]

More fully, we have, on this plan, the formulæ,

\[
ABC = -BAC = BCA = -CBA = CAB = -ACB;
\]

and any two complanar triangles, ABC, A'B'C', bear to each other a positive or a negative ratio, according as the two rotations, which may be conceived to be denoted by the same symbols ABC, A'B'C', are similarly or oppositely directed.

29. If A' and B' bisect respectively the sides BC and CA, then

\[
a = b = c,
\]

and C' bisects AB; whence the known theorem follows, that the three bisectors of the sides of a triangle concur, in a point which is often called the centre of gravity, but which we prefer to call the mean point of the triangle, and which
is here the origin $o$. At the same time, the first expressions in Art. 27 for $a', \beta', \gamma'$ become,

$$a' = -\frac{a}{2'}, \quad \beta' = -\frac{\beta}{2'}, \quad \gamma' = -\frac{\gamma}{2};$$

whence this other known theorem results, that the three bisectors trisect each other.

30. The linear equation between $a, \beta, \gamma$ reduces itself, in the case last considered, to the form,

$$a + \beta + \gamma = 0, \quad \text{or} \quad oA + OB + OC = 0;$$

the three vectors $a, \beta, \gamma$, or $oa, ob, oc$, are therefore, in this case, adapted (by Art. 10) to become the successive sides of a triangle, by transports without rotation; and accordingly, if we complete (as in fig. 19) the parallelogram $AOBD$, the triangle $OAD$ will have the property in question. It follows (by 11) that if we project the four points $o, a, b, c$, by any system of parallel ordinates, into four other points, $o', A', B', C'$, on any assumed plane, the sum of the three projected vectors, $a', \beta', \gamma'$, or $o'A', o'B', o'C'$, will be null; so that we shall have the new linear equation,

$$a' + \beta' + \gamma' = 0,$$

or,

$$o'A + o'B + o'C = 0;$$

and in fact it is evident (see fig. 20) that the projected mean point $o'$ will be the mean point of the projected triangle, $A', B', C'$. We shall have also the equation,

$$(a_r - a) + (\beta_r - \beta) + (\gamma_r - \gamma) = 0;$$

where

$$a_r - a = o'A - OA = (oA + AA) - (oo + oA) = AA - oo;$$

hence

$$oo_r = \frac{1}{3} (AA + BB + CC),$$

or the ordinate of the mean point of a triangle is the mean of the ordinates of the three corners.
SECTION 3.

On Plane Geometrical Nets.

31. Resuming the more general case of Art. 27, in which the coefficients a, b, c are supposed to be unequal, we may next inquire, in what points λ", β", γ" do the lines b'c', c'A', A'b' meet respectively the sides bc, ca, ab, of the triangle; or may seek to assign the vectors a", β", γ" of the points of intersection (comp. 27),

$$A'' = b'c' \cdot BC, \quad B'' = c'A' \cdot CA, \quad C'' = A'b' \cdot AB.$$ 

The first expressions in Art. 27 for β', γ', give the equations,

$$(c + a) \beta' + b\beta = 0, \quad (a + b) \gamma' + c\gamma = 0;$$

whence

$$\frac{b\beta - c\gamma}{b - c} = \frac{(a + b) \gamma' - (c + a) \beta'}{(a + b) - (c + a)};$$

but (by 25) one member is the vector of a point on bc, and the other of a point on b'c'; each therefore is a value for the vector a" of λ", and similarly for β" and γ". We may therefore write,

$$a'' = \frac{b\beta - c\gamma}{b - c}, \quad \beta'' = \frac{c\gamma - a\alpha}{c - a}, \quad \gamma'' = \frac{a\alpha - b\beta}{a - b};$$

and by comparing these expressions with the second set of values of a', β', γ' in Art. 27, we see (by 26) that the points λ", β", γ" are, respectively, the harmonic conjugates (as they are indeed known to be) of the points λ', β', γ',

Fig. 21.
with respect to the three pairs of points, $b, c; c, a; a, b$; so that, in the notation of Art. 25, we have the equations,
\[(ba'ca'') = (cb'ab'') = (ac'bc'') = -1.\]
And because the expressions for $a'', \beta'', \gamma''$ conduct to the following linear equation between those three vectors,
\[(b - c) a'' + (c - a) \beta'' + (a - b) \gamma'' = 0,
\]
with the relation
\[(b - c) + (c - a) + (a - b) = 0
\]
between its coefficients, we arrive (by 23) at this other known theorem, that the three points $a'', b'', c''$ are collinear, as indicated by one of the dotted lines in the recent fig. 21.

32. The line $a''b'c'$ may represent any rectilinear transversal, cutting the sides of a triangle $abc$; and because we have
\[
\frac{ba''}{a''c} = \frac{a'' - \beta}{\gamma - a''} = -\frac{c}{b'},
\]
while $\frac{cb'}{b'a} = \frac{a}{c'}$ and $\frac{ac'}{c'b} = \frac{b}{a'}$, as before, we arrive at this other equation of six segments, for any triangle cut by a right line (comp. 28),
\[
\frac{ba''}{a''c} \cdot \frac{cb'}{b'a} \cdot \frac{ac'}{c'b} = -1;
\]
which again agrees with known results.

33. Eliminating $\beta$ and $\gamma$ between either set of expressions (27) for $\beta'$ and $\gamma'$, with the help of the given linear equation, we arrive at this other equation, connecting the three vectors $a, \beta', \gamma'$:
\[
0 = -aa + (c + a) \beta' + (a + b) \gamma'.
\]
Treating this on the same plan as the given equation between $a, \beta, \gamma$, we find that if (as in fig. 21) we make,
\[
A''' = OA \cdot B'C', \quad B''' = OB \cdot C'A', \quad C''' = OC \cdot A'b',
\]
the vectors of these three new points of intersection may be expressed in either of the two following ways, whereof the first is shorter, but the second is, for some purposes (comp. 34, 36), more convenient:
\[
\text{I.} \ldots a''' = \frac{aa}{2a + b + c}, \quad \beta''' = \frac{b\beta}{2b + c + a}, \quad \gamma''' = \frac{c\gamma}{2c + a + b};
\]
or
\[
\text{II.} \ldots a''' = \frac{2aa + b\beta + c\gamma}{2a + b + c}, \quad \beta''' = \frac{2b\beta + c\gamma + aa}{2b + c + a}, \quad \gamma''' = \frac{2c\gamma + aa + b\beta}{2c + a + b}.
\]
And the three equations, of which the following is one,
\[(b - c) a'' - (2b + c + a) \beta'' + (2c + a + b) \gamma'' = 0,
\]
with the relations between their coefficients which are evident on inspection, show (by 23) that we have the three additional collineations, \(\Lambda'' \beta'' \gamma'\), \(\beta'' \gamma'' \Lambda'\), \(\gamma'' \Lambda'' \beta'\), as indicated by three of the dotted lines in the figure. Also, because we have the two expressions,
\[
\frac{a''}{(a + b) + (c + a)} = \frac{(a + b) \gamma' + (c + a) \beta'}{(a + b) + (c + a)}, \quad \frac{a''}{(a + b) - (c + a)} = \frac{(a + b) \gamma' - (c + a) \beta'}{(a + b) - (c + a)},
\]
we see (by 26) that the two points \(\Lambda'', \Lambda'''\) are harmonically conjugate with respect to \(\beta', \gamma'\); and similarly for the two other pairs of points, \(\beta'', \beta''',\) and \(\gamma'', \gamma''',\) compared with \(\gamma', \beta',\) and with \(\beta', \gamma':\) so that, in a notation already employed (25, 31), we may write,
\[
(\beta' \gamma'' \Lambda'') = (\gamma' \beta'' \Lambda''') = (\Lambda' \gamma'' \beta''') = -1.
\]

34. If we begin, as above, with any four complanar points, \(o, \alpha, \beta, \gamma,\) of which no three are collinear, we can (as in fig. 18), by what may be called a First Construction, derive from them six lines, connecting them two by two, and intersecting each other in three new points, \(\alpha', \beta', \gamma';\) and then by a Second Construction (represented in fig. 21), we may connect these by three new lines, which will give, by their intersections with the former lines, six new points, \(\alpha'', \ldots \gamma'''.\) We might proceed to connect these with each other, and with the given points, by sixteen new lines, or lines of a Third Construction, namely, the four dotted lines of fig. 21, and twelve other lines, whereof three should be drawn from each of the four given points: and these would be found to determine eighty-four new points of intersection, of which some may be seen, although they are not marked, in the figure.

But however far these processes of linear construction may be continued, so as to form what has been called* a plane geometrical net, the vectors of the points thus determined have all one common property: namely, that each can be represented by an expression of the form,
\[
\rho = \frac{xa + yb + zc}{xa + yb + zc},
\]
where the coefficients \(x, y, z\) are some whole numbers. In fact we see (by 27, 31, 33) that such expressions can be assigned for the nine derived vectors,

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* By Prof. A. F. Möbius, in page 274 of his Barycentric Calculus (der barycentrische Calcul, Leipzig, 1827).
a',...γ″″, which alone have been hitherto considered; and it is not difficult
to perceive, from the nature of the calculations employed, that a similar result
must hold good, for every vector subsequently deduced. But this and other
connected results will become more completely evident, and their geometrical
signification will be better understood, after a somewhat closer consideration
of anharmonic quotients, and the introduction of a certain system of anharmonic
co-ordinates, for points and lines in one plane, to which we shall next proceed:
reserving, for a subsequent Chapter, any applications of the same theory to
space.

SECTION 4.

On Anharmonic Co-ordinates and Equations of Points and
Lines in one Plane.

35. If we compare the last equations of Art. 33 with the corresponding
equations of Art. 31, we see that the harmonic group ba'ca″, on the side bc of
the triangle abc in fig. 21, has been simply reflected into another such group,
b'α″″c′α″, on the line b'c', by a harmonic pencil of four rays, all passing through
the point o; and similarly for the other groups. More generally, let oa, ob,
oc, od, or briefly o . abcd, be any pencil, with the point o for vertex; and let
the new ray od be cut, as in fig. 22, by the three sides of the triangle abc, in
the three points A1, B1, C1; let also

\[ OA_1 = a_1 = \frac{yb\beta + zc\gamma}{yb + zc} , \]

so that (by 25) we shall have the anharmonic quotients,

\[ \frac{(ba'ca_1)}{z} = \frac{y}{x} , \quad \frac{(ca'ba_1)}{y} = \frac{z}{x} ; \]

and let us seek to express the two other vectors of intersection, β, and γ, with
a view to determining the anharmonic ratios of
the groups on the two other sides. The given
equation (27),

\[ aa + b\beta + c\gamma = 0 , \]

shows us at once that these two vectors are,

\[ OB_1 = \beta_1 = \frac{(y - z) c\gamma + yaa}{(y - z) c + ya} ; \]

\[ OC_1 = \gamma_1 = \frac{(z - y) b\beta + zaa}{(z - y) b + za} ; \]

Fig. 22.
whence we derive (by 25) these two other anharmonics,

\[(CB'AB_1) = \frac{y - z}{y}; \quad (BC'AC_1) = \frac{z - y}{z};\]

so that we have the relations,

\[(CB'AB_1) + (CA'BA_1) = (BC'AC_1) + (BA'CA_1) = 1.\]

But in general, for any four collinear points \(A, B, C, D\), it is not difficult to prove that

\[\frac{AB}{BC} \cdot CD + \frac{AC}{CB} \cdot BD = DA;\]

whence by the definition (25) of the signification of the symbol \((ABCD)\), the following identity is derived,

\[(ABCD) + (ACBD) = 1.\]

Comparing this, then, with the recently found relations, we have, for fig. 22, the following anharmonic equations:

\[(CAB'B_1) = (CA'B'A_1) = \frac{z}{y}; \quad (BAC'C_1) = (BA'C'A_1) = \frac{y}{z};\]

and we see that (as was to be expected from known principles) the anharmonic of the group does not change, when we pass from one side of the triangle, considered as a transversal of the pencil, to another such side, or transversal. We may therefore speak (as usual) of such an anharmonic of a group, as being at the same time the Anharmonic of a Pencil; and, with attention to the order of the rays, and to the definition (25), may denote the two last anharmonics by the two following reciprocal expressions:

\[(O \cdot CABD) = \frac{z}{y}; \quad (O \cdot BACD) = \frac{y}{z};\]

with other resulting values, when the order of the rays is changed; it being understood that

\[(O \cdot CABD) = (C'A'B'D'),\]

if the rays \(OC, OA, OB, OD\) be cut, in the points \(C', A', B', D'\), by any one right line.

36. The expression (34),

\[\rho = \frac{xa + yb + ze}{xa + yb + ze},\]

may represent the vector of any point \(p\) in the given plane, by a suitable choice
of the coefficients $x, y, z$, or simply of their ratios. For since (by 22) the three complanar vectors $pa, pb, pc$ must be connected by some linear equation, of the form

$$a' \cdot pa + b' \cdot pb + c' \cdot pc = 0,$$

or

$$a' (a - \rho) + b' (\beta - \rho) + c' (\gamma - \rho) = 0,$$

which gives

$$\rho = \frac{a' a + b' \beta + c' \gamma}{a' + b' + c'},$$

we have only to write

$$\frac{a'}{a} = x, \quad \frac{b'}{b} = y, \quad \frac{c'}{c} = z,$$

and the proposed expression for $\rho$ will be obtained. Hence it is easy to infer, on principles already explained, that if we write (compare the annexed fig. 23),

$$P_1 = PA \cdot BC, \quad P_2 = PB \cdot CA, \quad P_3 = PC \cdot AB,$$

we shall have, with the same coefficients $xyz$, the following expressions for the vectors $op_1, op_2, op_3, or \rho_1, \rho_2, \rho_3$ of these three points of intersection, $p_1, p_2, p_3$:

$$\rho_1 = \frac{yb \beta + zc \gamma}{yb + zc}, \quad \rho_2 = \frac{zc \gamma + xa \alpha}{zc + xa}, \quad \rho_3 = \frac{xa \alpha + yb \beta}{xa + yb};$$

which give at once the following anharmonics of pencils, or of groups,

$$(a \cdot bocp) = (ba'cp) = \frac{y}{z};$$

$$(b \cdot coap) = (cb'aP) = \frac{z}{x};$$

$$(c \cdot aobp) = (ac'bP) = \frac{x}{y};$$

whereof we see that the product is unity. Any two of these three pencils suffice to determine the position of the point $p$, when the triangle $abc$, and the origin $o$ are given; and therefore it appears that the three coefficients $x, y, z$, or any scalars proportional to them, of which the quotients thus represent the anharmonics of those pencils, may be conveniently called the Anharmonic Co-ordinates of that point, $p$, with respect to the given triangle and origin: while the point $p$ itself may be denoted by the Symbol,

$$p = (x, y, z).$$

Hamilton's Elements of Quaternions.
With this notation, the thirteen points of fig. 21 come to be thus symbolized:

\[
\begin{align*}
\mathbf{a} &= (1, 0, 0), \quad \mathbf{b} = (0, 1, 0), \quad \mathbf{c} = (0, 0, 1), \quad \mathbf{o} = (1, 1, 1); \\
\mathbf{a}' &= (0, 1, 1), \quad \mathbf{b}' = (1, 0, 1), \quad \mathbf{c}' = (1, 1, 0); \\
\mathbf{a}'' &= (0, 1, -1), \quad \mathbf{b}'' = (-1, 0, 1), \quad \mathbf{c}'' = (1, -1, 0); \\
\mathbf{a}''' &= (2, 1, 1), \quad \mathbf{b}''' = (1, 2, 1), \quad \mathbf{c}''' = (1, 1, 2).
\end{align*}
\]

37. If \(e_1\) and \(e_2\) be any two points in the given plane,

\[
\mathbf{p}_1 = (x_1, y_1, z_1), \quad \mathbf{p}_2 = (x_2, y_2, z_2),
\]

and if \(t\) and \(u\) be any two scalar coefficients, then the following third point,

\[
\mathbf{p} = (tx_1 + uz_2, ty_1 + uy_2, tz_1 + uz_2),
\]
is collinear with the two former points, or (in other words) is situated on the right line \(e_1 e_2\). For, if we make

\[
\begin{align*}
x &= tx_1 + uz_2, \\
y &= ty_1 + uy_2, \\
z &= tz_1 + uz_2,
\end{align*}
\]

and

\[
\begin{align*}
\rho_1 &= \frac{x_1 a + \ldots}{x_1 a + \ldots}, \\
\rho_2 &= \frac{x_2 a + \ldots}{x_2 a + \ldots}, \\
\rho &= \frac{x a + \ldots}{x a + \ldots},
\end{align*}
\]

these vectors of the three points \(e_1 e_2 p\) are connected by the linear equation,

\[
t (x_1 a + \ldots) \rho_1 + u (x_2 a + \ldots) \rho_2 - (x a + \ldots) \rho = 0;
\]
in which (comp. 23), the sum of the coefficients is zero. Conversely, the point \(p\) cannot be collinear with \(e_1, e_2\), unless its co-ordinates admit of being thus expressed in terms of theirs. It follows that if a variable point \(p\) be obliged to move along a given right line \(e_1 e_2\), or if it have such a line (in the given plane) for its locus, its co-ordinates \(xyz\) must satisfy a homogeneous equation of the first degree, with constant coefficients; which, in the known notation of determinants, may be thus written,

\[
0 = \begin{vmatrix} x, & y, & z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix};
\]
or, more fully,

\[
0 = x (y_2 z_2 - z_1 y_2) + y (z_1 x_2 - x_1 z_2) + z (x_1 y_2 - y_1 x_2);
\]
or briefly

\[
0 = lx + my + nz,
\]
where \(l, m, n\) are three constant scalars, whereof the quotients determine the position of the right line \(\Lambda\), which is thus the locus of the point \(p\). It is natural to call the equation, which thus connects the co-ordinates of the point \(p\), the Anharmonic Equation of the Line \(\Lambda\); and we shall find it convenient also to speak of
Arts. 36–39. POINTS AND LINES IN A GIVEN PLANE.

the coefficients \( l, m, n \), in that equation, as being the Anharmonic Co-ordinates of that Line: which line may also be denoted by the Symbol,

\[ \Lambda = [l, m, n]. \]

38. For example, the three sides \( bc, ca, ab \) of the given triangle have thus for their equations,

\[ x = 0, \quad y = 0, \quad z = 0, \]

and for their symbols,

\[ [1, 0, 0], \quad [0, 1, 0], \quad [0, 0, 1]. \]

The three additional lines \( oa, ob, oc \), fig. 18, have, in like manner, for their equations and symbols,

\[ y - z = 0, \quad z - x = 0, \quad x - y = 0, \]

\[ [0, 1, -1], \quad [-1, 0, 1], \quad [1, -1, 0]. \]

The lines \( b'c''a'', c'a''b'', a'b'c'' \), of fig. 21, are

\[ y + z - x = 0, \quad z + x - y = 0, \quad x + y - z = 0, \]

or

\[ [-1, 1, 1], \quad [1, -1, 1], \quad [1, 1, -1]; \]

the lines \( A''b''c'' \), \( b''c''A'' \), \( c''A''b'' \), of the same figure, are in like manner represented by the equations and symbols,

\[ y + z - 3x = 0, \quad z + x - 3y = 0, \quad x + y - 3z = 0, \]

\[ [-3, 1, 1], \quad [1, -3, 1], \quad [1, 1, -3]; \]

and the line \( A''b''c'' \) is

\[ x + y + z = 0, \quad \text{or} \quad [1, 1, 1]. \]

Finally, we may remark that on the same plan, the equation and the symbol of what is often called the line at infinity, or of the locus of all the infinitely distant points in the given plane, are respectively,

\[ ax + by + cz = 0, \quad \text{and} \quad [a, b, c]; \]

because the linear function, \( ax + by + cz \), of the co-ordinates \( x, y, z \) of a point \( p \) in the plane, is the denominator of the expression (34, 36) for the vector \( p \) of that point: so that the point \( p \) is at an infinite distance from the origin \( o \), when, and only when, this linear function vanishes.

39. These anharmonic co-ordinates of a line, although above interpreted (37) with reference to the equation of that line, considered as connecting the co-ordinates of a variable point thereof, are capable of receiving an independent geometrical interpretation. For the three points \( l, m, n \), in which the line \( \Lambda \), or \( [l, m, n] \), or \( lx + my + nz = 0 \), intersects the three sides \( bc, ca, ab \) of the
given triangle \( \triangle abc \), or the three given lines \( x = 0 \), \( y = 0 \), \( z = 0 \) (38), may evidently (on the plan of 36) be thus denoted:

\[
\begin{align*}
L &= (0, n, -m); \\
M &= (-n, 0, l); \\
N &= (m, -l, 0).
\end{align*}
\]

But we had also (by 36),

\[
\begin{align*}
\alpha'' &= (0, 1, -1); \\
\beta'' &= (-1, 0, 1); \\
\gamma'' &= (1, -1, 0);
\end{align*}
\]

whence it is easy to infer, on the principles of recent articles, that

\[
\frac{n}{m} = \frac{(BA''CL)}{(CB''AM)}; \\
\frac{l}{n} = \frac{(CB''AM)}{(AC''BN)};
\]

with the resulting relation,

\[
\frac{(BA''CL)}{(CB''AM)} \cdot \frac{(CB''AM)}{(AC''BN)} = 1.
\]

40. Conversely, this last equation is easily proved, with the help of the known and general relation between segments (32), applied to any two transversals, \( \alpha''\beta''\gamma'' \) and \( L\!M\!N \), of any triangle \( \triangle abc \). In fact, we have thus the two equations,

\[
\frac{BA''}{A''C} \cdot \frac{CB''}{B''A} \cdot \frac{AC''}{C''B} = -1, \\
\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = -1;
\]

on dividing the former of which by the latter, the last formula of the last article results. We might therefore in this way have been led, without any consideration of a variable point \( P \), to introduce three auxiliary scalars, \( l, m, n \), defined as having their quotients \( \frac{n}{m}, \frac{l}{n}, \frac{m}{l} \) equal respectively, as in 39, to the three anharmonics of groups,

\[
\frac{(BA''CL)}{(CB''AM)} \cdot \frac{(CB''AM)}{(AC''BN)};
\]

and then it would have been evident that these three scalars, \( l, m, n \) (or any others proportional thereto), are sufficient to determine the position of the right line \( \Lambda \), or \( L\!M\!N \), considered as a transversal of the given triangle \( \triangle abc \): so that they might naturally have been called, on this account, as above, the anharmonic co-ordinates of that line. But although the anharmonic co-ordinates of a point and of a line may thus be independently defined, yet the geometrical utility of such definitions will be found to depend mainly on their combination: or on the formula \( lx + my + nz = 0 \) of 37, which may at pleasure be considered as expressing, either that the variable point \( (x, y, z) \) is situated somewhere upon the given right line \( [l, m, n] \); or else that the variable line \( [l, m, n] \) passes, in some direction, through the given point \( (x, y, z) \).
41. If \( \Lambda_1 \) and \( \Lambda_2 \) be any two right lines in the given plane,
\[ \Lambda_1 = [l_1, m_1, n_1], \quad \Lambda_2 = [l_2, m_2, n_2], \]
then any third right line \( \Lambda \) in the same plane, which passes through the intersection \( \Lambda_1 \cdot \Lambda_2 \), or (in other words) which concurs with them (at a finite or infinite distance), may be represented (comp. 37) by a symbol of the form,
\[ \Lambda = [t l_1 + u l_2, t m_1 + u m_2, t n_1 + u n_2], \]
where \( t \) and \( u \) are scalar coefficients. Or, what comes to the same thing, if \( l, m, n \) be the anharmonic co-ordinates of the line \( \Lambda \), then (comp. again 37), the equation
\[ 0 = l (m_1 n_2 - n_1 m_2) + \text{c}. = \begin{vmatrix} l, & m, & n \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}, \]
must be satisfied; because, if \( (X, Y, Z) \) be the supposed point common to the three lines, the three equations
\[ lX + mY + nZ = 0, \quad l_1X + m_1Y + n_1Z = 0, \quad l_2X + m_2Y + n_2Z = 0, \]
must co-exist. Conversely, this co-existence will be possible, and the three lines will have a common point (which may be infinitely distant), if the recent condition of concurrence be satisfied. For example, because \([a, b, c]\) has been seen (in 38) to be the symbol of the line at infinity (at least if we still retain the same significations of the scalars \( a, b, c \) as in Articles 27, &c.), it follows that
\[ \Lambda = [l, m, n], \quad \text{and} \quad \Lambda' = [l + ua, m + ub, n + uc], \]
are symbols of two parallel lines; because they concur at infinity. In general, all problems respecting intersections of right lines, collineations of points, &c., in the given plane, when treated by this anharmonic method, conduct to easy eliminations between linear equations (of the scalar kind), on which we need not here delay: the mechanism of such calculations being for the most part the same as in the known method of trilinear co-ordinates: although (as we have seen) the geometrical interpretations are altogether different.

SECTION 5.

On Plane Geometrical Nets, resumed.

42. If we now resume, for a moment, the consideration of those plane geometrical nets, which were mentioned in Art. 34; and agree to call those points and lines, in the given plane, rational points and rational lines, respectively, which have their anharmonic co-ordinates equal (or proportional) to whole
numbers; because then the anharmonic quotients, which were discussed in the last Section, are rational; but to say that a point or line is irrational, or that it is irrationally related to the given system of four initial points \( a, \ b, \ c, \) when its anharmonic co-ordinates are not thus all equal (or proportional) to integers; it is clear that whatever four points we may assume as initial, and however far the construction of the net may be carried, the net-points and net-lines which result will all be rational, in the sense just now defined. In fact, we begin with such; and the subsequent eliminations (41) can never afterwards conduct to any, that are of the contrary kind: the right line which connects two rational points being always a rational line; and the point of intersection of two rational lines being necessarily a rational point. The assertion made in Art. 34 is therefore fully justified.

43. Conversely, every rational point of the given plane, with respect to the four assumed initial points \( a, \ b, \ c, \) is a point of the net which those four points determine. To prove this, it is evidently sufficient to show that every rational point \( x' = (0, y, z), \) on any one side \( bc \) of the given triangle \( abc, \) can be so constructed. Making, as in fig. 22,

\[
B_1 = OA_1 \cdot CA, \quad \text{and} \quad C_1 = OA_1 \cdot AB,
\]

we have (by 35, 36) the expressions,

\[
B_1 = (y, 0, y - z), \quad C_1 = (z, z - y, 0);
\]

from which it is easy to infer (by 36, 37), that

\[
c'B_1 \cdot BC = (0, y, z - y), \quad b'C_1 \cdot BC = (0, y - z, z);
\]

and thus we can reduce the linear construction of the rational point \( (0, y, z), \) in which the two whole numbers \( y \) and \( z \) may be supposed to be prime to each other, to depend on that of the point \( (0, 1, 1), \) which has already been constructed as \( x'. \) It follows that although no irrational point \( q \) of the plane can be a net-point, yet every such point can be indefinitely approached to, by continuing the linear construction; so that it can be included within a quadrilateral interstice \( P_1P_2P_3P_4, \) or even within a triangular interstice \( P_1P_2P_3, \) which interstice of the net can be made as small as we may desire. Analogous remarks apply to irrational lines in the plane, which can never coincide with net-lines, but may always be indefinitely approximated to by such.

44. If \( P, \ P_1, \ P_2 \) be any three collinear points of the net, so that the formulae of 37 apply, and if \( x' \) be any fourth net-point \( (x', y', z') \) upon the same line, then writing

\[
x_1a + y_1b + z_1c = v_1, \quad x_2a + y_2b + z_2c = v_2,
\]
we shall have two expressions of the forms,

\[ \rho = \frac{tv_1p_1 + uv_2p_2}{tv_1 + uv_2}, \quad \rho' = \frac{t'v_1p_1 + u'v_2p_2}{t'v_1 + u'v_2}, \]

in which the coefficients \( tu't' \) are rational, because the co-ordinates \( xyz, \&c. \), are such, whatever the constants \( abc \) may be. We have therefore (by 25) the following rational expression for the anharmonic of this net-group:

\[ \left( \frac{p_1pp'p''}{p_1pp'} \right) = \frac{ut'}{tu'} = \frac{(yx_1 - xy_1)}{(xy_2 - yx_2)} \frac{(y'x_1 - y'x_2)}{(x'y_1 - y'x_1)}; \]

and similarly for every other group of the same kind. Hence every group of four collinear net-points, and consequently also every pencil of four concurrent net-lines, has a rational value for its anharmonic function; which value depends only on the processes of linear construction employed, in arriving at that group or pencil, and is quite independent of the configuration or arrangement of the four initial points: because the three initial constants, \( a, b, c, \) disappear from the expression which results. It was thus that, in fig. 21, the nine pencils, which had the nine derived points \( \lambda', \ldots, \rho'' \) for their vertices, were all harmonic pencils, in whatever manner the four points \( 0, \lambda, \beta, \gamma \) might be arranged. In general, it may be said that plane geometrical nets are all homographic figures;* and conversely, in any two such plane figures, corresponding points may be considered as either coinciding, or at least (by 43) as indefinitely approaching to coincidence, with similarly constructed points of two plane nets: that is, with points of which (in their respective systems) the anharmonic co-ordinates (36) are equal integers.

45. Without entering here on any general theory of transformation of anharmonic co-ordinates, we may already see that if we select any four net-points \( o_1, \alpha_1, \beta_1, \gamma_1 \), of which no three are collinear, every other point \( p \) of the same net is rationally related (42) to these; because (by 44) the three new anharmonics of pencils, \( (\alpha_1, \beta_1, \gamma_1, p) = \frac{y_1}{z_1}, \&c. \), are rational: and therefore (comp. 36) the new co-ordinates \( x_1, y_1, z_1 \) of the point \( p \), as well its old co-ordinates \( xyz \), are equal or proportional to whole numbers. It follows (by 43) that every point \( p \) of the net can be linearly constructed, if any four such points be given (no three being collinear, as above); or, in other words, that the whole net can be reconstructed,† if any one of its quadrilaterals (such as the

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* Compare the Géométrie Supérieure of M. Chasles, p. 362.
† This theorem (45) of the possible reconstruction of a plane net, from any one of its quadrilaterals, and the theorem (43) respecting the possibility of indefinitely approaching by net-lines to the points above called irrational (42), without ever reaching such points by any processes of linear construction.
interstice in fig. 24) be known. As an example, we may suppose that the four points \( \alpha \alpha' \beta' \gamma' \) in fig. 21 are given, and that it is required to recover from them the three points \( \alpha \beta \gamma \), which had previously been among the data of the construction. For this purpose, it is only necessary to determine first the three auxiliary points \( \alpha'' \beta'' \gamma'' \), as the intersections \( \alpha \alpha' \beta' \gamma' \cdot \beta'' \gamma'' \), &c.; and next the three other auxiliary points \( \alpha'' \beta'' \gamma'' \), as \( \alpha' \beta' \gamma' \cdot \beta'' \gamma'' \), &c. : after which the formulae, \( \alpha = \beta \beta'' \cdot \gamma \gamma'' \), &c., will enable us to return, as required, to the points \( \alpha, \beta, \gamma \), as intersections of known right lines.

SECTION 6.

On Anharmonic Equations, and Vector Expressions, for Curves in a given Plane.

46. When, in the expressions 34 or 36 for a variable vector \( \rho = \omega r \), the three variable scalars (or anharmonic co-ordinates) \( x, y, z \) are connected by any given algebraic equation, such as

\[ f_p(x, y, z) = 0, \]

supposed to be rational and integral, and homogeneous of the \( p^{th} \) degree, then the locus of the term \( r \) (Art. 1) of that vector is a plane curve of the \( p^{th} \) order; because (comp. 37) it is cut in \( p \) points (distinct or coincident, and real or imaginary), by any given right line, \( lx + my + nz = 0 \), in the given plane.

For example, if we write

\[ f = t^2 + u^2 + v^2 = 2uv - 2ux - 2xy; \]

where \( t, u, v \) are three new variable scalars, of which we shall suppose that the sum is zero, then, by eliminating these between the four equations,

\[ x = t^2, \quad y = u^2, \quad z = v^2, \quad t + u + v = 0, \]

we are conducted to the following equation of the second degree,

\[ 0 = f_p = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy; \]

so that here \( p = 2 \), and the locus of \( r \) is a conic section. In fact, it is the conic which touches the sides of the given triangle \( \alpha \beta \gamma \), at the points above called \( \alpha', \beta', \gamma' \); for if we seek its intersections with the side \( \beta \gamma \), by making \( x = 0 \) (38),

of the kind here considered, have been taken, as regards their substance (although investigated by a totally different analysis), from that highly original treatise of Möbius, which was referred to in a former note (p. 22). Compare the remarks in the following Chapter, upon nets in space.
we obtain a quadratic with equal roots, namely, \( (y - z)^2 = 0 \); which shows that there is contact with this side at the point \((0, 1, 1)\), or \(A'\) (36): and similarly for the two other sides.

47. If the point \(o\), in which the three right lines \(AA', BB', CC'\) concur, be (as in fig. 18, &c.) interior to the triangle \(ABC\), the sides of that triangle are then all cut internally, by the points \(A', B', C'\) of contact with the conic; so that in this case (by 28) the ratios of the constants \(a, b, c\) are all positive, and the denominator of the recent expression (46) for \(\rho\) cannot vanish, for any real values of the variable scalars \(t, u, v\); and consequently no such values can render infinite that vector \(\rho\). The conic is therefore generally in this case, as in fig. 25, an inscribed ellipse; which becomes however the inscribed circle, when

\[
\frac{1}{a^2} : \frac{1}{b^2} : \frac{1}{c^2} = s - a : s - b : s - c;
\]

\(a, b, c\) denoting here the lengths of the sides of the triangle, and \(s\) being their semi-sum.

48. But if the point of concourse \(o\) be exterior to the triangle of tangents \(ABC\), so that two of its sides are cut externally, then two of the three ratios of segments (28) are negative; and therefore one of the three constants \(a, b, c\) may be treated as < 0, but each of the two others as > 0. Thus if we suppose that

\[
b > 0, \quad c > 0, \quad a < 0, \quad a + b > 0, \quad a + c > 0,
\]

\(A'\) will be a point on the side \(bc\) itself; but the points \(B', C', o\) will be on the lines \(AC, AB, AA'\) prolonged, as in fig. 26; and then the conic \(A'B'C'\) will be an ellipse (including the case of a circle), or a parabola, or an hyperbola, according as the roots of the quadratic,

\[
(a + c) t^2 + 2ctu + (b + c) u^2 = 0,
\]

obtained by equating the denominator (46) of the vector \(\rho\) to zero, are either, 1st, imaginary; or IIInd, real and equal; or IIIrd, real and unequal: that is, according as we have

\[
bc + ca + ab > 0, \quad \text{or} \quad 0, \quad \text{or} \quad < 0;
\]

or (because the product \(abc\) is here negative), according as

\[
a^{-1} + b^{-1} + c^{-1} < 0, \quad \text{or} \quad 0, \quad \text{or} \quad > 0.
\]
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For example, if the conic be what is often called the escribed circle, the known ratios of segments give the proportion,

$$a^{-1} : b^{-1} : c^{-1} = -s : -s : s - b;$$

and

$$-s + s - c + s - b < 0.$$  

49. More generally, if $c_1$ be (as in fig. 26) a point upon the side $ab$, or on that side prolonged, such that $cc_1$ is parallel to the chord $b'c'$, then

$$c_1c' : ac' = cb' : ab' = -a : c, \quad \text{and} \quad ab : ac' = a + b : b;$$

writing then the condition (48) of ellipticity (or circularity) under the form,

$$-a < \frac{a + b}{c},$$

we see that the conic is an ellipse, parabola, or hyperbola, according as $c_1c' < or = or > ab$; the arrangement being still, in other respects, that which is represented in fig. 26. Or, to express the same thing more symmetrically, if we complete the parallelogram $cabc$, then according as the point $d$ falls, 1st, beyond the chord $b'c'$, with respect to the point $a$; or 2nd, on that chord; or 3rd, within the triangle $ab'a$, the general arrangement of the same figure being retained, the curve is elliptic, or parabolic, or hyperbolic. In that other arrangement or configuration, which answers to the system of inequalities, $b > 0, c > 0, a + b + c < 0$, the point $a'$ is still upon the side $bc$ itself, but $o$ is on the line $a'a$ prolonged through $A$; and then the inequality,

$$a (b + c) + bc < -(b^2 + bc + c^2) < 0,$$

shows that the conic is necessarily an hyperbola; whereof it is easily seen that one branch is touched by the side $bc$ at $a'$, while the other branch is touched in $b'$ and $c'$, by the sides $ca$ and $ba$ prolonged through $a$. The curve is also hyperbolic, if either $a + b$ or $a + c$ be negative, while $b$ and $c$ are positive as before.

50. When the quadratic (48) has its roots real and unequal, so that the conic is an hyperbola, then the directions of the asymptotes may be found, by substituting those roots, or the values of $t, u, v$ which correspond to them (or any scalars proportional thereto), in the numerator of the expression (46) for $\rho$; and similarly we can find the direction of the axis of the parabola, for the case when the roots are real but equal: for we shall thus obtain the directions, or direction, in which a right line $op$ must be drawn from $o$, so as to meet the conic at infinity. And the same conditions as before, for distinguishing the species of the conic, may be otherwise obtained by combining the anharmonic equation, $f = 0$ (46), of that conic, with the corresponding equation $ax + by + cz = 0$ (38) of the line at infinity; so as to inquire (on known principles of
modern geometry) whether that line meets that curve in two imaginary points, or touches it, or cuts it, in points which (although infinitely distant) are here to be considered as real.

51. In general, if \( f(x, y, z) = 0 \) be the anharmonic equation (46) of any plane curve, considered as the locus of a variable point \( r \); and if the differential* of this equation be thus denoted,

\[
0 = df(x, y, z) = Xdx + Ydy + Zdz;
\]
then because, by the supposed homogeneity (46) of the function \( f \), we have the relation

\[
Xx + Yy + Zz = 0,
\]
we shall have also this other but analogous relation,

\[
Xdx' + Ydy' + Zdz' = 0,
\]
if

\[
x' : x = y' : y = z' : z = dx : dy : dz;
\]
that is (by the principles of Art. 37), if \( r' = (x', y', z') \) be any point upon the tangent to the curve, drawn at the point \( r = (x, y, z) \), and regarded as the limit of a secant. The symbol (37) of this tangent at \( r \) may therefore be thus written,

\[
[X, Y, Z], \text{ or } [d_1, f, d_2, f, d_3, f];
\]
where \( d_1, d_2, d_3 \) are known characteristics of partial derivation.

52. For example, when \( f \) has the form assigned in 46, as answering to the conic lately considered, we have \( d_1f = 2(x - y - z), \&c. \); whence the tangent at any point \( (x, y, z) \) of this curve may be denoted by the symbol,

\[
[x - y - z, \quad y - z - x, \quad z - x - y];
\]
in which, as usual, the co-ordinates of the line may be replaced by any others proportional to them. Thus at the point \( x', \text{ or } (36) \text{ at } (0, 1, 1), \) which is evidently (by the form of \( f \)) a point upon the curve, the tangent is the line \([-2, 0, 0], \text{ or } [1, 0, 0]\); that is (by 38), the side \( bc \) of the given triangle, as was otherwise found before (46). And in general it is easy to see that the recent symbol denotes the right line, which is (in a well known sense) the polar of the point \((x, y, z)\), with respect to the same given conic; or that the line \([X', Y', Z']\) is the polar of the point \((x', y', z')\): because the equation

\[
Xdx' + Ydy' + Zdz' = 0,
\]

* In the theory of quaternions, as distinguished from (although including) that of vectors, it will be found necessary to introduce a new definition of differentials, on account of the non-commutative property of quaternion-multiplication: but, for the present, the usual significations of the signs \( d \) and \( \partial \) are sufficient.
which for a conic may be written as \( X'x + Y'y + Z'z = 0 \), expresses (by 51) the condition requisite, in order that a point \((x, y, z)\) of the curve* should belong to a tangent which passes through the point \((x', y', z')\). Conversely, the point \((x, y, z)\) is (in the same well-known sense) the pole of the line \([X, Y, Z]\); so that the centre of the conic, which is (by known principles) the pole of the line at infinity (38), is the point which satisfies the conditions \(a^{-1}X = b^{-1}Y = c^{-1}Z\); it is therefore, for the present conic, the point \(\kappa = (b+c, c+a, a+b)\), of which the vector \(\mathbf{o}\kappa\) is easily reduced, by the help of the linear equation, \(aa + bj + c\gamma = 0\) (27), to the form,

\[
\kappa = \frac{-a^2a + b^2\beta + c^2\gamma}{2(bc + ca + ab)};
\]

with the verification that the denominator vanishes, by 48, when the conic is a parabola. In the more general case, when this denominator is different from zero, it can be shown that every chord of the curve, which is drawn through the extremity \(\kappa\) of the vector \(\kappa\), is bisected at that point \(\kappa\); which point would therefore in this way be seen again to be the centre.

53. Instead of the inscribed conic (46), which has been the subject of recent articles, we may, as another example, consider that exscribed (or circumscribed) conic, which passes through the three corners \(A, B, C\) of the given triangle, and touches there the lines \(AA'', BB'', CC''\) of fig. 21. The anharmonic equation of this new conic is easily seen to be,

\[
yz + zx + xy = 0;
\]

the vector of a variable point \(p\) of the curve may therefore be expressed as follows,

\[
\rho = \frac{t^{-1}aa + u^{-1}b\beta + v^{-1}c\gamma}{t^{-1}a + u^{-1}b + v^{-1}c},
\]

with the condition \(t + u + v = 0\), as before. The vector of its centre \(\kappa'\) is found to be

\[
\kappa' = \frac{2(a^2a + b^2\beta + c^2\gamma)}{a^2 + b^2 + c^2 - 2bc - 2ca - 2ab};
\]

and it is an ellipse, a parabola, or an hyperbola, according as the denominator of this last expression is negative, or null, or positive. And because these two recent vectors, \(\kappa, \kappa'\), bear a scalar ratio to each other, it follows (by 19) that the three points \(o, \kappa, \kappa'\) are collinear; or in other words, that the line of

* If the curve \(f = 0\) were of a degree higher than the second, then the two equations above written would represent what are called the first polar, and the last or the line-polar, of the point \((x', y', z')\), with respect to the given curve.
centres \( k, k' \), of the two conics here considered, passes through the point of concourse \( o \) of the three lines \( AA', BB', CC' \). More generally, if \( l \) be the pole of any given right line \( \Delta = [l, m, n] \) (37), with respect to the inscribed conic (46), and if \( l' \) be the pole of the same line \( \Delta \) with respect to the exscribed conic of the present article, it can be shown that the vectors \( ol, ol' \), or \( \lambda, \lambda' \), of these two poles are of the forms,

\[
\lambda = k (laa + mb\beta + ncy), \quad \lambda' = k' (laa + mb\beta + ncy),
\]

where \( k \) and \( k' \) are scalars; the three points \( o, l, l' \) are therefore ranged on one right line.

54. As an example of a vector-expression for a curve of an order higher than the second, the following may be taken:

\[
op = \rho = \frac{t^3a + u^3b\beta + v^3c}{t^3a + u^3b\beta + v^3c}\]

with \( t + u + v = 0 \), as before. Making \( x = t^3, y = u^3, z = v^3 \), we find here by elimination of \( t, u, v \) the anharmonic equation,

\[
(x + y + z)^3 - 27xyz = 0;
\]

the locus of the point \( p \) is therefore, in this example, a curve of the third order, or briefly a cubic curve. The mechanism (41) of calculations with anharmonic co-ordinates is so much the same as that of the known trilinear method, that it may suffice to remark briefly here that the sides of the given triangle \( \triangle abc \) are the three (real) tangents of inflexion; the points of inflexion being those which are marked as \( \Lambda'', B'', C'' \) in fig. 21; and the origin of vectors \( o \) being a conjugate point.\(^*\) If \( a = b = c \), in which case (by 29) this origin \( o \) becomes (as in fig. 19) the mean point of the triangle, the chord of inflexion \( \Lambda''B''C'' \) is then the line at infinity, and the curve takes the form represented in fig. 27; having three infinite branches, inscribed within the angles vertically opposite to those of the given triangle \( \triangle abc \), of which the sides are the three asymptotes.

55. It would be improper to enter here into any details of discussion of such cubic curves, for which the reader will naturally turn to other works.\(^†\)

\(^*\) Answering to the values \( t = 1, u = \theta, v = \theta^2 \), where \( \theta \) is one of the imaginary cube-roots of unity; which values of \( t, u, v \) give \( x = y = z \), and \( \rho = 0 \).

But it may be remarked, in passing, that because the general cubic may be represented, on the present plan, by combining the general expression of Art. 34 or 36 for the vector $\mathbf{p}$, with the scalar equation

$$s^3 = 27 kxyz, \quad \text{where} \quad s = x + y + z;$$

$k$ denoting an arbitrary constant, which becomes equal to unity, when the origin is (as in 54) a conjugate point; it follows that if $\mathbf{p} = (x, y, z)$ and $\mathbf{p}' = (x', y', z')$ be any two points of the curve, and if we make $s' = x' + y' + z'$, we shall have the relation,

$$xyzs^3 = x'y'z's'^3, \quad \text{or} \quad \frac{xs'}{sx} \cdot \frac{ys'}{sy} \cdot \frac{zs'}{sz} = 1;$$

in which it is not difficult to prove that

$$\frac{xs'}{sx} = (A'' \cdot \mathbf{p} \cdot \mathbf{p}' \cdot B''), \quad \frac{ys'}{sy} = (B'' \cdot \mathbf{p} \cdot \mathbf{p}' \cdot C''), \quad \frac{zs'}{sz} = (C'' \cdot \mathbf{p} \cdot \mathbf{p}' \cdot A'');$$

the notation (35) of anharmonics of pencils being retained. We obtain therefore the following Theorem:—"If the sides of any given plane* triangle $\triangle ABC$ be cut (as in fig. 21) by any given rectilinear transversal, $\Delta'' \Delta' \Delta'''$, and if any two points $\mathbf{p}$ and $\mathbf{p}'$ in its plane be such as to satisfy the anharmonic relation

$$(A'' \cdot \mathbf{p} \cdot \mathbf{p}' \cdot B'') \cdot (B'' \cdot \mathbf{p} \cdot \mathbf{p}' \cdot C'') \cdot (C'' \cdot \mathbf{p} \cdot \mathbf{p}' \cdot A'') = 1,$$

then these two points $\mathbf{p}, \mathbf{p}'$ are on one common cubic curve, which has the three collinear points, $A'', B'', C''$ for its three real points of inflexion, and has the sides $BC, CA, AB$ of the triangle for its three tangents at those points"; a result which seems to offer a new geometrical generation for curves of the third order.

56. Whatever the order of a plane curve may be, or whatever may be the degree $p$ of the function $f$ in 46, we saw in 51 that the tangent to the curve at any point $\mathbf{p} = (x, y, z)$ is the right line

$$\Lambda = [l, m, n], \quad \text{if} \quad l = D_x f, \quad m = D_y f, \quad n = D_z f;$$

expressions which, by the supposed homogeneity of $f$, give the relation $lx + my + nz = 0$, and therefore enable us to establish the system of the two following differential equations,

$$l dx + m dy + n dz = 0, \quad x dl + y dm + z dn = 0.$$  

If then, by elimination of the ratios of $x, y, z$, we arrive at a new homogeneous equation of the form,

$$0 = F (D_x f', D_y f', D_z f'),$$  

* This Theorem may be extended, with scarcely any modification, from plane to spherical curves, of the third order.
as one that is true for all values of $x, y, z$ which render the function $f = 0$ (although it may require to be cleared of factors, introduced by this elimination), we shall have the equation

$$ f(l, m, n) = 0, $$

as a condition that must be satisfied by the tangent $A$ to the curve, in all the positions which can be assumed by that right line. And, by comparing the two differential equations.

$$ dF(l, m, n) = 0, \quad xdl + ydm + zdn = 0, $$

we see that we may write the proportion,

$$ x : y : z = DlF : DmF : DnF, \quad \text{and the symbol } P = (DlF, DmF, DnF), \quad \text{if } (x, y, z) $$

be, as above, the point of contact $P$ of the variable line $[l, m, n]$ in any one of its positions, with the curve which is its envelope. Hence we can pass (or return) from the tangential equation $f = 0$, of a curve considered as the envelope of a right line $A$, to the local equation $f = 0$, of the same curve considered (as in §46) as the locus of a point $P$: since, if we obtain, by elimination of the ratios of $l, m, n$, an equation of the form

$$ 0 = f(DlF, DmF, DnF), $$

(cleared, if it be necessary, of foreign factors) as a consequence of the homogeneous equation $F = 0$, we have only to substitute for these partial derivatives, $DlF$, &c., the anharmonic co-ordinates $x, y, z$, to which they are proportional. And when the functions $f$ and $P$ are not only homogeneous (as we shall always suppose them to be), but also rational and integral (which it is sometimes convenient not to assume them as being), then, while the degree of the function $f$, or of the local equation, marks (as before) the order of the curve, the degree of the other homogeneous function $P$, or of the tangential equation $f = 0$, is easily seen to denote, in this anharmonic method (as, from the analogy of other and older methods, it might have been expected to do), the class of the curve to which that equation belongs: or the number of tangents (distinct or coincident, and real and imaginary), which can be drawn to that curve, from an arbitrary point in its plane.

57. As an example (comp. 52), if we eliminate $x, y, z$ between the equations,

$$ l = x - y - z, \quad m = y - z - x, \quad n = z - x - y, \quad lx + my + nz = 0, $$

where $l, m, n$ are the co-ordinates of the tangent to the inscribed conic of Art. 46, we are conducted to the following tangential equation of that conic, or curve of the second class,

$$ f(l, m, n) = mn + nl + lm = 0; $$
with the verification that the sides \([1, 0, 0], \text{&c.} (38)\), of the triangle \(abc\) are among the lines which satisfy this equation. Conversely, if this tangential equation were given we might (by 56) derive from it expressions for the co-ordinates of contact \(x, y, z\), as follows:

\[x = D_1 F = m + n, \quad y = n + l, \quad z = l + m;\]

with the verification that the side \([1, 0, 0]\) touches the conic, considered now as an envelope, in the point \((0, 1, 1)\), or \(\alpha'\), as before: and then, by eliminating \(l, m, n\), we should be brought back to the local equation, \(f = 0\), of 46. In like manner, from the local equation \(f = yz + zx + xy = 0\) of the escribed conic (53), we can derive by differentiation the tangential co-ordinates,*

\[l = D_2 f = y + z, \quad m = z + x, \quad n = x + y,\]

and so obtain by elimination the tangential equation, namely,

\[F (l, m, n) = l^2 + m^2 + n^2 - 2mn - 2nl - 2lm = 0;\]

from which we could in turn deduce the local equation. And (comp. 40), the very simple formula

\[lx + my + nz = 0,\]

which we have so often had occasion to employ, as connecting two sets of anharmonic co-ordinates, may not only be considered (as in 37) as the local equation of a given right line \(\Lambda\), along which a point \(p\) moves, but also as the tangential equation of a given point, round which a right line turns: according as we suppose the set \(l, m, n\), or the set \(x, y, z\), to be given. Thus, while the right line \(a'b'c'\), or \([1, 1, 1]\), of fig. 21, was represented in 38 by the equation \(x + y + z = 0\), the point \(o\) of the same figure, or the point \((1, 1, 1)\), may be represented by the analogous equation,

\[l + m + n = 0;\]

because the co-ordinates \(l, m, n\) of every line, which passes through this point \(o\), must satisfy this equation of the first degree, as may be seen exemplified, in the same Art. 38, by the lines \(oa, ob, oc\).

58. To give an instance or two of the use of forms, which, although homogeneous, are yet not rational and integral (56) we may write the local equation of the inscribed conic (46) as follows:

\[x^2 + y^2 + z^2 = 0;\]

* This name of "tangential co-ordinates" appears to have been first introduced by Dr. Booth in a Tract published in 1840, to which the author of the present Elements cannot now more particularly refer: but the system of Dr. Booth was entirely different from his own. See the reference in Salmon's Higher Plane Curves, note to page 16.
Arts. 57-59. | LOCAL AND TANGENTIAL EQUATIONS.

and then (suppressing the common numerical factor 1/2), the partial derivatives are

\[ l = x^{-1}, \quad m = y^{-1}, \quad n = z^{-1}; \]

so that a form of the tangential equation for this conic is,

\[ l^1 + m^1 + n^1 = 0; \]

which evidently, when cleared of fractions, agrees with the first form of the last Article: with the verification (48), that \( a^{-1} + b^{-1} + c^{-1} = 0 \) when the curve is a parabola; that is, when it is touched (50) by the line at infinity (38). For the exscribed conic (53), we may write the local equation thus,

\[ x^{-1} + y^{-1} + z^{-1} = 0; \]

whence it is allowed to write also,

\[ l = x^{-2}, \quad m = y^{-2}, \quad n = z^{-2}, \]

and

\[ l^2 + m^2 + n^2 = 0; \]

a form of the tangential equation which, when cleared of radicals, agrees again with 57. And it is evident that we could return, with equal ease, from these tangential to these local equations.

59. For the cubic curve with a conjugate point (54), the local equation may be thus written,*

\[ x^1 + y^1 + z^1 = 0; \]

we may therefore assume for its tangential co-ordinates the expressions,

\[ l = x^{-1}, \quad m = y^{-1}, \quad n = z^{-1}; \]

and a form of its tangential equation is thus found to be,

\[ l^1 + m^1 + n^1 = 0. \]

Conversely, if this tangential form were given, we might return to the local equation, by making

\[ x = l^2, \quad y = m^2, \quad z = n^2, \]

which would give \( x^1 + y^1 + z^1 = 0 \), as before. The tangential equation just now found becomes, when it is cleared of radicals,

\[ 0 = l^2 + m^2 + n^2 - 2m^{-1}n^{-1} - 2n^{-1}l^{-1} - 2l^{-1}m^{-1}; \]

or, when it is also cleared of fractions,

\[ 0 = r = m^3n^3 + n^3l^3 + l^3m^3 - 2m^2n^2 - 2l^2m^2 - 2mn^2l; \]

* Compare Salmon's Higher Plane Curves, page 172 [Art. 216, new ed.].

Hamilton's Elements of Quaternions.
of which the biquadratic form shows (by 56) that this cubic is a curve of the fourth class, as indeed it is known to be. The inflexional character (54) of the points \( \lambda'' \), \( \beta'' \), \( \gamma'' \) upon this curve is here recognised by the circumstance, that when we make \( m - n = 0 \), in order to find the four tangents from \( \lambda'' = (0, 1, -1) \) (36), the resulting biquadratic, \( 0 = m^4 - 4lm^2 \), has three equal roots; so that the line \( [1, 0, 0] \), or the side \( bc \), counts as three, and is therefore a tangent of inflexion: the fourth tangent from \( \lambda'' \) being the line \( [1, 4, 4] \), which touches the cubic at the point \( (-8, 1, 1) \).

60. In general, the two equations (56),

\[
nd_x^2 f - nd_z f = 0, \quad nd_y f - mD_x f = 0,
\]

may be considered as expressing that the homogeneous equation,

\[
f (nx, ny, -lx - my) = 0,
\]

which is obtained by eliminating \( z \) with the help of the relation \( lx + my + nz = 0 \), from \( f (x, y, z) = 0 \), and which we may denote by \( \phi (x, y) = 0 \), has two equal roots \( x : y \), if \( l, m, n \) be still the co-ordinates of a tangent to the curve \( f \); an equality which obviously corresponds to the coincidence of two intersections of that line with that curve. Conversely, if we seek by the usual methods the condition of equality of two roots \( x : y \) of the homogeneous equation of the \( p^{th} \) degree,

\[
0 = \phi (x, y) = f (nx, ny, -lx - my),
\]

by eliminating the ratio \( x : y \) between the two derived homogeneous equations, \( 0 = D_x \phi, 0 = D_y \phi \), we shall in general be conducted to a result of the dimension \( 2p(p - 1) \) in \( l, m, n \), and of the form,

\[
0 = n^{p(p-1)} \Phi (l, m, n);
\]

and so, by the rejection of the foreign factor \( n^{p(p-1)} \), introduced by this elimination,* we shall obtain the tangential equation \( \Phi = 0 \), which will be in general of the degree \( p(p - 1) \); such being generally the known class (56) of the curve of which the order (46) is denoted by \( p \): with (of course) a similar mode of passing, reciprocally, from a tangential to a local equation.

61. As an example, when the function \( f \) has the cubic form assigned in 54, we are thus led to investigate the condition for the existence of two equal roots in the cubic equation,

\[
0 = \phi (x, y) = (n - l) x + (m - l) y + 27n^2xy (lx + my),
\]

---

* Compare the method employed in Salmon's Higher Plane Curves, page 98 [Art. 91, new ed.] to find the equation of the reciprocal of a given curve, with respect to the imaginary conic, \( x^2 + y^2 + z^2 = 0 \). In general, if the function \( \nu \) be deduced from \( f \) as above, then \( \nu (xyz) = 0 \), and \( f (xyz) = 0 \) are equations of two reciprocal curves.
by eliminating $x : y$ between two derived and quadratic equations; and the
result presents itself, in the first instance, as of the twelfth dimension in the
tangential co-ordinates $l, m, n$; but it is found to be divisible by $n^6$, and when
this division is effected, it is reduced to the sixth degree, thus appearing to
imply that the curve is of the sixth class, as in fact the general cubic is well
known to be. A further reduction is however possible in the present case, on
account of the conjugate point $o$ (54), which introduces (comp. 57) the
quadratic factor,

$$(l + m + n)^2 = 0;$$

and when this factor also is set aside, the tangential equation is found to be
reduced to the biquadratic form* already assigned in 59; the algebraic division,
last performed, corresponding to the known geometric depression of a cubic curve
with a double point, from the sixth to the fourth class. But it is time to close
this Section on Plane Curves; and to proceed, as in the next Chapter we
propose to do, to the consideration and comparison of vectors of points in space.

* If we multiply that form $r = 0$ (59) by $z^2$, and then change $nz$ to $-lx - my$, we obtain a
biquadratic equation in $l : m$, namely,

$$0 = \psi(l, m) = (l - m)^2 (lx + my)^2 + 2lm(l + m)(lx + my)z + m^2z^2;$$

and if we then eliminate $l : m$ between the two derived cubics, $0 = n\psi, 0 = m\psi$, we are conducted to
the following equation of the twelfth degree, $0 = x^3y^2z^2f(x, y, z)$, where $f$ has the same cubic form as
in 54. We are therefore thus brought back (comp. 59) from the tangential to the local equation of
the cubic curve (54); complicated, however, as we see, with the factor $x^3y^2z^2$, which corresponds to
the system of the three real tangents of inflexion to that curve, each tangent being taken three times.
The reason why we have not here been obliged to reject also the foreign factor, $z^2$, as by the general
theory (60) we might have expected to be, is that we multiplied the biquadratic function $r$ only by $z^2$,
and not by $z^4$. 
CHAPTER III.
APPLICATIONS OF VECTORS TO SPACE.

SECTION 1.

On Linear Equations between Vectors not Complanar.

62. When three given and actual vectors $\mathbf{oa}, \mathbf{ob}, \mathbf{oc}$, or $\mathbf{a}, \mathbf{b}, \mathbf{c}$, are not contained in any common plane, and when the three scalars $a, b, c$ do not all vanish, then (by 21, 22) the expression $aa + b\beta + c\gamma$ cannot become equal to zero; it must therefore represent some actual vector (1), which we may, for the sake of symmetry, denote by the symbol $-d\delta$: where the new (actual) vector $\delta$, or $\mathbf{od}$, is not contained in any one of the three given and distinct planes, $\mathbf{boc}$, $\mathbf{coa}$, $\mathbf{aob}$, unless some one, at least, of the three given coefficients $a, b, c$, vanishes; and where the new scalar, $d$, is either greater or less than zero. We shall thus have a linear equation between four vectors,

$$aa + b\beta + c\gamma + d\delta = 0;$$

which will give

$$\delta = \frac{-aa}{d} + \frac{-b\beta}{d} + \frac{-c\gamma}{d}, \text{ or } \mathbf{od} = \mathbf{oa}' + \mathbf{ob}' + \mathbf{oc}';$$

where $\mathbf{oa}', \mathbf{ob}', \mathbf{oc}'$, or $\frac{-aa}{d}, \frac{-b\beta}{d}, \frac{-c\gamma}{d}$, are the vectors of the three points $A', B', C'$, into which the point $D$ is projected, on the three given lines $\mathbf{oa}$, $\mathbf{ob}$, $\mathbf{oc}$, by planes drawn parallel to the three given planes, $\mathbf{boc}$, &c.; so that they are the three co-initial edges of a parallelepiped, whereof the sum, $\mathbf{od}$ or $\delta$, is the internal and co-initial diagonal (comp 6).

Or we may project $\mathbf{d}$ on the three planes, by lines $\mathbf{oa''}$, $\mathbf{ob''}$, $\mathbf{oc''}$ parallel to the three given lines, and then shall have

$$\mathbf{oa''} = \mathbf{ob'} + \mathbf{oc'} = \frac{b\beta + c\gamma}{-d}, \text{ &c.}, \text{ and } \delta = \mathbf{od} = \mathbf{oa'} + \mathbf{oa''} = \mathbf{ob'} + \mathbf{ob''} = \mathbf{oc'} + \mathbf{oc''}.$$
And it is evident that this construction will apply to any fifth point \( d \) of space, if the four points \( oabc \) be still supposed to be given, and not complanar: but that some at least of the three ratios of the four scalars \( a, b, c, d \) (which last letter is not here used as a mark of differentiation) will vary with the position of the point \( d \), or with the value of its vector \( \delta \). For example, we shall have \( a = 0 \), if \( d \) be situated in the plane \( boc \); and similarly for the two other given planes through \( o \).

63. We may inquire (comp. 23), what relation between these scalar coefficients must exist, in order that the point \( d \) may be situated in the fourth given plane \( abc \); or what is the condition of complanarity of the four points, \( a, b, c, d \). Since the three vectors \( da, db, dc \) are now supposed to be complanar, they must (by 22) be connected by a linear equation, of the form

\[
a (a - \delta) + b (\beta - \delta) + c (\gamma - \delta) = 0;
\]

comparing which with the recent and more general form (62), we see that the required condition is,

\[
a + b + c + d = 0.
\]

This equation may be written (comp. again 23) as

\[
\frac{-a}{d} + \frac{-b}{d} + \frac{-c}{d} = 1, \quad \text{or} \quad \frac{oa'}{oa} + \frac{ob'}{ob} + \frac{oc'}{oc} = 1;
\]

and, under this last form, it expresses a known geometrical property of a plane \( abcd \), referred to three co-ordinate axes \( oa, ob, oc \), which are drawn from any common origin \( o \), and terminate upon the plane. We have also, in this case of complanarity (comp. 28), the following proportion of coefficients and areas:

\[
a : b : c : d = dbc : dca : dab : abc;
\]

or, more symmetrically, with attention to signs of areas,

\[
a : b : c : d = bcd : - cda : dab : - abc;
\]

where fig. 18 may serve for illustration, if we conceive \( o \) in that figure to be replaced by \( d \).

64. When we have thus at once the two equations,

\[
aa + b\beta + c\gamma + d\delta = 0, \quad \text{and} \quad a + b + c + d = 0,
\]

so that the four co-initial vectors, \( a, \beta, \gamma, \delta \) terminate (as above) on one common plane, and may therefore be said (comp. 24) to be termino-complanar, it is evident that the two right lines, \( da \) and \( bc \), which connect two pairs of the four complanar points, must intersect each other in some point \( a' \) of the plane, at a finite or infinite distance. And there is no difficulty in perceiving, on
the plan of 31, that the vectors of the three points, \( a', b', c' \) of intersection, which thus result, are the following:

\[
\begin{align*}
\text{for } a' &= BC \cdot DA, \quad a' = \frac{b\beta + c\gamma}{b + c} = \frac{aa + d\delta}{a + d}; \\
\text{for } b' &= CA \cdot DB, \quad b' = \frac{c\gamma + a\alpha}{c + a} = \frac{bb + d\delta}{b + d}; \\
\text{for } c' &= AB \cdot DC, \quad c' = \frac{a\alpha + b\beta}{a + b} = \frac{cc + d\delta}{c + d};
\end{align*}
\]

expressions which are independent of the position of the arbitrary origin \( o \), and which accordingly coincide with the corresponding expressions in 27, when we place that origin in the point \( d \), or make \( \delta = 0 \). Indeed, these last results hold good (comp. 31), even when the four vectors, \( a, \beta, \gamma, \delta \), or the five points \( o, A, B, C, D \), are all complanar. For, although there then exist two linear equations between those four vectors, which may in general be written thus,

\[
d' a + b' \beta + c' \gamma + d' \delta = 0, \quad a'' a + b'' \beta + c'' \gamma + d'' \delta = 0,
\]

without the relations, \( a' + \&c. = 0, a'' + \&c. = 0 \), between the coefficients, yet if we form from these another linear equation, of the form,

\[
(a'' + t a') a + (b'' + t b') \beta + (c'' + t c') \gamma + (d'' + t d') \delta = 0,
\]

and determine \( t \) by the condition,

\[
t = -\frac{a'' + b'' + c'' + d''}{a' + b' + c' + d'},
\]

we shall only have to make \( a = a'' + td' \), \&c., and the two equations written at the commencement of the present article will then both be satisfied; and will conduct to the expressions assigned above, for the three vectors of intersection: which vectors may thus be found, without its being necessary to employ those processes of scalar elimination, which were treated of in the foregoing Chapter.

As an Example, let the two given equations be (comp. 27, 33),

\[
aa + b\beta + c\gamma = 0, \quad (2a + b + c) a'' - aa = 0;
\]

and let it be required to determine the vectors of the intersections of the three pairs of lines \( BC, AA'' \); \( CA, BA'' \); and \( AB, CA'' \). Forming the combination,

\[
(2a + b + c) a'' - aa + t(aa + b\beta + c\gamma) = 0,
\]

and determining \( t \) by the condition,

\[
(2a + b + c) - a + t(a + b + c) = 0,
\]
which gives \( t = -1 \), we have for the three sought vectors the expressions,

\[
\frac{b\beta + cy}{b + c}, \quad \frac{cy + 2ac}{c + 2a}, \quad \frac{2ac + b\beta}{2a + b};
\]

whereof the first = \( a' \), by 27. Accordingly, in fig. 21, the line \( \triangle A'' \) intersects \( bc \) in the point \( A' \); and although the two other points of intersection here considered, which belong to what has been called (in 34) a Third Construction, are not marked in that figure, yet their anharmonic symbols (36), namely, \((2, 0, 1)\) and \((2, 1, 0)\), might have been otherwise found by combining the equations \( y = 0 \) and \( x = 2z \) for the two lines \( ca, ba'' \); and by combining \( z = 0, x = 2y \) for the remaining pair of lines.

65. In the more general case, when the four given points \( A, B, C, D \), are not in any common plane, let \( E \) be any fifth given point of space, not situated on any one of the four faces of the given pyramid \( ABCD \), nor on any such face prolonged; and let its vector \( OE = \varepsilon \). Then the four co-initial vectors, \( EA, EB, EC, ED \), whereof (by supposition) no three are complanar, and which do not terminate upon one plane, must be (by 62) connected by some equation of the form

\[
aa + b\beta + cy + d\delta + e\varepsilon = 0;
\]

where the four scalars, \( a, b, c, d \), and their sum, which we shall denote by \(-\varepsilon\), are all different from zero. Hence, because \( EA = a - \varepsilon \), \&c., we may establish the following linear equation between five co-initial vectors, \( a, \beta, \gamma, \delta, \varepsilon \), whereof no four are termino-complanar (64),

\[
aa + b\beta + cy + d\delta + e\varepsilon = 0;
\]

with the relation, \( a + b + c + d + e = 0 \), between the five scalars \( a, b, c, d, e \), whereof no one now separately vanishes. Hence also,

\[
\varepsilon = (aa + b\beta + cy + d\delta) : (a + b + c + d), \ \&c.
\]

66. Under these conditions, if we write

\[
D_1 = DE \cdot ABC, \quad \text{and} \quad OD_1 = \delta_1,
\]

that is, if we denote by \( \delta_1 \) the vector of the point \( D_1 \), in which the right line \( DR \) intersects the plane \( ABC \), we shall have

\[
\delta_1 = \frac{aa + b\beta + cy}{a + b + c} = \frac{d\delta + e\varepsilon}{d + e}.
\]

In fact, these two expressions are equivalent, or represent one common vector, in virtue of the given equations; but the first shows (by 63) that this vector
§ 1. \( \delta \), terminates on the plane \( \text{abc} \), and the second shows (by 25) that it terminates on the line \( \text{de} \); its extremity \( d_1 \) must therefore be, as required, the intersection of this line with that plane. We have therefore the two equations,

I. \[ a(\alpha - \delta_1) + b(\beta - \delta_1) + c(\gamma - \delta_1) = 0; \]

II. \[ d(\delta - \delta_1) + e(\varepsilon - \delta_1) = 0; \]

whence (by 28 and 24) follow the two proportions,

I'. \[ a : b : c = D_1BC : D_1CA : D_1AB; \]

II'. \[ d : e = ED_1 : D_1D; \]

the arrangement of the points, in the annexed fig. 29, answering to the case where all the four coefficients \( a, b, c, d \) are positive (or have one common sign), and when therefore the remaining coefficient \( e \) is negative (or has the opposite sign).

67. For the three complanar triangles, in the first proportion, we may substitute any three pyramidal volumes, which rest upon those triangles as their bases, and which have one common vertex, such as \( D \) or \( E \); and because the collineation \( DDE_1 \) gives \( D_1BC - ED_1BC = debc, \) &c., we may write this other proportion,

I''. \[ a : b : c = DEBC : DECA : DEAB. \]

Again, the same collineation gives

\[ ED_1 : DD_1 = EABC : DABC; \]

we have therefore, by II', the proportion,

II''. \[ d : - e = EABC : DABC. \]

But

\[ DEBC + DECA + DEAB + EABC = DABC, \]

and

\[ a + b + c + d = - e; \]

we may therefore establish the following fuller formula of proportion, between coefficients and volumes:

III. \[ a : b : c : d : - e = DEBC : DECA : DEAB : EABC : DABC; \]

the ratios of all these five pyramids to each other being considered as positive, for the particular arrangement of the points which is represented in the recent figure.

68. The formula III. may however be regarded as perfectly general, if we agree to say that a pyramidal volume changes sign, or rather that it changes its
algebraical character, as positive or negative, in comparison with a given pyramid, and with a given arrangement of points, in passing through zero (comp. 28); namely when, in the course of any continuous change, any one of its vertices crosses the corresponding base. With this convention* we shall have, generally,

\[ \text{DABC} = - \text{ADBC} = \text{ABDC} = - \text{ABCD}, \quad \text{DEBC} = \text{BCDE}, \quad \text{DECA} = \text{CDEA}; \]

the proportion III. may therefore be expressed in the following more symmetric, but equally general form:

\[ \text{III’...} \quad a : b : c : d : e = \text{bcde} : \text{cdea} : \text{deab} : \text{eabc} : \text{abcd}; \]

the sum of these five pyramids being always equal to zero, when signs (as above) are attended to.

69. We saw (in 24) that the two equations,

\[ aa + bj + cy = 0, \quad a + b + c = 0, \]
gave the proportion of segments,

\[ a : b : c = bc : ca : ab, \]

whatever might be the position of the origin o. In like manner we saw (in 63) that the two other equations,

\[ aa + bj + cy + dz = 0, \quad a + b + c + d = 0, \]
gave the proportion of areas,

\[ a : b : c : d = bcd : - cda : dab : - abc; \]

where again the origin is arbitrary. And we have just deduced (in 68) a corresponding proportion of volumes from the two analogous equations (65),

\[ aa + bj + cy + dz + ez = 0, \quad a + b + c + d + e = 0, \]

with an equally arbitrary origin. If then we conceive these segments, areas, and volumes to be replaced by the scalars to which they are thus proportional, we may establish the three general formulae:

\[ \text{I.} \quad \text{OA} \cdot \text{BC} + \text{OB} \cdot \text{CA} + \text{OC} \cdot \text{AB} = 0; \]

\[ \text{II.} \quad \text{OA} \cdot \text{BCD} - \text{OB} \cdot \text{CDA} + \text{OC} \cdot \text{DAB} - \text{OD} \cdot \text{ABC} = 0; \]

\[ \text{III.} \quad \text{OA} \cdot \text{BCDE} + \text{OB} \cdot \text{CDEA} + \text{OC} \cdot \text{DEAB} + \text{OD} \cdot \text{EABC} + \text{OE} \cdot \text{ABCD} = 0; \]

where in I., \( A, B, C \) are any three collinear points;

in II., \( A, B, C, D \) are any four complanar points;

and in III., \( A, B, C, D, E \) are any five points of space;

* Among the consequences of this convention respecting signs of volumes, which has already been adopted by some modern geometers, and which indeed is necessary (comp. 28) for the establishment of general formulae, one is that any two pyramids, \( ABCD \) and \( A'B'C'D' \), bear to each other a positive or a negative ratio, according as the two rotations, \( BCD \) and \( B'C'D' \), supposed to be seen respectively from the points \( A \) and \( A' \), have similar or opposite directions, as right-handed or left-handed.

\text{Hamilton's Elements of Quaternions.}
while \( o \) is, in each of the three formulæ, an entirely arbitrary point. It must, however, be remembered, that the additions and subtractions are supposed to be performed according to the rules of vectors, as stated in the First Chapter of the present Book; the segments, or areas, or volumes, which the equations indicate, being treated as coefficients of those vectors. We might still further abridge the notations, while retaining the meaning of these formulæ, by omitting the symbol of the arbitrary origin \( o \); and by thus writing,*

\[
I'. \quad A \cdot BC + B \cdot CA + C \cdot AB = 0,
\]

for any three collinear points; with corresponding formulæ \( II' \) and \( III' \), for any four complanar points, and for any five points of space.

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SECTION 2.

On Quinary Symbols for Points and Planes in Space.

70. The equations of Art. 65 being still supposed to hold good, the vector \( \rho \) of any point \( p \) of space may, in indefinitely many ways, be expressed (comp. 36) under the form:

\[
I. \quad \mathbf{OP} = \rho = \frac{xa + yb + zc + wd + ve}{xa + yb + zc + wd + ve};
\]

in which the ratios of the differences of the five coefficients, \( xyzwv \), determine the position of the point. In fact, because the four points \( abcd \) are not in any common plane, there necessarily exists (comp. 65) a determined linear relation between the four vectors drawn to them from the point \( p \), which may be written thus,

\[
\rho = \frac{x'a \cdot PA + y'b \cdot PB + z'c \cdot PC + w'd \cdot PD = 0},
\]

giving the expression,

\[
\Pi. \quad \rho = \frac{x'a + y'b + z'c + w'd}{x'a + y'b + z'c + w'd},
\]

in which the ratios of the four scalars \( x'y'z'w' \), depend upon, and conversely determine, the position of \( p \); writing, then,

\[
x = tx' + v, \quad y = ty' + v, \quad z = tx' + v, \quad w = tw' + v,
\]

where \( t \) and \( v \) are two new and arbitrary scalars, and remembering that \( aa + \ldots + ee = 0 \), and \( a + \ldots + e = 0 \) (65), we are conducted to the form for \( \rho \), assigned above.

* We should thus have some of the notations of the Barycentric Calculus, but employed here with different interpretations.
71. When the vector \( \mathbf{r} \) is thus expressed, the point \( \mathbf{r} \) may be denoted by the Quinary Symbol \((x, y, z, w, v)\); and we may write the equation,

\[
\mathbf{r} = (x, y, z, w, v).
\]

But we see that the same point \( \mathbf{r} \) may also be denoted by this other symbol, of the same kind, \((x', y', z', w', v')\), provided that the following proportion between differences of coefficients (70) holds good:

\[
\frac{x' - v'}{y' - v'} : \frac{z' - v'}{w' - v'} : x - v : y - v : z - v : w - v.
\]

Under this condition, we shall therefore write the following formula of congruence,

\[
(x', y', z', w', v') \equiv (x, y, z, w, v),
\]

to express that these two quinary symbols, although not identical in composition, have yet the same geometrical signification, or denote one common point. And we shall reserve the symbolic equation,

\[
(x', y', z', w', v') = (x, y, z, w, v),
\]

to express that the five coefficients, \( x' \ldots v' \), of the one symbol, are separately equal to the corresponding coefficients of the other, \( x' = x, \ldots v' = v \).

72. Writing also, generally,

\[
(t, v', t', t', v') = t (x, y, z, w, v),
\]

and abridging the particular symbol* \((1, 1, 1, 1, 1)\) to \((\mathcal{U})\), while \((Q), (Q')\) may briefly denote the quinary symbols \((x, v), (x', v')\), we may thus establish the congruence (71),

\[
(Q') = (Q), \text{ if } (Q) = t (Q') + u (\mathcal{U});
\]

in which \( t \) and \( u \) are arbitrary coefficients. For example,

\[
(0, 0, 0, 0, 1) = (1, 1, 1, 1, 0), \quad (0, 0, 0, 1, 1) = (1, 1, 1, 0, 0);
\]

each symbol of the first pair denoting (65) the given point \( e \); and each symbol of the second pair denoting (66) the derived point \( v \). When the coefficients are so simple as in these last expressions, we may occasionally omit the commas, and thus write, still more briefly,

\[
(00001) = (11110); \quad (00011) = (11100).
\]

* This quinary symbol \((\mathcal{U})\) denotes no determined point, since it corresponds (by 70, 71) to the indeterminate vector \( \mathbf{p} = 0 \); but it admits of useful combinations with other quinary symbols, as above.
73. If three vectors, \( \rho, \rho', \rho'' \), expressed each under the first form (70), be termino-collinear (24) and if we denote their denominators, \( xa + \ldots + v'e \), \( x'a + \ldots + v'e \), \( x''a + \ldots + v'e \), by \( m, m', m'' \), they must then (23) be connected by a linear equation with a null sum of coefficients, which may be written thus:

\[
tmp + t'm'p' + t''m''p'' = 0; \quad tm + t'm' + t''m'' = 0.
\]

We have, therefore, the two equations of condition,

\[
t (xa + \ldots + ve) + t' (x'a + \ldots + v'e) + t'' (x''a + \ldots + v'e) = 0; \\
t (xa + \ldots + ve) + t' (x'a + \ldots + v'e) + t'' (x''a + \ldots + v'e) = 0;
\]

where \( t, t', t'' \) are three new scalars, while the five vectors \( a \ldots e \), and the five scalars \( a \ldots e \), are subject only to the two equations (65) : but these equations of condition are satisfied by supposing that

\[
tx + t'x' + t''x'' = \ldots = tv + t'v' + t''v'' = - u,
\]

where \( u \) is some new scalar, and they cannot be satisfied otherwise. Hence the condition of collinearity of the three points \( \rho, \rho', \rho'' \), in which the three vectors \( \rho, \rho', \rho'' \) terminate, and of which the quinary symbols are \( (Q), (Q'), (Q'') \), may briefly be expressed by the equation,

\[
t (Q) + t' (Q') + t'' (Q'') = u (U);
\]

so that if any four scalars, \( t, t', t'', u \), can be found, which satisfy this last symbolic equation, then, but not in any other case, those three points \( \rho \rho' \rho'' \) are ranged on one right line. For example, the three points \( d, e, d', \) which are denoted (72) by the quinary symbols, \( (00010), (00001), (11100) \), are collinear; because the sum of these three symbols is \( (U) \). And if we have the equation,

\[
(Q') = t (Q) + t' (Q') + u (U),
\]

where \( t, t', u \) are any three scalars, then \( (Q') \) is a symbol for a point \( \rho'' \), on the right line \( \rho \rho' \). For example, the symbol \( (0, 0, 0, t, t') \) may denote any point on the line \( de \).

74. By reasonings precisely similar it may be proved, that if \( (Q) (Q') (Q'') \) be quinary symbols for any four points \( \rho \rho' \rho'' \rho''' \) in any common plane, so that the four vectors \( \rho \rho' \rho'' \rho''' \) are termino-complanar (64), then an equation, of the form

\[
t (Q) + t' (Q') + t'' (Q'') + t''' (Q''') = - u (U),
\]

must hold good; and conversely, that if the fourth symbol can be expressed as follows,

\[
(Q''') = t (Q) + t' (Q') + t'' (Q'') + u (U),
\]
with any scalar values of \(t, t', t'', u\), then the fourth point \(p''\) is situated in the plane \(pp'p''\) of the other three. For example, the four points,

\[
(10000), \quad (01000), \quad (00100), \quad (11100),
\]
or \(A, B, C, D_1\) (66), are complanar; and the symbol \((t, t', t'', 0, 0)\) may represent any point in the plane \(ABC\).

75. When a point \(p\) is thus complanar with three given points, \(p_0, p_1, p_2\), we have therefore expressions of the following forms, for the five coefficients \(x, \ldots v\) of its quinary symbol, in terms of the fifteen given coefficients of their symbols, and of four new and arbitrary scalars:

\[
x = t_0x_0 + t_1x_1 + t_2x_2 + u; \quad \ldots \quad v = t_0v_0 + t_1v_1 + t_2v_2 + u.
\]

And hence, by elimination of these four scalars, \(t_0, \ldots u\), we are conducted to a linear equation of the form

\[
I (x - v) + m (y - v) + n (z - v) + r (w - v) = 0,
\]

which may be called the Quinary Equation of the Plane \(p_0p_1p_2\), or of the supposed locus of the point \(p\): because it expresses a common property of all the points of that locus; and because the three ratios of the four new coefficients \(I, m, n, r\), determine the position of the plane in space. It is, however, more symmetrical, to write the quinary equation of a plane \(\Pi\) as follows,

\[
lx + my + nz + rw + sv = 0,
\]

where the fifth coefficient, \(s\), is connected with the others by the relation,

\[
l + m + n + r + s = 0;
\]

and then we may say that \([I, m, n, r, s]\) is (comp. 37) the Quinary Symbol of the Plane \(\Pi\), and may write the equation,

\[
\Pi = [I, m, n, r, s].
\]

For example, the coefficients of the symbol for a point \(p\) in the plane \(ABC\) may be thus expressed (comp. 74):

\[
x = t_0 + u, \quad y = t_1 + u, \quad z = t_2 + u, \quad w = u, \quad v = u;
\]

between which the only relation, independent of the four arbitrary scalars \(t_0, \ldots u\), is \(w - v = 0\); this therefore is the equation of the plane \(ABC\), and the symbol of that plane is \([0, 0, 0, 1, -1]\); which may (comp. 72) be sometimes written more briefly, without commas, as \([00011]\). It is evident that, in any such symbol, the coefficients may all be multiplied by any common factor.
76. The symbol of the plane \( p_0 p_1 p_2 \) having been thus determined, we may next propose to find a symbol for the point, \( p \), in which that plane is intersected by a given line \( p_3 p_4 \); or to determine the coefficients \( x \ldots v \), or at least the ratios of their differences (70), in the quinary symbol of that point,

\[
(x, y, z, w, v) = p = p_0 p_1 p_2 \cdot p_3 p_4.
\]

Combining, for this purpose, the expressions,

\[
x = t_3 x_3 + t_4 x_4 + u', \quad v = t_3 v_3 + t_4 v_4 + u',
\]

(which are included in the symbolical equation (73),

\[
(Q) = t_3 (Q_3) + t_4 (Q_4) + u' (U),
\]

and express the collinearity \( p p_3 p_4 \), with the equations (75),

\[
lx + \ldots + sv = 0, \quad l + \ldots + s = 0,
\]

(which express the complanarity \( p p_3 p_4 \)), we are conducted to the formula,

\[
t_3 (lx_3 + \ldots + sv_3) + t_4 (lx_4 + \ldots + sv_4) = 0;
\]

which determines the ratio \( t_3 : t_4 \), and contains the solution of the problem. For example, if \( p \) be a point on the line \( d e \), then (comp. 73),

\[
x = y = z = u', \quad w = t_3 + u', \quad v = t_4 + u';
\]

but if it be also a point in the plane \( a b c \), then \( w - v = 0 \) (75), and therefore \( t_3 - t_4 = 0 \); hence

\[
(Q) = t_3 (00011) + u' (11111), \quad \text{or} \quad (Q) = (00011);
\]

which last symbol had accordingly been found (72) to represent the intersection (66), \( d_1 = a b c \cdot d e \).

77. When the five coefficients, \( xyzuv \), of any given quinary symbol \( (Q) \) for a point \( p \), or those of any congruent symbol (71), are any whole numbers (positive or negative, or zero), we shall say (comp. 42) that the point \( p \) is rationally related to the five given points, \( a \ldots e \); or briefly, that it is a Rational Point of the System, which those five points determine. And in like manner, when the five coefficients, \( lmnrs \), of the quinary symbol (75) of a plane \( \Pi \) are either equal or proportional to integers, we shall say that the plane is a Rational Plane of the same System; or that it is rationally related to the same five points. On the contrary, when the quinary symbol of a point, or of a plane, has not thus already whole coefficients, and cannot be transformed (comp. 72) so as to have them, we shall say that the point or plane is irrationally related to the given points; or briefly, that it is irrational. A right line which connects two
rational points, or is the intersection of two rational planes, may be called, on
the same plan, a Rational Line; and lines which cannot in either of these
two ways be constructed, may be said by contrast to be Irrational Lines. It
is evident from the nature of the eliminations employed (comp. again 42), that
a plane, which is determined as containing three rational points, is necessarily a
rational plane; and in like manner, that a point, which is determined as the
common intersection of three rational planes, is always a rational point: as is
also every point which is obtained by the intersection of a rational line with a
rational plane; or of two rational lines with each other (when they happen to be
complanar).

78. Finally, when two points, or two planes, differ only by the arrangement
(or order) of the coefficients in their quinary symbols, those points or planes
may be said to have one common type; or briefly to be syntypical. For ex­
ample, the five given points, A, . . . E, are thus syntypical, as being represented
by the quinary symbols (10000), . . . (00001); and the ten planes, obtained by
taking all the ternary combinations of those five points, have in like manner
one common type. Thus, the quinary symbol of the plane ABC has been seen
(75) to be [00011]; and the analogous symbol [11000] represents the plane
CDE, &c. Other examples will present themselves, in a shortly subsequent
Section, on the subject of Nets in Space. But it seems proper to say here a
few words, respecting those Anharmonic Co-ordinates, Equations, Symbols, and
Types, for Space, which are obtained from the theory and expressions of the
present Section, by reducing (as we are allowed to do) the number of the
coefficients, in each symbol or equation, from five to four.

SECTION 3.

On Anharmonic Co-ordinates in Space.

79. When we adopt the second form (70) for \( \rho \), or suppose (as we may)
that the fifth coefficient in the first form vanishes, we get this other general
expression (comp. 34, 36), for the vector of a point in space:

\[
op = \rho = \frac{x\alpha + y\beta + z\gamma + \omega\delta}{xa + yb + zc + \omega \delta};
\]

and may then write the symbolic equation (comp. 36, 71),

\[ r = (x, y, z, w), \]

and call this last the Quaternary Symbol of the Point \( r \); although we shall
soon see cause for calling it also the Anharmonic Symbol of that point. Meanwhile we may remark, that the only congruent symbols (71), of this last form, are those which differ merely by the introduction of a common factor: the three ratios of the four coefficients, \( x \ldots w \), being all required, in order to determine the position of the point; whereof those four coefficients may accordingly be said (comp. 36) to be the Anharmonic Co-ordinates in Space.

80. When we thus suppose that \( v = 0 \), in the quinary symbol of the point \( p \), we may suppress the fifth term \( sv \), in the quinary equation of a plane \( \Pi \),

\[
Ix + \ldots + sv = 0 \tag{75}
\]

and therefore may suppress also (as here unnecessary) the fifth coefficient, \( s \), in the quinary symbol of that plane, which is thus reduced to the quaternary form,

\[
\Pi = [l, m, n, r].
\]

This last may also be said (37, 79), to be the Anharmonic Symbol of the Plane, of which the Anharmonic Equation is

\[
lx + my + nz + rw = 0;
\]

the four coefficients, \( lmnr \), which we shall call also (comp. again 37) the Anharmonic Co-ordinates of that Plane \( \Pi \), being not connected among themselves by any general relation (such as \( l + \ldots + s = 0 \)): since their three ratios (comp. 79) are all in general necessary, in order to determine the position of the plane in space.

81. If we suppose that the fourth coefficient, \( w \), also vanishes, in the recent symbol of a point, that point \( p \) is in the plane \( \text{abc} \); and may then be sufficiently represented (as in 36) by the Ternary Symbol \( (x, y, z) \). And if we attend only to the points in which an arbitrary plane \( \Pi \) intersects the given plane \( \text{abc} \), we may suppress its fourth coefficient, \( r \), as being for such points unnecessary. In this manner, then, we are reconducted to the equation, \( lx + my + nz = 0 \), and to the symbol, \( \Delta = [l, m, n] \), for a right line (37) in the plane \( \text{abc} \), considered here as the trace, on that plane, of an arbitrary plane \( \Pi \) in space. If this plane \( \Pi \) be given by its quinary symbol (75), we thus obtain the ternary symbol for its trace \( \Delta \), by simply suppressing the two last coefficients, \( r \) and \( s \).

82. In the more general case, when the point \( p \) is not confined to the plane \( \text{abc} \), if we denote (comp. 72) its quaternary symbol by \( (Q) \), the lately established formulae of collineation and complanarity (73, 74) will still hold good: provided that we now suppress the symbol \( (U) \), or suppose its coefficient to be zero. Thus, the formula,

\[
(Q) = t'(Q') + t''(Q'') + t'''(Q'''),
\]
expresses that the point $p$ is in the plane $p'p''p'''$; and if the coefficient $t''$ vanish, the equation which then remains, namely,

$$(Q) = t'(Q') + t''(Q''),$$

signifies that $p$ is thus complanar with the two given points $p'$, $p''$, and with an arbitrary third point; or, in other words, that it is on the right line $p'p''$ whence (comp. 76) problems of intersections of lines with planes can easily be resolved. In like manner, if we denote briefly by $[R]$ the quaternary symbol $[I, m, n, r]$ for a plane $\Pi$, the formula

$$[R] = t'[R'] + t''[R''],$$

expresses that the plane $\Pi$ passes through the intersection of the three planes $\Pi'$, $\Pi''$, $\Pi'''$; and if we suppose $t'' = 0$, so that

$$[R] = t'[R'],$$

the formula thus found denotes that the plane $\Pi$ passes through the point of intersection of the two planes, $\Pi'$, $\Pi''$, with any third plane; or (comp. 41), that this plane $\Pi$ contains the line of intersection of $\Pi'$, $\Pi''$; in which case the three planes, $\Pi$, $\Pi'$, $\Pi''$, may be said to be collinear. Hence it appears that either of the two expressions,

$$I \ldots t'(Q') + t''(Q''), \quad \Pi \ldots t'[R'] + t''[R''],$$

may be used as a Symbol of a Right Line in Space: according as we consider that line $A$ either, Ist, as connecting two given points, or IInd, as being the intersection of two given planes. The remarks (77) on rational and irrational points, planes, and lines require no modification here; and those on types (78) adapt themselves as easily to quaternary as to quinary symbols.

83. From the foregoing general formulæ of collineation and complanarity, it follows that the point $p'$, in which the line $ab$ intersects the plane $cdp$ through $cd$ and any proposed point $p = (xyzw)$ of space, may be denoted thus:

$$p' = ab \cdot cdp = (xy00);$$

for example, $e = (1111)$, and $c' = ab \cdot cde = (1100)$. In general, if $abcdee$ be any six points of space, the four collinear planes (82), $abc, abd, abe, abf$, are said to form a pencil through $ab$; and if this be cut by any rectilinear transversal, in four points, $c', d', e', f'$, then (comp. 35) the anharmonic function of this group of points (25) is called also the Anharmonic of the Pencil of Planes: which may be thus denoted,

$$(ab \cdot cdef) = (c'd'e'f').$$

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Hence (comp. again 25, 35), by what has just been shown respecting \( c' \) and \( p' \), we may establish the important formula:

\[
(cd \cdot aebp) = (ac'bp') = \frac{x}{y};
\]

so that this ratio of coefficients, in the symbol \((xyzw)\) for a variable point \( p \) (79), represents the anharmonic of a pencil of planes, of which the variable plane \( cdp \) is one; the three other planes of this pencil being given. In like manner,

\[
(ad \cdot becp) = \frac{y'}{z'}, \quad \text{and} \quad (bd \cdot cраП) = \frac{z}{x};
\]

so that (comp. 36) the product of these three last anharmonics is unity. On the same plan we have also,

\[
(bc \cdot ardP) = \frac{x}{w'}, \quad (ca \cdot bedtimeP) = \frac{y}{w'}, \quad (ab \cdot cedP) = \frac{z}{w'};
\]

so that the three ratios, of the three first coefficients \( xyz \) to the fourth coefficient \( w \), suffice to determine the three planes, \( bcp, cap, abp \), whereof the point \( p \) is the common intersection, by means of the anharmonics of three pencils of planes, to which the three planes respectively belong. And thus we see a motive (besides that of analogy to expressions already used for points in a given plane), for calling the four coefficients, \( xyzw \), in the quaternary symbol (79) for a point in space, the Anharmonic Co-ordinates of that Point.

84. In general, if there be any four collinear points, \( p_0, \ldots, p_3 \), so that (comp. 82) their symbols are connected by two linear equations, such as the following,

\[
\begin{align*}
(Q_1) &= t (Q_0) + u (Q_3), \\
(Q_3) &= t' (Q_0) + u' (Q_3),
\end{align*}
\]

then the anharmonic of their group may be expressed (comp. 25, 44) as follows:

\[
\left(P_0P_1P_2P_3\right) = \frac{ut'}{tu'};
\]

as appears by considering the pencil \((cd \cdot p_0p_1p_2p_3)\), and the transversal \( ab \) (83).

And in like manner, if we have (comp. again 82) the two other symbolic equations, connecting four collinear planes \( \Pi_0 \ldots \Pi_3 \),

\[
\begin{align*}
[R_1] &= t [R_0] + u [R_3], \\
[R_3] &= t' [R_0] + u' [R_3],
\end{align*}
\]

the anharmonic of their pencil (83) is expressed by the precisely similar formula,

\[
\left(\Pi_0\Pi_1\Pi_2\Pi_3\right) = \frac{ut'}{tu'};
\]

as may be proved by supposing the pencil to be cut by the same transversal line \( ab \).
85. It follows that if \( f (xyzw) \) and \( f_1 (xyzw) \) be any two homogeneous and linear functions of \( x, y, z, w \); and if we determine four collinear planes \( \Pi_1 \ldots \Pi_3 \) (82), by the four equations,
\[
f = 0, \quad f_1 = f, \quad f_1 = 0, \quad f_1 = kf,
\]
where \( k \) is any scalar; we shall have the following value of the anharmonic function, of the pencil of planes thus determined:
\[
\frac{f_1}{f} = (\Pi_1\Pi_1\Pi_3\Pi_2) = k.
\]
Hence we derive this Theorem, which is important in the application of the present system of co-ordinates to space:

"The Quotient of any two given homogeneous and linear Functions, of the anharmonic Co-ordinates (79) of a variable Point \( p \) in space, may be expressed as the Anharmonic \( (\Pi_0\Pi_1\Pi_2\Pi_3) \) of a Pencil of Planes; whereof three are given, while the fourth passes through the variable point \( p \), and through a given right line \( \Lambda \) which is common to the three former planes."

86. And in like manner may be proved this other but analogous Theorem:

"The Quotient of any two given homogeneous and linear Functions, of the anharmonic Co-ordinates (80) of a variable Plane \( \Pi \), may be expressed as the Anharmonic \( (p_0p_1p_2p_3) \) of a Group of Points; whereof three are given and collinear; and the fourth is the intersection, \( \Lambda \cdot \Pi \), of their common and given right line \( \Lambda \) with the variable plane \( \Pi \)."

More fully, if the two given functions of \( lnmr \) be \( f \) and \( f_1 \), and if we determine three points \( p_0p_1p_2 \) by the equations (comp. 57) \( f = 0, f_1 = f, f_1 = 0 \), and denote by \( p_3 \) the intersection of their common line \( \Lambda \) with \( \Pi \), we shall have the quotient,
\[
\frac{f_1}{f} = (p_0p_1p_2p_3).
\]

For example, if we suppose that
\[
A_2 = (1001), \quad B_2 = (0101), \quad C_2 = (0011),
\]
\[
A'_2 = (100\overline{1}), \quad B'_2 = (01\overline{1}), \quad C'_2 = (001\overline{1}),
\]
so that \( A_2 = DA \cdot BCE, \&c. \), and \( (DA_2AA'_2) = -1, \&c. \),

we find that the three ratios of \( l, m, n \) to \( r \), in the symbol \( \Pi = [lnmr] \), may be expressed (comp. 39) under the form of anharmonics of groups, as follows:
\[
\frac{l}{r} = (DA'_2AQ); \quad \frac{m}{r} = (DB'_2BR); \quad \frac{n}{r} = (DC'_2CS);
\]
where \( A, R, S \) denote the intersections of the plane \( \Pi \) with the three given
right lines, \(da, db, dc\). And thus we have a motive (comp. 83) besides that of analogy to lines in a given plane (37), for calling (as above) the four coefficients \(l, m, n, r\), in the quaternary symbol (80) for a plane \(\Pi\), the Anharmonic Co-ordinates of that Plane in Space.

87. It may be added, that if we denote by \(l, m, n\) the points in which the same plane \(\Pi\) is cut by the three given lines \(bc, ca, ab\), and retain the notations \(\lambda'', \mu'', \nu''\) for those other points on the same three lines which were so marked before (in 31, &c.), so that we may now write (comp. 36)

\[
\lambda'' = (01\bar{1}0), \quad \mu'' = (\bar{1}010), \quad \nu'' = (\bar{1}\bar{1}00),
\]

we shall have (comp. 39, 83) these three other anharmonics of groups, with their product equal to unity:

\[
\frac{m}{n} = (ca''bl) ; \quad \frac{n}{\lambda} = (ab''cm) ; \quad \frac{l}{m} = (bc''an);
\]

and the six given points, \(\lambda'', \mu'', \nu'', \lambda'_2, \mu'_2, \nu'_2\), are all in one given plane \([\varepsilon]\), of which the equation and symbol are:

\[
x + y + z + w = 0; \quad [\varepsilon] = [1111].
\]

The six groups of points, of which the anharmonic functions thus represent the six ratios of the four anharmonic co-ordinates, \(lmnr\), of a variable plane \(\Pi\), are therefore situated on the six edges of the given pyramid, \(ABCD\); two points in each group being corners of that pyramid, and the two others being the intersections of the edge with the two planes, \([\varepsilon]\) and \(\Pi\). Finally, the plane \([\varepsilon]\) is (in a known modern sense) the plane of homology,\(^*\) and the point \(e\) is the centre of homology, of the given pyramid \(ABCD\), and of an inscribed pyramid \(A_1B_1C_1D_1\), where \(A_1 = ea \cdot bcd, \&c.; so that \(d_1\) retains its recent signification (66, 76), and we may write the anharmonic symbols,

\[
A_1 = (0111), \quad B_1 = (1011), \quad C_1 = (1101), \quad D_1 = (1110).
\]

And if we denote by \(A'_1, B'_1, C'_1, D'_1\) the harmonic conjugates to these last points, with respect to the lines \(EA, EB, EC, ED\), so that

\[
(EA_1A'_1) = \ldots = (ED_1D'_1) = - 1,
\]

we have the corresponding symbols,

\[
A'_1 = (2111), \quad B'_1 = (1211), \quad C'_1 = (1121), \quad D'_1 = (1112).
\]

Many other relations of position exist, between these various points, lines, and planes, of which some will come naturally to be noticed, in that theory of nets in space to which in the following Section we shall proceed.

\* See Poncelet's Traité des Propriétés Projectives (Paris, 1822).
SECTION 4.

On Geometrical Nets in Space.

88. When we have (as in 65) five given points \( A \ldots E \), whereof no four are complanar, we can connect any two of them by a right line, and the three others by a plane, and determine the point in which these last intersect one another: deriving thus a system of ten lines \( \Lambda_1 \), ten planes \( \Pi_1 \), and ten points \( P_1 \), from the given system of five points \( P_0 \), by what may be called (comp. 34) a First Construction. We may next propose to determine all the new and distinct lines, \( \Lambda_2 \), and planes, \( \Pi_2 \), which connect the ten derived points \( P_1 \) with the five given points \( P_0 \), and with each other; and may then inquire what new and distinct points \( P_2 \) arise (at this stage) as intersections of lines with planes, or of lines in one plane with each other: all such new lines, planes, and points being said (comp. again 34) to belong to a Second Construction. And then we might proceed to a Third Construction of the same kind, and so on for ever: building up thus what has been called* a Geometrical Net in Space. To express this geometrical process by quinary symbols (71, 75, 82) of points, planes, and lines, and by quinary types (78), so far at least as to the end of the second construction, will be found to be an useful exercise in the application of principles lately established: and therefore ultimately in that Method of Vectors, which is the subject of the present Book. And the quinary form will here be more convenient than the quaternary, because it will exhibit more clearly the geometrical dependence of the derived points and planes on the five given points, and will thereby enable us, through a principle of symmetry, to reduce the number of distinct types.

89. Of the five given points, \( P_0 \), the quinary type has been seen (78) to be \((10000)\); while of the ten derived points \( P_1 \), of first construction, the corresponding type may be taken as \((00011)\); in fact, considered as symbols, these two represent the points \( A \) and \( D \). The nine other points \( P_1 \) are \( A'B'C' \), \( A_1B_1C_1D_1 \); and we have now (comp. 83, 87, 86) the symbols,

\[
A' = BC \cdot ADE = (01100), \quad A_1 = EA \cdot BCD = (10001),
\]

\[
A_2 = DA \cdot BCD = (10010);
\]

also, in any symbol or equation of the present form, it is permitted to change \( A, B, C \) to \( B, C, A \), provided that we at the same time write the third, first,
and second co-efficients, in the places of the first, second, and third: thus, 
\[ b' = c \cdot a \cdot bde = (10100) \], \&c. The symbol \([xy000]\) represents an arbitrary point on the line \(ab\); and the symbol \([00nrs]\), with \(n + r + s = 0\), represents an arbitrary plane through that line: each therefore may be regarded (comp. 82) as a symbol also of the line \(ab\) itself, and at the same time as a type of the ten lines \(\Lambda_1\); while the symbol \([00011]\), of the plane \(abc\) (75), may be taken (78) as a type of the ten planes \(\Pi_1\). Finally, the five pyramids,
\[ \text{BCDE, CADE, ABDE, ABCE, ABCD}, \]
and the ten triangles, such as \(abc\), where each is a common face of two such pyramids, may be called pyramids \(R_1\), and triangles \(T_1\), of the First Construction.

90. Proceeding to a Second Construction (88), we soon find that the lines \(\Lambda_2\) may be arranged in two distinct groups; one group consisting of fifteen lines \(\Lambda_{2,1}\), such as the line* \(\Lambda_2\Lambda_5\), whereof each connects two points \(p_1\), and passes also through one point \(p_0\), being the intersection of two planes \(\Pi_1\) through that point, as here of \(abc\), \(ade\); while the other group consists of thirty lines \(\Lambda_{2,2}\), such as \(b'c'\), each connecting two points \(p_1\), but not passing through any point \(p_0\), and being one of the thirty edges of five new pyramids \(R_2\), namely,
\[ c'b'\Lambda_2A_1, A'c'b_1b_2, b'a'c_2c_1, A_2B_2C_2D_1, A_1B_1C_1D_1; \]
which pyramids \(R_2\) may be said (comp. 87) to be inscribed homologues of the five former pyramids \(R_1\), the centres of homology for these five pairs of pyramids being the five given points \(A\ldots E\); and the planes of homology being five planes \([A]\ldots [E]\), whereof the last has been already mentioned (87), but which belong properly to a third construction (88). The planes \(\Pi_2\), of second construction, form in like manner two groups; one consisting of fifteen planes \(\Pi_{2,1}\), such as the plane of the five points, \(AB_1B_2C_2\), whereof each passes through one point \(p_0\), and through four points \(p_1\), and contains two lines \(\Lambda_{2,1}\), as here the lines \(AB_1C_2, AC_2B_2\), besides containing four lines \(\Lambda_{2,2}\), as here \(B_1B_2\), \&c.; while the other group is composed of twenty planes \(\Pi_{2,2}\), such as \(A_1B_1C_1\), namely, the twenty faces of the five recent pyramids \(R_2\), whereof each contains three points \(p_1\), and three lines \(\Lambda_{2,2}\), but does not pass through any point \(p_0\). It is now required to express these geometrical conceptions† of the forty-five lines \(\Lambda_2\); the thirty-five planes \(\Pi_2\); and the five planes of homology of pyramids, \([A]\ldots [E]\), by

* \(AB_1C_2, AB_2C_1, DA'A_1, EA'A_2\), are other lines of this group.
† Möbius (in his Barycentric Calculus, p. 284, \&c.) has very clearly pointed out the existence and chief properties of the foregoing lines and planes; but besides that his analysis is altogether different from ours, he does not appear to have aimed at enumerating, or even at classifying, all the points of what has been above called (88) the second construction, as we propose shortly to do.
quinary symbols and types, before proceeding to determine the points \( p_2 \) of second construction.

91. An arbitrary point on the right line \( \Lambda \Lambda' \Omega \Omega' (90) \) may be represented by the symbol \( (\text{tuu}000) \); and an arbitrary plane through that line by this other symbol, \([\text{omm}mr]\), where \( m \) and \( r \) are written (to save commas) instead of \(-m\) and \(-r\); hence these two symbols may also (comp. 82) denote the line \( \Lambda \Lambda' \Omega \Omega' \) itself, and may be used as types (78) to represent the group of lines \( \Lambda_{2,1} \). The particular symbol \([01\overline{1}1\overline{1}]\), of the last form, represents that particular plane through the last-mentioned line, which contains also the line \( \Lambda_{2,1} \) of the same group; and may serve as a type for the group of planes \( \Pi_{2,1} \). The line \( \text{n'c'} \), and the group \( \Lambda_{2,2} \), may be represented by \([\text{stu}000]\) and \([\text{ttt}us]\), if we agree* to write \( s = t + u \), and \( s = -s \); while the plane \( \text{n'c'} \Lambda_{2,2} \), and the group \( \Pi_{2,2} \), may be denoted by \([\overline{1}\overline{1}\overline{1}\overline{1}]\). Finally, the plane \([\text{e}]\) has for its symbol \([11114]\); and the four other planes \([\Lambda], \&c., \) of homology of pyramids (90), have this last for their common type.

92. The points \( p_2 \), of second construction (88), are more numerous than the lines \( \Lambda_{2} \) and planes \( \Pi_{2} \) of that construction: yet with the help of types, as above, it is not difficult to classify and to enumerate them. It will be sufficient here to write down these types, which are found to be eight, and to offer some remarks respecting them; in doing which we shall avail ourselves of the eight following typical points, whereof the two first have already occurred, and which are all situated in the plane of \( \Lambda \Omega \Omega' \):

\[
\begin{align*}
\Lambda'' &= (0\overline{1}100); & \Lambda''' &= (21\overline{1}00); & \Lambda^v &= (\overline{2}1\overline{1}00); & \Lambda^v &= (02100); \\
\Lambda^{vi} &= (02\overline{1}00); & \Lambda^{vii} &= (12\overline{1}00); & \Lambda^{viii} &= (32100); & \Lambda^{ix} &= (23\overline{1}00);
\end{align*}
\]

the second and third of these having \((10011)\) and \((30011)\) for congruent symbols (71). It is easy to see that these eight types represent, respectively, ten, thirty, thirty, twenty, twenty, sixty, sixty, and sixty distinct points, belonging to eight groups, which we shall mark as \( p_{2,1}, \ldots p_{2,8} \); so that the total number of the points \( p_2 \) is 290. If then we consent (88) to close the present inquiry, at the end of what we have above defined to be the Second Construction, the total number of the net points, \( p_1, p_2 \), which are thus derived by lines and planes from the five given points \( p_0 \), is found to be exactly three hundred: while the joint number of the net-lines, \( \Lambda_1, \Lambda_2 \), and of the net-planes, \( \Pi_1, \Pi_2 \), has been seen to be one hundred, so far.

* With this convention, the line \( \Lambda \Omega \), and the group \( \Lambda_1 \), may be denoted by the plane-symbol \([00\text{tu}0]\), their point-symbol being \((\text{tuu}000)\).
(1.) To the type \( p_{21} \) belong the ten points,

\[
A''B''C'', \quad A'_2B'_2C'_2, \quad A'_1B'_1C'_1D'_1,
\]

with the quinary symbols,

\[
A'' = (01100), \quad A'_2 = (10010), \quad A'_1 = (10001), \quad D'_1 = (00011),
\]

which are the harmonic conjugates of the ten points \( p_i \), namely, of

\[
A'B'C', \quad A_2B_2C_2, \quad A_1B_1C_1D_1,
\]

with respect to the ten lines \( \Lambda_{1i} \), on which those points are situated; so that we have ten harmonic equations, \((BA'CA'') = -1, \&c., as already seen (31, 86, 87). Each point \( p_{21} \) is the common intersection of a line \( \Lambda_{1i} \) with three lines \( \Lambda_{22} \); thus we may establish the four following formulae of concurrence (equivalent, by 89, to ten such formulae):

\[
A'' = BC \cdot B'C' \cdot B_1C_1 \cdot B_2C_2; \quad A'_2 = DA \cdot D_1A_1 \cdot B'C_2 \cdot C'B_2; \\
A'_1 = EA \cdot E_1A_2 \cdot B'C_1 \cdot C'B_1; \quad D'_1 = DE \cdot A_1A_2 \cdot B_1B_2 \cdot C_1C_2.
\]

Each point \( p_{21} \) is also situated in three planes \( \Pi_1 \); in three other planes, of the group \( \Pi_{21} \); and in six planes \( \Pi_{22} \); for example, \( A'' \) is a point common to the twelve planes,

\[
ABC, \quad BCD, \quad BCR; \quad AB_1C_2B_3, \quad DB'B_1C'O_1, \quad EB'B_2C'O_2; \\
B'C'A_1, \quad B_1C_2A_1, \quad B_2C_2A_2, \quad B'C'A_2, \quad B_1C_1D_1, \quad B_2C_2D_1.
\]

Each line, \( \Lambda_1 \), or \( \Lambda_{22} \), contains one point \( p_{21} \); but no line \( \Lambda_{21} \) contains any. Each plane, \( \Pi_1 \), or \( \Pi_{22} \), contains three such points; and each plane \( \Pi_{21} \) contains two, which are the intersections of opposite sides of a quadrilateral \( Q_2 \) in that plane, whereof the diagonals intersect in a point \( p_i \); for example, the diagonals \( B_1C_2, B_2C_1 \) of the quadrilateral \( B_2B_1C_2C_1 \) which is (by 90) in one of the planes \( \Pi_{21} \), intersect* each other in the point \( A \); while the opposite sides \( C_1B_1, B_2C_2 \) intersect in \( A'' \); and the two other opposite sides, \( B_1B_2, C_2C_1 \) have the point \( D'_1 \) for their intersection. The ten points \( p_{21} \) are also ranged, three by three, on ten lines of third construction \( \Lambda_3 \), namely, on the axes of homology,

\[
A''B'_1C'_1, \quad A''B'_2C'_2, \quad A'_1A'_2D'_1, \quad A''B''C''.
\]

of ten pairs of triangles \( T_1, T_2 \), which are situated in the ten planes \( \Pi_1 \), and of which the centres of homology are the ten points \( p_i \); for example, the dotted line \( A''C'' \), in fig. 21, is the axis of homology of the two triangles, \( ABC, A'B'C' \), whereof the latter is inscribed in the former, with the point \( o \) in that figure (replaced by \( D_1 \) in fig. 29), to represent their centre of homology. The same ten points \( p_{21} \) are also ranged six by six, and the ten last lines \( \Lambda_3 \) are ranged

* Compare the first Note to page 62.
four by four, in five planes $\Pi_3$, namely in the planes of homology of five pairs of pyramids, $R_1, R_2$, already mentioned (90): for example, the plane $[E]$ contains (87) the six points $A''B''C''A'2B'2C'2$, and the four right lines,

$$A''B'2C'2, B''C'2A'2, C''A'2B'2, A''B''C'';$$

which latter are the intersections of the four faces, $DCB$, $DAC$, $DBA$, $ABC$,
of the pyramid $ABCD$, with the corresponding faces,

$$D_1C_1B_1, D_1A_1C_1, D_1B_1A_1, A_1B_1C_1,$$

of its inscribed homologue $A_1B_1C_1D_1$; and are contained, besides, in the four other planes,

$$A_2B'2C', B_2C'A', C_2A'B', A_2B'C_2;$$

the three triangles, $ABC, A_1B_1C_1, A_2B_2C_2$, for instance, being all homologous, although in different planes, and having the line $A''B''C''$ for their common axis of homology. We may also say, that this line $A''B''C''$ is the common trace (81) of two planes $\Pi_{2,i}$, namely of $A_1B_1C_1$ and $A_2B_2C_2$, on the plane $ABC$; and in like manner, that the point $A''$ is the common trace, on that plane $\Pi_1$, of two lines $A_2, A_2$, namely of $B_1C_1$ and $B_2C_2$: being also the common trace of the two lines $B'_2C'_2$ and $B'_2C'_2$, which belong to the third construction.

(2.) On the whole, these ten points, of second construction, $A'' \ldots$, may be considered to be already well known to geometers, in connexion with the theory of transversal* lines and planes in space: but it is important here to observe, with what simplicity and clearness their geometrical relations are expressed (88), by the quinary symbols and quinary types employed. For example, the collinearity (82) of the four planes, $ABC, A_1B_1C_1, A_2B_2C_2$, and $[E]$, becomes evident from mere inspection of their symbols,

$$[00011], [11121], [11112], [11114],$$

which represent (75) the four quinary equations,

$$w - v = 0, \ x + y + z - 2w - v = 0, \ x + y + z - w - 2v = 0, \ x + y + z + w - 4v = 0;$$

with this additional consequence, that the ternary symbol (81) of the common trace, of the three latter on the former, is $[111]$ : so that this trace is (by 38) the line $A''B''C''$ of fig. 21, as above. And if we briefly denote the quinary symbols of the four planes, taken in the same form and order as above, by

* The collinear, complanar, and harmonic relations between the ten points, which we have above marked as $p_2,i$, and which have been considered by Möbius also, in connexion with his theory of nets in space, appear to have been first noticed by Carnot, in a Memoir upon transversals.
we see that they are connected by the two relations,

\[ [R_3] = -[R_0] + [R_2] ; \quad [R_3] = 2[R_0] + [R_2] ; \]

whence if we denote the planes themselves by \( \Pi_1, \Pi_2, \Pi_3, \Pi_4 \), we have (comp. 84) the following value for the anharmonic of their pencil,

\[ (\Pi_1 \Pi_2 \Pi_3 \Pi_4) = -2 ; \]

a result which can be very simply verified, for the case when \( ABCD \) is a regular pyramid, and \( E \) (comp. 29) is its mean point: the plane \( \Pi_3 \), or \( [E] \), becoming in this case (comp. 38) the plane at infinity, while the three other planes, \( \Pi_1, \Pi_2, \Pi_3, \Pi_4 \), are parallel; the second being intermediate between the other two, but twice as near to the third as to the first.

(3.) We must be a little more concise in our remarks on the seven other types of points \( p_2 \), which indeed, if not so well known,* are perhaps also, on the whole, not quite so interesting: although it seems that some circumstances of their arrangement in space may deserve to be noted here, especially as affording an additional exercise (88), in the present system of symbols and types. The type \( p_{2,2} \) represents, then, a group of thirty points, of which \( \lambda'' \), in fig. 21, is an example; each being the intersection of a line \( A_{2,i} \) with a line \( A_{2,j} \) as \( \lambda'' \) is the point in which \( A_{2,2} \) intersects \( b'c' \): but each belonging to no other line, among those which have been hitherto considered. But without aiming to describe here all the lines, planes, and points, of what we have called the third construction, we may already see that they must be expected to be numerous: and that the planes \( \Pi_3 \), and the lines \( \Lambda_3 \), of that construction, as well as the pyramids \( R_2 \), and the triangles \( T_2 \), of the second construction, above noticed, can only be regarded as specimens, which in a closer study of the subject, it becomes necessary to mark more fully, on the present plan, as \( \Pi_{3,1} \ldots T_{3,1} \). Accordingly it is found that not only is each point \( p_{2,2} \) one of the corners of a triangle \( T_{3,1} \) of third construction (as \( \lambda'' \) is of \( \Lambda''B''C'' \) in fig. 21), the sides of which new triangle are lines \( A_{3,2} \), passing each through one point \( p_{3,1} \) and through two points \( p_{2,2} \) (like the dotted line \( A''B''C'' \) of fig. 21); but also each such point \( p_{2,2} \) is the intersection of two new lines of

* It does not appear that any of these other types, or groups, of points \( p_2 \), have hitherto been noticed, in connexion with the net in space, except the one which we have ranked as the fifth, \( p_{2,5} \), and which represents two points on each line \( A_1 \), as the type \( p_{2,1} \) has been seen to represent one point on each of those ten lines of first construction: but that fifth group, which may be exemplified by the intersections of the line \( DM \) with the two planes \( A_1B_C \) and \( A_2B_2C_2 \), has been indicated by Möbius (in page 290 of his already cited work), although with a different notation, and as the result of a different analysis.
third construction, \( A_{22} \), whereof each connects a point \( p_0 \) with a point \( p_{21} \). For example, the point \( A''' \) is the common trace (on the plane \( abc \)) of the two new lines, \( DA', EA_2' \): because, if we adopt for this point \( A''' \) the second of its two congruent symbols, we have (comp. 73, 82) the expressions,
\[
A''' = (\overline{1}0011) = (D) - (A_1') = (E) - (A_2').
\]
We may therefore establish the formula of concurrence (comp. the first sub-article):
\[
A''' = AA' \cdot B'C' \cdot DA_1' \cdot EA_2';
\]
which represents a system of thirty such formulae.

(4) It has been remarked that the point \( A''' \) may be represented, not only by the quinary symbol \((21100)\), but also by the congruent symbol, \((\overline{1}0011)\); if then we write,
\[
A_0 = (111100), \quad B_0 = (111100), \quad C_0 = (111100),
\]
these three new points \( A_0B_0C_0 \) in the plane of \( abc \), must be considered to be syntypical, in the quinary sense (78), with the three points \( A''''B''''C'''' \), or to belong to the same group \( p_{22} \), although they have (comp. 88) a different ternary type. It is easy to see that, while the triangle \( A''''B''''C'''' \) is (comp. again fig. 21) an inscribed homologue \( T_{51} \) of the triangle \( A'B'C' \), which is itself (comp. sub-article 1) an inscribed homologue \( T_{21} \) of a triangle \( T_1 \), namely of \( ABC \), with \( A''''B''''C'''' \) for their common axis of homology, the new triangle \( A_0B_0C_0 \) is on the contrary an exscribed homologue \( T_{32} \), with the same axis \( A_{32} \), of the same given triangle \( T_1 \). But from the syntypical relation existing as above for space between the points \( A''' \) and \( A_0 \), we may expect to find that these two points \( p_{22} \) admit of being similarly constructed, when the five points \( p_0 \) are treated as entering symmetrically (or similarly), as geometrical elements, into the constructions. The point \( A_0 \) must therefore be situated, not only on a line \( A_{32} \), namely, on \( AA' \), but also on a line \( A_{31} \), which is easily found to be \( A_1A_2 \), and on two lines \( A_{22} \), each connecting a point \( p_0 \) with a point \( p_{21} \); which latter lines are soon seen to be \( BB'' \) and \( CC'' \). We may therefore establish the formula of concurrence (comp. the last sub-article):
\[
A_0 = AA' \cdot A_1A_2 \cdot BB'' \cdot CC'';
\]
and may consider the three points \( A_0, B_0, C_0 \) as the traces of the three lines \( A_1A_2, B_1B_2, C_1C_2 \): while the three new lines \( AA'', BB'', CC'' \), which coincide in position with the sides of the exscribed triangle \( A_0B_0C_0 \), are the traces \( A_{32} \) of three planes \( \Pi_{32} \), such as \( AB_1C_2B_2C_1 \), which pass through the three given points \( A, B, C \).
but do not contain the lines $A_{2,1}$ whereon the six points $p_{2,2}$ in their plane $\Pi_2$ are situated. Every other plane $\Pi_1$ contains, in like manner, six points $p_2$ of the present group; every plane $\Pi_{2,1}$ contains eight of them; and every plane $\Pi_{2,2}$ contains three; each line $A_{2,1}$ passing through two such points, but each line $A_{2,2}$ only through one. But besides being (as above) the intersection of two lines $A_2$, each point of this group $p_{2,2}$ is common to two planes $\Pi_1$, four planes $\Pi_{2,1}$, and two planes $\Pi_{2,2}$; while each of these thirty points is also a common corner of two different triangles of third construction, of the lately mentioned kinds $T_{2,1}$ and $T_{2,2}$, situated respectively in the two planes of first construction which contain the point itself. It may be added that each of the two points $p_{2,2}$, on a line $A_{2,1}$, is the harmonic conjugate of one of the two points $P_1$, with respect to the point $P_0$, and to the other point $P_1$ on that line; thus we have here the two harmonic equations,

$$(AA'D_1A'\prime') = (AD_1A'A_0) = -1,$$

by which the positions of the two points $A''$ and $A_0$ might be determined.

(5.) A third group, $p_{3,3}$, of second construction, consists (like the preceding group) of thirty points, ranged two by two on the fifteen lines $A_{2,1}$, and six by six on the ten planes $\Pi_1$, but so that each is common to two such planes; each is also situated in two planes $\Pi_{2,1}$, in two planes $\Pi_{2,2}$, and on one line $A_{3,1}$, in which (by sub-art. 1) these two last planes intersect each other, and two of the five planes $\Pi_{3,1}$; each plane $\Pi_{2,1}$ contains four such points, and each plane $\Pi_{2,2}$ contains three of them; but no point of this group is on any line $A_1$, or $A_{2,2}$. The six points $p_{3,3}$, which are in the plane $ABC$, are represented (like the corresponding points of the last group) by two ternary types, namely by $(211)$ and $(311)$; and may be exemplified by the two following points, of which these last are the ternary symbols:

$$A_{IV} = AA' \cdot A''B''C'' = AA' \cdot A_1B_1C_1 \cdot A_2B_2C_2;$$

$$A_{IV} = AA' \cdot D_1A_2A_1 = AA' \cdot B'C_2C_2 \cdot C'B_1B_2.$$

The three points of the first sub-group $A_{IV}$... are collinear; but the three points $A_{IV}$... of the second sub-group are the corners of a new triangle, $T_{3,3}$, which is homologous to the triangle $ABC$, and to all the other triangles in its plane which have been hitherto considered, as well as to the two triangles $A_1B_1C_1$ and $A_2B_2C_2$; the line of the three former points being their common axis of homology; and the sides of the new triangle, $A_{IV}B_{IV}C_{IV}$, being the traces of the three planes (comp., 90) of homology of pyramids, $[A]$ [B] [C]; as (comp. sub-art. 2) the line $A_{IV}B_{IV}C_{IV}$ or $A''B''C''$ is the common trace of the two other planes of the same
group $\Pi_{31}$, namely of $[d]$ and $[e]$. We may also say that the point $\alpha_{iv}$ is the trace of the line $\alpha', \alpha''_2$; and because the lines $b'c''_0$, $c'b''_0$ are the traces of the two planes $\Pi_{32}$ in which that point is contained, we may write the formula of concurrence,

$$A_{iv} = AA' \cdot A'_{iv}A'' \cdot B'c''_0 \cdot c'b''_0.$$

(6.) It may be also remarked, that each of the two points $\rho_{23}$, on any line $\Lambda_{21}$, is the harmonic conjugate of a point $\rho_{22}$, with respect to the point $\rho_0$, and to one of the two points $\rho_1$ on that line; being also the harmonic conjugate of this last point, with respect to the same point $\rho_0$, and the other point $\rho_{22}$: thus, on the line $\alpha\alpha'D_1$, we have the four harmonic equations, which are not however all independent, since two of them can be deduced from the two others, with the help of the two analogous equations of the fourth sub-article:

$$AA''A'_{iv} = (AA_0\alpha A_{iv}) = (AA_0D_1A_{iv}) = (AD_1A''_{iv}A_{iv}) = -1.$$ 

And the three pairs of derived points $\rho_1$, $\rho_{22}$, $\rho_{23}$, on any such line $\Lambda_{21}$, will be found (comp. 26) to compose an involution, with the given point $\rho_0$ on the line for one of its two double points (or foci): the other double point of this involution being a point $\rho_0$ of third construction; namely, the point in which the line $\Lambda_{21}$ meets one of the five planes of homology $\Pi_{31}$, which corresponds (comp. 90) to the particular point $\rho_0$ as centre. Thus, in the present example, if we denote by $\alpha^x$ the point in which the line $\alpha\alpha'$ meets the plane $[A]$, of which (by 81, 91) the trace on $\alpha\beta\gamma$ is the line $[411]$, and therefore is (as has been stated) the side $b''_0c''_1\gamma''_0$ of the lately mentioned triangle $T_{29}$, so that

$$\alpha^x = (122) = AA' \cdot BC'' \cdot CB'' \cdot B_0\gamma''_1\gamma''_0,$$

we shall have the three harmonic equations,

$$(AA'\alpha^x D_1) = (AA''\alpha^x A_0) = (AA''\alpha^x A_{iv}) = -1;$$

which express that this new point $\alpha^x$ is the common harmonic conjugate of the given point $\alpha$, with respect to the three pairs of points, $\alpha D_1$, $\alpha'' A_0$, $\alpha''_{iv} A_{iv}$; and therefore that these three pairs form (as has been said) an involution, with $\alpha$ and $\alpha^x$ for its two double points.

(7.) It will be found that we have now exhausted all the types of points of second construction, which are situated upon lines $\Lambda_{21}$; there being only four such points on each such line. But there are still to be considered two new groups of points $\rho_2$ on lines $\Lambda_1$, and three others on lines $\Lambda_{32}$. Attending first to the former set of lines, we may observe that each of the two new types, $\rho_{24}$, $\rho_{25}$, represents twenty points, situated two by two on the ten
lines of first construction, but not on any line \( \Lambda_2 \); and therefore six by six in the ten planes \( \Pi_1 \), each point however being common to three such planes: also each point \( p_{24} \) is common to three planes \( \Pi_{25} \), and each point \( p_{25} \) is situated in one such plane; while each of these last planes contains three points \( p_{24} \), but only one point \( p_{25} \). If we attend only to points in the plane \( abc \), we can represent these two new groups by the two ternary types \((021)\) and \((02\overline{1})\), which as symbols denote the two typical points,

\[
A^y = BC \cdot C' A_1 A_2 \cdot D_1 A_1 B_1 \cdot D_2 A_2 B_2; \quad A^y_i = BC \cdot C'B_1 B_2 = BC \cdot C'B_0;
\]

we have also the concurrence,

\[
A^y = BC \cdot C' A_0 \cdot D_1 C'' \cdot AB'''.
\]

It may be noted that \( A^y \) is the harmonic conjugate of \( C' \), with respect to \( A_0 \) and \( B_1 \), which last point is on the same trace \( C' A_0 \) of the plane \( C'A_1 A_2 \); and that \( A^y_i \) is harmonically conjugate to \( B_1 \), with respect to \( C' \) and \( B_0 \), on the trace of the plane \( C'B_1 B_2 \), where \( B_1 \) denotes (by an analogy which will soon become more evident) the intersection of that trace with the line \( ca \); so that we have the two equations,

\[
(A_0 C'B_1 A^y) = (B_0 B_1 A C'A^y) = -1.
\]

(8.) Each line \( \Lambda_1 \) contains thus two points \( p_2 \), of each of the two last new groups, besides the point \( p_{211} \), the point \( p_1 \), and the two points \( p_0 \), which had been previously considered: it contains therefore eight points in all, if we still abstain (88) from proceeding beyond the Second Construction. And it is easy to prove that these eight points can, in two distinct modes, be so arranged as to form (comp. sub-art 6) an involution, with two of them for the two double points thereof. Thus, if we attend only to points on the line \( bc \), and represent them by ternary symbols, we may write,

\[
B = (010), \quad C = (001), \quad A' = (011), \quad A'' = (0\overline{1});
\]

\[
A^y = (021), \quad A^y_i = (02\overline{1}), \quad A^y = (012), \quad A^y = (0\overline{1});
\]

and the resulting harmonic equations

\[
\text{I. } \ldots (BA'CA'') = (BA^yCA^y_i) = (BA^yCA_1^y) = -1,
\]

\[
\text{II. } \ldots (A'BA''c) = (A^yA'A_1^y) = (A^yA'A''A_1^y) = -1,
\]

will then suffice to show: Ist, that the two points \( p_0 \), on any line \( \Lambda_1 \), are the double points of an involution, in which the points \( p_1 \), \( p_{211} \) form one pair of conjugates, while the two other pairs are of the common form, \( p_{24}, \ p_{25} \); and IIInd, that the two points \( p_1 \) and \( p_{211} \), on any such line \( \Lambda_1 \), are the double points of a second involution, obtained by pairing the two points of each of the three other
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Also each of the two points \( p_0 \), on a line \( \Lambda_1 \), is the harmonic conjugate of one of the two points \( p_{2,6} \) on that line, with respect to the other point of the same group, and to the point \( p_1 \) on the same line; thus,

\[
(b\Lambda_1 A_1 V) = (c\Lambda_1 A_1 V) = -1.
\]

(9.) It remains to consider briefly three other groups of points \( p_2 \), each group containing sixty points, which are situated, two by two, on the thirty lines \( \Lambda_{2,6} \), and six by six in the ten planes \( \Pi_i \). Confining our attention to those which are in the plane \( \Lambda_{2,6} \), and denoting them by their ternary symbols, we have thus, on the line \( b'c' \), the three new typical points, of the three remaining groups, \( p_{2,6}, p_{2,7}, p_{2,8} \):

\[
A_{11} = (121) ; \quad A_{12} = (321) ; \quad A_{13} = (231) ;
\]

with which may be combined these three others, of the same three types, and on the same line \( b'c' \):

\[
A_{11} = (112) ; \quad A_{12} = (312) ; \quad A_{13} = (213).
\]

Considered as intersections of a line \( \Lambda_{2,6} \) with lines \( \Lambda_2 \) in the same plane \( \Pi_i \), or with planes \( \Pi_2 \) (in which latter character alone they belong to the second construction), the three points \( A_{11}, A_{12}, A_{13} \), \( A_{14} \), may be thus denoted:

\[
A_{11} = B'C' \cdot BB' \cdot CB' \cdot AA' = B'C' \cdot BC_1 A_2 A_1 C_2 ;
\]

\[
A_{12} = B'C' \cdot D_1 B' \cdot AB' \cdot A' = B'C' \cdot D_1 C_1 A_1 \cdot D_1 C_2 A_2 ;
\]

\[
A_{13} = B'C' \cdot A'C_5 B_1 V C_1 V B' \cdot BA' V B_1 V_{11} = B'C' \cdot A'C_1 C_2 ;
\]

with the harmonic equation,

\[
(c_0 A_1 V A_{13}) = -1,
\]

and with analogous expressions for the three other points, \( A_{11}, A_{12}, A_{13} \). The line \( b'b' \) thus intersects one plane \( \Pi_{2,1} \) (or its trace \( b'b' \) on the plane \( \Lambda_{2,6} \)), in the point \( A_{11} \); it intersects two planes \( \Pi_{2,2} \) (or their common trace \( d_1 b' \)) in \( A_{12} \); and one other plane \( \Pi_{2,2} \) (or its trace \( A'B' \)) in \( A_{13} \), and similarly for the other points, \( A_{11}, A_{12}, A_{13} \), of the same three groups. Each plane \( \Pi_{2,1} \) contains twelve points \( p_{2,6} \), eight points \( p_{2,7} \), and eight points \( p_{2,8} \); while every plane \( \Pi_{2,2} \) contains six points \( p_{2,6} \), twelve points \( p_{2,7} \), and nine points \( p_{2,8} \). Each point \( p_{2,6} \) is contained in one plane \( \Pi_1 \); in three planes \( \Pi_{2,1} \); and in two planes \( \Pi_{2,2} \). Each point \( p_{2,7} \) is in one plane \( \Pi_1 \), in two planes \( \Pi_{2,1} \), and in four planes \( \Pi_{2,2} \). And each point \( p_{2,8} \) is situated in one plane \( \Pi_1 \), in two planes \( \Pi_{2,1} \), and in three planes \( \Pi_{2,2} \).

(10.) The points of the three last groups are situated only on lines \( \Lambda_{2,6} \); but, on each such line, two points of each of those three groups are situated;
which, along with one point of each of the two former groups, \( p_{21} \) and \( p_{32} \), and with the two points \( p_1 \), whereby the line itself is determined, make up a system of ten points upon that line. For example, the line \( b'c' \) contains, besides the six points mentioned in the last sub-article, the four others:

\[
\begin{align*}
b' &= (101) ; & c' &= (110) ; & \lambda' &= (011) ; & \lambda'' &= (211).
\end{align*}
\]

Of these ten points, the two last mentioned, namely the points \( p_{21} \) and \( p_{32} \) upon the line \( A_{2,2} \), are the double points (comp. sub-art. 8) of a new involution, in which the two points of each of the four other groups compose a conjugate pair, as is expressed by the harmonic equations,

\[
(A''B'A''C') = (A''A''A''A_{1,11}) = (A''A''A''A_{1,1}) = -1.
\]

And the analogous equations,

\[
(B'A''C'A'') = (B'A''C'A_{1,11}) = (B'A''C'A_{1,1}) = -1,
\]

show that the two points \( p_1 \) on any line \( A_{2,2} \) are the double points of another involution (comp. again sub-art. 8), whereof the two points \( p_{21}, p_{32} \) on that line form one conjugate pair, while each of the two points \( p_{21} \) is paired with one of the points \( p_{32} \) as its conjugate. In fact, the eight-rayed pencil \((\alpha \cdot c'B'A''A''A_{1,11})\) coincides in position with the pencil \((\alpha \cdot bca'A''A''A_{1,11})\), and may be said to be a pencil in double involution; the third and fourth, the fifth and sixth, and the seventh and eighth rays forming one involution, whereof the first and second are the two double* rays; while the first and second, the fifth and seventh, and the sixth and eighth rays compose another involution, whereof the double rays are the third and fourth of the pencil.

(11.) If we proceeded to connect systematically the points \( p_n \) among themselves, and with the points \( p_1 \) and \( p_0 \), we should find many remarkable lines and planes of third construction (88), besides those which have been incidentally noticed above; for example, we should have a group \( \Pi_{2,2} \) of twenty new planes, exemplified by the two following,

\[
[b_n] = [11103], \quad [d_n] = [11130],
\]

which have the same common trace \( A_{2,1} \), namely the line \( A''B''C'' \), on the plane \( \alpha bc \), as the two planes \( A_2B_1C_1, A_2B_2C_2 \), and the two planes \([d], [e]\), of the groups \( \Pi_{2,2} \) and \( \Pi_{2,1} \), which have been considered in former sub-articles; and each of these new planes \( \Pi_{2,2} \) would be found to contain one point \( p_0 \), three points

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p_21, six points p_22, and three points p_23. It might be proved also that these twenty new planes are the twenty faces of five new pyramids r_5, which are the exscribed homologues of the five old pyramids r_1 (89), with the five given points p_0 for the corresponding centres of homology. But it would lead us beyond the proposed limits, to pursue this discussion further: although a few additional remarks may be useful, as serving to establish the completeness of the enumeration above given, of the lines, planes, and points of second construction.

93. In general, if there be any n given points, whereof no four are situated in any common plane, the number N of the derived points, which are immediately obtained from them, as intersections A · II of line with plane (each line being drawn through two of the given points, and each plane through three others), or the number of points of the form AB · CDE, is easily seen to be,

\[ N = f(n) = \frac{n(n-1)(n-2)(n-3)(n-4)}{2 \cdot 3} \]

so that \( N = 10 \), as before, when \( n = 5 \). But if we were to apply this formula to the case \( n = 15 \), we should find, for that case, the value,

\[ N = f(15) = 15 \cdot 14 \cdot 13 \cdot 11 = 30030; \]

and thus fifteen given and independent points of space would conduct, by what might (relatively to them) be called a First Construction (comp. 88), to a system of more than thirty thousand points. Yet it has been lately stated (92), that from the fifteen points above called p_0, p_1, there can be derived, in this way, only two hundred and ninety points p_2, as intersections of the form* A · II; and therefore fewer than three hundred. That this reduction of the number of derived points, at the end of what has been called (88) the Second Construction for the net in space, arising from the dependence of the ten points p_1 on the five points p_0, would be found to be so considerable, might not perhaps have been anticipated; and although the foregoing examination proves that all the eight types (92) do really represent points p_2, it may appear possible, at this stage, that some other type of such points has been omitted. A study of the manner in which the types of points result, from those of the lines and planes of which they are the intersections, would indeed decide this question; and it was, in fact, in that way that the eight types, or groups, p_21, ..., p_28, of points of second construction for space, were investigated, and found to be sufficient: yet it

* The definition (88) of the points p_2 admits, indeed, intersections A · A of complanar lines, when they are not already points p_0 or p_1; but all such intersections are also points of the form A · II; so that no generality is lost, by confining ourselves to this last form, as in the present discussion we propose to do.

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may be useful (compare the last sub-art.) to verify, as below, the completeness of the foregoing enumeration.

(1.) The fifteen points, \( p_0, p_1 \), admit of 105 binary, and of 455 ternary combinations; but these are far from determining so many distinct lines and planes. In fact, those 15 points are connected by 25 collineations, represented by the 25 lines \( \Lambda_1, \Lambda_2, \ldots \); which lines therefore count as 75, among the 105 binary combinations of points: and there remain only 30 combinations of this sort, which are constructed by the 30 other lines, \( \Lambda_3 \). Again, there are 25 ternary combinations of points, which are represented (as above) by lines, and therefore do not determine any plane. Also, in each of the ten planes \( \Pi_1 \), there are 29 \((= 35 - 6)\) triangles \( T_1, T_2 \), because each of those planes contains 7 points \( p_0, p_1 \), connected by 6 relations of collinearity. In like manner, each of the fifteen planes \( \Pi_2, \Pi_3 \) contains 8 \((= 10 - 2)\) other triangles \( T_3 \), because it contains 5 points \( p_0, p_1 \), connected by two collineations. There remain therefore only 20 \((= 455 - 25 - 290 - 120)\) ternary combinations of points to be accounted for; and these are represented by the 20 planes \( \Pi_4 \). The completeness of the enumeration of the lines and planes of the second construction is therefore verified; and it only remains to verify that the 305 points, \( p_0, p_1, p_2 \), above considered, represent all the intersections \( A \cdot \Pi \), of the 55 lines \( \Lambda_1, \Lambda_2 \), with the 45 planes \( \Pi_1, \Pi_2 \).

(2.) Each plane \( \Pi_i \) contains three lines of each of the three groups, \( \Lambda_1 \), \( \Lambda_2, \ldots \); each plane \( \Pi_2, \Pi_3 \) contains two lines \( \Lambda_2, \ldots \), and four lines \( \Lambda_3 \); and each plane \( \Pi_4 \) contains three lines \( \Lambda_3 \). Hence (or because each line \( \Lambda_1 \) is contained in three planes \( \Pi_1 \); each line \( \Lambda_2, \ldots \), in two planes \( \Pi_1 \), and in two planes \( \Pi_2 \); and each line \( \Lambda_3 \), in one plane \( \Pi_1 \), in two planes \( \Pi_2, \ldots \), and in two planes \( \Pi_4 \)), it follows that, without going beyond the second construction, there are 240 \((= 30 + 30 + 30 + 30 + 60 + 60)\) cases of coincidence of line and plane; so that the number of cases of intersection is reduced, hereby, from \( 55 \cdot 45 = 2475 \), to 2235 \((= 2475 - 240)\).

(3.) Each point \( p_0 \) represents twelve intersections of the form \( \Lambda_1 \cdot \Pi \); because it is common to four lines \( \Lambda_1 \), and to six planes \( \Pi_1 \), each plane containing two of those four lines, but being intersected by the two others in that point \( p_0 \); as the plane \( \Lambda_0 \), for example, is intersected in \( \Lambda \) by the two lines, \( \Lambda_0 \) and \( \Lambda_0 \). Again, each point \( p_0 \) is common to three planes \( \Pi_1 \), no one of which contains any of the four lines \( \Lambda_1 \) through that point; it represents therefore a system of twelve other intersections, of the form \( \Lambda_1 \cdot \Pi_2 \). Again, each point \( p_0 \) is common to three lines \( \Lambda_2, \ldots \), each of which is contained in two of the six planes \( \Pi_1 \), but intersects the four others in that point \( p_0 \); which
therefore counts as twelve intersections, of the form $A_{21} \cdot \Pi_1$. Finally, each of the points $p_0$ represents three intersections, $A_{21} \cdot \Pi_{21}$; and it represents no other intersection, of the form $A \cdot \Pi$, within the limits of the present inquiry. Thus, each of the five given points is to be considered as representing, or constructing, thirty-nine ($= 12 + 12 + 12 + 3$) intersections of line with plane; and there remain only 2040 ($= 2235 - 195$) other cases of such intersection $A \cdot \Pi$, to be accounted for (in the present verification) by the 300 derived points, $p_1, p_2$.

(4.) For this purpose, the nine columns, headed as I. to IX. in the following Table, contain the numbers of such intersections which belong respectively to the nine forms,

$$A_1 \cdot \Pi_1, A_1 \cdot \Pi_{21}, A_1 \cdot \Pi_{22}; \quad A_{21} \cdot \Pi_1, A_{21} \cdot \Pi_{21}, A_{21} \cdot \Pi_{22};$$

$$A_{22} \cdot \Pi_1, A_{22} \cdot \Pi_{21}, A_{22} \cdot \Pi_{22},$$

for each of the nine typical derived points, $A' \cdots A''$, of the nine groups $p_1, p_{21}, \ldots p_{29}$. Column X. contains, for each point, the sum of the nine numbers, thus tabulated in the preceding columns; and expresses therefore the entire number of intersections, which any one such point represents. Column XI. states the number of the points for each type; and column XII. contains the product of the two last numbers, or the number of intersections $A \cdot \Pi$ which are represented (or constructed) by the group. Finally, the sum of the numbers in each of the two last columns is written at its foot; and because the 300 derived points, of first and second constructions, are thus found to represent the 2040 intersections which were to be accounted for, the verification is seen to be complete: and no new type, of points $p_2$, remains to be discovered.

(5.)

**Table of Intersections $A \cdot \Pi$.**

<table>
<thead>
<tr>
<th>Type</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
<th>IX</th>
<th>X</th>
<th>XI</th>
<th>XII</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A'$</td>
<td>1</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>12</td>
<td>18</td>
<td>18</td>
<td>24</td>
<td>24</td>
<td>115</td>
<td>10</td>
<td>1150</td>
</tr>
<tr>
<td>$A''$</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>3</td>
<td>12</td>
<td>30</td>
<td>10</td>
<td>300</td>
</tr>
<tr>
<td>$A'''$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>0</td>
<td>30</td>
<td>210</td>
</tr>
<tr>
<td>$A''''$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>30</td>
<td>60</td>
</tr>
<tr>
<td>$A'''''$</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>20</td>
<td>60</td>
</tr>
<tr>
<td>$A''''''$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>20</td>
<td>60</td>
</tr>
<tr>
<td>$A'''''''$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>$A''''''''$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>60</td>
<td>60</td>
</tr>
</tbody>
</table>

| | | | | | | | | | | 300 | 2040 |

L 2
(6.) It is to be remembered that we have not admitted, by our definition (88), any points which can only be determined by intersections of three planes \( \Pi_1, \Pi_2 \), as belonging to the second construction: nor have we counted, as lines \( \Lambda_2 \) of that construction, any lines which can only be found as intersections of two such planes. For example, we do not regard the traces \( \Lambda A'' \), &c., of certain planes \( \Lambda_{31} \) considered in recent sub-articles, as among the lines of second construction, although they would present themselves early in an enumeration of the lines \( \Lambda_3 \) of the third. And any point in the plane ABC, which can only be determined (at the present stage) as the intersection of two such traces, is not regarded as a point \( p_2 \). A student might find it however to be not useless, as an exercise, to investigate the expressions for such intersections; and for that reason it may be noted here, that the ternary types (comp. 81) of the forty-four traces of planes \( \Pi_1, \Pi_2 \), on the plane ABC which are found to compose a system of only twenty-two distinct lines in that plane, whereof nine are lines \( \Lambda_1, \Lambda_2 \), are the seven following (comp. 38):
\[
\begin{align*}
[100], & [011], [\bar{1}11], [111], [011], \bar{[}211], [2\bar{1}1];
\end{align*}
\]
which, as ternary symbols, represent the seven lines,
\[
BC, \ AA', \ B'C', \ AA'', \ D_1A'', \ A'C_0.
\]

(7.) Again, on the same principle, and with reference to the same definition, that new point, say \( f \), which may be denoted by either of the two congruent quinary symbols (71),
\[
f = (43210) = (01234),
\]
and which, as a quinary type (78), represents a new group of sixty points of space (and of no more, on account of this last congruence, whereas a quinary type, with all its five coefficients unequal, represents generally a group of 120 distinct points), is not regarded by us as a point \( p_2 \); although this new point \( f \) is easily seen to be the intersection of three planes of second construction, namely, of the three following, which all belong to the group \( \Pi_{31} \):
\[
[01111], \ \bar{[}11011], \ [1\bar{1}110].
\]
or \( \Lambda A'D_1C_2, \ CC'D_1B_1A_2, \ BB'D_1C_2 \). It may, however, be remarked in passing, that each plane, \( \Pi_{31} \), contains twelve points \( p_3 \) of this new group: every such point being common (as is evident from what has been shown) to three such planes.

94. From the foregoing discussion it appears that the five given points \( p_0 \), and the three hundred derived points \( p_1, p_2 \), are arranged in space, upon the fifty-
five lines \( \Lambda_1, \Lambda_2 \), and in the forty-five planes \( \Pi_1, \Pi_2 \), as follows. Each line \( \Lambda_1 \) contains eight of the 305 points, forming on it what may be called (see the sub-article (8.) to 92) a double involution. Each line \( \Lambda_2 \), contains seven points, whereof one, namely the given point, \( p_0 \), has been seen (in the earlier sub-art. (6.)) to be a double point of another involution, to which the three derived pairs of points, \( p_1, p_2 \), on the same line belong. And each line \( \Lambda_2 \) contains ten points, forming on it a new involution; while eight of these ten points, with a different order of succession, compose still another involution* (92, (10.)).

Again, each plane \( \Pi_1 \) contains fifty-two points, namely three given points, four points of first, and 45 points of second construction. Each plane \( \Pi_2 \), contains forty-seven points, whereof one is a given point, four are points \( p_1 \), and 42 are points \( p_2 \): of which last, 38 are situated on the six lines \( \Lambda_2 \) in the plane,

* These theorems respecting the relations of involution, of given and derived points on lines of first and second constructions, for a net in space, are perhaps new; although some of the harmonic relations, above mentioned, have been noticed under other forms by Möbius: to whom, indeed, as has been stated, the conception of such a net is due. Thus, if we consider (compare the note to page 66) the two intersections,

\[ E_1 = DE \cdot A_1B_1C_1, \quad E_2 = DE \cdot A_2B_2C_2, \]

we easily find that they may be denoted by the quinary symbols,

\[ E_1 = (00012), \quad E_2 = (00021); \]

they are, therefore, by Art. 92, the two points \( p_{2,5} \) on the line \( DE \): and consequently, by the theorem stated at the end of sub-art. (8.), the harmonic conjugate of each, taken with respect to the other and to the point \( A_1 \), must be one of the two points \( a, b \) on that line. Accordingly, we soon derive, by comparison of the symbols of these five points, \( D_1E_1E_2B_2 \), the two following harmonic equations, which belong to the same type as the two last of that sub-art. (8.):

\[ (D_1E_1E_2B_2) = -1; \quad (D_1E_1E_2E_2) = -1; \]

but these two equations have been assigned (with notations slightly different) in the formerly cited page 290 of the Barycentric Calculus. (Comp. again the recent note to page 66.) The geometrical meaning of the last equation may be illustrated, by conceiving that \( ABCD \) is a regular pyramid, and that \( b \) is its mean point; for then (comp. 92, sub-art. (2.), \( d_1 \) is the mean point of the base \( ABC \); \( d_1b \) is the altitude of the pyramid; and the three segments \( d_1b, d_1b_1, d_1b_2 \) are, respectively, the quarter, the third part, and the half of that altitude; they compose therefore (as the formula expresses) a harmonic progression; or \( d_1 \) and \( b_1 \) are conjugate points, with respect to \( b \) and \( b_2 \). But in order to exemplify the double involution of the same sub-art. (8.), it would be necessary to consider three other points \( p_2 \), on the same line \( DE \); whereof one, above called \( d_1 \), belongs to a known group \( p_{2,1} \) (92, (2.)); but the other two are of the group \( p_{2,4} \), and do not seem to have been previously noticed. As an example of an involution on a line of third construction, it may be remarked that on each line of the group \( \Lambda_3, \) or on each of the sides of any one of the ten triangles \( T_{3,2} \), in addition to one given point \( p_0 \), and one derived point \( p_{2,1} \), there are two points \( p_{2,3} \) and two points \( p_{2,5} \); and that the two first points are the double points of an involution, to which the two last pairs belong: thus, on the side \( A_2B_2C_2 \) of the exscribed triangle \( A_2B_2C_2 \), or on the trace of the plane \( BC_1A_0A_0C_0 \), we have the two harmonic equations,

\[ (BA_2B_2')C_0) = (BA_2''B_2'C_0') = -1. \]

Again, on the trace \( A'CO \) of the plane \( A'O_1C_0 \) (which latter trace is a line not passing through any one of the given points), \( c_0 \) and \( n'' \) are the double points of an involution, wherein \( A' \) is conjugate to \( C_0 \) and \( A'X \) to \( n'' \). But it would be tedious to multiply such instances.
but four are intersections of that plane $\Pi_{21}$ with four other lines of second construction. Finally, each plane $\Pi_{22}$ passes through no given point, but contains forty-three derived points, whereof 40 are points of second construction. And because the planes of first construction alone contain specimens of all the ten groups of points, $p_0, p_1, p_{21}, \ldots p_{28}$, given or derived, and of all the three groups of lines, $A_1, A_{21}, A_{32}$, at the close of that second construction (since the types $p_{24}, p_{28}, A_1$ are not represented by any points or lines in any plane $\Pi_{21}$, nor are the types $p_9, A_1, A_{21}$ represented in a plane $\Pi_{22}$), it has been thought convenient to prepare the annexed diagram (fig. 30), which may serve to illustrate, by some selected instances, the arrangement of the fifty-two points $p_0, p_1, p_2$ in a plane $\Pi$, namely, in the plane $\text{ABC}$; as well as the arrangement of the nine lines $A_1, A_2$ in that plane, and the traces $\Lambda_3$ of other planes upon it.

View of the Arrangement of the Principal Points and Lines in a Plane of First Construction.

In this figure, the triangle $\text{ABC}$ is supposed, for simplicity, to be the equilateral base of a regular pyramid $\text{ABCD}$ (comp. sub-art. (2.) to 92); and $\text{D_1}$, again replaced by $\text{O}$, is supposed to be its mean point (29). The first inscribed triangle, $\triangle \text{A'B'C'}$, therefore, bisects the three sides; and the axis of homology $\Lambda'' \text{A'BC}''$ is the line at infinity (38): the number 1, on the line $\text{C'B'}$ prolonged, being designed to suggest that the point $\Lambda''$, to which that line tends, is of the type $r_{21}$, or belongs to the first group of points of second construction. A
second inscribed triangle, \( \triangle \overline{a''b''c''} \), for which fig. 21 may be consulted, is only indicated by the number 2 placed at the middle of the side \( \overline{b'c'} \), to suggest that this bisecting point \( \overline{a''} \) belongs to the second group of \( p_3 \). The same number 2, but with an accent, \( 2' \), is placed near the corner \( A_0 \) of the exscribed triangle \( A_0B_0C_0 \), to remind us that this corner also belongs (by a syntypical relation in space) to the group \( p_{3_2} \). The point \( A_{1''} \), which is now infinitely distant, is indicated by the number 3, on the dotted line at the top; while the same number with an accent, lower down, marks the position of the point \( A_{1'''} \).

Finally, the ten other numbers, unaccented or accented, \( 4, 4', 5, 5', 6, 6', 7, 7', 8, 8' \), denote the places of the ten points, \( A'_{1}, A_{1}'_{1}, A''_{1',1}, A_{1''}, A_1''_{1}, A_1''_{1}, A_1''_{1}, A_1''_{1}, A_1''_{1} \). And the principal harmonic relations, and relations of involution, above mentioned, may be verified by inspection of this Diagram.

95. However far the series of construction of the net in space may be continued, we may now regard it as evident, at least on comparison with the analogous property \((42)\) of the plane net, that every point, line, or plane, to which such constructions can conduct, must necessarily be rational \((77)\); or that it must be rationally related to the system of the five given points: because the anharmonic co-ordinates \((79, 80)\) of every net-point, and of every net-plane, are equal or proportional to whole numbers. Conversely \((\text{comp. } 43)\) every point, line, or plane, in space, which is thus rationally related to the system of points \( ABCDE \), is a point, line, or plane of the net, which those five points determine. Hence \((\text{comp. again } 43)\), every irrational point, line, or plane \((77)\), is indeed incapable of being rigorously constructed, by any processes of the kind above described: but it admits of being indefinitely approximated to, by points, lines, or planes of the net. Every anharmonic ratio, whether of a group of net-points, or of a pencil of net-lines, or of net-planes, has a rational value \((\text{comp. } 44)\), which depends only on the processes of linear construction employed, in the generation of that group or pencil, and is entirely independent of the arrangement, or configuration, of the five given points in space. Also, all relations of collineation, and of complanarity, are preserved, in the passage from one net to another, by a change of the given system of points: so that it may be briefly said \((\text{comp. again } 44)\) that all geometrical nets in space are homographic figures.

Finally, any five points* of such a net, of which no four are in one plane, are

* These general properties \((95)\) of the space-net are in substance taken from Möbius, although (as has been remarked before) the analysis here employed appears to be new: as do also most of the theorems above given, respecting the points of second construction \((92)\), at least after we pass beyond the first group \( p_{3_1} \) of ten such points, which (as already stated) have been known comparatively long.
sufficient (comp. 45) for the determination of the whole net, or for the linear
construction of all its points, including the five given ones.

(I.) As an Example, let the five points $A_1B_1C_1D_1$ and $e$ be now supposed to
be given; and let it be required to derive the four points $abcd$, by linear con­
structions, from these new data. In other words, we are now required to
escribe a pyramid $ABCD$ to a given pyramid $A_1B_1C_1D_1$, so that it may be homolo­
gous thereto, with the point $e$ for their given centre of homology. An
obvious process is (comp. 45) to inscribe another homologous pyramid, $A_3B_3C_3D_3$,
so as to have $A_3 = EA_1 \cdot B_1C_1D_1$, &c.; and then to determine the intersections of
corresponding faces, such as $A_1B_1C_1$ and $A_3B_3C_3$; for these four lines of intersection
will be in the common plane $[E]$ of homology of the three pyramids, and will be
the traces on that plane of the four sought planes, $ABC$, &c., drawn through the
four given points $D_1$, &c. If it were only required to construct one corner $A$
of the exscribed pyramid, we might find the point above called $A^{IV}$ as the
common intersection of three planes, as follows,

$$A^{IV} = A_1B_1C_1 \cdot A_1D_1E \cdot A_3B_3C_3;$$

and then should have this other formula of intersection,

$$A = EA_1 \cdot D_1A^{IV}.$$

Or the point $A$ might be determined by the anharmonic equation,

$$\left( EA_1A_3 \right) = 3,$$

which for a regular pyramid is easily verified.

(2.) As regards the general passage from one net in space to another, let
the symbols $P_1 = (x_1 y_1 z_1 v_1)$, \ldots $P_5 = (x_5 y_5 z_5 v_5)$ denote any five given points, whereof
no four are co-planar; and let $a'b'c'd'e'$ and $u'$ be six coefficients, of which
the five ratios are such as to satisfy the symbolical equation (comp. 71, 72),

$$(a' (P_1) + b' (P_2) + c' (P_3) + d' (P_4) + e' (P_5) = -u' (U));$$

or the five ordinary equations which it includes, namely,

$$a'x_1 + \ldots + e'x_5 = \ldots a'v_1 + \ldots + e'v_5 = -u'.$$

Let $\mathbf{P}'$ be any sixth point of space, of which the quinary symbol satisfies the
equation,

$$(\mathbf{P}') = xa' (P_1) + yb' (P_2) + ze' (P_3) + w^d (P_4) + ve' (P_5) + u' (U);$$

then it will be found that this last point $\mathbf{P}'$ can be derived from the five
points $P_1 \ldots P_5$ by precisely the same constructions, as those by which the
point $\mathbf{P} = (xyzuv)$ is derived from the five points $ABCD$. As an example, if
$$v' = x + y + z + w - 3v,$$
then the point $(xyzuv')$ is derived from $A_1B_1C_1D_1E$,  

$$A_1 = A_3 = EA_1 \cdot B_1C_1D_1E = EA_2 \cdot B_2C_2D_2E;$$
and then should have this other formula of intersection,

$$A = EA_1 \cdot D_1A'^{IV}.$$
by the same constructions as \((xyzwv)\) from \(ABCDE\); thus \(A\) itself may be constructed from \(A_1 \ldots \, \ell\), as the point \(P = (30001)\) is from \(A \ldots \, F\); which would conduct anew to the anharmonic equation of the last sub-article.

(3.) It may be briefly added here, that instead of anharmonic ratios, as connected with a net in space, or indeed generally in relation to spatial problems, we are permitted (comp. 68) to substitute products (or quotients) of quotients of volumes of pyramids; as a specimen of which substitution, it may be remarked, that the anharmonic relation, just referred to, admits of being replaced by the following equation, involving one such quotient of pyramids, but introducing no auxiliary point:

\[
EA : A_1A = 3EB_1C_1D_1 : A_1B_1C_1D_1.
\]

In general, if \(xyzw\) be (as in 79, 83) the anharmonic co-ordinates of a point \(P\) in space, we may write,

\[
\frac{x}{y} = \frac{PBCD}{PCDA} : \frac{EBCD}{ECDA};
\]

with other equations of the same type, on which we cannot here delay.

SECTION 5.

On Barycentres of Systems of Points; and on Simple and Complex Means of Vectors.

96. In general, when the sum \(\Sigma a\) of any number of co-initial vectors,

\[
a_1 = OA_1, \ldots \quad a_m = OA_m,
\]

is divided (16) by their number, \(m\), the resulting vector,

\[
\mu = OM = \frac{1}{m} \Sigma a = \frac{1}{m} \Sigma OA,
\]

is said to be the Simple Mean of those \(m\) vectors; and the point \(M\), in which this mean vector terminates, and of which the position (comp. 18) is easily seen to be independent of the position of the common origin \(O\), is said to be the Mean Point (comp. 29), of the system of the \(m\) points, \(A_1, \ldots A_m\). It is evident that we have the equation,

\[
0 = (a_1 - \mu) + \ldots + (a_m - \mu) = \Sigma (a - \mu) = \Sigma MA;
\]

or that the sum of the \(m\) vectors, drawn from the mean point \(M\), to the points \(A\) of the system, is equal to zero. And hence (comp. 10, 11, 30), it follows 1st, that these \(m\) vectors are equal to the \(m\) successive sides of a closed polygon; 2nd, that if the system and its mean point be projected, by any parallel
ordinates, on any assumed plane (or line), the projection $M'$, of the mean point $M$, is the mean point of the projected system: and IIIrd, that the ordinate $MM'$, of the mean point, is the mean of all the other ordinates, $A_1A'_1, \ldots A_mA'_m$. It follows, also, that if $N$ be the mean point of another system, $b_1, \ldots b_n$; and if $S$ be the mean point of the total system, $A_1 \ldots b_m$, of the $m + n = s$ points obtained by combining the two former, considered as partial systems; while $\nu$ and $\sigma$ may denote the vectors, on and $os$, of these two last mean points: then we shall have the equations,

$$m\nu = \Sigma a, \quad n\nu = \Sigma b, \quad s\sigma = \Sigma a + \Sigma b = m\nu + n\nu,$$

$$m(\sigma - \mu) = n(\nu - \sigma), \quad m \cdot ms = n \cdot sn;$$

so that the general mean point, $S$, is situated on the right line $MN$, which connects the two partial mean points, $M$ and $N$; and divides that line (internally), into two segments $ms$ and $sn$, which are inversely proportional to the two whole numbers, $m$ and $n$.

(1.) As an Example, let $ABCDEFG$ be a gauche quadrilateral, and let $E$ be its mean point; or more fully, let

$$OE = \frac{1}{4} (OA + OB + OC + OD),$$

or

$$e = \frac{1}{4} (a + \beta + \gamma + \delta);$$

that is to say, let $a = b = c = d$, in the equations of Art. 65. Then, with notations lately used, for certain derived points $d_1$, &c., if we write the vector formula,

$$OA_1 = a_1 = \frac{1}{2} (\beta + \gamma + \delta), \ldots \quad \delta_1 = \frac{1}{2} (a + \beta + \gamma),$$

$$OA_2 = a_2 = \frac{1}{2} (a + \delta), \ldots \quad \gamma_2 = \frac{1}{2} (\gamma + \delta),$$

$$OA' = a' = \frac{1}{2} (\beta + \gamma), \ldots \quad \gamma' = \frac{1}{2} (a + \beta),$$

we shall have seven different expressions for the mean vector, $e$; namely, the following:

$$e = \frac{1}{4} (a + 3a_1) = \ldots \frac{1}{4} (\delta + 3\delta_1)$$

$$= \frac{1}{2} (a' + a_2) = \ldots \frac{1}{2} (\gamma' + \gamma_2).$$

And these conduct to the seven equations between segments,

$$AE = 3EA_1, \ldots \quad DB = 3ED_1;$$

$$A'E = EA_2, \ldots \quad C'E = EC_2;$$

which prove (what is otherwise known) that the four right lines, here denoted by $AA_1, \ldots DD_1$, whereof each connects a corner of the pyramid $ABCD$ with the mean point of the opposite face, intersect and quadrisect each other, in one common point, $E$; and that the three common bisectors $A'A_2, B'B_2, C'C_2$, of pairs of
opposite edges, such as BC and DA, intersect and bisect each other, in the same mean point: so that the four middle points, c', a', c2, a2, of the four successive sides AB, &c., of the gauche quadrilateral ABCD, are situated in one common plane, which bisects also the common bisector, B'B2, of the two diagonals, AC and BD.

(2.) In this example, the number s of the points A . . . D being four, the number of the derived lines, which thus cross each other in their general mean point e is seen to be seven; and the number of the derived planes through that point is nine: namely, in the notation lately used for the net in space, four lines A1, three lines A2, six planes II, and three planes II2. Of these nine planes, the six former may (in the present connexion) be called triple planes, because each contains three lines (as the plane ABE, for instance, contains the lines AA1, BB1, c'c2), all passing through the mean point e; and the three latter may be said, by contrast, to be non-triple planes, because each contains only two lines through that point, determined on the foregoing principles.

(3.) In general, let \( \phi (s) \) denote the number of the lines, through the general mean points s of a total system of s given points, which is thus, in all possible ways, decomposed into partial systems; let \( f(s) \) denote the number of the triple planes, obtained by grouping the given points into three such partial systems; let \( \psi (s) \) denote the number of non-triple planes, each determined by grouping those s points in two different ways into two partial systems; and let \( f(s) = f(s) + \psi (s) \) represent the entire number of distinct planes through the point s: so that

\[
\phi (4) = 7, \quad f(4) = 6, \quad \psi (4) = 3, \quad f(4) = 9.
\]

Then it is easy to perceive that if we introduce a new point c, each old line mn furnishes two new lines, according as we group the new point with one or other of the two old partial systems, (M) and (N); and that there is, besides, one other new line, namely cs: we have, therefore, the equation in finite differences,

\[
\phi (s + 1) = 2\phi (s) + 1;
\]

which, with the particular value above assigned for \( \phi (4) \), or even with the simpler and more obvious value, \( \phi (2) = 1 \), conducts to the general expression,

\[
\phi (s) = 2^{s-1} - 1.
\]

(4.) Again, if (M) (N) (P) be any three partial systems, which jointly make up the old or given total system (S); and if, by grouping a new point a with each of these in turn, we form three new partial systems, (M') (N') (P'); then each old triple plane such as MNP, will furnish three new triple planes,

\[ M'NP, \quad MN'P, \quad MNP'. \]
while each old line, \( \kappa \ell \), will give one new triple plane, \( \kappa \ell \kappa \ell \); nor can any new triple plane be obtained in any other way. We have, therefore, this new equation in differences:

\[
f(s + 1) = 3f(s) + \phi(s).
\]

But we have seen that
\[
\phi(s + 1) = 2\phi(s) + 1;
\]
if then we write, for a moment,
\[
f(s) + \phi(s) = \chi(s),
\]
we have this other equation in finite differences,
\[
\chi(s + 1) = 3\chi(s) + 1.
\]

Also,
\[
f(3) = 1, \quad \phi(3) = 3, \quad \chi(3) = 4:
\]
therefore,
\[
2\chi(s) = 3^{s-1} - 1,
\]
and
\[
2f(s) = 3^{s-1} - 2^s + 1.
\]

(5.) Finally, it is clear that we have the relation,

\[
3f(s) + \psi(s) = \frac{1}{2}\phi(s) \cdot (\phi(s) - 1) = (2^{s-1} - 1) (2^{s-2} - 1);
\]

because the triple planes, each treated as three, and the non-triple planes, each treated as one, must jointly represent all the binary combinations of the lines, drawn through the mean point \( s \) of the whole system. Hence,

\[
2\psi(s) = 2^{s-2} + 3 \cdot 2^{s-1} - 3^s - 1;
\]

and
\[
x(s + 1) - 4x(s) = 3^{s-1} - 2^{s-1},
\]
and
\[
\psi(s + 1) - 4\psi(s) = 3f(s);
\]

which last equation in finite differences admits of an independent geometrical interpretation.

(6.) For instance, these general expressions give,

\[
\phi(5) = 15; \quad f(5) = 25; \quad \psi(5) = 30; \quad x(5) = 55;
\]

so that if we assume a gauche pentagon, or a system of five points in space, \( \mathbf{A} \ldots \mathbf{E} \), and determine the mean point \( x \) of this system, there will in general be a set of fifteen lines, of the kind above considered, all passing through this sixth point \( x \): and these will be arranged generally in fifty-five distinct planes, whereof twenty-five will be what we have called triple, the thirty others being of the non-triple kind.

97. More generally, if \( a_1 \ldots a_m \) be, as before, a system of \( m \) given and co-initial vectors, and if \( a_1 \ldots a_m \) be any system of \( m \) given scalars (17), then that
new co-initial vector $\beta$, or $ob$, which is deduced from these by the formula,

$$\beta = \frac{a_1 a + \ldots + a_m a_m}{a_1 + \ldots + a_m} = \frac{\sum a_a}{\sum a}, \quad \text{or} \quad ob = \frac{\sum a_o A}{\sum A},$$

or by the equation

$$\sum a (a - \beta) = 0, \quad \text{or} \quad \sum ABA = 0,$$

may be said to be the Complex Mean of those $m$ given vectors $a$, or $oa$, considered as affected (or combined) with that system of given scalars, $a$, as coefficients, or as multipliers (12, 14). It may also be said that the derived point $b$, of which (comp. 96) the position is independent of that of the origin $o$, is the Barycentre (or centre of gravity) of the given system of points $a_1, \ldots$, considered as loaded with the given weights $a_1, \ldots$; and theorems of intersections of lines and planes arise, from the comparison of these complex means, or barycentres, of partial and total systems, which are entirely analogous to those lately considered (96), for simple means of vectors and of points.

(1.) As an example, in the case of Art. 24, the point $c$ is the barycentre of the system of the two points, $a$ and $b$, with the weights $a$ and $b$; while, under the conditions of 27, the origin $o$ is the barycentre of the three points $a, b, c$, with the three weights $a, b, c$; and if we use the formula for $p$, assigned in 34 or 36, the same three given points $a, b, c$, when loaded with $xa, yb, zc$ as weights, have the point $p$ in their plane for their barycentre. Again, with the equations of 65, $e$ is the barycentre of the system of the four given points, $a, b, c, d$, with the weights $a, b, c, d$; and if the expression of 79 for the vector $op$ be adopted, then $xa, yb, zc, wd$ are equal (or proportional) to the weights with which the same four points $a, b, c, d$ must be loaded, in order that the point $p$ of space may be their barycentre. In all these cases, the weights are thus proportional (by 69) to certain segments, or areas, or volumes, of kinds which have been already considered; and what we have called the anharmonic co-ordinates of a variable point $p$, in a plane (36), or in space (79), may be said, on the same plan, to be quotients of quotients of weights.

(2.) The circumstance that the position of a barycentre (97), like that of a simple mean point (96), is independent of the position of the assumed origin of vectors, might induce us (comp. 69) to suppress the symbol $o$ of that arbitrary and foreign point; and therefore to write* simply, under the lately supposed conditions,

$$b = \frac{\sum a A}{\sum a} \quad \text{or} \quad ob = \sum a A, \quad \text{if} \quad b = \sum a.$$

* We should thus have some of the principal notations of the Barycentric Calculus: but used mainly with a reference to vectors. Compare the note to page 50.
It is easy to prove (comp. 96), by principles already established, that the ordinate of the barycentre of any given system of points is the complex mean (in the sense above defined, and with the same system of weights), of the ordinates of the points of that system, with reference to any given plane: and that the projection of the barycentre, on any such plane, is the barycentre of the projected system.

(3.) Without any reference to ordinates, or to any foreign origin, the barycentric notation \( \mathbf{b} = \frac{\Sigma a_A}{\Sigma a} \) may be interpreted, by means of our fundamental convention (Art. 1) respecting the geometrical signification of the symbol \( \mathbf{b} - \mathbf{a} \), considered as denoting the vector from \( \mathbf{a} \) to \( \mathbf{b} \): together with the rules for multiplying such vectors by scalars (14, 17), and for taking the sums (6, 7, 8, 9) of those (generally new) vectors, which are (15) the products of such multiplications. For we have only to write the formula as follows,

\[
\Sigma a (\mathbf{A} - \mathbf{b}) = 0,
\]

in order to perceive that it may be considered as signifying, that the system of the vectors from the barycentre \( \mathbf{b} \), to the system of the given points \( \mathbf{A}_1, \mathbf{A}_2, \ldots \) when multiplied respectively by the scalars (or coefficients) of the given system \( a_1, a_2, \ldots \) becomes (generally) a new system of vectors with a null sum: in such a manner that these last vectors, \( a_1 . \mathbf{BA}_1, a_2 . \mathbf{BA}_2, \ldots \) can be made (10) the successive sides of a closed polygon, by transports without rotation.

(4.) Thus if we meet the formula,

\[
\mathbf{B} = \frac{1}{2} (\mathbf{A}_1 + \mathbf{A}_2),
\]

we may indeed interpret it as an abridged form of the equation,

\[
\mathbf{OB} = \frac{1}{2} (\mathbf{oA}_1 + \mathbf{oA}_2);
\]

which implies that if \( o \) be any arbitrary point, and if \( o' \) be the point which completes (comp. 6) the parallelogram \( \mathbf{A}_1 o \mathbf{A}_2 o' \), then \( \mathbf{B} \) is the point which bisects the diagonal \( oo' \), and therefore also the given line \( \mathbf{A}_1 \mathbf{A}_2 \), which is here the other diagonal. But we may also regard the formula as a mere symbolical transformation of the equation,

\[
(\mathbf{A}_2 - \mathbf{B}) + (\mathbf{A}_1 - \mathbf{B}) = 0;
\]

which (by the earliest principles of the present Book) expresses that the two vectors, from \( \mathbf{B} \) to the two given points \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \), have a null sum; or that they are equal in length, but opposite in direction: which can only be, by \( \mathbf{B} \) bisecting \( \mathbf{A}_1 \mathbf{A}_2 \), as before.
(5.) Again, the formula, $b_1 = \frac{1}{3} (a_1 + a_2 + a_3)$, may be interpreted as an abridgment of the equation,

$$OB_1 = \frac{1}{3} (OA_1 + OA_2 + OA_3),$$

which expresses that the point $b$ trisects the diagonal $oo'$ of the parallelepiped (comp. 62), which has $OA_1$, $OA_2$, $OA_3$ for three co-initial edges. But the same formula may also be considered to express, in full consistency with the foregoing interpretation, that the sum of the three vectors, from $b$ to the three points $A_1, A_2, A_3$, vanishes: which is the characteristic property (30) of the mean point of the triangle $A_1A_2A_3$. And similarly in more complex cases: the legitimacy of such transformations being here regarded as a consequence of the original interpretation (1) of the symbol $b - a$, and of the rules for operations on vectors, so far as they have been hitherto established.

SECTION 6.

On Anharmonic Equations, and Vector Expressions of Surfaces and Curves in Space.

98. When, in the expression 79 for the vector $p$ of a variable point $p$ of space, the four variable scalars, or anharmonic co-ordinates, $xyzw$, are connected (comp. 46) by a given algebraic equation,

$$f_p (x, y, z, w) = 0,$$

supposed to be rational and integral, and homogeneous of the $p^{th}$ dimension, then the point $p$ has for its locus a surface of the $p^{th}$ order, whereof $f = 0$ may be said (comp. 56) to be the local equation. For if we substitute instead of the co-ordinates $x, y, z, w$, expressions of the forms,

$$X = tx_0 + UX_1, \quad \ldots \quad W = tw_0 + UW_1,$$

to indicate (82) that $p$ is collinear with two given points $p_0, p_1$, the resulting algebraic equation in $t; u$ is of the $p^{th}$ degree; so that (according to a received modern mode of speaking), the surface may be said to be cut in $p$ points (distinct or coincident, and real or imaginary*), by any arbitrary right line $p_0 p_1$.

* It is to be observed, that no interpretation is here proposed, for imaginary intersections of this kind, such as those of a sphere with a right line, which is wholly external thereto. The language of modern geometry requires that such imaginary intersections should be spoken of, and even that they should be enumerated: exactly as the language of algebra requires that we should count what are called the imaginary roots of an equation. But it would be an error to confound geometrical imaginaries, of this sort, with those square roots of negatives, for which it will soon be seen that the Calculus of Quaternions supplies, from the outset, a definite and real interpretation.
And in like manner, when the four anharmonic co-ordinates $l m n r$ of a variable plane $\Pi$ (80) are connected by an algebraical equation of the form,

$$f_q (l, m, n, r) = 0,$$

where $f$ denotes a rational and integral function, supposed to be homogenous of the $q^{th}$ dimension, then this plane $\Pi$ has for its envelope (comp. 56) a surface of the $q^{th}$ class, with $f = 0$ for its tangential equation: because if we make

$$l = t_{l_0} + u_{l_1}, \ldots \quad r = t_{r_0} + u_{r_1},$$

to express (comp. 82) that the variable plane $\Pi$ passes through a given right line $\Pi_0 \cdot \Pi_1$, we are conducted to an algebraical equation of the $q^{th}$ degree, which gives $q$ (real or imaginary) values for the ratio $t : u$, and thereby assigns $q$ (real or imaginary*) tangent planes to the surface, drawn through any such given but arbitrary right line. We may add (comp. 51, 56), that if the functions $f$ and $f$ be only homogeneous (without necessarily being rational and integral), then

$$[D_x f, D_y f, D_z f, D_w f]$$

is the anharmonic symbol (80) of the tangent plane to the surface $f = 0$, at the point $(xyzw)$; and that

$$(D_x F, D_y F, D_z F, D_w F)$$

is in like manner, a symbol for the point of contact of the plane $[l m n r]$, with its enveloped surface, $f = 0$; $D_x, \ldots D_z, \ldots$ being characteristics of partial derivation.

(1.) As an example, the surface of the second order, which passes through the nine points called lately

$$A, A', B, C, C_2, D, A_2, E,$$

has for its local equation,

$$0 = f = xz - yw;$$

which gives, by differentiation,

$$l = D_x f = z; \quad m = D_y f = - w;$$

$$n = D_z f = x; \quad r = D_w f = - y;$$

so that

$$[z, - w, x, - y]$$

is a symbol for the tangent plane, at the point $(x, y, z, w)$.

* As regards the uninterpreted character of such imaginary contacts in geometry, the preceding note to the present Article, respecting imaginary intersections, may be consulted.
(2.) In fact, the surface here considered is the ruled (or hyperbolic) hyperboloid, on which the gauche quadrilateral $ABCD$ is superscribed, and which passes also through the point $e$. And if we write

$$p = (xyzw), \quad q = (xy00), \quad r = (0yz0), \quad s = (00zw), \quad t = (x00w),$$

then $qs$ and $rt$ (see the annexed figure 31), namely, the lines drawn through $p$ to intersect the two pairs, $AB$, $CD$, and $BC$, $DA$, of opposite sides of that quadrilateral $ABCD$, are the two generating lines, or generatrices, through that point; so that their plane, $qrst$, is the tangent plane to the surface, at the point $p$. If, then, we denote that tangent plane by the symbol $[lmnr]$, we have the equations of condition,

$$0 = lx + my = my + nz = nz + rw = rw + lx;$$

whence follows the proportion,

$$l : m : n : r = x^{-1} : y^{-1} : z^{-1} : w^{-1};$$

or, because $xz = yw$,

$$l : m : n : r = z : w : y : x,$$

as before.

(3.) At the same time we see that

$$-w \quad (ac'bq) \quad y \quad -w$$

so that the variable generatrix $qs$ divides (as is known) the two fixed generatrices $AB$ and $DC$ homographically*; $AD$, $BC$, and $c'c_2$ being three of its positions. Conversely, if it were proposed to find the locus of the right line $qs$, which thus divides homographically (comp. 26) two given right lines in space, we might take $AB$ and $DC$ for those two given lines, and $AD$, $BC$, $c'c_2$ (with the recent meanings of the letters) for three given positions of the variable line; and then should have, for the two variable but corresponding (or homologous) points $q$, $s$ themselves, and for any arbitrary point $p$ collinear with them, anharmonic symbols of the forms,

$$q = (s, u, 0, 0), \quad s = (0, 0, u, s), \quad p = (st, tu, uv, vs);$$

because, by 82, we should have, between these three symbols, a relation of the form

$$(p) = \tau (q) + \nu (s);$$

if then we write $p = (x, y, z, w)$, we have the anharmonic equation $xz = yw$, as

* Compare 298 of the Géométrie Supérieure.

Hamilton's Elements of Quaternions.
before; so that the locus, whether of the line qs, or of the point p, is (as is known) a ruled surface of the second order.

(4.) As regards the known double generation of that surface, it may suffice to observe that if we write, in like manner,

\[ r = (0tv0), \quad \mathbf{T} = (t0w0), \quad (\mathbf{p}) = u (r) + s (t), \]

we shall have again the expression,

\[ \mathbf{p} = (st, tu, uv, vs), \quad \text{giving} \quad xz = yw, \]

as before: so that the same hyperboloid is also the locus of that other line \( rt \), which divides the other pair of opposite sides \( bc, ad \) of the same gauche quadrilateral \( abcd \) homographically; \( ba, cd, \) and \( \alpha' \alpha_2 \) being three of its positions; and the lines \( \alpha' \alpha_2, \alpha' \alpha_2 \) being still supposed to intersect each other in the given point \( e \).

(5.) The symbol of an arbitrary point on the variable line \( rt \) is (by sub-art. 2) of the form, \( t (0, y, z, 0) + u (x, 0, 0, w) \), or \( (ux, ty, tz, uw) \); while the symbol of an arbitrary point on the given line \( c'c_2 \) is \( (t', t', u', u') \). And these two symbols represent one common point (comp. fig. 31),

\[ \mathbf{p'} = rt \cdot c'c_2 = (y, y, z, z), \]

when we suppose \( t' = y, \quad u' = z, \quad t = 1, \quad u = \frac{y}{w} = \frac{z}{w}. \)

Hence the known theorem results, that a variable generatrix, \( rt \), of one system, intersects three fixed lines, \( bc, ad, c'c_2 \), which are generatrices of the other system. Conversely, by the same comparison of symbols, for points on the two lines \( rt \) and \( c'c_2 \), we should be conducted to the equation \( xz = yw \), as the condition for their intersection; and thus should obtain this other known theorem, that the locus of a right line, which intersects three given right lines in space, is generally an hyperboloid with those three lines for generatrices. A similar analysis shows that qs intersects \( \alpha' \alpha_2 \), in a point (comp. again fig. 31) which may be thus denoted:

\[ \mathbf{p''} = qs \cdot \alpha' \alpha_2 = (xyyx). \]

(6.) As another example of the treatment of surfaces by their anharmonic and local equations, we may remark that the recent symbols for \( \mathbf{p'} \) and \( \mathbf{p''} \), combined with those of sub-art. (2.) for \( r, q, s, t \); with the symbols of 83, 86 for \( c', \alpha', c_2, \alpha_2, e \); and with the equation \( xz = yw \), give the expressions:

\[ (\mathbf{p}) = (q) + (s) = (r) + (t); \quad (\mathbf{p'}) = y (c') + z (c_2) = (r) + \frac{y}{w} (t); \]

\[ (\mathbf{q}) = (c') + (c_2) = (\alpha') + (\alpha_2); \quad (\mathbf{p''}) = y (\alpha') + x (\alpha_2) = (q) + \frac{y}{w} (s); \]
whence it follows (84) that the two points $P', P''$, and the sides of the quadrilateral $ABCD$, divide the four generating lines through $P$ and $E$ in the following anharmonic ratios:

$$
\frac{c'(EA_2P')}{c'(EA_2P'')} = \frac{q'(FP')}{q'(FP'')} = \frac{b(A'CER)}{b(A'CER)} = (AA_2DT);
$$

$$
\frac{a'(EA_2P'')}{a'(EA_2P')} = \frac{r'(FP'')}{r'(FP')} = \frac{b(C'AQ)}{b(C'AQ)} = (CC_2DS);
$$

so that (as again is known) the variable generatrices, as well as the fixed ones, of the hyperboloid, are all divided homographically.

(7.) The tangential equation of the present surface is easily found, by the expressions in sub-art. (1.) for the co-ordinates $lmnr$ of the tangent plane, to be the following:

$$
0 = f = ln - mr;
$$

which may be interpreted as expressing, that this hyperboloid is the surface of the second class, which touches the nine planes,

$[1000], [0100], [0010], [0001], [1100], [0110], [0011], [1001], [1111]$;

or with the literal symbols lately employed (comp. 86, 87),

$$
BCD, CD_a, DA_b, AB_c, CD'_a, DA_a, AB'_a, BC_a, and [E].
$$

Or we may interpret the same tangential equation $f = 0$ as expressing (comp. again 86, 87, where $q, l, n$ are now replaced by $t, r, q$), that the surface is the envelope of a plane $qrst$, which satisfies either of the two connected conditions of homography:

$$
\frac{(BC'AQ)}{m} = -\frac{l}{n} = -\frac{r}{m} = (CC_2DS);
$$

$$
\frac{(CA'BR)}{n} = -\frac{m}{n} = -\frac{l}{r} = (DA_2AT);
$$

a double generation of the hyperboloid thus showing itself in a new way. And as regards the passage (or return), from the tangential to the local equation (comp. 56), we have in the present example the formulæ:

$$
\begin{align*}
  x &= D_2F = n; & y &= D_4F = -r; & z &= D_4F = l; & w &= D_4F = -m; \\
  xz - yw &= 0, & \text{as before.}
\end{align*}
$$

(8.) More generally, when the surface is of the second order, and therefore also of the second class, so that the two functions $f$ and $F$, when presented...
under rational and integral forms, are both homogeneous of the second dimension, then whether we derive \( l \ldots r \) from \( x \ldots w \) by the formulæ,
\[
l = D_aF, \quad m = D_yF, \quad n = D_zF, \quad r = D_wF,
\]
or \( x \ldots w \) from \( l \ldots r \) by the converse formulæ,
\[
x = D_lF, \quad y = D_mF, \quad z = D_nF, \quad w = D_rF,
\]
the point \( p = (xyzw) \) is, relatively to that surface, what is usually called (comp. 52) the pole of the plane \( II = [lmnr] \); and conversely, the plane \( II \) is the polar of the point \( p \); wherever in space the point \( p \) and plane \( II \), thus related to each other, may be situated. And because the centre of a surface of the second order is known to be (comp. again 52) the pole of (what is called) the plane at infinity; while (comp. 38) the equation and the symbol of this last plane are, respectively,
\[
ax + by + cz + dw = 0, \quad \text{and} \quad [a, b, c, d],
\]
if the four constants \( abcd \) have still the same significations as in 65, 70, 79, &c., with reference to the system of the five given points \( ABCDE \); it follows that we may denote this centre by the symbol,
\[
K = (D_aF_0, D_bF_0, D_cF_0, D_dF_0);
\]
where \( F_0 \) denotes, for abridgment, the function \( F(abcd) \), and \( d \) is still a scalar constant.

\( 9. \) In the recent example, we have \( F_0 = ac - bd \); and the anharmonic symbol for the centre of the hyperboloid becomes thus,
\[
K = (c, -d, a, -b).
\]
Accordingly if we assume (comp. sub-arts. (3.), (4.),)
\[
p = (st, tu, uv, vs), \quad p' = (s't', -t'u', u'v', -v's'),
\]
where \( s, t, u, v \) are any four scalars, and \( p' \) is a new point, while
\[
s' = bt + cv, \quad t' = cu + ds, \quad u' = dv + at, \quad v' = as + bu;
\]
if also we write, for abridgment,
\[
e' = ac - bd, \quad w' = ast + btu + cuv + dvs;
\]
we shall then have the symbolic relations,
\[
e'(p) + (p') = w'(K), \quad e'(p) - (p') = (p''),
\]
if \( p'' = (x'y'z''w'') \) be that new point, of which the co-ordinates are,
\[
x'' = 2e'st - cw', \quad y'' = 2e'tu + dw', \quad z'' = 2e'uv - aw', \quad w'' = 2e'vs + bw',
\]
and therefore,
\[
ax'' + by'' + cz'' + dw'' = 0.
\]
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That is to say, if \( pp' \) be any chord of the hyperboloid, which passes through the fixed point \( k \), and if \( p'' \) be the harmonic conjugate of that fixed point, with respect to that variable chord, so that \((pkp'p'') = -1\), then this conjugate point \( p'' \) is on the infinitely distant plane \([abcd]\); or in other words, the fixed point \( k \) bisects all the chords \( pp' \) which pass through it, and is therefore (as above asserted) the centre of the surface.

(10.) With the same meanings (65, 79) of the constants \( a, b, c, d \), the mean point (96) of the quadrilateral \( abcd \), or of the system of its corners, may be denoted by the symbol,

\[ m = (a^{-1}, b^{-1}, c^{-1}, d^{-1}) ; \]

if then this mean point be on the surface, so that

\[ ac - bd = 0, \]

the centre \( k \) is on the plane \([a, b, c, d]\); or in other words, it is infinitely distant: so that the surface becomes, in this case, a ruled (or hyperbolic) paraboloid. In general (comp. sub-art. (8.)), if \( f_0 = 0 \), the surface of the second order is a paraboloid of some kind, because its centre is then at infinity, in virtue of the equation

\[ (ad_a + bD_b + cD_c + dD_d) f_0 = 0 ; \]

or because (comp. 50, 58) the plane \([abcd]\) at infinity is then one of its tangent planes, as satisfying its tangential equation, \( f = 0 \).

(11.) It is evident that a curve in space may be represented by a system of two anharmonic and local equations; because it may be regarded as the intersection of two surfaces. And then its order, or the number of points (real or imaginary*), in which it is cut by an arbitrary plane, is obviously the product of the orders of those two surfaces; or the product of the degrees of their two local equations, supposed to be rational and integral.

(12.) A curve of double curvature may also be considered as the edge of regression (or arête de rebroussement) of a developable surface, namely of the locus of the tangents to the curve; and this surface may be supposed to be circumscribed at once to two given surfaces, which are envelopes of variable planes (98), and are represented, as such, by their tangential equations. In this view, a curve of double curvature may itself be represented by a system of two anharmonic and tangential equations; and if the class of such a curve be defined to be the number of its osculating planes, which pass through an arbitrary point of space, then this class is the product of the classes of the two curved sur-

* Compare the notes to pages 87, 88.
faces just now mentioned: or (what comes to the same thing) it is the product of the dimensions of the two tangential equations, by which the curve is (on this plan) symbolized. But we cannot enter further into these details; the mechanism of calculation respecting which would indeed be found to be the same, as that employed in the known method (comp. 41) of quadriplanar co-ordinates.

99. Instead of anharmonic co-ordinates, we may consider any other system of \( n \) variable scalars, \( x_1, \ldots, x_n \), which enter into the expression of a variable vector, \( \rho \); for example, into an expression of the form (comp. 96, 97),

\[
\rho = x_1 a_1 + x_2 a_2 + \ldots = \Sigma x a.
\]

And then, if those \( n \) scalars \( x \) be all functions of one independent and variable scalar, \( t \), we may regard this vector \( \rho \) as being itself a function of that single scalar; and may write,

\[
I. \ldots \rho = \phi (t).
\]

But if the \( n \) scalars \( x \) be functions of two independent and scalar variables, \( t \) and \( u \), then \( \rho \) becomes a function of those two scalars, and we may write accordingly,

\[
II. \ldots \rho = \phi (t, u).
\]

In the 1st case, the term \( \rho \) (comp. 1) of the variable vector \( \rho \) has generally for its locus a curve in space, which may be plane or of double curvature, or may even become a right line, according to the form of the vector-function \( \phi \); and \( \rho \) may be said to be the vector of this line, or curve. In the 2nd case, \( \rho \) is the vector of a surface, plane or curved, according to the form of \( \phi (t, u) \); or to the manner in which this vector \( \rho \) depends on the two independent scalars that enter into its expression.

(1.) As examples (comp. 25, 63), the expressions,

\[
\begin{align*}
I. & \ldots \rho = \frac{a + t \beta}{1 + t} ; \\
II. & \ldots \rho = \frac{a + t \beta + u \gamma}{1 + t + u},
\end{align*}
\]

signify, 1st, that \( \rho \) is the vector of a variable point \( P \) on the right line \( AB \); or that it is the vector of that line itself, considered as the locus of a point; and 2nd, that \( \rho \) is the vector of the plane \( ABC \), considered in like manner as the locus of an arbitrary point \( P \) thereon.

(2.) The equations,

\[
\begin{align*}
I. & \ldots \rho = xa + y\beta, \\
II. & \ldots \rho = xa + y\beta + z\gamma,
\end{align*}
\]

with \( x^2 + y^2 = 1 \) for the 1st, and \( x^2 + y^2 + z^2 = 1 \) for the 2nd, signify 1st, that \( \rho \) is the vector of an ellipse, and 2nd, that it is the vector of
an ellipsoid, with the origin o for their common centre, and with oA, oB, or oA, oB, oo, for conjugate semi-diameters.

(3.) The equation (comp. 46),
\[ \rho = t^2a + u^2\beta + (t + u)^2\gamma, \]
expresses that ρ is the vector of a cone of the second order, with o for its vertex (or centre), which is touched by the three planes obc, oca, oab; the section of this cone, made by the plane abc, being an ellipse (comp. fig. 25), which is inscribed in the triangle ABC; and the middle points A', B', C', of the sides of that triangle, being the points of contact of those sides with that cone.

(4.) The equation (comp. 53),
\[ \rho = t^3a + u^3\beta + v^3\gamma, \quad \text{with} \quad t + u + v = 0, \]
expresses that ρ is the vector of another cone of the second order, with o still for vertex, but with oA, oB, oo for three of its sides (or rays). The section by the plane abc is a new ellipse, circumscribed to the triangle abc, and having its tangents at the corners of that triangle respectively parallel to the opposite sides thereof.

(5.) The equation (comp. 54),
\[ \rho = t^2a + u^2\beta + v^2\gamma, \quad \text{with} \quad t + u + v = 0, \]
signifies that ρ is the vector of a cone of the third order, of which the vertex is still the origin; its section (comp. fig. 27) by the plane abc being a cubic curve, whereof the sides of the triangle ABC are at once the asymptotes, and the three (real) tangents of inflexion; while the mean point (say o') of that triangle is a conjugate point of the curve; and therefore the right line oo', from the vertex o to that mean point, may be said to be a conjugate ray of the cone.

(6.) The equation (comp. 98, sub-art. (3.)),
\[ \rho = \frac{s'ta + t'u'b + u'v'c + v's'd}{sta + t'u'b + u'v'c + v's'd}, \]
in which \( \frac{s}{u} \) and \( \frac{t}{v} \) are two variable scalars, while a, b, c, d are still four constant scalars, and \( a, \beta, \gamma, \delta \) are four constant vectors, but ρ is still a variable vector, expresses that ρ is the vector of a ruled (or single-sheeted) hyperboloid, on which the gauche quadrilateral abcd is superscribed and which passes through the given point e, whereof the vector e is assigned in 65.

(7.) If we make (comp. 98, sub-art. (9.)),
\[ \rho' = \frac{s't'a - t'u'b + u'v'c - v's'd}{s't'a - t'u'b + u'v'c - v's'd}, \]
where \( s' = bt + cv, \quad t' = cu + ds, \quad u' = dv + at, \quad v' = as + bu, \)
then $\rho' = \mathbf{op}'$ is the vector of another point $\mathbf{r}'$ on the same hyperboloid; and because it is found that the sum of these two last vectors is constant,

$$\rho + \rho' = 2\kappa,$$

it follows that $\kappa$ is the vector of a fixed point $\mathbf{k}$, which bisects every chord $\mathbf{r}\mathbf{p}'$ that passes through it; or in other words (comp. 52), that this point $\mathbf{k}$ is the centre of the surface.

(8.) The three vectors, $\kappa, \frac{\alpha + \gamma}{2}, \frac{\beta + \delta}{2}$, are termino-collinear (24); if then a gauche quadrilateral $\mathbf{abcd}$ be superscribed on a ruled hyperboloid, the common bisector of the two diagonals, $\mathbf{ac}$, $\mathbf{bd}$, passes through the centre $\mathbf{k}$.

(9.) When $\alpha c = \beta d$, or when we have the equation,

$$p = \frac{sta + tuj + uv\gamma + vs\delta}{st + tu + uv + vs},$$

or simply,

$$p = sta + tuj + uv\gamma + vs\delta,$$

with $s + u = t + v = 1$, $\rho$ is then the vector of a ruled paraboloid, of which the centre (comp. 52, and 98, sub-art. (10.)), is infinitely distant, but upon which the quadrilateral $\mathbf{abcd}$ is still superscribed. And this surface passes through the mean point $\mathbf{m}$ of that quadrilateral, or of the system of the four given points $\mathbf{a} \ldots \mathbf{d}$; because, when $s = t = u = v = \frac{1}{2}$, the variable vector $\rho$ takes the value (comp. 96, sub-art. (1.)),

$$\mu = \frac{1}{2} (\alpha + \beta + \gamma + \delta).$$

(10.) In general, it is easy to prove, from the last vector-expression for $\rho$, that this paraboloid is the locus of a right line, which divides similarly the two opposite sides $\mathbf{ab}$ and $\mathbf{dc}$ of the same gauche quadrilateral $\mathbf{abcd}$; or the other pair of opposite sides, $\mathbf{bc}$ and $\mathbf{ad}$.

SECTION 7.

On Differentials of Vectors.

100. The equation (99, I.),

$$\rho = \phi (t),$$

in which $\rho = \mathbf{op}$ is generally the vector of a point $\mathbf{p}$ of a curve in space, $\mathbf{pq} \ldots$, gives evidently, for the vector $\mathbf{oq}$ of another point $\mathbf{q}$ of the same curve, an expression of the form

$$\rho + \Delta \rho = \phi (t + \Delta t);$$
so that the chord $pq$, regarded as being itself a vector, comes thus to be represented (4) by the finite difference,

\[ pq = \Delta \rho = \Delta \phi (t) = \phi (t + \Delta t) - \phi (t). \]

Suppose now that the other finite difference, $\Delta t$, is the $n^{th}$ part of a new scalar, $u$; and that the chord $\Delta \rho$, or $pq$, is in like manner (comp. fig. 32), the $n^{th}$ part of a new vector, $\sigma_n$, or $pr$; so that we may write,

\[ n \Delta t = u, \quad \text{and} \quad n \Delta \rho = n \cdot pq = \sigma_n = pr. \]

Then, if we treat the two scalars, $t$ and $u$, as constant, but the number $n$ as variable (the form of the vector-function $\phi$, and the origin $o$, being given), the vector $\rho$ and the point $r$ will be fixed; but the two points $q$ and $k$, the two differences $\Delta t$ and $\Delta \rho$, and the multiple vector $n \Delta \rho$, or $\sigma_n$, will (in general) vary together. And if this number $n$ be indefinitely increased, or made to tend to infinity, then each of the two differences $\Delta t$, $\Delta \rho$ will in general tend to zero; such being the common limit, of $n^{-1}u$, and of $\phi (t + n^{-1}u) - \phi (t)$: so that the variable point $q$ of the curve will tend to coincide with the fixed point $p$. But although the chord $pq$ will thus be indefinitely shortened, its $n^{th}$ multiple, $pr$ or $\sigma_n$, will tend (generally) to a finite limit,* depending on the supposed continuity of the function $\phi (t)$; namely, to a certain definite vector, $pt$, or $\sigma_w$, or (say) $r$, which vector $pt$ will evidently be (in general) tangential to the curve: or, in other words, the variable point $r$ will tend to a fixed position $t$, on the tangent to that curve at $p$. We shall thus have a limiting equation, of the form

\[ \tau = pt = \lim_{n \to \infty} pr = \sigma_w = \lim_{n \to \infty} n \Delta \phi (t), \quad \text{if} \quad n \Delta t = u; \]

$t$ and $u$ being, as above, two given and (generally) finite scalars. And if we then agree to call the second of these two given scalars the differential of the first, and to denote it by the symbol $dt$, we shall define that the vector-limit, $\tau$ or $\sigma_w$, is the (corresponding) differential of the vector $\rho$, and shall denote it by the corresponding symbol, $d\rho$; so as to have, under the supposed conditions,

\[ u = dt, \quad \text{and} \quad \tau = d\rho. \]

Or, eliminating the two symbols $u$ and $\tau$, and not necessarily supposing that $p$ is a point of a curve, we may express our Definition† of the Differential of a vector.

---

* Compare Newton's Principia. † Compare the Note to page 35.
Vector $\rho$, considered as a function $\phi$ of a scalar $t$, by the following General Formula:

$$d\rho = d\phi(t) = \lim_{n \to \infty} n \left\{ \phi(t + \frac{dt}{n}) - \phi(t) \right\},$$

in which $t$ and $dt$ are two arbitrary and independent scalars, both generally finite; and $d\rho$ is, in general, a new and finite vector, depending on those two scalars, according to a law expressed by the formula, and derived from that given law, whereby the old or former vector, $\rho$ or $\phi(t)$ depends upon the single scalar, $t$.

(1.) As an example, let the given vector-function have the form,

$$\rho = \phi(t) = \frac{1}{2} t^2 a,$$

where $a$ is a given vector.

Then, making $\Delta t = \frac{u}{n}$, where $u$ is any given scalar, and $n$ is a variable whole number, we have

$$\Delta \rho = \Delta \phi(t) = \frac{a}{2} \left[ \left( t + \frac{u^2}{n} \right)^2 - t^2 \right] = \frac{au}{n} \left( t + \frac{u}{2n} \right);$$

$$\sigma_\omega = n \Delta \rho = au \left( t + \frac{u}{2n} \right); \quad \sigma_\omega = atu;$$

and finally, writing $dt$ and $d\rho$ for $u$ and $\sigma_\omega$.

$$d\rho = d\phi(t) = \frac{d(\frac{t^2 a}{2})}{dt} = atdt.$$

(2.) In general, let $\phi(t) = af(t)$, where $a$ is still a given or constant vector, and $f(t)$ denotes a scalar function of the scalar variable, $t$. Then because $a$ is a common factor within the brackets $\{ \}$ of the recent general formula (100) for $d\rho$, we may write,

$$d\rho = d\phi(t) = d . af(t) = adf(t);$$

provided that we now define that the differential of a scalar function, $f(t)$, is a new scalar function of two independent scalars, $t$ and $dt$, determined by the precisely similar formula:

$$df(t) = \lim_{n \to \infty} n \left\{ f(t + \frac{dt}{n}) - f(t) \right\};$$

which can easily be proved to agree, in all its consequences, with the usual rules for differentiating functions of one variable.

(3.) For example, if we write $dt = nh$, where $h$ is a new variable scalar, namely, the $n^{th}$ part of the given and (generally) finite differential, $dt$, we shall thus have the equation,

$$\frac{df(t)}{dt} = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h};$$

in which the first member is here considered as the actual quotient of two finite
scalars, \( df(t) : dt \), and not merely as a differential coefficient. We may, however, as usual, consider this quotient, from the expression of which the differential \( dt \) disappears, as a derived function of the former variable, \( t \); and may denote it, as such, by either of the two usual symbols,

\[ f'(t) \quad \text{and} \quad \frac{df(t)}{dt}. \]

(4.) In like manner we may write, for the derivative of a vector-function, \( \phi(t) \), the formula:

\[
\rho' = \phi'(t) = \frac{d\rho}{dt} = \frac{d\phi(t)}{dt};
\]

these two last forms denoting that actual and finite vector, \( \rho' \) or \( \phi'(t) \), which is obtained, or derived, by dividing (comp. 16) the not less actual (or finite) vector, \( d\rho \) or \( d\phi(t) \), by the finite scalar, \( dt \). And if again we denote the \( n^{th} \) part of this last scalar by \( h \), we shall thus have the equally general formula:

\[
d_{n}\phi = d_{n}\phi(t) = \lim_{h \to 0} \frac{\phi(t+h) - \phi(t)}{h};
\]

with the equations

\[
d\rho = d_{n}\rho \cdot dt = \rho' \cdot dt; \quad d\phi(t) = d_{n}\phi(t) \cdot dt = \phi'(t) \cdot dt,
\]

exactly as if the vector-function, \( \rho \) or \( \phi \), were a scalar function, \( f \).

(5.) The particular value, \( dt = 1 \), gives thus \( d\rho = \rho' \); so that the derived vector \( \rho' \) is (with our definitions) a particular but important case of the differential of a vector. In applications to mechanics, if \( t \) denote the time, and if the term \( \rho \) of the variable vector \( \rho \) be considered as a moving point, this derived vector \( \rho' \) may be called the Vector of Velocity: because its length represents the amount, and its direction is the direction of the velocity. And if, by setting off vectors \( \rho = \rho' \) (comp. again fig. 32) from one origin, to represent thus the velocities of a point moving in space according to any supposed law, expressed by the equation \( \rho = \phi(t) \), we construct a new curve \( \rho' = \phi'(t) \), then this new curve has been defined to be the Hodograph,† as the old \( \rho \) may be called the orbit of the motion, or of the moving point.

* In the theory of Differentials of Functions of Quaternions, a definition of the differential \( d\phi(q) \) will be proposed, which is expressed by an equation of precisely the same form as those above assigned, for \( df(q) \), and for \( d\phi(t) \); but it will be found that, for quaternions, the quotient \( d\phi(q) : dq \) is not generally independent of \( dq \); and consequently that it cannot properly be called a derived function, such as \( \phi'(q) \), of the quaternion \( q \) alone. (Compare again the Note to page 39.) [See 327.]

† The subject of the Hodograph will be resumed at a subsequent stage of this work. In fact, it almost requires the assistance of Quaternions, to connect it, in what appears to be the best mode, with Newton's Law of Gravitation. [Compare 419.]
(6.) We may differentiate a vector-function twice (or oftener), and so obtain its successive differentials. For example, if we differentiate the derived vector $\rho'$, we obtain a result of the form,

$$dp' = p'' dt,$$

where $p'' = \frac{D^2p'}{dt^2}$, by an obvious extension of notation; and if we suppose that the second differential, $d^2t$ or $dt^2$, of the scalar $t$ is zero, then the second differential of the vector $\rho$ is,

$$d^2\rho = dd\rho = d . \rho' dt = dp' . dt = \rho'' . dt^2;$$

where $d^2\rho$, as usual, denotes $(d\rho)^2$; and where it is important to observe that, with the definitions adopted, $d^2\rho$ is as finite a vector as $d\rho$, or as $\rho$ itself. In applications to motion, if $t$ denote the time, $\rho''$ may be said to be the Vector of Acceleration.

(7.) We may also say that, in mechanics, the finite differential $d\rho$, of the Vector of Position $\rho$, represents, in length and in direction, the right line (suppose $pt$ in fig. 32) which would have been described, by a freely moving point $p$, in the finite interval of time $dt$, immediately following the time $t$, if at the end of this time $t$ all foreign forces had ceased to act.*

(8.) In geometry, if $\rho = \phi(t)$ be the equation of a curve of double curvature, regarded as the edge of regression (comp. 98, (12.)) of a developable surface, then the equation of that surface itself, considered as the locus of the tangents to the curve, may be thus written (comp. 99, II.):

$$\rho = \phi(t) + u\phi'(t);$$

or simply, $\rho = \phi(t) + d\phi(t)$,

if it be remembered that $u$, or $dt$, may be any arbitrary scalar.

(9.) If any other curved surface (comp. again 99, II.) be represented by an equation of the form, $\rho = \phi(x,y)$, where $\phi$ now denotes a vector-function of two independent and scalar variables, $x$ and $y$, we may then differentiate this equation, or this expression for $\rho$, with respect to either variable separately, and so obtain what may be called two partial (but finite) differentials, $d_x\rho$, $d_y\rho$, and two partial derivatives, $d_x\rho$, $d_y\rho$, whereof the former are connected with the latter, and with the two arbitrary (but finite) scalars, $dx$, $dy$, by the relations,

$$d_x\rho = d_x\rho . dx;$$
$$d_y\rho = d_y\rho . dy.$$
that surface at the point \( p \) : so that their plane is (or is parallel to) the tangent plane at that point.

(10.) The mechanism of all such differentiations of vector-functions is, at the present stage, precisely the same as in the usual processes of the Differential Calculus; because the most general form of such a vector-function, which has been considered in the present Book, is that of a sum of products (comp. 99) of the form \( xa \), where \( a \) is a constant vector, and \( x \) is a variable scalar: so that we have only to operate on these scalar coefficients \( x \), by the usual rules of the calculus, the vectors \( a \) being treated as constant factors (comp. sub-art. 2).

But when we shall come to consider quotients or products of vectors, or generally those new functions of vectors which can only be expressed (in our system) by Quaternions, then some few new rules of differentiation become necessary, although deduced from the same (or nearly the same) definitions, as those which have been established in the present section.

(11.) As an example of partial differentiation (comp. sub-art. 9) of a vector function (the word "vector" being here used as an adjective) of two scalar variables, let us take the equation

\[
\rho = \phi(x, y) = \frac{1}{2}(x^2a + y^2\beta + (x + y)^2\gamma);
\]

in which \( \rho \) (comp. 99, (3.)) is the vector of a certain cone of the second order; or, more precisely, the vector of one sheet of such a cone, if \( x \) and \( y \) be supposed to be real scalars. Here, the two partial derivatives of \( \rho \) are the following:

\[
D_x\rho = xa + (x + y)\gamma; \quad D_y\rho = y\beta + (x + y)\gamma;
\]

and therefore,

\[
2\rho = xD_x\rho + yD_y\rho;
\]

so that the three vectors, \( \rho \), \( D_x\rho \), \( D_y\rho \), if drawn (18) from one common origin, are contained (22) in one common plane; which implies that the tangent plane to the surface, at any point \( r \), passes through the origin \( o \); and thereby verifies the conical character of the locus of that point \( r \), in which the variable vector \( \rho \), or \( or \), terminates.

(12.) If, in the same example, we make \( x = 1, y = -1 \), we have the values,

\[
\rho = \frac{1}{2}(a + \beta), \quad D_x\rho = a, \quad D_y\rho = -\beta;
\]

whence it follows that the middle point, say \( o' \), of the right line \( AB \), is one of the points of the conical locus; and that (comp. again the sub-art. 3 to Art. 99, and the recent sub-art. 9) the right lines \( OA \) and \( OB \) are parallel to two of the tangents to the surface at that point; so that the cone in question is
touched by the plane $AOB$, along the side (or ray) $OC'$. And in like manner it may be proved, that the same cone is touched by the two other planes $BOC$ and $COA$, at the middle points $A'$ and $B'$ of the two other lines $BC$ and $CA$; and therefore along the two other sides (or rays), $OA'$ and $OB'$: which again agrees with former results.

(13.) It will be found that a vector function of the sum of two scalar variables, $t$ and $dt$, may generally be developed, by an extension of Taylor’s Series, under the form,

$$
\phi(t + dt) = \phi(t) + d\phi(t) + \frac{1}{2!} d^2\phi(t) + \ldots
$$

$$
= (1 + d + \frac{d^2}{2!} + \frac{d^3}{3!} + \ldots) \phi(t) = e^d \phi(t);
$$

it being supposed that $d^2t = 0$, $d^2t = 0$, &c. (comp. sub-art. 6). Thus, if 

$$
\phi = \frac{1}{2}a^t (\text{as in sub-art. 1}),
$$

where $a$ is a constant vector, we have $d\phi = adt$, $d^2\phi = ad^2t$, $d^3\phi = 0$, &c.; and

$$
\phi(t + dt) = \frac{1}{2}a(t + dt)^2 = \frac{1}{2}at^2 + adt + \frac{1}{2}ad^2t,
$$

rigorously, without any supposition that $dt$ is small.

(14.) When we thus suppose $\Delta t = dt$, and develop the finite difference, $\Delta \phi(t) = \phi(t + dt) - \phi(t)$, the first term of the development so obtained, or the term of first dimension relatively to $dt$, is hence (by a theorem, which holds good for vector-functions, as well as for scalar functions) the first differential $d\phi(t)$ of the function; but we do not choose to define that this Differential is (or means) that first term: because the Formula (100), which we prefer, does not postulate the possibility, nor even suppose the conception, of any such development. Many recent remarks will perhaps appear more clear, when we shall come to connect them, at a later stage, with that theory of Quaternions, to which we next proceed.

[Compare generally III. ii. Two elementary illustrations of Hamilton’s method are given in § 2 of the Chapter cited. It may be of interest to refer to Art. xxviii. of J. Clerk Maxwell’s “Matter and Motion.” Another mode of obtaining the diagram of velocities of a system at a given instant is to take a small interval of time, say the $n$th part of the unit of time, so that the middle of this interval corresponds to the given instant. Take the diagram of displacements corresponding to this interval and magnify all its dimensions $n$ times. The result will be a diagram of the mean velocities of the system during the interval. If we now suppose the number $n$ to increase without limit the interval will diminish without limit, and the mean velocities]
will approximate to the actual velocities at the given instant. Finally, when 
n becomes infinite the diagram will represent accurately the velocities at the 
given instant." The unit of time is of course not necessarily small: compare 
sub-art. (5). In a letter to De Morgan, dated April 26th, 1852 (Graves’s Life, 
vol. iii., p. 629), Hamilton says:—‘‘I lay no stress on the infinitely great 
value of n. It would suit me almost as well to define

$$df/dq = \lim_{x \to 0} x^{-1}[f(q + xdq) - f(q)],$$

though I think the other form a little clearer. But the important thing is 
that I avoid—1st, commutation of factors; 2nd, development in series; 
3rd, smallness of differentials.”]
BOOK II.

ON QUATERNIONS, CONSIDERED AS QUOTIENTS OF VECTORS, AND AS INVOLVING ANGULAR RELATIONS.

Hamilton's Elements of Quaternions.
CHAPTER I.

FUNDAMENTAL PRINCIPLES RESPECTING QUOTIENTS OF VECTORS.

SECTION 1.

Introductory Remarks; First Principles adopted from Algebra.

Art. 101.—The only angular relations, considered in the foregoing Book, have been those of parallelism between vectors (Art. 2, &c.) ; and the only quotients, hitherto employed, have been of the three following kinds:

I. Scalar quotients of scalars, such as the arithmetical fraction \( \frac{n}{m} \) in Art. 14;

II. Vector quotients, of vectors divided by scalars, as \( \frac{\beta}{x} = \alpha \) in Art. 16;

III. Scalar quotients of vectors, with directions either similar or opposite, as \( \frac{\beta}{\alpha} = \alpha \) in the last cited Article. But we now propose to treat of other geometric Quotients (or geometric Fractions, as we shall also call them), such as

\[
\frac{OB}{OA} = \frac{\beta}{\alpha} = q, \text{ with } \beta \text{ not } \parallel \alpha \text{ (comp. 15)};
\]

for each of which the Divisor (or denominator), \( \alpha \) or \( OA \), and the Dividend (or numerator), \( \beta \) or \( OB \), shall not only both be Vectors, but shall also be inclined to each other at an Angle, distinct (in general) from zero, and from two* right angles.

102. In introducing this new conception, of a General Quotient of Vectors, with Angular Relations in a given plane, or in space, it will obviously be necessary to employ some properties of circles and spheres, which were not wanted for

* More generally speaking, from every even multiple of a right angle.
the purpose of the former Book. But, on the other hand, it will be possible
and useful to suppose a much less degree of acquaintance with many important
theories* of modern geometry, than that of which the possession was assumed,
in several of the foregoing sections. Indeed it is hoped that a very moderate
amount of geometrical, algebraical, and trigonometrical preparation will be
found sufficient to render the present Book, as well as the early parts of the
preceeding one, fully and easily intelligible to any attentive reader.

103. It may be proper to premise a few general principles respecting
quotients of vectors, which are indeed suggested by algebra, but are here adopted
by definition. And 1st, it is evident that the supposed operation of division
(whatever its full geometrical import may afterwards be found to be), by which
we here conceive ourselves to pass from a given divisor-line \(a\), and from a given
dividend-line \(\beta\), to what we have called (provisionally) their geometric quotient,
\(q\), may (or rather must) be conceived to correspond to some converse act (as yet
not fully known) of geometrical multiplication: in which new act the former
quotient, \(q\), becomes a Factor, and operates on the line \(a\) so as to produce (or
generate) the line \(\beta\). We shall therefore write, as in algebra,

\[
\beta = q \cdot a,
\]

or simply, \(\beta = qa\), when \(\beta : a = q\);
even if the two lines \(a\) and \(\beta\), or \(oa\) and \(ob\), be supposed to be inclined to each
other, as in fig. 33. And this very simple and natural notation (comp. 16)
will then allow us to treat as identities the two following formulae:

\[
\left( \frac{\beta}{a} \cdot a = \right) \frac{\beta}{a} a = \beta; \quad \frac{qa}{a} = q;
\]

although we shall, for the present, abstain from writing also such formulæ† as
the following:

\[
\frac{\beta a}{a} = \beta, \quad \frac{q a}{a} = q;
\]

where \(a, \beta\) still denote two vectors, and \(q\) denotes their geometrical quotient:

* Such as homology, homography, inversion, and generally whatever depends on anharmonic ratio:
although all that is needful to be known respecting such ratio, for the applications subsequently made,
may be learned, without reference to any other treatise, from the definitions incidentally given, in
Art. 25, &c. It was, perhaps, not strictly necessary to introduce any of these modern geometrical
theories, in any part of the present work; but it was thought that it might interest one class, at least,
of students, to see how they could be combined with that fundamental conception of the Vector, which
the First Book was designed to develop.

† It will be seen, however, at a later stage, that these two formulæ are permitted, and even
required, in the development of the Quaternion System.
because we have not yet even begun to consider the multiplication of one vector by another, or the division of a quotient by a line.

104. As a second general principle, suggested by algebra, we shall next lay it down, that if
\[
\frac{\beta'}{a'} = \frac{\beta}{a}, \quad \text{and} \quad a' = a, \quad \text{then} \quad \beta' = \beta;
\]
or in words, and under a slightly varied form, that unequal vectors, divided by equal vectors, give unequal quotients. The importance of this very natural and obvious assumption will soon be seen in its applications.

105. As a third principle, which indeed may be considered to pervade the whole of mathematical language, and without adopting which we could not usefully speak, in any case, of equality as existing between any two geometrical quotients, we shall next assume that two such quotients can never be equal to the same third* quotient, without being at the same time equal to each other:

or in symbols, that
\[
\text{if} \quad q' = q, \quad \text{and} \quad q'' = q, \quad \text{then} \quad q'' = q'.
\]

106. In the fourth place, as a preparation for operations on geometrical quotients, we shall say that any two such quotients, or fractions (101), which have a common divisor-line, or (in more familiar words) a common denominator, are added, subtracted, or divided, among themselves, by adding, subtracting, or dividing their numerators: the common denominator being retained, in each of the two former of these three cases. In symbols, we thus define (comp. 14) that, for any three (actual) vectors, \(a, \beta, \gamma\),
\[
\frac{\gamma + \beta}{a} = \frac{\gamma + \beta}{a}; \quad \frac{\gamma - \beta}{a} = \frac{\gamma - \beta}{a};
\]
and
\[
\frac{\gamma}{a} \cdot \frac{\beta}{a} = \frac{\gamma}{\beta};
\]
aiming still at agreement with algebra.

107. Finally, as a fifth principle, designed (like the foregoing) to assimilate, so far as can be done, the present Calculus to Algebra, in its operations on geometrical quotients, we shall define that the following formula holds good:
\[
\left(\frac{\gamma}{a} \cdot \beta\right) = \frac{\gamma \beta}{a} = \frac{\gamma}{a};
\]

* It is scarcely necessary to add, what is indeed included in this third principle, in virtue of the identity \(q = q\), that if \(q' = q\), then \(q = q'\); or in words, that we shall never admit that any two geometrical quotients, \(q\) and \(q'\), are equal to each other in one order, without at the same time admitting that they are equal, in the opposite order also.
or that if two geometrical fractions, \( q \) and \( q' \), be so related, that the denominator, \( \beta \), of the multiplier \( q' \) (here written towards the left-hand) is equal to the numerator of the multiplicand \( q \), then the product, \( q' \cdot q \) or \( q'q \), is that third fraction, whereof the numerator is the numerator \( \gamma \) of the multiplier, and the denominator is the denominator \( \alpha \) of the multiplicand: all such denominators, or divisor-lines, being still supposed (16) to be actual (and not null) vectors.

SECTION 2.

First Motive for naming the Quotient of two Vectors a Quaternion.

108. Already we may see grounds for the application of the name, Quaternion, to such a Quotient of two Vectors as has been spoken of in recent articles. In the first place, such a quotient cannot generally be what we have called (17) a Scalar: or in other words, it cannot generally be equal to any of the (so-called) reals of algebra, whether of the positive or of the negative kind. For let \( * \) denote any such (actual*) scalar, and let \( a \) denote any (actual) vector; then we have seen (15) that the product \( xa \) denotes another (actual) vector, say \( \beta' \), which is either similar or opposite in direction to \( a \), according as the scalar coefficient, or factor, \( x \), is positive or negative; in neither case, then, can it represent any vector, such as \( j3 \), which is inclined to \( a \), at any actual angle, whether acute, or right, or obtuse: or in other words (comp. 2), the equation \( j3' = j3 \), or \( xa = j3 \), is impossible, under the conditions here supposed. But we have agreed (16, 103) to write, as in algebra, \( - = x \); we must, therefore (by the 11nd principle of the foregoing section, stated in Art. 104), abstain from writing also \( - = x \), under the same conditions: \( x \) still denoting a scalar. Whatever else a quotient of two inclined vectors may be found to be, it is thus, at least, a Non-Scalar.

109. Now, in forming the conception of the scalar itself, as the quotient of two parallel\* vectors (17), we took into account not only relative length, or ratio of the usual kind, but also relative direction, under the form of similarity or opposition. In passing from \( a \) to \( xa \), we altered generally (15) the length of

---

* By an actual scalar, as by an actual vector (comp. 1), we mean here one that is different from zero. An actual vector, multiplied by a null scalar, has for product (15) a null vector; it is therefore unnecessary to prove that the quotient of two actual vectors cannot be a null scalar, or zero.

† It is to be remembered that we have proposed (15) to extend the use of this term parallel, to the case of two vectors which are (in the usual sense of the word) parallel to one common line, even when they happen to be parts of one and the same right line.
the line $a$, in the ratio of $\pm x$ to 1; and we preserved or reversed the direction of that line, according as the scalar coefficient $x$ was positive or negative. And in like manner, in proceeding to form, more definitely than we have yet done, the conception of the non-scalar quotient (108), $q = \beta : a = ob : oa$, of two inclined vectors, which for simplicity may be supposed (18) to be co-initial, we have still to take account both of the relative length, and of the relative direction, of the two lines compared. But while the former element of the complex relation here considered, between these two lines or vectors, is still represented by a simple Ratio (of the kind commonly considered in geometry), or by a number* expressing that ratio; the latter element of the same complex relation is now represented by an Angle, $aob$: and not simply (as it was before) by an algebraical sign, $+$ or $-$.  

110. Again, in estimating this angle, for the purpose of distinguishing one quotient of vectors from another, we must consider not only its magnitude (or quantity), but also its Plane: since otherwise, in violation of the principle stated in Art. 104, we should have $ob' : oa = ob : oa$, if $ob$ and $ob'$ were two distinct rays or sides of a cone of revolution, with $oa$ for its axis; in which case (by 2) they would necessarily be unequal vectors. For a similar reason, we must attend also to the contrast between two opposite angles, of equal magnitudes, and in one common plane. In short, for the purpose of knowing fully the relative direction of two co-initial lines $oa$, $ob$ in space, we ought to know not only how many degrees, or other parts of some angular unit, the angle $aob$ contains; but also (comp. fig. 33) the direction of the rotation from $oa$ to $ob$: including a knowledge of the plane, in which the rotation is performed; and of the hand (as right or left, when viewed from a known side of the plane), towards which the rotation is directed.

111. Or, if we agree to select some one fixed hand (suppose the right† hand), and to call all rotations positive when they are directed towards this selected

---

* This number, which we shall presently call the tensor of the quotient, may be whole or fractional, or even incommensurable with unity; but it may always be equated, in calculation, to a positive scalar: although it might perhaps more properly be said to be a signless number, as being derived solely from comparison of lengths, without any reference to directions.

† If right-handed rotation be thus considered as positive, then the positive axis of the rotation $aob$ in fig. 33, must be conceived to be directed downward, or below the plane of the paper. [Compare the Note to 295 (2), and Art. 23 of Clerk Maxwell’s Electricity and Magnetism. Hamilton compared the positive axis to a handle or screw used in screwing a right-handed screw into a nut. It is now usual to regard the positive axis as drawn in the direction of the translation of a right-handed screw moving in a fixed nut, or Hamilton’s left-handed rotation is now called right-handed.]
hand, but all rotations negative when they are directed towards the other hand, then, for any given angle \( \angle \text{aob} \), supposed for simplicity to be less than two right angles, and considered as representing a rotation in a given plane from \( \text{oa} \) to \( \text{ob} \), we may speak of one perpendicular \( \text{oc} \) to that plane \( \angle \text{aob} \) as being the positive axis of that rotation; and of the opposite perpendicular \( \text{oc}' \) to the same plane as being the negative axis thereof: the rotation round the positive axis being itself positive, and vice versa. And then the rotation \( \angle \text{aob} \) may be considered to be entirely known, if we know, 1st, its quantity, or the ratio which it bears to a right rotation; and 2nd, the direction of its positive axis, \( \text{oc} \): but not without a knowledge of these two things, or of some data equivalent to them. But whether we consider the direction of an Axis, or the aspect of a Plane, we find (as indeed is well known) that the determination of such a direction, or of such an aspect, depends on two polar co-ordinates, or other angular elements.

112. It appears, then, from the foregoing discussion, that for the complete determination, of what we have called the geometrical Quotient of two co-initial Vectors, a System of Four Elements, admitting each separately of numerical expression, is generally required. Of these four elements, one serves (109) to determine the relative length of the two lines compared; and the other three are in general necessary, in order to determine fully their relative direction. Again, of these three latter elements, one represents the mutual inclination, or elongation, of the two lines; or the magnitude (or quantity) of the angle between them; while the two others serve to determine the direction of the axis, perpendicular to their common plane, round which a rotation through that angle is to be performed, in a sense previously selected as the positive one (or towards a fixed and previously selected hand), for the purpose of passing (in the simplest way, and therefore in the plane of the two lines) from the direction of the divisor-line, to the direction of the dividend-line. And no more than four numerical elements are necessary, for our present purpose: because the relative length of two lines is not changed, when their two lengths are altered proportionally, nor is their relative direction changed, when the angle which they form is merely turned about, in its own plane. On account, then, of this essential connexion of that complex relation (109) between two lines, which is compounded of a relation of lengths, and of a relation of directions, and to which we have given (by an extension from the theory of scalars) the name of a

* The actual (or at least the frequent) use of such co-ordinates is foreign to the spirit of the present System: but the mention of them here seems likely to assist a student, by suggesting an appeal to results, with which his previous reading can scarcely fail to have rendered him familiar.
geometrical quotient, with a System of Four numerical Elements, we have already a motive* for saying, that “the Quotient of two Vectors is generally a Quaternion.”

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SECTION 3.

Additional Illustrations.

113. Some additional light may be thrown, on this first conception of a Quaternion, by the annexed figure 34. In that figure, the letters cdef are designed to indicate corners of a prismatic desk, resting upon a horizontal table. The angle hcd (supposed to be one of thirty degrees) represents a (left-handed) rotation, whereby the horizontal ledge cd of the desk is conceived to be elongated (or removed) from a given horizontal line ch, which may be imagined to be an edge of the table. The angle gce (supposed here to contain forty degrees) represents the slope† of the desk, or the amount of its inclination to the table. On the face cdef of the desk are drawn two similar and similarly turned triangles, aob and a’o’b’, which are supposed to be halves of two equilateral triangles; in such a manner that each rotation, aob or a’o’b’ is one of sixty degrees, and is directed towards one common hand (namely the right hand in the figure): while if lengths alone be attended to, the side ob is to the side oa, in one triangle, as the side o’b’ is to the side o’a’, in the other; or as the number two to one.

114. Under these conditions of construction, we consider the two quotients, or the two geometric fractions,

\[
\frac{ob}{oa} \text{ and } \frac{ob'}{oa'}, \text{ or } \frac{ob}{oa} \text{ and } \frac{o'b'}{o'a'},
\]

as being equal to each other; because we regard the two lines, oa and ob, as having the same relative length, and the same relative direction, as the two other

* Several other reasons for thus speaking will offer themselves, in the course of the present work.
† These two angles, hcd and gce, may thus be considered to correspond to longitude of node, and inclination of orbit, of a planet or comet in astronomy.
114 ELEMENTS OF QUATERNIONS. [II. i. §§ 3, 4.

lines, o'α' and o'β'. And we consider and speak of each Quotient, or Fraction, as a Quaternion: because its complete construction (or determination) depends, for all that is essential to its conception, and requisite to distinguish it from others, on a system of four numerical elements (comp. 112); which are, in this Example, the four numbers,

2, 60, 30, and 40.

115. Of these four elements (to recapitulate what has been above supposed), the 1st, namely the number 2, expresses that the length of the dividend-line, oβ or o'β', is double of the length of the divisor-line, oα or o'α'. The IInd numerical element, namely 60, expresses here that the angle αoβ or α'o'β', is one of sixty degrees; while the corresponding rotation, from oα to oβ, or from o'α' to o'β', is towards a known hand (in this case the right hand, as seen by a person looking at the face cdef of the desk), which hand is the same for both of these two equal angles. The IIIrd element, namely 30, expresses that the horizontal ledge cd of the desk makes an angle of thirty degrees with a known horizontal line ch, being removed from it, by that angular quantity, in a known direction (which in this case happens to be towards the left hand, as seen from above). Finally, the IVth element, namely 40, expresses here that the desk has an elevation of forty degrees as before.

116. Now an alteration in any one of these Four Elements, such as an alteration of the slope or aspect of the desk would make (in the view here taken) an essential change in the Quaternion, which is (in the same view) the Quotient of the two lines compared; although (as the figure is in part designed to suggest) no such change is conceived to take place, when the triangle αoβ is merely turned about, in its own plane, without being turned over (comp. fig. 36); or when the sides of that triangle are lengthened or shortened proportionally, so as to preserve the ratio (in the old sense of that word), of any one to any other of those sides. We may then briefly say, in this mode of illustrating the notion of a Quaternion* in geometry, by reference to an angle on a desk, that the Four Elements which it involves are the following:

Ratio, Angle, Ledge, and Slope;

although the two latter elements are in fact themselves angles also, but are not immediately obtained as such, from the simple comparison of the two lines, of which the Quaternion is the Quotient.

* As to the mere word, Quaternion, it signifies primarily (as is well known), like its Latin original, "Quaternio," or the Greek noun τετρακτύς, a Set of Four: but it is obviously used here, and elsewhere in the present work, in a technical sense.
SECTION 4.

On Equality of Quaternions; and on the Plane of a Quaternion.

117. It is an immediate consequence of the foregoing conception of a Quaternion, that two quaternions, or two quotients of vectors, supposed for simplicity to be all co-initial (18), are regarded as being equal to each other, or that the equation,

$$\frac{\delta}{\gamma} = \frac{\beta}{\alpha}, \text{ or } \frac{OD}{OC} = \frac{OB}{OA},$$

is by us considered and defined to hold good, when the two triangles, $\triangle aob$ and $\triangle cod$, are similar and similarly turned, and in one common plane, as represented in the annexed fig. 35: the relative length (109), and the relative direction (110), of the two lines, $oa$, or, being then in all respects the same as the relative length and the relative direction of the two other lines, $oc$, $od$.

118. Under the same conditions, we shall write the following formula of direct similitude,

$$\Delta aob \propto cod;$$

reserving this other formula,

$$\Delta aob \propto a'ob', \text{ or } \Delta a'ob \propto a'ob',$$

which we shall call a formula of inverse similitude, to denote that the two triangles, $\triangle aob$ and $\triangle a'ob'$, or $\triangle a'ob$ and $\triangle a'ob'$, although otherwise similar (and even, in this case, equal,* on account of their having a common side, $oa$ or $oa'$), are oppositely turned (comp. fig. 36), as if one were the reflection of the other in a mirror; or as if the one triangle were derived (or generated) from the other, by a rotation of its plane through two right angles. We may therefore write,

$$\frac{OB}{OA} = \frac{OD}{OC}, \text{ if } \Delta aob \propto cod.$$

119. When the vectors are thus all drawn from one common origin $o$, the plane $aob$ of any two of them may be called the Plane of the Quaternion.

---

* That is to say, equal in absolute amount of area, but with opposite algebraic signs (28). The two quotients $oa : oa$, and $oa' : oa$, although not equal (110), will soon be defined to be conjugate quaternions. Under the same conditions, we shall write also the formula,

$$\Delta aob' \propto cod.$$
(or of the Quotient), $ob : oa$; and of course also the plane of the inverse (or reciprocal) quaternion (or of the inverse quotient), $oa : ob$. And any two quaternions, which have a common plane (through $o$), may be said to be Complanar* Quaternions, or complanar quotients, or fractions; but any two quaternions (or quotients), which have different planes (intersecting therefore in a right line through the origin), may be said, by contrast, to be Diplanar.

120. Any two quaternions, considered as geometric fractions (101), can be reduced to a common denominator without change of the value† of either of them, as follows. Let $\frac{ob}{oa}$ and $\frac{od}{oc}$ be the two given fractions, or quaternions; and if they be complanar (119), let $oe$ be any line in their common plane; but if they be diplanar (see again 119), then let $oe$ be any assumed part of the line of intersection of the two planes: so that, in each case, the line $oe$ is situated at once in the plane $aob$, and also in the plane $cod$. We can then always conceive two other lines, $of$, $og$, to be determined so as to satisfy the two conditions of direct similitude (118),

$$\Delta eof \propto aob, \quad \Delta eog \propto cod;$$

and therefore also the two equations between quotients (117, 118),

$$\frac{of}{oe} = \frac{ob}{oa}, \quad \frac{og}{oe} = \frac{od}{oc};$$

and thus the required reduction is effected, $oe$ being the common denominator sought, while $of$, $og$ are the new or reduced numerators. It may be added that if $h$ be a new point in the plane $aob$, such that $\Delta hoe \propto aob$, we shall have also,

$$\frac{oh}{oa} = \frac{of}{oe};$$

and therefore, by 106, 107,

$$\frac{od}{oa} = \frac{og}{oe}; \quad \frac{ob}{oa} = \frac{og}{oe}; \quad \frac{od}{oa} = \frac{og}{oh};$$

whatever two geometric quotients (complanar or diplanar) may be represented by $ob : oa$ and $od : oc$.

* It is, however, convenient to extend the use of this word, complanar, so as to include the case of quaternions represented by angles in parallel planes. Indeed, as all vectors which have equal lengths, and similar directions, are equal (2), so the quaternion, which is a quotient of two such vectors, ought not to be considered as undergoing any change, when either vector is merely changed in position, by a transport without rotation.

† That is to say, the new or transformed quaternions will be respectively equal to the old or given ones.
121. If now the two triangles $AOB$, $COD$ are not only complanar but directly similar (118), so that $\Delta AOB \propto \Delta COD$, we shall evidently have $\Delta EOF \propto EOG$; so that we may write $OF = OG$ (or $F = G$, by 20), the two new lines $OF$, $OG$ (or the two new points $F$, $G$) in this case coinciding. The general construction (120), for the reduction to a common denominator, gives therefore here only one new triangle, $EOF$, and one new quotient, $OF : OG$, to which in this case each (comp. 105) of the two given equal and complanar quotients, $OB : OA$ and $OD : OC$, is equal.

122. But if these two latter symbols (or the fractional forms corresponding) denote two diplanar* quotients, then the two new numerator-lines, $OF$ and $OG$, have different directions, as being situated in two different planes, drawn through the new denominator-line $OE$, without having either the direction of that line itself, or the direction opposite thereto; they are therefore (by 2) unequal vectors, even if they should happen to be equally long; whence it follows (by 104) that the two new quotients, and therefore also (by 105) that the two old or given quotients, are unequal, as a consequence of their diplanarity. It results, then, from this analysis, that diplanar quotients of vectors, and therefore that Diplanar Quaternions (119), are always unequal; a new and comparatively technical process thus confirming the conclusion, to which we had arrived by general considerations, and in (what might be called) a popular way before, and which we had sought to illustrate (comp. fig. 34) by the consideration of angles on a desk: namely, that a Quaternion, considered as the quotient of two mutually inclined lines in space, involves generally a Plane, as an essential part (comp. 110) of its constitution, and as necessary to the completeness of its conception.

123. We propose to use the mark
\[ ||| \]
as a Sign of Complanarity, whether of lines or of quotients; thus we shall write the formula,
\[ \gamma \ ||| \ a, \beta, \]
to express that the three vectors, $a$, $\beta$, $\gamma$, supposed to be (or to be made) co-initial (18), are situated in one plane; and the analogous formula,
\[ q' \ ||| \ q, \quad \text{or} \quad \frac{\delta}{\gamma} \ ||| \ \frac{\beta}{a}, \]

* And therefore non-scalar (108); for a scalar, considered as a quotient (17), has no determined plane, but must be considered as complanar with every geometric quotient; since it may be represented (or constructed) by the quotient of two similarly or oppositely directed lines, in any proposed plane whatever.
to express that the two quaternions, denoted here by \( q \) and \( q' \), and therefore that the four vectors, \( a, \beta, \gamma, \delta \), are complanar (119). And because we have just found (122) that diplanar quotients are unequal, we see that one equation of quaternions includes two complanarities of vectors; in such a manner that we may write,

\[
\gamma \parallel a, \beta, \quad \text{and} \quad \delta \parallel a, \beta, \quad \text{if} \quad \frac{\delta}{\gamma} = \frac{\beta}{\alpha};
\]

the equation of quotients, \( \frac{\overline{OD}}{\overline{OC}} = \frac{\overline{OB}}{\overline{OA}} \) being impossible, unless all the four lines from \( o \) be in one common plane. We shall also employ the notation

\[
\gamma \parallel q,
\]

to express that the vector \( \gamma \) is in (or parallel to) the plane of the quaternion \( q \).

124. With the same notation for complanarity, we may write generally,

\[
xa \parallel a, \beta;
\]

\( a \) and \( \beta \) being any two vectors, and \( x \) being any scalar; because, if \( a = o\alpha \) and \( \beta = o\beta \) as before, then (by 15, 17) \( xa = o\alpha' \), where \( \alpha' \) is some point on the indefinite right line through the points \( o \) and \( a \); so that the plane \( oab \) contains the line \( o\alpha' \). For a similar reason, we have generally the following formula of complanarity of quotients,

\[
\frac{y\beta}{xa} \parallel \frac{\beta}{a},
\]

whatever two scalars \( x \) and \( y \) may be; \( a \) and \( \beta \) still denoting any two vectors.

125. It is evident (comp. fig. 35) that

if \( \Delta aob \propto cod \), then \( \Delta boa \propto doc \), and \( \Delta aoc \propto bod \);

whence it is easy to infer that for quaternions, as well as for ordinary or algebraic quotients,

\[
\text{if} \quad \frac{\beta}{a} = \frac{\delta}{\gamma}, \quad \text{then, inversely,} \quad \frac{a}{\beta} = \frac{\gamma}{\delta}, \quad \text{and alternately,} \quad \frac{\gamma}{\alpha} = \frac{\delta}{\beta};
\]

it being permitted now to establish the converse of the last formula of 118, or to say that

\[
\text{if} \quad \frac{\overline{OB}}{\overline{OA}} = \frac{\overline{OD}}{\overline{OC}}, \quad \text{then} \quad \Delta aob \propto cod.
\]

Under the same condition, by combining inversion with alternation, we have also this other equation, \( \frac{a}{\gamma} = \frac{\beta}{\delta} \).
126. If the sides, $oa$, $ob$, of a triangle $aob$, or those sides either way prolonged, be cut (as in fig. 37) by any parallel, $a'b'$ or $a''b''$, to the base $ab$, we have evidently the relations of direct similarity (118),

$$\Delta a'ob' \propto aob, \quad \Delta a''ob'' \propto aob;$$

whence (comp. Art. 13 and fig. 12) it follows that we may write, for quaternions as in algebra, the general equation, or identity,

$$\frac{x\beta}{xa} = \frac{\beta}{a};$$

where $x$ is again any scalar, and $a$, $\beta$ are any two vectors. It is easy also to see, that for any quaternion $q$, and any scalar $x$, we have the product (comp. 107),

$$xq = \frac{x\beta}{\beta} \cdot \frac{\beta}{a} = \frac{x\beta}{x^2a} = \frac{\beta}{a} \cdot \frac{a}{x^2a} = qx;$$

so that, in the multiplication of a quaternion by a scalar (as in the multiplication of a vector by a scalar, 15), the order of the factors is indifferent.

SECTION 5.

On the Axis and Angle of a Quaternion; and on the Index of a Right Quotient, or Quaternion.

127. From what has been already said (111, 112), we are naturally led to define that the axis, or more fully that the positive axis, of any quaternion (or geometric quotient) $ob : oa$, is a right line perpendicular to the plane $aob$ of that quaternion; and is such that the rotation round this axis, from the divisor-line $oa$, to the dividend-line $ob$, is positive: or (as we shall henceforth assume) directed towards the right-hand,* like the motion of the hands of a watch.

128. To render still more definite this conception of the axis of a quaternion, we may add, Ist, that the rotation, here spoken of, is supposed (112) to be the simplest possible, and therefore to be in the plane of the two lines (or of the quaternion), being also generally less than a semi-revolution in that plane; IIInd, that the axis shall be usually supposed to be a line $ox$ drawn

* This is, of course, merely conventional, and the reader may (if he pleases) substitute the left-hand throughout. [The axis is supposed to be drawn outwards from the face of the watch. See Note, page 111.]
from the assumed origin $o$; and IIIrd, that the length of this line shall be supposed to be given, or fixed, and to be equal to some assumed unit of length: so that the term $x$, of this axis $ox$, is situated (by its construction) on a given spheric surface described about the origin $o$ as centre, which surface we may call the surface of the unit-sphere.

129. In this manner, for every given non-scalar quotient (108), or for every given quaternion $q$ which does not reduce itself (or degenerate) to a mere positive or negative number, the axis will be an entirely definite vector, which may be called an unit-vector, on account of its assumed length, and which we shall denote,* for the present, by the symbol $Ax.q$. Employing then the usual sign of perpendicularity, $\perp$, we may now write, for any two vectors $a, \beta$, the formula:

$$Ax.a = \beta \perp a; \quad Ax.a = \beta \perp \beta; \quad \text{or briefly,} \quad Ax.a = \beta \perp \frac{\beta}{a}.$$ 

130. The Angle of a quaternion, such as $ob : oa$, shall simply be, with us, the angle $aob$ between the two lines, of which the quaternion is the quotient; this angle being supposed here to be one of the usual kind (such as are considered by Euclid): and therefore being acute, or right, or obtuse (but not of any class distinct from these), when the quaternion is a non-scalar (108). We shall denote this angle of a quaternion $q$, by the symbol, $\angle q$; and thus shall have, generally, the two inequalities† following:

$$\angle q > 0; \quad \angle q < \pi;$$

where $\pi$ is used as a symbol for two right angles.

131. When the general quaternion, $q$, degenerates into a scalar, $x$, then the axis (like the plane) becomes entirely indeterminate in its direction; and the angle takes, at the same time, either zero or two right angles for its value, according as the scalar is positive or negative. Denoting then, as above, any such scalar by $x$, we have:

$$Ax.x = \text{an indeterminate unit-vector;}$$

$$\angle x = 0, \text{ if } x > 0; \quad \angle x = \pi, \text{ if } x < 0.$$

---

* At a later stage, reasons will be assigned for denoting this axis, Ax. $q$, of a quaternion $q$, by the less arbitrary (or more systematic) symbol, UV$q$; but for the present, the notation in the text may suffice. [See 291.]

† In some investigations respecting complanar quaternions, and powers or roots of quaternions, it is convenient to consider negative angles, and angles greater than two right angles: but these may then be called amplitudes; and the word "Angle," like the word "Ratio," may thus be restricted, at least for the present, to its ordinary geometrical sense. [See 235.]

‡ Compare the Note to page 117. The angle, as well as the axis, becomes indeterminate, when the quaternion reduces itself to zero; unless we happen to know a law, according to which the dividend-line tends to become null, in the transition from $\frac{\beta}{a}$ to $-\frac{\beta}{a}$. 
132. Of non-scalar quaternions, the most important are those of which the angle is right, as in the annexed figure 38; and when we have thus,

\[ q = \frac{OB}{OA}, \text{ and } OB \perp OA, \text{ or } \angle q = \frac{\pi}{2}, \]

the quaternion \( q \) may then be said to be a Right Quotient; or sometimes, a Right Quaternion.

(1.) If then \( a = OA \) and \( \rho = OP \), where \( O \) and \( A \) are two given (or fixed) points, but \( P \) is a variable point, the equation

\[ \angle \frac{\rho}{a} = \frac{\pi}{2} \]

expresses that the locus of this point \( P \) is the plane through \( O \), perpendicular to the line \( OA \); for it is equivalent to the formula of perpendicularity \( \rho \perp a \) (129).

(2.) More generally, if \( \beta = OB \), \( B \) being any third given point, the equation,

\[ \angle \frac{\rho}{a} = \angle \frac{\beta}{a} \]

expresses that the locus of \( P \) is one sheet of a cone of revolution, with \( O \) for vertex, and \( OA \) for axis, and passing through the point \( B \); because it implies that the angles \( \angle AOB \) and \( \angle AOP \) are equal in amount, but not necessarily in one common plane.

(3.) The equation (comp. 128, 129),

\[ \text{Ax.} \cdot \frac{\rho}{a} = \text{Ax.} \cdot \frac{\beta}{a}, \]

expresses that the locus of the variable point \( P \) is the given plane \( AOB \); or rather the indefinite half-plane, which contains all the points \( P \) that are at once complanar with the three given points \( O, A, B \), and are also at the same side of the indefinite right line \( OA \), as the point \( B \).

(4.) The system of the two equations,

\[ \angle \frac{\rho}{a} = \angle \frac{\beta}{a}, \quad \text{Ax.} \cdot \frac{\rho}{a} = \text{Ax.} \cdot \frac{\beta}{a}, \]

expresses that the point \( P \) is situated, either on the finite right line \( OB \), or on that line prolonged through \( B \), but not through \( O \); so that the locus of \( P \) may in this case be said to be the indefinite half-line, or ray, which sets out from \( O \) in the

* Reasons will afterwards be assigned, for equating such a quotient, or quaternion, to a Vector; namely to the line which will presently (133) be called the Index of the Right Quotient. [See 290.]

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direction of the vector $ob$ or $\beta$; and we may write $\rho = x\beta$, $x > 0$ ($x$ being understood to be a scalar), instead of the equations assigned above.

(5.) This other system of two equations,

$$\frac{\rho}{a} = \pi - \frac{\beta}{a}, \quad Ax \cdot \frac{p}{a} = -Ax \cdot \frac{\beta}{a},$$

expresses that the locus of $p$ is the opposite ray from $o$; or that $p$ is situated on the prolongation of the revector $bo$ (1); or that $\rho = x\beta$, $x < 0$; or that

$$\rho = x\beta', \quad x > 0, \text{ if } \beta' = ob' = -\beta.$$  

(Comp. fig. 33, bis.)

(6.) Other notations, for representing these and other geometric loci, will be found to be supplied, in great abundance, by the Calculus of Quaternions; but it seemed proper to point out these, at the present stage, as serving already to show that even the two symbols of the present section, $Ax.$ and $\zeta$, when considered as Characteristics of Operation on quotients of vectors, enable us to express, very simply and concisely, several useful geometrical conceptions.

133. If a third line, $oi$, be drawn in the direction of the axis $ox$ of such a right quotient (and therefore perpendicular, by 127, 129, to each of the two given rectangular lines, $oa$, $ob$) ; and if the length of this new line or bear to the length of that axis $ox$ (and therefore also, by 128, to the assumed unit of length) the same ratio, which the length of the dividend-line, $ob$, bears to the length of the divisor-line, $oa$; then the line $oi$, thus determined, is said to be the Index of the Right Quotient. And it is evident, from this definition of such an Index, combined with our general definition (117, 118) of Equality between Quaternions, that two right quotients are equal or unequal to each other, according as their two index-lines (or indices) are equal or unequal vectors.

SECTION 6.

On the Reciprocal, Conjugate, Opposite, and Norm of a Quaternion; and on Null Quaternions.

134. The Reciprocal (or the Inverse, comp. 119) of a quaternion, such as $q = \frac{\beta}{a}$, is that other quaternion,

$$q' = \frac{a}{\beta'},$$

which is formed by interchanging the divisor-line and the dividend-line; and in thus passing from any non-scalar quaternion to its reciprocal, it is evident that
the angle (as lately defined in 130) remains unchanged, but that the axis (127, 128) is reversed in direction: so that we may write generally,

\[ \angle \frac{a}{\beta} = \angle \frac{\beta}{a}; \quad \text{Ax} \cdot \frac{a}{\beta} = - \text{Ax} \cdot \frac{\beta}{a}. \]

135. The product of two reciprocal quaternions is always equal to positive unity; and each is equal to the quotient of unity divided by the other; because we have, by 106, 107,

\[ 1 : \frac{\beta}{a} = \frac{a}{\beta} : \frac{a}{a} = \frac{a}{\beta} ; \quad \text{and} \quad \frac{a}{\beta} : \frac{\beta}{a} = \frac{\beta}{a} = 1. \]

It is therefore unnecessary to introduce any new or peculiar notation, to express the mutual relation existing between a quaternion and its reciprocal; since, if one be denoted by the symbol \( q \), the other may (in the present System, as in Algebra) be denoted by the connected symbol, \( *q\), or \( \frac{1}{q}\). We have thus the two general formulae (comp. 134):

\[ \angle \frac{1}{q} = \angle q; \quad \text{Ax} \cdot \frac{1}{q} = - \text{Ax} \cdot q. \]

136. Without yet entering on the general theory of multiplication and divisions of quaternions, beyond what has been done in Art. 120, it may be here remarked that if any two quaternions \( q \) and \( q' \) be (as in 134) reciprocal to each other, so that \( q' \cdot q = 1 \) (by 135), and if \( q'' \) be any third quaternion, then (as in algebra), we have the general formula,

\[ q'' : q = q'' \cdot q' = q'' \cdot \frac{1}{q}; \]

because if (by 120) we reduce \( q \) and \( q'' \) to a common denominator \( a \), and denote the new numerators by \( \beta \) and \( \gamma \), we shall have (by the definitions in 106, 107),

\[ q'' : q = \frac{\gamma}{a} : \frac{\beta}{a} = \frac{\gamma}{\beta} = \frac{\gamma \cdot a}{a \cdot \beta} = q'' \cdot q'. \]

137. When two complanar triangles \( \triangle aob, aob' \), with a common side \( oa \), are (as in fig. 36) inversely similar (118), so that the formula \( \Delta aob' \propto aob \) holds good, then the two unequal quotients, \( \frac{ob}{oa} \) and \( \frac{ob'}{oa} \), are said to be Conjugate.

* The symbol \( q^{-1} \), for the reciprocal of a quaternion \( q \), is also permitted in the present Calculus; but we defer the use of it, until its legitimacy shall have been established, in connexion with a general theory of powers of Quaternions. [See 234.]

† Compare the Note to page 115.
Quaternions; and if the first of them be still denoted by \( q \), then the second, which is thus the conjugate of that first, or of any other quaternion which is equal thereto, is denoted by the new symbol, \( Kq \); in which the letter \( K \) may be said to be the Characteristic of Conjugation. Thus, with the construction above supposed (comp. again fig. 36), we may write,

\[
\frac{OB}{OA} = q; \quad \frac{OB'}{OA} = Kq = K\frac{OB}{OA}.
\]

138. From this definition of conjugate quaternions, it follows, 1st, that if the equation \( \frac{OB'}{OA} = K\frac{OB}{OA} \) hold good, then the line \( OB' \) may be called (118) the reflection of the line \( OB \) (and conversely, the latter line the reflection of the former), with respect to the line \( OA \); 2nd, that, under the same condition, the line \( OA \) (prolonged if necessary) bisects perpendicularly the line \( BB' \), in some point \( A' \) (as represented in fig. 36); and 3rd, that any two conjugate quaternions (like any two reciprocal quaternions, comp. 134, 135) have equal angles, but opposite axes: so that we may write, generally,

\[
\angle Kq = \angle q; \quad A_k \cdot Kq = -A_k \cdot q;
\]

and therefore* (by 135),

\[
\angle Kq = \angle \frac{1}{q}; \quad A_k \cdot Kq = A_k \cdot \frac{1}{q}.
\]

139. The reciprocal of a scalar, \( x \), is simply another scalar, \( \frac{1}{x} \), or \( x^{-1} \), having the same algebraic sign, and in all other respects related to \( x \) as in algebra. But the conjugate \( Kx \), of a scalar \( x \), considered as a limit of a quaternion, is equal to that scalar \( x \) itself; as may be seen by supposing the two equal but opposite angles, \( AOB \) and \( AOB' \), in fig. 36, to tend together to zero or to two right angles. We may therefore write, generally,

\[
Kx = x, \text{ if } x \text{ be any scalar;}
\]

and conversely,†

\[
q = \text{a scalar, if } Kq = q;
\]

because then (by 104) we must have \( OB = OB' \), \( BB' = 0 \); and therefore each of the two (now coincident) points \( B, B' \), must be situated somewhere on the indefinite right line \( OA \).

* It will soon be seen that these two last equations (138) express, that the conjugate and the reciprocal, of any proposed quaternion \( q \), have always equal versors, although they have in general unequal tensors. [See 157.]

† Somewhat later it will be seen that the equation \( Kq = q \) may also be written as \( Vq = 0 \); and that this last is another mode of expressing that the quaternion, \( q \), degenerates (131) into a scalar. [See 204, xiv.]
140. In general, by the construction represented in the same figure, the sum (comp. 6) of the two numerators (or dividend-lines, ob and ob'), of the two conjugate fractions (or quotients, or quaternions), q and Kq (137), is equal to the double of the line oa'; whence (by 106), the sum of those two conjugate quaternions themselves is,

\[ Kq + q = q + Kq = \frac{2oa'}{oa}; \]

this sum is therefore always scalar, being positive if the angle \( \angle q \) be acute, but negative if that angle be obtuse.

141. In the intermediate case, when the angle AOB is right, the interval OA' between the origin O and the line BB' vanishes; and the two lately mentioned numerators, OB, OB', become two opposite vectors, of which the sum is null (5). Now, in general, it is natural, and will be found useful, or rather necessary (for consistency with former definitions), to admit that a null vector, divided by an actual vector, gives always a NULL QUATERNION as the quotient; and to denote this null quotient by the usual symbol for Zero. In fact, we have (by 106) the equation,

\[ \frac{0}{a} = \frac{a - a}{a} = \frac{a}{a} - \frac{a}{a} = 1 - 1 = 0; \]

the zero in the numerator of the left-hand fraction representing here a null line (or a null vector, 1, 2); but the zero on the right-hand side of the equation denoting a null quotient (or quaternion). And thus we are entitled to infer that the sum, \( Kq + q \), or \( q + Kq \), of a right-angled quaternion, or right quotient (132), and of its conjugate, is always equal to zero.

142. We have, therefore, the three following formulae, whereof the second exhibits a continuity in the transition from the first to the third:

I. \[ q + Kq > 0, \text{ if } \angle q < \frac{\pi}{2}; \]

II. \[ q + Kq = 0, \text{ if } \angle q = \frac{\pi}{2}; \]

III. \[ q + Kq < 0, \text{ if } \angle q > \frac{\pi}{2}. \]

And because a quaternion, or geometric quotient, with an actual and finite divisor-line (as here OA), cannot become equal to zero unless its dividend-line vanishes, because by (104) the equation

\[ \frac{\beta}{a} = 0 = \frac{0}{a} \]

requires the equation \( \beta = 0 \),
if $a$ be any actual and finite vector, we may infer, conversely, that the sum $q + Kq$ cannot vanish, without the line $oa'$ also vanishing; that is, without the lines $on$, $on'$ becoming opposite vectors, and therefore the quaternion $q$ becoming a right quotient (132). We are therefore entitled to establish the three following converse formulae (which indeed result from the three former):

I'. . . if $q + Kq > 0$, then $\angle q < \frac{\pi}{2}$;

II'. . . if $q + Kq = 0$, then $\angle q = \frac{\pi}{2}$;

III'. . . if $q + Kq < 0$, then $\angle q > \frac{\pi}{2}$.

143. When two opposite vectors (1), as $\beta$ and $-\beta$, are both divided by one common (and actual) vector, $a$, we shall say that the two quotients, thus obtained are opposite quaternions; so that the opposite of any quaternion $q$, or of any quotient $\beta : a$, may be denoted as follows (comp. 4):

$$-\frac{\beta}{a} = 0 - \frac{\beta}{a} = 0 - \frac{\beta}{a} = 0 - q = -q;$$

while the quaternion $q$ itself may, on the same plan, be denoted (comp. 7) by the symbol $0 + q$, or $+q$. The sum of any two opposite quaternions is zero, and their quotient is negative unity; so that we may write, as in algebra (comp. again 7),

$$(-q) + q = (+q) + (-q) = 0; \quad (-q) : q = -1; \quad -q = (-1) q;$$

because, by 106 and 141,

$$-\frac{\beta}{a} + \frac{\beta}{a} = \frac{\beta}{a} - \frac{\beta}{a} = 0, \quad -\frac{\beta}{a} : \frac{\beta}{a} = \frac{-\beta}{\beta} = -1, \text{ &c.}$$

The reciprocals of opposite quaternions are themselves opposite; or in symbols (comp. 126),

$$-\frac{1}{q} = -\frac{1}{q}, \quad \text{because} \quad \frac{a}{-\beta} = -\frac{a}{\beta};$$

Opposite quaternions have opposite axes, and supplementary angles (comp. fig. 33, bis); so that we may establish (comp. 132, (5.)) the two following general formulae,

$$\angle (-q) = \pi - \angle q; \quad \text{Ax.} (-q) = -\text{Ax.} . q.$$
and conversely* (comp. 138),

\[ \Pi'''' \ldots \text{if } Kq = -q, \text{ then } \angle Kq = \angle q = \frac{\pi}{2}. \]

In words, the conjugate of a right quotient, or of a right-angled (or right) quaternion (132), is the right quotient opposite thereto; and conversely, if an actual quaternion (that is, one which is not null) be opposite to its own conjugate it must be a right quotient.

(1) If then we meet the equation,

\[ \frac{Kp}{a} = -\frac{p}{a}, \text{ or } \frac{p}{a} + \frac{Kp}{a} = 0, \]

we shall know that \( \rho \perp a \); and therefore (if \( a = oA \), and \( \rho = op \), as before), that the locus of the point \( p \) is the plane through \( o \), perpendicular to the line \( oa \) (as in 132, (1.)).

(2.) On the other hand, the equation,

\[ \frac{Kp}{a} = +\frac{p}{a}, \text{ or } \frac{p}{a} - \frac{Kp}{a} = 0, \]

expresses (by 139) that the quotient \( \rho : a \) is a scalar; and therefore (by 131) that its angle \( \angle (\rho : a) \) is either 0 or \( \pi \); so that in this case, the locus of \( p \) is the indefinite right line through the two points \( o \) and \( a \).

145. As the opposite of the opposite, or the reciprocal of the reciprocal, so also the conjugate of the conjugate, of any quaternion, is that quaternion itself; or in symbols,

\[ \frac{K^2}{q} = q = 1, \]

and conversely, that the conjugates of opposite quaternions are themselves opposite quaternions; and that the conjugates of reciprocals are reciprocal: or in symbols, that

\[ I \ldots K(-q) = -Kq, \text{ or } Kq + K(-q) = 0; \]

and

\[ II \ldots K\frac{1}{q} = \frac{1}{Kq}, \text{ or } Kq \cdot K\frac{1}{q} = 1. \]

* It will be seen at a later stage, that the equation \( Kq = -q \), or \( q + Kq = 0 \), may be transformed to this other equation, \( Sq = 0 \); and that, under this last form, it expresses that the scalar part of the quaternion \( q \) vanishes: or that this quaternion is a right quotient (132). [See 196, II.]
(1.) The equation \( K(-q) = -Kq \) is included (comp. 143) in this more general formula, \( K(xq) = xKq \), where \( x \) is any scalar; and this last equation (comp. 126) may be proved, by simply conceiving that the two lines \( ob, ob' \), in fig. 36, are multiplied by any common scalar; or that they are both cut by any parallel to the line \( bb' \).

(2.) To prove that conjugates of reciprocals are reciprocal, or that \( Kq \cdot K\frac{1}{q} = 1 \), we may conceive that, as in the annexed figure 36, \( bis \), while we have still the relation of inverse similitude,

\[ \triangle AOB' \propto \triangle AOB \ (118, 137) \]

as in the former figure 36, a new point \( c \) is determined, either on the line \( oa \) itself, or on that line prolonged through \( a \), so as to satisfy either of the two following connected conditions of direct similitude:

\[ \triangle BOC \propto \triangle AOB ; \quad \triangle B'OC \propto \triangle AOB ; \]

or simply, as a relation between the four points \( o, a, b, c \), the formula,

\[ \triangle BOC \propto \triangle AOB. \]

For then we shall have the transformations,

\[ K \frac{1}{q} = K \frac{oa}{ob} = K \frac{ob'}{ob} = \frac{ob}{oa} = \frac{oa}{ob} = \frac{1}{Kq}. \]

(3.) The two quotients \( ob : oa \), and \( ob' : oc \), that is to say, the quaternion \( q \) itself and the conjugate of its reciprocal, or* the reciprocal of its conjugate, have the same angle, and the same axis; we may therefore write, generally,

\[ \angle K \frac{1}{q} = \angle q; \quad Ax \cdot K \frac{1}{q} = Ax \cdot q. \]

(4.) Since \( oa : ob \) and \( oa : ob' \) have thus been proved (by sub-art. 2) to be a pair of conjugate quotients, we can now infer this theorem, that any two geometric fractions, \( \frac{a}{\beta} \) and \( \frac{a}{\beta'} \), which have a common numerator \( a \), are conjugate

* It will be seen afterwards, that the common value of these two equal quaternions, \( K \frac{1}{q} \) and \( K \frac{1}{Kq} \), may be represented by either of the two new symbols, \( Uq : Tq \), or \( q : Nq \); or in words, that it is equal to the versor divided by the tensor; and also to the quaternion itself divided by the norm. [See 190, (3).]
quaternions, if the denominator $\beta'$ of the second be the reflexion of the denominator $\beta$ of the first, with respect to that common numerator (comp. 138, I.); whereas it had only been previously assumed, as a definition (137), that such conjugation exists, under the same geometrical condition, between the two other (or inverse) fractions, $\frac{\beta}{a}$ and $\frac{\beta'}{a}$; the three vectors $a$, $\beta$, $\beta'$ being supposed to be all co-initial (18).

(5.) Conversely, if we meet, in any investigation, the formula

$$\frac{oa}{ob'} = K \left(\frac{oa}{ob}\right),$$

we shall know that the point $b'$ is the reflexion of the point $b$, with respect to the line $oa$; or that this line, $oa$, prolonged if necessary in either of two opposite directions, bisects at right angles the line $bb'$, in some point $\lambda'$, as in either of the two figures 36 (comp. 138, II.).

(6.) Under the recent conditions of construction, it follows from the most elementary principles of geometry, that the circle, which passes through the three points $a$, $b$, $c$, is touched at $b$, by the right line $ob$; and that this line is, in length, a mean proportional between the lines $oa$, $oc$. Let then $on$ be such a geometric mean, and let it be set off from $o$ in the common direction of the two last mentioned lines, so that the point $d$ falls between $a$ and $c$; also let the vectors $oc$, $od$ be denoted by the symbols $\gamma$, $\delta$; we shall then have expressions of the forms,

$$\delta = aa, \quad \gamma = a^2a,$$

where $a$ is some positive scalar, $a > 0$; and the vector $\beta$ of $b$ will be connected (comp. sub-art. 2) with this scalar $a$, and with the vector $a$, by the formula

$$\frac{ob}{oc} = K \frac{oa}{ob}, \quad \text{or} \quad \frac{oc}{ob} = K \frac{ob}{oa}, \quad \text{or} \quad a^2a = K \frac{\beta}{a}.$$

(7.) Conversely, if we still suppose that $\gamma = a^2a$, this last formula expresses the inverse similitude of triangles, $\Delta$ $boc$ $\propto \Delta$ $aob$; and it expresses nothing more: or in other words, it is satisfied by the vector $\beta$ of every point $b$, which gives that inverse similitude. But for this purpose it is only requisite that the length of $ob$ should be (as above) a geometric mean between the lengths of $oa$, $oc$; or that the two lines, $ob$, $od$ (sub-art. 6), should be equally long: or finally, that $b$ should be situated somewhere on the surface of a sphere, which is described so as to pass through the point $d$ (in fig. 36, bis), and to have the origin $o$ for its centre.

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(8.) If then we meet an equation of the form,

$$\frac{a^2a}{p} = K \frac{p}{a}, \quad \text{or} \quad K \frac{p}{a} = a^2,$$

in which $a = OA$, $p = OP$, and $a$ is a scalar, as before, we shall know that the locus of the point $P$ is a spheric surface, with its centre at the point $O$, and with the vector $aa$ for a radius; and also that if we determine a point $C$ by the equation $OC = a^2a$, this spheric locus of $P$ is a common orthogonal to all the circles $APC$, which can be described, so as to pass through the two fixed points, $A$ and $C$: because every radius $OP$ of the sphere is a tangent, at the variable point $P$, to the circle $APC$, exactly as $OB$ is to $ABC$ in the recent figure.

(9.) In the same fig. 36, bis, the similar triangles show (by elementary principles) that the length of $BC$ is to that of $AB$ in the sub-duplicate ratio of $OC$ to $OA$; or in the simple ratio of $OD$ to $OA$; or as the scalar $a$ to 1. If then we meet, in any research, the recent equation in $p$ (sub-art. 8), we shall know that

$$\text{length of } (p - a^2a) = a \times \text{length of } (p - a),$$

while the recent interpretation of the same equation gives this other relation of the same kind:

$$\text{length of } p = a \times \text{length of } a.$$

(10.) At a subsequent stage [200 (3)], it will be shown that the Calculus of Quaternions supplies Rules of Transformation, by which we can pass from any one to any other of these last equations respecting $p$, without (at the time) constructing any Figure, or (immediately) appealing to Geometry: but it was thought useful to point out, already, how much geometrical meaning* is contained in so simple a formula, as that of the last sub-art. 8.

(11.) The product of two conjugate quaternions is said to be their common Norm,† and is denoted thus:

$$qKq = Nq.$$
It follows that \( \text{NKq} = Nq \); and that the norm of a quaternion is generally a positive scalar: namely, the square of the quotient of the lengths of the two lines of which (as vectors) the quaternion itself is the quotient (112). In fact we have, by sub-art. 6, and by the definition of a norm, the transformations:

\[
\frac{\text{N} \overrightarrow{OB}}{\overrightarrow{OA}} = \frac{\text{N} \overrightarrow{OB'}}{\overrightarrow{OA}} = \frac{\text{OC} \cdot \overrightarrow{OB'}}{\overrightarrow{OA}} = \frac{\text{OC} \cdot \overrightarrow{OB}}{\overrightarrow{OA}} = \frac{\text{OC} \cdot \overrightarrow{OD}}{\overrightarrow{OA}};
\]

\[
\text{Nq} = \text{N} \frac{\overrightarrow{\beta}}{a} = \frac{\overrightarrow{\beta}}{a} \frac{\text{T}}{a} = \left( \frac{\text{length of } \overrightarrow{\beta}}{\text{length of } a} \right)^2.
\]

As a limit, we may say that the norm of a null quaternion is zero; or in symbols, \( \text{N0} = 0 \).

(12.) With this notation, the equation of the spheric locus (sub-art. 8), which has the point o for its centre, and the vector \( aa \) for one of its radii, assumes the shorter form:

\[
\text{N} \frac{\rho}{a} = a^2; \quad \text{or } \frac{\rho}{aa} = 1.
\]

SECTION 7.

On Radial Quotients; and on the Square of a Quaternion.

146. It was early seen (comp. Art. 2, and fig. 4) that any two radii, \( ab, ac \), of any one circle, or sphere, are necessarily unequal vectors; because their directions differ. On the other hand, when we are attending only to relative direction (110), we may suppose that all the vectors compared are not merely co-initial (18), but are also equally long; so that if their common length be taken for the unit, they are all radii, \( oa, ob, .. \) of what we have called the Unit-Sphere (128), described round the origin as centre; and may all be said to be Unit-Vectors (129). And then the quaternion, which is the quotient of any one such vector divided by any other, or generally the quotient of any two equally long vectors, may be called a Radial Quotient; or sometimes simply a Radial. (Compare the annexed figure 39.)

and the characteristic, of the Tensor, \( Tq \), of a quaternion, which is of greater geometrical utility than the Norm, but of which it will be proved that this norm is simply the square,

\[
qKq = Nq = (Tq)^2.
\]

Compare the Note to sub-art. 3.
147. The two Unit-Scalars, namely, Positive and Negative Unity, may be considered as limiting cases of radial quotients, corresponding to the two extreme values, 0 and $\pi$, of the angle $\alpha \beta \delta$, or $\angle q$ (131). In the intermediate case, when $\alpha \beta \delta$ is a right angle, or $\angle q = \frac{\pi}{2}$, as in fig. 40, the resulting quotient, or quaternion, may be called (comp. 132) a Right Radial Quotient; or simply, a Right Radial. The consideration of such right radicals will be found to be of great importance, in the whole theory and practice of Quaternions.

148. The most important general property of the quotients last mentioned is the following: that the Square of every Right Radial is equal to Negative Unity; it being understood that we write generally, as in algebra,

$$q \cdot q = q^2,$$

and call this product of two equal quaternions the square of each of them.

For if, as in fig. 41, we describe a semicircle $\alpha \beta \alpha'$, with $\alpha$ for centre, and with $\alpha \beta$ for the bisecting radius, then the two right quotients, $\alpha \beta : \alpha \beta$, and $\alpha \beta' : \alpha \beta$, are equal (comp. 117); and therefore their common square is (comp. 107) the product,

$$\left( \frac{\alpha \beta}{\alpha \beta} \right)^2 = \frac{\alpha' \beta}{\beta} \cdot \frac{\alpha \beta'}{\alpha \beta} = \frac{\alpha' \beta}{\alpha \beta} = -1;$$

where $\alpha \beta$ and $\alpha \beta$ may represent any two equally long, but mutually rectangular lines. More generally, the Square of every Right Quotient (132) is equal to a Negative Scalar; namely, to the negative of the square of the number, which represents the ratio of the lengths* of the two rectangular lines compared; or to zero minus the square of the number which denotes (comp. 133) the length of the Index of that Right Quotient: as appears from fig. 41, bis, in which $\alpha \beta$ is

* Hence, by 145 (11.), $q^2 = -Nq$, if $\angle q = \frac{\pi}{2}$. 

only an ordinate, and not (as before) a radius, of the semicircle \( \text{ABA}' \); for we have thus,

\[
\left( \frac{\text{OB}}{\text{OA}} \right)^2 = \frac{\text{OA}'}{\text{OA}} = -\left( \frac{\text{length of OB}}{\text{length of OA}} \right)^2, \text{ if } \text{OB} \perp \text{OA}.
\]

149. Thus every Right Radial is, in the present System, one of the Square Roots of Negative Unity; and may therefore be said to be one of the Values of the Symbol \( \sqrt{-1} \); which celebrated symbol has thus a certain degree of vagueness, or at least of indetermination, of meaning in this theory, on account of which we shall not often employ it. For although it thus admits of a perfectly clear and geometrically real Interpretation, as denoting what has been above called a Right Radial Quotient, yet the Plane of that Quotient is arbitrary; and therefore the symbol itself must be considered to have (in the present system) indefinitely many values; or in other words the Equation, \( q^2 = -1 \), has (in the Calculus of Quaternions) indefinitely many Roots, which are all Geometrical Reals; besides any other roots of a purely symbolical character, which the same equation may be conceived to possess, and which may be called Geometrical Imaginaries. Conversely, if \( q \) be any real quaternion, which satisfies the equation \( q^2 = -1 \), it must be a right radial; for if, as in fig. 42, we suppose that \( \Delta \text{AOB} \propto \text{BOC} \), we shall have

\[
q^2 = \left( \frac{\text{OB}}{\text{OA}} \right)^2 = \frac{\text{OC} \cdot \text{OB}}{\text{OB} \cdot \text{OA}} = \frac{\text{OC}}{\text{OA}};
\]

and this square of \( q \) cannot become equal to negative unity, except by \( \text{OC} \) being \(-\text{OA} \), or \( \text{OA}' \) in fig. 41; that is, by the line \( \text{OA} \) being at right angles to the line \( \text{OA} \), and being at the same time equally long, as in fig. 40.

(1.) If then we meet the equation,

\[
\left( \frac{p}{a} \right)^2 = -1,
\]

where \( a = \text{OA} \), and \( p = \text{OP} \), as before, we shall know that the locus of the point

* It will be subsequently shown [in 222], that if \( x, y, z \) be any three scalars, of which the sum of the squares is unity, so that

\[ x^2 + y^2 + z^2 = 1; \]

and if \( i, j, k \) be any three right radials, in three mutually rectangular planes; then the expression,

\[ q = ix + jy + kz, \]

denotes another right radial, which satisfies (as such, and by symbolical laws to be assigned) the equation \( q^2 = -1 \); and is therefore one of the geometrically real values of the symbol \( \sqrt{-1} \).

† Such imaginaries will be found to offer themselves, in the treatment by Quaternions (or rather by what will be called Biquaternions), of ideal intersections, and of ideal contacts, in geometry (see 214); but we confine our attention, for the present, to geometrically reals alone. Compare the Notes to pages 87 and 88.
\( r \) is the circumference of a circle, with \( o \) for its centre, and with a radius which has the same length as the line \( oA \); while the plane of the circle is perpendicular to that given line. In other words, the locus of \( r \) is a great circle, on a sphere of which the centre is the origin; and the given point \( A \), on the same spheric surface, is one of the poles of that circle.

(2.) In general, the equation \( q^2 = -a^2 \), where \( a \) is any (real) scalar, requires that the quaternion \( q \) (if real) should be some right quotient (132); the number \( a \) denoting the length of the index (133), of that right quotient or quaternion (comp. Art. 148, and fig. 41, bis). But the plane of \( q \) is still entirely arbitrary; and therefore the equation

\[
q^2 = -a^2,
\]

like the equation \( q^2 = -1 \), which it includes, must be considered to have (in the present system) indefinitely many geometrically real roots.

(3.) Hence the equation,

\[
\left(\frac{\rho}{a}\right)^2 = -a^2,
\]

in which we may suppose that \( a > 0 \), expresses that the locus of the point \( r \) is a (new) circular circumference, with the line \( oA \) for its axis, and with a radius of which the length = \( a \times \) the length of \( oA \).

150. It may be added that the index (133), and the axis (128), of a right radial (147), are the same; and that its reciprocal (134), its conjugate (137), and its opposite (143), are all equal to each other. Conversely, if the reciprocal of a given quaternion \( q \) be equal to the opposite of that quaternion, then \( q \) is a right radial; because its square, \( q^2 \), is then equal (comp. 136) to the quaternion itself, divided by its opposite; and therefore (by 143) to negative unity. But the conjugate of every radial quotient is equal to the reciprocal of that quotient; because if, in fig. 36 [p. 115], we conceive that the three lines \( oA, oB, oB' \) are equally long, or if, in fig. 39, we prolong the arc \( ba \), by an equal arc \( AB' \), we have the equation,

\[
Kq = \frac{oB'}{oA} = \frac{oA}{oB} = \frac{1}{q}.
\]

And conversely,\(^+\)

if \( Kq = \frac{1}{q} \), or if \( qKq = 1 \),

then the quaternion \( q \) is a radial quotient.

---

* It being understood, that the axis of a circle is a right line perpendicular to the plane of that circle, and passing through its centre.

+ Hence, in the notation of norms (145, (11)), if \( Nq = 1 \), then \( q \) is a radial; and conversely, the norm of a radial quotient, is always equal to positive unity.
SECTION 8.

On the Versor of a Quaternion, or of a Vector; and on some General Formulae of Transformation.

151. When a quaternion \( q = \beta : a \) is thus a radial quotient (146), or when the lengths of the two lines \( a \) and \( \beta \) are equal, the effect of this quaternion \( q \), considered as a Factor (103), in the equation \( qa = \beta \), is simply the turning of the multiplicand-line \( a \), in the plane of \( q \) (119), and towards the hand determined by the direction of the positive axis \( Ax \cdot q \) (129), through the angle denoted by \( \angle q \) (130); so as to bring that line \( a \) (or a revolving line which had coincided therewith) into a new direction: namely, into that of the product-line \( \beta \). And with reference to this conceived operation of turning, we shall now say that every Radial Quotient is a Versor.

152. A Versor has thus, in general, a plane, an axis, and an angle; namely, those of the Radial (146) to which it corresponds, or is equal: the only difference between them being a difference in the points of view* from which they are respectively regarded; namely, the radial as the quotient, \( q \), in the formula, \( q = \beta : a \); and the versor as the (equal) factor, \( q \), in the converse formula, \( \beta = q \cdot a \); where it is still supposed that the two vectors, \( a \) and \( \beta \), are equally long.

153. A versor, like a radial (147), cannot degenerate into a scalar, except by its angle acquiring one or other of the two limit-values, 0 and \( \pi \). In the first case, it becomes positive unity; and in the second case, it becomes negative unity: each of these two unit-scalars (147) being here regarded as a factor (or coefficient, comp. 12), which operates on a line, to preserve or to reverse its direction. In this view, we may say that \(-1\) is an Inversor; and that every Right Versor (or versor with an angle=\( \pi \)) is a Semi-inversor:* because it half-inverts the line on which it operates, or turns it through half of two right angles.

---

* In a slightly metaphysical mode of expression it may be said, that the radial quotient is the result of an analysis, wherein two radii of one sphere (or circle) are compared, as regards their relative direction; and that the equal versor is the instrument of a corresponding synthesis, wherein one radius is conceived to be generated, by a certain rotation, from the other.

† This word, "semi-inversor," will not be often used; but the introduction of it here, in passing, seems adapted to throw light on the view taken, in the present work, of the symbol \( \sqrt{-1} \), when regarded as denoting a certain important class (149) of Reals in Geometry. There are uses of that symbol, to denote Geometrical Imaginaries (comp. again Art. 149, and the Notes to pages 87 and 88), considered as connected with ideal intersections, and with ideal contacts; but with such uses of \( \sqrt{-1} \) we have, at present, nothing to do.
(comp. fig. 41). For the same reason, we are led to consider every right versor (like every right radial, 149, from which indeed we have just seen, in 152, that it differs only as factor differs from quotient), as being one of the square roots of negative unity: or as one of the values of the symbol \( \sqrt{-1} \).

154. In fact we may observe that the effect of a right versor, considered as operating on a line (in its own plane), is to turn that line, towards a given hand, through a right angle. If then \( q \) be such a versor, and if \( qa = \beta \), we shall have also (comp. fig. 41), \( q\beta = -a \); so that, if \( a \) be any line in the plane of a right versor \( q \), we have the equation,

\[
q \cdot qa = -a ;
\]

whence it is natural to write, under the same condition,

\[
q^2 = -1,
\]

as in 149. On the other hand, no versor, which is not right-angled, can be a value of \( \sqrt{-1} \); or can satisfy the equation \( q^2a = -a \), as fig. 42 may serve to illustrate. For it is included in the meaning of this last equation, as applied to the theory of versors, that a rotation through \( 2\gamma q \), or through the double of the angle of \( q \) itself, is equivalent to an inversion of direction; and therefore to a rotation through two right angles.

155. In general, if \( a \) be any vector, and if \( a \) be used as a temporary* symbol for the number expressing its length; so that \( a \) is here a positive scalar, which bears to positive unity, or to the scalar \( +1 \), the same ratio as that which the length of the line \( a \) bears to the assumed unit of length (comp. 128); then the quotient \( a : a \) denotes generally (comp. 16) a new vector, which has the same direction as the proposed vector \( a \), but has its length equal to that assumed unit: so that it is (comp. 146) the Unit-Vector in the direction of \( a \). We shall denote this unit-vector by the symbol, \( Ua \); and so shall write, generally,

\[
Ua = \frac{a}{a}, \quad \text{if } a = \text{length of } a;
\]

that is, more fully, if \( a \) be, as above supposed, the number (commensurable or incommensurable, but positive) which represents that length, with reference to some selected standard.

156. Suppose now that \( q = \beta : a \) is (as at first) a general quaternion, or the quotient of any two vectors, \( a \) and \( \beta \), whether equal or unequal in length. Such a Quaternion will not (generally) be a Versor (or at least not simply such),

* We shall soon propose [in 185] a general notation for representing the lengths of vectors, according to which the symbol \( Ta \) will denote what has been above called \( a \); but are unwilling to introduce more than one new characteristic of operation, such as \( K \), or \( T \), or \( U \), &c., at one time.
Arts. 153–157.] VERSOR OF A QUATERNION, OR OF A VECTOR. 137

according to the definition lately given; because its effect, when operating as a factor (103) on \( a \), will not in general be simply to turn that line (151): but will (generally) alter the length,* as well as the direction. But if we reduce the two proposed vectors, \( a \) and \( \beta \), to the two unit-vectors \( Ua \) and \( U\beta \) (155), and form the quotient of these, we shall then have taken account of relative direction alone: and the result will therefore be a versor, in the sense lately defined (151). We propose to call the quotient, or the versor, thus obtained, the versor-element, or briefly, the Versor, of the Quaternion \( q \); and shall find it convenient to employ the same† Characteristic, \( U \), to denote the operation of taking the versor of a quaternion, as that employed above to denote the operation (155) of reducing a vector to the unit of length, without any change of its direction. On this plan, the symbol \( Uq \) will denote the versor of \( q \); and the foregoing definitions will enable us to establish the General Formula:

\[
Uq = \frac{U\beta}{a} = \frac{U\beta}{Ua};
\]

in which the two unit-vectors, \( Ua \) and \( U\beta \), may be called, by analogy, and for other reasons which will afterwards appear, the versors‡ of the vectors, \( a \) and \( \beta \).

157. In thus passing from a given quaternion, \( q \), to its versor, \( Uq \), we have only changed (in general) the lengths of the two lines compared, namely, by reducing each to the assumed unit of length (155, 156), without making any change in their directions. Hence the plane (119), the axis (127, 128), and the angle (130), of the quaternion, remain unaltered in this passage; so that we may establish the two following general formulæ:

\[
\angle Uq = \angle q; \quad Ax \cdot Uq = Ax \cdot q.
\]

* By what we shall soon call an act of tension, which will lead us to the consideration of the tensor of a quaternion.

† For the moment, this double use of the characteristic \( U \), to assist in denoting both the unit-vector \( Ua \) derived from a given line \( a \), and also the versor \( Uq \) derived from a quaternion \( q \), may be regarded as established here by arbitrary definition; but as permitted, because the difference of the symbols, as here \( a \) and \( q \), which serve for the present to denote vectors and quaternions, considered as the subjects of these two operations \( U \), will prevent such double use of that characteristic from giving rise to any confusion. But we shall further find that several important analogies are by anticipation expressed, or at least suggested, when the proposed notation is employed. Thus it will be found (comp. the Note to page 121), that every vector \( a \) may usefully be equated to that right quotient, of which it is (133) the index; and that then the unit-vector \( Ua \) may be, on the same plan, equated to that right radial (147), which is (in the sense lately defined) the versor of that right quotient. We shall also find ourselves led to regard every unit-vector as the axis of a quadrantal (or right) rotation, in a plane perpendicular to that axis; which will supply another inducement, to speak of every such vector as a versor. On the whole, it appears that there will be no inconvenience, but rather a prospective advantage, in our already reading the symbol \( Ua \) as "versor of \( a \)"; just as we may read the analogous symbol \( Uq \), as "versor of \( q \)." [Compare 286 and 290.]

‡ Compare the Note immediately preceding.

Hamilton's Elements of Quaternions.
More generally we may write,

$$\angle q' = \angle q, \text{ and } \text{Ax} \cdot q' = \text{Ax} \cdot q, \text{ if } Uq' = Uq;$$

the versor of a quaternion depending solely on, but conversely being sufficient to determine, the relative direction (156) of the two lines, of which (as vectors) the quaternion itself is the quotient (112); or the axis and angle of the rotation, in the plane of those two lines, from the divisor to the dividend (128): so that any two quaternions, which have equal versors, must also have equal angles, and equal (or coincident) axes, as is expressed by the last written formula. Conversely, from this dependence of the versor $Uq$ on relative direction* alone, it follows that any two quaternions, of which the angles and the axes are equal, have also equal versors; or in symbols, that

$$Uq' = Uq, \text{ if } \angle q' = \angle q, \text{ and } \text{Ax} \cdot q' = \text{Ax} \cdot q.$$  

For example, we saw (in 138) that the conjugate and the reciprocal of any quaternion have thus their angles and their axes the same; it follows, therefore, that the versor of the conjugate is always equal to the versor of the reciprocal; so that we are permitted to establish the following general formula,†

$$UKq = \frac{U}{q}.$$

158. Again, because

$$U\left(1; \frac{\beta}{a}\right) = U\frac{a}{\beta} = \frac{Ua}{U\beta} = 1; \frac{U\beta}{Ua} = 1; \frac{U\beta}{a},$$

it follows that the versor of the reciprocal of any quaternion is, at the same time, the reciprocal of the versor; so that we may write,

$$U\frac{1}{q} = \frac{1}{Uq}; \text{ or } Uq \cdot U\frac{1}{q} = 1.$$

Hence, by the recent result (157), we have also, generally,

$$UKq = \frac{1}{Uq}; \text{ or, } Uq \cdot UKq = 1.$$

* The unit-vector $Ua$, which we have recently proposed (156) to call the versor of the vector $a$, depends in like manner on the direction of that vector alone; which exclusive reference, in each of these two cases, to direction, may serve as an additional motive for employing, as we have lately done, one common name, versor, and one common characteristic, $U$, to assist in describing or denoting both the Unit-Vector $Ua$ itself, and the Quotient of two such Unit-Vectors, $Uq = Ua : U\beta$; all danger of confusion being sufficiently guarded against (comp. the Note to Art. 156), by the difference of the two symbols, $a$ and $q$, employed to denote the vector and the quaternion, which are respectively the subjects of the two operations $U$; while those two operations agree in this essential point, that each serves to eliminate the quantitative element, of absolute or relative length.

† Compare the Note to Art. 138.
Also, because the versor \( Uq \) is always a radial quotient (151, 152), it is (by 150) the conjugate of its own reciprocal; and therefore, at the same time (comp. 145), the reciprocal of its own conjugate; so that the product of two conjugate versors, or what we have called (145, (11.)) their common Norm, is always equal to positive unity; or in symbols (comp. 150),

\[
NUq = Uq \cdot KUq = 1.
\]

For the same reason, the conjugate of the versor of any quaternion is equal to the reciprocal of that versor, or (by what has just been seen) to the versor of the reciprocal of that quaternion; and therefore also (by 157), to the versor of the conjugate; so that we may write generally, as a summary of recent results, the formula:

\[
KUq = \frac{1}{Uq} = U^{\frac{1}{q}} = UKq;
\]

each of these four symbols denoting a new versor, which has the same plane, and the same angle, as the old or given versor \( Uq \), but has an opposite axis, or an opposite direction of rotation: so that, with respect to that given versor, it may naturally be called a Reversor.

159. As regards the versor itself; whether of a vector or of a quaternion, the definition (155) of \( Ua \) gives,

\[
Uxa = + Ua, \quad \text{or} \quad = - Ua, \quad \text{according as } x > \text{or} < 0;
\]

because (by 15) the scalar coefficient \( x \) preserves, in the first case, but reverses, in the second case, the direction of the vector \( a \); whence also, by the definition (156) of \( Uq \), we have generally (comp. 126, 143),

\[
Uxq = + Uq, \quad \text{or} \quad = - Uq, \quad \text{according as } x > \text{or} < 0.
\]

The versor of a scalar, regarded as the limit of a quaternion (131, 139), is equal to positive or negative unity (comp. 147, 153), according as the scalar itself is positive or negative; or in symbols,

\[
Ux = + 1, \quad \text{or} \quad = - 1, \quad \text{according as } x > \text{or} < 0;
\]

the plane and axis of each of these two unit scalars (147), considered as versors (153), being (as we have already seen) indeterminate. The versor of a null quaternion (141) must be regarded as wholly arbitrary, unless we happen to know a law,* according to which the quaternion tends to zero, before actually reaching that limit; in which latter case, the plane, the axis, and the angle of

* Compare the Note to Art. 131.
the versor* \( \text{U}_0 \) may all become determined, as limits deduced from that law. The versor of a right quotient (132), or of a right-angled quaternion (141), is always a right radial (147), or a right versor (153); and therefore is, as such, one of the square roots of negative unity (149), or one of the values of the symbol \( \sqrt{-1} \); while (by 150) the axis and the index of such a versor coincide; and in like manner its reciprocal, its conjugate, and its opposite are all equal to each other.

160. It is evident that if a proposed quaternion \( q \) be already a versor (151), in the sense of being a radial (146), the operation of taking its versor (156) produces no change; and in like manner that, if a given vector \( a \) be already an unit-vector, it remains the same vector, when it is divided (155) by its own length; that is, in this case, by the number one. For example, we have assumed (128, 129), that the axis of every quaternion is an unit-vector; we may therefore write, generally, in the notation of 155, the equation,

\[
\text{U} (\text{Ax} \cdot q) = \text{Ax} \cdot q.
\]

A second operation \( \text{V} \) leaves thus the result of the first operation \( \text{U} \) unchanged, whether the subject of such successive operations be a line, or a quaternion; we have therefore the two following general formulæ, differing only in the symbols of that subject:

\[
\text{UU}a = \text{U}a; \quad \text{UU}q = \text{U}q;
\]

whence, by abstracting (comp. 145) from the subject of the operation, we may write, briefly and symbolically,

\[
\text{U}^2 = \text{UU} = \text{U}.
\]

161. Hence, with the help of 145, 158, 159, we easily deduce the following (among other) transformations of the versor of a quaternion:

\[
\text{U}q = \frac{1}{\text{U}_q} = \frac{1}{\text{KU}_q} = \frac{1}{\text{UK}_q} = \text{KU} = \frac{1}{\text{q}} = \text{KU} \frac{1}{\text{q}} = \text{KU}q
\]

\[
= \text{U} \frac{1}{\text{Kq}} = \text{UK} = \text{U}q = \text{UKU} = \text{UK} \frac{1}{\text{q}} = \text{UK} \frac{1}{\text{q}} = (\text{UK})q;
\]

\[
\text{U}q = \text{U}xq, \text{ if } x > 0; \quad = - \text{U}xq, \text{ if } x < 0.
\]

We may also write, generally,

\[
\frac{q}{\text{Kq}} = \frac{\text{U}q}{\text{KUq}} = (\text{U}q)^2 = \text{U}(q^2) = \text{U}q^2;
\]

* When the zero in this symbol, \( \text{U}_0 \), is considered as denoting a null vector (2), the symbol itself denotes generally, by the foregoing principles, an indeterminate unit-vector; although the direction of this unit-vector may, in certain questions, become determined, as a limit resulting from a law.
the parentheses being here unnecessary, because (as will soon be more fully seen) the symbol $Uq^2$ denotes one common versor, whether we interpret it as denoting the square of the versor, or as the versor of the square, of $q$. The present Calculus will be found to abound in General Transformations of this sort; which all (or nearly all), like the foregoing, depend ultimately on very simple geometrical conceptions; but which, notwithstanding (or rather, perhaps, on account of) this extreme simplicity of their origin, are often useful, as elements of a new kind of Symbolical Language in Geometry: and generally, as instruments of expression, in all those mathematical or physical researches to which the Calculus of Quaternions can be applied. It is, however, by no means necessary that a student of the subject, at the present stage, should make himself familiar with all the recent transformations of $Uq$; although it may be well that he should satisfy himself of their correctness, in doing which the following remarks will perhaps be found to assist.

(1.) To give a geometrical illustration, which may also serve as a proof, of the recent equation,

$$q : Kq = (Uq)^2,$$

we may employ fig. 36, bis [p. 128]; in which, by 145, (2.), we have

$$1 \cdot \frac{OB}{OA} \cdot \frac{OA}{OB} = \frac{OB}{OD} = \left(\frac{OB}{OA}\right)^2 = (U_{OB})^2 = (Uq)^2.$$

(2.) As regards the equation, $\sqrt{q^2} = (Uq)^2$, we have only to conceive that the three lines $oa, ob, oc$, of fig. 42, are cut (as in fig. 42, bis) in three new points, $a', b', c'$, by an unit-circle (or by a circle with a radius equal to the unit of length), which is described about their common origin $o$ as centre, and in their common plane; for then if these three lines be called $a, b, c$, the three new lines $oa', ob', oc'$ are (by 155) the three unit-vectors denoted by the symbols, $Ua, Ub, Uc$; and we have the transformations (comp. 148, 149),

$$\sqrt{q^2} = U \cdot \left(\frac{B}{a}\right)^2 = U \frac{U}{a} \frac{U_{a}}{oa} = \frac{oc}{oa} = \left(\frac{ob}{oa}\right)^2 = (Uq)^2.$$

(3.) As regards other recent transformations (161), although we have seen (135) that it is not necessary to invent any new or peculiar symbol, to represent the reciprocal of a quaternion, yet if, for the sake of present convenience, and as a merely temporary notation, we write

$$Rq = \frac{1}{q},$$

employing thus, for a moment, the letter $R$ as a characteristic of reciprocation,
or of the operation of taking the reciprocal, we shall then have the symbolic equations (comp. 145, 158):

\[ R^2 = K^2 = 1; \quad RK = KR; \quad RU = UR = KU = UK; \]

but we have also (by 160), \( U^3 = U \); whence it easily follows that

\[ U = RUR = RKU = RUK = KUR = KRU = KUK \]
\[ = UKR = UK = UKUR = UKRU = (UK)^2 = \&c. \]

(4.) The equation

\[ U^{\rho} = U^{\beta}, \quad \text{or simply,} \quad U^\rho = U^\beta, \]

expresses that the locus of the point \( p \) is the indefinite right line, or ray (comp. 132, (4.)), which is drawn from \( o \) in the direction of \( ob \), but not in the opposite direction; because it is equivalent to

\[ U^{\rho} = 1; \quad \text{or} \quad \omega^{\rho} = 0; \quad \text{or} \quad \rho = x\beta, \quad x > 0. \]

(5.) On the other hand the equation,

\[ U^{\rho} = - U^{\beta}, \quad \text{or} \quad U^\rho = - U^\beta, \]

expresses (comp. 132, (5.)) that the locus of \( p \) is the opposite ray from \( o \); or that it is the indefinite prolongation of the revector \( bo \); because it may be transformed to

\[ U^{\rho} = - 1; \quad \text{or} \quad \omega^{\rho} = \pi; \quad \text{or} \quad \rho = x\beta, \quad x < 0. \]

(6.) If \( a, \beta, \gamma \) denote (as in sub-art. 2) the three lines \( oa, ob, oc \) of fig. 42 (or of fig. 42, bis), so that (by 149) we have the equation \( \frac{\gamma}{a} = \left(\frac{b}{a}\right)^2 \), then this other equation,

\[ \left(\frac{U^\rho}{a}\right)^2 = \left(\frac{U^\gamma}{a}\right), \]

expresses generally that the locus of \( p \) is the system of the two last loci; or that it is the whole indefinite right line, both ways prolonged, through the two points \( o \) and \( b \) (comp. 144, (2.)).

(7.) But if it happen that the line \( \gamma \), or \( oc \), like \( oa' \) in fig. 41 (or in fig. 41, bis), has the direction opposite to that of \( a \), or of \( oa \), so that the last equation takes the particular form,

\[ \left(\frac{U^\rho}{a}\right)^2 = - 1, \]
then $U \frac{\theta}{a}$ must be (by 154) a right versor; and reciprocally, every right versor, with a plane containing $a$, will be (by 153) a value satisfying the equation. In this case, therefore, the locus of the point $r$ is (as in 132, (1.), or in 144, (1.)) the plane through $o$, perpendicular to the line $oa$; and the recent equation itself, if supposed to be satisfied by a real* vector $p$, may be put under either of these two earlier but equivalent forms:

$$L \frac{P}{a} = \frac{\pi}{2} ; \quad p \perp a.$$ 

SECTION 9.

On Vector-Arcs, and Vector-Angles, considered as Representatives of Versors of Quaternions; and on the Multiplication and Division of any one such Versor by another.

162. Since every unit-vector $oa$ (129), drawn from the origin $o$, terminates in some point $a$ on the surface of what we have called the unit-sphere (128), that term $a$ (1) may be considered as a Representative Point, of which the position on that surface determines, and may be said to represent, the direction of the line $oa$ in space; or of that line multiplied (12, 17) by any positive scalar. And then the Quaternion which is the quotient (112) of any two such unit-vectors, and which is in one view a Radial (146), and in another view a Versor (151), may be said to have the arc of a great circle, $ab$, upon the unit sphere, which connects the terms of the two vectors, for its Representative Arc. We may also call this arc a Vector Arc, on account of its having a definite direction (comp. Art. 1), such as is indicated (for example) by a curved arrow in fig. 39 [p. 131]; and as being thus contrasted with its own opposite, or with what may be called by analogy the Revector Arc $ba$ (comp. again 1): this latter arc representing, on the present plan, at once the reciprocal (134), and the conjugate (137), of the former versor; because it represents the corresponding Reversor (158).

163. This mode of representation, of versors of quaternions by vector arcs, would obviously be very imperfect, unless equals were to be represented by equals. We shall therefore define, as it is otherwise natural to do, that a vector arc, $ab$, upon the unit sphere, is equal to every other vector arc $cd$ which can be

* Compare 149, (2.); also the second Note to the same Article; and the Notes to pages 87 and 88.
derived from it, by simply causing (or conceiving) it to slide* in its own great circle, without any change of length, or reversal of direction. In fact, the two isosceles and plane triangles AOB, COD, which have the origin o for their common vector, and rest upon the chords of these two arcs as bases, are thus complanar, similar, and similarly turned; so that (by 117, 118) we may here write,

$$\Delta AOB \propto COD, \quad \frac{OB}{OA} = \frac{OD}{OC};$$

the condition of the equality of the quotients (that is, here, of the versors), represented by the two arcs, being thus satisfied. We shall sometimes denote this sort of equality of two vector arcs, AB and CD, by the formula,

$$\diamond AB = \diamond CD;$$

and then it is clear (comp. 125, and the earlier Art. 3) that we shall also have, by what may be called inversion and alternation, these two other formulae of arcual equality,

$$\diamond BA = \diamond DC; \quad \diamond AC = \diamond BD.$$

(Compare the annexed figure 35, bis.)

164. Conversely, unequal versors ought to be represented (on the present plan) by unequal vector arcs; and accordingly, we purpose to regard any two such arcs, as being, for the present purpose, unequal (comp. 2), even when they agree in quantity, or contain the same number of degrees, provided that they differ in direction: which may happen in either of two principal ways, as follows. For, 1st, they may be opposite arcs of one great circle; as, for example, a vector arc AB, and the corresponding revector ARO BA; and so may represent (162) a versor, OB : OA, and the corresponding reversor, OA : OB, respectively. Or, II nd, the two arcs may belong to different great circles, like AB and BC in fig. 43; in which latter case, they represent two radial quotients (146) in different planes; or (comp. 119) two diplanar versors, OB : OA, and OC : OB; but it has been shown generally (122), that diplanar quaternions are always unequal: we consider therefore, here again the arcs, AB and BC, themselves, to be (as has been said) unequal vectors.

* Some aid to the conception may here be derived from the inspection of fig. 34 [p. 113]; in which two equal angles are supposed to be traced on the surface of one common desk. Or the four lines OA, OB, OC, OD, of fig. 35, may now be conceived to be equally long; or to be cut by a circle with O for centre as in the modification of that figure, which is given in Article 163, a little lower down.
165. In this manner, then, we may be led (comp. 122) to regard the conception of a plane, or of the position of a great circle on the unit sphere, as entering, essentially, in general,* into the conception of a vector-arc, considered as the representative of a versor (162). But even without expressly referring to versors, we may see that if, in fig. 43, we suppose that \( b \) is the middle point of an arc \( AA' \) of a great circle, so that in a recent notation (163) we may establish the *arcual equation*,

\[ \rho AB = \rho BA', \]

we ought then (comp. 105) not to write also,

\[ \rho AB = \rho BO; \]

because the two co-initial arcs, \( BA' \) and \( BC \), which terminate differently, must be considered (comp. 2) to be, as vector-arcs, unequal. On the other hand, if we should refuse to admit (as in 163) that any two complanar arcs, if equally long, and similarly (not oppositely) directed, like \( AB \) and \( CD \) in the recent fig. 35, bis, are equal vectors, we could not usefully speak of equality between vector-arcs as existing under any circumstances. We are then thus led again to include, generally, the conception of a plane, or of one great circle as distinguished from another, as an element in the conception of a Vector-Arc. And hence an equation between two such ares must in general be conceived to include two relations of co-arcuality. For example, the equation \( \rho AB = \rho CD \), of Art. 163, includes generally, as a part of its signification, the assertion (comp. 123) that the four points \( A, B, C, D \) belong to one common great circle of the unit-sphere; or that each of the two points, \( C \) and \( D \), is co-arcual with the two other points, \( A \) and \( B \).

166. There is, however, a remarkable case of exception, in which two vector arcs may be said to be equal, although situated in different planes: namely, when they are both great semicircles. In fact, upon the present plan, every great semicircle, \( AA' \), considered as a vector arc, represents an *inversor* (153) or it represents *negative unity* \((\alpha A' : \alpha A = -1 : a = -1)\), considered as one limit of a versor; but we have seen (159) that such a versor has in general an indeterminate plane. Accordingly, whereas the initial and final points, or (comp. 1) the origin \( A \) and the term \( B \), of a vector arc \( AB \), are in general sufficient to determine the plane of that arc, considered as the shortest or the most direct path (comp. 112, 128) from the one point to the other on the sphere; in the particular case when one of the two given points is diametrically opposite to

\* We say, in general; for it will soon be seen that there is a sense in which all great semicircles, considered as vector arcs, may be said to be equal to each other.

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the other, as \(A'\) to \(A\), the direction of this path becomes, on the contrary, indeterminate. If then we only attend to the effect produced, in the way of change of position of a point, by a conceived vection (or motion) upon the sphere, we are permitted to say that all great semicircles are equal vector arcs; each serving simply, in the present view, to transport a point from one position to the opposite; and thereby to reverse (like the factor \(-1\), of which it is here the representative) the direction of the radius which is drawn to that point of the unit sphere.

(1.) The equation, \(\bigcirc AA' = \bigcirc BB'\),
in which it is here supposed that \(A'\) is opposite to \(A\), and \(B'\) to \(B\), satisfies evidently the general conditions of co-arcuality (165); because the four points \(ABA'B'\) are all on one great circle. It is evident that the same arcual equation admits (as in 163) of inversion and alternation; so that
\[\bigcirc A'A = \bigcirc B'B, \quad \text{and} \quad \bigcirc AB = \bigcirc A'B'.\]

(2.) We may also say (comp. 2) that all null arcs are equal, as producing no effect on the position of a point upon the sphere; and thus may write generally,
\[\bigcirc AA = \bigcirc BB = 0,\]
with the alternate equation, or identity, \(\bigcirc AB = \bigcirc AB\).

(3.) Every such null vector arc \(AA\) is a representative, on the present plan, of the other unit scalar, namely positive unity, considered as another limit of a versor (153); and its plane is again indeterminate (159), unless some law be given, according to which the arcual vection may be conceived to begin, from a given point \(A\), to an indefinitely near point \(B\) upon the sphere.

167. The principal use of Vector Arcs, in the present theory, is to assist in representing, and (so to speak) in constructing, by means of a Spherical Triangle, the Multiplication and Division of any two Diplanar Versors (comp. 119, 164). In fact, any two such versors of quaternions (156), considered as radial quotients (152), can easily be reduced (by the general process of Art. 120) to the forms,
\[q = \beta : a = ob : oa, \quad q' = \gamma : \beta = oc : ob,\]
where \(A, B, C\) are corners of such a triangle on the unit sphere; and then (by 107), the former quotient multiplied by the latter will give for product:
\[q' \cdot q = \gamma : a = oc : oa.\]
If then (on the plan of Art. 1) any two successive arcs, as \(AB\) and \(BC\) in fig. 43, be called (in relation to each other) vector and provector; while that third arc
which is drawn from the initial point of the first to the final point of the second, shall be called (on the same plan) the transvector: we may now say that in the multiplication of any one versor (of a quaternion) by any other, if the multiplicand* q be represented (162) by a vector-arc AB, and if the multiplier q' be in like manner represented by a provector-arc BC, which mode of representation is always possible, by what has been already shown, then the product q'q, or q'q, is represented, at the same time, by the transvector-arc AC corresponding.

168. One of the most remarkable consequences of this construction of the multiplication of versors is the following: that the value of the product of two diplanar versors (164) depends upon the order of the factors; or that q'q and q'q are unequal, unless q' be complanar (119) with q. For let AA' and cc' be any two arcs of great circles, in different planes, bisecting each other in the point b, as fig. 43 is designed to suggest; so that we have the two arcual equations

\[ \text{aab = a ba', and \ bac = c cb;} \]

then one or other of the two following alternatives will hold good. Either, 1st, the two mutually bisecting arcs will both be semicircles, in which case the two new arcs, AC and c'A', will indeed both belong to one great circle, namely to that of which b is a pole, but will have opposite directions therein; because, in this case, A' and c' will be diametrically opposite to A and c, and therefore (by 166, (1.)) the equation

\[ \text{n ac = n a'c',} \]

but not the equation

\[ \text{n ac = n c'a',} \]

will be satisfied. Or, 2nd, the arcs AA' and cc', which are supposed to bisect each other in b, will not both be semicircles, even if one of them happen to be such; and in this case, the arcs AC, c'A' will belong to two distinct great circles, so that they will be diplanar, and therefore unequal, when considered as vectors. (Compare the 1st and 2nd cases of Art. 164.) In each case, therefore, AC and c'A' are unequal vector arcs; but the former has been seen (167) to represent the product q'q; and the latter represents, in like manner, the other product, q'q', of the same two versors taken in the opposite order, because it is the new transvector arc, when c'B (= BC) is treated as the new vector arc, and BA' (= AB) as the new provector arc, as is indicated by the curved arrows in

* Here, as in 107, and elsewhere, we write the symbol of the multiplier towards the left-hand, and that of the multiplicand towards the right.
fig. 43. The two products, $q'q$ and $qq'$, are therefore themselves unequal, as above asserted, under the supposed condition of diplanarity.

169. On the other hand, when the two factors, $q$ and $q'$, are complanar versors, it is easy to prove, in several different ways, that their products, $q'q$ and $qq'$, are equal, as in algebra. Thus we may conceive that the arc $cc'$, in fig. 43, is made to turn round its middle point $b$, until the spherical angle $cba'$ vanishes; and then the two new transvector-arcs, $ac$ and $c'a'$, will evidently become not only complanar but equal, in the sense of Art. 163, as being still equally long, and being now similarly directed. Or, in fig. 35, bis, of the last cited Article, we may conceive a point $e$, bisecting the arc $bc$, and therefore also the arc $ad$, which is commedial therewith (comp. Art. 2, and the second figure 3 of that Article); and then, if we represent the one versor $q$ by either of the two equal arcs, $ae$, $ed$, we may at the same time represent the other versor $q'$ by either of the two other equal arcs, $ec$, $be$; so that the one product, $q'q$, will be represented by the arc $ac$, and the other product, $qq'$, by the equal arc $bd$. Or, without reference to vector arcs, we may suppose that the two factors are,

$$q = \beta : a = ob : oa, \quad q' = \gamma : a = oc : oa,$$

$oa$, $ob$, $oc$ being any three complanar and equally long right lines (see again fig. 35, bis); for thus we have only to determine a fourth line, $\delta$ or $od$, of the same length, and in the same plane, which shall satisfy the equation $\delta : \gamma = \beta : a$ (117), and therefore also (by 125) the alternate equation, $\delta : \beta = \gamma : a$; and it will then immediately follow* (by 107) that

$$q'q = \frac{\delta}{\beta} \cdot \frac{\beta}{a} = \frac{\delta}{a} = \frac{\delta}{\gamma} \cdot \frac{\gamma}{a} = q \cdot q'.$$

We may therefore infer, for any two versors of quaternions, $q$ and $q'$, the two following reciprocal relations:

I. .. $q'q = qq'$, if $q' || q$ (123);

II. .. if $q'q = qq'$, then $q' || q$ (168);

convertibility of factors (as regards their places in the product) being thus at once a consequence and a proof of complanarity.

170. In the 1st case of Art. 168, the factors $q$ and $q'$ are both right versors (153); and because we have seen that then their two products, $q'q$ and $qq'$, are

* It is evident that, in this last process of reasoning, we make no use of the supposed equality of lengths of the four lines compared; so that we might prove, in exactly the same way, that $q'q = qq'$ if $q' || q$ (123), without assuming that these two complanar factors, or quaternions, $q$ and $q'$, are versors.
versors represented by equally long but oppositely directed arcs of one great circle, as in the 1st case of 164, it follows (comp. 162) that these two products are at once reciprocal (134), and conjugate (137), to each other; or that they are related as versor and reversor (158). We may therefore write, generally,

\[ I \ldots \vec{q}\vec{q}' = K\vec{q}\vec{q}', \quad \text{and} \quad II \ldots \vec{q}\vec{q}' = \frac{1}{\vec{q}'\vec{q}}. \]

if \( q \) and \( q' \) be any two right versors; because the multiplication of any two such versors, in two opposite orders, may always be represented or constructed by a figure such as that lately numbered 43, in which the bisecting arcs \( \mathcal{A}\mathcal{A}' \) and \( \mathcal{C}\mathcal{C}' \) are semicircles. The IIInd formula may also be thus written (comp. 135, 154):

\[ III \ldots \text{if } \vec{q} = -1, \text{ and } \vec{q}' = -1, \text{ then } \vec{q}'\vec{q} \cdot \vec{q}\vec{q}' = +1; \]

and under this form it evidently agrees with ordinary algebra, because it expresses that, under the supposed conditions,

\[ \vec{q}'\vec{q} \cdot \vec{q}\vec{q}' = \vec{q}^2; \]

but it will be found that this last equation is not an identity in the general theory of quaternions.

171. If the two bisecting semicircles cross each other at right angles, the conjugate products are represented by two quadrants, oppositely turned, of one great circle. It follows that if two right versors, in two mutually rectangular planes, be multiplied together in two opposite orders, the two resulting products will be two opposite right versors, in a third plane, rectangular to the two former; or in symbols, that

\[ \text{if } a^2 = -1, \text{ and } a'^2 = -1, \text{ and } Ax \cdot \vec{q}' \perp Ax \cdot \vec{q}; \]

then

\[ (\vec{q}'\vec{q})^2 = (\vec{q}\vec{q}')^2 = -1, \quad \vec{q}'\vec{q} = -\vec{q}\vec{q}'; \]

and

\[ Ax \cdot \vec{q}'\vec{q} \perp Ax \cdot \vec{q}, \quad Ax \cdot \vec{q}' \perp Ax \cdot \vec{q}'. \]

In this case, therefore, we have what would be in algebra a paradox, namely the equation,

\[ (\vec{q}'\vec{q})^2 = -\vec{q}^2; \]

if \( q \) and \( q' \) be any two right versors, in two rectangular planes; but we see that this result is not more paradoxical, in appearance, than the equation

\[ \vec{q}'\vec{q} = -\vec{q}\vec{q}', \]

which exists, under the same conditions. And when we come to examine what, in the last analysis, may be said to be the meaning of this last equation, we find it to be simply this: that any two quadrantal or right rotations, in planes
perpendicular to each other, compound themselves into a third right rotation, as
their resultant, in a plane perpendicular to each of them: and that this third or
resultant rotation has one or other of two opposite directions, according to the
order in which the two component rotations are taken, so that one shall be
successive to the other.

172. We propose to return, in the next section, to the consideration of
such a System of Right Versors as that which we have here briefly touched
upon: but desire at present to remark (comp. 167) that a spherical triangle
ABC may serve to construct, by means of representative arcs (162), not only the
multiplication, but also the division, of any one of two diplanar versors (or
diplanar quotients) by the other. In fact, we have only to conceive (comp.
fig. 43) that the vector arc AB represents a given divisor, say q, or β : α, and
that the transevector arc AC (167) represents a given dividend, suppose q", or
γ : α; for then the provector arc BC (comp. again 167) will represent, on
the same plan, the quotient of these two versors, namely q": q, or γ : β (106), or the
versor lately called q'; since we have generally, by 106, 107, 120, for
quaternions, as in algebra, the two identities:

\[(q": q) \cdot q = q"; q'q : q = q'.\]

173. It is however to be observed that, for reasons already assigned, we
must not employ, for diplanar versors, such an equation as \(q \cdot (q" : q) = q"\);
because we have found (168) that, for such versors, the ordinary algebraic
identity, \(qq' = q'q\), ceases to be true. In fact by 169, we may now establish
the two converse formulæ:

I. . . . \(q (q" : q) = q"\), if \(q" ||| q\) (123);

II. . . . if \(q (q" : q) = q"\), then \(q" ||| q\).

Accordingly, in fig. 43, if \(q, q', q''\) be still represented by the arcs AB, BC,
AC, the product \(q (q" : q)\), or \(qq'\), is not represented by AC, but by the different
arc \(c'\alpha'\) (168), which as a vector arc has been seen to be unequal thereto:
although it is true that these two last arcs, AC and \(c'\alpha'\), are always equally
long, and therefore subtend equal angles at the centre \(O\) of the unit sphere; so
that we may write, generally, for any two versors (or indeed for any two
quaternions)\,* \(q\) and \(q'\), the formula,

\[\angle q (q" : q) = \angle q".\]

* It will soon be seen [see 191] that several of the formulæ of the present section, respecting the
multiplication and division of versors, considered as radial quotients (151), require little or no modi-
fication, in the passage to the corresponding operations on quaternions, considered as general quotients
of vectors (112).
174. Another mode of Representation of Versors, or rather two such new modes, although intimately connected with each other, may be briefly noticed here.

Ist. We may consider the angle $\angle aob$, at the centre $o$ of the unit-sphere, when conceived to have not only a definite quantity, but also a determined plane (110), and a given direction therein (as indicated by one of the curved arrows in fig. 39 [p. 131], or by the arrow in fig. 33 [p. 111]), as being what may be called by analogy a Vector-Angle; and may say that it represents, or that it is the Representative Angle of, the Versor $ob : oa$, where $oa, ob$ are radii of the unit-sphere.

2nd. Or we may replace this rectilinear angle $\angle aob$ at the centre, by the equal Spherical Angle $\angle ac'b$, at what may be called the Positive Pole of the representative arc $ab$; so that $c'a$ and $c'b$ are quadrants; and the rotation, at this pole $c'$, from the first of these two quadrants to the second (as seen from a point outside the sphere), has the direction which has been selected (111, 127) for the positive one, as indicated in the annexed figure 44: and then we may consider this spherical angle as a new Angular Representative of the same versor $q$, or $ob : oa$, as before.

175. Conceive now that after employing a first spherical triangle $abc$, to construct (as in 167) the multiplication of any one given versor $q$, by any other given versor $q'$, we form a second or polar triangle, of which the corners $\alpha', b', c'$ shall be respectively (in the sense just stated) the positive poles of the three successive sides, $bc, ca, ab$, of the former triangle; and that then we pass to a third triangle $\alpha'b''c'$, as part of the same lune $b'b''$ with the second, by taking for $b''$ the point diametrically opposite to $b'$; so that $b''$ shall be the negative pole of the arc $ca$, or the positive pole of what was lately called (167) the transvector-arc $ac$: also let $c''$ be, in like manner, the point opposite to $c'$ on the unit sphere. Then we may not only write (comp. 129),

$$\Delta x \cdot q = oc', \quad \Delta x \cdot q' = oa', \quad \Delta x \cdot q'q = ob''$$

but shall also have the equations,

$$\angle q = b''c'a', \quad \angle q' = c'a'b'', \quad \angle q'q = c''b''a';$$

these three spherical angles, namely the two base-angles at $c'$ and $\alpha'$, and the external vertical angle at $b''$, of the new or third triangle $\alpha'b''c'$, will therefore
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represent, respectively, on the plan of 174, II., the multiplicand, \( q \), the multiplier, \( q' \), and the product, \( q'q \). (Compare the annexed figure 45.)

176. Without expressly referring to the former triangle \( ABC \), we can connect this last construction of multiplication of versors (175) with the general formula (107), as follows.

Let \( a \) and \( \beta \) be now conceived to be two unit-tangents* to the sphere at \( c' \), perpendicular respectively to the two arcs \( c'b'' \) and \( c'a' \), and drawn towards the same sides of those arcs as the points \( a' \) and \( b' \) respectively; and let two other unit-tangents, equal to these, and denoted by the same letters, be drawn (as in the annexed figure 45, bis) at the points \( b'' \) and \( a' \), so as to be normal there to the same arcs \( c'b'' \) and \( c'a' \), and to fall towards the same sides of them as before. Let also two other unit-tangents, equal to each other, and each denoted by \( \gamma \), be drawn at the two last points \( b'' \) and \( a' \), so as to be both perpendicular to the arc \( a'b'' \), and to fall towards the same side of it as the point \( c' \). Then (comp. 174, II.) the two quotients, \( \beta : a \) and \( \gamma : \beta \), will be equal to the two versors, \( q \) and \( q' \), which were lately represented (in fig. 45) by the two base angles, at \( c' \) and \( a' \), of the spherical triangle \( a'b''c' \); the product, \( q'q \), of these two versors, is therefore (by 107) equal to the third quotient, \( \gamma : a \); and consequently it is represented, as before, by the external vertical angle \( c''b''a' \) of the same triangle, which is evidently equal in quantity to the angle of this third quotient, and has the same axis \( ob'' \), and the same direction of rotation, as the arrows in fig. 45, bis, may assist to show.

177. In each of the two last figures, the internal vertical angle at \( b'' \) is thus equal to the Supplement, \( \pi - \angle q'q \), of the angle of the product; and it is important to observe that the corresponding rotation at the vertex \( b'' \), from the side \( b''a' \) to the side \( b''c' \), or (as we may briefly express it) from the point \( a' \) to the point \( c' \), is positive; a result which is easily seen to be a general one, by the reasoning of the foregoing Article.† We may then infer, generally, that when the multiplication of any two versors is constructed by a spherical triangle, of which the two base angles represent (as in the two last Articles) the factors,

* By an unit tangent is here meant simply an unit line (or unit vector, 129) so drawn as to be tangential to the unit-sphere, and to have its origin, or its initial point (1), on the surface of that sphere, and not (as we have usually supposed) at the centre thereof.

† If a person be supposed to stand on the sphere at \( n'' \), and to look towards the arc \( a'c' \), it would appear to him to have a right-handed direction, which is the one here adopted as positive (127).
while the external vertical angle represents the product, then the rotation round the axis (ob") of that product q'q, from the axis (oa') of the multiplier q', to the axis (oc") of the multiplicand q, is positive: whence it follows that the rotation round the axis Ax. q' of the multiplier, from the axis Ax. q of the multiplicand, to the axis Ax. q'q of the product, is also positive. Or, to express the same thing more fully, since the only rotations hitherto considered have been plane ones (as in 128, &c.), we may say that if the two latter axes be projected on a plane perpendicular to the former, so as still to have a common origin o, then the rotation round Ax. q', from the projection of Ax. q to the projection of Ax. q'q, will be directed (with our conventions) towards the right hand.

178. We have therefore thus a new mode of geometrically exhibiting the inequality of the two products, q'q and qq', of two diplanar versors (168), when taken as factors in two different orders. For this purpose, let

$$Ax. q = op, \quad Ax. q' = oq, \quad Ax. q'q = or;$$

and prolong to some point s the arc pr of a great circle on the unit sphere. Then, for the spherical triangle pqr, by principles lately established, we shall have (comp. 175) the following values of the two internal base angles at p and q, and of the external vertical angle at r:

$$RPQ = \angle q; \quad PQR = \angle q'; \quad SRQ = \angle q'q;$$

and the rotation at q, from the side qp to the side qr will be right-handed. Let fall an areal perpendicular, rt, from the vertex r on the base pq, and prolong this perpendicular to r', in such a manner as to have

$$\angle RT = \angle TR';$$

also prolong pr' to some point s'. We shall then have a new triangle pqr', which will be a sort of reflection (comp. 138) of the old one with respect to their common base rq; and this new triangle will serve to construct the new product, qq'. For the rotation at p from pq to pr' will be right-handed, as it ought to be; and we shall have the equations,

$$QPR' = \angle q; \quad r'QP = \angle q'; \quad QR's' = \angle qq'; \quad or' = Ax. qq';$$

so that the new external and spherical angle, qr's', will represent the new versor, qq', as the old angle srq represented the old versor, q'q, obtained from a different order of the factors. And although, no doubt, these two angles, at r and r',
are always equal in quantity, so that we may establish (comp. 173) the general formula,
\[ \angle q'q = \angle qq', \]
yet as vector angles (174), and therefore as representatives of versors, they must be considered to be unequal: because they have different planes, namely, the tangent planes to the sphere at the two vertices \( r \) and \( r' \); or the two planes respectively parallel to these, which are drawn through the centre \( o \).

179. Division of Versors (comp. 172) can be constructed by means of Representative Angles (174), as well as by representative arcs (162). Thus to divide \( q'' \) by \( q \), or rather to represent such division geometrically, on a plan entirely similar to that last employed for multiplication, we have only to determine the two points \( p \) and \( r \), in fig. 46, by the two conditions,
\[ op = \Delta x \cdot q, \quad or = \Delta x \cdot q'', \]
and then to find a third point \( q \) by the two angular equations,
\[ RPQ = \angle q, \quad QRP = \pi - \angle q'', \]
the rotation round \( p \) from \( PR \) towards \( PQ \) being positive; after which we shall have,
\[ \Delta x \cdot (q'' : q) = oq; \quad \angle (q'' : q) = PQR. \]

(1.) Instead of conceiving, in fig. 46, that the dotted line \( rtr' \), which connects the vertices of the two triangles, with \( pq \) for their common base (178), is an arc of a great circle, perpendicularly bisected by that base, we may imagine it to be an arc of a small circle, described with the point \( p \) for its positive pole (comp. 174, II.). And then we may say that the passage (comp. 173) from the versor \( q'' \), or \( q'q \), to the unequal versor \( q \) (\( q'' : q \)), or \( qq' \), is geometrically performed by a Conical Rotation of the Axis \( Ax \cdot q'' \), round the axis \( Ax \cdot q \), through an angle \( = 2 \angle q \), without any (quantitative) change of the angle \( \angle q'' \); so that we have, as before, the general formula (comp. again 173),
\[ \angle q (q'' : q) = \angle q''. \]

(2.) Or if we prefer to employ the construction of multiplication and division by representative arcs, which fig. 43 [p. 144] was designed to illustrate, and conceive that a new point \( c'' \) is determined in that figure by the condition \( \angle A'c'' = \angle c'A' \), we may then say that in the passage from the versor \( q'' \), which is represented by \( Ac \), to the versor \( q (q'' : q) \), represented by \( c'A' \) or by \( A'c'' \), the representative arc of \( q'' \) is made to move, without change of length, so as
to preserve a constant inclination* to the representative arc $AB$ of $q$, while its initial point describes the double of that arc $AB$, in passing from $A$ to $A'$.

(3.) It may be seen, by these few examples, that if, even independently of some new characteristics of operation, such as $K$ and $U$, new combinations of old symbols, such as $q(q'':q)$, occur in the present Calculus, which are not wanted in algebra, they admit for the most part of geometrical interpretations, of an easy and interesting kind; and in fact represent conceptions, which cannot well be dispensed with, and which it is useful to be able to express, with so much simplicity and conciseness. (Compare the remarks in Art. 161; and the sub-articles to 132, 145.)

180. In connexion with the construction indicated by the two figures 45, it may be here remarked, that if $ABC$ be any spherical triangle, and if $A'$, $B'$, $C'$ be (as in 175) the positive poles of its three successive sides, $BC$, $CA$, $AB$, then the rotation (comp. 177, 179) round $A'$ from $B'$ to $C'$, or that round $B'$ from $C'$ to $A'$, &c., is positive. The easiest way, perhaps, of seeing the truth of this assertion is to conceive that if the rotation round $A$ from $B$ to $C$ be not already positive, we make it such, by passing to the diametrically opposite triangle on the sphere, which will not change the poles $A'$, $B'$, $C'$. Assuming then that these poles are thus the near ones to the corresponding corners of the given triangle, we arrive without any difficulty at the conclusion stated above: which has been virtually employed in our construction of multiplication (and division) of versors, by means of Representative Angles (175, 176); and which may be otherwise justified (as before), by the consideration of the unit-tangents of fig. 45, bis.

(1.) Let then $\alpha, \beta, \gamma$ be any three given unit vectors, such that the rotation round the first, from the second to the third, is positive (in the sense of Art. 177); and let $\alpha', \beta', \gamma'$ be three other unit vectors, derived from these by the equations,

$$a' = Ax \cdot (\gamma : \beta), \quad \beta' = Ax \cdot (\alpha : \gamma), \quad \gamma' = Ax \cdot (\beta : \alpha);$$

then the rotation round $\alpha'$, from $\beta'$ to $\gamma'$, will be positive also; and we shall have the converse formulae,

$$a = Ax \cdot (\gamma' : \beta'), \quad \beta = Ax \cdot (\alpha' : \gamma'), \quad \gamma = Ax \cdot (\beta' : \alpha').$$

(2.) If the rotation round $\alpha$ from $\beta$ to $\gamma$ were given to be negative, $\alpha', \beta', \gamma'$ being still deduced from those three vectors by the same three equations as before, then the signs of $\alpha, \beta, \gamma$ would all require to be changed, in the three

* In a manner analogous to the motion of the equator on the ecliptic, by luni-solar precession, in astronomy.
last (or reciprocal) formulae; but the rotation round \( \alpha' \), from \( \beta' \) to \( \gamma' \), would still be positive.

(3.) Before closing this section, it may be briefly noticed, that it is sometimes convenient, from motives of analogy (comp. Art. 5), to speak of the Transvector-Arc \((167)\), which has been seen to represent a product of two versors, as being the Arcual Sum of the two successive vector-arcs, which represent (on the same plan) the factors; Provector being still said to be added to Vector: but the Order of such Addition of Diplanar Arcs being not now indifferent \((168)\), as the corresponding order had been early found (in 7) to be, when the vectors to be added were right lines. [Thus in fig. 43, \( \angle AB = \angle AC \) and \( \angle BA' + \angle c'B = \angle c'A' \). But \( \angle BA' = \angle AB \) and \( \angle c'B = \angle BC \), consequently \( \angle AB + \angle BC = \angle c'A' \). If \( \alpha \) and \( \beta \) are any two vector arcs, and if \( x \) is any scalar, \( x(\alpha \pm \beta) \) is not equal to \( xa \pm x\beta \). Compare 14, and notice that the property there proved depends on the possibility of constructing similar plane triangles of different sizes.]

(4.) We may also speak occasionally, by an extension of the same analogy, of the External Vertical Angle of a spherical triangle, as being the Spherical Sum of the two Base Angles of that triangle, taken in a suitable order of summation (comp. fig. 46); the Angle which represents \((174)\) the Multiplier being then said to be added (as a sort of Angular Provector) to that other Vector-Angle which represents the Multiplicand; whilst what is here called the sum of these two angles (and is, with respect to them, a species of Transvector-Angle) represents, as has been proved, the Product.

(5.) This conception of angular transvection becomes perhaps a little more clear, when (on the plan of 174, I.) we assume the centre o as the common vertex of three angles \( \angle AOB \), \( \angle BOC \), \( \angle AOC \), situated generally in three different planes. For then we may conceive a revolving radius to be either carried by two successive angular motions, from \( OA \) to \( OB \), and thence to \( OC \); or to be transported immediately, by one such motion, from the first to the third position.

(6.) Finally, as regards the construction indicated by fig. 45, bis, in which tangents instead of radii were employed, it may be well to remark distinctly here, that \( \alpha'b''c' \), in that figure, may be any given spherical triangle, for which the rotation round \( b'' \) from \( \alpha' \) to \( c' \) is positive \((177)\); and that then, if the two factors \( q \) and \( q' \) be defined to be the two versors, of which the internal angles at \( c' \) and \( \alpha' \) are (in the sense of 174, II.) the representatives, the reasonings of Art. 176 will prove, without necessarily referring, even in thought, to any other triangle (such as \( ABC \)), that the external angle at \( b'' \) is (in the same sense) the representative of the product, \( q'q \), as before.
SECTION 10.

On a System of Three Right Versors, in Three Rectangular Planes; and on the Laws of the Symbols, i, j, k.

181. Suppose that $o_1, o_j, o_k$ are any three given and co-initial but rectangular unit-lines, the rotation round the first from the second to the third being positive; and let $o_1', o_j', o_k'$ be the three unit-vectors respectively opposite to these, so that

$$o_1' = -o_1, \quad o_j' = -o_j, \quad o_k' = -o_k.$$ 

Let the three new symbols $i, j, k$ denote a system (comp. 172) of three right versors, in three mutually rectangular planes, with the three given lines for their respective axes; so that

$$Ax \cdot i = o_1, \quad Ax \cdot j = o_j, \quad Ax \cdot k = o_k,$$

and

$$i = o_k : o_j, \quad j = o_i : o_k, \quad k = o_j : o_i,$$

as figure 47 may serve to illustrate. We shall then have these other expressions for the same three versors:

$$i = o_j : o_k = o_k : o_j = o_j : o_k;$$
$$j = o_k' : o_i = o_i' : o_k = o_k' : o_i';$$
$$k = o_i' : o_j = o_j' : o_i = o_i' : o_j'.$$

while the three respectively opposite versors may be thus expressed:

$$-i = o_j : o_k = o_k' : o_j = o_j' : o_k = o_k : o_j';$$
$$-j = o_k : o_i = o_i' : o_k = o_k' : o_i';$$
$$-k = o_i : o_j = o_j' : o_i = o_i' : o_j' = o_j : o_i'.$$

And from the comparison of these different expressions several important symbolical consequences follow, which it will be worth while to enunciate separately here, although some of them are virtually included in the results of former sections.

182. In the first place, since

$$i^2 = (o_j' : o_k) \cdot (o_k' : o_j) = o_j' : o_j, \&c.,$$

we deduce (comp. 148) the following equal values for the squares of the new symbols:

$$i^2 = -1; \quad j^2 = -1; \quad k^2 = -1;$$
as might indeed have been at once inferred (154), from the circumstance that the three radial quotients, (146), denoted here by $i, j, k$, are all right versors (181).

In the second place, since

$$i \cdot j = (oj : ok') \cdot (ok' : oi) = oj : oi, \&c.,$$

we have the following values for the products of the same three symbols, or versors, when taken two by two, and in a certain order of succession (comp. 168, 171):

$$\text{II. } ij = k; \quad jk = i; \quad ki = j.$$

But in the third place (comp. again 171), since

$$j \cdot i = (oi : ok) \cdot (ok : oj) = oi : oj, \&c.,$$

we have these other and contrasted formulae, for the binary products of the same three right versors, when taken as factors with an opposite order:

$$\text{III. } ji = -k; \quad kj = -i; \quad ik = -j.$$

Hence, while the square of each of the three right versors, denoted by these three new symbols, $ijk$, is equal (154) to negative unity, the product of any two of them is equal either to the third itself, or to the opposite (171) of that third versor, according as the multiplier precedes or follows the multiplicand, in the cyclical succession,

$$i, j, k, i, j, \ldots$$

which the annexed figure 47, bis, may give some help towards remembering.

(1.) To connect such multiplications of $i, j, k$ with the theory of representative arcs (162), and of representative angles (174), we may regard any one of the four quadrantal arcs, $jk, k'j, j'k', k'j$, in fig. 47, or any one of the four spherical right angles, $jik, k'ij', j'ik', k'ij$, which those arcs subtend at their common pole $i$, as representing the versor $i$; and similarly for $j$ and $k$, with the introduction of the point $i'$ opposite to $i$, which is to be conceived as being at the back of the figure.

(2.) The squaring of $i$, or the equation $i^2 = -1$, comes thus to be geometrically constructed by the doubling (comp. Arts. 148, 154, and figs. 41, 42) of an arc, or of an angle. Thus, we may conceive the quadrant $ki'$ to be added to the equal arc $jk$, their sum being the great semicircle $jj'$, which (by 166) represents an inversor (153), or negative unity considered as a factor. Or we may add the right angle $kij'$ to the equal angle $jik$, and so obtain a rotation.
through two right angles at the pole \( \mathbf{i} \), or at the centre \( \mathbf{o} \); which rotation is equivalent (comp. 154, 174) to an inversion of direction, or to a passage from the radius \( \mathbf{o} \mathbf{j} \), to the opposite radius \( \mathbf{o} \mathbf{j}' \).

(3.) The multiplication of \( \mathbf{j} \) by \( \mathbf{i} \), or the equation \( \mathbf{i} \mathbf{j} = \mathbf{k} \), may in like manner be arcually constructed, by the addition of \( \mathbf{i} \mathbf{k} \), as a provector-arc (167), to \( \mathbf{j} \mathbf{k} \) as a vector-arc (162), giving \( \mathbf{i} \mathbf{j} \), which is a representative of \( \mathbf{k} \), as the transvector-arc, or arcual-sum (180, (3.)). Or the same multiplication may be angularly constructed, with the help of the spherical triangle \( \mathbf{i} \mathbf{j} \mathbf{k} \); in which the base-angles at \( \mathbf{i} \) and \( \mathbf{j} \) represent respectively the multiplier, \( \mathbf{i} \), and the multiplicand, \( \mathbf{j} \), the rotation round \( \mathbf{i} \) from \( \mathbf{j} \) to \( \mathbf{k} \) being positive: while their spherical sum (180, (4.)), or the external vertical angle at \( \mathbf{k} \) (comp. 175, 176), represents the same product, \( \mathbf{k} \), as before.

(4.) The contrasted multiplication of \( \mathbf{i} \) by \( \mathbf{j} \), or of \( \mathbf{j} \) into* \( \mathbf{i} \), may in like manner be constructed, or geometrically represented, either by the addition of the arc \( \mathbf{k} \mathbf{i} \), as a new provector, to the arc \( \mathbf{j} \mathbf{k} \) as a new vector, which new process gives \( \mathbf{j} \mathbf{i} \) (instead of \( \mathbf{u} \)) as the new transvector; or with the aid of the new triangle \( \mathbf{i} \mathbf{j} \mathbf{k}' \) (comp. figs. 46, 47), in which the rotation round \( \mathbf{i} \) from \( \mathbf{j} \) to the new vertex \( \mathbf{k}' \) is negative, so that the angle at \( \mathbf{i} \) represents now the multiplicand, and the resulting angle at the new pole \( \mathbf{k}' \) represents the new and opposite product, \( \mathbf{j} \mathbf{i} = - \mathbf{k} \).

183. Since we have thus \( \mathbf{j} \mathbf{i} = - \mathbf{i} \mathbf{j} \) (as we had \( \mathbf{q}' \mathbf{q} = - \mathbf{q} \mathbf{q}' \) in 171), we see that the laws of combination of the new symbols, \( \mathbf{i}, \mathbf{j}, \mathbf{k} \), are not in all respects the same as the corresponding laws in algebra; since the Commutative Property of Multiplication, or the convertibility (169) of the places of the factors without change of value of the product, does not here hold good: which arises (168) from the circumstance, that the factors to be combined are here diplanar versors (181). It is therefore important to observe, that there is a respect in which the laws of \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) agree with usual and algebraic laws: namely, in the Associative Property of Multiplication; or in the property that the new symbols always obey the associative formula (comp. 9),

\[ \mathbf{i} \cdot \mathbf{k} \mathbf{\lambda} = \mathbf{\lambda} \cdot \mathbf{k} \mathbf{\lambda}, \]

whichever of them may be substituted for \( \mathbf{i} \), for \( \mathbf{k} \), and for \( \mathbf{\lambda} \); in virtue of which equality of values we may omit the point, in any such symbol of a

* A multiplicand is said to be multiplied by the multiplier; while, on the other hand, a multiplier is said to be multiplied into the multiplicand: a distinction of this sort between the two factors being necessary, as we have seen, for quaternions, although it is not needed for algebra.
ternary product (whether of equal or of unequal factors), and write it simply as $ik\lambda$. In particular we have thus,

$$i \cdot jk = i \cdot i = i^2 = -1; \quad ij \cdot k = k \cdot k = k^2 = -1;$$

or briefly,

$$ijk = -1.$$

We may, therefore, by 182, establish the following important Formula:

$$i^2 = j^2 = k^2 = ijk = -1; \quad (A)$$

to which we shall occasionally refer, as to "Formula A," and which we shall find to contain (virtually) all the laws of the symbols $ijk$, and therefore to be a sufficient symbolical basis for the whole Calculus of Quaternions,* because it will be shown that every quaternion can be reduced to the Quadrinomial Form,

$$q = w + ix + jy + kz,$$

where $w, x, y, z$ compose a system of four scalars, while $i, j, k$ are the same three right versors as above. [See 221.]

(1.) A direct proof of the equation, $ijk = -1$, may be derived from the definitions of the symbols in Art. 181. In fact, we have only to remember that those definitions were seen to give,

$$i = oj': ok, \quad j = ok : oj', \quad k = oj': oj;$$

and to observe that, by the general formula of multiplication (107), whatever four lines may be denoted by $a, \beta, \gamma, \delta$, we have always,

$$\frac{\delta}{\gamma} \cdot \frac{\beta}{a} = \frac{\delta}{\gamma} \cdot \frac{\beta}{a} = \frac{\delta}{\gamma} \cdot \frac{\beta}{a} = \frac{\delta}{\gamma} \cdot \frac{\beta}{a};$$

or briefly, as in algebra,

$$\frac{\delta}{\gamma} \cdot \frac{\beta}{a} = \frac{\delta}{\gamma} \cdot \frac{\beta}{a},$$

the point being thus omitted without danger of confusion: so that

$$ijk = oj' : oj = -1,$$

as before.

* This formula (A) was accordingly made the basis of that Calculus in the first communication on the subject, by the present writer, to the Royal Irish Academy in 1843; and the letters $i, j, k$, continued to be, for some time, the only peculiar symbols of the Calculus in question. But it was gradually found to be useful to incorporate with these a few other notations (such as $K$ and $U$, &c.), for representing Operations on Quaternions. It was also thought to be instructive to establish the principles of that Calculus, on a more geometrical (or less exclusively symbolical) foundation than at first; which was accordingly afterwards done, in the volume entitled: Lectures on Quaternions (Dublin, 1853); and is again attempted in the present work, although with many differences in the adopted plan of exposition, and in the applications brought forward, or suppressed.
Similarly, we have these two other ternary products:
\[ jki = (ok' : o1) (o1 : oj') (oj' : ok) = ok' : ok = - 1; \]
\[ kij = (or' : or) (or : ok') (ok' : or) = or' : or = - 1. \]

(2.) On the other hand,
\[ kji = (oj : oi) (oi : ok) (ok : oj) = oj : oj = + 1; \]
and in like manner,
\[ ikj = + 1, \text{ and } jik = + 1. \]

(3.) The equations in 182 give also these other ternary products, in which the law of association of factors is still obeyed:
\[ i . ij = ik = - j = i'j = ii . j, \quad ii = j; \]
\[ i . ji = i . - k = - ik = j = k = ij . i, \quad ij = + j; \]
\[ i . jj = i . - 1 = - i = kj = ij . j, \quad ij = - i; \]
with others deducible from these, by mere cyclical permutation of the letters, on the plan illustrated by fig. 47, bis.

(4.) In general, if the Associative Law of Combination exist for any three symbols whatever of a given class, and for a given mode of combination, as for addition of lines in Art. 9, or for multiplication of ijk in the present Article, the same law exists for any four (or more) symbols of the same class, and combinations of the same kind. For example, if each of the four letters i, *, \( \lambda \), \( \mu \) denote some one of the three symbols i, j, k (but not necessarily the same one), we have the formula,
\[ i . \kappa \lambda \mu = i . \kappa . \lambda \mu = i . \lambda \mu = i . \kappa \lambda . \mu = i \kappa \lambda \mu. \]

(5.) Hence, any multiple (or complex) product of the symbols ijk, in any manner repeated, but taken in one given order, may be interpreted, with one definite result, by any mode of association, or of reduction to partial factors, which can be performed without commutation, or change of place of the given factors. For example, the symbol ijkki may be interpreted in either of the two following (among other) ways:
\[ ij . kk . ji = ij . - ji = i . - j^2 . i = ii = - 1; \quad ijk . kji = - 1 . 1 = - 1. \]

184. The formula (a) of 183 includes obviously the three equations (I.) of 182. To show that it includes also the six other equations, (II.), (III.), of the last cited Article, we may observe that it gives, with the help of the

*Hamilton's Elements of Quaternions,*
associative principle of multiplication (which may be suggested to the memory by the absence of the point in the symbol $ijk$),

\[
ij = -ij, \quad kk = -ijk, \quad k = +k; \quad jk = -i, \quad jk = -i;
\]
\[
ji = j, \quad jk = j^2k = -k; \quad ik = i, \quad ij = -jk = -j;
\]
\[
kj = ij, \quad j = i^2k = -i; \quad ki = -k, \quad k = -ji = -jk = +j.
\]

And then it is easy to prove, without any reference to geometry, if the foregoing laws of the symbols be admitted, that we have also,

\[
jki = kij = -1, \quad kji = jik = ikj = +1,
\]
as otherwise and geometrically shown in recent sub-articles. It may be added that the mere inspection of the formula (\lambda) is sufficient to show that the three* square roots of negative unity, denoted in it by $i, j, k$, cannot be subject to all the ordinary rules of algebra: because that formula gives, at sight,

\[
ij^2k^2 = (-1)^3 = -1 = -(ijk)^2;
\]

the non-commutative character (183), of the multiplication of such roots among themselves, being thus put in evidence.

Conversely if three symbols $i, j, k$ satisfy the equations

\[
jk + kj = 0, \quad ki + ik = 0, \quad ij + ji = 0;
\]

and if the associative property hold good,

\[
i^2 \cdot j = i \cdot ij = -i \cdot ji = -ij \cdot i = j \cdot i^2.
\]

$i^2$ is therefore commutative in multiplication with $i, j, k$ and with products formed from them, and cannot be distinguished from a scalar. Assuming therefore that the squares of the symbols are scalars, and that the symbols have been multiplied by suitable numerical coefficients so that their squares are equal,

\[
i^2 = j^2 = k^2 = P.
\]

Again \[i \cdot jk = -i \cdot kj = kij = -kji = jki = -jik = Q \text{ suppose,}\]

and \[iQ = i \cdot jk = P \cdot jk = jki \cdot i = Qi.\]

The product $Q$ is likewise commutative with $i, j, k$, and is indistinguishable from a scalar. Also \[Q^2 = -ijk \cdot jki = -P^2,\] so if $P = -1, Q = \pm 1$.

* It is evident that $-i, -j, -k$ are also, on the same principles, values of the symbol $\sqrt{-1}$; because they also are right versors (153); or because $(-g)^2 = g^2$. More generally (comp. a Note to page 133), if $x, y, z$ be any three scalars which satisfy the condition $x^2 + y^2 + z^2 = 1$, it will be proved, at a later stage, that

\[
(ix + jy + kz)^2 = -1.
\]
Mr. Oliver Heaviside takes $P = +1$ but $i = jk$, $j = ki$ and $k = ij$, and consequently $i^2 = j$ but $i = ij = ik = -j$, and in his system the associative property does not hold, or the product of three symbols has no definite meaning (see Art. 25 of a paper “On the Forces in the Electro-magnetic Field,” Trans. Roy. Soc. A., 1892).

Grassmann supposes $P = 0$, and his progressive multiplication is associative, but his regressive multiplication is not. $Q$ is taken as a scalar differing from zero. Hamilton (p. 61 of the preface to the “Lectures on Quaternions”) refers to the octaves of Messrs. J. T. Graves and Arthur Cayley as not obeying the associative principle. See Prof. Cayley’s paper “On the 8-square Imaginaries,” Am. Jour. of Math., 1881. When the associative principle does not hold, a distinct operation of grouping must be combined with multiplication to render a product definite.]

SECTION 11.

On the Tensor of a Vector, or of a Quaternion; and on the Product or Quotient of any two Quaternions.

185. Having now sufficiently availed ourselves, in the two last sections, of the conceptions (alluded to, so early as in the First Article of these Elements) of a vector-arc (162), and of a vector-angle (174) in illustration* of the laws of multiplication and division of versors of quaternions; we propose to return to that use of the word, Vector, with which alone the First Book, and the first eight sections of this First Chapter of the Second Book, have been concerned: and shall therefore henceforth mean again, exclusively, by that word “vector,” a Directed Right Line (as in 1). And because we have already considered and expressed the Direction of any such line, by introducing the conception and notation (155) of the Unit-Vector, $Ua$, which has the same direction with the line $a$, and which we have proposed (156) to call the Versor of that Vector, $a$; we now propose to consider and express the Length of the same line $a$, by introducing the new name Tensor, and the new symbol,† $Ta$;

* One of the chief uses of such vectors, in connexion with those laws, has been to illustrate the non-commutative property (168) of multiplication of versors, by exhibiting a corresponding property of what has been called, by analogy to the earlier operation of the same kind on linear vectors (5), the addition of arcs and angles on a sphere. Compare 180, (3.), (4.).

† Compare the Note to Art. 155.
which latter symbol we shall read, as the Tensor of the Vector \( a \); and shall define it to be, or to denote, the Number (comp. again 155) which represents the Length of that line \( a \), by expressing the Ratio which that length bears to some assumed standard, or Unit (128).

186. To connect more closely these two conceptions, of the versor and the tensor of a vector, we may remember that when we employed (in 155) the letter \( a \) as a temporary symbol for the number which thus expresses the length of the line \( a \), we had the equation, \( Ua = a : a \), as one form of the definition of the unit-vector denoted by \( Ua \). We might therefore have written also these two other forms of equation (comp. 15, 16),

\[
a = a \cdot Ua, \quad a = a : Ua,
\]

to express the dependence of the vector, \( a \), and of the scalar, \( a \), on each other, and on what has been called (156) the versor, \( Ua \). For example, with the construction of fig. 42, bis (comp. 161, (2.)), we may write the three equations,

\[
a = o\alpha : o\alpha', \quad b = ob : ob', \quad c = oc : oc',
\]

if \( a, b, c \) be thus the three positive scalars, which denote the lengths of the three lines, \( o\alpha, ob, oc \); and these three scalars may then be considered as factors, or as coefficients (12), by which the three unit-vectors \( U\alpha, U\beta, U\gamma \), or \( o\alpha', ob', oc' \) (in the cited figure), are to be respectively multiplied (15), in order to change them into the three other vectors \( a, \beta, \gamma \), or \( o\alpha, ob, oc \), by altering their lengths, without any change in their directions. But such an exclusive Operation, on the Length (or on the extension) of a line, may be said to be an Act of Tension;* as an operation on direction alone may be called (comp. 151) an act of version. We have then thus a motive for the introduction of the name, Tensor, as applied to the positive number which (as above) represents the length of a line. And when the notation \( Ta \) (instead of \( a \)) is employed for such a tensor, we see that we may write generally, for any vector \( a \), the equations (compare again 15, 16):

\[
Ua = a : Ta; \quad Ta = a : Ua; \quad a = Ta \cdot Ua = Ua \cdot Ta.
\]

For example, if \( a \) be an unit-vector, so that \( Ua = a \) (160), then \( Ta = 1 \); and therefore, generally, whatever vector may be denoted by \( a \), we have always,

\[
TUa = 1.
\]

* Compare the first Note in page 137.
For the same reason, whatever quaternion may be denoted by \( q \), we have always (comp. again 160) the equation,

(1.) Hence the equation

\[ T(Ax \cdot q) = 1. \]

where \( \rho = op \), expresses that the locus of the variable point \( p \) is the surface of the unit sphere (128).

(2.) The equation \( T\rho = Ta \) expresses that the locus of \( p \) is the spheric surface with \( o \) for centre, which passes through the point \( \lambda \).

(3.) On the other hand, for the sphere through \( o \), which has its centre at \( \lambda \), we have the equation,

\[ T(\rho - a) = Ta; \]

which expresses that the lengths of the two lines, \( ap, ao \), are equal.

(4.) More generally, the equation,

\[ T(\rho - a) = T(\beta - a), \]

expresses that the locus of \( p \) is the spheric surface through \( \beta \), which has its centre at \( \lambda \).

(5.) The equation of the Apollonian* Locus, 145, (8.), (9.), may be written under either of the two following forms:

\[ T(\rho - a^2a) = aT(\rho - a); \quad T\rho = aTa; \]

from each of which we shall find ourselves able to pass to the other, at a later stage, by general Rules of Transformation, without appealing to geometry (comp. 145, (10.) [and 200 (3.), (4.)]).

(6.) The equation,

\[ T(\rho + a) = T(\rho - a), \]

expresses that the locus of \( p \) is the plane through \( o \), perpendicular to the line \( oa \); because it expresses that if \( o\alpha' = -oa \), then the point \( p \) is equally distant from the two points \( \lambda \) and \( \lambda' \). It represents therefore the same locus as the equation,

\[ L\frac{\rho}{a} = \frac{\pi}{2} \] of 132, (1.); \\

or as the equation,

\[ \frac{\rho}{a} + K\frac{\rho}{a} = 0, \] of 144, (1.); \\

or as

\[ (U\frac{\rho}{a})^2 = -1, \] of 161, (7.); \\

* Compare the first Note to page 130.
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or as the simple geometrical formula, \( \rho \perp \alpha \) (129). And in fact it will be found possible, by General Rules of this Calculus, to transform any one of these five formulæ into any other of them; or into this sixth form,

\[
S^\rho_a = 0,
\]

which expresses that the scalar part* of the quaternion \( \frac{\rho}{a} \) is zero, and therefore that this quaternion is a right quotient (132).

(7.) In like manner, the equation

\[
T(\rho - \beta) = T(\rho - \alpha)
\]

expresses that the locus of \( \rho \) is the plane which perpendicularly bisects the line \( \alpha \beta \); because it expresses that \( \rho \) is equally distant from the two points \( \alpha \) and \( \beta \).

(8.) The tensor, \( T\alpha \), being generally a positive scalar, but vanishing (as a limit) with \( \alpha \), we have,

\[
T\alpha x = \pm xT\alpha, \quad \text{according as} \quad x > 0 \text{ or } x < 0;
\]

thus, in particular,

\[
T(-\alpha) = T\alpha; \quad \text{and} \quad T0\alpha = T0 = 0.
\]

(9.) That

\[
T(\beta + \alpha) = T\beta + T\alpha, \quad \text{if} \quad U\beta = U\alpha,
\]

but not otherwise (\( \alpha \) and \( \beta \) being any two actual vectors), will be seen, at a later stage, to be a symbolical consequence from the rules of the present Calculus; but in the mean time it may be geometrically proved, by conceiving that while \( \alpha = \alpha \alpha \), as usual, we make \( \beta + \alpha = \alpha \alpha \), and therefore \( \beta = \alpha \alpha - \alpha = \alpha \alpha \) (4); for thus we shall see that while, in general, the three points \( \alpha, \alpha, \alpha \) are corners of a triangle, and therefore the length of the side \( \alpha \alpha \) is less than the sum of the lengths of the two other sides \( \alpha \alpha \) and \( \alpha \alpha \), the former length becomes, on the contrary, equal to the latter sum, in the particular case when the triangle vanishes, by the point \( \alpha \) falling on the finite line \( \alpha \alpha \); in which case, \( \alpha \alpha \) and \( \alpha \alpha \), or \( \alpha \) and \( \beta \), have one common direction, as the equation \( U\alpha = U\beta \) implies.

(10.) If \( \alpha \) and \( \beta \) be any actual vectors, and if their versors be unequal (\( U\alpha \neq U\beta \)), then

\[
T(\beta + \alpha) < T\beta + T\alpha;
\]

an inequality which results at once from the consideration of the recent

* Compare the Note to page 127; and the following Section of the present Chapter.
triangle oac; but which (as it will be found) may also be symbolically proved,
by rules of the calculus of quaternions. [See 210 (15.)]

(11.) If \( U\beta = - Ua \), then \( T(\beta + a) = \pm (T\beta - Ta) \), according as \( T\beta > \) or \( < Ta \); but

\[
T(\beta + a) > \pm (T\beta - Ta), \text{ if } U\beta \not= - Ua.
\]

187. The quotient, \( U\beta : Ua \), of the versors of the two vectors, \( a \) and \( \beta \), has been called (156) the *Versor of the Quotient*, or quaternion, \( q = \beta : a \); and has been denoted, as such, by the symbol, \( Uq \). On the same plan, we propose now to call the quotient, \( T\beta : Ta \), of the *tensors* of the same two vectors, the *Tensor* of the Quaternion \( q \), or \( \beta : a \), and to denote it by the corresponding symbol, \( Tq \). And then, as we have called the letter \( U \) (in 156) the characteristic of the operation of taking the versor, so we may now speak of \( T \) as the Characteristic of the (corresponding) Operation of taking the Tensor, whether of a Vector, \( a \), or of a Quaternion, \( q \). We shall thus have, generally,

\[
T(\beta : a) = T\beta : Ta, \text{ as we had } U(\beta : a) = U\beta : Ua \ (156);
\]

and may say that as the versor \( Uq \) depended solely on, but conversely was sufficient to determine, the relative direction (157), so the tensor \( Tq \) depends on and determines the relative length† (109), of the two vectors, \( a \) and \( \beta \), of which the quaternion \( q \) is the quotient (112).

(1.) Hence the equation \( T\rho a \overset{\rho}{a} = 1 \), like \( T\rho = Ta \), to which it is equivalent, expresses that the locus of \( p \) is the sphere with \( o \) for centre, which passes through the point \( a \).

(2.) The equation (comp. 186, (6.)),

\[
T \overset{\rho + a}{\rho - a} = 1,
\]

expresses that the locus of \( p \) is the plane through \( o \), perpendicular to the line \( oa \).

* Compare the Note to Art. 109, in page 111; and the first Note in page 137.
† It has been shown, in Art. 112, and in the Additional Illustrations of the third section of the present Chapter (113–116), that Relative Length, as well as relative direction, enters as an essential element into the very Conception of a Quaternion. Accordingly, in Art. 117, an agreement of relative lengths (as well as an agreement of relative directions) was made one of the conditions of equality, between any two quaternions, considered as quotients of vectors: so that we may now say, that the tensors (as well as the versors) of equal quaternions are equal. Compare the first Note to page 138, as regards what was there called the *quantitative element*, of absolute or relative length, which was eliminated from \( a \), or from \( q \), by means of the characteristic \( U \); whereas, the new characteristic, \( T \), of the present section, serves on the contrary to retain that element alone, and to eliminate what may be called by contrast the *qualitative element*, of absolute or relative direction.
(3.) Other examples of the same sort may easily be derived from the sub-articles to 186, by introducing the notation (187) for the tensor of a quotient, or quaternion, as additional to that for the tensor of a vector (185).

(4.) \( T(\beta : \alpha) >, =, \) or \(<1, \) according as \( T\beta >>=, =, \) or \(<Ta. \)

(5.) The tensor of a right quotient (132) is always equal to the tensor of its index (133).

(6.) The tensor of a radial (146) is always positive unity; thus we have, generally, by 156,
\[ TUq = 1; \]
and in particular, by 181,
\[ Ti = Tj = Tk = 1. \]

(7.) \( Txq = \pm xTq, \) according as \( x > \) or \( <0; \)
thus, in particular, \( T(-q) = Tq, \) or the tensors of opposite quaternions are equal.

(8.) \( Tx = \pm x, \) according as \( x > \) or \( <0; \)
thus, the tensor of a scalar is that scalar taken positively.

(9.) Hence,
\[ TTa = Ta, \quad TTq = Tq; \]
so that, by abstracting from the subject of the operation \( T (\text{comp. 145, 160}), \)
we may establish the symbolical equation,
\[ T^2 = TT = T. \]

(10.) Because the tensor of a quaternion is generally a positive scalar, such a tensor is its own conjugate (139); its angle is zero (131); and its versor (159) is positive unity; or in symbols,
\[ KTq = Tq; \quad \angle Tq = 0; \quad UTq = 1. \]

(11.) \( T(1 : q) = T(\alpha : \beta) = Ta : T\beta = 1 : Tq; \)
or in words, the tensor of the reciprocal of a quaternion is equal to the reciprocal of the tensor.

(12.) Again, since the two lines, \( ob \) and \( ob', \) in fig. 36 [p. 115], are equally long, the definition (137) of a conjugate gives
\[ TKq = Tq; \]
or in words, the tensors of conjugate quaternions are equal.

(13.) It is scarcely necessary to remark, that any two quaternions which have equal tensors, and equal versors, are themselves equal: or in symbols, that
\[ q' = q, \] if \( Tq' = Tq, \) and \( Uq' = Uq. \]
188. Since we have, generally,
\[
\frac{\beta}{a} = \frac{T\beta}{Ua} \cdot \frac{Ua}{Ta} = \frac{T\beta}{Ta} \cdot \frac{Ua}{Ua} = \frac{T\beta}{Ta}
\]
we may establish the two following general formulæ of decomposition of a quaternion into two factors, of the tensor and versor kinds:

I. . . \(q = T_q \cdot U_q\);  \(II. . . q = U_q \cdot T_q\);

which are exactly analogous to the formulæ (186) for the corresponding decomposition of a vector, into factors of the same two kinds: namely,

I'. . . \(a = T_a \cdot U_a\);  \(II'. . . a = U_a \cdot T_a\).

To illustrate this last decomposition of a quaternion, \(q\), or \(ob : oa\), into factors, we may conceive that \(AA'\) and \(BB'\) are two concentric and circular, but oppositely directed arcs, which terminate respectively on the two lines \(ob\) and \(oa\), or rather on the longer of those two lines itself, and on the shorter of them prolonged, as in the annexed figure 48; so that \(oa'\) has the length of \(oa\), but the direction of \(ob\), while \(ob'\), on the contrary, has the length of \(ob\), but the direction of \(oa\); and that therefore we may write, by what has been defined respecting versors and tensors of vectors (155, 156, 185, 186),

\[oa' = T_a \cdot U\beta; \quad ob' = T\beta \cdot Ua.\]

Then, by the definitions in 156, 187, of the versor and tensor of a quaternion,

\[Uq = U (ob : oa) = oa' : oa = ob : ob';\]
\[Tq = T (ob : oa) = ob' : oa = ob : oa';\]

whence, by the general formula of multiplication of quotients (107),

I. . . \(q = ob : oa = (ob : oa') \cdot (oa' : oa) = Tq \cdot Uq\);

and

II. . . \(q = ob : oa = (ob : ob') \cdot (ob' : oa) = Uq \cdot Tq,

as above.

189. In words, if we wish to pass from the vector \(a\) to the vector \(\beta\), or from the line \(oa\) to the line \(ob\), we are at liberty either, 1st, to begin by turning, from \(oa\) to \(oa'\), and then to end by stretching, from \(oa'\) to \(ob\), as fig. 48 may serve to illustrate; or, 2nd, to begin by stretching, from \(oa\) to \(ob'\), and end...
by turning, from $oa'$ to $ob$. The act of multiplication of a line $a$ by a quaternion $q$, considered as a factor (103), which affects both length and direction (109), may thus be decomposed into two distinct and partial acts, of the kinds which we have called Version and Tension; and these two acts may be performed, at pleasure, in either of two orders of succession. And although, if we attended merely to lengths, we might be led to say that the tensor of a quaternion was a signless number;* expressive of a geometrical ratio of magnitudes, yet when the recent construction (fig. 48) is adopted, we see, by either of the two resulting expressions (188) for $Tq$, that there is a propriety in treating this tensor as a positive scalar, as we have lately done, and propose systematically to do.

190. Since $TKq = Tq$, by 187, (12.), and $UKq = 1 : Uq$, by 158, we may write, generally, for any quaternion and its conjugate, the two connected expressions:

I. $q = Tq \cdot Uq$;  
II. $Kq = Tq : Uq$;

whence, by multiplication and division,

III. $q \cdot Kq = (Tq)^2$;  
IV. $q : Kq = (Uq)^2$.

This last formula had occurred before; and we saw (161) that in it the parentheses might be omitted, because $(Uq)^2 = U(q^2)$. In like manner (comp. 161, (2.), ) we have also

$$(Tq)^2 = T(q^2) = Tq^2,$$

parentheses being again omitted; or in words, the tensor of the square of a quaternion is always equal to the square of the tensor: as appears (among other ways) from inspection of fig. 42, bis [p. 141], in which the lengths of $oa$, $ob$, $oc$ form a geometrical progression; whence

$$T \cdot \left(\frac{ob}{oa}\right)^2 = T \cdot \frac{oc}{oa} = T \cdot \frac{oc}{oa} = \left(T \cdot \frac{ob}{oa}\right)^2 = \left(T \cdot \frac{ob}{oa}\right)^2.$$

At the same time, we see again that the product $qKq$ of two conjugate quaternions, which has been called (145, (11.)) their common Norm, and denoted by the symbol $Nq$, represents geometrically the square of the quotient of the lengths of the two lines, of which (when considered as vectors) the quaternion $q$ is itself the quotient (112). We may therefore write generally,†

V. $qKq = Tq^2 = Nq$;  
VI. $Tq = \sqrt{Nq} = \sqrt{(qKq)}$.

* Compare the Note, in page 111, to Art. 109.  
† Compare the second Note in page 180.
(1.) We have also, by II., the following other general transformations for the tensor of a quaternion:

\[
\text{VII. } T_q = Kq \cdot Uq; \quad \text{VIII. } T_q = Uq \cdot Kq;
\]

of which the geometrical significations might easily be exhibited by a diagram, but of which the validity is sufficiently proved by what precedes.

(2.) Also (comp. 158),

\[
\frac{1}{Uq} = \frac{Kq}{Tq} = K \frac{q}{Tq} = KUq; \quad K \frac{1}{Uq} = \frac{q}{Tq} = Uq.
\]

(3.) The reciprocal of a quaternion, and the conjugate* of that reciprocal, may now be thus expressed:

\[
\frac{1}{q} = \frac{Kq}{Tq^2} = \frac{q}{Nq} = \frac{KUq}{Tq} = \frac{1}{Uq} \cdot \frac{1}{Tq} = \frac{1}{Uq} \cdot 1;
\]

\[
\frac{1}{q} = \frac{q}{Nq} = \frac{q}{Tq^2} = \frac{Uq}{Tq} = \frac{1}{Kq}.
\]

(4.) We may also write, generally,

\[
\text{IX. } Kq = Tq, \quad KUq = Nq : q.
\]

191. In general, let any two quaternions, \(q\) and \(q'\), be considered as multiplicand and multiplier, and let them be reduced (by 120) to the forms \(\beta : a\) and \(\gamma : \beta\); then the tensor and versor of that third quaternion, \(\gamma : a\), which is (by 107) their product \(q'q\), may be thus expressed:

\[
\text{I. } T(q'q) = T(\gamma : a) = T\gamma : Ta = (T\gamma : T\beta) \cdot (T\beta : Ta) = Tq' \cdot Tq;
\]

\[
\text{II. } U(q'q) = U(\gamma : a) = U\gamma : Ua = (U\gamma : U\beta) \cdot (U\beta : Ua) = Uq' \cdot Uq;
\]

where \(Tq'q\) and \(Uq'q\) are written, for simplicity, instead of \(T(q' q)\) and \(U(q' q)\).

Hence, in any such multiplication, the tensor of the product is the product of the tensor; and the versor of the product is the product of the versors; the order of the factors being generally retained for the latter (comp. 168, &c.), although it may be varied for the former, on account of the scalar character of a tensor.

In like manner, for the division of any one quaternion \(q'\), by any other \(q\), we have the analogous formulæ:

\[
\text{III. } T(q' : q) = Tq' : Tq; \quad \text{IV. } U(q' : q) = Uq' : Uq;
\]

or in words, the tensor of the quotient of any two quaternions is equal to the

* Compare Art. 145, and the Note to page 128.
quotient of the tensors; and similarly, the versor of the quotient is equal to the
quotient of the versors. And because multiplication and division of tensors are
performed according to the rules of algebra, or rather of arithmetic (a tensor
being always, by what precedes, a positive number), we see that the difficulty
(whatever it may be) of the general multiplication and division of quaternions is
thus reduced to that of the corresponding operations on versors: for which
latter operations geometrical constructions have been assigned, in the ninth
section of the present Chapter.

(1.) The two products, \( q'q \) and \( qq' \), of any two quaternions taken as factors
in two different orders, are equal or unequal, according as those two factors are
complanar or diplanar; because such equality (169), or inequality (168), has
been already proved to exist, for the case* when each tensor is unity: but we
have always (comp. 178),

\[
Tq'q = Tqq', \quad \text{and} \quad \angle q'q = \angle qq'.
\]

(2.) If \( \angle q = \angle q' = \frac{\pi}{2} \) then \( qq' = Kq'q \) (170); so that the products of two
right quotients, or right quaternions (132), taken in opposite orders, are always
conjugate quaternions.

(3.) If \( \angle q = \angle q' = \frac{\pi}{2} \) and \( \Delta x . q' \perp \Delta x . q \), then \( qq' = - q'q \),

\[
\angle qq' = \angle q'q = \frac{\pi}{2}, \quad \Delta x . q' \perp \Delta x . q, \quad \Delta x . q' \perp \Delta x . q' \quad (171);
\]

so that the product of two right quaternions, in two rectangular planes, is a third
right quaternion, in a plane rectangular to both; and is changed to its own oppo­
site, when the order of the factors is reversed: as we had \( ij = k = - ji \) (182).

(4.) In general, if \( q \) and \( q' \) be any two diplanar quaternions, the rotation
round \( \Delta x . q' \), from \( \Delta x . q \) to \( \Delta x . q'q \), is positive (177).

(5.) Under the same condition, \( q (q': q) \) is a quaternion with the same
tensor, and same angle, as \( q' \), but with a different axis; and this new axis,
\( \Delta x . q(q': q) \), may be derived (179, (1.)) from the old axis, \( \Delta x . q' \), by a conical
rotation (in the positive direction) round \( \Delta x . q \), through an angle = 2 \( \angle q \).

(6.) The product or quotient of two complanar quaternions is, in general,
a third quaternion complanar with both; but if they be both scalar, or both
right, then this product or quotient degenerates (131) into a scalar.

* Compare the Notes to pages 148, 150.
(7.) Whether $q$ and $q'$ be complanar or diplanar, we have always as in
algebra (comp. 106, 107, 136) the two identical equations:

$$V\.\ (q':q)q = q'; \quad VI.\ (q'q):q = q'.$$

(8.) Also, by 190, V., and 191, I., we have this other general formula:

$$VII.\ Nq'q = Nq'.Nq;$$
or in words, the norm of the product is equal to the product of the norms.

192. Let $q = \beta : a$, and $q' = \gamma : \beta$, as before; then

$$1 : q'q = 1 : (\gamma : a) = a : \gamma = (a : \beta) \cdot (\beta : \gamma) = (1 : q) \cdot (1 : q');$$

so that the reciprocal of the product of any two quaternions is equal to the product of the reciprocals, taken in an inverted order: or briefly,

$$1.\ .\ .\ Rq'q = Rq \cdot Rq',$$

if $R$ be again used (as in 161, (3,.) as a (temporary) characteristic of reciprocation. And because we have then (by the same sub-article) the symbolical equation, $KU = UR$, or in words, the conjugate of the versor of any quaternion $q$ is equal (158) to the versor of the reciprocal of that quaternion; while the versor of a product is equal (191) to the product of the versors: we see that

$$KUq'q = URq'q = URq \cdot URq' = KUq \cdot KUq'.$$

But

$$Kq = Tq \cdot KUq,$$

by 190, IX.; and

$$Tq'q = Tq' \cdot Tq = Tq \cdot Tq',$$

by 191; we arrive then thus at the following other important and general formula:

$$II.\ .\ .\ Kq'q = Kq \cdot Kq';$$
or in words, the conjugate of the product of any two quaternions is equal to the product of the conjugates, taken (still) in an inverted order.

(1.) These two results, I., II., may be illustrated, for versors ($Tq = Tq' = 1$), by the consideration of a spherical triangle $abc$ (comp. fig. 43 [p. 144]); in which the sides $ab$ and $bc$ (comp. 167) may represent $q$ and $q'$, the arc $ac$ then representing $q'q$. For then the new multiplier $Rq = Kq$ (158) is represented (162) by $ba$, and the new multiplicand $Rq' = Kq'$ by $cb$; whence the new product, $Rq \cdot Rq' = Kq \cdot Kq'$, is represented by the inverse arc $ca$, and is therefore at once the reciprocal $Rq'q$, and the conjugate $Kq'q$, of the old product $q'q$.

(2.) If $q$ and $q'$ be right quaternions, then $Kq = -q$, $Kq' = -q'$ (by 144); and the recent formula II. becomes, $Kq'q = qq'$, as in 170.
(3.) In general, that formula II. (of 192) may be thus written:

$$III. \ldots K \frac{\gamma}{a} = K \frac{\beta}{a} \cdot K \frac{\gamma}{\beta};$$

where $\alpha, \beta, \gamma$ may denote any three vectors.

(4.) Suppose then that, as in the annexed fig. 49, we have the two following relations of inverse similitude of triangles (118),

$$\Delta AOB \propto BOC, \quad \Delta BOE \propto DOB;$$

and therefore (by 137) the two equations,

$$\frac{\gamma}{\beta} = K \frac{\beta}{a}, \quad \frac{\beta}{\delta} = K \frac{\epsilon}{\beta};$$

we shall have, by III.,

$$\frac{\gamma}{\delta} = K \frac{\epsilon}{a}, \quad \Delta DOC \propto AOE;$$

so that this third formula of inverse similitude is a consequence from the other two.

(5.) If then (comp. 145, 6.) any two circles, whether in one plane or in space, touch one another at a point $b$: and if from any point $o$, on the common tangent $bo$, two secants $oac, oed$ be drawn, to these two circles; the four points of section, $a, c, d, e$, will be on one common circle: for such concircularity is an easy consequence (through equal angles, &c.), from the last inverse similitude.

(6.) The same conclusion (respecting concircularity, &c.) may be otherwise and geometrically drawn, from the equality of the two rectangles, $aoc$ and $doe$, each being equal to the square of the tangent $ob$; which may serve as an instructive verification of the recent formula III., and as an example of the consistency of the results, to which calculations with quaternions conduct.

(7.) It may be noticed that the construction would in general give three circles, although only one is drawn in the figure; but that if the two triangles $abc$ and $dbe$ be situated in different planes, then these three circles, and of course the five points $a, b, c, d, e$, are situated on one common sphere.

193. An important application of the foregoing general theory of Multiplication and Division, is the case of Right Quaternions (132), taken in connection with their Index-Vectors, or Indices (133).

Considering division first, and employing the general formula of 106, let $\beta$ and $\gamma$ be each $\perp a$; and let $\beta'$ and $\gamma'$ be the respective indices of the two right quotients, $q = \beta : a$, and $q' = \gamma : a$. We shall thus have the two complanarities, $\beta' \parallel \beta, \gamma,$ and $\gamma' \parallel \beta, \gamma$ (comp. 123), because the four lines $\beta, \gamma,$
\(\beta', \gamma'\) are all perpendicular to \(a\); and within their common plane it is easy to see, from definitions already given, that these four lines form a proportion of vectors, in the same sense in which \(a, \beta, \gamma, \delta\) did so, in the fourth section of the present Chapter: so that we may write the equation of quotients,
\[
\gamma' : \beta' = \gamma : \beta.
\]

In fact, we have (by 133, 185, 187) the following relations of length,
\[
T\beta' = T\beta : Ta, \quad T\gamma' = T\gamma : Ta, \quad \therefore T(\gamma' : \beta') = T(\gamma : \beta);
\]
while the relation of directions, expressed by the formula,
\[
U(\gamma' : \beta') = U(\gamma : \beta), \quad \text{or} \quad U\gamma' : U\beta' = U\gamma : U\beta,
\]
is easily established by means of the equations,
\[
\angle(\gamma' : \gamma) = \angle(\beta' : \beta) = \frac{\pi}{2}; \quad \text{Ax.} \quad (\gamma' : \gamma) = \text{Ax.} \quad (\beta' : \beta) = Ua.
\]

We arrive, then, at this general Theorem (comp. again 133) : that "the Quotient of any two Right Quaternions is equal to the Quotient of their Indices."*

(1.) For example (comp. 150, 159, 181), the indices of the right versors \(i, j, k\) are the axes of those three versors, namely, the lines \(oi, oj, ok\); and we have the equal quotients,
\[
j : i = oi : oj' = k = oj' : oi', \&c.
\]

(2.) In like manner, the indices of \(-i, -j, -k\) are \(oi', oj', ok'\); and
\[
i : -j = oj' : o\gamma = k = o\gamma : oj', \&c.
\]

(3.) In general the quotient of any two right versors is equal to the quotient of their axes; as the theory of representative arcs, and of their poles, may easily serve to illustrate.

194. As regards the multiplication of two right quaternions, in connexion with their indices, it may here suffice to observe that, by 106 and 107, the product \(\gamma : a = (\gamma : \beta) \cdot (\beta : a)\) is equal (comp. 136) to the quotient, \((\gamma : \beta) : (a : \beta)\);
whence it is easy to infer that "the product, $q'q$, of any two Right Quaternions, is equal to the Quotient of the Index of the Multiplier, $q'$, divided by the Index of the Reciprocal of the Multiplier, $q$.”

It follows that the plane, whether of the product or of the quotient of two right quaternions, coincides with the plane of their indices; and therefore also with the plane of their axes; because we have, generally, by principles already established, the transformation,

$$\text{if } \angle q = \frac{\pi}{2}, \text{ then } \text{Index of } q = Tq \cdot Ax \cdot q.$$  

SECTION 12.

On the Sum or Difference of any two Quaternions; and on the Scalar (or Scalar Part) of a Quaternion.

195. The Addition of any given quaternion $q'$, considered as a geometrical quotient or fraction (101), to any other given quaternion $q$, considered also as a fraction, can always be accomplished by the first general formula of Art. 106,* when these two fractions have a common denominator; and if they be not already given as having such, they can always be reduced so as to have one, by the process of Art. 120. And because the addition of any two lines was early seen to be a commutative operation (7, 9), so that we have always $\gamma + \beta = \beta + \gamma$, it follows (by 106) that the addition of any two quaternions is likewise a commutative operation, or in symbols, that

$$I. \ldots q + q' = q' + q;$$

so that the Sum of any two† Quaternions has a Value, which is independent of their Order: and which (by what precedes) must be considered to be given, or at least known, or definite, when the two summand quaternions are given. It is easy also to see that the conjugate of any such sum is equal to the sum of the conjugates, or in symbols, that

$$II. \ldots K(q' + q) = Kq' + Kq.$$  

(1.) The important formula last written becomes geometrically evident, when it is presented under the following form. Let $O\text{bdc}$ be any parallelogram, and let $oa$ be any right line, drawn from one corner of it, but not

* [This formula is a definition.]
† It will be found [in 207] that this result admits of being extended to the case of three (or more) quaternions; but, for the moment, we content ourselves with two. [As an example of non-commutative addition contrast Art. 180 (3).]
generally in its plane. Let the three other corners, b, c, d, be reflected (in the sense of 145, (5)) with respect to that line oa, into three new points, b', c', d'; or let the three lines ob, oc, od be reflected (in the sense of 138) with respect to the same line oa; which thus bisects at right angles the three joining lines, bb', cc', dd', as it does BB' in fig. 36 [p. 115]. Then each of the lines ob, oc, od, and therefore also the whole plane figure obdc, may be considered to have simply revolved round the line oa as an axis, by a conical rotation through two right angles; and consequently the new figure ob'd'c', like that old one obdc, must be a parallelogram. Thus (comp. 106, 137), we have

\[ od' = oc' + ob', \quad S' = \gamma' + \beta', \quad S' : a = (\gamma' : a) + (\beta' : a); \]

and the recent formula II. is justified.

(2.) Simple as this last reasoning is, and unnecessary as it appears to be to draw any new diagram to illustrate it, the reader's attention may be once more invited to the great simplicity of expression, with which many important geometrical conceptions, respecting space of three dimensions, are stated in the present Calculus: and are thereby kept ready for future application, and for easy combination with other results of the same kind. Compare the remarks already made in 132, (6.); 145, (10.); 161; 179, (3.); 192, (6.); and some of the shortly following sub-articles to 196, respecting properties of an oblique cone with circular base.

196. One of the most important cases of addition, is that of two conjugate summands, q and Kq; of which it has been seen (in 140) that the sum is always a scalar. We propose now to denote the half of this sum by the symbol, Sg; thus writing generally,

\[ I. \ldots q + Kq = Kq + q = 2Sg; \]

or defining the new symbol Sg by the formula,

\[ II. \ldots Sg = \frac{1}{2} (q + Kq); \quad \text{or briefly, II'.} \ldots S = \frac{1}{2} (1 + K). \]

For reasons which will soon more fully appear, we shall also call this new quantity, Sg, the scalar part, or simply the Scalar, of the Quaternion, q; and shall therefore call the letter S, thus used, the Characteristic of the Operation of taking the Scalar of a quaternion. (Comp. 132, (6.); 137; 156; 187.) It follows that not only equal quaternions, but also conjugate quaternions, have equal scalars; or in symbols,

\[ III. \ldots Sg' = Sg, \quad \text{if} \quad q' = q; \quad \text{and IV.} \ldots SKq = Sg; \]

or briefly,

\[ IV'. \ldots SK = S. \]
And because we have seen that \( Kq = +q \), if \( q \) be a scalar (139), but that \( Kq = -q \), if \( q \) be a right quotient (144), we find that the scalar of a scalar (considered as a degenerate quaternion, 131) is equal to that scalar itself; but that the scalar of a right quaternion is zero. We may therefore now write (comp. 160):

V. \( Sx = x \), if \( x \) be a scalar;
VI. \( S^2q = Sq \), \( S^3 = SS = S \);

and

VII. \( Sq = 0 \), if \( \angle q = \frac{\pi}{2} \).

Again, because \( \alpha \alpha' \) in fig. 36 [p. 115] is multiplied by \( x \), when \( \alpha \beta \) is multiplied thereby, we may write, generally,

VIII. \( Sxq = xS\alpha \), if \( x \) be any scalar;

and therefore in particular (by 188),
IX. \( Sq = S(Tq \cdot Uq) = Tq \cdot SUq \).

Also because \( SKq = Sq \), by IV., while \( KUq = Uq = \frac{1}{q} \), by 158, we have the general equation,

X. \( SUq = SU\frac{1}{q} \); or \( SU\frac{\beta}{a} = SU\frac{a}{\beta} \);

whence, by IX.,
XI. \( Sq = Tq \cdot SU\frac{1}{q} \); or \( S\frac{\beta}{a} = T\frac{\beta}{a} \cdot SU\frac{a}{\beta} \);

and therefore also, by 190, (V.), since \( Tq \cdot Tq = 1 \),

XII. \( Sq = Tq^2 \cdot S\frac{1}{q} = Nq \cdot S\frac{1}{q} \); XII'. \( S\frac{\beta}{a} = N\frac{\beta}{a} \cdot S\frac{a}{\beta} \).

The results of 142, combined with the recent definition I. or II., enable us to extend the recent formula VII., by writing,

XIII. \( Sq >, =, \) or \(< 0 \), according as \( \angle q <, =, \) or \( > \frac{\pi}{2} \);

and conversely,

XIV. \( \angle q <, =, \) or \( > \frac{\pi}{2} \), according as \( Sq >, =, \) or \(< 0 \).

In fact, if we compare that definition I. with the formula of 140, and with fig. 36, we see at once that because, in that figure,

\( S(\alpha \beta : \alpha \alpha) = \alpha \alpha' : \alpha \alpha \),

we may write, generally,

XV. \( Sq = Tq \cdot \cos \angle q \); or \( XVI. \cdot SUq = \cos \angle q \);
equations which will be found of great importance, as serving to connect quaternions with Trigonometry; and which show that

**XVII.** \( \angle q' = \angle q \), if \( SUq' = SUq \),

the angle \( \angle q \) being still taken (as in 130), so as not to fall outside the limits 0 and \( \pi \); whence also,

**XVIII.** \( \angle q' = \angle q \), if \( S\gamma' = S\gamma \), and \( T\gamma' = T\gamma \),

the angle of a quaternion being thus given, when the scalar and the tensor of that quaternion are given, or known. Finally because, in the same figure 36 (comp. 15, 103), the line,

\[ OA' = (OA' : OA) \cdot OA = OA \cdot S(\beta B : OA), \]

may be said to be the projection of \( OB \) on \( OA \), since \( A' \) is the foot of the perpendicular let fall from the point \( B \) upon this latter line \( OA \), we may establish this other general formula:

**XIX.** \( aS^\beta_a = S \frac{\beta}{a} \cdot a = \text{projection of } \beta \text{ on } a; \)

a result which will be found to be of great utility, in investigations respecting geometrical loci, and which may be also written thus:

**XX.** \( \text{Projection of } \beta \text{ on } a = Ua \cdot T\beta \cdot SU\frac{\beta}{a}; \)

with other transformations deducible from principles stated above. It is scarcely necessary to remark that, on account of the scalar character of \( S\gamma \), we have, generally, by 159, and 187, (8.), the expressions,

**XXI.** \( US\gamma = \pm 1; \)

**XXII.** \( TS\gamma = \pm S\gamma; \)

while, for the same reason, we have always, by 139, the equation (comp. IV.),

**XXIII.** \( K8\gamma = S\gamma; \) or **XXIII'.** \( K\gamma = S; \)

and, by 131,

**XXIV.** \( \angle S\gamma = 0, \) or \( = \pi, \) unless \( \angle q = \frac{\pi}{2}; \)

in which last case \( S\gamma = 0, \) by VII., and therefore \( \angle S\gamma \) is indeterminate: \( US\gamma \) becoming at the same time indeterminate, by 159, but \( TS\gamma \) vanishing, by 186, 187.

* Compare the Note in page 120, to Art. 131.
(1.) The equation, \( S \frac{\rho}{a} = 0 \),
is now seen to be equivalent to the formula, \( \rho \perp a \); and therefore to denote
the same plane locus for \( \rho \), as that which is represented by any one of the four
other equations of 186, (6.); or by the equation,

\[
T \frac{\rho + a}{\rho - a} = 1, \text{ of 187, (2.).}
\]

(2.) The equation,

\[
S \frac{\rho - \beta}{a} = 0, \quad \text{or} \quad S \frac{\rho}{a} = S \frac{\beta}{a},
\]
expresses that \( b \rho \perp o a \); or that the points \( b \) and \( \rho \) have the same projection
on \( o a \); or that the locus of \( \rho \) is the plane through \( b \), perpendicular to the line \( o a \).

(3.) The equation,

\[
S U \frac{\rho}{a} = S U \frac{\beta}{a},
\]
expresses (comp. 132, (2.)) that \( \rho \) is on one sheet of a cone of revolution, with \( o \)
for vertex, and \( o a \) for axis, and passing through the point \( b \).

(4.) The other sheet of the same cone is represented by this other equation,

\[
S U \frac{\rho}{a} = - S U \frac{\beta}{a},
\]
and both sheets jointly by the equation,

\[
\left( S U \frac{\rho}{a} \right)^2 = \left( S U \frac{\beta}{a} \right)^2.
\]

(5.) The equation,

\[
S \frac{\rho}{a} = 1, \quad \text{or} \quad S U \frac{\rho}{a} = T \frac{a}{\rho},
\]
expresses that the locus of \( \rho \) is the plane through \( a \), perpendicular to the line \( o a \);
because it expresses (comp. XIX.) that the projection of \( o a \) on \( o a \) is the line
\( o a \) itself; or that the angle \( o a \) is right; or that \( S \frac{\rho - a}{a} = 0 \).

(6.) On the other hand the equation,

\[
S \frac{\beta}{\rho} = 1, \quad \text{or} \quad S U \frac{\beta}{\rho} = T \frac{\rho}{\beta},
\]
expresses that the projection of \( o b \) on \( o p \) is \( o b \) itself; or that the angle \( o p b \) is
right; or that the locus of \( \rho \) is that spheric surface which has the line \( o b \) for a
diameter.
(7.) Hence the system of the two equations,

\[ \frac{S^a}{\rho} = 1, \quad \frac{S^\rho}{\rho} = 1, \]

represents the circle, in which the sphere (6.), with \( ob \) for a diameter, is cut by the plane (5.), with \( oa \) for the perpendicular let fall on it from \( o \).

(8.) And therefore this new equation,

\[ \frac{S^\rho}{a} \cdot \frac{S^\beta}{\rho} = 1, \]

obtained by multiplying the two last, represents the Cyclic* Cone (or cone of the second order, but not generally of revolution), which rests on this last circle (7.) as its base, and has the point \( o \) for its vertex. In fact, the equation (8.) is evidently satisfied, when the two equations (7.) are so; and therefore every point of the circular circumference, denoted by those two equations, must be a point of the locus, represented by the equation (8.). But the latter equation remains unchanged, at least essentially, when \( \rho \) is changed to \( xp, x \) being any scalar; the locus (8.) is, therefore, some conical surface, with its vertex at the origin, \( o \); and consequently it can be none other than that particular cone (both ways prolonged), which rests (as above) on the given circular base (7.).

(9.) The system of the two equations,

\[ \frac{S^\rho}{a} \cdot \frac{S^\beta}{\rho} = 1, \quad \frac{S^\rho}{\gamma} = 1, \]

(in writing the first of which the point may be omitted), represents a conic section; namely that section, in which the cone (8.) is cut by the new plane, which has \( oc \) for the perpendicular let fall upon it, from the origin of vectors \( o \).

(10.) Conversely, every plane ellipse (or other conic section) in space, of which the plane does not pass through the origin, may be represented by a system of two equations, of this last form (9.); because the cone which rests on any such conic as its base, and has its vertex at any given point \( o \), is known to be a cyclic cone.

(11.) The curve (or rather the pair of curves), in which an oblique but cyclic cone (8.) is cut by a concentric sphere (that is to say, a cone resting on a circular

---

* Historically speaking, the oblique cone with circular base may deserve to be named the Apollonian Cone, from Apollonius of Perga, in whose great work on Conics (Kôniks), already referred to in a Note to page 130, the properties of such a cone appear to have been first treated systematically; although the cone of revolution had been studied by Euclid. But the designation “cyclic cone” is shorter; and it seems more natural, in geometry, to speak of the above-mentioned oblique cone thus, for the purpose of marking its connexion with the circle, than to call it, as is now usually done, a cone of the second order, or of the second degree: although these phrases also have their advantages.
base by a sphere which has its centre at the vertex of that cone), has come, in modern times, to be called a Spherical Conic. And any such conic may, on the foregoing plan, be represented by the system of the two equations,

\[ S_a^\rho S_\beta^\rho = 1, \quad T\rho = 1; \]

the length of the radius of the sphere being here, for simplicity, supposed to be the unit of length. But, by writing \( T\rho = a \), where \( a \) may denote any constant and positive scalar, we can at once remove this last restriction, if it be thought useful or convenient to do so.

(12.) The equation (8.) may be written, by XII. or XII', under the form (comp. 191, VII.):

\[ S_a^\rho S_\beta^-\beta = N_a^\alpha = \left( T\frac{\alpha}{\beta} \right)^2; \]

or briefly,

\[ S_a^\rho S_\beta^-\beta = 1, \]

if \( a' = \beta T\frac{\alpha}{\beta} = Ta \cdot U\beta \), and \( \beta' = aT\frac{\beta}{a} = T\beta \cdot Ua \);

so that \( a' \) and \( \beta' \) are here the lines \( o\alpha' \) and \( o\beta' \), of Art. 188, and fig. 48.

(13.) Hence the cone (8.) is cut, not only by the plane (5.) in the circle (7.), which is on the sphere (6.), but also by the (generally) new plane, \( S_a^\rho \rho = 1 \), in the (generally) new circle, in which this new plane cuts the (generally) new sphere, \( S_\beta^-\beta = 1 \); or in the circle which is represented by the system of the two equations,

\[ S_a^\rho = 1, \quad S_\beta^-\beta = 1. \]

(14.) In the particular case when \( \beta \parallel a \) (15.), so that the quotient \( \beta : a \) is a scalar, which must be positive and greater than unity, in order that the plane (5.) may (really) cut the sphere (6.), and therefore that the circle (7.) and the cone (8.) may be real, we may write

\[ \beta = a^\beta a, \quad a > 1, \quad T(\beta : a) = a^\beta, \quad a' = a, \quad \beta' = \beta; \]

and the circle (13.) coincides with the circle (7.).

(15.) In the same case, the cone is one of revolution; every point \( r \) of its circular base (that is, of the circumference thereof) being at one constant distance
from the vertex $o$, namely at a distance $= aTa$. For, in the case supposed, the equations (7.) give, by XII.,

$$
\frac{N\rho}{a} = \frac{S\rho}{\rho} : \frac{a}{\rho} = 1 : \frac{a}{\rho} = a^2 : \frac{S\beta}{\rho} = a^2 ; \text{ or } T\rho = aTa.
$$

(Compare 145, (12.), and 186, (5.).)

(16.) Conversely, if the cone be one of revolution, the equations (7.) must conduct to a result of the form,

$$
\alpha^2 = \frac{N\rho}{a} = \frac{S\rho}{\rho} : \frac{a}{\rho} = \frac{S\beta}{\rho} : \frac{a}{\rho}, \text{ or (comp. (2.)) } \frac{S\beta - a^2a}{\rho} = 0;
$$

which can only be by the line $\beta = a^2a$ vanishing, or by our having $\beta = a^2a$, as in (14.) ; since otherwise we should have, by XIV., $\rho \perp \beta = a^2a$, and all the points of the base would be situated in one plane passing through the vertex $o$, which (for any actual cone) would be absurd.

(17.) Supposing, then, that we have not $\beta \parallel a$, and therefore not $a' = a$, $\beta' = \beta$, as in (14.), nor even $a' \parallel a$, $\beta' \parallel \beta$, we see that the cone (8.) is not a cone of revolution (or what is often called a right cone); but that it is, on the contrary, an oblique (or scalene) cone, although still a cyclic one. And we see that such a cone is cut in two distinct series* of circular sections, by planes parallel to the two distinct (and mutually non-parallel) planes, (5.) and (13.); or to two new planes, drawn through the vertex $o$, which have been called† the two Cyclic Planes of the cone, namely, the two following:

$$
\frac{S\rho}{a} = 0; \quad \frac{S\rho}{\beta} = 0;
$$

while the two lines from the vertex, $oa$ and $ob$, which are perpendicular to these two planes respectively, may be said to be the two Cyclic Normals.

(18.) Of these two lines, $a$ and $\beta$, the second has been seen to be a diameter of the sphere (6.), which may be said to be circumscribed to the cone (8.), when that cone is considered as having the circle (7.) for its base; the second cyclic plane (17.) is therefore the tangent plane at the vertex of the cone, to that first circumscribed sphere (6.).

(19.) The sphere (13.) may in like manner be said to be circumscribed to

---

* These two series of sub-contrary (or antiparallel) but circular sections of a cyclic cone, appear to have been first discovered by Apollonius: see the Fifth Proposition of his First Book, in which he says, καλείσθω δὲ ἡ τοιαύτη τομὴ ἑπεράτη (page 22 of Halley's Edition).

† By M. Chasles.
the cone, if the latter be considered as resting on the new circle (13.), or as terminated by that circle as its new base; and the diameter of this new sphere is the line $on'$, or $\beta'$, which has by (12.) the direction of the line $a$, or of the first cyclic normal (17.); so that (comp. (18.)) the first cyclic plane is the tangent plane at the vertex, to the second circumscribed sphere (13.).

(20.) Any other sphere through the vertex, which touches the first cyclic plane, and which therefore has its diameter from the vertex $= b'\beta'$, where $b'$ is some scalar co-efficient, is represented by the equation,

$$S \frac{b'\beta'}{\rho} = 1, \text{ or } S \frac{\beta'}{\rho} = \frac{1}{b'};$$

it therefore cuts the cone in a circle, of which (by (12.)) the equation of the plane is

$$S \frac{\rho}{a} = b', \text{ or } S \frac{\rho}{b'a} = 1,$$

so that the perpendicular from the vertex is $b'a || \beta$ (comp. (5.)); and consequently this plane of section of sphere and cone is parallel to the second cyclic plane (17.).

(21.) In like manner any sphere, such as

$$S \frac{b\beta}{\rho} = 1, \text{ where } b \text{ is any scalar},$$

which touches the second cyclic plane at the vertex, intersects the cone (8.) in a circle, of which the plane has for equation,

$$S \frac{\rho}{ba} = 1,$$

and is therefore parallel to the first cyclic plane.

(22.) The equation of the cone (by IX., X., XVI.) may also be thus written:

$$SU \frac{\rho}{a} . SU \frac{\beta}{\rho} = T \frac{\alpha}{\beta}; \text{ or, } \cos \frac{\rho}{a} . \cos \frac{\rho}{\beta} = T \frac{\alpha}{\beta};$$

it expresses, therefore, that the product of the cosines of the inclinations, of any variable side ($\rho$) of an oblique cyclic cone, to two fixed lines ($a$ and $\beta$), namely to the two cyclic normals (17.), is constant; or that the product of the sines of the inclinations, of the same variable side (or ray, $\rho$) of the cone, to two fixed planes, namely to the two cyclic planes, is thus a constant quantity.
(23.) The two great circles, in which the concentric sphere $T_p = 1$ is cut by the two cyclic planes, have been called the two Cyclic Arcs* of the Spherical Conic (11.), in which that sphere is cut by the cone. It follows (by (22.)) that the product of the sines of the (arcual) perpendiculars, let fall from any point $r$ of a given spherical conic, on its two cyclic arcs, is constant.

(24.) These properties of cyclic cones, and of spherical conics, are not put forward as new; but they are of importance enough, and have been here deduced with sufficient facility, to show that we are already in possession of a Calculus, with its own Rules† of Transformation, whereby one enunciation of a geometrical theorem, or problem, or construction, can be translated into several others, of which some may be clearer, or simpler, or more elegant than the one first proposed.

197. Let $\beta, \gamma, \delta$ be any three co-initial vectors, $oa, &c$, and let $od = \delta = \gamma + \beta$, so that $obdc$ is a parallelogram (6); then, if we write

$$\beta : a = q, \quad \gamma : a = q', \quad \delta : a = q'' = q' + q \quad (106),$$

and suppose that $b', c', d'$ are the feet of perpendiculars let fall from the points $b, c, d$ on the line $oa$, we shall have, by 196, XIX., the expressions,

$$ob' = aSg, \quad y' = aSg', \quad \delta' = aSg'' = aS (g' + q).$$

But also $ob = cd$, and therefore $ob' = oc' + od'$, the similar projections of equal lines being equal; hence (comp. 11) the sum of the projections of the lines $\beta, \gamma$ must be equal to the projection of the sum, or in symbols,

$$od' = oc' + ob', \quad \delta' = \gamma' + \beta', \quad \delta : a = (\gamma : a) + (\beta : a).$$

Hence, generally, for any two quaternions, $q$ and $q'$, we have the formula:

$$\therefore S (q' + q) = S q' + S q;$$

or in words, the scalar of the sum is equal to the sum of the scalars. It is easy to extend this result to the case of any three (or more) quaternions, with their respective scalars; thus, if $q''$ be a third arbitrary quaternion, we may write

$$S (q'' + (q' + q)) = S q'' + S (q' + q) = S q'' + (S q' + S q);$$

where, on account of the scalar character of the summands, the last parentheses may be omitted. We may therefore write, generally,

$$\Pi \ldots S S q = \Sigma S q, \quad \text{or briefly,} \quad S \Sigma = \Sigma S;$$

where $\Sigma$ is used as a sign of Summation: and may say that the Operation of

* By M. Chasles.
† Comp. 145, (10.), &c.

Hamilton's Elements of Quaternions.
taking the Scalar of a Quaternion is a Distributive Operation (comp. 13). As to the general Subtraction of any one quaternion from any other, there is no difficulty in reducing it, by the method of Art. 120, to the second general formula of 106; nor in proving that the Scalar of the Difference* is always equal to the Difference of the Scalars. In symbols,

\[ \text{III. } S(g' - g) = Sg' - Sg; \]

or briefly,

\[ \text{IV. } SAg = ASg, \quad S\Delta = \Delta S; \]

when \( \Delta \) is used as the characteristic of the operation of taking a difference, by subtracting one quaternion, or one scalar, from another.

(1.) It has not yet been proved (comp. 195) that the Addition of any number of Quaternions, \( q, q', q'', \ldots \) is an associative and a commutative operation (comp. 9). But we see, already, that the scalar of the sum of any such set of quaternions has a value, which is independent of their order, and of the mode of grouping them.

(2.) If the summands be all right quaternions (132), the scalar of each separately vanishes, by 196, VII.; wherefore the scalar of their sum vanishes also, and that sum is consequently itself, by 196, XIV., a right quaternion: a result which it is easy to verify. In fact, if \( \beta \perp a \) and \( \gamma \perp a \), then \( \gamma + \beta \perp a \), because \( a \) is then perpendicular to the plane of \( \beta \) and \( \gamma \); hence, by 106, the sum of any two right quaternions is a right quaternion, and therefore also the sum of any number of such quaternions.

(3.) Whatever two quaternions \( q \) and \( q' \) may be, we have always, as in algebra, the two identities (comp. 191, (7.)):

\[ \text{V. } (q' - q) + q = q'; \quad \text{VI. } (q' + q) - q = q'. \]

198. Without yet entering on the general theory of scalars of products or quotients of quaternions, we may observe here that because, by 196, XV., the scalar of a quaternion depends only on the tensor and the angle, and is independent of the axis, we are at liberty to write generally (comp. 173, 178, and 191, (1.), (5.)):

\[ \text{I. } Sqq' = Sqp; \quad \text{II. } S. q (q' : q) = Sqp; \]

the two products, \( qq' \) and \( q'q \), having thus always equal scalars, although they have been seen to have unequal axes, for the general case of diplanarity (168, 191). It may also be noticed that, in virtue of what was shown in 193,

* Examples have already occurred in 196, (2.), (5.), (16.).
respecting the quotient, and in 194 respecting the product, of any two right quaternions (132), in connexion with their indices (133), we may now establish, for any such quaternions, the formulæ:

III. \( S(q' : q) = S(Iq' : Ig) = T(q' : q) \cdot \cos \angle (Ax \cdot q' : Ax \cdot q) \);

IV. \( S(q' \cdot q) = S(Iq' \cdot Ig) = S(Iq' : Ig) = -Tq'q \cdot \cos \angle (Ax \cdot q' : Ax \cdot q) \);

where the new symbol \( Ig \) is used, as a temporary abridgment, to denote the Index of the quaternion \( q \), supposed here (as above) to be a right one. With the same supposition, we have therefore also these other and shorter formulæ:

V. \( SU(q' : q) = + \cos \angle (Ax \cdot q' : Ax \cdot q) \);

VI. \( SUq'q = - \cos \angle (Ax \cdot q' : Ax \cdot q) \);

which may, by 126, XVI., be interpreted as expressing that, under the same condition of rectangularity of \( q \) and \( q' \),

VII. \( L(q' : q) = L(Ax \cdot q' : Ax \cdot q) \);

VIII. \( Uq'q = \pi - L(Ax \cdot q' : Ax \cdot q) \).

In words, the Angle of the Quotient of two Right Quaternions is equal to the Angle of their Axes; but the Angle of the Product, of two such quaternions, is equal to the Supplement of the Angle of the Axes. There is no difficulty in proving these results otherwise, by constructions such as that employed in Art. 193; nor in illustrating them by the consideration of isosceles quadrantal triangles, upon the surface of a sphere.

199. Another important case of the scalar of a product is the case of the scalar of the square of a quaternion. On referring to Art. 149, and to fig. 42 [p. 133], we see that while we have always \( T(q^2) = (Tq)^2 \); as in 190, and \( U(q^2) = (Uq)^2 \); as in 161, we have also,

I. \( \angle (q^2) = 2 \angle q, \) and \( Ax \cdot (q^2) = Ax \cdot q, \) if \( \angle q < \frac{\pi}{2} \);

but, by the adopted definitions of \( \angle q \) (130), and of \( Ax \cdot q \) (127, 128),

II. \( \angle (q^2) = 2(\pi - \angle q), \) \( Ax \cdot (q^2) = -Ax \cdot q, \) if \( \angle q > \frac{\pi}{2} \).

In each case, however, by 196, XVI., we may write,

III. \( SU(q^2) = \cos \angle (q^2) = \cos 2 \angle q; \)

a formula which holds even when \( \angle q \) is 0, or \( \frac{\pi}{2} \), or \( \pi \), and which gives,

IV. \( SU(q^2) = 2(SUq)^2 - 1. \)
Hence, generally, the scalar of $q^2$ may be put under either of the two following forms:

V. $S(q^2) = Tq^2 \cdot \cos 2 \angle q$;  
VI. $S(q^2) = 2(Sq)^2 - Tq^2$;

where we see that it would not be safe to omit the parentheses, without some convention previously made, and to write simply $Sq^2$, without first deciding whether this last symbol shall be understood to signify the scalar of the square, or the square of the scalar of $q$: these two things being generally unequal. The latter of them, however, occurring rather oftener than the former, it appears convenient to fix on $it$ as that which is to be understood by $Sq^2$, while the other may occasionally be written with a point thus, $S \cdot q^2$; and then, with these conventions respecting notation, we may write:

VII. $Sq^2 = (Sq)^2$;  
VIII. $S \cdot q^2 = S(q^2)$.

But the square of the conjugate of any quaternion is easily seen to be the conjugate of the square; so that we have generally (comp. 190, II.) the formula:

IX. $Kq^2 = K(q^2) = (Kq)^2 = Tq^2 : Uq^2$.

(1.) A quaternion, like a positive scalar, may be said to have in general two opposite square roots; because the squares of opposite quaternions are always equal (comp. (3.)). But of these two roots the principal (or simpler) one, and that which we shall denote by the symbol $\sqrt[q]{q}$ or $\sqrt[\mathbb{R}]q$, and shall call by eminence the Square Root of $q$, is that which has its angle acute, and not obtuse. We shall therefore write, generally,

X. $\angle \sqrt[\mathbb{R}]q = \frac{1}{2} \angle q$;  
Ax. $\sqrt[\mathbb{R}]q = Ax \cdot q$;

with the reservation that, when $\angle q = 0$, or $= \pi$, this common axis of $q$ and $\sqrt[q]{q}$ becomes (by 131, 149) an indeterminate unit-line.

(2.) Hence,

XI. $S \sqrt[q]{q} > 0$, if $\angle q < \pi$;

while this scalar of the square root of a quaternion may, by VI., be thus transformed:

XII. $S \sqrt[q]{q} = \sqrt[\mathbb{R}]\left\{ \frac{1}{2}(Tq + Sq) \right\}$;

a formula which holds good, even at the limit $\angle q = \pi$.

* As, in the Differential Calculus, it is usual to write $dx^2$ instead of $(dx)^2$; while $d(x^2)$ is sometimes written as $d.x^2$. But as $d^2x$ denotes a second differential, so it seems safest not to denote the square of $Sq$ by the symbol $S^2q$, which properly signifies $SSq$, or $Sg$, as in 196, VI.; the second scalar (like the second tensor, 187, (9.), or the second versor, 160) being equal to the first. Still every calculator will of course use his own discretion; and the employment of the notation $S^2q$ for $S(q^2)$, as $\cos^2x$ is often written for $(\cos x)^2$, may sometimes cause a saving of space.
(3.) The principle* (1.) that, in quaternions, as in algebra, the equation,

\[ (-q)^2 = q^2, \]

is an identity, may be illustrated by conceiving that, in fig. 42, a point \( b' \) is determined by the equation \( ob' = bo \); for then we shall have (comp. fig. 33, bis [p. 122]),

\[ (-q)^2 = \left( \frac{ob'}{oa} \right)^2 = \frac{oc}{oa} = q^2, \]

because \( \Delta aob' \propto b'oc. \)

200. Another useful connexion between scalars and tensors (or norms) of quaternions may be derived as follows. In any plane triangle \( aob \), we have† the relation,

\[ (T \cdot ab)^2 = (T \cdot oa)^2 - 2 (T \cdot oa) \cdot (T \cdot ob) \cdot \cos aob + (T \cdot ob)^2; \]

in which the symbols \( T \cdot oa, \&c. \), denote (by 185, 186) the lengths of the sides \( oa, \&c.; \) but if we still write \( q = ob : oa \), we have \( q - 1 = ab : oa \); dividing therefore by \( (T \cdot oa)^2 \), the formula becomes (by 196, \&c.),

\[ I. \ldots T(q-1)^2 = 1 - 2Tq \cdot Suq + Tq^2 = Tq^2 - 2S_q + 1; \]

or

\[ II. \ldots N(q-1) = Nq - 2S_q + 1. \]

But \( q \) is here a perfectly general quaternion; we may therefore change its sign, and write,

\[ III. \ldots T(1+q)^2 = 1 + 2S_q + Tq^2; \quad IV. \ldots N(1+q) = 1 + 2S_q + N_q. \]

And since it is easy to prove (by 106, 107) that

\[ V. \ldots \left( \frac{q'}{q} + 1 \right) q = q' + q, \]

whatever two quaternions \( q \) and \( q' \) may be, while

\[ VI. \ldots S \frac{q'}{q} \cdot N_q = S \cdot q' K_q = S \cdot q K_q', \]

we easily infer this other general formula,

\[ VII. \ldots N(q' + q) = Nq' + 2S \cdot q K_q' + N_q; \]

which gives, if \( x \) be any scalar,

\[ VIII. \ldots N(q + x) = Nq + 2xS_q + x^2. \]

* Compare the Note to page 162.
† By the Second Book of Euclid, or by plane trigonometry.
(1.) We are now prepared to effect, by *rules* of transformation, some other *passages* from one mode of *expression* to another, of the kind which has been alluded to, and partly exemplified, in former sub-articles. Take, for example, the formula,

\[ T \frac{\rho + a}{\rho - a} = 1, \quad \text{of 187, (2.)} \]

or the equivalent formula,

\[ T (\rho + a) = T (\rho - a), \quad \text{of 186, (6.)} \]

which has been seen, on geometrical grounds, to represent a certain locus, namely the plane through o, perpendicular to the line oa; and therefore the same locus as that which is represented by the equation

\[ S \frac{\rho}{a} = 0, \quad \text{of 196, (1.)} \]

To pass now from the former equations to the latter, by *calculation*, we have only to denote the quotient \( \rho : a \) by \( g \), and to observe that the first or second form, as just now cited, becomes then,

\[ T (g + 1) = T (g - 1); \quad \text{or} \quad N (g + 1) = N (g - 1); \]

or finally, by II. and IV.,

\[ S g = 0, \]

which gives the third form of equation, as required.

(2.) Conversely, from \( S \frac{\rho}{a} = 0 \), we can *return* by the same general formulae II. and IV., to the equation \( N \left( \frac{\rho - 1}{a} \right) = \left( \frac{\rho + 1}{a} \right) \), or by I. and III.,

\[ T \left( \frac{\rho - 1}{a} \right) = T \left( \frac{\rho + 1}{a} \right) \]

or to \( T (\rho - a) = T (\rho + a) \), or to \( T \frac{\rho + a}{\rho - a} = 1 \), as above; and generally,

\[ S g = 0 \quad \text{gives} \quad T (g - 1) = T (g + 1), \quad \text{or} \quad T \frac{g + 1}{g - 1} = 1; \]

while the latter equation, in turn, involve, as has been seen, the former.

(3.) Again, if we take the Apollonian Locus, 145, (8.), (9.), and employ the *first* of the two forms 186, (5.) of its equation, namely,

\[ T (\rho - a^2 a) = a T (\rho - a), \]

* Compare 146, (10); and several subsequent sub-articles.*
where $a$ is a given positive scalar different from unity, we may write it as

$$T(q - a^2) = aT(q - 1), \text{ or as } N(q - a^2) = a^2N(q - 1);$$

or by VIII.,

$$Nq - 2a^2Sq + a^4 = a^2(Nq - 2Sq + 1);$$

or, after suppressing $-2a^2Sq$, transposing, and dividing by $a^2 - 1$,

$$Nq = a^2; \text{ or, } Np = a^2Na; \text{ or, } Tp = aTa;$$

which last is the second form 186, (5.), and is thus deduced from the first, by calculation alone, without any immediate appeal to geometry, or the construction of any diagram.

(4.) Conversely if we take the equation,

$$N^2 = a^2, \text{ of 145, (12.),}$$

which was there seen to represent the same locus, considered as a spheric surface, with $o$ for centre, and $aa$ for one of its radii, and write it as $Nq = a^2$,

we can then by calculation return to the form

$$N(q - a^2) = a^2N(q - 1), \text{ or } T(q - a^2) = aT(q - 1),$$

or finally,

$$T(p - a^2a) = aT(p - a), \text{ as in 186, (5.);}$$

this first form of that sub-article being thus deduced from the second, namely from $Tp = aTa$, or $T^a = a$.

(5.) It is far from being the intention of the foregoing remarks, to discourage attention to the geometrical interpretation of the various forms of expression, and general rules of transformation, which thus offer themselves in working with quaternions; on the contrary, one main object of the present Chapter has been to establish a firm geometrical basis, for all such forms and rules. But when such a foundation has once been laid, it is, as we see, not necessary that we should continually recur to the examination of it, in building up the superstructure. That each of the two forms, in 186, (5.), involves the other may be proved, as above, by calculation; but it is interesting to inquire what is the meaning of this result: and in seeking to interpret it, we should be led anew to the theorem of the Apollonian Locus.

(6.) The result (4.) of calculation, that

$$N(q - a^2) = a^2N(q - 1), \text{ if } Nq = a^2;$$

may be expressed under the form of an identity, as follows:

IX. . . $N(q - Nq) = Nq \cdot N(q - 1);$ 

in which $q$ may be any quaternion.
(7.) Or, by 191, VII., because it will soon be seen that
\[ q (q - 1) = q^2 - q, \]
we may write it as this other identity:
\[ X \cdots N(q - Ng) = N(q^2 - q). \]

(8.) If \( T(q - 1) = 1 \), then \( S \frac{1}{q} = \frac{1}{2} \); and conversely, the former equation follows from the latter; because each may be put under the form (comp. 196, XII.), \( Nq = 2Sq \).

(9.) Hence, if \( T(\rho - \alpha) = Ta \), then \( S \frac{2\alpha}{\rho} = 1 \), and reciprocally. In fact (comp. 196, (6.)), each of these two equations expresses that the locus of \( r \) is the sphere which passes through \( o \), and has its centre at \( \alpha \); or which has \( ob = 2\alpha \) for a diameter.

(10.) By changing \( q \) to \( q + 1 \) in (8), we find that
\[ \text{if } Tq = 1, \text{ then } S \frac{q - 1}{q + 1} = 0, \text{ and reciprocally.} \]

(11.) Hence if \( T\rho = Ta \), then \( S \frac{\rho - \alpha}{\rho + a} = 0 \), and reciprocally; because (by 106)
\[ \frac{\rho - \alpha}{\rho + a} = \frac{\rho - a}{\alpha} \cdot \frac{\rho + a}{\alpha} = \left( \frac{\rho}{\alpha} - 1 \right) \cdot \left( \frac{\rho}{\alpha} + 1 \right). \]

(12.) Each of these two equations (11.) expresses that the locus of \( r \) is the sphere through \( \alpha \), which has its centre at \( o \); and their proved agreement is a recognition, by quaternions, of the elementary geometrical theorem, that the angle in a semicircle is a right angle.

SECTION 13.

On the Right Part (or Vector Part) of a Quaternion; and on the Distributive Property of the Multiplication of Quaternions.

201. A given vector \( ob \) can always be decomposed, in one but in only one way, into two component vectors, of which it is the sum (6); and of which one, as \( ob' \) in fig. 50, is parallel (15) to another given vector \( oa \), while the other, as \( ob'' \) in the same figure, is perpendicular to that given line \( oa \); namely, by letting fall the perpendicular \( bb' \) on \( oa \), and drawing \( ob'' = b'b \), so that \( ob'bb'' \) shall be a rectangle. In other words, if \( a \) and \( \beta \) be any two given, actual, and co-initial vectors, it is always possible to deduce from them, in one definite way, two other
co-initial vectors, $\beta'$ and $\beta''$, which need not however both be actual (1); and which shall satisfy (comp. 6, 15, 129) the conditions,

$$\beta = \beta' + \beta'' = \beta'' + \beta'; \quad \beta' \parallel a, \quad \beta'' \perp a;$$

$\beta'$ vanishing, when $\beta \perp a$; and $\beta''$ being null, when $\beta \parallel a$; but both being (what we may call) determinate vector-functions of $a$ and $\beta$. And of these two functions, it is evident that $\beta'$ is the orthographic projection of $\beta$ on the line $a$; and that $\beta''$ is the corresponding projection of $\beta$ on the plane through $o$, which is perpendicular to $a$.

202. Hence it is easy to infer, that there is always one, but only one way, of decomposing a given quaternion,

$$q = ob : oa = \beta : a,$$

into two parts or summands (195), of which one shall be, as in 196, a scalar, while the other shall be a right quotient (132). Of these two parts, the former has been already called (196) the scalar part, or simply the Scalar of the Quaternion, and has been denoted by the symbol $S_q$; so that, with reference to the recent figure 50, we have

I. . . $S_q = S (ob : oa) = ob' : oa$; or, $S (\beta : a) = \beta' : a$.

And we now propose to call the latter part the Right Part* of the same quaternion, and to denote it by the new symbol

$$V_q;$$

writing thus, in connexion with the same figure,

II. . . $V_q = V (ob : oa) = ob'' : oa$; or, $V (\beta : a) = \beta'' : a$.

The System of Notations, peculiar to the present Calculus, will thus have been completed; and we shall have the following general Formula of Decomposition of a Quaternion into two Summands (comp. 188), of the Scalar and Right kinds:

III. . . $q = S_q + V_q = V_q + S_q$,

or, briefly and symbolically,

IV. . . $1 = S + V = V + S$.

(1.) In connexion with the same fig. 50, we may write also,

$$V (ob : oa) = v'v : oa,$$

because, by construction, $v'v = ob''$.

* This Right Part, $V_q$, will come to be also called the Vector Part, or simply the Vector, of the Quaternion; because it will be found possible and useful to identify such part with its own Index-Vector (133). Compare the Notes to pages 121, 137, 175 [and Art. 286].
(2.) In like manner, for fig. 36 [p. 115], we have the equation,
\[ V(\mathbf{ob} : \mathbf{oa}) = A'B : OA. \]

(3.) Under the recent conditions,
\[ V(\beta' : a) = 0, \quad \text{and} \quad S(\beta'' : a) = 0. \]

(4.) In general, it is evident that
\[ V\ldots \mathbf{q} = 0, \quad \text{if} \quad S\mathbf{q} = 0, \quad \text{and} \quad V\mathbf{q} = 0; \quad \text{and reciprocally.} \]

(5.) More generally,
\[ VI.\ldots \mathbf{q}' = \mathbf{q}, \quad \text{if} \quad S\mathbf{q}' = S\mathbf{q}, \quad \text{and} \quad V\mathbf{q}' = V\mathbf{q}; \quad \text{with the converse.} \]

(6.) Also
\[ VII.\ldots V\mathbf{q} = 0, \quad \text{if} \quad \angle \mathbf{q} = 0, \quad \text{or} \quad \pi; \]

or
\[ VIII.\ldots V(\beta : a) = 0, \quad \text{if} \quad \beta \parallel a; \]
the right part of a scalar being zero.

(7.) On the other hand,
\[ IX.\ldots V\mathbf{q} = \mathbf{q}, \quad \text{if} \quad \angle \mathbf{q} = \frac{\pi}{2}; \]
a right quaternion being its own right part.

203. We had (196, XIX.) a formula which may now be written thus,
\[ I.\ldots \mathbf{ob}' = S(\mathbf{ob} : \mathbf{oa}) \cdot \mathbf{oa}, \quad \text{or} \quad \beta' = S\frac{\beta}{a} \cdot a, \]
to express the projection of \( \mathbf{ob} \) on \( \mathbf{oa} \), or of the vector \( \beta \) on \( a \); and we have evidently, by the definition of the new symbol \( V\mathbf{q} \), the analogous formula,
\[ II.\ldots \mathbf{ob}'' = V(\mathbf{ob} : \mathbf{oa}) \cdot \mathbf{oa}, \quad \text{or} \quad \beta'' = V\frac{\beta}{a} \cdot a, \]
to express the projection of \( \beta \) on the plane (through \( a \)), which is drawn so as to be perpendicular to \( a \); and which has been considered in several former subarticles (comp. 186, (6.), and 196, (1.)). It follows (by 186, &c.) that
\[ III.\ldots T\beta'' = TV\frac{\beta}{a} \cdot Ta = \text{perpendicular distance of} \ b \ \text{from} \ \mathbf{oa}; \]
this perpendicular being here considered with reference to its length alone, as the characteristic \( T \) of the tensor implies. It is to be observed that because the factor, \( V\frac{\beta}{a} \), in the recent formula \( II. \) for the projection \( \beta'' \), is not a scalar, we must write that factor as a multiplier, and not as a multiplicand; although we were at liberty, in consequence of a general convention (15), respecting the
multiplication of vectors and scalars, to denote the other projection $\beta'$ under the form,

$$I' \ldots \beta' = a \frac{\beta}{a}$$

(1.) The equation,

$$\nabla \frac{\rho}{a} = 0,$$

expresses that the locus of $p$ is the indefinite right line $oa$.

(2.) The equation,

$$\nabla \frac{\rho - \beta}{a} = 0 \text{ or } \nabla \frac{\rho}{a} = \nabla \frac{\beta}{a},$$

expresses that the locus of $p$ is the indefinite right line $bb'$, in fig. 50, which is drawn through the point $b$, parallel to the line $oa$.

(3.) The equation

$$S \frac{\rho - \beta}{a} = 0, \text{ or } S \frac{\rho}{a} = S \frac{\beta}{a},$$

has been seen to express that the locus of $p$ is the plane through $b$, perpendicular to the line $oa$; if then we combine it with the recent equation (2.), we shall express that the point $p$ is situated at the intersection of the two last mentioned loci; or that it coincides with the point $b$.

(4.) Accordingly, whether we take the two first or the two last of these recent forms (2.), (3.), namely,

$$\nabla \frac{\rho - \beta}{a} = 0, \text{ or } S \frac{\rho - \beta}{a} = 0, \text{ or } \nabla \frac{\rho}{a} = \nabla \frac{\beta}{a}, \text{ or } S \frac{\rho}{a} = S \frac{\beta}{a},$$

we can infer this position of the point $p$: in the first case by inferring, through 202, $V.$, that $\frac{\rho - \beta}{a} = 0$, whence $\rho - \beta = 0$, by 142; and in the second case by inferring, through 202, VI., that $\frac{\rho}{a} = \frac{\beta}{a}$; so that we have in each case (comp. 104), or as a consequence from each system, the equality $\rho = \beta$, or $op = ob$; or finally (comp. 20) the coincidence, $p = b$.

(5.) The equation

$$TV \frac{\rho}{a} = TV \frac{\beta}{a},$$

expresses that the locus of the point $p$ is the cylindric surface of revolution, which passes through the point $b$, and has the line $oa$ for its axis; for it expresses, by III., that the perpendicular distances of $p$ and $b$, from this latter line, are equal.
(6.) The system of the two equations,
\[ TV^\alpha = TV^\beta, \quad S^\gamma = 0, \]
expresses that the locus of \( \rho \) is the (generally) elliptic section of the cylinder (5.), made by the plane through \( o \), which is perpendicular to the line \( oc \).

(7.) If we employ an analogous decomposition of \( \rho \), by supposing that
\[ \rho = \rho' + \rho'', \quad \rho' \parallel \alpha, \quad \rho'' \perp \alpha, \]
the three rectilinear or plane loci, (1.), (2.), (3.), may have their equations thus briefly written:
\[ \rho'' = 0; \quad \rho'' = \beta''; \quad \rho' = \beta': \]
while the combination of the two last of these gives \( \rho = \beta \), as in (4.).

(8.) The equation of the cylindric locus, (5.), takes at the same time the form
\[ T\rho'' = T\beta''; \]
which last equation expresses that the projection \( \rho'' \) of the point \( \rho \), on the plane through \( o \) perpendicular to \( oA \), falls somewhere on the circumference of a circle, with \( o \) for centre, and \( oA'' \) for radius: and this circle may accordingly be considered as the base of the right cylinder, in the sub-article last cited.

204. From the mere circumstance that \( Vq \) is always a right quotient (132), whence \( UVq \) is a right versor (153), of which the plane (119), and the axis (127), coincide with those of \( q \), several general consequences easily follow. Thus we have generally, by principles already established, the relations:

I. \[ \angle Vq = \frac{\pi}{2}; \]
II. \[ Ax \cdot Vq = Ax \cdot UVq = Ax \cdot q; \]
III. \[ KVq = -Vq, \quad KV = -V \] (144);
IV. \[ SVq = 0, \quad SV = 0 \] (196, VII.);
V. \[ (UVq)^2 = -1 \] (153, 159);
and therefore, VI. \[ (Vq)^2 = -(TVq)^2 = -NVq,^* \]
because, by the general decomposition (188) of a quaternion into factors, we have

VII. \[ Vq = TVq \cdot UVq. \]
We have also (comp. 196, VI.),

VIII. \[ VSq = 0, \quad VS = 0 \] (202, VII.);

^* Compare the Note to page 132.
IX. \[ VV_q = V_q, \text{ or } V^2 = VV = V \text{ (202, IX.)} \]

and

X. \[ VK_q = -V_q, \text{ or } VK = -V, \]

because conjugate quaternions have opposite right parts, by the definitions in 137, 202, and by the construction of fig. 36 [p. 115]. For the same reason, we have this other general formula,

XI. \[ Kq = S_q - V_q, \text{ or } K = S - V; \]

but we had

\[ q = S_q + V_q, \text{ or } 1 = S + V, \] by 202, III., IV.;

hence not only, by addition,

\[ q + Kq = 2S_q, \text{ or } 1 + K = 2S, \] as in 196, I.,

but also, by subtraction,

XII. \[ g - Kq = 2V_q, \text{ or } 1 - K = 2V; \]

whence the Characteristic, \( V \), of the Operation of taking the Right Part of a Quaternion (comp. 132, (6); 137; 156; 187; 196), may be defined by either of the two following symbolical equations:

XIII. \[ V = 1 - S \text{ (202, IV.);} \]

XIV. \[ V = \frac{1}{2} (1 - K); \]

whereof the former connects it with the characteristic \( S \), and the latter with the characteristic \( K \); while the dependence of \( K \) on \( S \) and \( V \) is expressed by the recent formula XI.; and that of \( S \) on \( K \) by 196, II'. Again, if the line \( ob \), in fig. 50, be multiplied (15) by any scalar coefficient, the perpendicular \( bb' \) is evidently multiplied by the same; hence, generally,

XV. \[ Vxq = xVq, \text{ if } x \text{ be any scalar;} \]

and therefore, by 188, 191,

XVI. \[ Vq = T_q \cdot VU_q, \text{ and} \]

XVII. \[ TV_q = T_q \cdot TVU_q. \]

But the consideration of the right-angled triangle, \( ob' \)\( b' \), in the same figure, shows that

XVIII. \[ TV_q = T_q \cdot \sin \angle q, \]

because, by 202, II., we have

\[ TV_q = T (ob'': oA) = T \cdot ob'': T \cdot oA, \]

and

\[ T \cdot ob' = T \cdot ob \cdot \sin AOB; \]

we arrive then thus at the following general and useful formula, connecting quaternions with trigonometry anew:

XIX. \[ TVU_q = \sin \angle q; \]
by combining which with the formula,

\[ \sin \theta = \cos \theta \cdot (196, \text{XVI.}), \]

we arrive at the general relation:

\[ \text{XX.} \ldots (\sin \theta)^2 + (\cos \theta)^2 = 1; \]

which may also (by XVII., and by 196, IX.) be written thus:

\[ \text{XXI.} \ldots (\sin \theta)^2 + (\cos \theta)^2 = \sin \theta^2; \]

and might have been immediately deduced, without sines and cosines, from the right-angled triangle, by the property of the square of the hypotenuse, under the form,

\[ (\sin \theta)^2 + (\cos \theta)^2 = \sin \theta^2. \]

The same important relation may be expressed in various other ways; for example, we may write,

\[ \text{XXII.} \ldots \sin \theta = \cos \theta^2 = \sin \theta^2 - \cos \theta^2, \]

where it is assumed, as an abridgment of notation (comp. 199, VII., VIII.), that

\[ \text{XXIII.} \ldots \sin \theta^2 = \sin \theta^2; \] \text{XXIV.} \ldots \cos \theta^2 = \cos \theta^2, \]

the import of this last symbol remaining to be examined. And because, by the definition of a norm, and by the properties of \( \sin \theta \) and \( \cos \theta \),

\[ \text{XXV.} \ldots \sin \sin \theta^2 = \sin \sin \theta^2, \] \text{XXVI.} \ldots \cos \sin \theta^2 = - \cos \sin \theta^2, \]

we may write also,

\[ \text{XXVII.} \ldots \sin \sin \theta^2 = \sin \sin \theta^2, \]

a result which is indeed included in the formula 200, VIII., since that equation gives, generally,

\[ \text{XXVIII.} \ldots \sin (\theta + x) = \sin \theta + \sin x, \] \text{if} \ \theta = \frac{\pi}{2}; \]

\( x \) being, as usual, any scalar. It may be added that because (by 106, 143) we have, as in algebra, the identity,

\[ \text{XXIX.} \ldots -(\theta' + \theta) = -\theta' - \theta, \]

the opposite of the sum of any two quaternions being thus equal to the sum of the opposites, we may (by XI.) establish this other general formula:

\[ \text{XXX.} \ldots -(\theta q) = -\theta \cdot q, \]

the opposite of the conjugate of any quaternion \( q \) having thus the same right part as that quaternion, but an opposite scalar part.
(1.) From the last formula it may be inferred, that

\[ g' = -Kg, \quad \text{then} \quad Vg' = +Vg, \quad \text{but} \quad Sg' = -Sg; \]

and therefore that

\[ Tg' = Tg, \quad \text{and} \quad Ax \cdot g' = Ax \cdot g, \quad \text{but} \quad \angle g' = \pi - \angle g; \]

which two last relations might have been deduced from 138 and 143, without the introduction of the characteristics \( S \) and \( V \).

(2.) The equation,

\[ \left( \frac{V^p}{a} \right)^2 = \left( \frac{V^\beta}{a} \right)^2, \quad \text{or} \quad (\text{by XXVI}), \quad NV^p = NV^\beta, \]

like the equation of 203, (5.), expresses that the locus of \( p \) is the right cylinder, or cylinder of revolution, with \( oa \) for its axis, which passes through the point \( b \).

(3.) The system of the two equations,

\[ \left( \frac{V^p}{a} \right)^2 = \left( \frac{V^\beta}{a} \right)^2, \quad S \frac{p}{\gamma} = 0, \]

like the corresponding system in 203, (6.), represents generally an elliptic section of the same right cylinder; but if it happened that \( \gamma \parallel a \), the section then becomes circular.

(4.) The system of the two equations,

\[ S \frac{p}{a} = x, \quad \left( \frac{V^p}{a} \right)^2 = x^2 - 1, \quad \text{with} \quad x > -1, \quad x < 1, \]

represents the circle,* in which the cylinder of revolution, with \( oa \) for axis, and with \( (1 - x^2)^{\frac{1}{2}}Ta \) for radius, is perpendicularly cut by a plane at a distance \( = \pm xTa \) from \( o \); the vector of the centre of this circular section being \( xa \).

(5.) While the scalar \( x \) increases (algebraically) from \(-1\) to \(0\), and thence to \(+1\), the connected scalar \( \sqrt{(1 - x^2)} \) at first increases from \(0\) to \(1\), and then decreases from \(1\) to \(0\); the radius of the circle (4.) at the same time enlarging from zero to a maximum \( = Ta \), and then again diminishing to zero; while the position of the centre of the circle varies continuously, in one constant direction, from a first limit-point \( a' \), if \( oa' = -a \), to the point \( a \), as a second limit.

* By the word "circle," in these pages, is usually meant a circumference, and not an area; and in like manner, the words "sphere," "cylinder," "cone," &c., are usually here employed to denote surfaces, and not volumes.
(6.) The locus of all such circles is the sphere, with \(\mathbf{A}\mathbf{A}'\) for a diameter, and therefore with 0 for centre; namely, the sphere which has already been represented by the equation \(T\rho = Ta\) of 186, (2.); or by \(T\frac{\rho}{\alpha} = 1\), of 187, (1.); or by
\[
S\frac{\rho - \alpha}{\rho + \alpha} = 0, \text{ of } 200, (11.);
\]
but which now presents itself under the new form,
\[
\left(\frac{S\rho}{\alpha}\right)^2 - \left(\frac{V\rho}{\alpha}\right)^2 = 1,
\]
obtained by eliminating \(x\) between the two recent equations (4.).

(7.) It is easy, however, to return from the last form to the second, and thence to the first, or to the third, by rules of calculation already established, or by the general relations between the symbols used. In fact, the last equation (6.) may be written, by XXII., under the form,
\[
N\frac{\rho}{\alpha} = 1; \text{ whence } T\frac{\rho}{\alpha} = 1, \text{ by } 190, \text{ VI.};
\]
and therefore also \(T\rho = Ta\), by 187, and \(S\frac{\rho - \alpha}{\rho + \alpha} = 0\), by 200, (11.).

(8.) Conversely, the sphere through \(A,\) with 0 for centre, might already have been seen, by the first definition and property of a norm, stated in 145, (11.), to admit (comp. 145, (12.)) of being represented by the equation \(N\frac{\rho}{\alpha} = 1\); and therefore, by XXII., under the recent form (6.); in which if we write \(x\) to denote the variable scalar \(S\frac{\rho}{\alpha}\), as in the first of the two equations (4.), we recover the second of those equations: and thus might be led to consider, as in (6.), the sphere in question as the locus of a variable circle, which is (as above) the intersection of a variable cylinder, with a variable plane perpendicular to its axis.

(9.) The same sphere may also, by XXVII., have its equation written thus,
\[
N\left(\frac{S\rho}{\alpha} + V\frac{\rho}{\alpha}\right) = 1; \text{ or } T\left(\frac{S\rho}{\alpha} + V\frac{\rho}{\alpha}\right) = 1.
\]

(10.) If, in each variable plane represented by the first equation (4.), we conceive the radius of the circle, or that of the variable cylinder, to be multiplied by any constant and positive scalar \(\alpha\), the centre of the circle and the axis of the cylinder remaining unchanged, we shall pass thus to a new system of circles, represented by this new system of equations,
\[
S\frac{\rho}{\alpha} = x, \quad \left(\frac{V\frac{\rho}{\alpha}}{aa}\right)^2 = x^2 - 1.
\]
(11.) The locus of these new circles will evidently be a Spheroid of Revolution; the centre of this new surface being the centre o, and the axis of the same surface being the diameter AA', of the sphere lately considered: which sphere is therefore either inscribed or circumscribed to the spheroid, according as the constant a > or < 1; because the radii of the new circles are in the first case greater, but in the second case less, than the radii of the old circles; or because the radius of the equator of the spheroid = a Ta, while the radius of the sphere = Ta.

(12.) The equations of the two coaxal cylinders of revolution, which envelope respectively the sphere and spheroid (or are circumscribed thereto) are:

\[ \left( \frac{V}{a} \right)^2 = -1, \quad \left( \frac{V}{a} \right)^2 = -1; \quad \text{or} \quad NV \frac{V}{a} = 1, \quad NV \frac{V}{a} = a^2; \]

or

\[ TV \frac{V}{a} = 1, \quad TV \frac{V}{a} = a. \]

(13.) The system of the two equations,

\[ S \frac{V}{a} = x, \quad \left( \frac{V}{\beta} \right)^2 = x^2 - 1, \quad \text{with} \ \beta \ \text{not} \ || \ a, \]

represents (comp. (3.)) a variable ellipse, if the scalar x be still treated as a variable.

(14.) The result of the elimination of x between the two last equations, namely this new equation,

\[ \left( \frac{S}{a} \right)^2 - \left( \frac{V}{\beta} \right)^2 = 1; \quad \text{or} \quad NS \frac{V}{a} + NV \frac{V}{\beta} = 1, \quad \text{by XXV., XXVI.;} \]

or

\[ N \left( S \frac{V}{a} + V \frac{V}{\beta} \right) = 1, \quad \text{by XXVII.; or finally,} \quad T \left( S \frac{V}{a} + V \frac{V}{\beta} \right) = 1, \quad \text{by 190, VI.,} \]

represents the locus of all such ellipses (13.), and will be found to be an adequate representation, through quaternions, of the general Ellipsoid (with three unequal axes): that celebrated surface being here referred to its centre, as the origin o of vectors to its points; and the six scalar (or algebraic) constants, which enter into the usual algebraic equation (by co-ordinates) of such a central ellipsoid, being here virtually included in the two independent vectors, a and \( \beta \), which may be called its two Vector-Constants.*

* It will be found, however, that other pairs of vector-constants, for the central ellipsoid, may occasionally be used with advantage.

Hamilton’s Elements of Quaternions.
(15.) The equation (comp. (12.)),

\[
\left( \frac{V}{\beta} \right)^2 = -1, \quad \text{or} \quad NV \frac{\rho}{\beta} = 1, \quad \text{or} \quad TV \frac{\rho}{\beta} = 1,
\]

represents a cylinder of revolution, circumscribed to the ellipsoid, and touching it along the ellipse which answers to the value \( x = 0 \), in (13.); so that the plane of this ellipse of contact is represented by the equation,

\[
S \frac{\rho}{a} = 0;
\]

the normal to this plane being thus (comp. 196, (17.)) the vector \( a \), or \( oa \); while the axis of the lately mentioned enveloping cylinder is \( \beta \), or \( ob \).

(16.) Postponing any further discussion of the recent quaternion equation of the ellipsoid (14.), it may be noted here that we have generally, by XXII., the two following useful transformations for the squares, of the scalar \( Sg \), and of the right part \( Vg \), of any quaternion \( g \):

\[
\begin{align*}
\text{XXXI.} & \quad Sg^2 = Tg^2 + Vg^2; & \text{XXXII.} & \quad Vg^2 = Sg^2 - Tg^2.
\end{align*}
\]

(17.) In referring briefly to these, and to the connected formula XXII., upon occasion, it may be somewhat safer to write,

\[
\begin{align*}
(S)^2 &= (T)^2 + (V)^2, & (V)^2 &= (S)^2 - (T)^2, & (T)^2 &= (S)^2 - (V)^2,
\end{align*}
\]

than \( S^2 = T^2 + V^2 \); because these last forms of notation, \( S^2 \), \&c., have been otherwise interpreted already, in analogy to the known Functional Notation, or Notation of the Calculus of Functions, or of Operations (comp. 187, (9.)); 196, VI.; and 204, IX.).

(18.) In pursuance of the same analogy, any scalar may be denoted by the general symbol,

\[
V^{-1}0;
\]

because scalars are the only quaternions of which the right parts vanish.

(19.) In like manner, a right quaternion, generally, may be denoted by the symbol,

\[
S^{-1}0;
\]

and since this includes (comp. 204, I.) the right part of any quaternion, we may establish this general symbolic transformation of a Quaternion:

\[
q = V^{-1}0 + S^{-1}0.
\]
(20.) With this form of notation, we should have generally, at least for real* quaternions, the inequalities,

\[(V^{-1}0)^2 > 0; \quad (S^{-1}0)^2 < 0;\]

so that a (geometrically real) Quaternion is generally of the form:

Square-root of a Positive, plus Square-root of a Negative.

(21.) The equations 196, XVI., and 204, XIX., give, as a new link between quaternions and trigonometry, the formula:

XXXIII. \( \tan \angle q = TVUq : SUq = TVq : Sg. \)

(22.) It may not be entirely in accordance with the theory of that Functional (or Operational) Notation to which allusion has lately been made, but it will be found to be convenient in practice, to write this last result under one or other of the abridged forms:

XXXIV. \( \tan \angle q = \frac{TV}{S} q; \) or \( XXXIV' \ldots \tan \angle q = (TV : S) q; \)

which have the advantage of saving the repetition of the symbol of the quaternion, when that symbol happens to be a complex expression, and not, as here, a single letter, \( q. \)

(23.) The transformation 194, for the index of a right quotient, gives generally, by II., for any quaternion \( q, \) the formulæ:

XXXV. \( IVg = TVq \cdot Ax.q; \) XXXVI. \( IUVg = Ax.q; \)

so that we may establish generally the symbolical † equation,

XXXVI'. \( IUV = Ax. \)

(24.) And because \( Ax. (1 : Vq) = - Ax.Vq, \) by 135, and therefore \( = - Ax.q, \) by II., we may write also, by XXXV.,

XXXV'. \( I (1 : Vq) = - Ax.q : TVq. \)

* Compare Art. 149; and the Notes to pages 87, 135.
† Compare the Note to Art. 199.
‡ At a later stage [286] it will be found possible (comp. the Note to page 175, &c.), to write, generally,

\[ IVg = Vg, \quad IUVg = UVq; \]

and then (comp. the Note in page 120 to Art. 129) the recent equations, XXXVI., XXXV', will take these shorter forms [291]:

\[ Ax.q = UVq; \quad Ax. = UV.\]
205. If any parallelogram \(\text{obdc}\) (comp. 197) be projected on the plane through \(o\), which is perpendicular to \(oa\), the projected figure \(\text{ob}''\text{c}''\) (comp. 11) is still a parallelogram; so that
\[
\text{od}'' = \text{oc}'' + \text{ob}'' \quad (6), \quad \text{or} \quad \delta'' = \gamma'' + \beta'';
\]
and therefore, by 106,
\[
\delta'' : a = (\gamma'' : a) + (\beta'' : a).
\]
Hence, by 120, 202, for any two quaternions, \(q\) and \(q'\), we have the general formula,
\[
\text{I.} \quad \text{V}(q' + q) = \text{V}q' + \text{V}q;
\]
with which it is easy to connect this other,
\[
\text{II.} \quad \text{V}(q' - q) = \text{V}q' - \text{V}q.
\]
Hence also, for any three quaternions, \(q, q', q''\),
\[
\text{III.} \quad \text{V}(q'' + (q' + q)) = \text{V}q'' + \text{V}(q' + q) = \text{V}q'' + (\text{V}q' + \text{V}q);
\]
and similarly for any greater number of summands: so that we may write generally (comp. 197, II.),
\[
\text{III.} \quad \text{V}2q = 2\text{V}q, \quad \text{or briefly} \quad \text{III.'} \quad \text{V}S = \text{SV};
\]
while the formula II. (comp. 197, IV.) may, in like manner, be thus written,
\[
\text{IV.} \quad \text{V}Aq = \text{AV}q, \quad \text{or IV.'} \quad \text{V}A = \text{AV}.
\]
The order of the terms added, and the mode of grouping them, in III., being as yet supposed to remain unaltered, although both those restrictions will soon be removed. We conclude then, that the characteristic \(V\), of the operation of taking the right part (202, 204) of a quaternion, like the characteristic \(S\) of taking the scalar (196, 197), and the characteristic \(K\) of taking the conjugate (137, 195*), is a Distributive Symbol, or represents a distributive operation: whereas the characteristics, \(A\), \(\angle\), \(N\), \(U\), \(T\), of the operations of taking respectively the axis (128, 129), the angle (130), the norm (145, (11.)), the versor (156), and the tensor (187), are not thus distributive symbols (comp. 186, (10.), and 200, VII.); or do not operate upon a whole (or sum), by operating on its parts (or summands).

(1.) We may now recover the symbolical equation \(K^2 = 1\) (145), under the form (comp. 196, VI.; 202, IV.; and 204, IV., VIII., IX., XI.):
\[
\text{VIII.} \quad K^2 = (S - V)^2 = S^2 - SV - VS + V^2 = S + V = 1.
\]

* Indeed, it has only been proved as yet (comp. 195, (1.)), that \(K2q = 2Kq\), for the case of two summands; but this result will soon be extended [207].
(2.) In like manner we can recover each of the expressions for $S^2$, $V^2$ from the other, under the forms (comp. again 202, IV.):

VI. $S^2 = (1 - V)^2 = 1 - 2V + V^2 = 1 - V = S$, as in 196, VI.;

VII. $V^2 = (1 - S)^2 = 1 - 2S + S^2 = 1 - S = V$, as in 204, IX.;

or thus (comp. 196, II., and 204, XIV.), from the expressions for $S$ and $V$ in terms of $K$:

VIII. $S^2 = \frac{1}{2} (1 + K)^2 = \frac{1}{2} (1 + 2K + K^2) = \frac{1}{2} (1 + K) = S$;

IX. $V^2 = \frac{1}{2} (1 - K)^2 = \frac{1}{2} (1 - 2K + K^2) = \frac{1}{2} (1 - K) = V$.

(3.) Similarly,

X. $SV = \frac{1}{2} (1 + K)(1 - K) = \frac{1}{2} (1 - K^2) = 0$, as in 204, IV.;

and XI. $VS = \frac{1}{2} (1 - K)(1 + K) = \frac{1}{2} (1 - K^2) = 0$, as in 204, VIII.*

206. As regards the addition (or subtraction) of such right parts, $Vq$, $Vq'$, or generally of any two right quaternions (132), we may connect it with the addition (or subtraction) of their indices (133), as follows. Let $obdc$ be again any parallelogram (197, 205), but let $oa$ be now an unit-vector (129) perpendicular to its plane; so that

$$Ta = 1, \quad \angle (\beta : a) = \angle (\gamma : a) = \angle (\delta : a) = \frac{\pi}{2}; \quad \delta = \gamma + \beta.$$

Let $ob'd'c'$ be another parallelogram in the same plane, obtained by a positive rotation of the former, through a right angle, round $oa$ as an axis; so that

$$Ax. (\beta' : \beta) = Ax. (\gamma' : \gamma) = Ax. (\delta' : \delta) = a.$$

Then the three right quotients, $\beta : a$, $\gamma : a$, and $\delta : a$, may represent any two right quaternions, $q$, $q'$, and their sum, $q' + q$, which is always (by 197, (2.)) itself a right quaternion; and the indices of these three right quotients are (comp. 133, 193) the three lines $\beta'$, $\gamma'$, $\delta'$, so that we may write, under the foregoing conditions of construction,

$$\beta' = I (\beta : a), \quad \gamma' = I (\gamma : a), \quad \delta' = I (\delta : a).$$

* [It may be instructive to the student to form symbolical equations analogous to those in 161 (3.) from the six symbols $S$, $V$, $K$ and $T$, $U$, $R$. He may compare the equations obtained from the distributive symbols $S$, $V$ and $K$, with those obtained from $T$, $U$ and $R$, and may notice the pairs of symbols commutative in order of operation, &c. It is well to combine the symbols as in a multiplication table.]
But this third index is (by the second parallelogram) the sum of the two former indices, or in symbols, \( \delta' = \gamma' + \beta' \); we may therefore write,

\[
I \ldots I (q' + q) = Iq' + Iq, \quad \text{if} \quad \angle q = \angle q' = \frac{\pi}{2};
\]

or in words the Index of the Sum* of any two Right Quaternions is equal to the Sum of their Indices. Hence, generally, for any two quaternions, \( q \) and \( q' \), we have the formula,

\[
II \ldots IV (q' + q) = IVq' + IVq,
\]

because \( Vq, Vq' \) are always right quotients (202, 204), and \( V (q' + q) \) is always their sum (205, I.); so that the index of the right part of the sum of any two quaternions is the sum of the indices of the right parts. In like manner, there is no difficulty in proving that

\[
III \ldots I (q' - q) = Iq' - Iq, \quad \text{if} \quad \angle q' = \angle q = \frac{\pi}{2};
\]

and generally, that

\[
IV \ldots IV (q' - q) = IVq' - IVq;
\]

the Index of the Difference of any two right quotients, or of the right parts of any two quaternions, being thus equal to the Difference of the Indices.* We may then reduce the addition or subtraction of any two such quotients, or parts, to the addition or subtraction of their indices; a right quaternion being always (by 133) determined, when its index is given, or known.

207. We see, then, that as the Multiplication of any two Quaternions was (in 191) reduced to (1st) the arithmetical operation of multiplying their tensors, and (2nd) the geometrical operation of multiplying their versors, which latter was constructed by a certain composition of rotations, and was represented (in either of two distinct but connected ways, 167, 175) by sides or angles of a spherical triangle: so the Addition of any two Quaternions may be reduced (by 197, I., and 206, II.) to, 1st, the algebraical addition of their scalar parts, considered as two positive or negative numbers (16.); and, 2nd, the geometrical addition of the indices of their right parts, considered as certain vectors (1.): this latter Addition of Lines being performed according to the Rule of the Parallelogram (6.).† In like manner, as the general Division of Quaternions

* Compare the Note to page 175.
† It does not fall within the plan of these Notes to allude often to the history of the subject; but it ought to be distinctly stated that this celebrated Rule, for what may be called Geometrical Addition of right lines, considered as analogous to composition of motions (or of forces), had occurred to several writers before the invention of the quaternions: although the method adopted, in the present and in a former work, of deducing that rule, by algebraical analogies, from the symbol \( b - a \) (1.) for the line \( AB \), may possibly not have been anticipated. The reader may compare the Notes to the Preface to the author's Volume of Lectures on Quaternions (Dublin, 1863).
Aets. 206-208.]

GENERAL ADDITION OF QUATERNIONS.

was seen (in 191) to admit of being reduced to an arithmetical division of
tensors, and a geometrical division of versors, so we may now (by 197, III., and
206, IV.) reduce, generally, the Subtraction of Quaternions to (1st) an alge­
braical subtraction of scalars, and (2nd) a geometrical subtraction of vectors:
this last operation being again constructed by a parallelogram, or even by a
plane triangle (comp. Art. 4, and fig. 2). And because the sum of any given
set of vectors was early seen to have a value (9.), which is independent of their
order, and of the mode of grouping them, we may now infer that the Sum of
any number of given Quaternions has, in like manner, a Value (comp. 197, (I.)),
which is independent of the Order, and of the Grouping of the Summands: or in
other words, that the general Addition of Quaternions is a Commutative* and an
Associative Operation.

(1.) The formula, \( V\Sigma g = \Sigma Vg \), of 205, III.,
is now seen to hold good, for any number of quaternions, independently of the
arrangement of the terms in each of the two sums, and of the manner in
which they may be associated.

(2.) We can infer anew that
\[ K (q' + q) = Kq' + Kq, \]
as in 195, II.,
under the form of the equation or identity,
\[ S (q' + q) - V (q' + q) = (S q' - V q') + (S q - V q). \]

(3.) More generally, it may be proved, in the same way, that
\[ K \Sigma g = \Sigma K g, \]
or briefly, \( K \Sigma = \Sigma K \),
whatever the number of the summands may be.

208. As regards the quotient or product of the right parts, \( Vg \) and \( Vg' \), of
any two quaternions, let \( t \) and \( t' \) denote the tensors of those two parts, and
let \( x \) denote the angle of their indices, or of their axes, or the mutual inclination
of the axes, or of the planes, \( \dagger \) of the two quaternions \( g \) and \( g' \) themselves, so
that (by 204, XVIII.),
\[ t = TVg = Tg \cdot \sin \angle g, \quad t' = TVg' = Tg' \cdot \sin \angle g', \]
and
\[ w = \angle (TVg': TVg) = \angle (Ax.g': Ax.g). \]

* Compare the Note to page 176.

† Two planes, of course, make with each other, in general, two unequal and supplementary angles:
but we here suppose that these are mutually distinguished, by taking account of the aspect of each
plane, as distinguished from the opposite aspect: which is most easily done (111), by considering the
axes as above.
Then, by 193, 194, and by 204, XXXV., XXXV':,

\[I. \quad Vq' : Vq = IVq' : IVq = + (TVq' \cdot TVq). (Ax. q' : Ax. q); \]

\[II. \quad Vq' \cdot Vq = IVq' = I \frac{1}{Vq} = - (TVq' \cdot TVq). (Ax. q' : Ax. q); \]

and therefore (comp. 198), with the temporary abridgments proposed above,

\[III. \quad S (Vq' : Vq) = t't^{-1} \cos x; \quad IV. \quad SU (Vq' : Vq) = + \cos x; \]

\[V. \quad S (Vq' \cdot Vq) = - t't \cos x; \quad VI. \quad SU (Vq' \cdot Vq) = - \cos x; \]

\[VII. \quad \angle (Vq' : Vq) = x; \quad VIII. \quad \angle (Vq' \cdot Vq) = \pi - x. \]

We have also generally (comp. 204, XVIII., XIX.),

\[IX. \quad TV (Vq' : Vq) = t't^{-1} \sin x; \quad X. \quad TVU (Vq' : Vq) = \sin x; \]

\[XI. \quad TV (Vq' \cdot Vq) = t't \sin x; \quad XII. \quad TVU (Vq' \cdot Vq) = \sin x; \]

and in particular,

\[XIII. \quad V (Vq' : Vq) = 0, \quad \text{and} \quad XIV. \quad V (Vq' \cdot Vq) = 0, \]

if \( q' \parallel q \) (123);

because (comp. 191, (6.), and 204, VI.) the quotient or product of the right parts of two complanar quaternions (supposed here to be both non-scalar (108), so that \( t \) and \( t' \) are each > 0) degenerates (131) into a scalar, which may be thus expressed:

\[XV. \quad Vq' : Vq = + t't^{-1}, \quad \text{and} \quad XVI. \quad Vq' \cdot Vq = - t't, \quad \text{if} \quad x = 0; \]

but

\[XVII. \quad Vq' : Vq = - t't^{-1}, \quad \text{and} \quad XVIII. \quad Vq' \cdot Vq = + t't, \quad \text{if} \quad x = \pi; \]

the first case being that of coincident, and the second case that of opposite axes. \( \sqrt{\text{In the more general case of diplanarity (119), if we denote by} \delta \text{the unit-line which is perpendicular to both their axes, and therefore common to their two planes, or in which those planes intersect, and which is so directed that the rotation round it from Ax. q to Ax. q' is positive (comp. 127, 128), the recent formulae I., II., give easily,}} \]

\[XIX. \quad Ax. (Vq' : Vq) = + \delta; \quad XX. \quad Ax. (Vq' \cdot Vq) = - \delta; \]

and therefore (by IX., XI., and by 204, XXXV.), the indices of the right parts, of the quotient and product of the right parts of any two diplanar quaternions, may be expressed as follows:

\[XXI. \quad IV (Vq' : Vq) = + \delta \cdot t't^{-1} \sin x; \]

\[XXII. \quad IV (Vq' \cdot Vq) = - \delta \cdot t't \sin x. \]
(1.) Let \(\triangle ABC\) be any triangle upon the unit-sphere \((128)\), of which the spherical angles and the corners may be denoted by the same letters \(a, b, c\), while the sides shall as usual be denoted by \(a, b, c\); and let it be supposed that the rotation \((\text{comp. 177})\) round \(a\) from \(c\) to \(b\), and therefore that round \(b\) from \(a\) to \(c\), \(\&c\.), is positive, as in fig. 43 \([p. 144]\). Then writing, as we have often done,
\[
q = \alpha : \beta, \quad \text{and} \quad q' = \gamma : \beta, \quad \text{where} \quad a = oa, \&c.,
\]
we easily obtain the following expressions for the three scalars \(t, t', x\), and for the vector \(\delta\):
\[
t = \sin c; \quad t' = \sin a; \quad x = \pi - b; \quad \delta = - \beta.
\]

(2.) In fact we have here,
\[
Tq = Tq' = 1, \quad \angle q = c, \quad \angle q' = a;
\]
whence \(t\) and \(t'\) are as just stated. Also if \(A', B', C'\) be \((\text{as in 175})\) the positive poles of the three successive sides \(BC, CA, AB\), of the given triangle, and therefore the points \(A, B, C\) the negative poles \((\text{comp. 180, (2.)})\) of the new arcs \(B'C', C'A', A'B'\), then
\[
Ax.q = oc'; \quad Ax.q' = OA';
\]
but \(x\) and \(\delta\) are the angle and the axis of the quotient of these two axes, or of the quaternion which is represented \((162)\) by the arc \(C'A'\); therefore \(x\) is, as above stated, the supplement of the angle \(b\), and \(\delta\) is directed to the point upon the sphere, which is diametrically opposite to the point \(B\).

(3.) Hence, by III. V. VII. VIII. IX. XI., for any triangle \(\triangle ABC\) on the unit-sphere, with \(a = oa, \&c.,\) we have the formulæ:
\[
\begin{align*}
XXIII. \ldots S \left( V_{\beta}^{\gamma} : V_{a}^{\beta} \right) &= - \sin a \cosec c \cos b; \\
XXIV. \ldots S \left( V_{\beta}^{\gamma} : V_{a}^{\beta} \right) &= + \sin a \sin c \cos b; \\
XXV. \ldots \angle \left( V_{\beta}^{\gamma} : V_{a}^{\beta} \right) &= \pi - b; \quad XXVI. \ldots \angle \left( V_{\beta}^{\gamma} : V_{a}^{\beta} \right) = b; \\
XXVII. \ldots TV \left( V_{\beta}^{\gamma} : V_{a}^{\beta} \right) &= + \sin a \cosec c \sin b; \\
XXVIII. \ldots TV \left( V_{\beta}^{\gamma} : V_{a}^{\beta} \right) &= + \sin a \sin c \sin b.
\end{align*}
\]
(4.) Also, by XIX. XX. XXI. XXII., if the rotation round \( b \) from \( a \) to \( c \) be still positive,

\[
XXIX...\text{Ax. } \left( V \frac{\gamma}{\beta} : V \frac{\beta}{a} \right) = -\beta; \quad XXX...\text{Ax. } \left( V \frac{\gamma}{\beta} \cdot V \frac{\beta}{a} \right) = +\beta;
\]

\[
XXXI...\, IV \left( V \frac{\gamma}{\beta} : V \frac{\beta}{a} \right) = -\beta \sin a \cosec c \sin b;
\]

\[
XXXII...\, IV \left( V \frac{\gamma}{\beta} \cdot V \frac{\beta}{a} \right) = +\beta \sin a \sin c \sin b.
\]

(5.) If, on the other hand, the rotation round \( b \) from \( a \) to \( c \) were negative, then writing for a moment \( a_1 = -a, \beta_1 = -\beta, \gamma_1 = -\gamma \), we should have a new and opposite triangle, \( A_1B_1C_1 \), in which the rotation round \( B_1 \) from \( A_1 \) to \( C_1 \) would be positive, but the angle at \( B_1 \) equal in magnitude to that at \( B \); so that by treating (as usual) all the angles of a spherical triangle as positive, we should have \( B_1 = B \), as well as \( C_1 = C \), and \( A_1 = A \); and therefore, for example, by XXXXI,

\[
IV \left( V \frac{\gamma_1}{\beta_1} : V \frac{\beta_1}{a_1} \right) = -\beta_1 \sin a_1 \cosec c_1 \sin b_1;
\]

or \( IV \left( V \frac{\gamma}{\beta} : V \frac{\beta}{a} \right) = +\beta \sin a \cosec c \sin b; \)

the four formulae of (4.) would therefore still subsist, provided that, for this new direction of rotation in the given triangle, we were to change the sign of \( \beta \), in the second member of each.

(6.) Abridging, generally \( IV g : S g \) to \( (IV : S) g \), as \( TV g : S g \) was abridged, in 204, XXXIV., to \( (TV : S) g \), we have by (5.), and by XXIV., XXXII., this other general formula, for any three unit-vectors \( a, \beta, \gamma \), considered still as terminating at the corners of a spherical triangle \( ABC \):

\[
XXXIII...\, (IV : S) \left( V \frac{\gamma}{\beta} \cdot V \frac{\beta}{a} \right) = \pm\beta \tan b;
\]

the upper or the lower sign being taken, according as the rotation round \( B \) from \( A \) to \( C \), or that round \( \beta \) from \( a \) to \( \gamma \), which might perhaps be denoted by the symbol \( a\beta\gamma \), and which in quantity is equal to the spherical angle \( B \), is positive or negative.

209. When the planes of any three quaternions \( g, g', g'' \), considered as all passing through the origin \( o \) (119), contain any common line, those three may then be said to be Collinear* Quaternions; and because the axis of each is then

* Quaternions of which the planes are parallel to any common line may also be said to be collinear. Compare the first Note to page 116.
perpendicular to that line, it follows that the \textit{Axes of Collinear Quaternions are Complanar}: while conversely, the \textit{complanarity of the axes insures the collinearity of the quaternions}, because the \textit{perpendicular to the plane of the axes} is a line common to the planes of the quaternions.

(1.) Complanar quaternions are always collinear; but the converse proposition does not hold good, collinear quaternions being not necessarily complanar.

(2.) Collinear quaternions, considered as \textit{fractions} (101), can always be reduced to a \textit{common denominator} (120); and conversely, if three or more quaternions \textit{can} be so reduced, as to appear under the form of fractions with a common denominator \(\varepsilon\), those quaternions must be \textit{collinear}: because the line \(\varepsilon\) is then common to all their planes.

(3.) \textit{Any two quaternions are collinear with any scalar}; the plane of a scalar being indeterminate* (131).

(4.) Hence the \textit{scalar and right parts}, \(S\), \(S'\), \(V\), \(V'\), of any two quaternions, are always \textit{collinear} with each other.

(5.) The \textit{conjugates} of collinear quaternions are themselves collinear.

210. Let \(q\), \(q'\), \(q''\) be any three collinear quaternions; and let \(a\) denote a line common to their planes. Then we may determine (comp. 120) three other lines \(\beta\), \(\gamma\), \(\delta\), such that

\[
q = \frac{\beta}{a}, \quad q' = \frac{\gamma}{a}, \quad q'' = \frac{a}{\delta};
\]

and thus may conclude that (as in algebra),

\[
I. \ldots (q' + q) q'' = q'q'' + qq''
\]

because, by 106, 107,

\[
\left(\frac{\gamma + \beta}{a}\right) = \frac{\gamma + \beta}{a} \cdot \frac{a}{\delta} = \frac{\gamma + \beta}{\delta} = \frac{\gamma}{\delta} + \frac{\beta}{a} + \frac{\beta a}{\delta};
\]

In like manner, at least under the same condition of collinearity,† it may be proved that

\[
II. \ldots (q' - q) q'' = q'q'' - qq''
\]

Operating by the characteristic \(K\) upon these two equations, and attending to 192, II., and 195, II., we find that

\[
III. \ldots Kq''(Kq' + Kq) = Kq''Kq' + Kq'. Kq;
\]

\[
IV. \ldots Kq''(Kq' - Kq) = Kq''Kq' - Kq'. Kq;
\]

where (by 209, (5.)) the three \textit{conjugates} of arbitrary collinears, \(Kq, Kq', Kq''\),

* Compare the Note to page 117.
† It will soon be seen, however, that this condition is unnecessary.
may represent any three collinear quaternions. We have, therefore, with the same degree of generality as before,

\[ \mathbf{V}. \quad q''(q' + q) = q''q' + q''q ; \quad \mathbf{VI}. \quad q''(q' - q) = q''q' - q''q. \]

If, then, \( q, q', q'', q''' \) be any four collinear quaternions, we may establish the formula (again agreeing with algebra):

\[ \mathbf{VII}. \quad (q''' + q'')(q' + q) = q'''q' + q'''q + q''q + q''q ; \]

and similarly for any greater number, so that we may write briefly,

\[ \mathbf{VIII}. \quad \sum q' \cdot \sum q = \sum q'q, \]

where \( \sum q' = q_1 + q_2 + \ldots + q_m \), \( \sum q' = q'_1 + q'_2 + \ldots + q'_n \),

and

\[ \sum q'q = q'_1 q_1 + q'_2 q_2 + \ldots + q'_n q_m, \]

\( m \) and \( n \) being any positive whole numbers. In words (comp. 13.), the Multiplication of Collinear* Quaternions is a Doubly Distributive Operation.

(1.) Hence, by 209, (4.), and 202, III., we have this general transformation, for the product of any two quaternions:

\[ \mathbf{IX}. \quad q'q = \sum q' \cdot \sum q + q'q' + q''q' + q''q. \]

(2.) Hence also, for the square of any quaternion, we have the transformation† (comp. 126; 199, VII.; and 204, XXIII.):

\[ \mathbf{X}. \quad q^2 = \sum q^2 + 2\sum q'q. \]

(3.) Separating the scalar and right parts of this last expression, we find these other general formulae:

\[ \mathbf{XI}. \quad S.q^2 = S.q^2 + V.q^2; \quad \mathbf{XII}. \quad V.q^2 = 2S.q \cdot V.q; \]

whence also, dividing by \( T.q^2 \), we have

\[ \mathbf{XIII}. \quad SU(q^2) = (SUq)^2 + (VUq)^2; \quad \mathbf{XIV}. \quad VU(q^2) = 2SUq \cdot VUq. \]

(4.) By supposing \( q' = Kq \), in IX., and therefore \( S.q' = S.q, \ V.q' = -V.q \), and transposing the two conjugate and therefore complanar factors (comp. 191, (1.)), we obtain this general transformation for a norm, or for the square of a tensor (comp. 190, V.; 202, III.; and 204, XI.):

\[ \mathbf{XV}. \quad T.q^2 = N.q = qKq = (S.q + V.q)(S.q - V.q) = S.q^2 - V.q^2; \]

which had indeed presented itself before (in 204, XXII.), but is now obtained

---

* This distributive property of multiplication will soon be found (compare the last Note) to extend to the more general case, in which the quaternions are not collinear.

† [By means of the formula of 204 many different transformations involving \( K, S, V, \) and \( T \) may be effected on a square or product.]
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in a new way, and without any employment of sines, or cosines, or even of the well-known theorem respecting the square of the hypotenuse.

(5.) Eliminating $Vg^2$, by XV., from XI., and dividing by $Tg^2$, we find that

\[
\text{XVI. } S \cdot q^2 = 2Sg^2 - Tg^2; \quad \text{XVII. } SU(\cdot q^2) = 2(SUg)^2 - 1;
\]

agreeing with 199, VI. and IV., but obtained here without any use of the known formula for the cosine of the double of an angle.

(6.) Taking the scalar and right parts of the expression IX., we obtain these other general expressions:

\[
\begin{align*}
\text{XVIII. } Sg'q &= Sg' \cdot Sg + S(Vg' \cdot Vg); \\
\text{XIX. } Vg'q &= Vg' \cdot Sg + Vg \cdot Sg' + V(Vg' \cdot Vg);
\end{align*}
\]

in the latter of which we may (by 126) transpose the two factors $Vg'$, $Sg$, or $Vg$, $Sg'$. We may also (by 206, 207) write, instead of XIX., this other formula:

\[
\text{XIX'. } IVg'q = IVg' \cdot Sg + IVg \cdot Sg' + IV(Vg' \cdot Vg).
\]

(7.) If we suppose, in VII., that $g'' = Kg$, $g''' = Kg'$, and transpose (comp. (4.)) the two complanar (because conjugate) factors, $q' + q$ and $K(q' + q)$, we obtain the following general expression for the norm of a sum:

\[
(q' + q)K(q' + q) = q'Kq' + qKq' + q'Kq + qKq;
\]

or briefly,

\[
\text{XX. } N(q' + q) = Nq' + 2S \cdot qKq' + Nq,
\]

because

\[
q'Kq = K \cdot qKq', \text{ by 192, II.}, \text{ and } (1 + K) \cdot qKq' = 2S \cdot qKq', \text{ by 196, II}'.
\]

(8.) By changing $q'$ to $x$ in XX., or by forming the product of $q + x$ and $Kq + x$, where $x$ is any scalar, we find that

\[
\text{XXI. } N(q + x) = Nq + 2xSg + x^2,
\]

whence, in particular,

\[
\text{XXI'. } N(q - 1) = Nq - 2Sg + 1,
\]

(9.) Changing $q$ to $\beta : a$, and multiplying by the square of $Ta$, we get, for any two vectors, $a$ and $\beta$, the formula,

\[
\text{XXII. } T(\beta - a)^2 = T\beta^2 - 2T\beta \cdot Ta \cdot SU_{a}^{\beta} + Ta^2,
\]
in which \( Ta^2 \) denotes* \((Ta)^2\); because (by 190, and by 196, IX.),

\[
N\left( \frac{\beta}{a} - 1 \right) = N\left( \frac{\beta - a}{a} \right) = N\left( \frac{T(\beta - a)}{Ta} \right), \quad \text{and} \quad S\left( \frac{\beta}{a} \right) = \frac{T\beta}{Ta} S\frac{\beta}{a}.
\]

(10.) In any plane triangle, \( \triangle abc \), with sides of which the lengths are as usual denoted by \( a, b, c \), let the vertex \( c \) be taken as the origin \( o \) of vectors; then

\[
a = ca, \quad \beta = cb, \quad \beta - a = ab, \quad Ta = b, \quad T\beta = a, \quad T(\beta - a) = c, \quad SU\frac{\beta}{a} = \cos c;
\]

we recover therefore, from XXII., the fundamental formula of plane trigonometry, under the form

\[
XXIII. \ldots c^2 = a^2 - 2ab \cos c + b^2.
\]

(11.) It is important to observe that we have not here been arguing in a circle; because although, in Art. 200, we assumed, for the convenience of the student, a previous knowledge of the last written formula, in order to arrive more rapidly at certain applications, yet in these recent deductions from the distributive property VIII. of multiplication of (at least) collinear quaternions, we have founded nothing on the results of that former Article; and have made no use of any properties of \textit{oblique-angled triangles}, or even of \textit{right-angled} ones, since the theorem of the square of the hypotenuse has been virtually proved anew in (4.): nor is it necessary to the argument, that any properties of trigonometric functions should be known, beyond the mere definition of a cosine, as a certain projecting factor, from which the formula 196, XVI. was derived, and which justifies us in writing \( \cos c \) in the last equation (10.). The geometrical Examples, in the sub-articles to 200, may therefore be read again, and their validity be seen anew, without any appeal to even plane trigonometry being now supposed.

(12.) The formula XV. gives \( S\phi^2 = T\phi^2 + V\phi^2 \), as in 204, XXXI.; and we know that \( V\phi^2 \), as being generally the \textit{square} of a \textit{right quaternion}, is equal to a \textit{negative scalar} (comp. 204, VI.), so that

\[
XXIV. \ldots V\phi^2 < 0, \quad \text{unless} \quad \angle\phi = 0, \quad \text{or} \quad = \pi,
\]

in each of which two cases \( V\phi = 0 \), by 202, (6.), and therefore its square vanishes; hence,

\[
XXV. \ldots S\phi^2 < T\phi^2, \quad (SU\phi)^2 < 1,
\]

in every other case.

* * * We are not yet at liberty to interpret the symbol \( Ta^2 \) as denoting also \( T(a^2) \); because we have not yet assigned any meaning to the \textit{square} of a \textit{vector}, or generally to the \textit{product} of two \textit{vectors}. In the Third Book of these Elements [282 (3.)] it will be shown, that such a square or product can be interpreted as being a \textit{quaternion} and then it will be found (comp. 190), that

\[
T(a^2) = (Ta)^2 = Ta^2,
\]

whatever vector \( a \) may be.
(13.) It might therefore have been thus proved, without any use of the transformation $SUq = \cos \angle q$ (196, XVI.), that (for any real quaternion $q$) we may have the inequalities,

$$\text{XXVI. } SUq < +1, \quad SUq > -1, \quad \text{and } Sg < +Tq, \quad Sg > -Tq,$$

unless it happen that $\angle q = 0$, or $\pi$; $SUq$ being $= + 1$, and $Sg = + Tq$, in the first case; whereas $SUq = -1$, and $Sg = - Tq$, in the second case.

(14.) Since $Tq^2 = Nq$, and $Tq . Tq' = T . qKq' = T . qKq = Nq . T (q' : q)$, while $S . qKq' = S . qKq = Nq . S (q' : q)$, the formula XX. gives, by XXVI.,

$$\text{XXVII. } (Tq' + Tq)^2 - T(q' + q)^2 = 2 (T - S) qKq' = 2Nq . (T - S)(q' : q) > 0,$$

if we adopt the abridged notation,

$$\text{XXVIII. } Tg - Sg = (T - S) g,$$

and suppose that the quotient $q' : q$ is not a positive scalar; hence,

$$\text{XXIX. } Tg + Tg > T(q' + q), \quad \text{unless } q' = xq, \quad \text{and } x > 0;$$

in which excepted case, each member of this last inequality becomes $= (1 + x) Tq$.

(15.) Writing $q = \beta : a$, $q' = \gamma : a$, and multiplying by $Ta$, the formula XXIX. becomes,

$$\text{XXX. } T\gamma + T\beta > T(\gamma + \beta), \quad \text{unless } \gamma = x\beta, \quad x > 0;$$

in which latter case, but not in any other, we have $U\gamma = U\beta$ (155). We therefore arrive anew at the results of 186, (9.), (10.), but without its having been necessary to consider any triangle, as was done in those former subarticles.

(16.) On the other hand, with a corresponding abridgment of notation, we have, by XXVI.,

$$\text{XXXI. } Tq + Sg = (T + S) g > 0, \quad \text{unless } \angle q = \pi;$$

also, by XX., &c,

$$\text{XXXII. } T(q' + q)^2 - (Tq' - Tq)^2 = 2 (T + S) qKq' = 2Nq . (T + S)(q' : q);$$

hence,

$$\text{XXXIII. } T(q' + q) > \pm (Tq' - Tq), \quad \text{unless } q' = -xq, \quad x > 0;$$

where either sign may be taken.

(17.) And hence, on the plan of (15.), for any two vectors $\beta, \gamma$,

$$\text{XXXIV. } T(\gamma + \beta) > \pm (T\gamma - T\beta), \quad \text{unless } U\gamma = -U\beta,$$
whichever sign he adopted; but, on the contrary,

\[ T(\gamma + \beta) = \pm (T\gamma - T\beta), \quad \text{if} \quad U\gamma = - U\beta, \]

the upper or the lower sign being taken, according as \( T\gamma > \) or \( < T\beta \): all which agrees with what was inferred, in 186, (11.), from geometrical considerations alone, combined with the definition of \( T\alpha \). In fact, if we make \( \beta = \alpha b, \gamma = \alpha c, \) and \( - \gamma = \alpha c' \), then \( \alpha bc' \) will be in general a plane triangle, in which the length of the side \( bc' \) exceeds the difference of the lengths of the two other sides; but if it happen that the directions of the two lines \( \alpha b, \alpha c' \) coincide, or in other words that the lines \( \alpha b, \alpha c \) have opposite directions, then the difference of lengths of these two lines becomes equal to the length of the line \( bc' \).

(18.) With the representations of \( \alpha \) and \( \alpha' \), assigned in 208, (1.), by two sides of a spherical triangle \( \triangle abc \), we have the values,

\[ S\alpha = \cos c, \quad S\alpha' = \cos a, \quad S\alpha'\alpha = S(\gamma : a) = \cos b; \]

the equation XVIII. gives therefore, by 208, XXIV., the fundamental formula of spherical trigonometry (comp. (10.)), as follows:

\[ \cos b = \cos a \cos c + \sin a \sin c \cos b. \]

(19.) To interpret, with reference to the same spherical triangle, the connected equation XIX., or XIX', let it be now supposed, as in 208, (5.), that the rotation round \( \beta \) from \( c \) to \( a \) is positive, so that \( \beta \) and \( \beta' \) are situated at the same side of the arc \( ca \), if \( \beta' \) be still, as in 208, (2.), the positive pole of that arc. Then writing \( \alpha' = \alpha a', \&c., \) we have

\[ IV\alpha = \gamma' \sin c; \quad IV\alpha' = \alpha' \sin a; \quad IV\alpha'\alpha = - \beta' \sin b; \]

and \( IV (V\alpha', V\alpha) = - \beta \sin a \sin c \sin b \) (comp. 208, (5.)),

with the recent values (18), for \( S\alpha \) and \( S\alpha' \); thus the formula XIX'. becomes, by transposition of the two terms last written:

\[ \beta \sin a \sin c \sin b = \alpha' \sin a \cos c + \beta' \sin b + \gamma' \sin c \cos a. \]

(20.) Let \( \rho = op \) be any unit-vector; then, dividing each term of the last equation by \( \rho \), and taking the scalar of each of the four quotients, we have, by 196, XVI., this new equation:

\[ \sin a \sin c \sin b \cos \rho \beta = \sin a \cos c \cos \rho a' + \sin b \cos \rho b' \]

\[ + \sin c \cos a \cos \rho c'; \]
where \( a, b, c \) are as usual the sides of the spherical triangle \( \triangle abc \), and \( a', b', c' \) are still, as in 208, (2.), the positive poles of those sides; but \( p \) is an arbitrary point, upon the surface of the sphere. Also \( \cos pa', \cos pb', \cos pc' \) are evidently the sines of the arcual perpendiculars let fall from that point upon those sides; being positive when \( p \) is, relatively to them, in the same hemispheres as the opposite corners of the triangle, but negative in the contrary case; so that \( \cos aa', \&c., \) are positive, and are the sines of the three altitudes of the triangle.

(21.) If we place \( p \) at \( b \), two of these perpendiculars vanish, and the last formula becomes, by 208, XXVIII.,

\[ \text{XXXIX. } \sin b \cos bb' = \sin a \sin c \sin b = TV \left( V_{\beta} V_{\alpha} \right); \]

such then is the quaternion expression for the product of the sine of the side \( ca \), multiplied by the sine of the perpendicular let fall upon that side, from the opposite vertex \( b \).

(22.) Placing \( p \) at \( a \), dividing by \( \sin a \cos c \), and then interchanging \( b \) and \( c \), we get this other fundamental formula of spherical trigonometry,

\[ \text{XLI. } \cos aa' = \sin c \sin b = \sin b \sin c; \]

and we see that this is included in the interpretation of the quaternion equation XIX., or XIX', as the formula XXXVI. was seen in (18.) to be the interpretation of the connected equation XVIII.

(23.) By assigning other positions to \( p \), other formulæ of spherical trigonometry may be deduced, from the recent equation XXXVIII. Thus if we suppose \( p \) to coincide with \( b' \), and observe that (by the supplementary* triangle),

\[ b'c' = \pi - a, \quad c'a' = \pi - b, \quad a'b' = \pi - c, \]

while

\[ \cos bb' = \sin a \sin c = \sin c \sin a, \text{ by XLI.}, \]

we easily deduce the formula,

\[ \text{XLI. } \sin a \sin c \sin a \sin b \sin c = \sin b = \cos c \cos c \sin a - \cos a \cos a \sin c; \]

which obviously agrees, at the plane limit, with the elementary relation,

\[ a + b + c = \pi. \]

* No previous knowledge of Spherical Trigonometry, properly so called, is here supposed; the supplementary relations of two polar triangles to each other forming rather a part, and a very elementary one, of spherical geometry.
(24.) Again, by placing \( p \) at \( \lambda' \), the general equation becomes,

\[
\sin a \cos c = \sin b \cos c + \sin c \cos a \cos b;
\]

with the verification that, at the plane limit,

\[ a = b \cos c + c \cos b. \]

But we cannot here delay on such deductions, or verifications: although it appeared to be worth while to point out, that the whole of spherical trigonometry may thus be developed, from the fundamental equation of multiplication of quaternions (107), when that equation is operated on by the two characteristics \( S \) and \( V \), and the results interpreted as above.

211. It may next be proved, as follows, that the distributive formula I. of the last Article holds good, when the three quaternions, \( q, q', q'' \), which enter into it, without being now necessarily \textit{collinear}, are right; in which case their \textit{reciprocals} (135), and their \textit{sums} (197, (2.)), will be right also. Let then

\[
\angle q = \angle q' = \angle q'' = \frac{\pi}{2}, \quad qq_1 = 1;
\]

and therefore,

\[
\angle q_1 = \angle (q'' + q') = \frac{\pi}{2}.
\]

We shall then have, by 106, 194, 206,

\[
(q'' + q')q = I (q'' + q') : Iq,
\]

\[
= (Iq'' : Iq_1) + (Iq' : Iq_1) = q''q + q'q;
\]

and the distributive property in question is proved.

(1.) By taking conjugates, as in 210, it is easy hence to infer, that the \textit{other} distributive formula, 210, \textit{V.}, holds good for any three right quaternions; or that

\[
q (q'' + q') = qq'' + qq', \quad \text{if} \quad \angle q = \angle q' = \angle q'' = \frac{\pi}{2}.
\]

(2.) For \textit{any} three quaternions, we have therefore the two equations:

\[
(Vq'' + Vq') . Vq = Vq'' . Vq + Vq' . Vq;
\]

\[
Vq . (Vq'' + Vq') = Vq . Vq'' + Vq . Vq'.
\]

(3.) The quaternions \( q, q', q'' \) being still arbitrary, we have thus, by 210, \textit{IX.},

\[
(q'' + q')q = (Sq'' + Sq') . Sq + (Vq'' + Vq') . S_q + Vq . (Sq'' + Sq') + (Vq'' + Vq') . Vq
\]

\[
= (Sq'' . Sq + Vq'' . Sq + Vq . Sq' + Vq'' . Vq) + (Sq' . Sq + Vq' . Sq + Vq . Sq' + Vq' . Vq)
\]

\[
= q''q + q'q;
\]
so that the formula 210, I., and therefore also (by conjugates) the formula 210, V., is valid generally.

212. The General* Multiplication of Quaternions is therefore (comp. 13, 210) a Doubly Distributive Operation; so that we may extend, to quaternions generally, the formula (comp. 210, VIII.),

\[ \sum q' \cdot \sum q = \sum q'q ; \]

however many the summands of each set may be, and whether they be, or be not, collinear (209), or right (211).

(1.) Hence, as an extension of 210, XX., we have now,

\[ \sum q = \sum \sum q + 2 \sum \sum q \cdot q' ; \]

where the second sign of summation refers to all possible binary combinations of the quaternions \( q, q' \ldots \)

(2.) And, as an extension of 210, XXIX., we have the inequality,

\[ \sum T q > T \sum q, \]

unless all the quaternions \( q, q' \ldots \) bear scalar and positive ratios to each other, in which case the two members of this inequality become equal: so that the sum of the tensors, of any set of quaternions, is greater than the tensor of the sum, in every other case.

(3.) In general, as an extension of 210, XXVII.,

\[ \left( \sum T q \right)^2 - \left( T \sum q \right)^2 = 2 \sum (T - S) q \cdot q' ; \]

(4.) The formulae, 210, XVIII., XIX., admit easily of analogous extensions.

(5.) We have also (comp. 168) the general equation,

\[ \sum (q \cdot q) - \sum (q^2) = \sum (qq' + q'q) ; \]

in which, by 210, IX.,

\[ q q' + q'q = 2 \sum (q \cdot q' + V \cdot V q) \cdot \sum (q \cdot q' + V \cdot V q) ; \]

because, by 208, we have generally

\[ V (V q' \cdot V q) = - V (V q \cdot V q) ; \]

or

\[ V q'q = - V q q', \text{ if } \angle q = \angle q' = \frac{\pi}{2} . \]

(Comp. 191, (2.), and 204, X.)

* Compare the Notes to pages 211 and 212. [On page (35) of the Preface to the "Lectures on Quaternions," Hamilton refers to an early speculation of his (1831) on the multiplication of lines for which the product of sums was not equal to the sum of products. When addition is not commutative, multiplication even by a scalar is not distributive. See 180 (3.)]
213. Besides the advantage which the Calculus of Quaternions gains, from the general establishment (212) of the Distributive Principle, or Distributive Property of Multiplication, by being, so far, assimilated to Algebra, in processes which are of continual occurrence, this principle or property will be found to be of great importance, in applications of that calculus to Geometry; and especially in questions respecting the (real or ideal*) intersections of right lines with spheres, or other surfaces of the second order, including contacts (real or ideal), as limits of such intersections. The following examples may serve to give some notion, how the general distributive principle admits of being applied to such questions: in some of which however the less general principle (210), respecting the multiplication of collinear quaternions (209), would be sufficient. And first we shall take the case of chords of a sphere, drawn from a given point upon its surface.

(1.) From a point \( A \), of a sphere with \( O \) for centre, let it be required to draw a chord \( AP \), which shall be parallel to a given line \( OB \); or more fully, to assign the vector, \( P = OP \), of the extremity of the chord so drawn, as a function of the two given vectors, \( A = OA \), and \( B = OB \); or rather of \( A \) and \( U B \), since it is evident that the length of the line \( B \) cannot affect the result of the construction, which fig. 51 may serve to illustrate.

(2.) Since \( AP \parallel OB \), or \( P - A \parallel B \), we may begin by writing the expression,

\[
\rho = a + xB \quad (15),
\]

which may be considered (comp. 23, 99) as a form of the equation of the right line \( AP \); and in which it remains to determine the scalar coefficient \( x \), so as to satisfy the equation of the sphere,

\[
T \rho = Ta \quad (186, (2.)).
\]

In short, we are to seek to satisfy the equation,

\[
T (a + xB) = Ta,
\]

by some scalar \( x \) which shall be (in general) different from zero; and then to substitute this scalar in the expression \( \rho = a + xB \), in order to determine the required vector \( \rho \).

* Compare the Notes to pages 87, 88, &c.
Art. 213.] INTERSECTIONS OF RIGHT LINES AND SPHERES. 221

(3.) For this purpose, an obvious process is, after dividing both sides by
$T\beta$, to square, and to employ the formula 210, XXI., which had indeed
occurred before, as 200, VIII., but not then as a consequence of the distribu­
tive property of multiplication. In this manner we are conducted to a
quadratic equation, which admits of division by $x$, and gives then,

$$x = -2S\frac{a}{\beta}; \quad \rho = a - 2\beta S\frac{a}{\beta};$$

the problem (1.) being thus resolved, with the verification that $\beta$ may be
replaced by $U\beta$, in the resulting expression for $\rho$.

(4.) As a mere exercise of calculation, we may vary the last process (3.),
by dividing the last equation (2.) by $Ta$, instead of $T\beta$, and then going on
as before. This last procedure gives

$$1 = N\left(1 + x\frac{\beta}{a}\right) = 1 + 2xS\frac{\beta}{a} + x^2N\frac{\beta}{a};$$

and therefore

$$x = -2S\frac{\beta}{a}: N\frac{\beta}{a} = -2S\frac{a}{\beta} \quad \text{(by 196, XII.)},$$
as before.

(5.) In general, by 196, II',

$$1 - 2S = -K;$$
hence, by (3.),

$$\frac{\rho}{\beta} = -K\frac{a}{\beta};$$
and finally,

$$\rho = -K\frac{a}{\beta} \cdot \beta;$$
a new expression for $\rho$, in which it is not permitted generally, as it was in
(3.), to treat the vector $\beta$ as the multiplier,* instead of the multiplicant.

(6.) It is now easy to see that the second equation of (2.) is satisfied; for
the expression (5.) for $\rho$ gives (by 186, 187, &c),

$$T\rho = T\frac{a}{\beta}, T\beta = Ta,$$
as was required.

(7.) To interpret the solution (3.), let $c$ in fig. 51 be the middle point of
the chord $ap$, and let $d$ be the foot of the perpendicular let fall from $A$ on $ob$;
then the expression (3.) for $\rho$ gives, by 196, XIX.,

$$ca = \frac{1}{2}(a - \rho) = \beta S\frac{a}{\beta} = od;$$
and accordingly, $oCAD$ is a parallelogram.

* Compare the Note to page 159.
(8.) To interpret the expression (5.), which gives
\[ \frac{-\beta}{\alpha} = K \frac{\alpha'}{\beta'} \quad \text{or} \quad \frac{\mathbf{OP'} \cdot \mathbf{OA}}{\mathbf{OB}} = K \frac{\mathbf{OA}}{\mathbf{OB}}, \quad \text{if} \quad \mathbf{OP'} = \mathbf{PO}, \]
we have only to observe (comp. 138) that the angle \( \angle aop' \) is bisected internally, or the supplementary angle \( \angle aop \) externally, by the indefinite right line \( \mathbf{ob} \) (see again, fig. 51).

(9.) Conversely, the geometrical considerations which have thus served in (7.) and (8.) to interpret or to verify the two forms of solution (3.), (5.), might have been employed to deduce those two forms, if we had not seen how to obtain them, by rules of calculation, from the proposed conditions of the question. (Comp. 145, (10.), &c.)

(10.) It is evident, from the nature of that question, that \( \alpha \) ought to be deducible from \( \beta \) and \( \rho \), by exactly the same processes as those which have served us to deduce \( \rho \) from \( \beta \) and \( \alpha \). Accordingly, the form (3.) of \( \rho \) gives
\[ \frac{\rho}{\beta} = -\frac{\alpha}{\beta}, \quad \alpha = \rho + 2\beta \frac{\alpha}{\beta} = \rho - 2\beta \frac{\rho}{\beta}; \]
and the form (5.) gives
\[ K \frac{\rho}{\beta} = -\frac{\alpha}{\beta'}, \quad \alpha = -K \frac{\rho}{\beta} \cdot \beta. \]
And since the first form can be recovered from the second, we see that each leads us back to the parallelism, \( \rho - \alpha \parallel \beta \) (2.).

(11.) The solution (3.) for \( x \) shows that
\[ x = 0, \quad \rho = a, \quad \mathbf{p} = \mathbf{A}, \quad \text{if} \quad S (\mathbf{a}: \beta) = 0, \quad \text{or if} \quad \beta \perp a. \]
And the geometrical meaning of this result is obvious; namely, that a right line drawn at the extremity of a radius \( \mathbf{OA} \) of a sphere, so as to be perpendicular to that radius, does not (in strictness) intersect the sphere, but touches it: its second point of meeting the surface coinciding, in this case, as a limit, with the first.

(12.) Hence we may infer that the plane represented by the equation,
\[ S \frac{\rho - a}{a} = 0, \quad \text{or} \quad S \frac{\rho}{a} = 1, \]
is the tangent plane (comp. 196, (5.)) to the sphere here considered, at the point \( \mathbf{A} \).

(13.) Since \( \beta \) may be replaced by any vector parallel thereto, we may substitute for it \( \gamma - a \), if \( \gamma = \infty \) be the vector of any given point \( \mathbf{C} \) upon the chord
Arts. 213, 214. IMAGINARY INTERSECTIONS.

213. Whether (as in fig. 51) the middle point, or not; we may therefore write, by (3.) and (5.),

\[ \rho = a - 2 (\gamma - a) S \frac{a}{\gamma - a} = - K \frac{a}{\gamma - a} \cdot (\gamma - a). \]

214. In the examples of the foregoing Article, there was no room for the occurrence of imaginary roots of an equation, or for ideal intersections of line and surface. To give now a case in which such imaginary intersections may occur, we shall proceed to consider the question of drawing a secant to a sphere, in a given direction, from a given external point; the recent figure 51 still serving us for illustration.

(1.) Suppose then that \( e \) is the vector of any given point \( e \), through which it is required to draw a chord or secant \( e f_0 \), parallel to the same given line \( \beta \) as before. We have now, if \( p_0 = o p_0 \),

\[ p_0 = e + x_0 \beta, \quad \beta a = T p_0 = T (e + x_0 \beta), \]

\[ x_0^2 + 2x_0 S \frac{e}{\beta} + N \frac{e}{\beta} = 0, \]

\[ x_0 = - S \frac{e}{\beta} \pm \sqrt{\left(\left(\frac{T a}{\beta}\right)^2 + \left(\frac{V e}{\beta}\right)^2\right)}, \]

\( x_0 \) being a new scalar; and similarly, if \( p_1 = o p_1 \),

\[ p_1 = e + x_1 \beta, \quad x_1 = - S \frac{e}{\beta} \pm \sqrt{\left(\left(\frac{T a}{\beta}\right)^2 + \left(\frac{V e}{\beta}\right)^2\right)}, \]

by transformations* which will easily occur to any one who has read recent articles with attention. And the points \( p_0, p_1 \) will be together real, or together imaginary, according as the quantity under the radical sign is positive or negative; that is, according as we have one or other of the two following inequalities,

\[ T \frac{a}{\beta} > \text{or}< TV \frac{e}{\beta}. \]

(2.) The equation (comp. 203, (5.)),

\[ TV \frac{p}{\beta} = T \frac{a}{\beta} \text{or} \left(\frac{T a}{\beta}\right)^2 + \left(\frac{V p}{\beta}\right)^2 = 0, \]

represents a cylinder of revolution, with \( o \) for its axis, and with \( T a \) for the radius of its base. If \( e \) be a point of this cylindric surface, the quantity

* It does not seem to be necessary, at the present stage, to supply so many references to former Articles, or sub-articles, as it has hitherto been thought useful to give; but such may still, from time to time, be given.
under the radical sign (1.) vanishes; and the two roots \(x_0, x_1\) of the quadratic become equal. In this case, then, the line through \(x\), which is parallel to \(ob\), touches the given sphere; as is otherwise evident geometrically, since the cylinder envelopes the sphere (comp. 204, (12.)), and the line is one of its generatrices. If \(x\) be internal to the cylinder, the intersections \(p_0, p_1\) are real; but if \(x\) be external to the same surface, those intersections are ideal, or imaginary.

(3.) In this last case, if we make, for abridgment,

\[
S = \frac{e}{\beta}, \quad \text{and} \quad t = \sqrt{\left( \frac{TV}{\beta} \right)^2 - \left( \frac{T\alpha}{\beta} \right)^2},
\]

\(s\) and \(t\) being thus two given and real scalars, we may write,

\[
\begin{align*}
x_0 &= s - t\sqrt{-1} - 1; \\
x_1 &= s + t\sqrt{-1} - 1;
\end{align*}
\]

where \(\sqrt{-1}\) is the old and ordinary imaginary symbol of Algebra, and is not invested here with any sort of Geometrical Interpretation. We merely express thus the fact of calculation, that (with these meanings of the symbols \(\alpha, \beta, \epsilon, s\) and \(t\)) the formula \(T\alpha = T(\epsilon + x\beta)\), (1.), when treated by the rules of quaternions, conducts to the quadratic equation,

\[
(x - s)^2 + t^2 = 0,
\]

which has no real root; the reason being that the right line through \(x\) is, in the present case, wholly external to the sphere, and therefore does not really intersect it at all; although, for the sake of generalization of language, we may agree to say, as usual, that the line intersects the sphere in two imaginary points.

(4.) We must however agree, then, for consistency of symbolical expression, to consider these two ideal points as having determinate but imaginary vectors, namely, the two following:

\[
\begin{align*}
\rho_0 &= \epsilon + s\beta - t\beta\sqrt{-1}; \\
\rho_1 &= \epsilon + s\beta + t\beta\sqrt{-1}.
\end{align*}
\]

in which it is easy to prove, 1st, that the real part \(\epsilon + s\beta\) is the vector \(\epsilon'\) of the foot \(x'\) of the perpendicular let fall from the centre \(o\) on the line through \(x\) which is drawn (as above) parallel to \(ob\); and 2nd, that the real tensor \(t\beta\) of the coefficient of \(\sqrt{-1}\) in the imaginary part of each expression, represents the length of a tangent \(x'x''\) to the sphere, drawn from that external point, or foot, \(x'\).

* Compare again the Note to page 87, and Art. 149.
(5.) In fact, if we write \( \mathbf{oe}' = \varepsilon' = \varepsilon + s\beta' \), we shall have
\[
\mathbf{e}'\mathbf{e} = \varepsilon - \varepsilon' = -s\beta = \beta\varepsilon + s\beta = \text{projection of \( \mathbf{oe} \) on \( \mathbf{ob} \)};
\]
which proves the 1st assertion (4.), whether the points \( \mathbf{p}_0, \mathbf{p}_1 \) be real or imaginary. And because
\[
\left( T \varepsilon \right)^2 = N \left( \varepsilon \right) = N \left( \varepsilon + s \right) = N \frac{\varepsilon}{\beta} + 2s\frac{\varepsilon}{\beta} + s^2
\]
\[
= \left( T \frac{\varepsilon}{\beta} \right)^2 - \left( S \frac{\varepsilon}{\beta} \right)^2 = \left( TV \frac{\varepsilon}{\beta} \right)^2 = \varepsilon^2 + \left( T \frac{\alpha}{\beta} \right)^2,
\]
we have, for the case of imaginary intersections,
\[
\mathbf{e}'\mathbf{e} = \sqrt{\left( T\varepsilon^2 - T\alpha^2 \right)} = T \cdot \mathbf{e}'\mathbf{e}'',
\]
and the 2nd assertion (4.) is justified.

(6.) An expression of the form (4.), or of the following,
\[
\rho' = \beta + \gamma \sqrt{-1},
\]
in which \( \beta \) and \( \gamma \) are two real vectors, while \( \sqrt{-1} \) is the (scalar) imaginary of algebra, and not a symbol for a geometrically real right versor (149, 153), may be said to be a Bivector.

(7.) In like manner, an expression of the form (3.), or \( \varepsilon' = s + t\sqrt{-1} \), where \( s \) and \( t \) are two real scalars, but \( \sqrt{-1} \) is still the ordinary imaginary of algebra, may be said by analogy to be a Biscalar. Imaginary roots of algebraic equations are thus, in general, biscalars.

(8.) And if a bivector (6.) be divided by a (real) vector, the quotient,
\[
\boldsymbol{g'} = \frac{\rho'}{\alpha} = \frac{\beta}{\alpha} + \frac{\gamma}{\alpha} \sqrt{-1} = g_0 + g_1 \sqrt{-1},
\]
in which \( g_0 \) and \( g_1 \), are two real quaternions, but \( \sqrt{-1} \) is, as before, imaginary, may be said to be a Biquaternion.*

215. The same distributive principle (212) may be employed in investigations respecting circumscribed cones, and the tangents (real or ideal), which can be drawn to a given sphere from a given point.

(1.) Instead of conceiving that \( \mathbf{o}, \mathbf{a}, \mathbf{b} \) are three given points, and that limits of position of the point \( \mathbf{e} \) are sought, as in 214, (2.), which shall allow the points of intersection \( \mathbf{p}_0, \mathbf{p}_1 \) to be real, we may suppose that \( \mathbf{o}, \mathbf{a}, \mathbf{b} \) (which may be assumed to be collinear, without loss of generality, since \( \mathbf{a} \) enters only by its tensor) are now the data of the question; and that limits of direction of

* Compare the second Note to page 133. [This word is used in a different sense by W. K. Clifford.]

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the line \( ob \) are to be assigned, which shall permit the same reality: \( \varepsilon \), \( \theta \), being still drawn parallel to \( ob \), as in 214, (1.).

(2.) Dividing the equation \( Ta = T(\varepsilon + x\beta) \) by \( T_\varepsilon \), and squaring, we have

\[
N\frac{a}{\varepsilon} = \left(N\left(1 + x\frac{\beta}{\varepsilon}\right)\right) + 2xS\frac{\beta}{\varepsilon} + x^2N\frac{\beta}{\varepsilon};
\]

the quadratic in \( x \) may therefore be thus written,

\[
\left(xT\frac{\beta}{\varepsilon} + SU\frac{\beta}{\varepsilon}\right)^2 = \left(T\frac{a}{\varepsilon}\right)^2 + \left(VU\frac{\beta}{\varepsilon}\right)^2;
\]

and its roots are real and unequal, or real and equal, or imaginary, according as

\[
TVU\frac{\beta}{\varepsilon} < \text{or} = \text{or} > T\frac{a}{\varepsilon};
\]

that is, according as

\[
\sin e_\varepsilon o_\beta < \text{or} = \text{or} > T \cdot o_\alpha : T \cdot o_\beta.
\]

(3.) If \( \varepsilon \) be interior to the sphere, then \( T_\varepsilon < T_\alpha, \ T(\alpha : \varepsilon) > 1 \); but \( TVU_\varepsilon \) can never exceed unity (by 204, XIX., or by 210, XV., &c.); we have, therefore, in this case, the first of the three recent alternatives, and the two roots of the quadratic are necessarily real and unequal, whatever the direction of \( \beta \) may be. Accordingly it is evident, geometrically, that every indefinite right line, drawn through an internal point, must cut the spheric surface in two distinct and real points.

(4.) If the point \( \varepsilon \) be superficial, so that \( T_\varepsilon = T_\alpha, \ T(\alpha : \varepsilon) = 1 \), then the first alternative (2.) still exists, except at the limit for which \( \beta \perp \varepsilon \), and therefore \( TVU(\beta : \varepsilon) = 1 \), in which case we have the second alternative. One root of the quadratic in \( x \) is now \( = 0 \), for every direction of \( \beta \); and the other root, namely \( x = -2S(\varepsilon : \beta) \), is likewise always real, but vanishes for the case when the angle \( e_\varepsilon o_\beta \) is right. In short, we have here the same system of chords and of tangents, from a point upon the surface, as in 213; the only difference being, that we now write \( \varepsilon \) for \( a \), or \( \varepsilon \) for \( a \).

(5.) But finally, if \( \varepsilon \) be an external point, so that \( T_\varepsilon > T_\alpha, \ T(\alpha : \varepsilon) < 1 \), then \( TVU(\beta : \varepsilon) \) may either fall short of this last tensor, or equal, or exceed it; so that any one of the three alternatives (2.) may come to exist, according to the varying direction of \( \beta \).

(6.) To illustrate geometrically the law of passage from one such alternative to another, we may observe that the equation

\[
TVU\frac{\beta}{\varepsilon} = T\frac{a}{\varepsilon},
\]

or

\[
\sin \varepsilon o_\beta \theta = T \cdot o_\alpha : T \cdot o_\beta,
\]
CIRCUMSCRIBED CONES.

represents (when \( \varepsilon \) is thus external) a real cone of revolution, with its vertex at the centre \( o \) of the sphere; and according as the line \( ob \) lies inside this cone, or on it, or outside it, the first or the second or the third of the three alternatives (2.) is to be adopted; or in other words, the line through \( e \), drawn parallel (as before) to \( ob \), either cuts the sphere, or touches it, or does not (really) meet it at all. (Compare the annexed fig. 52.)

(7.) If \( e \) be still an external point, the cone of tangents which can be drawn from it to the sphere is real; and the equation of this enveloping or circumscribed cone, with its vertex at \( e \), may be obtained from that of the recent cone (6.), by simply changing \( \rho \) to \( \rho - \varepsilon \); it is, therefore, or at least one form of it is,

\[ TVU \frac{\rho - \varepsilon}{\varepsilon} = T \frac{a}{\varepsilon}; \quad \text{or} \quad \sin oep = T \cdot oa : T \cdot oe. \]

(8.) In general, if \( q \) be any quaternion, and \( x \) any scalar,

\[ VU (q + x) = Vq : T (q + x); \]

the recent equation (7.) may therefore be thus written:

\[ T \frac{V(\rho : \varepsilon) \cdot \varepsilon}{\rho - \varepsilon} = T \frac{a}{\varepsilon}; \]

or

\[ T \cdot \frac{r'p}{T} : T \cdot ep = T \cdot oa : T \cdot oe, \]

if \( r' \) be the foot of the perpendicular let fall from \( p \) on \( oe \); and in fact the first quotient is evidently \( = \sin oep \).

(9.) We may also write,

\[ TV \frac{\rho}{\varepsilon} = T \frac{a}{\varepsilon} \cdot T \left( \frac{\rho}{\varepsilon} - 1 \right); \quad \text{or} \quad 0 = \left( S \frac{\rho}{\varepsilon} \right)^2 - N \frac{\rho}{\varepsilon} + N \frac{a}{\varepsilon} \left( N \frac{\rho}{\varepsilon} - 2S \frac{\rho}{\varepsilon} + 1 \right); \]

or

\[ \left( S \frac{\rho}{\varepsilon} - N \frac{a}{\varepsilon} \right)^2 = \left( 1 - N \frac{a}{\varepsilon} \right) \left( N \frac{\rho}{\varepsilon} - N \frac{a}{\varepsilon} \right), \]

as another form of the equation of the circumscribed cone.

(10.) If then we make also

\[ N \frac{\rho}{a} = 1, \quad \text{or} \quad N \frac{\rho}{\varepsilon} = N \frac{a}{\varepsilon}, \]

to express that the point \( p \) is on the enveloped sphere, as well as on the enveloping
cone, we find the following equation of the plane of contact, or of what is called the polar plane of the point \( e \), with respect to the given sphere:

\[
\left( S_{\rho \epsilon} - N_{\epsilon} \right)^2 = 0; \quad \text{or} \quad S_{\rho \epsilon} - N_{\epsilon} = 0;
\]

while the fact that it is a plane of contact* is exhibited by the occurrence of the exponent 2, or by its equation entering through its square.

(11.) The vector,

\[
\epsilon' = e S_{\rho \epsilon} = e N_{\epsilon} = o\epsilon',
\]

is that of the point \( e' \) in which the polar plane (10.) of \( e \) cuts perpendicularly the right line \( o\epsilon \); and we see that

\[
T_\epsilon . T_{\epsilon'} = T_a^2, \quad \text{or} \quad T . o\epsilon . T . o\epsilon' = (T . o\alpha)^2;
\]

as was to be expected from elementary theorems, of spherical or even of plane geometry.

(12.) The equation (10.), of the polar plane of \( e \), may easily be thus transformed:

\[
S_{\rho \epsilon} = \left( S_{\rho \epsilon} . N_{\rho} \right) N_{\rho}, \quad \text{or} \quad S_{\rho \epsilon} - N_{\rho} = 0;
\]

it continues therefore to hold good, when \( \epsilon \) and \( \rho \) are interchanged. If then we take, as the vertex of a new enveloping cone, any point \( c \) external to the sphere, and situated on the polar plane \( \pi \phi' \ldots \) of the former external point \( e \), the new plane of contact, or the polar plane \( \pi \phi' \ldots \) of the new point \( c \), will pass through the former vertex \( e \): a geometrical relation of reciprocity, or of conjugation, between the two points \( c \) and \( e \), which is indeed well known, but which it appeared useful for our purpose to prove by quaternions† anew.

(13.) In general, each of the two connected equations,

\[
S_{\rho'} = N_{\rho}, \quad S_{\rho'} = N_{\rho},
\]

which may also be thus written,

\[
1 = \left( S_{\rho'} a . N_{\rho} a \right) S_{\rho'} a K_{\rho'} a, \quad 1 = S_{\rho'} a K_{\rho'} a,
\]

* In fact a modern geometer would say, that we have here a case of two coincident planes of intersection, merged into a single plane of contact.
† In fact, it will easily be seen that the investigations in recent sub-articles are put forward, almost entirely, as exercises in the Language and Calculus of Quaternions, and not as offering any geometrical novelty of result.
may be said to be a form of the *Equation of Conjugation* between any two points \( \mathbf{r} \) and \( \mathbf{r}' \) (not those so marked in fig. 52), of which the vectors satisfy it: because it expresses that those two points are, in a well-known sense, *conjugate* to each other, with respect to the given sphere, \( \mathbf{T}_p = \mathbf{T}_a \).

(14.) If one of the two points, as \( \mathbf{r}' \), be given by its vector \( \mathbf{r}' \), while the other point \( \mathbf{r} \) and vector \( \mathbf{r} \) are variable, the equation then represents a *plane locus*; namely, what is still called the polar plane of the given point, whether that point be external or internal, or on the surface of the sphere.

(15.) Let \( \mathbf{r}, \mathbf{r}' \) be thus two conjugate points; and let it be proposed to find the points \( \mathbf{s}, \mathbf{s}' \), in which the right line \( \mathbf{r}\mathbf{r}' \) intersects the sphere. Assuming (comp. 25) that

\[
\mathbf{os} = \sigma = x\mathbf{r} + y\mathbf{r}', \quad x + y = 1, \quad \mathbf{T}_\sigma = \mathbf{T}_a,
\]

and attending to the equation of conjugation (13.), we have, by 210, XX., or by 200, VII., the following quadratic equation in \( y : x \),

\[
(x + y)^2 = \mathbf{N} \left( \frac{x\mathbf{r}}{a} + \frac{y\mathbf{r}'}{a} \right) = x^2 \mathbf{N} \frac{\mathbf{r}}{a} + 2xy + y^2 \mathbf{N} \frac{\mathbf{r}'}{a};
\]

which gives

\[
x^2 \left( \frac{\mathbf{N} \mathbf{r}}{a} - 1 \right) = y^2 \left( 1 - \frac{\mathbf{N} \mathbf{r}'}{a} \right).
\]

(16.) Hence it is evident that, if the points of intersection \( \mathbf{s}, \mathbf{s}' \) are to be *real*, one of the two points \( \mathbf{r}, \mathbf{r}' \) must be interior, and the other must be exterior to the sphere; because, of the two norms here occurring, one must be greater and the other less than unity. And because the *two roots* of the quadratic, or the *two values* of \( y : x \), *differ only by their signs*, it follows (by 26) that the right line \( \mathbf{r}\mathbf{r}' \) is *harmonically divided* (as indeed it is well known to be), at the two points \( \mathbf{s}, \mathbf{s}' \) at which it meets the sphere: or that in a notation already several times employed (25, 31, &c.), we have the harmonic

\[
(psp's') = -1.
\]

(17.) From a real but internal point \( \mathbf{r} \), we can still *speak of a cone of tangents*, as being drawn to the sphere: but if so, we must say that those tangents are *ideal*, or *imaginary*;* and must consider them as terminating on an *imaginary circle of contact*; of which the *real* but wholly *external plane* is, by quaternions, as by modern geometry, recognised as being (comp. (14.)) the polar plane of the supposed internal point.

---

* Compare again the Note to page 88, and others formerly referred to.
216. Some readers may find it useful, or at least interesting, to see here a few examples of the application of the General Distributive Principle (212) of multiplication to the Ellipsoid, of which some forms of the Quaternion Equation were lately assigned (in 204, (14.)); especially as those forms have been found to conduct* to a Geometrical Construction, previously unknown, for that celebrated and important Surface: or rather to several such constructions. In what follows, it will be supposed that any such reader has made himself already sufficiently familiar with the chief formulae of the preceding Articles; and therefore comparatively few references† will be given, at least upon the present subject.

(1.) To prove, first, that the locus of the variable ellipse,

\[ \frac{S^2}{a} \equiv x, \quad \left( \frac{V}{\beta} \right)^2 = x^2 - 1, \tag{204, (13.)} \]

whose locus is represented by the equation,

\[ \left( \frac{S^2}{a} \right)^2 - \left( \frac{V}{\beta} \right)^2 = 1, \tag{204, (14.)} \]

the two constant vectors \(a, \beta\) being supposed to be real, and to be inclined to each other at some acute or obtuse (but not right‡) angle, is a surface of the second order, in the sense that it is cut by an arbitrary rectilinear transversal in two (real or imaginary) points, and in no more than two, let us assume two points \(l, m\), or their vectors \(\lambda = ol, \mu = om\), as given; and let us seek to determine the points \(p\) (real or imaginary), in which the indefinite right line \(lm\) intersects the locus \(\Pi\); or rather the number of such intersections, which will be sufficient for the present purpose.

(2.) Making then \(p = \frac{y\lambda + z\mu}{y + z} \tag{25}\), we have, for \(y: z\), the following quadratic equation,

\[ \left( y\frac{\lambda}{a} + z\frac{\mu}{a} \right)^2 - \left( y\frac{\lambda}{\beta} + z\frac{\mu}{\beta} \right)^2 = (y + z)^2; \] without proceeding to resolve which, we see already, by its mere degree, that

* See the Proceedings of the Royal Irish Academy for the year 1846.
† Compare the Note to page 223.
‡ If \(\beta \perp a\), the system \(I.\) represents (not an ellipse but) a pair of right lines, real or ideal, in which the cylinder of revolution, denoted by the second equation of that system, is cut by a plane parallel to its axis, and represented by the first equation.
the number sought is two; and therefore that the locus II. is, as above stated, a surface of the second order.

(3.) The equation II. remains unchanged, when \( -p \) is substituted for \( p \); the surface has therefore a centre, and this centre is at the origin \( o \) of vectors.

(4.) It has been seen that the equation of the surface may also be thus written:

\[
\text{IV} \ldots T\left(\frac{S^2}{a} + V \frac{p}{\beta}\right) = 1;
\]

it gives therefore, for the reciprocal of the radius vector from the centre, the expression,

\[
\text{V} \ldots \frac{1}{T_p} = T\left(\frac{S U_p}{a} + V \frac{U_p}{\beta}\right);
\]

and this expression has a real value, which never vanishes, whatever real value may be assigned to the versor \( U_p \), that is, whatever direction may be assigned to \( p \): the surface is therefore closed, and finite.

(5.) Introducing two new constant and auxiliary vectors, determined by the two expressions,

\[
\text{VI} \ldots \gamma = \frac{2\beta}{\beta + a} \cdot a, \quad \delta = \frac{2\beta}{\beta - a} \cdot a,
\]

which give (by 125) these other expressions,

\[
\text{VI'} \ldots \gamma = \frac{2a}{\beta + a} \cdot \beta, \quad \delta = \frac{2a}{\beta - a} \cdot \beta,
\]

we have

\[
\text{VII} \ldots \frac{\gamma}{\beta} + \frac{\gamma}{\delta} = 2, \quad \frac{\delta}{\beta} - \frac{\delta}{\delta} = 2;
\]

\[
\text{VII'} \ldots \frac{a}{\gamma} + \frac{a}{\delta} = 1, \quad \frac{\beta}{\gamma} - \frac{\beta}{\delta} = 1
\]

and under these conditions, \( \gamma \) is said to be the harmonic mean between the two former vectors, \( a \) and \( \beta \); and in like manner, \( \delta \) is the harmonic mean between \( a \) and \( -\beta \); while \( 2a \) is the corresponding mean between \( \gamma \), \( \delta \); and \( 2\beta \) is so, between \( \gamma \) and \( -\delta \).

(6.) Under the same conditions, for any arbitrary vector \( p \), we have the transformations,

\[
\text{VIII} \ldots \frac{p}{\gamma} = \frac{1}{2} \left(\frac{p}{a} + \frac{p}{\beta}\right); \quad \frac{p}{\delta} = \frac{1}{2} \left(\frac{p}{a} - \frac{p}{\beta}\right);
\]

\[
\text{IX} \ldots \frac{p}{\gamma} + K \frac{p}{\delta} = S \frac{p}{a} + V \frac{p}{\beta};
\]

* It is to be remembered that we have excluded in (1.) the case where \( \beta \perp a \); in which case it can be shown that the equation II. represents an elliptic cylinder.
the equation IV. of the surface may therefore be thus written:

\[ X \ldots T \left( \frac{\rho}{\gamma} + \frac{K \rho}{\delta} \right) = 1; \quad \text{or thus,} \quad X' \ldots T \left( \frac{\rho}{\delta} + \frac{K \rho}{\gamma} \right) = 1; \]

the geometrical meaning of which new forms will soon be seen.

(7.) The system of the two planes through the origin, which are respectively perpendicular to the new vectors \( \gamma \) and \( \delta \), is represented by the equation,

\[ XI. \ldots S \left( \frac{\rho}{\gamma} \right) S \left( \frac{\rho}{\delta} \right) = 0, \quad \text{or} \quad XII. \ldots \left( S \frac{\rho}{\alpha} \right)^2 = \left( S \frac{\rho}{\beta} \right)^2; \]

combining which with the equation II. we get

\[ XIII. \ldots 1 = \left( S \frac{\rho}{\beta} \right)^2 - \left( S \frac{\rho}{\beta} \right)^2 = N \frac{\rho}{\beta}; \quad \text{or,} \quad XIV. \ldots T_\rho = T_\beta. \]

These two diametral planes therefore cut the surface in two circular sections, with \( T_\beta \) for their common radius; and the normals \( \gamma \) and \( \delta \), to the same two planes, may be called (comp. 196, (17.)) the cyclic normals of the surface; while the planes themselves may be called its cyclic planes.

(8.) Conversely, if we seek the intersection of the surface with the concentric sphere XIV., of which the radius is \( T_\beta \), we are conducted to the equation XII. of the system of the two cyclic planes, and therefore to the two circular sections (7.); so that every radius vector of the surface, which is not drawn in one or other of these two planes, has a length either greater or less than the radius \( T_\beta \) of the sphere.

(9.) By all these marks, it is clear that the locus II., or 204, (14.), is (as above asserted) an Ellipsoid; its centre being at the origin (3.), and its mean semiaxis being \( T_\beta \); while \( U_\beta \) has, by 204, (15.), the direction of the axis of a circumscribed cylinder of revolution, of which cylinder the radius is \( T_\beta \); and \( a \) is, by the last cited sub-article, perpendicular to the plane of the ellipse of contact.

(10.) Those who are familiar with modern geometry, and who have caught the notations of quaternions, will easily see that this ellipsoid, II. or IV., is a deformation of what may be called the mean sphere XIV., and is homologous thereto; the infinitely distant point in the direction of \( \beta \) being a centre of homology, and either of the two planes XI. or XII. being a plane of homology corresponding.

217. The recent form, \( X \). or \( X' \), of the quaternion equation of the ellipsoid admits of being interpreted in such a way as to conduct (comp. 216) to
a simple construction of that surface; which we shall first investigate by calculation, and then illustrate by geometry.

(1.) Carrying on the Roman numerals from the sub-articles to 216, and observing that (by 190, &c.),

$$\frac{\rho}{\gamma} = K \frac{\gamma}{\rho} \cdot N \frac{\rho}{\gamma}, \quad \text{and} \quad K \frac{\rho}{\delta} = \frac{\delta^2}{\rho} \cdot N \frac{\rho}{\delta},$$

the equation X. takes the form,

$$XV. \ldots I = T \left( \frac{\delta}{T\delta^2} + K \frac{\gamma}{\rho} \cdot \frac{\rho}{T\gamma^2} ; \frac{\rho}{T\rho^2} \right);$$

or

$$XVI. \ldots \frac{\rho}{T\rho} = T \left( \frac{\epsilon + K \frac{\kappa}{\rho} \cdot \rho}{T\rho^2} \right),$$

if we make

$$XVII. \ldots \frac{\delta}{T\delta^2} = \frac{\epsilon}{\epsilon^2} \quad \text{and} \quad \frac{\gamma}{T\gamma^2} = \frac{\kappa}{\kappa^2},$$

when \( \epsilon \) and \( \kappa \) are two new constant vectors, and \( \epsilon \) is a new constant scalar, which we shall suppose to be positive, but of which the value may be chosen at pleasure.

(2.) The comparison of the forms X. and X'. shows that \( \gamma \) and \( \delta \) may be interchanged, or that they enter symmetrically into the equation of the ellipsoid, although they may not at first seem to do so; it is therefore allowed to assume that

$$XVIII. \ldots T\gamma > T\delta, \quad \text{and therefore that} \quad XVIII'. \ldots T\epsilon > T\kappa;$$

for the supposition \( T\gamma = T\delta \) would give, by VI.,

$$T (\beta + a) = T (\beta - a), \quad \text{and} \quad \beta \perp a,$$

which latter case was excluded in 216, (1.).

(3.) We have thus,

$$XIX. \ldots U\epsilon = U\delta; \quad U\kappa = U\gamma;$$

$$XX. \ldots T\epsilon = \frac{T\epsilon}{T\delta}; \quad T\kappa = \frac{T\kappa}{T\gamma};$$

$$XXI. \ldots \frac{T\epsilon^2 - T\kappa^2}{T\delta^2} = \left( \frac{T\delta}{T\delta} \right)^2 - \left( \frac{T\gamma}{T\gamma} \right)^2.$$
Then if a sphere, which we shall call the diacentric sphere, be described round the point c as centre, with a radius = Tx, and therefore so as to pass through the centre a (here written instead of o) of the ellipsoid, and if d be the point in which the line ae meets this sphere again, we shall have, by 213, (5.), (13.),

\[
\text{XXIII. } \cd = -K \frac{k}{\rho} \cdot \rho,
\]

and therefore

\[
\text{XXIII'} \quad \db = t + K \frac{k}{\rho} \cdot \rho;
\]

so that the equation XVI. becomes

\[
\text{XXIV. } t^2 = T \cdot \AE \cdot T \cdot \db.
\]

(5.) The point b is external to the diacentric sphere (4.), by the assumption (2.); a real tangent (or rather cone of tangents) to this sphere can therefore be drawn from that point; and if we select the length of such a tangent as the value (1.) of the scalar t, that is to say, if we make each member of the formula XXI. equal to unity, and denote by d' the second intersection of the right line bd with the sphere, as in fig. 53, we shall have (by Euclid III.) the elementary relation,

\[
\text{XXV. } t^2 = T \cdot \db \cdot T \cdot \db';
\]

whence follows this Geometrical Equation of the Ellipsoid,

\[
\text{XXVI. } T \cdot \AE = T \cdot \db';
\]

or in somewhat more familiar notation,

\[
\text{XXVII. } \AE = \db';
\]

where \(\AE\) denotes the length of the line \(\AE\), and similarly for \(\db'\).

(6.) The following very simple Rule of Construction (comp. the recent fig. 53) results therefore from our quaternion analysis:

From a fixed point a, on the surface of a given sphere, draw any chord ad; let d' be the second point of intersection of the same spheric surface with the secant bd, drawn from a fixed external* point b; and take a radius vector \(\AE\), equal in

* It is merely to fix the conceptions, that the point b is here supposed to be external (5.); the calculations and the construction would be almost the same, if we assumed b to be an internal point, or \(T1 < Tx, T7 < T8\).
length to the line $bd'$, and in direction either coincident with, or opposite to, the chord $ad$: the locus of the point $e$ will be an ellipsoid, with $a$ for its centre, and with $b$ for a point of its surface.

(7.) Or thus—

If, of a plane but variable quadrilateral $abed'$, of which one side $ab$ is given in length and in position, the two diagonals $ae, bd'$ be equal to each other in length, and if their intersection $d$ be always situated upon the surface of a given sphere, whereof the side $ad'$ of the quadrilateral is a chord, then the opposite side $be$ is a chord of a given ellipsoid.

218. From either of the two foregoing statements, of the Rule of Construction for the Ellipsoid to which quaternions have conducted, many geometrical consequences can easily be inferred, a few of which may be mentioned here, with their proofs by calculation annexed: the present Calculus being, of course, still employed.

(1.) That the corner $b$, of what may be called the Generating Triangle $abc$, is in fact a point of the generated surface, with the construction, 217, (6.), may be proved, by conceiving the variable chord $ad$ of the given diacentric sphere to take the position $ag$; where $g$ is the second intersection of the line $ab$ with that spheric surface.

(2.) If $d$ be conceived to approach to $A$ (instead of $g$), and therefore $d'$ to $g$ (instead of $a$), the direction of $ae$ (or of $ad$) then tends to become tangential to the sphere at $a$, while the length of $ae$ (or of $bd'$) tends, by the construction, to become equal to the length of $bg$; the surface has therefore a diametral and circular section, in a plane which touches the diacentric sphere at $A$, and with a radius $=bg$.

(3.) Conceive a circular section of the sphere through $a$, made by a plane perpendicular to $bc$; if $d$ move along this circle, $d'$ will move along a parallel circle through $g$, and the length of $bd'$, or that of $ae$, will again be equal to $bg$; such then is the radius of a second diametral and circular section of the ellipsoid, made by the lately mentioned plane.

(4.) The construction gives us thus two cyclic planes through $a$; the perpendiculars to which planes, or the two cyclic normals (216, (7.)) of the ellipsoid, are seen to have the directions of the two sides, $ca, cb$, of the generating triangle $abc$ (1.).
(5.) Again, since the rectangle
\[ \overline{BA} \cdot \overline{BG} = \overline{BD} \cdot \overline{BD'} = \overline{BD} \cdot \overline{AE} = \text{double area of triangle } \triangle ABE : \sin \angle BDE, \]
we have the equation,
\[ \text{XXVIII. } \ldots \text{perpendicular distance of } E \text{ from } \overline{AB} = \overline{BG} \cdot \sin \angle BDE; \]
the third side, AB, of the generating triangle (1.), is therefore the \textit{axis of revolution} of a cylinder, which \textit{envelops} the ellipsoid, and of which the radius has the same length, \overline{BG}, as the radius of each of the two diametral and circular sections.

(6.) For the points of contact of ellipsoid and cylinder, we have the geometrical relation,
\[ \text{XXIX. } \ldots \angle BDE = \text{a right angle}; \text{ or } \text{XXIX'. } \ldots \angle ADB = \text{a right angle}; \]
the point d is therefore situated on a \textit{second spheric surface}, which has the line AB for a diameter, and intersects the diacentric sphere in a circle, whereof the plane passes through a, and cuts the enveloping cylinder in an \textit{ellipse of contact} (comp. 204, (15.), and 216, (9.)), of that cylinder with the ellipsoid.

(7.) Let AC meet the diacentric sphere again in \( v \), and let \( BF \) meet it again in \( v' \) (as in fig. 53); the \textit{common plane} of the last-mentioned circle and ellipse (6.) can then be easily proved to cut perpendicularly the plane of the generating triangle \( \triangle ABC \) in the line \( AB' \); so that the line \( \overline{F'B} \) is \textit{normal} to this \textit{plane of contact}; and therefore also (by conjugate diameters, &c.) to the ellipsoid, at B.

(8.) These geometrical consequences of the construction (217), to which many others might be added, can all be shown to be consistent with, and confirmed by, the quaternion analysis from which that construction itself was derived. Thus, the two \textit{circular sections} (2.), (3.) had presented themselves in 216, (7.); and their two \textit{cyclic normals} (4.), or the sides \( CA, CB \) of the triangle, being (by 217, (4.)) the two vectors \( \kappa, \iota \), have (by 217, (1.) or (3.)) the directions of the two former vectors \( \gamma, \delta \); which again agrees with 216, (7.).

(9.) Again, it will be found that the assumed relations between the \textit{three pairs of constant vectors}, \( \alpha, \beta \); \( \gamma, \delta \); and \( \iota, \kappa \), any one of which \textit{pairs} is sufficient to determine the ellipsoid, conduct to the following expressions (of which the investigation is left to the student, as an exercise):
\[ \text{XXX. } \ldots \alpha = \frac{\delta}{\delta + \gamma} \gamma + \frac{\gamma}{\delta + \gamma} \delta = \frac{+ \iota^2}{T (\iota + \kappa)} U (\iota + \kappa) = F'B; \]
\[ \text{XXXI. } \ldots \beta = \frac{\delta}{\delta - \gamma} \gamma + \frac{\gamma}{\delta - \gamma} \delta = \frac{- \iota^2}{T (\iota - \kappa)} U (\iota - \kappa) = BG; \]
the letters b, f', e referring here to fig. 53, while αβγδ retain their former meanings (216), and are not interpreted as vectors of the points ABCD in that figure. Hence the recent geometrical inferences, that AB (or BE) is the axis of revolution of an enveloping cylinder (5.), and that F'B is normal to the plane of the ellipse of contact (7.), agree with the former conclusions (216, (9.), or 204, (15.),) that β is such an axis, and that α is such a normal.

(10.) It is easy to prove, generally, that

\[
\frac{q - 1}{q + 1} = \frac{(q-1)(Kq+1)}{(q+1)(Kq+1)} = \frac{Nq-1}{N(q+1)}, \quad \frac{q + 1}{q - 1} = \frac{Nq-1}{N(q-1)};
\]

whence

\[
\begin{align*}
XXXII. & \quad S\frac{t - \kappa}{t + \kappa} = \frac{Tt^2 - Tk^2}{T(t+\kappa)^2}, \\
XXXIII. & \quad t^2 = Tt^2 - Tk^2, \text{ by 217, (5.);}
\end{align*}
\]

the recent expressions (9.) for α and β become, therefore,

\[
XXXIV. \quad \alpha = + (t + \kappa) S\frac{t - \kappa}{t + \kappa}; \quad \beta = - (t - \kappa) S\frac{t + \kappa}{t - \kappa}.
\]

The last form 204, (14.), of the equation of the ellipsoid, may therefore be now thus written:

\[
XXXV. \quad T\left(S\frac{p}{t + \kappa} : S\frac{t - \kappa}{t + \kappa} - V\frac{p}{t - \kappa} : S\frac{t + \kappa}{t - \kappa}\right) = 1;
\]

in which the sign of the right part may be changed. And thus we verify by calculation the recent result (1.) of the construction, namely, that b is a point of the surface; for we see that the last equation is satisfied, when we suppose

\[
XXXVI. \quad p = AB = t - \kappa = \beta : S\frac{\beta}{\alpha};
\]

a value of p which evidently satisfies also the form 216, IV.

(11.) From the form 216, II., combined with the value XXXIV. of α, it is easy to infer that the plane,

\[
XXXVII. \quad S\frac{p}{\alpha} = 1, \quad \text{or} \quad XXXVII'. \quad S\frac{p}{t + \kappa} = S\frac{t - \kappa}{t + \kappa},
\]

which corresponds to the value x = 1 in 216, I., touches the ellipsoid at the point b, of which the vector p has been thus determined (10.); the normal to the surface, at that point, has therefore the direction of t + κ, or of α, that is, of FB, or of F'B: so that the last geometrical inference (7.) is thus confirmed, by calculation with quaternions.
219. A few other consequences of the construction (217) may be here noted; especially as regards the geometrical determination of the three principal semiaxes of the ellipsoid, and the major and minor semiaxes of any elliptic and diametral section; together with the assigning of a certain system of spherical conics, of which the surface may be considered to be the locus.

(1.) Let \(a, b, c\) denote the lengths of the greatest, the mean, and the least semiaxes of the ellipsoid, respectively; then if the side \(bc\) of the generating triangle cut the diacentric sphere in the points \(h\) and \(h'\), the former lying (as in fig. 53) between the points \(b\) and \(c\), we have the values,

\[
XXXVIII. \quad a = \overline{bh'}; \quad b = \overline{bg}; \quad c = \overline{bh};
\]

so that the lengths of the sides of the triangle \(abc\) may be thus expressed, in terms of these semiaxes,

\[
XXXIX. \quad \overline{bc} = T_k = \frac{a+c}{2}; \quad \overline{ca} = T_\kappa = \frac{a-c}{2}; \quad \overline{ab} = T\left(\iota - \kappa\right) = \frac{ac}{b};
\]

and we may write,

\[
XL. \quad a = T\iota + T\kappa; \quad b = \frac{T\iota^2 - T\kappa^2}{T\left(\iota - \kappa\right)}; \quad c = T\iota - T\kappa.
\]

(2.) If, in the respective directions of the two supplementary chords \(AH, AH'\) of the sphere, or in the opposite directions, we set off lines \(AI, AN\), with the lengths of \(bh', bh\), the points \(I, N\), thus obtained, will be respectively a major and a minor summit of the surface. And if we erect, at the centre \(A\) of that surface a perpendicular \(AM\) to the plane of the triangle, with a length \(= BG\), the point \(M\) (which will be common to the two circular sections, and will be situated on the enveloping cylinder) will be a mean summit thereof.

(3.) Conceive that the sphere and ellipsoid are both cut by a plane through \(A\), on which the points \(B'\) and \(C'\) shall be supposed to be the projections of \(B\) and \(C\); then \(C'\) will be the centre of the circular section of the sphere; and if the line \(B'C'\) cut this new circle in the points \(D_1, D_2\), of which \(D_1\) may be supposed to be the nearer to \(B'\), the two supplementary chords \(AD_1, AD_2\) of the circle have the directions of the major and minor semiaxes of the elliptic section of the ellipsoid; while the lengths of those semiaxes are, respectively, \(\overline{BA} : \overline{BG} : \overline{BD_1}\), and \(\overline{BA} : \overline{BG} : \overline{BD_2}\); or \(\overline{BD_1}\) and \(\overline{BD_2}\), if the secants \(BD_1\) and \(BD_2\) meet the sphere again in \(D'_1\) and \(D'_2\).

(4.) If these two semiaxes of the section be called \(a'\) and \(c'\); and if we still denote by \(t\) the tangent from \(B\) to the sphere, we have thus,

\[
XLI. \quad \overline{BD_1} = t^\circ : a' = aca'^{-1}; \quad \overline{BD_2} = t^\circ : c' = ace'^{-1};
\]
but if we denote by \( p_1 \) and \( p_2 \) the inclinations of the plane of the section to the two cyclic planes of the ellipsoid, where \( cA \) and \( cB \) are perpendicular, so that the projections of these two sides of the triangle are

\[
\begin{align*}
\text{XLII} & \ldots \quad \left\{ \begin{array}{l}
\overrightarrow{CA} = cA \cdot \sin p_1 = \frac{1}{2} (a - c) \sin p_1, \\
\overrightarrow{CB} = cB \cdot \sin p_2 = \frac{1}{2} (a + c) \sin p_2,
\end{array} \right.
\end{align*}
\]

we have

\[
\begin{align*}
\text{XLIII} & \ldots \quad \overrightarrow{BD}^2 - \overrightarrow{DI}^2 = \overrightarrow{BD'}^2 - \overrightarrow{DI'}^2 = 4\overrightarrow{C'D'} \cdot \overrightarrow{C'A} = (a^2 - c^2) \sin p_1 \sin p_2;
\end{align*}
\]

whence follows the important formula,

\[
\begin{align*}
\text{XLIV} & \ldots \quad c_1^{-2} - a_1^{-2} = (c^2 - a^2) \sin p_1 \sin p_2;
\end{align*}
\]

or in words, the known and useful theorem, that "the difference of the inverse squares of the semiaxes, of a plane and diametral section of an ellipsoid, varies as the product of the sines of the inclinations of the cutting plane, to the two planes of circular section."

(5.) As verifications, if the plane be that of the generating triangle \( ABC \), we have

\[
\begin{align*}
p_1 = p_2 = \frac{\pi}{2}, \quad \text{and} \quad a_1 = a, \quad c_1 = c;
\end{align*}
\]

but if the plane be perpendicular to either of the two sides, \( cA, cB \), then either \( p_1 \) or \( p_2 = 0 \), and \( c_1 = a_1 \).

(6.) If the ellipsoid be cut by any concentric sphere, distinct from the mean sphere \( \text{XIV.} \), so that

\[
\begin{align*}
\text{XLV} & \ldots \quad \overrightarrow{AE} = T\rho = r < b, \quad \text{where} \ r \ \text{is a given positive scalar};
\end{align*}
\]

then

\[
\begin{align*}
\text{XLVI} & \ldots \quad \overrightarrow{BD} = t r^{-1} < ace^{-1}, \quad \text{that is,} \quad < \overrightarrow{BA};
\end{align*}
\]

so that the locus of what may be called the guide-point \( d \), through which, by the construction, the variable semidiameter \( AE \) of the ellipsoid (or one of its prolongations) passes, and which is still at a constant distance from the given external point \( b \), is now again a circle of the diacentric sphere, but one of which the plane does not pass (as it did in \( 218, (3.) \)) through the centre \( a \) of the ellipsoid. The point \( e \) has therefore here, for one locus, the cyclic cone which has \( a \) for vertex, and rests on the last-mentioned circle as its base; and since it is also on the concentric sphere \( \text{XLV.} \), it must be on one or other of the two spherical conics, in which (comp. 196, (11.)) the cone and sphere last mentioned intersect.
(7.) The intersection of an ellipsoid with a concentric sphere is therefore, generally, a system of two such conics, varying with the value of the radius \( r \), and becoming, as a limit, the system of the two circular sections, for the particular value \( r = b \); and the ellipsoid itself may be considered as the locus of all such spherical conics, including those two circles.

(8.) And we see, by (6.), that the two cyclic planes (comp. 196, (17.), &c.) of any one of the concentric cones, which rest on any such conic, coincide with the two cyclic planes of the ellipsoid: all this resulting, with the greatest ease, from the construction (217) to which quaternions had conducted.

(9.) With respect to the figure 53, which was designed to illustrate that construction, the significance of the letters \( \text{abcdd'eff'ghh'ln} \) has been already explained. But as regards the other letters we may here add, 1st, that \( n' \) is a second minor summit of the surface, so that \( an' = na \); IIInd, that \( k \) is a point in which the chord \( \text{agf} ' \), of what we may here call the diacentric circle \( \text{agf} \), intersects what may be called the principal ellipse,* of the ellipsoid, made by the plane of the greatest and least axes, that is by the plane of the generating triangle \( \text{abc} \), so that the lengths of \( \text{ak} \) and \( \text{bf} \) are equal; IIIrd, that the tangent, \( \text{vkv}' \), to this ellipse at this point, is parallel to the side \( \text{ab} \) of the triangle, or to the axis of revolution of the enveloping cylinder 218, (5.), being in fact one side (or generatrix) of that cylinder; IVth, that \( \text{ak} \), \( \text{ab} \) are thus two conjugate semidiameters of the ellipse, and therefore the tangent \( \text{tbtf}' \), at the point \( b \) of that ellipse, is parallel to the line \( \text{akf}' \), or perpendicular to the line \( \text{bfp}' \); Vth, this latter line is thus the normal (comp. 218, (7.), (11.)) to the same elliptic section, and therefore also to the ellipsoid, at \( b \); VIth, that the least distance \( kk' \) between the parallels \( \text{ab}, \text{kv} \), being = the radius \( b \) of the cylinder, is equal in length to the line \( \text{bc} \), and also to each of the two semidiameters, \( \text{as} \), \( \text{as}' \) of the ellipse, which are radii of the two circular sections of the ellipsoid, in planes perpendicular to the plane of the figure; VIIth, that \( \text{as} \) touches the circle at \( \lambda \); and VIIIth, that the point \( s' \) is on the chord \( \text{ai} \) of that circle, which is drawn at right angles to the side \( \text{bc} \) of the triangle.

220. The reader will easily conceive that the quaternion equation of the ellipsoid admits of being put under several other forms; among which, however, it may here suffice to mention one, and to assign its geometrical interpretation.

* In the plane of what is called, by many modern geometers, the focal hyperbolae of the ellipsoid.
(1.) For any three vectors, \(i, \kappa, \rho\), we have the transformations,

\[
\begin{align*}
\text{XLVII.} \quad N\left(\frac{i}{\rho} + \frac{\kappa}{\rho}\right) &= N\left(\frac{i}{\rho} + \frac{\kappa}{\rho}\right) + 2S\left(\frac{i}{\rho} + \frac{\kappa}{\rho}\right) \\
&= N\left(\frac{i}{\rho} + \frac{\kappa}{\rho}\right) + 2S\left(\frac{i}{\rho} + \frac{\kappa}{\rho}\right)
\end{align*}
\]

whence follows this other general transformation:

\[
\begin{align*}
\text{XLVIII.} \quad T\left(i + \frac{\kappa}{\rho}\right) &= T\left(U_i + \frac{K}{\rho}\right).
\end{align*}
\]

(2.) If then we introduce two new auxiliary and constant vectors, \(i'\) and \(\kappa'\), defined by the equations,

\[
\begin{align*}
\text{XLIX.} \quad i' &= -U_i, \quad \kappa' = -U_i.
\end{align*}
\]

which give

\[
\begin{align*}
\text{L.} \quad T'i' &= T_i, \quad T'\kappa' = T_k, \quad T'(i' - \kappa') = T(i - \kappa), \quad T'i'^2 - T'\kappa'^2 = i^2,
\end{align*}
\]

we may write the equation XVI. (in 217) of the ellipsoid under the following precisely similar form:

\[
\begin{align*}
\text{LI.} \quad \frac{i'^2}{T_\rho} &= T\left(i' + \frac{\kappa'}{\rho}\right);
\end{align*}
\]

in which \(i'\) and \(\kappa'\) have simply taken the places of \(i\) and \(\kappa\).

(3.) Retaining then the centre \(A\) of the ellipsoid, construct a new diacentric sphere, with a new centre \(c'\), and a new generating triangle, \(AB'c'\), where \(B'\) is a new fixed external point, but the lengths of the sides are the same, by the conditions,

\[
\begin{align*}
\text{LII.} \quad A\delta' = -\delta', \quad c'\delta' = +\delta', \quad \text{and therefore} \quad AB' = c' - \delta';
\end{align*}
\]

draw any secant \(B'D''D'''\) (instead of \(BDD'\)), and set off a line \(A\delta\) in the direction of \(AD''\), or in the opposite direction, with a length equal to that of \(BD'''\); the locus of the point \(\delta\) will be the same ellipsoid as before.

(4.) The only inference which we shall here* draw from this new construction is, that there exists (as is known) a second enveloping cylinder of

* If room shall allow, a few additional remarks may be made, on the relations of the constant vectors \(i, \kappa, \&c.,\) to the ellipsoid, and on some other constructions of that surface, when, in the following Book, its equation shall come to be put under the new form, \(T(i\rho + \rho i) = \kappa^2 - i^2\). [See 404.]
revolution, and that its axis is the side $AB'$ of the new triangle $AB'O'$; but that the radius of this second cylinder is equal to that of the first, namely to the mean semiaxis, $b$, of the ellipsoid; and that the major semiaxis, $a$, or the line $AL$ in fig. 53, bisects the angle $BAB'$, between the two axes of revolution of these two circumscribed cylinders: the plane of the new ellipse of contact being geometrically determined by a process exactly similar to that employed in 218, (7.); and being perpendicular to the new vector, $i + k'$, as the old plane of contact was (by 218, (11.) ) to $i + k$.

---

**SECTION 14.**

On the Reduction of the General Quaternion to a Standard Quadrinomial Form; with a First Proof of the Associative Principle of Multiplication of Quaternions.

221. Retaining the significations (181) of the three rectangular unit-lines $o_i, o_j, o_k$, as the axes, and therefore also the indices (159), of three given right versors, $i, j, k$, in three mutually rectangular planes, we can express the index $o_q$ of any other right quaternion, such as $V_q$, under the trinomial form (comp. 62),

$$I. \ldots IVq = o_q = x \cdot o_i + y \cdot o_j + z \cdot o_k;$$

where $xyz$ are some three scalar coefficients, namely, the three rectangular co-ordinates of the extremity $q$ of the index, with respect to the three axes $o_i, o_j, o_k$. Hence we may write also generally, by 206 and 126,

$$II. \ldots Vq = xi + yj + zk = ix + jy + kz;$$

and this last form, $ix + jy + kz$, may be said to be a **Standard Trinomial Form**, to which *every right quaternion, or the right part $Vq$ of any proposed quaternion $q$, can be (as above) reduced*. If then we denote by $w$ the scalar part, $S_q$, of the same general quaternion $q$, we shall have, by 202, the following **General Reduction of a Quaternion to a Standard Quadrinomial Form** (183):

$$III. \ldots q = (S_q + Vq =) w + ix + jy + kz;$$

in which the four scalars, $wxyz$, may be said to be the **Four Constituents of the Quaternion**. And it is evident (comp. 202, (5.), and 133), that if we write in like manner,

$$IV. \ldots q' = w' + ix' + jy' + kz',$$
where $ijk$ denote the same three given right versors (181) as before, then the equation

$$V \ldots q' = q,$$

between these two quaternions, $q$ and $q'$, includes the four following scalar equations between the constituents:

$$VI. \ldots \begin{align*} &w' = w, \\
&x' = x, \\
&y' = y, \\
&z' = z; \\
\end{align*}$$

which is a new justification (comp. 112, 116) of the propriety of naming, as we have done throughout the present Chapter, the General Quotient of two Vectors (101) a Quaternion.

222. When the Standard Quadrinomial Form (221) is adopted, we have then not only

$$I. \ldots S_q = w, \quad and \quad V_q = ix + jy + kz,$$

as before, but also, by 204, XI.,

$$II. \ldots K_q = (S_q - V_q) = w - ix - jy - kz.$$

And because the distributive property of multiplication of quaternions (212), combined with the laws of the symbols $ijk$ (182), or with the General and Fundamental Formula of this whole Calculus (183), namely with the formula,

$$i^2 = j^2 = k^2 = ijk = -1,$$

(A)

gives the transformation,

$$III. \ldots (ix + jy + kz)^2 = -(x^2 + y^2 + z^2),$$

we have, by 204, &c., the following new expressions:

$$IV. \ldots N V_q = (T V_q)^2 = - V_q^2 = x^2 + y^2 + z^2;$$

$$V. \ldots T V_q = \sqrt{x^2 + y^2 + z^2};$$

$$VI. \ldots U V_q = (ix + jy + kz): \sqrt{x^2 + y^2 + z^2};$$

$$VII. \ldots N_q = T_q = S_q^2 - V_q^2 = w^2 + x^2 + y^2 + z^2;$$

$$VIII. \ldots T_q = \sqrt{w^2 + x^2 + y^2 + z^2};$$

$$IX. \ldots U_q = (w + ix + jy + kz): \sqrt{(w^2 + a^2 + y^2 + z^2)};$$

$$X. \ldots S U_q = w: \sqrt{(w^2 + a^2 + y^2 + z^2)};$$

$$XI. \ldots V U_q = (ix + jy + kz): \sqrt{(w^2 + a^2 + y^2 + z^2)};$$

$$XII. \ldots T V U_q = \sqrt{\frac{a^2 + y^2 + z^2}{w^2 + a^2 + y^2 + z^2}}.$$
(1.) To prove the recent formula III., we may arrange as follows the steps of the multiplication (comp. again 182):

\[ Vq = ia + jy + kz, \]
\[ Vq = ia + jy + kz; \]
\[ ix \cdot Vq = -x^2 + kxy - jxz; \]
\[ jy \cdot Vq = -y^2 - kyx + iyz, \]
\[ kz \cdot Vq = -z^2 + jzx - izy; \]
\[ Vq^2 = Vq \cdot Vq = -x^2 - y^2 - z^2. \]

(2.) We have, therefore,

\[ \text{XIII. } \ldots (ia + jy + kz)^2 = -1, \text{ if } x^2 + y^2 + z^2 = 1, \]
a result to which we have already alluded,* in connexion with the partial indeterminateness of signification, in the present calculus, of the symbol \( \sqrt{-1} \), when considered as denoting a right radial (149), or a right versor (153), of which the plane or the axis is arbitrary.

(3.) If \( q'' = q'q \), then \( Nq'' = Nq' \cdot Nq \), by 191, (8); but if \( q = w + \&c., q' = w' + \&c., q'' = w'' + \&c., \) then

\[ w'' = w'w - (x'x + y'y + z'z), \]
\[ x'' = (w'x + x'w) + (y'z - z'y), \]
\[ y'' = (w'y + y'w) + (z'x - x'z), \]
\[ z'' = (w'z + z'w) + (x'y - y'x); \]

and conversely these four scalar equations are jointly equivalent to, and may be summed up in, the quaternion formula,

\[ \text{XIV. } \ldots w'' + ia'' + jy'' + kz'' = (w' + ia' + jy' + kz')(w + ia + jy + kz); \]

we ought therefore, under these conditions XIV., to have the equation,

\[ \text{XVI. } \ldots w''^2 + x''^2 + y''^2 + z''^2 = (w'^2 + x'^2 + y'^2 + z'^2)(w^2 + x^2 + y^2 + z^2); \]

which can in fact be verified by so easy an algebraical calculation, that its truth may be said to be obvious upon mere inspection, at least when the terms in the four quadrinomial expressions \( w'' \ldots z'' \) are arranged† as above.

* Compare the first Note to page 133; and that to page 162.
† From having somewhat otherwise arranged those terms, the author had some little trouble at first, in verifying that the twenty-four double products, in the expansion of \( w''^2 + \&c., \) destroy each other, leaving only the sixteen products of squares, or that XVI. follows from XIV., when he was led to anticipate that result through quaternions, in the year 1843. He believes, however, that the algebraic theorem XVI., as distinguished from the quaternion formula XV., with which it is here connected, had been discovered by the celebrated Euler.
223. The principal use which we shall here make of the standard quadri­nomial form (221) is to prove by it the general associative property of multiplication of quaternions; which can now with great ease be done, the distributive* property (212) of such multiplication having been already proved. In fact, if we write, as in 222, (3.),

\[
\begin{align*}
q &= w + iw + jy + kz, \\
q' &= w' + iw' + jy' + kz', \\
q'' &= w'' + iw'' + jy'' + kz'', \\
\end{align*}
\]

without now assuming that the relation \(q'' = q'q\), or any other relation, exists between the three quaternions \(q, q', q''\), and inquire whether it be true that the associative formula,

\[
\begin{align*}
q'' q' q &= q'' q' q, \\
\end{align*}
\]

holds good, we see, by the distributive principle, that we have only to try whether this last formula is valid when the three quaternion factors \(q, q', q''\) are replaced, in any one common order on both sides of the equation, and with or without repetition, by the three given right versors \(ijk\); but this has already been proved, in Art. 183. We arrive then, thus, at the important conclusion, that the General Multiplication of Quaternions is an Associative Operation, as it had been previously seen (212) to be a Distributive one: although we had also found (168, 183, 191) that such Multiplication is not (in general) Commutative: or that the two products, \(q'q\) and \(qq'\), are generally unequal. We may therefore omit the point (as in 183), and may denote each member of the equation II. by the symbol \(q'q'q\).

(1.) Let \(v = Vq, v' = Vq', v'' = Vq''\); so that \(v, v', v''\) are any three right quaternions, and therefore, by 191, (2.), and 196, 204,

\[
\begin{align*}
Kv'v &= v'e, \\
Sv'v &= \frac{1}{2}(v'e + v'e), \\
Vv'v &= \frac{1}{2}(v'e - v'e).
\end{align*}
\]

Let this last right quaternion be called \(v\), and let \(Sv'v = s\), so that \(v'e = s + v\); we shall then have the equations,

\[
2Vv'v = v''v - v'v'; \quad 0 = v's - v''p'
\]

whence, by addition,

\[
2Vv''v = v''v - v'v'.v''
\]

\[
= (v''v + v'v')v - v' (v''v + v'v')
\]

\[
= 2v'Sv'' - 2v'Sv';
\]

* At a later stage [II. iii. § 2], a sketch will be given of at least one proof of this Associative Principle of Multiplication, which will not presuppose the Distributive Principle.
and therefore generally, if $v, v', v''$ be still right, as above,

$$III. \ldots V . v'' \overrightarrow{v} = v S \overrightarrow{v}'' - \overrightarrow{v} S \overrightarrow{v}';$$

a formula with which the student ought to make himself completely familiar, on account of its extensive utility.

(2.) With the recent notations,

$$V . v'' \overrightarrow{v} = V s'' s_2 = v'' s_2 = v'' S \overrightarrow{v};$$

we have therefore this other very useful formula,

$$IV. \ldots V . v'' \overrightarrow{v} = v S \overrightarrow{v}'' - v' S \overrightarrow{v}'' + v'' S \overrightarrow{v},$$

where the point in the first member may often for simplicity be dispensed with; and in which it is still supposed that

$$\angle v = \angle v' = \angle v'' = \frac{\pi}{2}.$$

(3.) The formula III. gives (by 206),

$$V. \ldots IV . v'' \overrightarrow{v} = Iv . S \overrightarrow{v}'' - Iv' . S \overrightarrow{v};$$

hence this last vector, which is evidently complanar with the two indices $Iv$ and $Iv'$, is at the same time (by 208) perpendicular to the third index $Iv''$, and therefore (by (1.)) complanar with the third quaternion $q''$.

(4.) With the recent notations, the vector,

$$VI. \ldots Iv_r = IVv = IV (Vq' . Vq),$$

is (by 208, XXII.) a line perpendicular to both $ Iv $ and $ Iv' $; or common to the planes of $ q $ and $ q' $; being also such that the rotation round it from $ Iv' $ to $ Iv $ is positive: while its length,

$$T_1 v = T_2 v = TV . \overrightarrow{v},$$

bears to the unit of length the same ratio, as that which the parallelogram under the indices, $ Iv $ and $ Iv' $, bears to the unit of area.

(5.) To interpret (comp. IV.) the scalar expression,

$$VII. \ldots S \overrightarrow{v} = S \overrightarrow{v} = S . v'' \overrightarrow{v},$$

(because $S \overrightarrow{v}'' = 0$), we may employ the formula 208, V.; which gives the transformation,

$$VIII. \ldots S \overrightarrow{v}'' = T_1 v'' . TV . \cos (\pi - x);$$

where $ T_1 v'' $ denotes the length of the line $ I v'' $, and $ TV $ represents by (4.) the area (positively taken) of the parallelogram under $ I v' $ and $ I v $; while $ x $ is (by
Art. 223.] ASSOCIATIVE PRINCIPLE OF MULTIPLICATION. 247

208), the angle between the two indices \(Iv', Iv\). This angle will be obtuse, and therefore the cosine of its supplement will be positive, and equal to the sine of the inclination of the line \(Iv''\) to the plane of \(Iv\) and \(Iv'\), if the rotation round \(Iv''\) from \(Iv\) to \(Iv'\) be negative, that is, if the rotation round \(Iv\) from \(Iv'\) to \(Iv''\) be positive; but that cosine will be equal the negative of this sine, if the direction of this rotation be reversed. We have therefore the important interpretation:

**IX.** \(Sv''v'v = \pm \text{volume of parallelepiped under } Iv, Iv', Iv''\);

the upper or the lower sign being taken, according as the rotation round \(Iv\), from \(Iv'\) to \(Iv''\), is positively or negatively directed.

(6.) For example, we saw that the ternary products \(ijk\) and \(kji\) have scalar values, namely,

\[ijk = -1, \quad kji = +1\] by 183, (1.), (2.);

and accordingly the parallelepiped of indices becomes, in this case, an unit-cube; while the rotation round the index of \(i\), from that of \(j\) to that of \(k\), is positive (181).

(7.) In general, for any three right quaternions \(vv'v''\), we have the formula,

**X.** \(Svv'v'' = -Sv''v'v\);

and when the three indices are complanar, so that the volume mentioned in IX. vanishes, then each of these two last scalars becomes zero; so that we may write, as a new Formula of Complanarity;

**XI.** \(Sv''v'v = 0\), if \(Iv'' ||| Iv', Iv\) (123):

while, on the other hand, this scalar cannot vanish in any other case, if the quaternions (or their indices) be still supposed to be actual (1, 144); because it then represents an actual volume.

(8.) Hence also we may establish the following Formula of Collinearity, for any three quaternions:

**XII.** \(S(Vq'' . Vq' . Vq) = 0\), if \(IVq'' ||| IVq', IVq\);

that is, by 209, if the planes of \(q, q', q''\) have any common line.

(9.) In general, if we employ the standard trinomial form, 221, II., namely,

\[v = Vq = ix + jy + kx, \quad v' = ix' + &c, \quad v'' = ix'' + &c,\]

the laws (182, 183) of the symbols \(i, j, k\) give the transformation,

**XIII.** \(Sv''v'v = x'' (z'y - y'z) + y'' (x'z - z'x) + z'' (y'x - x'y)\);
and accordingly this is the known expression for the volume (with a suitable
sign) of the parallelepiped, which has the three lines $op, op', op''$ for three co-
initial edges, if the rectangular co-ordinates* of the four corners, $o, p, p', p''$,
be $000, xyz, x'y'z', x''y''z''$.

(10.) Again, as another important consequence of the general associative
property of multiplication, it may be here observed, that although products
of more than two quaternions have not generally equal scalars, for all possible
permutations of the factors, since we have just seen a case $X.$ in which such
a change of arrangement produces a change of sign in the result, yet cyclical
permutation is permitted, under the sign $S$; or in symbols, that for any three
quaternions (and the result is easily extended to any greater number of such
factors) the following formula holds good:

$Sg'g'g = Sgg'g',$

In fact, to prove this equality, we have only to write it thus,

$S(g'g'g') = S(g'g'g'),$

and to remember that the scalar of the product of any two quaternions
remains unaltered (198, I.), when the order of those two factors is changed.

(11.) In like manner, by 192, II., it may be inferred that

$Kg'g'g' = K(g'g'g') = Kg'g'g' = Kg'g'g'',$

with a corresponding result for any greater number of factors; whence by
192, I., if $Pq$ and $P'q$ denote the products of any one set of quaternions taken
in two opposite orders, we may write,

$Kpq = p'Kq; Kppq = p'ppq.$

(12.) But if $v$ be right, as above, then $Kv = -v$, by 144; hence,

$Kv = -v; Siv = -siv; XX... v = \pm v; XX... v = \pm \pm v; XX... v = \pm \pm v;$

upper or lower signs being taken, according as the number of the right factors
is even or odd; and under the same conditions,

$Siv = \frac{1}{2} (Iv \pm Iv); XX... v = \frac{1}{2} (Iv \mp Iv);$

as was lately exemplified (1.), for the case where the number is two.

* This result may serve as an example of the manner in which quaternions, although not based on
any usual doctrine of co-ordinates, may yet be employed to deduce, or to recover, and often with great
case, important co-ordinate expressions.
(13.) For the case where that number is three, the four last formulae give,

\[ XXIII \ldots Sv''v'v = -Sv''v'' = \frac{1}{2}(v''v'v - vv''v'); \]
\[ XXIV \ldots Vv''v'v = +Vv''v'' = \frac{1}{2}(v''v'v + vv''v'); \]

results which obviously agree with X. and IV.

224. For the case of Complanar Quaternions (119), the power of reducing each (120) to the form of a fraction (101) which shall have, at pleasure, for its denominator or for its numerator, any arbitrary line in the given plane, furnishes some peculiar facilities for proving the commutative and associative properties of Addition (207), and the distributive and associative properties of Multiplication (212, 223); while, for this case of multiplication of quaternions, we have already seen (191, (1.)) that the commutative property also holds good, as it does in algebraic multiplication. It may therefore be not irrevelant nor useless to insert here a short Second Chapter on the subject of such complanars: in treating briefly of which, while assuming as proved the existence of all the foregoing properties, we shall have an opportunity to say something of Powers and Roots and Logarithms; and of the connexion of Quaternions with Plane Trigonometry, and with Algebraical Equations. After which, in the Third and last Chapter of this Second Book, we propose to resume, for a short time, the consideration of Diplanar Quaternions; and especially to show how the Associative Principle of Multiplication can be established, for them, without* employing the Distributive Principle.

* Compare the Note to page 245.
CHAPTER II.

ON COMPLANAR QUATERNIONS, OR QUOTIENTS OF VECTORS IN ONE PLANE; AND ON POWERS, ROOTS, AND LOGARITHMS OF QUATERNIONS.

SECTION 1.

On Complanar Proportion of Vectors; Fourth Proportional to Three, Third Proportional to Two, Mean Proportional, Square Root; General Reduction of a Quaternion in a Given Plane, to a Standard Binomial Form.

225. The Quaternions of the present Chapter shall all be supposed to be complanar (119); their common plane being assumed to coincide with that of the given right versor * (181). And all the lines, or vectors, such as a, β, γ, &c., or α₀, α₁, α₂, &c., to be here employed, shall be conceived to be in that given plane of i; so that we may write (by 123), for the purposes of this Chapter, the formula of complanarity:

\[ q \parallel q' \parallel q'' \ldots \parallel i; \quad a \parallel i, \quad \beta \parallel i, \quad a₀ \parallel i, \quad &c. \]

226. Under these conditions, we can always (by 103, 117) interpret any symbol of the form \((\beta : a). \gamma\), as denoting a line \(\delta\) in the given plane; which line may also be denoted (125) by the symbol \((\gamma : a). \beta\), but not* (comp. 103) by either of the two apparently equivalent symbols, \((\beta . \gamma) : a, (\gamma . \beta) : a\); so that we may write,

\[ \delta = \frac{\beta}{a} \gamma = \frac{\gamma}{a} \beta, \]

and may say that this line \(\delta\) is the Fourth Proportional to the three lines \(a, \beta, \gamma\); or to the three lines \(a, \gamma, \beta\); the two Means, \(\beta\) and \(\gamma\), of any such

* In fact the symbols \(\beta . \gamma, \gamma . \beta, \) or \(\beta \gamma, \gamma \beta\), have not as yet received with us any interpretation; and even when they shall come to be interpreted as representing certain quaternions, it will be found (comp. 168) that the two combinations, \(\frac{\beta}{a} \gamma \) and \(\frac{\gamma}{a} \beta\), have generally different significations.
Complanar Proportion of Four Vectors, admitting thus of being interchanged, as in algebra. Under the same conditions we may write also (by 125),

\[ \Pi \ldots a = \frac{\beta}{\delta} \gamma = \gamma \frac{\beta}{\delta}; \quad \beta = \frac{a}{\gamma} = \frac{\gamma}{\delta}; \quad \gamma = \frac{\delta}{\beta} a = \frac{a}{\beta} \delta; \]

so that (still as in algebra) the two Extremes, \( a \) and \( \delta \), of any such proportion of four lines \( a, \beta, \gamma, \delta \), may likewise change places among themselves: while we may also make the means become the extremes, if we at the same time change the extremes to means. More generally, if \( a, \beta, \gamma, \delta, \ldots \) be any odd number of vectors in the given plane, we can always find another vector \( \rho \) in that plane, which shall satisfy the equation,

\[ \text{III.} \ldots \frac{\delta}{\beta} \frac{\gamma}{a} = \rho, \text{ or } \text{III'.} \ldots \frac{\delta}{\beta} \frac{\gamma}{a} = 1; \]

and when such a formula holds good, for any one arrangement of the numerator-lines \( a, \gamma, \varepsilon, \ldots \) and of the denominator-lines \( \rho, \beta, \delta \ldots \) it can easily be proved to hold good also for any other arrangement of the numerators, and any other arrangement of the denominators. For example, whatever four (complanar) vectors may be denoted by \( f \gamma \beta \varepsilon \), we have the transformations,

\[ \text{IV.} \ldots \frac{\varepsilon}{\delta} \frac{\gamma}{\beta} = \frac{\varepsilon}{\delta} \gamma; \beta = \frac{\gamma}{\delta} \varepsilon; \gamma = \frac{\delta}{\beta} \varepsilon, \]

the two numerators being thus interchanged. Again,

\[ \text{IV'.} \ldots \frac{\varepsilon}{\delta} \frac{\gamma}{\beta} = \frac{\varepsilon}{\delta} \gamma; \frac{\gamma}{\beta} \frac{\delta}{\gamma} = \frac{\varepsilon}{\beta} \gamma, \text{ by IV.}; \]

so that the two denominators also may change places. 

227. An interesting case of such proportion (22b*) is that in which the means coincide; so that only three distinct lines, such as \( a, \beta, \gamma \), are involved; and that we have (comp. Art. 149, and fig. 42 [p. 133]) an equation of the form,

\[ I. \ldots \gamma = \frac{\beta}{a} \beta, \text{ or } a = \frac{\beta}{\gamma} \beta; \]

but not* \( \gamma = \beta \beta : a \), nor \( a = \beta \beta : \gamma \). In this case, it is said that the three lines \( a \beta \gamma \) form a Continued Proportion; of which \( a \) and \( \gamma \) are now the Extremes, and \( \beta \) is the Mean; this line \( \beta \) being also said to be at† Mean Proportional

* Compare the Note to the foregoing Article.
† We say, a mean proportional; because we shall shortly see that the opposite line, \( -\beta \), is in the same sense another mean; although a rule will presently be given, for distinguishing between them, and for selecting one as that which may be called, by eminence, the mean proportional.
between the two others, \(a\) and \(\gamma\); while \(\gamma\) is the Third Proportional to the two lines \(a\) and \(\beta\); and \(a\) is, at the same time, the third proportional to \(\gamma\) and \(\beta\). Under the same conditions, we have

\[
\beta = \frac{a}{\beta} \gamma = \frac{\gamma}{\beta} a;
\]

so that this mean, \(\beta\), between \(a\) and \(\gamma\), is also the fourth proportional \((226)\) to itself, as first, and to those two other lines. We have also (comp. again 149),

\[
\beta = \frac{a}{\gamma} \gamma = \frac{\gamma}{\beta} a;
\]

whence it is natural to write,

\[
\beta = \frac{\gamma}{a} a,
\]

and therefore by \((103)\),

\[
\gamma = \frac{\beta}{a} a,
\]

although we are not here to write \(\beta = (\gamma a)^{\frac{1}{2}}\), nor \(\beta = (\alpha \gamma)^{\frac{1}{2}}\). But because we have always, as in algebra (comp. 199, (3.) ), the equation or identity, \((- q)^{2} = q^{2}\), we are equally well entitled to write,

\[
(- \beta)^{2} = \gamma,
\]

the symbol \(q^{2}\) denoting thus, in general, either of two opposite quaternions, whereof however one, namely that one of which the angle is acute, has been already selected in 199, (1.), as that which shall be called by us the Square Root of the quaternion \(q\), and denoted by \(\sqrt{q}\). We may therefore establish the formula,

\[
\beta = \pm \sqrt{\left(\frac{\gamma}{a}\right)} a = \pm \sqrt{\left(\frac{a}{\gamma}\right)} \gamma,
\]

if \(a\), \(\beta\), \(\gamma\) form, as above, a continued proportion; the upper signs being taken when (as in fig. 42) the angle \(\angle aoc\), between the extreme lines \(a\), \(\gamma\), is bisected by the line on, or \(\beta\), itself; but the lower signs, when that angle is bisected by the opposite line, \(- \beta\), or when \(\beta\) bisects the vertically opposite angle (comp. again 199, (3.) ) : but the proportion of tensors,

\[
T_{\gamma} : T\beta = T_{\beta} : T_{a},
\]

and the resulting formulæ

\[
T_{\beta} = T_{a} \cdot T_{\gamma}, \quad T\beta = \sqrt{(T_{a} \cdot T_{\gamma})},
\]

in each case holding good. And when we shall speak simply of the Mean
Proportional between two vectors, \( a \) and \( \gamma \), which make any acute, or right, or obtuse angle with each other, we shall always henceforth understand the former of these two bisectors; namely, the bisector \( ob \) of that angle \( aoc \) itself, and not that of the opposite angle: thus taking upper signs, in the recent formula VII. \( \sqrt{} \)

(1.) At the limit when the angle \( aoc \) vanishes, so that \( U\gamma = Ua \), then \( U\beta \) = each of these two unit-lines; and the mean proportional \( \beta \) has the same common direction as each of the two given extremes. This comes to our agreeing to write,

\[
X \ldots \sqrt{1} = + 1, \text{ and generally, } X' \ldots \sqrt{(a^2)} = + a,
\]

if \( a \) be any positive scalar.

(2.) At the other limit, when \( aoc = \pi \), or \( U\gamma = - Ua \), the length of the mean proportional \( \beta \) is still determined by IX., as the geometric mean (in the usual sense) between the lengths of the two given extremes (comp. the two figures 41 [p. 132]); but, even with the supposed restriction (225) on the plane in which all the lines are situated, an ambiguity arises in this case, from the doubt which of the two opposite perpendiculars at \( o \), to the line \( aoc \), is to be taken as the direction of the mean vector. To remove this ambiguity, we shall suppose that the rotation round the axis of \( i \) (to which axis all the lines considered in this Chapter are, by 225, perpendicular), from the first line \( oa \) to the second line \( ob \), is in this case positive; which supposition is equivalent to writing, for present purposes,

\[
XI. \ldots \sqrt{-1} = + i; \text{ and } XI'. \ldots \sqrt{(-a^2)} = ia, \text{ if } a > 0.
\]

And thus the mean proportional between two vectors (in the given plane) becomes, in all cases, determined: at least if their order (as first and third) be given.

(3.) If the restriction (225) on the common plane of the lines, were removed, we might then, on the recent plan (227), fix definitely the direction, as well as the length, of the mean ob, in every case but one: this excepted case being that in which, as in (2.), the two given extremes, \( oa, oc \), have exactly opposite directions; so that the angle \( (aoc = \pi) \) between them has no one definite bisector. In this case, the sought point \( b \) would have no one determined position, but only a locus: namely the circumference of a circle, with \( o \) for

* It is to be carefully observed that this square root of negative unity is not, in any sense, imaginary, nor even ambiguous, in its geometrical interpretation, but denotes a real and given right versor (181).
centre, and with a radius equal to the geometric mean between $\overline{OA}$, $\overline{OC}$, while
its plane would be perpendicular to the given right line $\overline{AC}$. (Comp. again
the figures 41; and the remarks in 148, 149, 153, 154, on the square of a
right radial, or versor, and on the partially indeterminate character of the
square root of a negative scalar, when interpreted, on the plan of this Calculus,
as a real in geometry.)

228. The quotient of any two complanar and right quaternions has been
seen (191, (6.)) to be a scalar; since then we here suppose (225) that $q \parallel i$, we
are at liberty to write,

I. $S q = x$;  $V q : i = y$;  $V q = yi = iy$;

and consequently may establish the following Reduction of a Quaternion in the
given plane (of $i$) to a Standard Binomial Form* (comp. 221):

II. $q = x + iy$, if $q \parallel i$;

$x$ and $y$ being some two scalars, which may be called the two constituents (comp.
again 221) of this binomial. And then an equation between two quaternions,
considered as binomials of this form, such as the equation,

III. $q' = q$, or III'. $x' + iy' = x + iy$,

breaks up (by 202, (5.)) into two scalar equations between their respective
constituents, namely,

IV. $x' = x$,  $y' = y$,

notwithstanding the geometrical reality of the right versor, $i$.

(1.) On comparing the recent equations II., III., IV., with those marked
as III., V., VI, in 221, we see that, in thus passing from general to complanar
quaternions, we have merely suppressed the coefficients of $j$ and $k$, as being for
our present purpose, null; and have then written $x$ and $y$, instead of $w$
and $x$.

(2.) As the word "binomial" has other meanings in algebra, it may be
convenient to call the form II. a Couple; and the two constituent scalars $x$
and $y$, of which the values serve to distinguish one such couple from another,
may not unnaturally be said to be the Co-ordinates of that Couple, for a reason
which it may be useful to state.

* It is permitted, by 227, XI., to write this expression as $x + y \sqrt{-1}$; but the form $x + iy$ is
shorter, and perhaps less liable to any ambiguity of interpretation.
(3.) Conceive, then, that the plane of fig. 50 [p. 192] coincides with that of \( i \), and that positive rotation round \( \text{Ax} \cdot i \) is, in that figure, directed towards the left-hand; which may be reconciled with our general convention (127), by imagining that this axis of \( i \) is directed from \( o \) towards the back of the figure; or below* it, if horizontal. This being assumed, and perpendiculars \( \text{bb}', \text{bb}'' \) being let fall (as in the figure) on the indefinite line \( \text{oa} \) itself, and on a normal to that line at \( o \), which normal we may call \( \text{oa}' \), and may suppose it to have a length equal to that of \( \text{oa} \), with a left-handed rotation \( \text{aoa}' \), so that

\[
V. \ldots \text{oa}' = i \cdot \text{oa}, \text{ or briefly, } V'. \ldots a' = ia,
\]
while

\[
\beta' = \text{ob}', \text{ and } \beta'' = \text{ob}'', \text{ as in 201, and } q = \beta : a, \text{ as in 202};
\]
then, on whichever side of the indefinite right line \( \text{oa} \) the point \( b \) may be situated, a comparison of the quaternion \( q \) with the binomial form \( II. \) will give the two equations,

\[
VI. \ldots x ( = \text{S}q) = \beta' : a; \quad y ( = \text{V}q : i = \beta'' : ia) = \beta' : a';
\]
so that these two scalars, \( x \) and \( y \), are precisely the two rectangular co-ordinates of the point \( b \), referred to the two lines \( \text{oa} \) and \( \text{oa}' \), as two rectangular unit-axes, of the ordinary (or Cartesian) kind. And since every other quaternion, \( q' = ax' + iy' \), in the given plane, can be reduced to the form \( \gamma : a \), or \( \text{oc is to } \text{oa} \), where \( c \) is a point in that plane, which can be projected into \( c' \) and \( c'' \) in the same way (comp. 197, 205), we see that the two new scalars, or constituents, \( x' \) and \( y' \), are simply (for the same reason) the co-ordinates of the new point \( c \), referred to the same pair of axes.

(4.) It is evident (from the principles of the foregoing Chapter), that if we thus express as couples (2.) any two complanar quaternions, \( q \) and \( q' \), we shall have the following general transformations for their sum, difference, and product:

\[
VII. \ldots q' + q = (x' + x) + i (y' + y);
\]
\[
VIII. \ldots q' \cdot q = (ax' - y'y) + i (x'y + y'x).
\]

(5.) Again, for any one such couple, \( q \), we have (comp. 222) not only \( \text{S}q = x \), and \( \text{V}q = iy \), as above, but also,

\[
IX. \ldots \text{K}q = x - iy; \quad X. \ldots \text{N}q = x^2 + y^2; \quad XI. \ldots \text{T}q = \sqrt{x^2 + y^2};
\]
\[
XII. \ldots \text{U}q = \frac{x + iy}{\sqrt{x^2 + y^2}}; \quad XIII. \ldots \frac{1}{q} = \frac{x - iy}{x^2 + y^2}; \text{ &c.}
\]

* Compare the second Note to page 111.
(6.) Hence, for the quotient of any two such couples, we have,

\[ \frac{q'}{q} = \frac{x' + iy'}{a + iy}, \quad \frac{x'' + iy''}{a^2 + y^2}, \quad \frac{x' + iy'}{q'Kq}, \]

\[ x'' = x'x + y'y, \quad y'' = y'x - x'y' \]

(7.) The law of the norms (191, (8.),) or the formula, \( Nq'q = Nq' \cdot Nq \), is expressed here (comp. 222, (3.)) by the well-known algebraic equation, or identity,

\[ \ldots (x' + y') (x' + y') = (x'x - y'y)^2 + (x'y + y'x)^2; \]

in which \( ax'x'y' \) may be any four scalars.

SECTION 2.

On Continued Proportion of Four or more Vectors; Whole Powers and Roots of Quaternions; and Roots of Unity.

229. The conception of continued proportion (227) may easily be extended from the case of three to that of four or more (complanar) vectors; and thus a theory may be formed of cubes and higher whole powers of quaternions, with a correspondingly extended theory of roots of quaternions, including roots of scalars, and in particular of unity. Thus if we suppose that the four vectors \( a\beta\gamma\delta \) form a continued proportion, expressed by the formula,

\[ \frac{\delta}{\gamma} = \frac{\beta}{a}, \quad \text{whence} \quad \frac{\delta}{a} = \frac{\gamma}{\beta} \cdot \frac{\beta}{a} = \left(\frac{\beta}{a}\right)^2 \]

(by an obvious extension of usual algebraic notation,) we may say that the quaternion \( \delta : a \) is the cube, or the third power, of \( \beta : a \); and that the latter quaternion is, conversely, a cube-root (or third root) of the former; which last relation may naturally be denoted by writing,

\[ \ldots \frac{\beta}{a} = \left(\frac{\delta}{a}\right)^{\frac{1}{3}}, \quad \text{or} \quad \ldots \beta = \left(\frac{\delta}{a}\right)^{\frac{1}{3}} \cdot a \quad \text{(comp. 227, IV., V.).} \]

230. But it is important to observe that as the equation \( q^2 = Q \), in which \( q \) is a sought and \( Q \) is a given quaternion, was found to be satisfied by two opposite quaternions \( q \), of the form \( \pm \sqrt{Q} \) (comp. 227, VII.), so the slightly less simple equation \( q^3 = Q \) is satisfied by three distinct and real quaternions, if \( Q \) be actual and real; whereof each, divided by either of the other two, gives for quotient a real quaternion, which is equal to one of the cube-roots of
positive unity. In fact, if we conceive (comp. the annexed fig. 54) that \(\beta'\) and \(\beta''\) are two other but equally long vectors in the given plane, obtained from \(\beta\) by two successive and positive rotations, each through the third part of a circumference, so that

\[
\text{IV}\ldots \frac{\beta}{\beta'} = \frac{\beta''}{\beta'} = \frac{\beta'}{\beta'},
\]

or

\[
\text{IV}'\ldots \frac{\beta}{\beta'} = \frac{\beta'}{\beta'} = \frac{\beta''}{\beta'},
\]

and therefore

\[
\text{V}\ldots \left(\frac{\beta'}{\beta}\right)^3 = \left(\frac{\beta''}{\beta}\right)^3 = 1, \quad \text{while} \quad \text{V}'.\ldots \frac{\beta'}{\beta} = \left(\frac{\beta'}{\beta}\right)^2, \quad \frac{\beta'}{\beta} = \left(\frac{\beta''}{\beta}\right)^3,
\]

we shall have

\[
\text{VI}\ldots \left(\frac{\beta''}{a}\right)^3 = \left(\frac{\beta'}{a}\right)^3 = \frac{\delta}{a}, \quad \text{and} \quad \text{VI}'\ldots \left(\frac{\beta}{a}\right)^3 = \frac{\delta}{a};
\]

so that we are equally entitled, at this stage, to write, instead of III. or III', these other equations:

\[
\text{VII}\ldots \frac{\beta'}{a} = \left(\frac{\delta}{a}\right)^\frac{1}{3}, \quad \frac{\beta'}{a} = \left(\frac{\delta}{a}\right)^\frac{1}{3} a;
\]

or

\[
\text{VII}'\ldots \frac{\beta''}{a} = \left(\frac{\delta}{a}\right)^\frac{1}{3}, \quad \frac{\beta''}{a} = \left(\frac{\delta}{a}\right)^\frac{1}{3} a.
\]

231. A (real and actual) quaternion \(Q\) may thus be said to have three (real, actual, and) distinct cube-roots; of which however only one can have an angle less than sixty degrees; while none can have an angle equal to sixty degrees, unless the proposed quaternion \(Q\) degenerates into a negative scalar. In every other case, one of the three cube-roots of \(Q\), or one of the three values of the symbol \(Q^\frac{1}{3}\), may be considered as simpler than either of the other two, because it has a smaller angle (comp. 199, (1.)); and if we, for the present, denote this one, which we shall call the Principal Cube-Root of the quaternion \(Q\), by the symbol \(\sqrt[3]{Q}\), we shall thus be enabled to establish the formula of inequality,

\[
\text{VIII}\ldots \angle \sqrt[3]{Q} < \frac{\pi}{3}, \quad \text{if} \quad \angle Q < \pi.
\]

232. At the limit, when \(Q\) degenerates, as above, into a negative scalar, one of its cube-roots is itself a negative scalar, and has therefore its angle = \(\pi\); while each of the two other roots has its angle = \(\frac{\pi}{3}\). In this case, among these two roots of which the angles are equal to each other, and are less than that
of the third, we shall consider as simpler, and therefore as principal, the one which answers (comp. 227, (2.)) to a positive rotation through sixty degrees; and so shall be led to write,

\[
\sqrt[3]{-1} = \frac{1 + i \sqrt{3}}{2}; \quad \text{and} \quad \sqrt[3]{\sqrt[3]{-1}} = \frac{\pi}{3};
\]

using thus the positive sign for the radical \(\sqrt[3]{3}\), by which \(i\) is multiplied in the expression IX. for \(2 \sqrt[3]{-1}\); with the connected formula,

\[
\sqrt[3]{(-a^3)} = \frac{a}{2} (1 + i \sqrt{3}), \quad \text{if} \quad a > 0;
\]

although it might at first have seemed more natural to adopt as principal the scalar value, and to write thus,

\[
\sqrt[3]{-1} = -1;
\]

which latter is in fact one value of the symbol, \((-1)^3\).

(1.) We have, however, on the present plan, as in arithmetic,

\[
\sqrt[3]{1} = 1; \quad \text{and} \quad \sqrt[3]{(a^3)} = a, \quad \text{if} \quad a > 0.
\]

(2.) The equations,

\[
\sqrt[3]{(1 + i \sqrt{3})^3} = -1, \quad \text{and} \quad \sqrt[3]{(-1 + i \sqrt{3})^3} = +1,
\]

can be verified in calculation, by actual cubing, exactly as in algebra; the only difference being, as regards the conception of the subject, that although \(i\) satisfies the equation \(i^2 = -1\), it is regarded here as altogether real; namely, as a real right versor* (181).

233. There is no difficulty in conceiving how the same general principles may be extended (comp. 229) to a continued proportion of \(n + 1\) complanar vectors,

\[
I \ldots a, a_1, a_2, \ldots a_n,
\]

when \(n\) is a whole number greater than three; nor in interpreting, in connexion therewith, the equations,

\[
\Pi \ldots \frac{a_n}{a} = \left(\frac{a_1}{a}\right)^n; \quad \text{III.} \ldots \frac{a_1}{a} = \left(\frac{a_2}{a}\right)^\frac{1}{2}; \quad \text{IV.} \ldots a_1 = \left(\frac{a_n}{a}\right)^\frac{n}{2} a.
\]

* This conception differs fundamentally from one which had occurred to several able writers, before the invention of the quaternions; and according to which the symbols 1 and \(\sqrt{-1}\) were interpreted as representing a pair of equally long and mutually rectangular right lines, in a given plane. In Quaternions, no line is represented by the number, One, except as regards its length; the reason being, mainly, that we require, in the present Calculus, to be able to deal with all possible planes; and that no one right line is common to all such.
Denoting, for the moment, what we shall call the principal \( n\)th root of a quaternion \( Q \) by the symbol \( \sqrt[n]{Q} \), we have, on this plan (comp. 231, VIII.),

\[
\text{V. } \angle (\sqrt[n]{Q}) < \frac{\pi}{n}, \text{ if } \angle Q < \pi;
\]

\[
\text{VI. } \angle (\sqrt[n]{-1}) = \frac{\pi}{n}; \quad \text{VII. } V (\sqrt[n]{-1}) : i > 0;
\]

this last condition, namely that there shall be a positive (scalar) coefficient \( y \) of \( i \), in the binomial (or couple) form \( x + iy \) (228), for the quaternion \( \sqrt[n]{-1} \), thus serving to complete the determination of that principal \( n\)th root of negative unity; or of any other negative scalar, since \(-1\) may be changed to \(-a\), if \( a > 0 \), in each of the two last formulae. And as to the general \( n\)th root of a quaternion, we may write, on the same principles,

\[
\text{VIII. } Q^{\frac{1}{n}} = 1^{\frac{1}{n}} \cdot \sqrt[n]{Q};
\]

where the factor \( 1^{\frac{1}{n}} \), representing the general \( n\)th root of positive unity, has \( n \) different values, depending on the division of the circumference of a circle into \( n \) equal parts, in the way lately illustrated, for the case \( n = 3 \), by figure 54; and only differing from ordinary algebra by the reality here attributed to \( i \). In fact, each of these \( n\)th roots of unity is with us a real versor; namely the quotient of two radii of a circle, which make with each other an angle, equal to the \( n\)th part of some whole number of circumferences.

(1.) We propose, however, to interpret the particular symbol \( i^{\frac{1}{n}} \), as always denoting the principal value of the \( n\)th root of \( i \); thus writing,

\[
\text{IX. } i^{\frac{1}{n}} = \sqrt[n]{i};
\]

whence it will follow that when this root is expressed under the form of a couple (228), the two constituents \( x \) and \( y \) shall both be positive, and the quotient \( y : x \) shall have a smaller value than for any other couple \( x + iy \) (with constituents thus positive), of which the \( n\)th power equals \( i \).

(2.) For example, although the equation

\[
q^2 = (x + iy)^2 = i,
\]

is satisfied by the two values, \( \pm (1 + i) : \sqrt{2} \), we shall write definitely,

\[
\text{X. } i^{\frac{1}{2}} = + \sqrt{i} = \frac{1 + i}{\sqrt{2}}.
\]

(3.) And although the equation,

\[
q^3 = (x + iy)^3 = i,
\]
is satisfied by the three distinct and real couples, \((i \pm \sqrt{3}) : 2\), and \(-i\), we shall adopt only the one value,

\[
\text{XI} \ldots i^k = \frac{2}{i} = \frac{i + \sqrt{3}}{2}.
\]

(4.) In general, we shall thus have the expression,

\[
\text{XII} \ldots i^n = \cos \frac{\pi}{2n} + i \sin \frac{\pi}{2n};
\]

which we shall occasionally abridge to the following:

\[
\text{XII'} \ldots i^n = \text{cis} \frac{\pi}{2n};
\]

and this root, \(i^n\), thus interpreted, denotes a versor, which turns any line on which it operates, through an angle equal to the \(n^{th}\) part of a right angle, in the positive direction of rotation, round the given axis of \(i\).

234. If \(m\) and \(n\) be any two positive whole numbers, and \(q\) any quaternion, the definition contained in the formula 233, II., of the whole power, \(q^n\), enables us to write, as in algebra, the two equations:

\[
\text{I} \ldots q^m q^n = q^{mn}; \quad \text{II} \ldots (q^n)^m = q^{mn};
\]

and we propose to extend the former to the case of null and negative whole exponents, writing therefore,

\[
\text{III} \ldots q^0 = 1; \quad \text{IV} \ldots q^{m-n} = q^m : q^n;
\]

and in particular,

\[
\text{V} \ldots q^{-1} = 1 : q = \frac{1}{q} = \text{reciprocal*} (134) \text{ of } q.
\]

We shall also extend the formula II., by writing,

\[
\text{VI} \ldots \left(q^n\right)^m = q^{nm},
\]

whether \(m\) be positive or negative; so that this last symbol, if \(m\) and \(n\) be still whole numbers, whereof \(n\) may be supposed to be positive, has as many distinct values as there are units in the denominator of its fractional exponent, when reduced to its least terms; among which values of \(q^n\), we shall naturally consider as the principal one, that which is the \(m^{th}\) power of the principal \(n^{th}\) root (233) of \(q\).

* Compare the first Note to page 123.
(1.) For example, the symbol $g^3$ denotes, on this plan, the *square of any cube-root* of $q$; it has therefore *three* distinct values, namely, the three values of the *cube-root of the square* of the same quaternion $q$; but among these we regard as principal, the *square of the principal cube-root* (231) of that proposed quaternion.

(2.) Again, the symbol $q^4$ is interpreted, on the same plan, as denoting the *square of any fourth root* of $q$; but because $(1^4)^2 = 1^8 = \pm 1$, this *square* has only *two* distinct values, namely those of the *square root* $q^4$, the fractional exponent $\frac{4}{4}$ being thus reduced to its least terms; and among these the principal value is the *square of the principal fourth root*, which square is, at the same time, the principal *square root* (199, (1.), or 227) of the quaternion $q$.

(3.) The symbol $g^{-1}$ denotes, as in algebra, the reciprocal of a square-root of $q$; while $g^{-2}$ denotes the reciprocal of the square, &c.

(4.) If the exponent $t$, in a symbol of the form $q^t$, be still a *scalar*, but a *surd* (or incommensurable), we may consider this *surd exponent*, $t$, as a *limit*, towards which a *variable fraction tends*; and the symbol itself may then be interpreted as the corresponding limit of a *fractional power* of a quaternion, which has however (in this case) *indefinitely many values*, and can therefore be of little or no *use*, until a *selection* shall have been made, of *one* value of this *surd power* as principal, according to a law which will be best understood by the introduction of the conception of the *amplitude* of a quaternion, to which in the next section we shall proceed.

(5.) Meanwhile (comp. 233, (4.)), we may already *definitely interpret* the symbol $i^t$ as denoting a *versor*, which *turns* any line in the given plane, *through* $t$ *right angles*, round $\text{Ax} \cdot i$, in the positive or negative direction, according as this *scalar exponent*, $t$, whether rational or irrational, is itself positive or negative; and thus may establish the formula,

$$\text{VII.}\ldots i^t = \cos \frac{t\pi}{2} + i \sin \frac{t\pi}{2};$$

or briefly (comp. 233, XII'),

$$\text{VIII.}\ldots i^t = \text{cis} \frac{t\pi}{2}. $$
SECTION 3.

On the Amplitudes of Quaternions in a given Plane; and on Trigonometric Expressions for such Quaternions, and for their Powers.

235. Using the binomial or couple form (228) for a quaternion in the plane of \( i \) (225), if we introduce two new and real scalars, \( r \) and \( z \), whereof the former shall be supposed to be positive, and which are connected with the two former scalars \( x \) and \( y \) by the equations,

\[
\begin{align*}
I. & \quad x = r \cos z, \quad y = r \sin z, \quad r > 0, \\
II. & \quad Tq = T(x + iy) = r; \\
III. & \quad U_q = U(x + iy) = \cos z + i \sin z;
\end{align*}
\]

which last expression may be conveniently abridged (comp. 233, XII', and 234, VIII.) to the following:

\[
\begin{align*}
IV. & \quad U_q = \text{cis} \, z; \quad \text{so that} \quad V. \quad q = r \text{cis} \, z.
\end{align*}
\]

And the areal or angular quantity, \( z \), may be called the Amplitude* of the quaternion \( q \); this name being here preferred by us to "Angle," because we have already appropriated the latter name, and the corresponding symbol \( \angle q \), to denote (130) an angle of the Euclidean kind, or at least one not exceeding, in either direction, the limits 0 and \( \pi \); whereas the amplitude, \( z \), considered as obliged only to satisfy the equations I., may have any real and scalar value. We shall denote this amplitude, at least for the present, by the symbol, \( \am \cdot q \),

* Compare the Note to Art. 130.

† The symbol \( V \) was spoken of, in 202, as completing the system of notations peculiar to the present Calculus; and in fact, besides the three letters, \( i, j, k \), of which the laws are expressed by the fundamental formula (A) of Art. 183, and which were originally (namely in the year 1843, and in the two following years) the only peculiar symbols of quaternions (see Note to page 160), that Calculus does not habitually employ, with peculiar significations, any more than the five characteristics of operation, \( K, S, T, U, V \), for conjugate, scalar, tensor, versor, and vector (or right part): although perhaps the mark \( N \) for norm, which in the present work has been adopted from the Theory of Numbers, will gradually come more into use than it has yet done, in connexion with quaternions also. As to the marks, \( \angle \), \( A \), \( R \), and now \( \am \cdot \), for angle, axis, index, reciprocal, and amplitude, they are to be considered as chiefly available for the present exposition of the system, and as not often wanted, nor employed, in the subsequent practice thereof; and the same remark applies to the recent abridgment cis, for \( \cos + i \sin \); to some notations in the present section for powers and roots, serving to express the conception of one \( n \)th root, &c., as distinguished from another; and to the characteristic \( P \), of what we shall call in the next section the potential of a quaternion, though not requiring that notation afterwards. No apology need be made for employing the purely geometrical signs, \( \perp, \parallel, \| \parallel \| \), for perpendicularity, parallelism, and complanarity: although the last of them was perhaps first introduced by the present writer, who has found it frequently useful.
or simply, \( \text{am } q \); and thus shall have the following formula, of connexion between amplitude and angle,

\[
\text{VI. } (z =) \text{am } q = 2n\pi \pm \angle q ;
\]

the upper or the lower sign being taken, according as \( \text{Ax. } q = \pm \text{Ax. } i \); and \( n \) being any whole number, positive or negative or null. We may then write also (for any quaternion \( q \parallel i \)) the general transformations following:

\[
\text{VII. } U_q = \text{cis am } q ; \quad \text{VIII. } q = T_q \cdot \text{cis am } q .
\]

(1.) Writing \( q = \beta : a \), the amplitude \( \text{am. } q \), or \( \text{am } (\beta : a) \), is thus a scalar quantity, expressing (with its proper sign) the amount of rotation, round \( \text{Ax. } i \), from the line \( a \) to the line \( \beta \); and admitting, in general, of being increased or diminished by any whole number of circumferences, or of entire revolutions, when only the directions of the two lines, \( a \) and \( \beta \), in the given plane of \( i \), are given.

(2.) But the particular quaternion, or right versor, \( i \) itself, shall be considered as having definitely, for its amplitude, one right angle; so that we shall establish the particular formula,

\[
\text{IX. } \text{am. } i = \angle i = \frac{\pi}{2} .
\]

(3.) When, for any other given quaternion \( q \), the generally arbitrary integer \( n \) in VI. receives any one determined value, the corresponding value of the amplitude may be denoted by either of the two following temporary symbols,* which we here treat as equivalent to each other,

\[
\text{am}_n q, \quad \text{or } \angle_n q ;
\]

so that (with the same rule of signs as before) we may write, as a more definite formula than VI., the equation:

\[
\text{X. } \text{am}_n q = \angle_n q = 2n\pi \pm \angle q ;
\]

and may say that this last quantity is the \( n^{th} \) value of the amplitude of \( q \); while the zero-value, \( \text{am}_0 q \), may be called the principal amplitude (or the principal value of the amplitude).

(4.) With these notations, and with the convention, \( \text{am}_0 (-1) = + \pi \), we may write,

\[
\text{XI. } \text{am}_0 q = \angle_0 q = \pm \angle q ; \quad \text{XII. } \text{am}_n a = \text{am}_n 1 = \angle_n 1 = 2n\pi, \quad \text{if } a > 0 ;
\]

and

\[
\text{XIII. } \text{am}_n (-a) = \text{am}_n (-1) = \angle_n (-1) = (2n + 1)\pi,
\]

if \( a \) be still a positive scalar.

* Compare the recent Note, respecting the notations employed.
236. From the foregoing definition of amplitude, and from the formerly established connexion of multiplication of versors with composition of rotations (207), it is obvious that (within the given plane, and with abstraction made of tensors) multiplication and division of quaternions answer respectively to (algebraical) addition and subtraction of amplitudes: so that, if the symbol \(\text{am} \cdot q\) be interpreted in the general (or indefinite) sense of the equation 235, VI., we may write:

\[
\text{I. am}(q' \cdot q) = \text{am} q' + \text{am} q; \quad \text{II. am}(q' : q) = \text{am} q' - \text{am} q;
\]

implying hereby that, in each formula, one of the values of the first member is among the values of the second member; but not here specifying which value. With the same generality of signification, it follows evidently that, for a product of any number of (complanar) quaternions, and for a whole power of any one quaternion, we have the analogous formulae:

\[
\text{III. am} \Pi g = \Sigma \text{am} q; \quad \text{IV. am} \cdot q^p = p \cdot \text{am} q;
\]

where the exponent \(p\) may be any positive or negative integer, or zero.

(1.) It was proved, in 191, II., that for any two quaternions, the formula \(Ug'q = Ug'.Uq\) holds good; a result which, by the associative principle of multiplication (223), is easily extended to any number of quaternion factors (complanar or diplanar), with an analogous result for tensors: so that we may write, generally,

\[
\text{V. } U^n g = nUg; \quad \text{VI. } T^n g = nTg.
\]

(2.) Confining ourselves to the first of these two equations, and combining it with III., and with 235, VII., we arrive at the important formula:

\[
\text{VII. } \Pi \text{cis am} q = \text{cis} 2\text{am} q; \quad \text{VIII. } \Pi \text{cis am} q^p = \text{cis}(p \cdot \text{am} q),
\]

whence in particular (comp. IV.),

\[
\text{IX. } \Pi \text{cis} z = \text{cis} \Sigma z; \quad \text{X. } (\text{cis} z)^p = \text{cis} pz;
\]

including thus, under abridged forms, some known and useful theorems, respecting cosines and sines of sums and multiples of arcs.
(4.) For example, if the number of factors of the form $\text{cis} z$ be two, we have thus,

\[ \text{IX'. cis} z' \cdot \text{cis} z = \text{cis} (z' + z) ; \quad \text{X'. (cis} z)^2 = \text{cis} 2z ; \]

whence

\[
\cos (z' + z) = \frac{\text{cis} z' \cdot \text{cis} z}{2} = \cos z' \cos z - \sin z' \sin z ;
\]

\[
\sin (z' + z) = \frac{\text{cis} z' \cdot \text{cis} z}{2i} = \cos z' \sin z + \sin z' \cos z ;
\]

\[
\cos 2z = (\cos z)^2 - (\sin z)^2 ; \quad \sin 2z = 2 \cos z \sin z ;
\]

with similar results for more factors than two.

(5.) Without expressly introducing the conception, or at least the notation of amplitude, we may derive the recent formula IX. and X., from the consideration of the power $i^t$ (234), as follows. That power of $i$, with a scalar exponent, $t$, has been interpreted in 234, (5.), as a symbol satisfying an equation which may be written thus:

\[ \text{XI. } i^t = \text{cis} z, \quad \text{if} \quad z = \frac{1}{2} t \pi ; \]

or geometrically as a versor, which turns a line through $t$ right angles, where $t$ may be any scalar. We see then at once, from this interpretation, that if $t'$ be either the same or any other scalar, the formula,

\[ \text{XII. } i \cdot i^t = i^{t'} ; \quad \text{XIII. } i^t = i^{2t}, \]

must hold good, as in algebra. And because the number of the factors $i^t$ is easily seen to be arbitrary in this last formula, we may write also,

\[ \text{XIV. } (i^t)^p = i^{pt}, \]

if $p$ be any whole* number. But the two last formulae may be changed by XI., to the equations IX. and X., which are therefore thus again obtained; although the later forms, namely XIII. and XIV., are perhaps somewhat simpler: having indeed the appearance of being mere algebraical identities, although we see that their geometrical interpretations, as given above, are important.

(6.) In connexion with the same interpretation XI. of the same useful symbol $i^t$, it may be noticed here that

\[ \text{XV. } K \cdot i^t = i^t ; \]

* It will soon be seen that there is a sense, although one not quite so definite, in which this formula holds good, even when the exponent $p$ is fractional, or surd; namely, that the second member is then one of the values of the first.

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and that therefore,

**XVI.** \( \cos \frac{t\pi}{2} = S \cdot i^t = \frac{1}{2} (i^t + i^{-t}) \);

**XVII.** \( \sin \frac{t\pi}{2} = i^{-1} V \cdot i^t = \frac{1}{2} i^{-1} (i^t - i^{-t}) \).

(7.) Hence, by raising the double of each member of XVI. to any positive whole power \( p \), halving, and substituting \( z \) for \( \frac{1}{2} t\pi \), we get the equation,

**XVIII.** \( 2^p \cdot (\cos z)^p = \frac{1}{2} (i^t + i^{-t})^p = \frac{1}{2} (i^{tp} + i^{-tp}) + \frac{1}{p} (i^{p-2})^t + i^{(p-1)} + \&c. \)

\( \cos pz + p \cos (p - 2) z + \frac{p(p - 1)}{2} \cos (p - 4) z + \&c. \),

with the usual rule for halving the coefficient of \( \cos 0z \), if \( p \) be an even integer; and with analogous processes for obtaining the known expansions of \( 2^{p-1} (\sin z)^p \), for any positive whole value, even or odd, of \( p \); and many other known results of the same kind.

237. If \( p \) be still a whole number, we have thus the transformation,

**I.** \( q^p = (r \cdot \text{cis } z)^p = r^p \cdot \text{cis } pz = (Tq)^p \cdot \text{cis } (p \cdot \text{am } q) \);

in which (comp. 190, 161) the two factors, of the tensor and versor kinds, may be thus written:

**II.** \( T (q^p) = (Tq)^p = Tq^p \);

**III.** \( U (q^p) = (Uq)^p = Uq^p \);

and any value (235) of the amplitude \( \text{am. } q \) may be taken, since all will conduct to one common value of this whole power \( q^p \). And if, for I., we substitute this slightly different formula (comp. 235, (3.),)

**IV.** \( (q^p)^n = Tq^n \cdot \text{cis } (p \cdot \text{am } n q) \), with \( p = \frac{m'}{n'} \), \( m', n' > 0 \),

\( m', n', n \) being whole numbers whereof the first is supposed to be prime to the second, so that the exponent \( p \) is here a fraction in its least terms, with a positive denominator \( n' \), while the factor \( Tq^n \) is interpreted as a positive scalar (of which the positive or negative logarithm, in any given system, is equal to \( p \times \) the logarithm of \( Tq \)), then the expression in the second member admits of \( n' \) distinct values, answering to different values of \( n \); which are precisely the \( n' \) values (comp. 234) of the fractional power \( q^p \), on principles already established: the principal value of that power corresponding to the value \( n = 0 \).

(1.) For any value of the integer \( n \), we may say that the symbol \( (q^p)^n \), defined by the formula IV., represents the \( n^{th} \) value of the power \( q^p \); such values, however, recurring periodically, when \( p \) is, as above, a fraction.
(2.) Abridging \((1^p)_n\) to \(1^p_n\), we have thus, generally, by 235, XII.,
\[ V. \ldots 1^p_n = \text{cis } 2pn\pi, \text{if } p \text{ be any fraction,} \]
a restriction which however we shall soon remove; and in particular,
\[ \text{VI.} \ldots \text{Principal value of } 1^p = 1^p_0 = 1. \]

(3.) Thus, making successively \(p = \frac{1}{2}, p = \frac{1}{3}\), we have
\[ \text{VII.} \ldots 1^i_n = \text{cis } n\pi, \quad 1^i_0 = +1, \quad 1^i_1 = -1, \quad 1^i_2 = +1, \text{ &c.}; \]
\[ \text{VIII.} \ldots 1^{\frac{2}{3}}_n = \text{cis } \frac{2n\pi}{3}, \quad 1^{\frac{2}{3}}_0 = 1, \quad 1^{\frac{2}{3}}_1 = \frac{-1 + i\sqrt{3}}{2}, \quad 1^{\frac{2}{3}}_2 = \frac{-1 - i\sqrt{3}}{2}, 1^{\frac{2}{3}}_3 = 1, \text{ &c.}; \]

(4.) Denoting in like manner the \(n\text{th value of } (-1)^p\) by the abridged symbol \((-1)^p_n\), we have, on the same plan (comp. 235, XIII.), for any fractional value of \(p\),
\[ \text{IX.} \ldots (-1)^p_n = \text{cis } p(2n+1)\pi; \text{ whence (comp. 232),} \]
\[ \text{X.} \ldots (-1)^i_n = \text{cis } \frac{\pi}{2} = +i, \quad (-1)^i_1 = \text{cis } \frac{3\pi}{2} = -i, \quad (-1)^i_2 = +i, \text{ &c.}; \]
and
\[ \text{XI.} \ldots (-1)^{\frac{1}{3}}_n = \frac{1 + i\sqrt{3}}{2}, \quad (-1)^{\frac{1}{3}}_1 = -1, \quad (-1)^{\frac{1}{3}}_2 = \frac{1 - i\sqrt{3}}{2}, \text{ &c.}, \]
these three values of \((-1)^i\) recurring periodically.

(5.) The formula IV. gives, generally, by V., the transformation,
\[ \text{XII.} \ldots (q^p)_n = (q^p)_0 \text{ cis } 2pn\pi = 1^n_n (q^p)_0; \]
so that the \(n\text{th value of } q^p\) is equal to the principal value of that power of \(q\), multiplied by the corresponding value of the same power of positive unity; and it may be remarked, that if the base \(a\) be any positive scalar, the principal \(p\text{th power, } (a^p)_0,\) is simply, by our definitions, the arithmetical value of \(a^p\).

(6.) The \(n\text{th value of the } p\text{th power of any negative scalar, } -a,\) is in like manner equal to the arithmetical \(p\text{th power of the positive opposite, } +a,\) multiplied by the corresponding value of the same power of negative unity; or in symbols,
\[ \text{XIII.} \ldots (-a)_n^p = (-1)_n^p (a^p)_0 = (a^p)_0 \text{ cis } p(2n+1)\pi. \]

(7.) The formula IV., with its consequences V. VI. IX. XII. XIII., may be extended so as to include, as a limit, the case when the exponent \(p\) being still scalar, becomes incommensurable, or surd; and although the number of values of the power \(q^p\) becomes thus unlimited (comp. 234, (4.)), yet we can still

* As before, this restriction is only a temporary one.
consider one of them as the principal value of this (now) surd power: namely
the value,

\[ (q^p)_o = Tq^p \cdot \cos(p \text{ am}_o q), \]

which answers to the principal amplitude (235, (3.)) of the proposed quaternion \( q \).

238. We may therefore consider the symbol,

\[ q^p, \]

in which the base, \( q \), is any quaternion, while the exponent, \( p \), is any scalar, as being now fully interpreted; but no interpretation has been as yet assigned to this other symbol of the same kind,

\[ q^p', \]

in which both the base \( q \), and the exponent \( q' \), are supposed to be (generally) quaternions, although for the purposes of this Chapter complanar (225).* To do this, in a way which shall be completely consistent with the foregoing conventions and conclusions, or rather which shall include and reproduce them, for the case where the new quaternion exponent, \( q' \), degenerates (131) into a scalar, will be one main object of the following section: which however will also contain a theory of logarithms of quaternions, and of the connexion of both logarithms and powers with the properties of a certain function, which we shall call the potential of a quaternion, and to consider which we next proceed.

SECTION 4.
On the Potential and Logarithm of a Quaternion; and on Powers of Quaternions, with Quaternions for their Exponents.

239. If we consider the polynomial function,

\[ P(q, m) = 1 + q_1 + q_2 + \ldots + q_m, \]

in which \( q \) is any quaternion, and \( m \) is any positive whole number, while it is supposed (for conciseness) that

\[ q_m = \frac{q^m}{1 \cdot 2 \cdot 3 \ldots m} \left(= \frac{q^m}{\Gamma(m + 1)}\right), \]

then it is not difficult to prove that however great, but finite and given, the

* [For the general case see 316.]
tensor $T_q$ may be, a finite number $m$ can be assigned, for which the inequality

\[ T (P (g, m + n) - P (g, m)) < a, \quad \text{if } a > 0, \]

shall be satisfied, however large the (positive whole) number $n$ may be, and however small the (positive) scalar $a$, provided that this last is given. In other words, if we write (comp. 228),

\[ q = x + iy, \quad P (g, m) = X_m + iY_m, \]

a finite value of the number $m$ can always be assigned, such that the following inequality,

\[ (X_{m+n} - X_m)^2 + (iY_{m+n} - iY_m)^2 < a^2, \]

shall hold good, however large the number $n$, and however small (but given and $> 0$) the scalar $a$ may be. It follows evidently that each of the two scalar series, or succession of scalar functions,

\[ X_0 = 1, \quad X_1 = 1 + x, \quad X_2 = 1 + x + \frac{x^2 - y^2}{2}, \ldots \quad X_m, \ldots \]

\[ Y_0 = 0, \quad Y_1 = y, \quad Y_2 = y + xy, \ldots \quad Y_m, \ldots ^t \]

converges ultimately to a fixed and finite limit, whereof the one may be called $X_o$, or simply $X$, and the latter $Y_o$, or $Y$, and of which each is a certain function of the two scalars, $x$ and $y$. Writing then

\[ Q = X_o + iY_o = X + iY, \]

we must consider this quaternion $Q$ (namely the limit to which the following series of quaternions,

\[ P (q, 0) = 1, \quad P (q, 1) = 1 + q, \quad P (q, 2) = 1 + q + \frac{q^2}{2}, \ldots \quad P (q, m), \ldots \]

converges ultimately) as being in like manner a certain function, which we shall call the ponential function, or simply the Ponential of $q$, in consequence of its possessing certain exponential properties; and which may be denoted by any one of the three symbols,

\[ P (q, \infty), \quad P (q), \quad \text{or simply } \quad Pq. \]

We have therefore the equation,

\[ X \ldots \text{Ponential of } q = Q = Pq = 1 + q_1 + q_2 + \ldots + q_m, \]

with the signification II. of the term $q_m$.

(1.) In connexion with the convergence of this ponential series, or with the
inequality III., it may be remarked that if we write (comp. 235) \( r = Tg \), and \( r_m = Tq_m \), we shall have, by 212, (2.),

\[
\text{XI. } T(P(q, m+n) - P(q, m)) \leq P(r, m+n) - P(r, m);
\]

it is sufficient then to prove that this last difference, or the sum of the \( n \) positive terms, \( r_{m+1}, \ldots r_{m+n} \), can be made < \( a \). Now if we take a number \( p > 2r - 1 \), we shall have \( r_{p+1} < \frac{1}{2}r_p, r_{p+2} < \frac{1}{2}r_{p+1}, &c. \), so that a finite number \( m > p > 2r - 1 \) can be assigned, such that \( r_m < a \); and then,

\[
\text{XII. } P(r, m+n) - P(r, m) < a (2^{-1} + 2^{-2} + \ldots + 2^{-n}) < a;
\]

the asserted inequality is therefore proved to exist.

(2.) In general, if an ascending series, with positive coefficients, such as

\[
\text{XIII. } a_0 + a_1 g + a_2 g^2 + &c., \text{ where } a_0 > 0, a_1 > 0, &c.,
\]

be convergent when \( g \) is changed to a positive scalar, it will à fortiori converge, when \( g \) is a quaternion.

240. Let \( q \) and \( q' \) be any two complanar quaternions, and let \( q'' \) be their sum, so that

\[
\text{I. } q'' = q' + q, \quad q'' || q' || q;
\]

then, as in algebra, with the signification 239, II. of \( q_m \), and with corresponding significations of \( q'_m \) and \( q''_m \), we have

\[
\text{II. } q''_m = \frac{(q' + q)^m}{1 \cdot 2 \cdot 3 \ldots m} = q'_m q_0 + q'_m q_1 + q'_m q_2 + \ldots + q'_m q_m,
\]

where \( q_0 = q'_0 = 1 \). Hence, writing again \( r = Tq, \quad r_m = Tq_m \), and in like manner \( r' = Tq' \), &c., the two differences,

\[
\text{III. } P(r', m) . P(r, m) - P(r + r', m),
\]

and

\[
\text{IV. } P(r + r', 2m) - P(r', m) . P(r, m),
\]

can be expanded as sums of positive terms of the form \( r'_p \cdot r_p \) (one sum containing \( \frac{1}{2}m(m+1) \), and the other containing \( m(m+1) \) such terms)*; but, by 239, III., the sum of these two positive differences can be made less than any given small positive scalar \( a \), since

\[
\text{V. } P(r + r', 2m) - P(r + r', m) < a, \text{ if } a > 0,
\]

* [For the total number of terms in \( P(r + r', m) \) is \( 1 + 2 + 3 + \ldots + (m+1) = \frac{1}{2} (m+1) (m+2) \). On expansion of III. the series is seen to be \( \sum r'_p r_p \) where \( p + p' > m \), and there are \( (m+1)^2 - \frac{1}{2} (m+1) (m+2) \) terms, all of which are positive; similarly for IV. From the expanded form of III. it is seen at once that]

\[
T(P(q', m) . P(q, m) - P(q + q', m)) \leq P(r', m) . P(r, m) - P(r + r', m).
\]
provided that the number $m$ is taken large enough; each difference, therefore, separately tends to 0, as $m$ tends to $\infty$; a tendency which must exist a fortiori, when the tensors, $r, r'$, are replaced by the quaternions, $q, q'$. The function $Pq$ is therefore subject to the Exponential Law,

\[ \text{VI. } P(q + q') = Pq' \cdot Pq = Pq' \cdot Pq', \text{ if } q' || q. \]

(1.) If we write (comp. 237, (5.)),

\[ \text{VII. } P1 = e, \text{ then } \text{VIII. } Px = (e^x)_0 = \text{arithmetical value of } e^x; \]

where $e$ is the known base of the natural system of logarithms, and $x$ is any scalar. We shall henceforth write simply $e^x$ to denote this principal (or arithmetical) value of the $x^{th}$ power of $e$, and so shall have the simplified equation,

\[ \text{VIII'. } Px = e^x. \]

(2.) Already we have thus a motive for writing, generally,

\[ \text{IX. } Pg = e^g; \]

but this formula is here to be considered merely as a definition of the sense in which we interpret this exponential symbol, $e^g$; namely as what we have lately called the ponential function, $Pg$, considered as the sum of the infinite but converging series, 239, X. It will however be soon seen to be included in a more general definition (comp. 238) of the symbol $q^g$.

(3.) For any scalar $x$, we have by VIII. the transformation:

\[ \text{X. } x = 1Px = \text{natural logarithm of ponential of } x. \]

241. The exponential law (240) gives the following general decomposition of a ponential into factors,

\[ \text{I. } Pg = P(x + iy) = Px \cdot Piy; \]

in which we have just seen that the factor $Px$ is a positive scalar. The other factor, $Piy$, is easily proved to be a versor, and therefore to be the versor of $Pg$, while $Px$ is the tensor of the same ponential; because we have in general,

\[ \text{II. } Pg \cdot P(-g) = P0 = 1, \text{ and } \text{III. } PKq = KPq, \]

since

\[ \text{IV. } (Kq)^m = K(q^m) = (\text{say}) Kq^m (\text{comp. } 199, \text{ IX.}); \]

and therefore, in particular (comp. 150, 158),

\[ \text{V. } 1 = Piy = P(-iy) = KPiy, \text{ or } \text{VI. } NPiy = 1. \]
We may therefore write (comp. 240, IX., X.),

\[ \text{VII.} \quad TPg = PSg = Px = \varepsilon^x ; \quad \text{VIII.} \quad x = Sg = 1TPq ; \]

\[ \text{IX.} \quad UPg = PVg = Piy = e^{iy} = \text{cis } y \text{ (comp. 235, IV.)} ; \]

this last transformation being obtained from the two series,

\[ \text{X.} \quad SPiy = 1 - \frac{y^2}{2} + \&c. = \cos y ; \]

\[ \text{XI.} \quad i^{-1} VPiy = y - \frac{y^3}{2} \cdot \frac{3}{3} + \&c. = \sin y. \]

Hence the potential \( Pq \) may be thus transformed:

\[ \text{XII.} \quad Pq = P(x + iy) = e^x \text{ cis } y. \]

(1.) If we had not chosen to assume as known the series for cosine and sine, nor to select (at first) any one unit of angle, such as that known one on which their validity depends, we might then have proceeded as follows. Writing

\[ \text{XIII.} \quad Piy = f(y + iy_1) + i\phi y_1, \quad f(- y) = + f_y, \quad \phi(- y) = - \phi y, \]

we should have, by the exponential law (240),

\[ \text{XIV.} \quad f(y + y') = S(Piy . Piy') = f_y . f_{y'} + \phi y . \phi y' ; \]

\[ \text{XV.} \quad f(y - y') = f_y . f_{y'} + \phi y . \phi y' ; \]

and then the functional equation, which results, namely,

\[ \text{XVI.} \quad f(y + y') + f(y - y') = 2f_y . f_{y'} , \]

would show that

\[ \text{XVII.} \quad f_y = \cos \left( \frac{y}{c} \times \text{a right angle} \right) , \]

whatever unit of angle may be adopted, provided that we determine the constant \( c \) by the condition,

\[ \text{XVIII.} \quad c = \text{least positive root of the equation } f_y (= SPiy) = 0 ; \]

or nearly,

\[ \text{XVIII} \quad c = 1.5708 , \text{ as the study of the series }^* \text{ would show.} \]

* In fact, the value of the constant \( c \) may be obtained to this degree of accuracy, by simple interpolation between the two approximate values of the function \( f \),

\[ f(1.6) = + 0.070737 , \quad f(1.6) = - 0.029200 ; \]

and of course there are artifices, not necessary to be mentioned here, by which a far more accurate value can be found.
(2.) A motive would thus arise for representing a right angle by this numerical constant, $c$; or for so selecting the angular unit, as to have the equation ($\pi$ still denoting two right angles),

\[ \text{XIX.} \quad \pi = 2c = \text{least positive root of the equation } \phi y = -1; \]
giving nearly,

\[ \text{XIX'}. \quad \pi = 3.14159, \text{ as usual}; \]
for thus we should reduce XVII. to the simpler form,

\[ \text{XX.} \quad \phi y = \cos y. \]

(3.) As to the function $\phi y$, since

\[ \text{XXI.} \quad (\phi y)^2 + (\phi y)^2 = \phi y \cdot P(-iy) = 1, \]
it is evident that $\phi y = \pm \sin y$; and it is easy to prove that the upper sign is to be taken. In fact, it can be shown (without supposing any previous knowledge of cosines or sines) that $\phi c$ is positive, and therefore that

\[ \text{XXII.} \quad \phi c = +1, \quad \text{or} \quad \text{XXIII.} \quad \phi c = i; \]
whence

\[ \text{XXIV.} \quad \phi y = \sin i \cdot \phi y = \sin (y - c) = f(y - c), \]
and

\[ \text{XXV.} \quad \phi y = fy + if(y - c). \]

If then we replace $c$ by $\pi/2$, we have

\[ \text{XXVI.} \quad \phi y = \cos \left(y - \frac{\pi}{2}\right) = \sin y; \quad \text{and} \quad \text{XXVII.} \quad \phi y = \text{cis } y, \text{ as in IX.} \]

(4.) The series X. XI. for cosine and sine might thus be deduced, instead of being assumed as known: and since we have the limiting value,

\[ \text{XXVIII.} \quad \lim_{y \to 0} y^{-1} \sin y = \lim_{y \to 0} y^{-1} \sqrt{y} i; \text{ for } P(iy) = 1, \]
it follows that the unit of angle, which thus gives $\phi y = \text{cis } y$, is (as usual) the angle subtended at the centre by the arc equal to radius; or that the number $\pi$ (or $2c$) is to 1, as the circumference is to the diameter of a circle.

(5.) If any other angular unit had been, for any reason, chosen, then a right angle would of course be represented by a different number, and not by 1.5708 nearly; but we should still have the transformation,

\[ \text{XXIX.} \quad \phi y = \text{cis} \left(\frac{y}{c} \times \text{a right angle}\right), \]
though not the same series as before, for $\cos y$ and $\sin y$. 

*Hamilton's Elements of Quaternions.*
242. The usual unit being retained, we see, by 241, XII., that

I. \[ P \cdot 2\sin \pi = 1, \quad \text{and} \quad \text{II.} \quad P \cdot (q + 2\sin \pi) = Pq, \]

if \( n \) be any whole number; it follows, then, that the inverseponential function, \( P^{-1}q \), or what we may call the Imponential, of a given quaternion \( q \), has indefinitely many values, which may all be represented by the formula,

III. \[ P_n^{-1}q = lTg + i\text{am}_n q; \]

and of which each satisfies the equation,

IV. \[ PP_n^{-1}q = q; \]

while the one which corresponds to \( n = 0 \), may be called the Principal Imponential. It will be found that when the exponent \( p \) is any scalar, the definition already given (237, IV., XII.) for the \( n^{th} \) value of the \( p^{th} \) power of \( q \) enables us to establish the formula,

V. \[ (q^p)_n = P(pP_n^{-1}q); \]

and we now propose to extend this last formula, by a new definition, to the more general case (238), when the exponent is a quaternion \( q' \); thus writing generally, for any two complanar quaternions, \( q \) and \( q' \) the General Exponential Formula,

VI. \[ (q^{q'})_n = P(q/P_n^{-1}q); \]

the principal value of \( q^{q'} \) being still conceived to correspond to \( n = 0 \), or to the principal amplitude of \( q \) (comp. 235, (3.)).

(1.) For example,

VII. \[ (e^x) = P(qP_n^{-1}e) = Pq, \quad \text{because} \quad P_n^{-1}e = l_e = 1; \]

the ponential \( Pq \), which we agreed, in 240, (2.), to denote simply by \( e^x \), is therefore now seen to be in fact, by our general definition, the principal value of that power, or exponential.

(2.) With the same notations,

VIII. \[ e^{iy} = \text{cis} y, \quad \cos y = \frac{1}{2} (e^{iy} + e^{-iy}), \quad \sin y = \frac{1}{2i} (e^{iy} - e^{-iy}); \]

these two last only differing from the usual imaginary expressions for cosine and sine, by the geometrical reality* of the versor \( i \).

* Compare 232, (2.), and the Notes to pp. 253, 258.
(3.) The cosine and sine of a quaternion (in the given plane) may now be defined by the equations:

\[ \cos q = \frac{1}{2} (\epsilon^q + \epsilon^{-q}) ; \quad \sin q = \frac{1}{2i} (\epsilon^q - \epsilon^{-q}) ; \]

and we may write (comp. 241, IX.),

\[ \text{cis } q = \epsilon^q = Pq. \]

(4.) With this interpretation of cis \( q \), the exponential properties, 236, IX., X., continue to hold good; and we may write,

\[ (q^r)_n = P (q'Tq) . P (iq' am_n q) = (Tq)_0^r \text{ cis } (q' am_n q) ; \]

a formula which evidently includes the corresponding one, 237, IV., for the \( n^{th} \) power of \( q \), when \( p \) is scalar.

(5.) The definitions III. and VI., combined with 235, XII., give generally,

\[ (q^r)_n = 1, \quad (q^r)_0 = 1, \quad (q^r)_1 = (q^r) ; \]

this last equation including the formula 237, XII.

(6.) The same definitions give,

\[ P^{-1} i = \frac{i \pi}{2} ; \quad \text{cis } i = \epsilon^\pi ; \]

which last equation agrees with a known interpretation of the symbol,

\[ \sqrt{-1} \]

considered as denoting in algebra a real quantity.

(7.) The formula VI. may even be extended to the case where the exponent \( q' \) is a quaternion, which is not in the given plane of \( i \), and therefore not complanar with the base \( q \); thus we may write,

\[ (i')_0 = P (jP_0^{-1} i) = P \left( -\frac{k\pi}{2} \right) = -k ; \]

but it would be foreign (225) to the plan of this Chapter to enter into any further details, on the subject of the interpretation of the exponential symbol \( q^r \), for this case of diplanar quaternions, though we see that there would be no difficulty in treating it, after what has been shown respecting complanars.

243. As regards the general logarithm \( q' \) of a quaternion \( q \) (in the given plane), we may regard it as any quaternion which satisfies the equation,

\[ \epsilon q' = Pq' = q ; \]

and in this view it is simply the Impenential \( P^{-1} q \), of which the \( n^{th} \) value is
expressed by the formula 242, III. But the principal imponential, which
answers (as above) to $n = 0$, may be said to be the principal logarithm, or
simply the Logarithm, of the quaternion $q$, and may be denoted by the symbol,

$$lq;$$

so that we may write,

I. $lq = P^{-1}q = lTq + i am_0 q$;

or still more simply,

II. $lq = l(Tq \cdot Uq) = lTq + lUq$,

because $lTUq = 11 = 0$, and therefore,

III. $lUq = i am_0 q$.

We have thus the two general equations,

IV. $Slq = lTq$; V. $Vlq = lUq$;

in which $lTq$ is still the scalar and natural logarithm of the positive scalar $Tq$.

(1.) As examples (comp. 235, (2.), and (4.)),

VI. $li = \frac{1}{2}i\pi$; VII. $1(-1) = i\pi$.

(2.) The general logarithm of $q$ may be denoted by any one of the symbols,

$\log q$, or $\log q$, or $(\log q)_n$,

this last denoting the $n^{th}$ value; and then we shall have,

VIII. $(\log q)_n = lq + 2in\pi$.

(3.) The formula,

IX. $\log q'q = \log q' + \log q$, if $q' \parallel q$,

holds good, in the sense that every value of the first member is one of the
values of the second (comp. 236).

(4.) Principal value of $q^{x'} = e^{r'q}$; and one value of $\log q^{x'} = q'lq$.

(5.) The quotient of two general logarithms,

X. $(\log q')_{n'} : (\log q)_n = \frac{lq' + 2in'\pi}{lq + 2in\pi}$;

may be said to be the general logarithm of the quaternion, $q'$, to the complanar
quaternion base, $q$; and we see that its expression involves* two arbitrary and
independent integers, while its principal value may be defined to be $lq' : lq$.

* As the corresponding expression in algebra, according to Graves and Ohm.
SECTION 5.

On Finite* (or Polynomial) Equations of Algebraic Form, involving Complanar Quaternions; and on the Existence of n Real Quaternion Roots, of any such Equation of the nth Degree.

244. We have seen (233) that an equation of the form,

I. . . \( q^n - Q = 0 \),

where \( n \) is any given positive integer, and \( Q \) is any† given, real, and actual quaternion (144), has always \( n \) real, actual, and unequal quaternion roots, \( q \), comptanar with \( Q \); namely, the \( n \) distinct and real values of the symbol \( Q^n \) (223, VIII.), determined on a plan lately laid down. This result is, however, included in a much more general Theorem, respecting Quaternion Equations of Algebraic Form; namely, that if \( q_1, q_2, \ldots, q_n \) be any \( n \) given, real, and comptanar quaternions, then the equation,

II. . . \( q^n + q_1q^{n-1} + q_2q^{n-2} + \ldots + q_n = 0 \),

has always \( n \) real quaternion roots, \( q', q'', \ldots, q^{(n)} \), and no more in the given plane; of which roots it is possible however that some, or all may become equal, in consequence of certain relations existing between the \( n \) given coefficients.

245. As another statement of the same Theorem, if we write,

I. . . \( F_nq = q^n + q_1q^{n-1} + \ldots + q_n \),

the coefficients \( q_1, \ldots, q_n \) being as before, we may say that every such polynomial function, \( F_nq \), is equal to a product of \( n \) real, comptanar, and linear (or binomial) factors, of the form \( q - q' \); or that an equation of the form,

II. . . \( F_nq = (q - q') (q - q'') \ldots (q - q^{(n)}) \),

can be proved in all cases to exist: although we may not be able, with our present methods, to assign expressions for the roots, \( q', \ldots, q^{(n)} \), in terms of the coefficients \( q_1, \ldots, q_n \).

* By saying finite equations, we merely intend to exclude here equations with infinitely many terms, such as \( Fq = 1 \), which has been seen (242) to have infinitely many roots, represented by the expression \( q = \sin \frac{n\pi}{n} \), where \( n \) may be any whole number.
† It is true that we have supposed \( Q \parallel i \) (225); but nothing hinders us, in any other case, from substituting for \( i \) the versor \( UVQ \), and then proceeding as before.
246. Or we may say that there is always a certain system of \( n \) real quaternions \( q', \&c., \) \( || i, \) which satisfies the system of equations, of known algebraic form,

\[
\begin{align*}
q' + q'' + \ldots + q^{(n)} &= -q_1 ; \\
q'q'' + q'q''' + q''q''' + \ldots &= + q_2 ; \\
q'q''q''' + \ldots &= -q_3 ; \&c.
\end{align*}
\]

247. Or because the difference \( F_nq - F_nq' \) is divisible by \( q - q' \), as in algebra, under the supposed conditions of complanarity (224), it is sufficient to say that at least one real quaternion \( q' \) always exists (whether we can assign it or not), which satisfies the equation,

\[F_nq' = 0,\]

with the foregoing form (245, I.) of the polynomial function \( F_n.\)

248. Or finally, because the theorem is evidently true for the case \( n = 1 \), while the case 244, I., has been considered, and the case \( q_n = 0 \) is satisfied by the supposition \( q = 0 \), we may, without essential loss of generality, reduce the enunciation to the following:

Every equation of the form,\( ^\dagger \)

\[
\begin{align*}
\ldots q(q - q') (q - q'') \ldots (q - q^{(n-1)}) &= Q, \\
in which \( q', q'', \ldots \) and \( Q \) are any \( n \) real and given quaternions in the given plane, whereof at least \( Q \) and \( q' \) may be supposed actual (144), is satisfied by at least one real, actual, and complanar quaternion, \( q \) [see 253 (1.)].
\]

249. Supposing that the \( m - 1 \) last of the \( n - 1 \) given quaternions \( q_1 \), \( q_2 \), \( \ldots \) \( q^{(n-1)} \) vanish, but that the \( n - m \) first of them are actual, where \( m \) may be any whole number, from 1 to \( n - 1 \), and introducing a new real, known, complanar, and actual quaternion \( q_0 \), which satisfies the condition,

\[\ldots q_0^m = \frac{Q}{q'q'' \ldots q^{(n-m)}},\]

\* [Thus \( \frac{F_nq}{q - q'} = F_{n-1}q = q^{n-1} + q_1 q^{n-2} + q_2 q^{n-3} + \ldots + q_{n-1} \), which is of the form 245, I. If then every equation of this form has a root, \( F_{n-1}q'' = 0 \), and \( q'' \) is a second root of \( F_{n-1}q.\)]

\† The corresponding form\( ^\dagger \) of the algebraical equation of the \( n^\text{th} \) degree, was proposed by Mourey, in his very ingenious and original little work, entitled \textit{La vraie théorie des Quantités Négatives, et des Quantités prétendues Imaginaires} (Paris, 1828). Suggestions also, towards the geometrical proof of the theorem in the text have been taken from the same work; in which, however, the curve here called (in 251) an ovale is not perhaps defined with sufficient precision: the inequality, here numbered as 251, XII., being not employed. It is to be observed that Mourey's book contains no hint of the present calculus, being confined, like the \textit{Double Algebra} of Prof. De Morgan (London, 1849), and like the earlier work of Mr. Warren (Cambridge, 1828), to questions within the plane; whereas the very conception of the Quaternion involves, as we have seen, a reference to Tridimensional Space.
we may write thus the recent equation I.,

\[ \frac{f_{e}}{q} = \left(\frac{q}{q_{o}}\right)^{m}\left(\frac{q}{q' - 1}\right)\left(\frac{q}{q'' - 1}\right)\left(\frac{q}{q^{n} - 1}\right) = 1; \]

and may (by 187, 159, 235) decompose it into the two following:

IV. \( F_{e}/q = 1; \) and V. \( U_{e}/q = 1, \) or VI. \( m_{f}/q = 2p\pi; \)

in which \( p \) is some whole number (negatives and zero included).

250. To give a more geometrical form to the equation, let \( \lambda \) be any given or assumed line \( \| i, \) and let it be supposed that \( a, \beta, \ldots \) and \( \rho, \sigma, \) or \( o_{A}, o_{B}, \ldots \) and \( o_{p}, o_{s}, \) are \( n - m + 2 \) other lines in the same planes, and that \( \phi_{p} \) is a known function of \( \rho, \) such that

VII. \( \phi_{p} = q'\lambda, \) \( \beta = q''\lambda, \) \( \ldots \) \( \rho = q\lambda, \) \( \sigma = q\lambda, \)

and

VIII. \( \phi_{p} = f_{q} = \left(\frac{\rho}{\sigma}\right)^{m}\left(\frac{\rho - a}{a}\right)\left(\frac{\rho - \beta}{\beta}\right) \ldots = \left(\frac{O_{F}}{O_{S}}\right)^{m}\frac{A_{P}}{O_{A}}\frac{B_{P}}{O_{B}} \ldots; \)

the theorem to be proved may then be said to be, that whatever system of points, \( o, a, b, \ldots \) and \( s, \) in a given plane, and whatever positive whole number \( m, \)

may be assumed, or given, there is always at least one real point \( v, \) in the same plane, which satisfies the two conditions:

\[ T_{\phi_{p}} = 1; \] \[ \phi_{p} = 2p\pi. \]

251. Whatever value \( \| i, \) we may assume for the versor (or unit-vector) \( U_{\rho}, \) there always exists at least one value of the tensor \( T_{\rho}, \) which satisfies the condition IX.; because the function \( T_{\phi_{p}} \) vanishes with \( T_{\rho}, \) and becomes infinite when \( T_{\rho} = \infty, \) having varied continuously (although perhaps with fluctuations) in the interval. Attending then only to the least value (if there be more than one) of \( T_{\rho}, \) which thus renders \( T_{\phi_{p}} \) equal to unity, we can conceive a real, unambiguous, and scalar function \( \psi_{\rho}, \) which shall have the two following properties:

\[ T_{\phi_{\rho}} = 1; \] \[ \phi_{\rho} = 2\pi. \]

And in this way the equation, or system of equations,

\[ T_{\phi} (\psi_{\rho}) = 1; \] \[ T_{\phi} (\psi_{\rho}) < 1, \] \( x > 0, < 1. \)

may be conceived to determine a real, finite, and plane closed curve, which we shall call generally an Oval, and which shall have the two following properties: 1st, every right line, or ray, drawn from the origin \( o, \) in any arbitrary
direction within the plane, meets the curve once, but once only; and \( \Pi \) nd, no one of the \( n - m \) other given points \( A, B, \ldots \) is on the oval, because \( \phi a = \phi \beta = \ldots = 0. * \)

252. This being laid down, let us conceive a point \( p \) to perform one circuit of the oval, moving in the positive direction relatively to the given interior point \( o \); so that, whatever the given direction of the line \( os \) may be, the amplitude \( am (p : o) \), if supposed to vary continuously, will have increased by four right angles, or by \( 2\pi \), in the course of this one positive circuit; and consequently, the amplitude of the left-hand factor \( (p : \sigma)^m \), of \( \phi p \), will have increased, at the same time, by \( 2m\pi \). Then, if the point \( A \) be also interior to the oval, so that the line \( OA \) must be prolonged to meet that curve, the ray \( AP \) will have likewise made one positive revolution, and the amplitude of the factor \( (p - a) : a \) will have increased by \( 2\pi \). But if \( A \) be an exterior point, so that the finite line \( OA \) intersects the curve in a point \( M \), and therefore never meets it again if prolonged, although the prolongation of the opposite line \( AO \) must meet it once in some point \( N \), then while the point \( p \) performs first what we may call the positive half-circuit from \( M \) to \( N \), and afterwards the other positive half-circuit from \( N \) to \( M \) again, the ray \( AP \) has only oscillated about its initial and final direction, namely that of the line \( AO \), without ever attaining the opposite direction; in this case, therefore, the amplitude \( am (AP : oA) \), if still supposed to vary continuously, has only fluctuated in its value, and has (upon the whole) undergone no change at all. And since precisely similar remarks apply to the other given points, \( B, \&c \), it follows that the amplitude, \( am \cdot (pp = m \cdot \cos \theta + 2(v - \cos \theta)) = \text{const.} \), of the product (VIII.) of all these factors, has (by 236) received a total increment \( = 2(m + t)\pi \), if \( t \) be the number (perhaps zero) of given internal points, \( A, B, \ldots \); while the number \( m \) is (by 249) at least \( = 1 \). Thus, while \( p \)

* [A curve traced out by a point moving so that the product of powers of its distances from fixed points is equal to a constant parameter, consists of closed curves or ovals surrounding the fixed points and enclosing all ovals corresponding to smaller parameters. If the parameter is small, each oval encloses but one fixed point, but as it increases, two ovals will combine into a curve with a “certain undulation” (254 (4.)). It is not generally true that a ray \( OP \) from one of the fixed points meets an undulatory oval only once. In this case \( OP \) will oscillate in its motion as \( P \) traces out the oval. But \( am \cdot \phi p = m \cdot \cos \theta + \cos (\pi - \cos \theta) = \text{const.} \), defines a set of curves diverging like half-lines or rays from the fixed points, and approximating to straight lines at great distances from them. By the properties of Conjugate Functions each of these curves which originates from \( O \) cuts at right angles each oval round \( O \) and does not meet it again. Near \( O \), \( am \cdot \phi p \) is nearly equal to \( m \cdot \cos \theta \) plus a constant. From this it appears that IX. and X. can always be satisfied, and that as \( P \) traces out an oval round \( O \) without oscillation, \( am \cdot \phi \) continually increases or diminishes without oscillation. The ovals are lines of magnetic force, and the orthogonal curves are traces of equipotential surfaces for a system of electric currents normal to the plane.]

† That is, so as not to receive any sudden increment, or decrement, of one or more whole circumferences (comp. 285, (1.)).
performs (as above) one positive circuit, the amplitude am \( \phi \rho \) has passed at least \( m \) times, and therefore at least once, through a value of the form \( 2m\pi \); and consequently the condition X. has been at least once satisfied. But the other condition, IX., is satisfied throughout, by the supposed construction of the oval: there is therefore at least one real position \( r \), upon that curve, for which \( \phi \rho \) or \( f_q = 1 \); so that, for this position of that point, the equation 249, III., and therefore also the equation 248, I., is satisfied. The theorem of Art. 248, and consequently also, by 247, the theorem of 244, with its transformations 245 and 246, is therefore in this manner proved.

253. This conclusion is so important, that it may be useful to illustrate the general reasoning, by applying it to the case of a quadratic equation, of the form,

\[
\text{I. } \ldots f_q = \frac{q}{q_0} \left( \frac{q}{q_1} - 1 \right) = 1;
\]

or

\[
\text{II. } \ldots \phi \rho = \frac{\rho}{\rho_0} \left( \frac{\rho}{\rho_1} - 1 \right) = \frac{op}{os} \frac{ap}{oa} = 1.
\]

We have now to prove (comp. 250, VIII.) that a (real) point \( p \) exists, which renders the fourth proportional (226) to the three lines \( oa, op, ap \) equal to a given line \( os \), or \( ab \), if this latter be drawn = \( os \); or which satisfies the following condition of similarity of triangles (118),

\[
\text{III. } \ldots \Delta\text{aop} \propto \text{pab};
\]

which includes the equation of rectangles,

\[
\text{IV. } \ldots \overline{op} \cdot \overline{ap} = \overline{oa} \cdot \overline{ab}.
\]

(Compare the annexed figures, 55, and 55, bis.) Conceive, then, that a continuous curve* is described as a locus (or as part of the locus) of \( p \), by means of this equality IV., with the additional condition when necessary, that \( o \) shall be within it; in such a manner that when (as in fig. 56) a right line from \( o \) meets the general or total locus in several points, \( m, m', n' \), we reject all but the

* This curve of the fourth degree is the well-known Cassinian; but when it breaks up, as in fig. 56, into two separate ovals, we here retain, as the oval of the proof, only the one round \( o \), rejecting for the present that round \( \lambda \).

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point \( M \) which is nearest to \( o \), as not belonging (comp. 251, XIX) to the oval here considered. Then while \( P \) moves upon that oval, in the positive direction relatively to \( o \), from \( M \) to \( N \), and from \( N \) to \( M \) again, so that the ray \( OP \) performs one positive revolution, and the amplitude of the factor \( OP : OS \) increases continuously by \( 2\pi \), the ray \( AP \) performs in like manner one positive revolution, or (on the whole) does not revolve at all, and the amplitude of the factor \( AP : OA \) increases by \( 2\pi \) or by 0, according as the point \( A \) is interior or exterior to the oval. In the one case, therefore, the amplitude \( am \phi \) of the product increases by \( 4\pi \) (as in fig. 55, bis); and in the other case, it increases by \( 2\pi \) (as in fig. 56); so that in each case, it passes at least once through a value of the form \( 2\pi n \), whatever its initial value may have been. Hence, for at least one real position, \( P \), upon the oval, we have

\[
\text{V. } \quad am \phi = 2\pi n, \quad \text{and therefore } \text{VI. } \quad U \phi = 1;
\]

but

\[
\text{VII. } \quad T \phi = 1,
\]

throughout, by the construction, or by the equation of the locus IV.; the geometrical condition \( \phi = 1 \) (II.) is therefore satisfied by at least one real vector \( \rho \); and consequently the quadratic equation \( f \rho = 1 \) (I.) is satisfied by at least one real quaternion root, \( q = \rho : \lambda \) (250, VII.). But the recent form I. has the same generality as the earlier form,

\[
\text{VIII. } \quad f_2 q = q^2 + q_1 q + q_2 = 0 \quad (\text{comp. 245}),
\]

where \( q_1 \) and \( q_2 \) are any two given, real, actual, and complanar quaternions; thus there is always a real quaternion \( q' \) in the given plane, which satisfies the equation,

\[
\text{VIII'. } \quad f_2 q' = q'^2 + q_1 q' + q_2 = 0 \quad (\text{comp. 247});
\]

subtracting, therefore, and dividing by \( q - q' \), as in algebra (comp. 224), we obtain the following depressed or linear equation \( q \),

\[
\text{IX. } \quad q + q' + q_1 = 0, \quad \text{or } \text{IX'. } \quad q = q'' = -q' - q_1 \quad (\text{comp. 246}).
\]

The quadratic VIII. has therefore a second real quaternion root, \( q'' \) related in this manner to the first; and because the quadratic function \( f_2 (q) \) (comp. again 245) is thus decomposable into two linear factors, or can be put under the form,

\[
\text{X. } \quad f_2 q = (q - q')(q - q''),
\]

it cannot vanish for any third real quaternion, \( q \); so that (comp. 244) the quadratic equation has no more than two such real roots.
(1.) The cubic equation may therefore be put under the form (comp. 248),

\[ x' \cdots f_3q = q^3 + q_2q + q_3q + q_4 = q(q' - q')(q' - q'') + q_4 = 0; \]

it has therefore one real root, say \( q' \), by the general proof (252), which has been above illustrated by the case of the quadratic equation; subtracting therefore (comp. 247) the equation \( f_3q' = 0 \), and dividing by \( q - q' \), we can depress the cubic to a quadratic, which will have two new real roots, \( q'' \) and \( q''' \); and thus the cubic function may be put under the form,

\[ x' \cdots f_3q = (q - q')(q - q'')(q - q'''), \]

which cannot vanish for any fourth real value of \( q \); the cubic equation \( x \) has therefore no more than three real quaternion roots (comp. 244); and similarly for equations of higher degrees.

(2.) The existence of two real roots \( q \) of the quadratic I., or of two real vectors, \( p \) and \( p' \), which satisfy the equation II., might have been geometrically anticipated, from the recently proved increase = irr of amplitude \( a \rho \), in the course of one circuit, for the case of fig. 55; bis, in consequence of which there must be two real positions, \( p \) and \( p' \), on the one oval of that figure, of which each satisfies the condition of similarity III.; and for the case of fig. 56, from the consideration that the second (or lighter) oval, which in this case exists, although not employed above, is related to \( a \) exactly as the first (or dark) oval of the figure is related to 0; so that, to the real position \( p \) on the first, there must correspond another real position \( p' \), upon the second.

(3.) As regards the law of this correspondence, if the equation II. be put under the form,

\[ x' \cdots \left( \frac{p}{a} \right)^2 - \left( \frac{p}{a} \right)^1 - \frac{\sigma}{a} = 0, \]

and if we now write

\[ x' \cdots p = qa, \]

we may write \( x' \cdots q_1 = -1, \ q_2 = -\sigma : a, \)

for comparison with the form VIII.; and then the recent relation IX'. (or 246) between the two roots will take the form of the following relation between vectors,

\[ x' \cdots p + p' = a; \]

so that the point \( p' \) completes (as in the cited figures) the parallelogram \( opa'p' \), and the line \( pr' \) is bisected by the middle point \( c \) of \( oa \). Accordingly, with this position of \( p' \), we have (comp. III.) the similarity, and (comp. II. and 226) the equation,

\[ x' \cdots \Delta aopa' \propto p'ab; \]

\[ x' \cdots \phi p' = \phi (a - \rho) = \phi \rho = 1. \]
(4.) The other relation between the two roots of the quadratic VIII.,
namely (comp. 246),
\[ XVIII. \ldots q'q'' = q_2, \]  
gives \[ XIX. \ldots \frac{\rho}{a} \rho' = - \sigma; \]
and accordingly, the line \( \sigma \), or \( os \), is a fourth proportional to the three lines
\( oA, op, \) and \( AP, \) or \( a, \rho, \) and \(- \rho'.\)

(5.) The actual solution, by calculation, of the quadratic equation VIII. in
complanar quaternions, is performed exactly as in algebra; the formula being,
\[ XX. \ldots q = - \frac{1}{2} q_1 \pm \sqrt{\left(\frac{1}{4} q_1^2 - q_2\right)}, \]
in which, however, the square root is to be interpreted as a real quaternion, on
principles already laid down.

(6.) Cubic and biquadratic equations, with quaternion coefficients of the
kind considered in 244, are in like manner resolved by the known formulae of
algebra; but we have now (as has been proved) three real (quaternion) roots
for the former, and four such real roots for the latter.

254. The following is another mode of presenting the geometrical reason­
ings of the foregoing Article, without expressly introducing the notation or
conception of amplitude. The equation \( \phi \rho = 1 \) of 253 being written as
follows,
\[ I. \ldots \sigma = \chi \rho = \frac{\rho}{a} (\rho - a), \]  
or \[ II. \ldots T \sigma = T \chi \rho, \]  
and \[ III. \ldots U \sigma = U \chi \rho, \]
we may thus regard the vector \( \sigma \) as a known function of the vector \( \rho, \) or the
point \( s \) as a function of the point \( r; \) in the sense that, while \( o \) and \( A \) are fixed,
\( r \) and \( s \) vary together: although it may (and does) happen, that \( s \) may return
to a former position without \( r \) having similarly returned. Now the essential
property of the oval (253) may be said to be this: that it is the locus of the
points \( r \) nearest to \( o, \) for which the tensor \( T \chi \rho \) has a given value, say \( b; \) namely
the given value of \( T \sigma, \) or of \( os, \) when the point \( s, \) like \( o \) and \( A, \) is given. If then
we conceive the point \( r \) to move, as before, along the oval, and the point \( s \) also
to move, according to the law expressed by the recent formula \( I, \) this latter
point must move (by \( II. \)) on the circumference of a given circle (comp. again
fig. 56), with the given origin \( o \) for centre; and the theorem is, that in so
moving, \( s \) will pass, at least once, through every position on that circle, while \( r \)
performs one circuit of the oval. And this may be proved by observing that
(by \( III. \)) the angular motion of the radius \( os \) is equal to the sum of the angular
motions of the two rays, \( op \) and \( AP; \) but this latter sum amounts to eight right
angles for the case of fig. 55, \( bie, \) and to four right angles for the case of fig. 56;
the radius \( os \), and the point \( s \), must therefore have revolved twice in the first case, and once in the second case, which proves the theorem in question.

(1.) In the first of these two cases, namely when \( A \) is an interior point, each of the three angular velocities is positive throughout, and the mean angular velocity of the radius \( os \) is double of that of each of the two rays \( op, \ AP \). But in the second case, when \( A \) is exterior, the mean angular velocity of the ray \( \AP \) is zero; and we might for a moment doubt, whether the sometimes negative velocity of that ray might not, for parts of the circuit, exceed the always positive velocity of the ray \( op \), and so cause the radius \( os \) to move backwards, for a while. This cannot be, however; for if we conceive \( p \) to describe, like \( p' \), a circuit of the other (or lighter) oval, in fig. 56, the point \( s \) (if still dependent on it by the law I.) would again traverse the whole of the same circumference as before; if then it could ever fluctuate in its motion, it would pass more than twice through some given series of real positions on that circle, during the successive description of the two ovals by \( p \); and thus, within certain limiting values of the coefficients, the quadratic equation would have more than two real roots: a result which has been proved to be impossible.*

(2.) While \( s \) thus describes a circle round \( o \), we may conceive the connected point \( b \) to describe an equal circle round \( A \); and in the case at least of fig. 56, it is easy to prove geometrically, from the constant equality (253, IV.) of the rectangles \( op \cdot \AP \) and \( oA \cdot AB \), that these two circles (with \( T'U \) and \( T'U' \) as diameters), and the two ovals (with \( MN \) and \( M'N' \) as axes), have two common tangents, parallel to the line \( oA \), which connects what we may call the two given foci (or focal points), \( o \) and \( a \): the new or third circle, which is described on this focal interval \( oA \) as diameter, passing through the four points of contact on the ovals, as the figure may serve to exhibit.

(3.) To prove the same things by quaternions, we shall find it convenient to change the origin (18), for the sake of symmetry, to the central point \( c \); and thus to denote now \( cp \) by \( p \), and \( ca \) by \( a \), writing also \( oA = Ta = a \), and representing still the radius of each of the two equal circles by \( b \). We shall then have, as the joint equation of the system of the two ovals, the following:

\[
IV. \ldots T(p + a) \cdot T(p - a) = 2ab; \\
V. \ldots T(q^2 - 1) = 2c, \quad \text{if} \quad q = \frac{b}{a} \quad \text{and} \quad c = \frac{b}{a}.
\]

But because we have generally (by 199, 204, &c.) the transformations,

\[
VI. \ldots S \cdot q^2 = 2Sq^2 - Tq^2 = Tq^2 + 2Vq^2 = 2NSq - Nq = Nq - 2NVq,
\]

* [See the Note to 251, page 280.]
the square of the equation \( V \) may (by 210, (8.)) be written under either of the two following forms:

\[
\text{VII.} \quad (Nq - 1)^2 + 4NVg = 4c^2; \quad \text{VIII.} \quad (Nq + 1)^2 - 4NSg = 4c^2;
\]

whereof the first shows that the maximum value of \( TVq \) is \( c \), at least if \( 2c < 1 \), as happens for this case of fig. 56; and that this maximum corresponds to the value \( Tq = 1 \), or \( Tp = a \): results which, when interpreted, reproduce those of the preceding sub-article.

(4.) When \( 2c > 1 \), it is permitted to suppose \( Sg = 0 \), \( NVg = Nq = 2c - 1 \); and then we have only one continuous oval, as in the case of fig. 55, bis; but if \( c < 1 \), though \( > \frac{1}{2} \), there exists a certain undulation in the form of the curve (not represented in that figure), \( TVq \) being a minimum for \( Sg = 0 \), or for \( \rho \perp a \), but becoming (as before) a maximum when \( Tq = 1 \), and vanishing when \( Sg^2 = 2c + 1 \), namely at the two summits \( m, n \), where the oval meets the axis.

(5.) In the intermediate case, when \( 2c = 1 \), the Cassinian curve IV. becomes (as is known) a lemniscata; of which the quaternion equation may, by \( V. \), be written (comp. 200, (8.)) under any one of the following forms:

\[
\text{IX.} \quad T(g^2 - 1) = 1; \quad \text{or X.} \quad Nq^2 = 2S \cdot q^2; \quad \text{or XI.} \quad Tq^2 = 2SU \cdot q^2;
\]

or finally,

\[
\text{XII.} \quad Tp^2 = 2Ta^2 \cos 2\angle \frac{p}{a};
\]

which last, when written as

\[
\text{XII'.} \quad cr^2 = 2ca^2 \cos 2\angle cAP,
\]

agrees evidently with known results.

(6.) This corresponds to the case when

\[
\text{XIII.} \quad \sigma = \frac{-a}{4}, \quad \text{and XIV.} \quad \rho = \rho' = \frac{a}{2} \text{ in 253, XII.},
\]

that quadratic equation having thus its roots equal; and in general, for all degrees, cases of equal roots answer to some interesting peculiarities of form of the ovals, on which we cannot here delay.

(7.) It may, however, be remarked, in passing, that if we remove the restriction that the vector \( \rho \), or \( cr \), shall be in a given plane (225), drawn through the line which connects the two foci, \( o \) and \( a \), the recent equation \( V. \) will then represent the surface (or surfaces) generated by the revolution of the oval (or ovals), or lemniscata, about that line \( oa \) as an axis.
255. If we look back, for a moment, on the formula of similarity, 253, III., we shall see that it involves not merely an equality of rectangles, 253, IV., but also an equality of angles, $aop$ and $pab$; so that the angle $oab$ represents (in the figures 55) a given difference of the base angles $aop$, $pao$ of the triangle $oap$: but to construct a triangle, by means of such a given difference, combined with a given base, and a given rectangle of sides, is a known problem of elementary geometry. To solve it briefly, as an exercise, by quaternions, let the given base be the line $aa'$, with $o$ for its middle point, as in the annexed figure 57; let $baa'$ represent the given difference of base angles, $paa' - aa'p$; and let $oa . ab$ be equal to the given rectangle of sides, $ap . a'p$. We shall then have the similarity and equation,

$$\Delta oap' \propto fab; \quad \frac{\rho + a}{a} = \frac{\beta - a}{\rho - a}$$

whence it follows by the simplest calculations, that

$$\frac{(\rho/a)^3}{\rho/a} = (\frac{\rho}{a} + 1)(\frac{\rho}{a} - 1) + 1 = \frac{\beta - a}{a} + 1 = \frac{\beta}{a};$$

or that $\rho$ is a mean proportional (227) between $a$ and $\beta$. Draw, therefore, a line $op$, which shall be in length a geometric mean between the two given lines, $oa$, $ob$, and shall also bisect their angle $aob$; its extremity will be the required vertex, $p$, of the sought triangle $aaa'$: a result of the quaternion analysis, which geometrical synthesis* easily confirms.

(1.) The equation III. is however satisfied also (comp. 227) by the opposite vector, $op' = po$, or $\rho' = -\rho$; and because $\beta = (\rho : a) \cdot \rho$, we have.

$$\frac{\rho + \beta}{\rho + a} = \frac{\rho}{\rho} = \frac{\beta}{\rho} = \frac{\rho'}{\rho'}, \quad \text{or} \quad \frac{p' b}{p'a} = \frac{op}{oa} = \frac{ob}{op} = \frac{op'}{oa'};$$

so that the four following triangles are similar (the two first of them indeed being equal):

$$\Delta aop' \propto aop \propto pob \propto a'p'b;$$

as geometry again would confirm.

(2.) The angles $a'p'b$, $bpa$, are therefore supplementary, their sum being equal to the sum of the angles in the triangle $oap$; whence it follows that

* In fact, the two triangles I. are similar, as required, because their angles at $o$ and $p$ are equal, and the sides about them are proportional.
the four points $A, B, C, D$ are concircular:** or in other words, the quadrilateral
$ABCD$ is inscribable in a circle, of which (we may add) the centre $C$ is on the
circle $OAB$ (see again fig. 57), because the angle $AOB$ is double of the angle
$AOP$, by what has been already proved.

(3.) Quadratic equations in quaternions may also be employed in the
solution of many other geometrical problems; for example, to decompose a
given vector into two others, which shall have a given geometrical mean, &c.

SECTION 6.

On the $n^2 - n$ Imaginary (or Symbolical) Roots of a Quaternion
Equation of the $n^h$ Degree, with coefficients of the kind
considered in the foregoing Section.

256. The polynomial function $F_nq$ (245), like the quaternions $q, q_1, \ldots q_n$
on which it depends, may always be reduced to the form of a couple (228); and
thus we may establish the transformation (comp. 239),

\[ F_nq = F_n(x + iy) = X_n + iY_n = G_n(x, y) + iH_n(x, y), \]

$X_n$ and $Y_n$, or $G_n$ and $H_n$, being two known, real, finite, and scalar functions
of the two sought scalars, $x$ and $y$; which functions, relatively to them, are
each of the $n^h$ dimension, but which involve also, though only in the first
dimension, the $2n$ given and real scalars, $x_1, y_1, \ldots x_n, y_n$. And since the one
quaternion (or couple) equation, $F_nq = 0$, is equivalent (by 228, IV.) to the
system of the two scalar equations,

\[ \begin{align*}
  I. & \quad X_n = 0, \\
  II. & \quad Y_n = 0, \\
  III. & \quad G_n(x, y) = 0, \\
  IV. & \quad H_n(x, y) = 0,
\end{align*} \]

we see (by what has been stated in 244, and proved in 252) that such a
system, of two equations of the $n^h$ dimension, can always be satisfied by $n$
systems (or pairs) of real scalars, and by not more than $n$, such as,

\[ \begin{align*}
  \text{IV.} & \quad x', y'; \quad x'', y''; \ldots \quad x^{(n)}, y^{(n)};
\end{align*} \]

* Geometrically, the construction gives at once the similarity,

\[ \Delta AOP \propto \Delta POB, \quad \text{whence} \quad \angle BPA = \angle PAB + \angle ABO = \angle PAB'; \]

and if we complete the parallelogram $APPA''$, the new similarity,

\[ \Delta OA'P \propto \Delta OP'B, \quad \text{gives} \quad \angle AAP'' = \angle OAP' + \angle ATO = \angle AOP; \]

thus the opposite angles $BPA, AP'B$ are supplementary, and the quadrilateral $APBP'$ is inscribable. It
will be shown, in a shortly subsequent section [261, (5.)], that these four points, $A, B, C, D$, form a
harmonic group upon their common circle.
although it may happen that two or more of these systems shall coincede with
(or become equal to) each other.

(1.) If $x$ and $y$ be treated as co-ordinates (comp. 228, (3.)), the two
equations II. or III. represent a system of two curves, in the given plane; and
then the theorem is, that these two curves intersect each other (generally*) in $n$
real points, and in no more: although two or more of these $n$ points may
happen to coincide with each other.

(2.) Let $h$ denote, as a temporary abridgment, the old or ordinary imaginary,
$\sqrt{-1}$, of algebra, considered as an uninterpreted symbol, and as not equal to
any real versor, such as $i$ (comp. 181, and 214, (3.)), but as following the rules
of scalars, especially as regards the commutative property of multiplication (126);
so that

$$V. \ldots h^2 + 1 = 0,$$

and $VI. \ldots hi = ih$, but $VII. \ldots h \neq \pm i$.

(3.) Let $q$ denote still a real quaternion, or real couple, $x + iy$; and with
the meaning just now proposed of $h$, let $[q]$ denote the connected but
imaginary algebraic quantity, or bi-scalar (214, (7.)), $x + hy$; so that

$$VIII. \ldots q = x + iy,$$  but $IX. \ldots [q] = x + hy$;

and let any bi quaternion (214, (8.)), or (as we may here call it) bi-couple, of
the form $[q'] + i [q'']$, be said to be complanar with $i$; with the old notation
(123) of complanarity.

(4.) Then, for the polynomial equation in real and complanar quaternions,
$F_n q = 0$ (244, 245), we may be led to substitute the following connected alge­
braical equation, of the same degree, $n$, and involving real scalars similarly:

$$X. \ldots [F_n q] = [q]^n + [q_1] [q]^n-1 + \ldots + [q_n] = 0;$$

which, after the reductions depending on the substitution $V.$ of $-1$ for $h^2$,
receives the form,

$$XI. \ldots [F_n q] = X_n + h Y_n = 0;$$

where $X_n$ and $Y_n$ are the same real and scalar functions as in I.

(5.) But we have seen in II., that these two real functions can be made to
vanish together, by selecting any one of $n$ real pairs IV. of scalar values, $x$ and $y$;

* Cases of equal roots may cause points of intersection, which are generally imaginary, to become
real, but coincident with each other, and with former real roots: for instance the hyperbola, $x^2 - y^2 = a$,
is intersected in two real and distinct points, by the pair of right lines $xy = 0$, if the scalar $a > 0$ or $< 0$;
but for the case $a = 0$, the two pairs of lines, $x^2 - y^2 = 0$ and $xy = 0$, may be considered to have four
coincident intersections at the origin.
the General Algebraical Equation $x^n + y \sqrt{-1} = 0$ has therefore $n$ Real or Imaginary Roots*, of the Form $x + y \sqrt{-1}$; and it has no more than $n$ such roots.

(6.) Elimination of $y$, between the two equations II. or III., conducts generally to an algebraic equation in $x$, of the degree $n^2$; which equation has therefore $n^2$ algebraic roots (5.), real or imaginary; namely, by what has been lately proved, $n$ real and scalar roots $x', \ldots, x^{(n)}$, with real and scalar values $y', \ldots, y^{(n)}$ (comp. IV.) of $y$ to correspond; and $n(n - 1)$ other roots, with the same number of corresponding values of $y$, which may be thus denoted,

$$\text{XII. } [x^{(n+1)}], \ldots, [x^{(n^2)}]$$
$$\text{XIII. } [y^{(n+1)}], \ldots, [y^{(n^2)}]$$

and which are either themselves imaginary (or bi-scalar, 214, (7.)), or at least correspond, by the supposed elimination, to imaginary or bi-scalar values of $y$; since if $x^{(n+1)}$ and $y^{(n+1)}$, for example, could both be real, the quaternion equation $F_n q = 0$, would then have an $(n + 1)$st real root, of the form, $q^{(n+1)} = x^{(n+1)} + iy^{(n+1)}$, contrary to what has been proved (252).

257. On the whole, then, it results that the equation $F_n q = 0$ in complanar quaternions, of the $n$th degree, with real coefficients, while it admits of only $n$ real quaternion roots,

$$\text{I. } q', q'', \ldots, q^{(n)} \text{ (244, &c.),}$$

is symbolically satisfied also (comp. 214, (3.)) by $n(n - 1)$ imaginary quaternion roots, or by $n^2 - n$ bi-quaternions (214, (8.)), or bi-couples (256, (3.)), which may be thus denoted,

$$\text{II. } [q^{(n+1)}], \ldots, [q^{(n^2)}];$$

and of which, the first, for example, has the form,

$$\text{III. } [q^{(n+1)}] = [x^{(n+1)}] + i[y^{(n+1)}] = x^{(n+1)} + h x^{(n+1)} + i(y^{(n+1)} + h y^{(n+1)});$$

where $x^{(n+1)}$, $x^{(n+1)}$, $y^{(n+1)}$, and $y^{(n+1)}$ are four real scalars, but $h$ is the imaginary of algebra (256, (2.)).

* This celebrated Theorem of Algebra has long been known, and has been proved in other ways; but it seemed necessary, or at least useful, for the purpose of the present work, to prove it anew, in connexion with Quaternions; or rather to establish the theorem (244, 252), to which in the present Calculus it corresponds. Compare the Note to page 278.
(1.) There must, for instance, be \( n(n - 1) \) imaginary \( n^{th} \) roots of unity, in the given plane of \( i \) (comp. 256, (3.)), besides the \( n \) real roots already determined (233, 237); and accordingly in the case \( n = 2 \), we have the four following square-roots of \( 1 \parallel i \), two real and two imaginary:

\[
\text{IV.} \quad +1, \quad -1; \quad +hi, \quad -hi;
\]

for, by 256, (2.), we have

\[
\text{V.} \quad (± hi)^2 = h^2i^2 = (-1)(-1) = +1.
\]

And the two imaginary roots of the quadratic equation \( Fq = 0 \), which generally exist, at least as symbols (214, (3.)), may be obtained by multiplying the square-root in the formula 253, XX. by \( hi \); so that in the particular case, when that radical vanishes, the four roots of the equation become real and equal: zero having thus only itself for a square-root.

(2.) Again, if we write (comp. 237, (3.)),

\[
\text{VI.} \quad \varphi = 1_1 = \frac{-1 + i \sqrt{3}}{2}, \quad \varphi^2 = 1_2 = \frac{-1 - i \sqrt{3}}{2},
\]

so that \( 1, \varphi, \varphi^2 \) are the three real cube-roots of positive unity, in the given plane; and if we write also,

\[
\text{VII.} \quad \theta = [\varphi] = \frac{-1 + h \sqrt{3}}{2}, \quad \theta^2 = [\varphi]^2 = \frac{-1 - h \sqrt{3}}{2},
\]

so that \( \theta \) and \( \theta^2 \) are (as usual) the two ordinary (or algebraical) imaginary cube-roots of unity; then the nine cube-roots of \( 1 \parallel i \) are the following:

\[
\text{VIII.} \quad 1; \quad \varphi, \varphi^2; \quad \theta, \theta^2; \quad \theta\varphi, \theta\varphi^2; \quad \theta^2\varphi, \theta^2\varphi^2;
\]

whereof the first is a real scalar; the two next are real couples, or quaternions \( \parallel i \); the two following are imaginary scalars, or biscalars; and the four that remain are imaginary couples, or bi-couples, or biquaternions.

(3.) The sixteen fourth roots of unity \( \parallel i \) are:

\[
\text{IX.} \quad \pm 1; \quad \pm i; \quad \pm h; \quad \pm hi; \quad \pm \frac{1}{2} (1 \pm h) (1 \pm i);
\]

the three ambiguous signs in the last expression being all independent of each other.
(4.) Imaginary roots, of this sort, are sometimes useful, or rather necessary, in calculations respecting ideal intersections,* and ideal contacts, in geometry: although in what remains of the present Volume, we shall have little or no occasion to employ them.

(5.) We may, however, here observe, that when the restriction (225) on the plane of the quaternion q is removed, the General Quaternion Equation of the nth Degree admits, by the foregoing principles, no fewer than \( n^4 \) Roots, real or imaginary; because, when that general equation is reduced, by 221, to the Standard Quadrinomial Form,

\[
X \ldots F_n q = \bar{W}_n + iX_n + jY_n + kZ_n = 0,
\]

it breaks up (comp. 221, VI.) into a System of Four Scalar Equations, each (generally) of the \( n^4 \) dimension, in \( v, x, y, z \); namely,

\[
XI \ldots \bar{W}_n = 0, \quad X_n = 0, \quad Y_n = 0, \quad Z_n = 0;
\]

and if \( x, y, z \) be eliminated between these four, the result is (generally) a scalar (or algebraical) equation of the degree \( n^4 \), relatively to the remaining constituent, \( w \); which therefore has \( n^4 \) (algebraical) values, real or imaginary; and similarly for the three other constituents, \( x, y, z \), of the sought quaternion \( q \).

(6.) It may even happen, when no plane is given, that the number of roots (or solutions) of a finite* equation in quaternions shall become infinite; as has been seen to be the case for the equation \( q^2 = -1 \) (149, 154), even when we confine ourselves to what we have considered as real roots. If imaginary roots be admitted, we may write, still more generally, besides the two bi-scalar values, \( \pm h \), the expression,

\[
XII. \ldots (-1)^{\frac{1}{2}} = v + hv', \quad Sv = Sv' = Svv' = 0, \quad Nv - Nv' = 1;
\]

\( v \) and \( v' \) being thus any two real and right quaternions, in rectangular planes, provided that the norm of the first exceeds that of the second by unity.

(7.) And in like manner, besides the two real and scalar values, \( \pm 1 \), we have this general symbolical expression for a square root of positive unity, with merely the difference of the norms reversed:

\[
XIII. \ldots 1^{\frac{1}{2}} = v + hv', \quad Sv = Sv' = Svv' = 0, \quad Nv' - Nv = 1.
\]

* Comp. Art. 214, and the Notes there referred to.
† Compare the Note to page 277.
SECTION 7.

On the Reciprocal of a Vector, and on Harmonic Means of Vectors; with Remarks on the Anharmonic Quaternion of a Group of Four Points, and on Conditions of Con-circularity.

258. When two vectors, \(a\) and \(a'\), are so related that

I. \(a' = -Ua: Ta\), and therefore II. \(a = -Ua': Ta\),

or that

III. \(Ta \cdot Ta' = 1\), and IV. \(Ua + Ua' = 0\),

we shall say that each of these two vectors is the Reciprocal* of the other; and shall (at least for the present) denote this relation between them, by writing

V. \(a' = Ra\), or VI. \(a = Ra'\);

so that for every vector \(a\), and every right quotient \(v\),

VII. \(Ra = -Ua: Ta\); VIII. \(R^2a = RRa = a\);

and

IX. \(RIv = IRv\) (comp. 161, (3.), and 204, XXXV').

259. One of the most important properties of such reciprocals is contained in the following theorem:

If any two vectors \(oa, ob\), have \(oa', ob'\) for their reciprocals, then (comp. fig. 58) the right line \(a'b'\) is parallel to the tangent \(od\), at the origin \(o\), to the circle \(oab\); and the two triangles, \(oab, ob'a'\), are inversely similar (118). Or in symbols,

I. if \(oa' = R . oa\), and \(ob' = R . ob\),

then

\[ \Delta oab \propto ob'a'. \]

(1.) Of course, under the same conditions, the tangent at \(o\) to the circle \(oa'b'\) is parallel to the line \(ab\).

(2.) The angles \(bao\) and \(ob'a'\) or \(bod\) being equal, the fourth proportional (226) to \(ab, ao,\) and \(ob\), or to \(ba, oa,\) and \(ob\), has the direction of \(od\), or the direction opposite to that of \(a'b'\); and its length is easily proved to be the

* Accordingly, under these conditions, we shall afterwards denote this reciprocal of a vector \(a\) by the symbol \(a^{-1}\); but we postpone the use of this notation, until we shall be prepared to connect it with a general theory of products and powers of vectors. Compare 234, V., and the first Note to page 123. And as regards the temporary use of the characteristic \(R\), compare the second Note to page 262.
reciprocal (or inverse) of the length of the same line \(\overline{a'b'}\), because the similar triangles give,

\[ \text{II. } (\overline{oa} : \overline{ba}) \cdot \overline{ob} = (\overline{ob'} : \overline{a'b'}) \cdot \overline{ob} = 1 : \overline{a'b'}, \]

it being remembered that

\[ \text{III. } \overline{oa} \cdot \overline{oa'} = \overline{ob} \cdot \overline{ob'} = 1; \]

we may therefore write,

\[ \text{IV. } (\overline{oa} : \overline{ba}) \cdot \overline{ob} = R \cdot \overline{a'b'}, \quad \text{or} \quad \text{V. } \beta = R (\overline{R\beta - Ra}), \]

wherever two vectors \(a\) and \(\beta\) may be.

(3.) Changing \(a\) and \(\beta\) to their reciprocals, the last formula becomes,

\[ \text{VI. } R (\beta - a) = \frac{Ra}{Ra - R\beta} \cdot R\beta; \quad \text{or} \quad \text{VII. } (\overline{oa'} : \overline{b'a'}) \cdot \overline{ob'} = R \cdot \overline{ab}. \]

(4.) The inverse similarity \(I\) gives also, generally, the relation,

\[ \text{VIII. } K \frac{\beta}{a} = \frac{Ra}{R\beta}. \]

(5.) Since, then, by 195, \(II\), or 207, (2),

\[ \text{IX. } K \frac{\beta}{a} \pm 1 = K \frac{\beta + a}{a}, \quad \text{we have} \quad \text{X. } \frac{Ra \pm R\beta}{R\beta} = \frac{Ra}{R(\beta + a)}; \]

the lower signs agreeing with \(VI\).

(6.) In general, the reciprocals of opposite vectors are themselves opposite; or in symbols,

\[ \text{XI. } R (-a) = -R\overline{a}. \]

(7.) More generally,

\[ \text{XII. } R\alpha a = \alpha^{-1} R\overline{a}, \]

if \(\alpha\) be any scalar.

(8.) Taking lower signs in \(X\), changing \(a\) to \(\gamma\), dividing, and taking conjugates, we find for any three vectors \(a, \beta, \gamma\) (complanar or diplanar) the formula:

\[ \text{XIII. } K \frac{R\gamma - R\beta}{Ra - R\beta} = K \left( \frac{R\gamma}{R(\beta - \gamma)} \cdot \frac{R(\beta - a)}{Ra} \right) = \frac{a}{\beta - a} \cdot \frac{\gamma - \beta}{\beta - a} = \frac{oA}{\overline{ab}} \cdot \frac{bc}{co}, \]

if \(a = oA, \beta = \overline{ob}, \text{and} \gamma = oo, \text{as usual.} \)

(9.) If then we extend, to any four points of space, the notation (25.),

\[ \text{XIV. } (\overline{abcd}) = \frac{AB}{BC} \cdot \frac{CD}{DA}, \]

interpreting each of these two factor-quotients as a quaternion, and defining that their product (in this order) is the anharmonic quaternion function, or simply the
Anharmonic, of the Group of four points $a, b, c, d$, or of the (plane or gauche) Quadrilateral $abcd$, we shall have the following general and useful formula of transformation:

$$ XV. \ (\text{oabc}) = K \frac{R_γ - R_β}{R_α - R_β} = K \frac{b'c'}{b'a'}, $$

where $oa', ob', oc'$ are supposed to be reciprocals of $oa, ob, oc$.

(10.) With this notation XIV., we have generally, and not merely for collinear groups (35.), the relations:

$$ XVI. \ (abcd) + (acbd) = 1; \quad XVII. \ (abcd) \cdot (adcb) = 1. $$

(11.) Let $o, a, b, c, d$ be any five points, and $oa', \ldots od'$ the reciprocals of $oa, \ldots od$; we shall then have, by XV,

$$ XVIII. \ \frac{b'a'}{b'c'} = K \ (ocba), \quad \frac{d'c'}{d'a'} = K \ (oadc); $$

and therefore,

$$ XIX. \ K \ (a'b'c'd') = (ocba), \quad K \ (oadc) = - (ocdab), $$

if we agree to write generally, for any six points, the formula,*

$$ XX. \ (abcdef) = \frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA}. $$

(12.) If then the five points $o \ldots d$ be complanar (225), we have, by 226, and by XIV.,

$$ XXI. \ K \ (a'b'c'd') = (abcd), \quad \text{or} \quad XXI'. \ (a'b'c'd') = K \ (abcd); $$

the anharmonic quaternion $(abcd)$ being thus changed to its conjugate, when the four rays $oa, \ldots od$ are changed to their reciprocals.

260. Another very important consequence from the definition (258) of reciprocals of vectors, or from the recent theorem (259), may be expressed as follows:

* There is a convenience in calling, generally, this product of three quotients, $(abcdef)$, the evolutionary quaternion, or simply the Evolutionary, of the Group of Six Points, $a \ldots f$, or (if they be not collinear) of the plane or gauche Hexagon $abcdef$: because the equation,

$$ (abca'b'c') = - 1, $$

expresses either Ist, that the three pairs of points, $aa', bb', cc'$, form a collinear involution (26.) of a well-known kind; or 2nd, that those three pairs, or the three corresponding diagonals of the hexagon, compose a complanar or a homospheric Involution, of a new kind suggested by quaternions (comp. 261, (11))).
collinear (24): or, in other words, if the four points o, A, B, C be concircular, then the three points A', B', C' are situated on one right line.

And conversely, if three coinitial vectors, OA', OB', OC', thus terminate on one right line, then their three coinitial reciprocals, OA, OB, OC, are chords of one circle; the tangent to which circle, at the origin, is parallel to the right line; while the anharmonic function (259, (9.)), of the inscribed quadrilateral OABC, reduces itself to a scalar quotient of segments of that line (which therefore is its own conjugate, by 139): namely,

I. . . (OABC) = B'C' : B'A' = (∞A'B'C') = (O . OABC),

if the symbol ∞ be used here to denote the point at infinity on the right line A'B'C'; and if, in thus employing the notation (35) for the anharmonic of a plane pencil, we consider the null chord, oo, as having the direction* of the tangent, od.

(1.) If ρ = op be the variable vector of a point p upon the circle OAB, the quaternion equation of that circle may be thus written:

II. . . Rρ = Rαβ + x(Rα - Rβ), where III. . . x = (OABp);

the coefficient x being thus a variable scalar (comp. 99, I.), which depends on the variable position of the point p on the circumference.

(2.) Or we may write,

IV. . . Rρ = \frac{tRα + uRβ}{t + u},

as another form of the equation of the same circle OAB; with which may usefully be contrasted the earlier form (comp. 25.), of the equation of the line AB,

V. . . ρ = \frac{tα + uβ}{t + u}.

(3.) Or, dividing the second member of IV. by the first, and taking conjugates, we have for the circle,

VI. . . \frac{tρ}{α} + \frac{uρ}{β} = t + u; while VII. . . \frac{tα}{ρ} + \frac{uβ}{ρ} = t + u,

for the right line.

(4.) Or we may write, by II.,

VIII. . . V \frac{Rρ - Rβ}{Ra - Rβ} = 0; or VIII'. . . \frac{Rρ - Rβ}{Ra - Rβ} = V^2 0;

this latter symbol, by 204, (18.), denoting any scalar.

* Compare the remarks in the Note to page 140, respecting the possible determinateness of signification of the symbol U0, when the zero denotes a line, which vanishes according to a law.
(5.) Or still more briefly,

\[
\mathbf{IX} \ldots \mathbf{V} (\mathbf{oabp}) = 0; \quad \text{or} \quad \mathbf{IX}' \ldots (\mathbf{oabp}) = \mathbf{V}^{-1} \mathbf{0}.
\]

(6.) If the four points \(o, \alpha, \beta, \gamma\) be still concircular, and if \(p\) be any fifth point in their plane, while \(\mathbf{po_1}, \ldots \mathbf{pc_1}\) are the reciprocals of \(\mathbf{po}, \ldots \mathbf{pc}\), then by 259, XXI, we have the relation,

\[
\mathbf{X} \ldots (\mathbf{oiAicC}) = K (\mathbf{oabc}) = (\mathbf{oabc}) = \mathbf{V}^{-1} \mathbf{0};
\]

the four new points \(\alpha_1 \ldots \gamma_1\) are therefore generally concircular.

(7.) If, however, the point \(p\) be again placed on the circle \(\mathbf{oabc}\), those four new points are (by the present Article) collinear; being the intersections of the pencil \(p \cdot \mathbf{oabc}\) with a parallel to the tangent at \(p\). In this case, therefore, we have the equation,

\[
\mathbf{XI} \ldots (\mathbf{p} \cdot \mathbf{oabc}) = (\mathbf{o^1a^1b^1c^1}) = (\mathbf{oabc}) ;
\]

so that the constant anharmonic of the pencil (35) is thus seen to be equal to what we have defined (259, (9.)) to be the anharmonic of the group.

(8.) And because the anharmonic of a circular group is a scalar, it is equal (by 187, (8.)) to its own tensor; either positively or negatively taken: we may therefore write, for any inscribed quadrilateral \(\mathbf{oabc}\), the formula,

\[
\mathbf{XII} \ldots (\mathbf{oabc}) = \mp T (\mathbf{oabc}) = \mp (\mathbf{oa} \cdot \mathbf{bc}) : (\mathbf{ab} \cdot \mathbf{co}) ,
\]

= \pm a quotient of rectangles of opposite sides; the upper or the lower sign being taken, according as the point \(b'\) falls, or does not fall, between the points \(a'\) and \(c'\): that is, according as the quadrilateral \(\mathbf{oabc}\) is an uncrossed or a crossed one.

(9.) Hence it is easy to infer that for any circular group \(o, \alpha, \beta, \gamma\), we have the equation,

\[
\mathbf{XIII} \ldots U \frac{\mathbf{oa}}{\mathbf{AB}} = \pm U \frac{\mathbf{co}}{\mathbf{CB}} ;
\]

the upper sign being taken when the succession \(\mathbf{oabc}\) is a direct one, that is, when the quadrilateral \(\mathbf{oabc}\) is uncrossed; and the lower sign, in the contrary case, namely, when the succession is (what may be called) indirect, or when the quadrilateral is crossed: while conversely this equation XIII. is sufficient to prove, whenever it occurs, that the anharmonic \((\mathbf{oabc})\) is a negative or a positive scalar, and therefore by (5.) that the group is circular (if not linear), as above.

(10.) If \(a, b, c, d, e\) be any five homospheric points (or points upon the surface of one sphere), and if \(o\) be any sixth point of space, while \(oa', \ldots oe'\)

Hamilton's Elements of Quaternions.
are the reciprocals of $oa, . . oe$, then the five new points $a', . . e'$ are generally homospheric (with each other); but if $o$ happens to be on the sphere $ABCDE$, then $a', . . e'$ are complanar, their common plane being parallel to the tangent plane to the given sphere at $o$: with resulting anharmonic relations, on which we cannot here delay.

261. An interesting case of the foregoing theory is that when the generally scalar anharmonic of a circular group becomes equal to negative unity: in which case (comp. 26), the group is said to be harmonic. A few remarks upon such circular and harmonic groups may here be briefly made: the student being left to fill up hints for himself, as what must be now to him an easy exercise of calculation.

(1.) For such a group (comp. again fig. 58), we have thus the equation,

$$ I. . . (oabc) = -1; $$
and therefore

$$ II. . . a'b' = b'o'; $$
or

$$ III. . . R\beta = \frac{1}{2} (Ra + R\gamma); $$
and under this condition, we shall say (comp. 216, (5.)) that the Vector $\beta$ is the \textit{Harmonic Mean} between the two vectors, $a$ and $\gamma$.

(2.) Dividing, and taking conjugates (comp. 260, (3.), and 216, (5.)), we thus obtain the equation,

$$ IV. . . \beta + \frac{\beta}{a} = 2; $$
or

$$ V . . . \beta = \frac{2a}{\gamma} \gamma = \frac{2a}{\gamma + a}; $$
or

$$ VI. . . \beta = \frac{a}{\varepsilon} \gamma = \frac{\gamma a}{\varepsilon}, $$
if

$$ VII. . . \varepsilon = \frac{1}{2} (\gamma + a); $$

$\varepsilon$ thus denoting here the vector $oh$ (fig. 58) of the middle point of the chord $ac$. We may then say that the \textit{harmonic mean} between any two lines is (as in algebra) the \textit{fourth proportional} to their semisum, and to themselves.

(3.) Geometrically, we have thus the similar triangles,

$$ VIII. . . \Delta aob \propto hoc; $$
$$ VIII'. . . \Delta aoe \propto hoc; $$
whence, either because the angles $obra$ and $oac$, or because the angles $oac$ and $oeb$ are equal, we may infer (comp. 260, (5.)) that, when the equation $I. .$ is satisfied, the four points $o, a, b, c$, if not collinear, are concircular.

(4.) We have also the similarities,

$$ IX. . . \Delta oeb \propto ceb, $$
$$ IX'. . . \Delta oea \propto aeb; $$
or the equations,

$$ X. . . \frac{\beta - \varepsilon}{\gamma - \varepsilon} = \frac{\gamma - \varepsilon}{-\varepsilon}, $$
and

$$ X'. . . \frac{\beta - \varepsilon}{a - \varepsilon} = \frac{\gamma - \varepsilon}{a - \varepsilon}; $$
in fact we have, by VI. and VII.,

\[ \begin{align*} 
\text{XI.} & \quad \frac{a}{\varepsilon} + \frac{\gamma}{\varepsilon} = 2; \\
\text{XII.} & \quad \frac{\beta - \varepsilon}{-\varepsilon} + \frac{\gamma}{\varepsilon} = 1 - \frac{\beta}{\alpha} - \frac{\varepsilon}{\varepsilon} = 1 - \frac{\gamma}{\varepsilon} = \left(1 - \frac{\alpha}{\varepsilon}\right)^2.
\end{align*} \]

(5.) Hence the line ec, in fig. 58, is the mean proportional (227) between the lines eo and eb; or in words, the semisum (oe), the semidifference (ec), and the excess (be) of the semisum over the harmonic mean (ob), form (as in algebra) a continued proportion (227).

(6.) Conversely, if any three coinitial vectors, eo, ec, eb, form thus a continued proportion, and if we take ea = ce, then the four points oabc will compose a circular and harmonic group; for example, the points apbr' of fig. 57 are arranged so as to form such a group.*

(7.) It is easy to prove that, for the inscribed quadrilateral oabc of fig. 58, the rectangles under opposite sides are each equal to half of the rectangle under the diagonals; which geometrical relation answers to either of the two anharmonic equations (comp. 259, (10.)):

\[ \begin{align*} 
\text{XIII.} & \quad (obac) = + 2; \\
\text{XIII'.} & \quad (ocab) = + \frac{1}{2}.
\end{align*} \]

(8.) Hence, or in other ways, it may be inferred that these diagonals, ob, ac, are conjugate chords of the circle to which they belong: in the sense that each passes through the pole of the other, and that thus the line db is the second tangent from the point d, in which the chord ac prolonged intersects the tangent at o.

(9.) Under the same conditions, it is easy to prove, either by quaternions or by geometry, that we have the harmonic equations:

\[ \begin{align*} 
\text{XIV.} & \quad (abco) = (bcoa) = (coab) = - 1;
\end{align*} \]

so that ac is the harmonic mean between ab and ao; bo is such a mean between bc and ba; and ca between co and cb.

(10.) In any such group, any two opposite points (or opposite corners of the quadrilateral), as for example o and b, may be said to be harmonically conjugate to each other, with respect to the two other points, a and c; and we see that when these two points a and c are given, then to every third point o (whether in a given plane, or in space) there always corresponds a fourth point b, which is in this sense conjugate to that third point: this fourth point being always complanar with the three points a, c, o, and being even concircular with them.

* Compare the Note to 255, (2.). In fig. 58, the centre of the circle oabc is concircular with the three points o, b, a.
unless they happen to be collinear with each other; in which extreme (or limiting) case, the fourth point b is still determined, but is now collinear with the others (as in 26, &c.).

(11.) When, after thus selecting two* points, a and c, or treating them as given or fixed, we determine (10.) the harmonic conjugates b, b', b'', with respect to them, of any three assumed points, o, o', o'', then the three pairs of points, o, b; o', b'; o'', b''; may be said to form an Involution,† either on the right line ac, (in which case it will only be one of an already well-known kind), or in a plane through that line, or even generally in space: and the two points a, c may in all these cases be said to be the two Double Points (or Foci) of this Involution. But the field thus opened, for geometrical investigation by Quaternions, is far too extensive to be more than mentioned here.

(12.) We shall therefore only at present add, that the conception of the harmonic mean between two vectors may easily be extended to any number of such, and need not be limited to the plane: since we may define that \( \eta \) is the harmonic mean of the \( n \) arbitrary vectors \( a_1, \ldots a_n \), when it satisfies the equation,

\[
\text{XV. } \eta = \frac{1}{n} (Ra_1 + \ldots + Ra_n); \quad \text{or, } \text{XVI. } n\eta = \Sigma Ra.
\]

(13.) Finally, as regards the notation \( Ra \), and the definition (258) of the reciprocal of a vector, it may be observed that if we had chosen to define reciprocal vectors as having similar (instead of opposite) directions, we should indeed have had the positive sign in the equation 258, VII.; but should have been obliged to write, instead of 258, IX., the much less simple formula,

\[
\text{RI}v = - \text{IR}v.
\]

* There is a sense in which the geometrical process here spoken of can be applied, even when the two fixed points, or foci, are imaginary. Compare the Géométrie Supérieure of M. Chasles, page 136.

† Compare the Note to 259, (11.).
CHAPTER III.

ON DIPLANAR QUATERNIONS, OR QUOTIENTS OF VECTORS IN SPACE: AND ESPECIALLY ON THE ASSOCIATIVE PRINCIPLE OF MULTIPLICATION OF SUCH QUATERNIONS.

SECTION 1.

On some Enunciations of the Associative Property, or Principle, of Multiplication of Diplanar Quaternions.

262. In the preceding chapter we have confined ourselves almost entirely, as had been proposed (224, 225), to the considerations of quaternions in a given plane (that of $i$); alluding only, in some instances, to possible extensions* of results so obtained. But we must now return to consider, as in the First Chapter of this Second Book, the subject of General Quotients of Vectors: and especially their Associative Multiplication (223), which has hitherto been only proved in connexion with the Distributive Principle (212), and with the Laws of the Symbols, $i, j, k$ (183). And first we shall give a few geometrical enunciations of that associative principle, which shall be independent of the distributive one, and in which it will be sufficient to consider (comp. 191) the multiplication of versors; because the multiplication of tensors is evidently an associative operation, as corresponding simply to arithmetical multiplication, or to the composition of ratios in geometry.† We shall therefore suppose, throughout the present chapter, that $q, r, s$ are some three given but arbitrary versors, in three given and distinct planes;‡ and our object will be to throw some

---

* As in 227, (3.); 242, (7.); 254, (7.); 257, (6.) and (7.); 259, (8.), (9.), (10.), (11.); 260, (10.); and 261, (11.) and (12.).
† Or, more generally, for any three pairs of magnitudes, each pair separately being homogeneous.
‡ If the factors $q, r, s$ were complanar, we could always (by 120) put them under the forms,
$$
q = \frac{\beta}{a}, \quad r = \frac{\gamma}{\beta}, \quad s = \frac{\delta}{\gamma};
$$
and then should have (comp. 183, (1.)) the two equal ternary products,
$$
sr \cdot q = \frac{\delta}{\beta} \frac{\beta}{a} = \frac{\delta}{a} \frac{\gamma}{\gamma} = s \cdot rq;
$$
so that in this case (comp. 224) the associative property would be proved without any difficulty.
additional light, by new enunciations in this section, and by new demonstra-
tions in the next, on the very important, although very simple, Associative
Formula (223, II.), which may be written thus:

\[ I \ldots sr.q = s.rq; \]

or thus, more fully,

\[ II \ldots q'q = t, \text{ if } q' = sr, \ s' = rq, \text{ and } t = ss'; \]

\( q', s', \text{ and } t \) being here three new and derived versors, in three new and derived planes.

263. Already we may see that this Associative Theorem of Multiplication,
in all its forms, has an essential reference to a System of Six Planes, namely
the planes of these six versors,

\[ IV \ldots q, r, s, rq, sr, srq, \text{ or } IV' \ldots q, r, s, s', q', t; \]
on the judicious selection and arrangement of which, the clearness and
elegance of every geometrical statement or proof of the theorem must very
much depend: while the versor character of the factors (in the only part of
the theorem for which proof is required) suggests a reference to a Sphere,
namely to what we have called the unit-sphere (128). And the three following
arrangements of the six planes appear to be the most natural and simple that
can be considered: namely, Ist, the arrangement in which the planes all pass
through the centre of the sphere; IInd, that in which they all touch its surface;
and IIIrd, that in which they are the six faces of an inscribed solid. We
proceed to consider successively these three arrangements.

264. When the first arrangement (263) is adopted, it is natural to employ
arcs of great circles, as representatives of the versors, on the plan of Art. 162.
Representing thus the factor \( q \) by the arc \( AB \), and \( r \) by the successive arc \( BC \),
we represent (167) their product \( rq \), or \( s' \), by \( AC \); or by any equal arc (165),
such as \( DE \), in fig. 59, may be supposed to be. Again, representing \( s \) by \( EF \),
we shall have \( DF \) as the representative of the ternary product \( sq.rq \), or \( ss' \), or \( t \), taken in one
order of association. To represent the other ternary product, \( sr.q \), or \( q'q \), we may first
determine three new points, \( G, H, I \), by arcal equations (165), between \( GH, BC \), and between
\( HI, EF \), so that \( BC, EF \) intersect in \( H \), as the
arcs representing \( s' \) and \( s \) had intersected in \( E \); and then, after thus finding
an arc \( GI \) which represents \( sr \) or \( q' \), may determine three other points \( K, L, M, \)
by equations between $KL, AB,$ and between $LM, GI,$ so that these two new arcs, $KL, LM,$ represent $q$ and $q',$ and that $AB, GI$ intersect in $L;$ for in this way we shall have an arc, namely $KM,$ which represents $q'q$ as required. And the theorem then is, that this last arc $KM$ is equal to the former arc $DF,$ in the full sense of Art. 165; or that when (as under the foregoing conditions of construction) the five arcual equations,

I. $\angle AB = \angle KL, \quad \angle BC = \angle GH, \quad \angle EF = \angle HI, \quad \angle AC = \angle DE, \quad \angle GI = \angle LM,$

exist, then this sixth equation of the same kind is satisfied also,

II. $\angle DF = \angle KM$:

the two points, $K$ and $M,$ being both on the same great circle as the two previously determined points, $D$ and $F;$ or $D$ and $M$ being on the great circle through $F$ and $K:$ and the two arcs, $DF$ and $KM,$ of that great circle, or the two dotted arcs, $DK,$ $FM$ in the figure, being equally long, and similarly directed (165).

(1.) Or, after determining the nine points $A, ..., I$ so as to satisfy the three middle equations I., we might determine the three other points $K, L, M,$ without any other arcual equations, as intersections of the three pairs of arcs $AB, DF;\ AB, GI;\ DF, GI;$ and then the theorem would be, that (if these three last points be suitably distinguished from their own opposites upon the sphere) the two extreme equations I., and the equation II., are satisfied.

(2.) The same geometrical theorem may also be thus enunciated: If the first, third, and fifth sides ($KL, GH, ED$) of a spherical hexagon $KLGHE$ be respectively and arcually equal (165) to the first, second, and third sides ($AB, BC, CA$) of a spherical triangle $ABC,$ then the second, fourth, and sixth sides ($LG, HE, DK$) of the same hexagon are equal to the three successive sides ($MI, IF, FM$) of another spherical triangle $MIF.

(3.) It may be also said, that if five successive sides ($KL, ..., ED$) of one spherical hexagon be respectively and arcually equal to the five successive diagonals ($AB, MI, BC, IF, CA$) of another such hexagon ($AMHICF$), then the sixth side ($DK$) of the first is equal to the sixth diagonal ($FM$) of the second.

(4.) Or, if we adopt the conception mentioned in 180, (3.), of an arcual sum, and denote such a sum by inserting + between the symbols of the two summands, that of the added arc being written to the left-hand, we may state the theorem, in connexion with the recent fig. 59, by the formula:

III. $\angle DF + \angle BA = \angle EF + \angle BC,$ if $\angle DA = \angle EC$;

where $B$ and $F$ may denote any two points upon the sphere.
(5.) We may also express* the same principle, although somewhat less simply, as follows (see again fig. 59, and compare sub-art. (2.)):

IV. if $\angle ED + \angle GH + \angle KL = 0$, then $\angle DX + \angle HE + \angle LG = 0$.

(6.) If, for a moment, we agree to write (comp. Art. 1),

V. $\angle AB = B - A$,

we may then express the recent statement IV. a little more lucidly thus:

VI. if $D - E + H - G + L - K = 0$, then $K - D + E - H + G - L = 0$.

(7.) Or still more simply, if $\alpha$, $\alpha'$, $\alpha''$ be supposed to denote any three diplanar arcs, which are to be added according to the rule (180, (3.)) above referred to, the theorem may be said to be, that

VII. $(\alpha'' + \alpha') + \alpha = \alpha'' + (\alpha' + \alpha)$;

or in words, that Addition of Arcs on a Sphere is an Associative Operation.

(8.) Conversely, if any independent demonstration be given, of the truth of any one of the foregoing statements, considered as expressing a theorem of spherical geometry,† a new proof will thereby be furnished, of the associative property of multiplication of quaternions.

265. In the second arrangement (263) of the six planes, instead of representing the three given versors, and their partial or total products, by arcs, it is natural to represent them (174, II.) by angles on the sphere. Conceive then that the two versors, $q$ and $r$, are represented, in fig. 60, by the two spherical angles, $EAB$ and $ABE$; and therefore (175) that their product, $rq$ or $s'$, is represented by the external vertical angle at $E$, of the triangle $ABE$. Let the second versor $r$ be also represented by the angle $FBC$, and the third versor $s$ by $BCF$; then the other binary product, $sr$ or $q'$, will be represented by the external angle at $F$, of the new triangle $BCF$. Again, to represent the first ternary product, $t = ss' = s \cdot rq$, we have only to take the

---

* Some of these formulae and figures, in connexion with the associative principle, are taken, though for the most part with modifications, from the author's Sixth Lecture on Quaternions, in which that whole subject is very fully treated. Comp. the Note to page 160.

† Such a demonstration, namely a deduction of the equation II. from the five equations I., by known properties of spherical conics, will be briefly given in the ensuing section.
Three arrangements of planes.

External angle at \( D \) of the triangle \( ECD \), if \( D \) be a point determined by the two conditions, that the angle \( ECD \) shall be equal to \( BCF \), and \( DEC \) supplementary to \( BEA \). On the other hand, if we conceive a point \( D' \) determined by the conditions that \( D'AF \) shall be equal to \( EAB \), and \( AFD' \) supplementary to \( CFB \), then the external angle at \( D' \), of the triangle \( AFD' \), will represent the second ternary product, \( q'q = sr \cdot q \), which (by the associative principle) must be equal to the first. Conceiving then that \( ED \) is prolonged to \( G \), and \( ED' \) to \( H \), the two spherical angles, \( GDC \) and \( AD'H \), must be equal in all respects; their vertices \( D \) and \( D' \) coinciding, and the rotations (174, 177) which they represent being not only equal in amount, but also similarly directed. Or, to express the same thing otherwise, we may enunciate (262) the Associative Principle by saying, that when the three angular equations,

\[
\begin{align*}
I \ldots & \ ABE = BFC, \quad BCF = ECD, \quad DEC = \pi - BEA, \\
II \ldots & \ DAF = EAB, \quad FDA = CDE, \quad AFD = \pi - CFB,
\end{align*}
\]

are satisfied, then these three other equations,

\[
\begin{align*}
I \ldots & \ c'ab'd, \quad ab'cd', \quad b'ca'd, \quad ca'bd', \quad a'bc'd,
\end{align*}
\]

are satisfied also. For not only is this theorem of spherical geometry a consequence of the associative principle of multiplication of quaternions, but conversely any independent demonstration* of the theorem is, at the same time, a proof of the principle.

266. The third arrangement (263) of the six planes may be illustrated by conceiving a gauche hexagon, \( AB'C'A'BC' \), to be inscribed in a sphere, in such a manner that the intersection \( D \) of the three planes, \( c'ab', b'ca', a'bc' \), is on the surface; and therefore that the three small circles, denoted by these three last triliteral symbols, concur in one point \( D \); while the second intersection of the two other small circles, \( AB'C, CA'B \), may be denoted by the letter \( D' \), as in the annexed fig. 61. Let it be also for simplicity at first supposed, that (as in the figure) the five circular successions,

\[
\begin{align*}
I \ldots & \ c'ab'd, \quad ab'cd', \quad b'ca'd, \quad ca'bd', \quad a'bc'd,
\end{align*}
\]

are all direct; or that the five inscribed quadrilaterals, denoted by these symbols

* Such as we shall sketch, in the following section, with the help of the known properties of the spherical cones. Compare the Note to the foregoing Article.

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I, are all uncrossed ones. Then (by 260, (9.)) it is allowed to introduce three versors, \( q, r, s \), each having two expressions, as follows:

\[
\begin{align*}
\mathbf{II} \ldots q &= U \frac{B'D}{DC'} + U \frac{AB'}{AC'}; \\
&= U \frac{DA'}{B'D} = U \frac{CA'}{CB}; \\
s &= U \frac{CD'}{CA'} + U \frac{BD'}{A'B};
\end{align*}
\]

although (by the cited sub-article) the last members of these three formulae should receive the negative sign, if the first, third, and fourth of the successions I. were to become indirect, or if the corresponding quadrilaterals were crossed ones. We have thus (by 191) the derived expressions,

\[
\begin{align*}
\mathbf{III} \ldots s' &= rq = U \frac{DA'}{DC'} = U \frac{A'B}{BC}; \\
q' &= sr = U \frac{CD'}{CB} = U \frac{D'A}{AB};
\end{align*}
\]

whereof, however, the two versors in the first formula would differ in their signs, if the fifth succession I. were indirect; and those in the second formula, if the second succession were such. Hence,

\[
\begin{align*}
\mathbf{IV} \ldots t &= ss' = s \cdot rq = U \frac{BD'}{BC}; \\
q'q &= sr \cdot q = U \frac{D'A}{AC};
\end{align*}
\]

and since, by the associative principle, these two last versors are to be equal, it follows that, under the supposed conditions of construction, the four points, \( B, C', A, D' \), compose a circular and direct succession; or that the quadrilateral, \( BC'AD' \), is plane, inscriptible, and uncrossed.

267. It is easy, by suitable changes of sign, to adapt the recent reasoning to the case where some or all of the successions I. are indirect; and thus to infer, from the associative principle, this theorem of spherical geometry: if \( AB'C'A'BC' \) be a spherical hexagon, such that the three small circles \( c'ab', b'ca', a'bc' \) concur in one point \( D \), then, 1st, the three other small circles, \( AB'C, CA'B, BC'A \), concur in another point, \( D' \); and 2nd, of the six circular successions, 266, I., and \( BC'AD' \), the number of those which are indirect is always even (including zero). And conversely, any independent demonstration† of this geometrical theorem will be a new proof of the associative principle.

268. The same fertile principle of associative multiplication may be enunciated in other ways, without limiting the factors to be versors, and

* Of course, since the four points \( BC'AD' \) are known to be homospheric (comp. 260, (10.) ), the inscriptibility of the quadrilateral in a circle would follow from its being plane, if the latter were otherwise proved: but it is here deduced from the equality of the two versors \( IV. \), on the plan of 260, (9.).

† An elementary proof, by stereographic projection, will be proposed in the following section.
without introducing the conception of a sphere. Thus we may say (comp. 264, (2.)), that if o . abedef (comp. 35) be any pencil of six rays in space, and o . a'b'c' any pencil of three rays, and if the three angles aob, cod, eof of the first pencil be respectively equal to the angles b'oc', c'oa', a'ob' of the second, then another pencil of three rays, o . a''b''c'', can be assigned, such that the three other angles boc, doe, foa of the first pencil shall be equal to the angles b''oc'', c''oa'', a''ob'' of the third: equality of angles (with one vertex) being here understood (comp. 165) to include complanarity, and similarity of direction of rotations.

(1.) Again (comp. 264, (4.)), we may establish the following formula, in which the four vectors $a\beta\gamma\delta$ form a complanar proportion (226), but $\varepsilon$ and $\zeta$ are any two lines in space:

$$I. \frac{\xi}{\varepsilon} = \frac{\zeta}{\zeta}, \frac{\delta}{\delta} = \frac{\beta}{\beta}, \frac{\alpha}{\alpha} = \frac{\varepsilon}{\varepsilon}, \frac{\gamma}{\gamma},$$

for, under this last condition, we have (comp. 125),

$$II. \frac{\xi}{\varepsilon} = \frac{\zeta}{\zeta}, \frac{\delta}{\delta} = \frac{\beta}{\beta}, \frac{\alpha}{\alpha} = \frac{\varepsilon}{\varepsilon}, \frac{\gamma}{\gamma}.$$

(2.) Another enunciation of the associative principle is the following:

$$III. \text{ if } \frac{\delta}{\varepsilon} = \frac{\beta}{\beta}, \text{ then } \frac{\varepsilon}{\alpha} = \frac{\zeta}{\zeta},$$

for if we determine (120) six new vectors, $\eta\theta\iota$, and $\kappa\lambda\mu$, so that

$$IV. \begin{align*}
\frac{\theta}{\eta} \frac{\delta}{\gamma} &= \frac{\beta}{\alpha}, \\
\frac{\eta}{\iota} &= \frac{\beta}{\alpha}, \\
\frac{\lambda}{\kappa} &= \frac{\beta}{\alpha}, \\
\frac{\kappa}{\lambda} &= \frac{\beta}{\alpha}, \\
\text{ whence } \frac{\theta}{\iota} &= \frac{\zeta}{\zeta}, \\
\text{ and } \frac{\lambda}{\mu} &= \frac{\beta}{\gamma},
\end{align*}$$

we shall have the transformations,

$$V. \frac{\lambda}{\zeta} = \frac{\lambda}{\varepsilon} \frac{\iota}{\theta} = \frac{\lambda}{\varepsilon} \frac{\eta}{\theta} = \frac{\lambda}{\varepsilon} \frac{\eta}{\theta} = \frac{\lambda}{\varepsilon} \frac{\eta}{\theta} = \frac{\lambda}{\varepsilon} \frac{\eta}{\theta} = \frac{\lambda}{\varepsilon} \frac{\eta}{\theta} = \frac{\lambda}{\varepsilon} \frac{\eta}{\theta}, \text{ or } VI. \frac{\lambda}{\mu} = \frac{\zeta}{\zeta}.$$

(3.) Conversely, the assertion that this last equation or proportion VI. is true, whenever the twelve vectors $a .. \mu$ are connected by the five proportions IV., is a form of enunciation of the associative principle; for it conducts (comp. IV. and V.) to the equation

$$VII. \frac{\lambda}{\varepsilon} \frac{\iota}{\eta} \frac{\delta}{\theta} = \frac{\lambda}{\varepsilon} \frac{\eta}{\theta} \frac{\delta}{\theta}, \text{ at least if } \varepsilon || \iota, \theta;$$

but, even with this last restriction, the three factor-quotients in VII. may represent any three quaternions.

2 R 2
SECTION 2.

On some Geometrical Proofs of the Associative Property of Multiplication of Quaternions, which are independent of the Distributive* Principle.

269. We propose, in this section, to furnish three geometrical Demonstrations of the Associative Principle, in connexion with the three figures (59-61) which were employed in the last section for its Enunciation; and with the three arrangements of six planes, which were described in Art. 263. The two first of these proofs will suppose the knowledge of a few properties of spherical conics (196, (11.)); but the third will only employ the doctrine of stereographic projection, and will therefore be of a more strictly elementary character. The Principle itself is, however, of such great importance in this Calculus, that its nature and its evidence can scarcely be put in too many different points of view.

270. The only properties of a spherical conic, which we shall in this Article assume as known,* are the three following: 1st, that through any three given points on a given sphere, which are not on a great circle, a conic can be described (consisting generally of two opposite ovals), which shall have a given great circle for one of its two cyclic arcs; 2nd, that if a transversal arc cut both these arcs, and the conic, the intercepts (suitably measured) on this transversal are equal; and 3rd, that if the vertex of a spherical angle move along the conic, while its legs pass always through two fixed points thereof, those legs intercept a constant interval, upon each cyclic arc, separately taken. Admitting these three properties, we see that if, in fig. 59, we conceive a spherical conic to be described, so as to pass through the three points \( b, f, h, \) 

* Compare 224 and 262; and the Note to page 245.
† The reader may consult the Translation (Dublin, 1841, pp. 46, 50, 55) by the present Dean Graves, of two Memoirs by M. Chasles, on Cones of the Second Degree, and Spherical Conics. [If a cone have one system of cyclic sections parallel to \( \alpha \beta \beta \), on inversion from \( o \), the vertex of the cone, it is seen to have a second system parallel to the tangent plane at \( o \) to the sphere through the vertex and the circle \( \alpha \beta \beta \). In the figure the circle \( oab \) is the section of this sphere by the plane through two edges of the cone \( o \alpha o \) and \( o \beta \), while \( o \alpha \) (parallel to \( \alpha \beta \)) and \( o \beta \) (tangent to \( o \alpha o \)) are the traces of the cyclic planes. \( \angle \alpha' o = \angle \beta o = \angle \alpha o \beta \) proves the 2nd property. Again, if \( \alpha o' \) and \( o \beta ' \) are the traces on the cyclic plane parallel to \( \alpha \beta \beta \) by the planes \( \alpha o \beta \) and \( o o' \) respectively, \( \angle \alpha \beta \beta = \angle \alpha' o = \angle \alpha o' \beta \), \( \alpha o' \) being parallel to \( \alpha \beta \) and \( o o' \) to \( o \beta \). But as \( p \) moves along the circle \( \alpha \beta \beta \), the angle \( \alpha \beta \beta \) is constant, and thus the 3rd property is also proved.]
and to have the great circle $DAEC$ for one cyclic arc, the second and third equations I. of 264 will prove that the arc $GLIM$ is the other cyclic arc for this conic; the first equation I. proves next that the conic passes through $K$; and if the arcual chord $FK$ be drawn and prolonged, the two remaining equations prove that it meets the cyclic arcs in $D$ and $M$; after which, the equation II. of the same Art. 264 immediately results, at least with the arrangement* adopted in the figure.

(1.) The 1st property is easily seen to correspond to the possibility of circumscribing a circle about a given plane triangle, namely that of which the corners are the intersections of a plane parallel to the plane of the given cyclic arc, with the three radii drawn to the three given points upon the sphere: but it may be worth while, as an exercise, to prove here the IInd property by quaternions.

(2.) Take then the equation of a cyclic cone, 196, (8.), which may (by 196, XII.) be written thus:

$$I \ldots \frac{\rho}{a} \frac{p}{\beta} = N \frac{p}{\beta}; \quad \text{and let} \quad II \ldots \frac{\rho'}{a} \frac{p'}{\beta} = N \frac{p'}{\beta},$$

$\rho$ and $\rho'$ being thus two rays (or sides) of the cone, which may also be considered to be the vectors of two points $p$ and $p'$ of a spherical conic, by supposing that their lengths are each unity. Let $\tau$ and $\tau'$ be the vectors of the two points $\tau$ and $\tau'$ on the two cyclic arcs, in which the arcual chord $pp'$ of the conic cuts them; so that

$$III \ldots S \frac{\tau}{a} = 0, \quad S \frac{\tau'}{\beta} = 0, \quad \text{and} \quad IV \ldots T\tau = T\tau' = 1.$$  

The theorem may then be stated thus: that

$$V \ldots \text{if } \rho = x\tau + x'\tau', \quad \text{then} \quad VI \ldots \rho' = x\tau + x'\tau';$$

or that this expression VI. satisfies II., if the equations I. III. IV. V. be satisfied.

Now, by III. V. VI., we have

$$VII \ldots S \frac{\rho}{a} = x S \frac{\tau'}{a} = \frac{x}{x'} S \frac{\rho'}{a}, \quad S \frac{\rho}{\beta} = x S \frac{\tau}{\beta} = \frac{x}{x'} S \frac{\rho'}{\beta};$$

whence it follows that the first members of I. and II. are equal, and it only

* Modifications of that arrangement may be conceived, to which however it would be easy to adapt the reasoning.
remains to prove that their second members are equal also, or that $T\rho' = T\rho$, if $T\tau' = T\tau$.

Accordingly we have, by $\nabla$ and $\nabla'$,

$$VIII. \frac{\rho' - \rho}{\rho' + \rho} = \frac{x'-x}{x'+x} \cdot \frac{\tau - \tau'}{\tau + \tau'} = S^{-0}, \text{by 200, (11.), and 204, (19.);}$$

and the property in question is proved.

271. To prove the associative principle, with the help of fig. 60, three other properties of a spherical conic shall be supposed known: * Ist, that for every such curve two focal points exist, possessing several important relations to it, one of which is, that if these two foci and one tangent arc be given, the conic can be constructed; IInd, that if, from any point upon the sphere, two tangents be drawn to the conic, and also two arcs to the foci, then one focal arc makes with one tangent the same angle as the other focal arc with the other tangent; and IIIrd, that if a spherical quadrilateral be circumscribed to such a conic (supposed here for simplicity to be a spherical ellipse, or the opposite ellipse being neglected), opposite sides subtend supplementary angles, at either of the two (interior) foci. Admitting these known properties, and supposing the arrangement to be as in fig. 60, we may conceive a conic described, which shall have $e$ and $f$ for its two focal points, and shall touch the arc $bc$; and then the two first of the equations I., in 265, will prove that it touches also the arcs $ab$ and $cd$, while the third of those equations proves that it touches $ad$, so that $abcd$ is a circumscribed† quadrilateral: after which the three equations II., of the same article, are consequences of the same properties of the curve.$\dagger$

272. Finally, to prove the same important Principle in a more completely elementary way, by means of the arrangement represented in fig. 61, or to prove the theorem of spherical geometry enunciated in Art. 267, we

---

* The reader may again consult pages 46 and 60 of the Translation lately cited. In strictness, there are of course four foci, opposite two by two.

† The writer has elsewhere proposed the notation, $ef (\ldots) abcd$, to denote the relation of the focal points $e, f$ to this circumscribed quadrilateral.

$\dagger$ [The two cyclic arcs and a point determine a spherical conic. Referring to the Note on 270, describe a sphere to touch one cyclic plane at the point $o$. Then if $oa$ is given, take the section $\triangle AB$ of the sphere by a plane parallel to the second cyclic plane, and the cone is determined. Reciprocating this, the 1st property follows. The 2nd property is the reciprocal of the 1nd of 270, and the 3rd is easily derived by reciprocating the 3rd of 270, remembering that for a point $v'$ on the remaining arc of the circle $\triangle AB$, $\angle A'vB + \angle A'v = \pi$.]
may assume the point $d$ as the pole of a stereographic projection, in which the three small circles through that point shall be represented by right lines, but the three others by circles, all being in one common plane. 

And then (interchanging accents) the theorem comes to be thus stated:

*If $a', b', c'$ be any three points (comp. fig. 62) on the sides $bc$, $ca$, $ab$ of any plane triangle, or on those sides prolonged, then, 1st, the three circles, I. . . $c'ab'$, $a'bc'$, $b'ca'$,

will meet in one point $d$; and 2nd, an even number (if any) of the six (linear or circular) successions,

II. . . $ab'c$ = $n\pi$, $bc'a$ = $n'\pi$, $ca'b$ = $n''\pi$,

will be direct; an even number therefore also (if any) being indirect. But, under this form, the theorem can be proved by very elementary considerations, and still without any employment of the distributive principle (224, 262).

1. The first part of the theorem, as thus stated, is evident from the Third Book of Euclid; but to prove both parts together, it may be useful to proceed as follows, admitting the conception (235) of amplitudes, or of angles as representing rotations, which may have any values, positive or negative, and are to be added with attention to their signs.

2. We may thus write the three equations,

III. . . $ab'c = n\pi$, $bc'a = n'\pi$, $ca'b = n''\pi$,

to express the three collineations, $ab'c$, &c. of fig. 62; the integer, $n$, being odd or even, according as the point $b'$ is on the finite line $ac$, or on a prolongation of that line; or in other words, according as the first succession II. is direct or indirect: and similarly for the two other coefficients, $n'$ and $n''$.

* [Invert figure 61 from the point $d$. The sphere becomes a plane, and the circles through $d$ right lines, the other circles remain circles.]

† The Associative Principle of Multiplication was stated nearly under this form, and was illustrated by the same simple diagram, in paragraph XXII. of a communication by the present author, which was entitled Letters on Quaternions, and has been printed in the First and Second Editions of the late Dr. Nichol's Cyclopaedia of the Physical Sciences (London and Glasgow, 1857 and 1860). The same communication contained other illustrations and consequences of the same principle, which it has not been thought necessary here to reproduce; and others may be found in the Sixth of the author's already cited Lectures on Quaternions (Dublin, 1853), from which (as already observed) some of the formulae and figures of this Chapter have been taken.

‡ [${\text{A}}'$ being the angle through which $n'{\text{A}}$ must be turned in the positive direction so as to coincide with $n'c$.]
(3.) Again, if \(opqr\) be \textit{any four points} in one plane, we may establish the formula,

\[
IV. \quad poq + qor = por + 2m\pi,
\]

with the same conception of addition of amplitudes; if then \(d\) be \textit{any point in the plane} of the triangle \(abc\), we may write,

\[
V. \quad ab'd + db'c = n\pi, \quad bc'd + dc'a = n'\pi, \quad ca'd + da'b = n''\pi;
\]

and therefore,

\[
VI. \quad (ab'd + dc'a) + (bc'd + da'b) + (ca'd + db'c) = (n + n' + n'')\pi.
\]

(4.) Again, if \(opqr\) be not merely complanar but concircular, we have the general formula,

\[
VII. \quad opq + qro = p\pi,
\]

the integer \(p\) being \textit{odd} or \textit{even}, according as the succession \(opqr\) is \textit{direct} or \textit{indirect}; if then we denote by \(d\) the \textit{second intersection} of the first and second circles \(I.\), whereof \(c'\) is a \textit{first intersection}, we shall have

\[
VIII. \quad ab'd + dc'a = p\pi, \quad bc'd + da'b = p'\pi,
\]

\(p\) and \(p'\) being \textit{odd}, when the two first successions \(II'.\) are direct, but \textit{even} in the contrary case.

(5.) Hence, by \(VI.\), we have,

\[
IX. \quad ca'd + db'c = p''\pi, \quad \text{where} \quad X. \quad p + p' + p'' = n + n' + n'';
\]

the \textit{third succession} \(II'.\) is therefore \textit{always circular}, or the \textit{third circle} \(I.\) passes through the \textit{intersection} \(d\) of the \textit{two first}; and it is \textit{direct} or \textit{indirect}, that is to say, \(p''\) is \textit{odd} or \textit{even}, according as the \textit{number of even coefficients}, among the \textit{five previously considered}, is \textit{even} or \textit{odd}; or in other words, according as the \textit{number of indirect successions}, among the \textit{five previously considered}, is \textit{even} (including \textit{zero}), or \textit{odd}.

(6.) In every case, therefore, the \textit{total number} of \textit{successions} of \textit{each kind} is \textit{even}, and \textit{both parts} of the \textit{theorem} are proved: the importance of the \textit{second part} of it (respecting the \textit{even partition}, if any, of the \textit{six successions} \(II.\) \(II'.\)) arising from the necessity of \textit{proving} that we have \textit{always}, as in algebra,

\[
XI. \quad sr.q = + s.rq, \quad \text{and never} \quad XII. \quad sr.q = - s.rq,
\]

if \(q, r, s\) be \textit{any three actual quaternions}.

(7.) The \textit{associative principle} of multiplication may also be proved, without the \textit{distributive principle}, by certain considerations of \textit{rotations of a system}, on which we cannot enter here.
SECTION 3.

On some Additional Formulae.

273. Before concluding the Second Book, a few additional remarks may be made, as regards some of the notations and transformations which have already occurred, or others analogous to them. And first as to notation, although we have reserved for the Third Book the interpretation of such expressions as $\beta a$, or $a^2$, yet we have agreed, in 210, (9.), to abridge the frequently occurring symbol $(Ta)^2$ to $Ta^2$; and we now propose to abridge it still further to $Na$, and to call this square of the tensor (or of the length) of a vector, $a$, the Norm of that Vector: as we had (in 190, &c.), the equation $Tq^2 = Nq$, and called $Nq$ the norm of the quaternion $q$ (in 145, (11.)). We shall therefore now write generally, for any vector $a$, the formula,

$$I \ldots (Ta)^2 = Ta^2 = Na.$$ 

(1.) The equations (comp. 186, (1.) (2.) (3.) (4.)),

II. . . $N\rho = 1$; III. . . $N\rho = Na$; IV. . . $N(\rho - a) = Na$;

V. . . $N(\rho - a) = N(\beta - a)$,

represent, respectively, the unit-sphere; the sphere through $\lambda$, with $o$ for centre; the sphere through $o$, with $\lambda$ for centre; and the sphere through $b$, with the same centre $\lambda$.

(2.) The equations (comp. 186, (6.) (7.)),

VI. . . $N(\rho + a) = N(\rho - a)$; VII. . . $N(\rho - \beta) = N(\rho - a)$,

represent, respectively, the plane through $o$, perpendicular to the line $oa$; and the plane which perpendicularly bisects the line $ab$.

274. As regards transformations, the few following may here be added, which relate partly to the quaternion forms (204, 216, &c.) of the Equation of the Ellipsoid.

(1.) Changing $K(k : \rho)$ to $R\rho : Rk$, by 259, VIII., in the equation 217, XVI., of the ellipsoid, and observing that the three vectors $\rho$, $R\rho$, and $Rk$ are complanar, while $I : T\rho = TR\rho$ by 258, that equation becomes, when divided

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by \( T \rho \), and when the value 217, (5.) for \( \ell^2 \) is taken, and the notation 273 is employed:

\[
I. \ldots T \left( \frac{\ell}{R \rho} + \frac{\rho}{R \kappa} \right) = N \ell - N \kappa;
\]

of which the first member will soon be seen to admit of being written* as

\[ T (\ell \rho + \rho \kappa), \]

and the second member as \( \kappa^2 - \ell^2 \).

(2.) If, in connexion with the earlier forms (204, 216) of the equation of the same surface, we introduce a new auxiliary vector, \( \sigma \) or \( \sigma \kappa \), such that (comp. 216, VIII.)

\[
II. \ldots \sigma = \left( \frac{S \rho}{\sigma} + \frac{\rho}{\beta} \right) \beta = \rho + 2 \beta S \frac{\rho}{\delta};
\]

the equation may, by 204, (14.), be reduced to the following extremely simple form:

\[
III. \ldots T \sigma = T \beta;
\]

which expresses that the locus of the new auxiliary point \( s \) is what we have called the mean sphere, 216, XIV.; while the line \( rs \), or \( \sigma - \rho \), which connects any two corresponding points, \( r \) and \( s \), on the ellipsoid and sphere, is seen to be parallel to the fixed line \( \beta \); which is one element of the homology, mentioned in 216, (10.).

(3.) It is easy to prove that

\[
IV. \ldots S \frac{\sigma}{\delta} = S \frac{\beta}{\alpha} S \frac{\rho}{\delta}, \quad \text{and therefore} \quad V. \ldots S \frac{\sigma'}{\delta} : S \frac{\sigma}{\delta} = S \frac{\rho'}{\delta} : S \frac{\rho}{\delta};
\]

if \( \rho' \) and \( \sigma' \) be the vectors of two new but corresponding points, \( r' \) and \( s' \), on the ellipsoid and sphere; whence it is easy to infer this other element of the homology, that any two corresponding chords, \( rr' \) and \( ss' \), of the two surfaces, intersect each other on the cyclic plane which has \( \delta \) for its cyclic normal (comp. 216, (7.)) : in fact, they intersect in the point \( t \) of which the vector is,

\[
VI. \ldots T = \frac{a \rho + a' \rho'}{a + a'}, \quad \text{if} \quad a = S \frac{\rho'}{\delta}, \quad \text{and} \quad a' = -S \frac{\rho}{\delta};
\]

and this point is on the plane just mentioned (comp. 216, XI.), because

\[
VII. \ldots S \frac{\tau}{\delta} = 0.
\]

* Compare the Note to page 241.
(4.) Quite similar results would have followed, if we had assumed

\[ \sigma = \left(-S^2 \frac{p}{\alpha} + V \frac{p}{\alpha^2}\beta\right) = \rho - 2\beta S^2 \frac{p}{\gamma}, \]

which would have given again, as in III.,

\[ \sigma = \frac{T\rho}{T\beta}, \text{ but with } \rho = -S^2 \frac{p}{\alpha} \gamma; \]

the other cyclic plane, with \(\gamma\) instead of \(\delta\) for its normal, might therefore have been taken (as asserted in 216, (10.), as another plane of homology of ellipsoid and sphere, with the same centre of homology as before: namely, the point at infinity on the line \(\beta\), or on the axis (204, (15.)) of one of the two circumscribed cylinders of revolution (comp. 220, (4.).)

(5.) The same ellipsoid is, in two other ways, homologous to the same mean sphere, with the same two cyclic planes as planes of homology, but with a new centre of homology, which is the infinitely distant point on the axis of the second circumscribed cylinder (or on the line \(AB'\) of the sub-article last cited).

(6.) Although not specially connected with the ellipsoid, the following general transformations may be noted here (comp. 199, XII., and 204, XXXIV.):

\[ TV \sqrt{q} = \sqrt{\left(\frac{1}{2}(Tq - S^2q)\right)}; \]

\[ \tan \frac{1}{2} q = (TV:S) \sqrt{q} = \sqrt{Tq - S^2q}. \]

(7.) The equations 204, XVI. and XXXV. give easily,

\[ UVq = UUVq; \]

\[ UTVq = Ax; \]

\[ TV = TTVq; \]

or the more symbolical forms,

\[ UVU = UV; \]

\[ UIV = Ax; \]

\[ TV = TIV; \]

and the identity 200, IX. becomes more evident, when we observe that

\[ XVI. \quad q - Nq = q(1 - Kq). \]

(8.) We have also generally (comp. 200, (10.) and 218, (10.),

\[ XVII. \quad \frac{q - 1}{q + 1} = \frac{(q - 1)(Kq + 1)}{(q + 1)(Kq + 1)} = \frac{Nq - 1 + 2Vq}{Nq + 1 + 2S^2}. \]

(9.) The formula,*

\[ XVIII. \quad U(rq + Kqr) = U(Sr \cdot S^2q + Vr \cdot Vq) = r^{-1}(r^2q)^{\frac{1}{2}}q^{-1}, \]

in which \(q\) and \(r\) may be any two quaternions, is not perhaps of any great

* This formula was given, but in like manner without proof, in page 587 of the author's Lectures on Quaternions. [It may be expressed in terms of \(p = (r^2q)^{\frac{1}{2}}\). Use 210, XI. and XII.]
importance in itself, but will be found to furnish a student with several useful exercises in transformation.

(10.) When it was said, in 257, (1.), that zero had only itself for a square-root, the meaning was (comp. 225), that no binomial expression of the form \( x + iy \) (228) could satisfy the equation,

\[
XIX. \quad 0 = q^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy,
\]

for any real or imaginary values of the two scalar coefficients \( x \) and \( y \), different from zero;* for if bi-quaternions (214, (8.)) be admitted, and if \( h \) again denote, as in 256, (2.), the imaginary of algebra, then (comp. 257, (6.) and (7.)) we may write, generally, besides the real value, \( 0^2 = 0 \), the imaginary expression,

\[
XX. \quad 0^2 = v + hv', \quad \text{if} \quad Sv = S v' = S v'' = N v'' - N v = 0;
\]

\( v \) and \( v' \) being thus any two real right quaternions, with equal norms (or with equal tensors), in planes perpendicular to each other.

(11.) For example, by 256, (2.) and by the laws (183) of \( ijk \), we have the transformations,

\[
XXI. \quad (i + hj)^2 = i^2 - j^2 + h (ij + ji) = 0 + h0 = 0;
\]

so that the biquaternion \( i + hj \) is one of the imaginary values of the symbol \( 0^2 \).

(12.) In general, when bi-quaternions are admitted into calculation, not only the square of one, but the product of two such factors may vanish, without either of them separately vanishing: a circumstance which may throw some light on the existence of those imaginary (or symbolical) roots of equations, which were treated of in 257.

(13.) For example, although the equation

\[
XXII. \quad q^2 - 1 = (q - 1) (q + 1) = 0.
\]

has no real roots except \( \pm 1 \), and therefore cannot be verified by the substitution of any other real scalar, or real quaternion, for \( q \), yet if we substitute for \( q \) the bi-quaternion \( v + hv' \), with the conditions 257, XIII., this equation XXII. is verified.

* Compare the Note to page 289.

† This includes the expression \( \pm hi \), of 257, (1.), for a symbolical square-root of positive unity. Other such roots are \( \pm hj \), and \( \pm hk \). [It is probable that Hamilton used the word Bi-quaternion in order to distinguish clearly the \( V - 1 \) of algebra from the geometrical reals \( i, j, \) and \( k \) of the new Calculus. In his earlier writings \( i, j, \) and \( k \) are called imaginaries; and in a Paper read before the Royal Irish Academy on November 11, 1844, the scalar of a quaternion is called the "real part," and the vector, the "imaginary part." See p. 3, vol. iii., of the Proc. R.I.A.]}
(14.) It will be found, however, that when two imaginary but non-evanescent factors give thus a null product, the norm of each is zero; provided that we agree to extend to bi-quaternions the formula \( Nq = S_q^2 - V_q^2 \) (204, XXII.); or to define that the Norm of a Biquaternion (like that of an ordinary or real quaternion) is equal to the Square of the Scalar Part, minus the Square of the Right Part: each of these two parts being generally imaginary, and the former being what we have called a Bi-scalar.

(15.) With this definition, if \( q \) and \( q' \) be any two real quaternions, and if \( h \) be, as above, the ordinary imaginary of algebra, we may establish the formula:

\[
XXIII. \quad N(q + hq') = (S_q + hS_{q'})^2 - (V_q + hV_{q'})^2;
\]
or (comp. 200, VII., and 210, XX.),

\[
XXIV. \quad N(q + hq') = Nq - Nq' + 2hS \cdot qKq'.
\]

(16.) As regards the norm of the sum of any two real quaternions, or real vectors (273), the following transformations are occasionally useful (comp. 220, (2.)):

\[
XXV. \quad N(q + q') = N(Tq' \cdot Uq + Tq \cdot Uq');
\]
\[
XXVI. \quad N(\beta + \alpha) = N(T\beta \cdot U\alpha + Ta \cdot U\beta);
\]
in each of which it is permitted to change the norms to the tensors of which they are the squares, or to write \( T \) for \( N \).
BOOK III.

ON QUATERNIONS, CONSIDERED AS PRODUCTS OR POWERS OF VECTORS;
AND ON SOME APPLICATIONS OF QUATERNIONS.
CHAPTER I.

ON THE INTERPRETATION OF A PRODUCT OF VECTORS, OR POWER OF A VECTOR, AS A QUATERNION.

SECTION 1.

On a First Method of interpreting a Product of Two Vectors as a Quaternion.

Art. 275. In the First Book of these Elements we interpreted, 1st, the difference of any two directed right lines in space (4.); IIInd, the sum of two or more such lines (5-9); IIIrd, the product of one such line, multiplied by or into a positive or negative number (15); IVth, the quotient of such a line, divided by such a number (16), or by what we have called generally a Scalar (17); and Vth, the sum of a system of such lines, each affected (97) with a scalar coefficient (99), as being in each case itself (generally) a Directed Line* in Space, or what we have called a Vector (1).

276. In the Second Book, the fundamental principle or pervading conception has been, that the Quotient of two such Vectors is, generally, a Quaternion (112, 116). It is however to be remembered, that we have included under this general conception, which usually relates to what may be called an Oblique Quotient, or the quotient of two lines in space making either an acute or an obtuse angle with each other (130), the three following particular cases: Ist, the limiting case, when the angle becomes null, or when the two lines are similarly directed, in which case the quotient degenerates (131) into a positive scalar; IIInd, the other limiting case, when the angle is equal to two right angles, or when the lines are oppositely directed, and when in consequence the quotient again degenerates, but now into a negative scalar; and IIIInd, the intermediate case, when the angle is right, or when the two lines are perpendicular (132), instead of being parallel (15), and when therefore their quotient

* The Fourth Proportional to any three complanar lines has also been since interpreted (226), as being another line in the same plane.

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becomes what we have called (132) a Right Quotient, or a Right Quaternion: which has been seen to be a case not less important than the two former ones.

277. But no Interpretation has been assigned, in either of the two foregoing Books, for a Product of two or more Vectors; or for the Square, or other Power of a Vector: so that the Symbols,

I. \( \ldots \beta a, \gamma \beta a, \ldots \) and II. \( \ldots a^2, a^3, \ldots a^{-1}, \ldots a^4 \),

in which \( a, \beta, \gamma \ldots \) denote vectors, but \( t \) denotes a scalar, remain as yet entirely uninterpreted; and we are therefore free to assign, at this stage, any meanings to these new symbols, or new combinations of symbols, which shall not contradict each other, and shall appear to be consistent with convenience and analogy. And to do so will be the chief object of this First Chapter of the Third (and last) Book of these Elements: which is designed to be a much shorter one than either of the foregoing.

278. As a commencement of such Interpretation we shall here define, that a vector \( a \) is multiplied by another vector \( \beta \), or that the latter vector is multiplied into the former, or that the product \( \beta a \) is obtained, when the multiplier-line \( \beta \) is divided by the reciprocal \( Ea \) (258) of the multiplicand-line \( a \); as we had proved (1.36) that one quaternion is multiplied into another, when it is divided by the reciprocal thereof. In symbols, we shall therefore write, as a first definition, the formula:

\[
\text{I. } \ldots \beta a = \beta : Ra; \quad \text{where II. } Ra = - Ua : Ta (258, VII.).
\]

And we proceed to consider, in the following section, some of the general consequences of this definition, or interpretation, of a Product of two Vectors, as being equal to a certain Quotient, or Quaternion.

SECTION 2.

On some Consequences of the foregoing Interpretation.

279. The definition (278) gives the formula:

\[
\text{I. } \ldots \beta a = \frac{\beta}{Ra}; \quad \text{and similarly, I' } \ldots a\beta = \frac{a}{Ra};
\]

it gives therefore, by 259, VIII., the general relation,

\[
\text{II. } \ldots \beta a = K\alpha \beta; \quad \text{or II' } \ldots a\beta = K\beta a.
\]

* Compare the Notes to pages 147, 159.
The Products of two Vectors, taken in two opposite orders, are therefore Conjugate Quaternions; and the Multiplication of Vectors, like that of Quaternions (168), is (generally) a Non-Commutative Operation.

(1.) It follows from II. (by 196, comp. 223, (1.)), that

\[ \text{III. } S \beta a = + S a \beta = \frac{1}{2} (\beta a + a \beta). \]

(2.) It follows also (by 204, comp. again 223, (1.)), that

\[ \text{IV. } V \beta a = - V a \beta = \frac{1}{2} (\beta a - a \beta). \]

280. Again, by the same general formula 259, VIII., we have the transformations,

\[ \frac{\beta}{R(a + a')} = K \frac{a + a'}{R \beta} = K \frac{a}{R \beta} + K \frac{a'}{R \beta} = \frac{\beta}{R a} + \frac{\beta}{R a'}; \]

it follows, then, from the definition (278), that

\[ \text{II. } \beta (a + a') = \beta a + \beta a'; \]

whence also, by taking conjugates (279), we have this other general equation,

\[ \text{III. } (a + a') \beta = a \beta + a' \beta. \]

Multiplication of Vectors is, therefore, like that of Quaternions (212), a Doubly Distributive Operation.

281. As we have not yet assigned any signification for a ternary product of vectors, such as \( y(\beta a) \), we are not yet prepared to pronounce, whether the Associative Principle (223) of Multiplication of Quaternions does or does not extend to Vector-Multiplication. But we can already derive several other consequences from the definition (278) of a binary product, \( \beta a \); among which, attention may be called to the Scalar character of a Product of two Parallel Vectors; and to the Right character of a Product of two Perpendicular Vectors, or of two lines at right angles with each other.

(1.) The definition (278) may be thus written,

\[ \text{I. } \beta a = - T \beta \cdot T a \cdot U(\beta : a); \]

it gives, therefore,

\[ \text{II. } T \beta a = T \beta \cdot T a; \quad \text{III. } U \beta a = - U(\beta : a) = U \beta \cdot U a; \]

the tensor and versor of the product of two vectors being thus equal (as for quaternions, 191) to the product of the tensors, and to the product of the versors, respectively.
(2.) Writing for abridgment (comp. 208),

\[ IV... a = Ta, \quad b = T\beta, \quad \gamma = \Delta x. (\beta : a), \quad x = \angle (\beta : a), \]

we have thus,

\[ V... T\beta a = ba; \quad VI... S\beta a = Sa\beta = - ba \cos x; \]
\[ VII... SU\beta a = SUa\beta = - \cos x; \quad VIII... \angle \beta a = \pi - x; \]

so that (comp. 198) the angle of the product of any two vectors is the supple­ment of the angle of the quotient.

(3.) We have next the transformations (comp. again 208),

\[ IX... TV\beta a = TVa\beta = ba \sin x; \quad X... TVU\beta a = TVUa\beta = \sin x; \]
\[ XI... IV\beta a = - \gamma ba \sin x; \quad XI'... IVa\beta = + \gamma db \sin x; \]
\[ XII... UV\beta a = Ax. \beta a = - \gamma; \quad XII'... UVa\beta = Ax. a\beta = + \gamma; \]

so that the rotation round the axis of a product of two vectors, from the multiplier to the multiplicand, is positive.

(4.) It follows also, by IX., that the tensor of the right part of such a product, \(\beta a\), is equal to the parallelogram under the factors; or to the double of the area of the triangle \(\text{oab}\), whereof those two factors \(a, \beta, \text{or oA, oB}\), are two coinitial sides: so that if we denote here this last-mentioned area by the symbol

\[ \Delta \text{oab}, \]

we may write the equation,

\[ XIII... TV\beta a = \text{parallelogram under } a, \beta, = 2\Delta \text{oab}; \]

and the index, \(TV\beta a\), is a right line perpendicular to the plane of this parallelo­gram, of which line the length represents its area, in the sense that they bear equal ratios to their respective units (of length and of area).

(5.) Hence, by 279, IV.,

\[ XIV... T(\beta a - a\beta) = 2 \times \text{parallelogram} = 4\Delta \text{oab}. \]

(6.) For any two vectors, \(a, \beta,\)

\[ XV... S\beta a = - Na. S(\beta : a); \quad XVI... V\beta a = - Na. V(\beta : a); \]

or briefly,*

\[ XVII... \beta a = - Na. (\beta : a); \]

with the signification (273) of \(Na,\) as denoting \((Ta)^2.\)

* All the consequences of the interpretation (278), of the product \(\beta a\) of two vectors, might be deduced from this formula XVII.; which, however, it would not have been so natural to have assumed for a definition of that symbol, as it was to assume the formula 278, I.
(7.) If the two factor-lines be perpendicular to each other, so that \( a \) is a right angle, then the parallelogram (4.) becomes a rectangle, and the product \( \beta a \) becomes a right quaternion (132); so that we may write,

\[
\text{XVIII. } S\beta a = Sa\beta = 0, \quad \text{if } \beta \perp a, \text{ and reciprocally.}
\]

(8.) Under the same condition of perpendicularity,

\[
\text{XIX. } \angle \beta a = \angle a\beta = \frac{\pi}{2}; \quad \text{XX. } I\beta a = -\gamma ba; \quad \text{XXI. } Ia\beta = +\gamma ab.
\]

(9.) On the other hand, if the two factor-lines be parallel, the right part of their product vanishes, or that product reduces itself to a scalar, which is negative or positive according as the two vectors multiplied have similar or opposite directions; for we may establish the formula,

\[
\text{XXII. } \text{if } \beta \parallel a, \text{ then } V\beta a = 0, \quad Va\beta = 0;
\]
and, under the same condition of parallelism,

\[
\text{XXIII. } \beta a = a\beta = S\beta a = Sa\beta = \mp ba,
\]
the upper or the lower sign being taken, according as \( x = 0 \), or \( x = \pi \).

(10.) We may also write (by 279, (1.) and (2.)) the following formula of perpendicularity and formula of parallelism:

\[
\text{XXIV. } \text{if } \beta \perp a, \text{ then } \beta a = -a\beta, \text{ and reciprocally;}
\]
\[
\text{XXV. } \text{if } \beta \parallel a, \text{ then } \beta a = +a\beta, \text{ with the converse.}
\]

(11.) If \( a, \beta, \gamma \) be any three unit-lines, considered as vectors of the corners \( \lambda, \beta, c \) of a spherical triangle, with sides equal to three new positive scalars, \( a, b, c \), then because, by XVII., \( \beta a = -\beta : a, \) and \( \gamma \beta = -\gamma : \beta \), the subarticles to 208 allow us to write,

\[
\text{XXVI. } S(V\gamma\beta \cdot V\beta a) = \sin a \sin c \cos b;
\]
\[
\text{XXVII. } IV(V\gamma\beta \cdot V\beta a) = \pm \beta \sin a \sin c \sin b;
\]
\[
\text{XXVIII. } (IV : S)(V\gamma\beta \cdot V\beta a) = \pm \beta \tan b;
\]
upper or lower signs being taken, in the two last formulæ, according as the rotation round \( \beta \) from \( a \) to \( \gamma \), or that round \( b \) from \( a \) to \( c \), is positive or negative.

(12.) The equation 274, I., of the Ellipsoid, may now be written thus:

\[
\text{XXIX. } T(\varphi + \rho\kappa) = T\varphi - T\kappa; \quad \text{or XXX. } T(\varphi + \rho\kappa) = N\tau - N\kappa.
\]
282. Under the general head of a product of two parallel vectors, two
interesting cases occur, which furnish two first examples of Powers of Vectors:
namely, 1st, the case when the two factors are equal, which gives this remark-
able result, that the Square of a Vector is always equal to a negative Scalar;
and 2nd, the case when the factors are (in the sense already defined, 258) reciprocal to each other, in which case it follows from the definition (278) that
their product is equal to Positive Unity: so that each may, in this case, be con-
sidered as equal to unity divided by the other, or to the Power of that other
which has Negative Unity for its Exponent.

(1.) When \( \beta = a \), the product \( \beta a \) reduces itself to what we may call the
square of \( a \), and may denote by \( a^2 \); and thus we may write, as a particular
but important case of 281, XXIII., the formula (comp. 273),

\[
I. \quad a^2 = -\mathbf{a}^2 = -(Ta)^2 = -Na;
\]

so that the square of any vector \( a \) is equal to the negative of the norm (273) of
that vector; or to the negative of the square of the number \( Ta \), which expresses
(185) the length of the same vector.

(2.) More immediately, the definition (278) gives,

\[
II. \quad a^2 = aa = a : \mathbf{a} = - (Ta)^2 = -Na, \text{ as before.}
\]

(3.) Hence (compare the notations 161, 190, 199, 204),

\[
III. \quad S \cdot a^2 = -Na; \quad IV. \quad V \cdot a^2 = 0;
\]

and

\[
V. \quad \mathbf{T} \cdot a^2 = T(a^2) = +Na = (Ta)^2 = Ta^2;
\]

the omission of the parentheses, or of the point, in this last symbol of a tensor,*
for the square of a vector, as well as for the square of a quaternion (190), being
thus justified: and in like manner we may write,

\[
VI. \quad U \cdot a^2 = U(a^2) = -1 = (Ua)^2 = Ua^2;
\]

the square of an unit-vector (129) being always equal to negative unity, and
parentheses (or points) being again omitted.

(4.) The equation

\[
VII. \quad \rho^2 = a^2, \quad \text{gives} \quad VII' \ldots N\rho = Na, \quad \text{or} \quad VII'' \ldots T\rho = Ta;
\]

it represents therefore, by 186, (2.), the sphere with \( o \) for centre, which passes
through the point \( a \).

* Compare the Note to page 214.
(5.) The more general equation,
\[ (\rho - a)^2 = (\beta - a)^2, \]
(comp. 186, (4.),)
represents the sphere with \( a \) for centre, which passes through the point \( b \).

(6.) For example, the equation,
\[ (\rho - a)^2 = a^2, \]
(comp. 186, (3.),)
represents the sphere with \( a \) for centre, which passes through the origin \( o \).

(7.) The equations (comp. 186, (6.), (7.),)
\[ (\rho + a)^2 = (\rho - a)^2; \quad (\rho - \beta)^2 = (\rho - a)^2, \]
represent, respectively, the plane through \( o \), perpendicular to the line \( oa \); and the plane which perpendicularly bisects the line \( ab \).

(8.) The distributive principle of vector-multiplication (280), and the formula 279, III., enable us to establish generally (comp. 210, (9.)) the formula,
\[ (\beta + a)^2 = \beta^2 + 2\beta a + a^2; \]
the recent equations IX. and X. may therefore be thus transformed:
\[ \rho^2 = 2\beta a; \quad \text{and} \quad Sap = 0. \]

(9.) The equations,
\[ \rho^2 + a^2 = 0; \quad \rho^2 + 1 = 0, \]
represent the spheres with \( o \) for centre, which have \( a \) and \( 1 \) for their respective radii; so that this very simple formula, \( \rho^2 + 1 = 0 \), is (comp. 186, (1.)) a form of the Equation of the Unit-Sphere (128), and is, as such, of great importance in the present Calculus.

(10.) The equation,
\[ \rho^2 - 2\beta a + c = 0, \]
may be transformed to the following,
\[ N(\rho - a) = - (\rho - a)^2 = c - a^2 = c + Na; \]
or
\[ T(\rho - a) = \sqrt{(c - a^2)} = \sqrt{(c + Na)}; \]
it represents therefore a (real or imaginary) sphere, with \( a \) for centre, and with this last radical (if real) for radius.

* Compare also the sub-articles to 273.
(11.) This sphere is therefore necessarily real, if \( c \) be a positive scalar; or if this scalar constant, \( c \), though negative, be (algebraically) greater than \( a^2 \), or than \(-Na\): but it becomes imaginary, if \( c + Na < 0 \).

(12.) The radical plane of the two spheres,

\[
XVII. \quad \rho^2 - 2S\alpha\rho + c = 0, \quad \rho^2 - 2S\alpha'\rho + c' = 0,
\]

has for equation,

\[
XVIII. \quad 2S (\alpha' - \alpha) \rho = c' - c;
\]

it is therefore always real, if the given vectors \( \alpha, \alpha' \) and the given scalars \( c, c' \) be such, even if one or both of the spheres themselves be imaginary.

(13.) The equation 281, XXIX., or XXX., of the Central Ellipsoid (or of the ellipsoid with its centre taken for the origin of vectors), may now be still further simplified, as follows:

\[
XIX. \quad T (\rho + \rho \kappa) = \kappa^2 - \tau^2.
\]

(14.) The definition (278) gives also,

\[
XX. \quad aRa = a : a = 1; \quad \text{or} \quad XX'. \quad Ra : a = Ra : Ra = 1;
\]

whence it is natural to write,†

\[
XXI. \quad Ra = 1 : a = a^{-1},
\]

if we so far anticipate here the general theory of powers of vectors, above alluded to (277), as to use this last symbol to denote the quotient, of unity divided by the vector \( a \); so as to have identically, or for every vector, the equation,

\[
XXII. \quad a \cdot a^{-1} = a^{-1} \cdot a = 1.
\]

(15.) It follows, by 258, VII., that

\[
XXIII. \quad a^{-1} = -Ua : Ta; \quad \text{and} \quad XXIV. \quad \beta a = \beta : a^{-1}.
\]

(16.) If we had adopted the equation XXIII. as a definition‡ of the symbol \( a^{-1} \), then the formula XXIV. might have been used, as a formula of interpretation for the symbol \( \beta a \). But we proceed to consider an entirely different method, of arriving at the same (or an equivalent) Interpretation of this latter symbol: or of a Binary Product of Vectors, considered as equal to a Quaternion.

* Compare the Note to page 241. † Compare the Note to page 293. ‡ Compare the Note to page 324.
SECTION 3.

On a Second Method of arriving at the same Interpretation, of a Binary Product of Vectors.

283. It cannot fail to have been observed by any attentive reader of the Second Book, how close and intimate a connexion* has been found to exist, between a Right Quaternion (132), and its Index, or Index-Vector (133). Thus, if \( v \) and \( v' \) denote (as in 223, (1), &c., any two right quaternions, and if \( Iv, Iv' \) denote, as usual, their indices, we have already seen that

\[
\text{I.} \quad Iv' = Iv, \quad \text{if} \quad v' = v, \quad \text{and conversely (133)};
\]
\[
\text{II.} \quad I(v' \pm v) = Iv' \pm Iv \quad (206);
\]
\[
\text{III.} \quad Iv': Iv = v': v \quad (193);
\]

to which may be added the more recent formula,

\[
\text{IV.} \quad RIv = IRv \quad (258, \text{IX}).
\]

284. It could not therefore have appeared strange, if we had proposed to establish this new formula of the same kind,

\[
\text{I.} \quad Iv'. Iv = v'. v = v'v,
\]

as a definition (supposing that the recent definition 278 had not occurred to us), whereby to interpret the product of any two indices of right quaternions, as being equal to the product of those two quaternions themselves. And then, to interpret the product \( \beta a \), of any two given vectors, taken in a given order, we should only have had to conceive (as we always may) that the two proposed factors, \( a \) and \( \beta \), are the indices of two right quaternions, \( v \) and \( v' \), and to multiply these latter, in the same order. For thus we should have been led to establish the formula,

\[
\text{II.} \quad \beta a = v'v, \quad \text{if} \quad a = Iv, \quad \text{and} \quad \beta = Iv';
\]
or we should have this slightly more symbolical equation,

\[
\text{III.} \quad \beta a = \beta . a = I^{-1}\beta . I^{-1}a;
\]
in which the symbols,

\[
I^{-1}a \quad \text{and} \quad I^{-1}\beta,
\]

are understood to denote the two right quaternions, whereof the two lines \( a \) and \( \beta \) are the indices.

* Compare the Note to page 175.
(1.) To establish now the substantial identity of these two interpretations, 278 and 284, of a binary product of vectors \( \beta a \), notwithstanding the difference of form of the definitional equations by which they have been expressed, we have only to observe that it has been found, as a theorem (194), that

\[
IV \ldots v'v = I v' : I (1 : v) = I v' : IRv;
\]

but the definition (258) of \( Ra \) gave us the lately cited equation, \( Rlv = IRv \);

we have therefore, by the recent formula II., the equation,

\[
V \ldots I v'.Iv = I v' : Rlv; \quad \text{or} \quad VI \ldots \beta .a = \beta : Ra,
\]

as in 278, I.; \( a \) and \( \beta \) still denoting any two vectors. The two interpretations therefore coincide, at least in their results, although they have been obtained by different processes, or suggestions, and are expressed by two different formulae.

(2.) The result 279, II., respecting conjugate products of vectors, corresponds thus to the result 191, (2.), or to the first formula of 223, (1.).

(3.) The two formulae of 279, (1.) and (2.), respecting the scalar and right parts of the product \( \beta a \), answer to the two other formulae of the same sub-article, 223, (1.), respecting the corresponding parts of \( v'v \).

(4.) The doubly distributive property (280), of vector-multiplication, is on this plan seen to be included in the corresponding but more general property (212), of multiplication of quaternions.

(5.) By changing \( IVq, IVq', t, t' \), and \( \delta \), to \( a, \beta, a, b, \) and \( \gamma \), in those formulae of Art. 208 which are previous to its sub-articles, we should obtain, with the recent definition (or interpretation) II. of \( \beta a \), several of the consequences lately given (in sub-arts. to 281), as resulting from the former definition, 278, I. Thus, the equations,

\[
VI., VII., VIII., IX., X., XI., XII., XXII., \text{and XXIII.}
\]

of 281, correspond to, and may (with our last definition) be deduced from, the formulae,

\[
V., VI., VII., XI., XII., XXII., XX., XIV., \text{and XVI., XVIII.}
\]

of 208. (Some of the consequences from the sub-articles to 208 have been already considered, in 281, (11.).)

(6.) The geometrical properties of the line \( IV\beta a \), deduced from the first definition (278) of \( \beta a \) in 281, (3.) and (4.), (namely, the positive rotation round that line, from \( \beta \) to \( a \); its perpendicularity to their plane; and the representation by the same line of the parallelogram under those two factors, regard being had to units of length and of area,) might also have been deduced from 223, (4.), by means of the second definition (284), of the same product, \( \beta a \).
SECT. 4.

On the Symbolical Identification of a Right Quaternion with its own Index: and on the Construction of a Product of two Rectangular Lines, by a Third Line, Rectangular to both.

285. It has been seen, then, that the recent formula 284, II. or III., may replace the formula 278, I., as a second definition of a product of two vectors, which conducts to the same consequences, and therefore ultimately to the same interpretation of such a product, as the first. Now, in the second formula, we have interpreted that product, $\beta a$, by changing the two factor-lines, $a$ and $\beta$, to the two right quaternions, $v$ and $v'$, or $I_1a$ and $I_1\beta$, of which they are the indices; and by then defining that the sought product $\beta a$ is equal to the product $v'v$, of those two right quaternions. It becomes, therefore, important to inquire, at this stage, how far such substitution, of $I_1a$ for $a$, or of $v$ for $Iv$, together with the converse substitution, is permitted in this Calculus, consistently with principles already established. For it is evident that if such substitutions can be shown to be generally legitimate, or allowable, we shall thereby be enabled to enlarge greatly the existing field of interpretation: and to treat, in all cases, Functions of Vectors, as being, at the same time, Functions of Right Quaternions.

286. We have first, by 133 (compare 283, I.), the equality,

$$I \beta = I_1a, \text{ if } \beta = a.$$

In the next place, by 206 (comp. 283, II.), we have the formula of addition or subtraction,

$$I_1 (\beta \pm a) = I_1\beta \pm I_1a;$$

with these more general results of the same kind (comp. 207 and 99),

$$I_1 \Sigma a = \Sigma I_1a; \quad I_1 \Sigma xa = \Sigma x I_1a.$$

In the third place, by 193 (comp. 283, III.), we have, for division, the formula,

$$V \beta : I_1a = \beta : a;$$

while the second definition (284) of multiplication of vectors, which has been proved to be consistent with the first definition (278), has given us the analogous equation,

$$VI \beta a = \beta a = \beta a.$$
It would seem, then, that we might at once proceed to define, for the purpose of interpreting any proposed Function of Vectors as a Quaternion, that the following general Equation exists:

\[ \text{VII. } I^{-1}a = a; \quad \text{or VII. } Iv = v, \quad \text{if } \angle v = \frac{\pi}{2}; \]

or still more briefly and symbolically, if it be understood that the subject of the operation \( I \) is always a right quaternion,

\[ \text{IX. } I = 1. \]

But, before finally adopting this conclusion, there is a case (or rather a class of cases), which it is necessary to examine, in order to be certain that no contradiction to former results can ever be thereby caused.

287. The most general form of a vector function, or of a vector regarded as a function of other vectors and of scalars, which was considered in the First Book, was the form (99, comp. 275),

\[ I \ldots p = \sum xa; \]

and we have seen that if we change, in this form, each vector \( a \) to the corresponding right quaternion \( I^{-1}a \), and then take the index of the new right quaternion which results, we shall thus be conducted to precisely the same vector \( p \), as that which had been otherwise obtained before; or in symbols, that

\[ \text{II. } \sum xa = I\sum I^{-1}a \text{ (comp. 286, IV.).} \]

But another form of a vector-function has been considered in the Second Book; namely, the form,

\[ \text{III. } . . p = \sum \varepsilon \gamma \delta \beta a \text{ (226, III.);} \]

in which \( a, \beta, \gamma, \delta, \varepsilon \ldots \) are any odd number of complanar vectors. And before we accept, as general, the equation VII. or VIII. or IX. of 286, we must inquire whether we are at liberty to write, under the same conditions of complanarity, and with the same signification of the vector \( p \), the equation,

\[ \text{IV. } . . p = I \left( \ldots \frac{I^{-1}\varepsilon}{I^{-1}\delta} \cdot \frac{I^{-1}\gamma}{I^{-1}\beta} \cdot I^{-1}a \right). \]

288. To examine this, let there be at first only three given complanar vectors, \( \gamma \parallel a, \beta \); in which case there will always be (by 226) a fourth vector \( \rho \), in the same plane, which will represent or construct the function \( (\gamma : \beta) \cdot a \); namely, the fourth proportional to \( \beta, \gamma, a \). Taking then what we may call
the Inverse Index-Functions, or operating on these four vectors $a, \beta, \gamma, \rho$ by the characteristic $I^{-1}$, we obtain four collinear and right quaternions (209), which may be denoted by $v, v', v'', v'''$; and we shall have the equation,

$$V \ldots v''' : v = (\rho : a = \gamma : \beta =) v'' : v';$$

or

$$\text{VI} \ldots v''' = (v'' : v') \cdot v;$$

which proves what was required. Or, more symbolically,

$$\text{VII} \ldots \frac{I^{-1} \beta}{I^{-1} a} = \frac{\rho}{a} = \frac{\gamma}{\beta} = \frac{I^{-1} \gamma}{I^{-1} \beta};$$

$$\text{VIII} \ldots \frac{\gamma}{\beta} a = \rho = I(I^{-1} \rho) = I \left( \frac{I^{-1} \gamma}{I^{-1} \beta} \cdot I^{-1} a \right).$$

And it is so easy to extend this reasoning to the case of any greater odd number of given vectors in one plane, that we may now consider the recent formula IV. as proved.

289. We shall therefore adopt, as general, the symbolical equations VII. VIII. IX. of 286; and shall thus be enabled, in a shortly subsequent section, to interpret ternary (and other) products of vectors, as well as powers and other Functions of Vectors, as being generally Quaternions; although they may, in particular cases, degenerate (131) into scalars, or may become right quaternions (132): in which latter event they may, in virtue of the same principle, be represented by, and equated to, their own indices (133), and so be treated as vectors. In symbols, we shall write generally, for any set of vectors $a, \beta, \gamma, \ldots$ and any function $f$, the equation,

$$I \ldots f(a, \beta, \gamma, \ldots) = f(I^{-1} a, I^{-1} \beta, I^{-1} \gamma, \ldots) = q,$$

$q$ being some quaternion; while in the particular case when this quaternion is right, or when

$$q = v = S^{-1} 0 = I^{-1} \rho,$$

we shall write also, and usually by preference (for that case), the formula,

$$\text{II} \ldots f(a, \beta, \gamma, \ldots) = I f(I^{-1} a, I^{-1} \beta, I^{-1} \gamma, \ldots) = \rho,$$

$\rho$ being a vector.

290. For example, instead of saying (as in 281) that the Product of any two Rectangular Vectors is a Right Quaternion, with certain properties of its Index, already pointed out (284, (6.)), we may now say that such a product is equal to that index. And hence will follow the important consequence, that
the Product of any two Rectangular Lines in Space is equal to (or may be con-structed by) a Third Line, rectangular to both; the Rotation round this Product-Line, from the Multiplier-Line to the Multiplicand-Line, being Positive: and the Length of the Product being equal to the Product of the Lengths of the Factors, or representing (with a suitable reference to units) the Area of the Rectangle under them. And generally we may now, for all purposes of calculation and expression, identify* a Right Quaternion with its own Index.

SECTION 5.

On some Simplifications of Notation, or of Expression, resulting from this Identification; and on the conception of an Unit-Line as a Right Versor.

291. An immediate consequence of the symbolical equation 286, IX., is that we may now suppress the Characteristic I, of the Index of a Right Quaternion, in all the formulæ into which it has entered; and so may simplify the Notation. Thus, instead of writing,

\[ Ax \cdot q = IUVq, \quad \text{or} \quad Ax. = IUV, \quad \text{as in 204, (23),} \]

\[ Ax \cdot g = UlVg, \quad \text{or} \quad Ax. = UIV, \quad \text{as in 274, (7),} \]

we may now write simply†,

\[ \text{I. . . } Ax \cdot q = UVq; \quad \text{or} \quad \text{II. . . } Ax. = UV. \]

The Characteristic Ax., of the Operation of taking the Axis of a Quaternion (132, (6.)), may therefore henceforth be replaced whenever we may think fit to dispense with it, by this combination of two other characteristics, U and V, which are of greater and more general utility, and indeed cannot‡ be dispensed with, in the practice of the present Calculus.

* Compare the Notes to pages 121, 137, 175, 193, 203.
† Compare the first Note to page 120, and the third Note to page 203.
‡ Of course, any one who chooses may invent new symbols, to denote the same operations on quaternions, as those which are denoted in these Elements, and in the elsewhere cited Lectures, by the letters U and V; but, under some form, such symbols must be used; and it appears to have been hitherto thought expedient, by other writers, not hastily to innovate on notations which have been already employed in several published researches, and have been found to answer their purpose. As to the type used for these, and for the analogous characteristics K, S, T, that must evidently be a mere affair of taste and convenience: and in fact they have all been printed as small italic capitals, in some examination-papers by the author.
292. We are now enabled also to diminish, to some extent, the number of technical terms, which have been employed in the foregoing Book. Thus, whereas we defined, in 202, that the right quaternion \( Vg \) was the Right Part of the Quaternion \( g \), or of the sum \( Sq + Vg \), we may now, by 290, identify that part with its own index-vector \( IVg \), and so may be led to call it the vector part, or simply the Vector,* of that Quaternion \( g \), without henceforth speaking of the right part: although the plan of exposition, adopted in the Second Book, required that we should do so for some time. And thus an enunciation, which was put forward at an early stage of the present work, namely, at the end of the First Chapter of the First Book, or the assertion (17) that

"Scalar plus Vector equals Quaternion,"

becomes entirely intelligible, and acquires a perfectly definite signification. For we are in this manner led to conceive a Number (positive or negative) as being added to a Line;† when it is added (according to rules already established) to that right quotient (132), of which the line is the Index. In symbols, we are thus led to establish the formula,

\[
I \ldots q = a + a, \quad \text{when} \quad II \ldots q = a + I^{-1}a;
\]

whatever scalar, and whatever vector, may be denoted by \( a \) and \( a \). And because either of these two parts, or summands, may vanish separately, we are entitled to say, that both Scalars and Vectors, or Numbers and Lines, are included in the Conception of a Quaternion, as now enlarged or modified.

293. Again, the same symbolical identification of \( Iv \) with \( v \) (286, VIII.) leads to the forming of a new conception of an Unit-Line, or Unit-Vector (129), as being also a Right Versor (153); or an Operator, of which the effect is to turn a line, in a plane perpendicular to itself, through a positive quadrant of rotation: and thereby to oblige the Operand-Line to take a new direction, at right angles to its old direction, but without any change of length. And then the remarks (154) on the equation \( g^2 = -1 \), where \( g \) was a right versor in the former sense (which is still a permitted one) of its being a right radial quotient

* Compare the Note to page 193.
† On account of this possibility of conceiving a quaternion to be the sum of a number and a line, it was at one time suggested by the present author, that a Quaternion might also be called a Grammarithm, by a combination of the two Greek words γραμμή and ἄρθρο, which signify respectively a Line and a Number.
(147), or the quotient of two equally long but mutually rectangular lines, become immediately applicable to the interpretation of the equation,

\[ \rho^2 = -1, \quad \text{or} \quad \rho^2 + 1 = 0 \quad (282, \text{XIV}). \]

where \( \rho \) is still an unit-vector.

(1.) Thus (comp. fig. 41, p. 132), if \( a \) be any line perpendicular to such a vector \( \rho \), we have the equations,

I. \( \rho a = \beta \); \quad II. \( \rho^* a = \rho \beta = a' = -a \);

\( \beta \) being another line perpendicular to \( \rho \), which is, at the same time, at right angles to \( a \), and of the same length with it; and from which a third line \( a' \), or \( -a \), opposite to the line \( a \), but still equally long, is formed by a repetition of the operation, denoted by (what we may here call) the characteristic \( \rho \); or having that unit-vector \( \rho \) for the operator, or instrument employed, as a sort of handle, or axis* of rotation.

(2.) More generally (comp. 290), if \( a, \beta, \gamma \) be any three lines at right angles to each other, and if the length of \( \gamma \) be numerically equal to the product of the lengths of \( a \) and \( \beta \), then (by what precedes) the line \( \gamma \) represents, or constructs, or is equal to, the product of the two other lines, at least if a certain order of the factors (comp. 279) be observed: so that we may write the equation (comp. 281, XXI.),

III. \( a \beta = \gamma \), \quad \text{if IV.} \beta \perp a, \ \gamma \perp a, \ \gamma \perp \beta, \quad \text{and V.} \ T_a \cdot T\beta = T\gamma,

provided that the rotation round \( a \), from \( \beta \) to \( \gamma \), or that round \( \gamma \) from \( a \) to \( \beta \), &c., has the direction taken as the positive one.

(3.) In this more general case, we may still conceive that the multiplier-line \( a \) has operated on the multiplicand-line \( \beta \), so as to produce (or generate) the product-line \( \gamma \); but not now by an operation of version alone, since the tensor of \( \beta \) is (generally) multiplied by that of \( a \), in order to form, by V., the tensor of the product \( \gamma \).

(4.) And if (comp. fig. 41, bis, in which \( a \) was first changed to \( \beta \), and then to \( a' \)) we repeat this compound operation, of tension and version combined (comp. 189), or if we multiply again by \( a \), we obtain a fourth line \( \beta' \), in the plane of \( \beta, \gamma \), but with a direction opposite to that of \( \beta \), and with a length generally different: namely the line,

VI. \( a \gamma = aa\beta = a^2\beta = \beta' = -a^2\beta \), \quad \text{if} \quad a = T_a.

* Compare the second Note to page 137.
The operator $a^2$, or $aa$, is therefore equivalent, in its effect on $\beta$, to the negative scalar, $-a^2$, or $-(Ta)^2$, or $-Na$, considered as a coefficient, or as a (scalar) multiplier (15): whence the equation,

$$VII. \ldots a^2 = -Na$$

may be again deduced, but now with a new interpretation, which is, however, as we see, completely consistent, in all its consequences, with the one first proposed (282).

---

SECTION 6.

On the Interpretation of a Product of Three or more Vectors, as a Quaternion.

294. There is now no difficulty in interpreting a ternary product of vectors (comp. 277, I.), or a product of more vectors than three, taken always in some given order; namely, as the result (289, I.) of the substitution of the corresponding right quaternions in that product: which result is generally what we have lately called (276) an Oblique Quotient, or a Quaternion with either an acute or an obtuse angle (130); but may degenerate (131) into a scalar, or may become itself a right quaternion (132), and so be constructed (289, II.) by a new vector. It follows (comp. 281), that Multiplication of Vectors, like that of Quaternions (223), in which indeed we now see that it is included, is an Associative Operation: or that we may write generally (comp. 223, II.), for any three vectors, $a, \beta, \gamma$, the Formula,

$$I. \ldots \gamma \beta \cdot a = \gamma \cdot \beta a.$$  

(1.) The formules 223, III. and IV., are now replaced by the following:

$$II. \ldots V. \gamma \nu \beta a = aS\beta \gamma - \beta S\gamma a;$$

$$III. \ldots V. \gamma \beta a = aS\beta \gamma - \beta S\gamma a + \gamma Sa\beta;*$$

in which $V. \gamma \beta a$ is written, for simplicity, instead of $V(\gamma \beta a)$, or $V. \gamma \beta a$; and with which, as with the earlier equations referred to, a student of this Calculus will find it useful to render himself very familiar.

* [On account of the importance of these formulæ, it is worth while to notice that, using the principles of the present Book,

$$V. \gamma \beta a = \frac{1}{2} (1 - K) \gamma \beta a = \frac{1}{2} (\gamma \beta a + a\beta \gamma) = \frac{1}{2} \gamma (\beta a + a\beta) - \frac{1}{2} (\gamma a + a\gamma) \beta$$

$$+ \frac{1}{2} a (\gamma \beta + \beta \gamma) = \gamma Sa\beta - \beta S\gamma a + aS\beta \gamma.]$$

Hamilton's Elements of Quaternions
(2.) Another useful form of the equation II. is the following:

\[ IV \ldots V (Va\beta \cdot \gamma) = a\beta \gamma - \beta \gamma a. \]

(3.) The equations IX. X. XIV. of 223 enable us now to write, for any three vectors, the formula:

\[ V \ldots S\gamma a = -Sa\beta \gamma = Sa\gamma \beta = -S\beta \gamma a = S\beta \gamma a = -S\gamma a \beta = S\gamma a \beta = S\beta a \gamma = S\gamma a \beta = S\gamma a \beta = \pm \text{volume of parallelepiped under } a, \beta, \gamma, \]

\[ = \pm 6 \times \text{volume of pyramid oabc}; \]

upper or lower signs being taken, according as the rotation round \( a \) from \( \beta \) to \( \gamma \) is positive or negative: or in other words, the scalar \( S\gamma a \beta \), of the ternary product \( \gamma \beta a \), being positive in the first case, but negative in the second.

(4.) The condition of complanarity of three vectors, \( a, \beta, \gamma \), is therefore expressed by the equation (comp. 223, XI.):

\[ VI \ldots S\gamma \beta a = 0; \quad \text{or} \quad VI' \ldots Sa\beta \gamma = 0; \; \text{&c.} \]

(5.) If \( a, \beta, \gamma \) be any three vectors, complanar or diplanar, the expression,

\[ VII \ldots \delta = a\beta \gamma - \beta \gamma a, \]

gives

\[ VIII \ldots S\gamma \delta = 0, \quad \text{and} \quad IX \ldots Sa\beta \delta = 0; \]

it represents therefore (comp. II. and IV.) a fourth vector \( \delta \), which is perpendicular to \( \gamma \), but complanar with \( a \) and \( \beta \); or in symbols,

\[ X \quad \delta \perp \gamma, \quad \text{and} \quad XI \ldots \delta \parallel \parallel a, \beta. \]

(Compare the notations 123, 129.)

(6.) For any four vectors, we have by II. and IV. the transformations,

\[ XII \ldots V (Va\beta \cdot V\gamma \delta) = S\delta a\beta \gamma - \gamma S\beta a \delta; \]
\[ XIII \ldots V (Va\beta \cdot V\gamma \delta) = a\beta \gamma \delta - \beta S\gamma a \delta; \]

and each of these three equivalent expressions represents a fifth vector \( \varepsilon \), which is at once complanar with \( a, \beta, \gamma \), and with \( \gamma, \delta \); or a line \( \omega \varepsilon \), which is in the intersection of the two planes, \( oab \) and \( ocd \).

(7.) Comparing them, we see that any arbitrary vector \( \rho \) may be expressed as a linear function of any three given diplanar vectors, \( a, \beta, \gamma \), by the formula:

\[ XIV \ldots \rho Sa\beta \gamma = a\beta \gamma \rho + \beta S\gamma a \rho + \gamma S\beta a \rho; \]

which is found to be one of extensive utility.
(8.) Another very useful formula, of the same kind, is the following:

\[ XV. \ldots \rho S_{\alpha \beta \gamma} = V_{\beta \gamma} \cdot S_{\alpha} + V_{\gamma \alpha} \cdot S_{\beta} + V_{\alpha \beta} \cdot S_{\gamma \rho} ; \]

in the second member of which, the points may be omitted.*

(9.) One mode of proving the correctness of this last formula XV., is to operate on both members of it, by the three symbols, or characteristics of operation,

\[ XVI. \ldots S \cdot a, S \cdot \beta, S \cdot \gamma ; \]

the common results on both sides being respectively the three scalar products,

\[ XVII. \ldots S_{\alpha} \cdot S_{\alpha \beta \gamma}, S_{\beta \rho} \cdot S_{\alpha \beta \gamma}, S_{\gamma \rho} \cdot S_{\alpha \beta \gamma} ; \]

where again the points may be omitted.

(10.) We here employ the principle, that if the three vectors \( a, \beta, \gamma \) be actual and diplanar, then no actual vector \( \lambda \) can satisfy all the three scalar equations,

\[ XVIII. \ldots S_{\alpha} \lambda = 0, S_{\beta} \lambda = 0, S_{\gamma} \lambda = 0 ; \]

because it cannot be perpendicular at once to those three diplanar vectors.

(11.) If, then, in any investigation with quaternions, we meet a system of this form XVIII., we can at once infer that

\[ XIX. \ldots \lambda = 0, \text{ if } XX. \ldots S_{\alpha \beta \gamma} > 0 ; \]

while, conversely, if \( \lambda \) be an actual vector, then \( a, \beta, \gamma \) must be complanar vectors, or \( S_{\alpha \beta \gamma} = 0 \), as in VI'.

(12.) Hence also, under the same condition XX., the three scalar equations,

\[ XXI. \ldots S_{\alpha} \lambda = S_{\alpha} \mu, S_{\beta} \lambda = S_{\beta} \mu, S_{\gamma} \lambda = S_{\gamma} \mu, \]

give

\[ XXII. \ldots \lambda = \mu. \]

(13.) Operating (comp. (9.)) on the equation XV. by the symbol, or characteristic, \( S \cdot \delta \), in which \( \delta \) is any new vector, we find a result which may be written thus (with or without the points):

\[ XXIII. \ldots 0 = S_{\alpha} \cdot S_{\beta} \delta - S_{\beta} \delta \cdot S_{\gamma} \rho + S_{\gamma} \rho \cdot S_{\delta} \alpha - S_{\delta} \alpha \cdot S_{\beta} \gamma ; \]

where \( a, \beta, \gamma, \delta, \rho \) may denote any five vectors.

* [Another method of proving XIV. is to assume \( \rho = x_{\alpha} + y_{\beta} + z_{\gamma} \). Operating by \( S \cdot V_{\beta \gamma}, S_{\beta} \gamma \rho \) \( = z S_{\alpha} \beta \gamma \); and similar expressions may be found for \( \gamma \) and \( z \). To prove XV. assume \( \rho = x' V_{\beta \gamma} + y' \gamma \alpha + z' \beta \alpha \), and operate in turn by \( S \cdot \alpha, S \cdot \beta, \) and \( S \cdot \gamma \).]
(14.) In drawing this last inference, we assume that the equation XV.
holds good, even when the three vectors, a, β, γ are complanar: which in fact
must be true, as a limit, since the equation has been proved, by (9.) and (12.),
to be valid, if γ be ever so little out of the plane of a and β.

(15.) We have therefore this new formula:

\[
\text{XXIV} \ldots V\beta\gamma S\alpha \rho + V\gamma a S\beta \rho + V a \beta S\gamma \rho = 0, \quad \text{if } S a \beta \gamma = 0;
\]
in which ρ may denote any fourth vector, whether in, or out of, the common
plane of a, β, γ.

(16.) If ρ be perpendicular to that plane, the last formula is evidently true,
each term of the first member vanishing separately, by 281, (7.); and if we
change ρ to a vector δ in the plane of a, β, γ, we are conducted to the follow­
ing equation, as an interpretation of the same formula XXIV., which expresses
a known theorem of plane trigonometry, including several others under it:

\[
\text{XXV} \ldots \sin \text{boc} \cdot \cos \text{aod} + \sin \text{coa} \cdot \cos \text{bod} + \sin \text{aob} \cdot \cos \text{cod} = 0,
\]
for any four complanar and co-initial lines, oa, ob, oc, od.

(17.) By passing from od to a line perpendicular thereto, but in their
common plane, we have this other known* equation:

\[
\text{XXVI} \ldots \sin \text{boc} \sin \text{aod} + \sin \text{coa} \sin \text{bod} + \sin \text{aob} \sin \text{cod} = 0;
\]
which, like the former, admits of many transformations, but is only mentioned
here as offering itself naturally to our notice, when we seek to interpret the
formula XXIV. obtained as above by quaternions.

(18.) Operating on that formula by S . δ, and changing ρ to ε, we have
this new equation:

\[
\text{XXVII} \ldots 0 = S a \varepsilon S \beta \gamma \delta + S \beta \varepsilon S \gamma a \delta + S \gamma \varepsilon S a \beta \delta, \quad \text{if } S a \beta \gamma = 0;
\]
which might indeed have been at once deduced from XXIII.

(19.) The equation XIV., as well as XV., must hold good at the limit,
when a, β, γ are complanar; hence

\[
\text{XXVIII} \ldots a S \beta \gamma \rho + \beta S \gamma a \rho + \gamma S a \beta \rho = 0, \quad \text{if } S a \beta \gamma = 0.
\]

* Compare page 20 of the Géométrie Supérieure of M. Chasles.
(20.) This last formula is evidently true, by (4.), if \( p \) be in the common plane of the three other vectors; and if we suppose it to be perpendicular to that plane, so that

\[
\text{XXIX} . . . p \parallel V\beta \gamma \parallel V\gamma a \parallel V\alpha \beta ,
\]

and therefore, by 281, (9.), since \( S (S\beta \gamma \cdot p) = 0 \).

\[
\text{XXX} . . . S\beta \gamma p = S (V\beta \gamma \cdot p) = V\beta \gamma \cdot p, \quad \&c.,
\]

we may divide each term by \( p \), and so obtain this other formula,

\[
\text{XXXI} . . . a\beta \gamma + \beta V\gamma a + \gamma V\alpha \beta = 0, \quad \text{if} \quad S\alpha \beta \gamma^2 = 0.
\]

(21.) In general, the vector (292) of this last expression vanishes by II.; the expression is therefore equal to its own scalar, and we may write,

\[
\text{XXXII} . . . a\beta \gamma + \beta V\gamma a + \gamma V\alpha \beta = 3S\alpha \beta \gamma,
\]

whatever three vectors may be denoted by \( a, \beta, \gamma \).

(22.) For the case of complanarity, if we suppose that the three vectors are equally long, we have the proportion,

\[
\text{XXXIII} . . . V\beta \gamma : V\gamma a : V\alpha \beta = \sin boc : \sin coa : \sin aob;
\]

and the formula XXXI. becomes thus,

\[
\text{XXXIV} . . . a\beta \gamma + \beta V\gamma a + \gamma V\alpha \beta = 0;
\]

where \( o\alpha, o\beta, o\gamma \) are any three radii of one circle, and the equation is interpreted as in Articles 10, 11, &c.

(23.) The equation XXIII. might have been deduced from XIV., instead of XV., by first operating with \( S \cdot S \), and then interchanging \( S \) and \( p \).

(24.) A vector \( p \) may in general be considered (221) as depending on three scalars (the co-ordinates of its term); it cannot then be determined by fewer than three scalar equations; nor can it be eliminated between fewer than four.

(25.) As an example of such determination of a vector, let \( a, \beta, \gamma \) be again any three given and diplanar vectors; and let the three given equations be,

\[
\text{XXXV} . . . Sa\rho = a, \quad S\beta \rho = b, \quad S\gamma \rho = c;
\]

in which \( a, b, c \) are supposed to denote three given scalars. Then the sought vector \( p \) has for its expression, by XV.,

\[
\text{XXXVI} . . . p = e^{-1} (aV\beta \gamma + bV\gamma a + cV\alpha \beta), \quad \text{if} \quad \text{XXXVII} . . . e = S\alpha \beta \gamma.
\]
(26.) As another example, let the three equations be,

\[ XXXVIII \ldots S\beta \gamma \rho = a', \quad S\gamma \rho a = b', \quad S\alpha \beta \rho = c'; \]

then, with the same signification of the scalar \( e \), we have, by XIV.,

\[ XXXIX \ldots \rho = e^{-1}(a' + b' + c'). \]

(27.) As an example of elimination of a vector, let there be the four scalar equations,

\[ XL \ldots S\rho a = a, \quad S\beta \rho = b, \quad S\gamma \rho = c, \quad S\delta \rho = d; \]

then, by XXIII., we have this resulting equation, into which \( \rho \) does not enter, but only the four vectors, \( a \ldots \delta \), and the four scalars, \( a \ldots d \):

\[ XLI \ldots a \cdot S\beta \gamma \delta - b \cdot S\gamma \delta a + c \cdot S\delta a \beta - d \cdot S\alpha \beta \gamma = 0. \]

(28.) This last equation may therefore be considered as the condition of concurrence of the four planes, represented by the four scalar equations XL., in one common point; for, although it has not been expressly stated before, it follows evidently from the definition 278 of a binary product of vectors, combined with 196, (5.), that every scalar equation of the linear form (comp. 282, XVIII.),

\[ XLI \ldots S\rho a = a, \quad S\rho a = a, \]

in which \( a = o \alpha \), and \( \rho = o \rho \), as usual, represents a plane locus of the point \( P \); the vector of the foot \( s \), of the perpendicular on that plane from the origin, being

\[ XLIII \ldots os = o = aRa = aa^{-1} (282, XXI.). \]

(29.) If we conceive a pyramidal volume (68) as having an algebraical (or scalar) character, so as to be capable of bearing either a positive or a negative ratio to the volume of a given pyramid, with a given order of its points, we may then omit the ambiguous sign, in the last expression (3.) for the scalar of a ternary product of vectors; and so may write, generally, \( oabc \) denoting such a volume, the formula,

\[ XLIV \ldots S\alpha \beta \gamma = 6 \cdot oabc, \]

= a positive or a negative scalar, according as the rotation round \( o \alpha \) from \( o \beta \) to \( o \gamma \) is negative or positive.

(30.) More generally, changing \( o \) to \( d \), and \( o \alpha \) or \( o \beta \) to \( o \alpha - \delta \), &c., we have thus the formula:

\[ XLV \ldots oabc = S(a - \delta)(\beta - \delta)(\gamma - \delta) = S\alpha \beta \gamma - S\beta \gamma \delta + S\gamma \delta a - S\delta a \beta; \]

in which it may be observed that the expression is changed to its own opposite,
or negative, or is multiplied by \(-1\), when any two of the four vectors, \(a, \beta, \gamma, \delta\), or when any two of the four points, \(A, B, C, D\), change places with each other; and therefore is restored to its former value, by a second such binary interchange.

(31.) Denoting then the new origin of \(a, \beta, \gamma, \delta\) by \(b\), we have first, by XLIV., XLV., the equation,

\[
\text{XLVI. } D_{ABC} = E_{ABC} - E_{BCD} + E_{CDA} - E_{DAB};
\]
and may then write the result (comp. 68) under the more symmetric form (because \(-1\) \(E_{BCD} = E_{CDE} = &c.):

\[
\text{XLVII. } B_{CDE} + C_{DEA} + D_{EAB} + E_{ABC} + A_{BCD} = 0;
\]
in which \(A, B, C, D, E\) may denote any five points of space.

(32.) And an analogous formula (69, III.) of the First Book, for any six points \(OABCDE\), namely the equation (comp. 65, 70),

\[
\text{XLVIII. } O_{ABC} + B_{CDE} + C_{DEA} + D_{EAB} + E_{ABC} + A_{BCD} = 0,
\]
in which the additions are performed according to the rules of vectors, the volumes being treated as scalar coefficients, is easily recovered from the foregoing principles and results. In fact, by XLVII., this last formula may be written as

\[
\text{XLIX. } E_{DAB} = E_{EABC} + E_{EBCD} + E_{CDEA} + E_{DEAB} + E_{EABC} + E_{ABCD};
\]
or, substituting \(a, \beta, \gamma, \delta\) for \(eA, eB, eC, eD\), as

\[
\text{L. } S_{a\beta\gamma} = aS_{\beta\gamma\delta} + \beta S_{\gamma\alpha\delta} + \gamma S_{\alpha\beta\delta};
\]
which is only another form of XIV., and ought to be familiar to the student.

(33.) The formula 69, II. may be deduced from XXXI. by observing that, when the three vectors \(a, \beta, \gamma\) are complanar, we have the proportion,

\[
\text{LI. } V_{\beta\gamma} : V_{\gamma a} : V_{a\beta} : V(\beta\gamma + a\gamma + a\beta) = OBC : OCA : OAB : ABC,
\]
if signs (or algebraic or scalar ratios) of areas be attended to (28, 63); and the formula 69, I., for the case of three collinear points \(A, B, C\), may now be written as follows:

\[
\text{LII. } a(\beta - \gamma) + \beta(\gamma - a) + \gamma(a - \beta) = 2V.(\beta\gamma + a\gamma + a\beta)
\]
\[
= 2V(\beta - a)(\gamma - a) = 0,
\]
if the three coinitial vectors \(a, \beta, \gamma\) be termino-collinear (24).
The case when four coinitial vectors \(a, \beta, \gamma, \delta\) are termino-complanar (64), or when they terminate in four compplanar points \(A, B, C, D\), is expressed by equating to zero the second or the third member of the formula XLV.*

Finally, for ternary products of vectors in general, we have the formula:

\[
\text{LIII. } a^2 \beta^2 \gamma^2 + (Sa\beta\gamma)^2 = (Va\beta\gamma)^2 = (aS\beta\gamma - \beta S\gamma a + \gamma Sa\beta)^2
\]

\[
= a^2 (S\beta\gamma)^2 + \beta^2 (S\gamma a)^2 + \gamma^2 (Sa\beta)^2 - 2S\beta\gamma S\gamma a Sa\beta. \dagger
\]

295. The identity (290) of a right quaternion with its index, and the conception (293) of an unit-line as a right versor, allow us now to treat the three important versors, \(i, j, k\), as constructed by, and even as (in our present view) identical with, their own axes; or with the three lines \(0i, 0j, 0k\) of 181, Considered as being each a certain instrument, or operator, or agent in a right rotation (293, (1.) ), which causes any line, in a plane perpendicular to itself, to turn in that plane, through a positive quadrant, without any change of its length. With this conception, or construction, the Laws of the Symbols \(ijk\) are still included in the Fundamental Formula of 183, namely,

\[
i^2=j^2=k^2=ijk=-1; \tag{A}
\]

and if we now, in conformity with the same conception, transfer the Standard Trinomial Form (221) from Right Quaternions to Vectors, so as to write generally an expression of the form,

I. \(p = ix + jy + kz\), or \(V\. a = ia + jb + kc\), &c.,

where \(xyz\) and \(abc\) are scalars (namely, rectangular co-ordinates), we can recover many of the foregoing results with ease: and can, if we think fit, connect them with co-ordinates.

(1.) As to the laws (182), included in the Fundamental Formula A, the law \(i^2 = -1\), &c., may be interpreted on the plan of 293, (1.), as representing the reversal which results from two successive quadrantal rotations.

(2.) The two contrasted laws, or formulæ,

\[
i\hat{j} = +k, \quad ji = -k, \tag{182, II. and III.}
\]

may now be interpreted as expressing, that although a positive rotation through a right angle, round the line \(i\) as an axis, brings a revolving line from the position \(j\) to the position \(k\), or +\(k\), yet, on the contrary, a positive quadrantal rotation round the line \(j\), as a new axis, brings a new revolving line from a new initial

* [And the equation of the plane \(ABC\) is \(SpV(\beta\gamma + \gamma a + a\beta) = Sa\beta\gamma\).]
† [Since \(K\alpha\beta\gamma = -\gamma\beta\alpha\).]
position, \( i \), to a new final position, denoted by \(-k\), or opposite\(^*\) to the old final position, \(+k\).

(3.) Finally, the law \( ijk = -1 \) (183) may be interpreted by conceiving, that we operate on a line \( a \), which has at first the direction of \(+j\), by the three lines, \( k, j, i \), in succession; which gives three new but equally long lines, \( \beta, \gamma, \delta \), in the directions of \(-i, +k, -j\), and so conducts at last to a line \(-a\), which has a direction opposite to the initial one.

(4.) The foregoing laws of \( ijk \), which are all (as has been said) included (184) in the Formula A, when combined with the recent expression I. for \( \rho \), give (comp. 222, (1.) ) for the square of that vector the value:

\[
\text{II. } \rho^2 = (ix + jy + kz)^2 = -(x^2 + y^2 + z^2);
\]

this square of the line \( \rho \) is therefore equal to the negative of the square of its length \( T_\rho \) (185), or to the negative of its norm \( N_\rho \) (273), which agrees with the former result\(\dagger\) 282, (1.) or (2.).

(5.) The condition of perpendicularity of the two lines \( \rho \) and \( a \), when they are represented by the two trinomials I. and I', may be expressed (281, XVIII.) by the formula,

\[
\text{III. } 0 = Sa\rho = -(ax + by + cz);
\]

which agrees with a well-known theorem of rectangular co-ordinates.

(6.) The condition of complanarity of three lines, \( \rho, \rho', \rho'' \), represented by the trinomial forms,

\[
\text{IV. } \rho = ix + jy + kz, \quad \rho' = ix' + \&c., \quad \rho'' = ix'' + \&c.
\]

is (by 294, VI.) expressed by the formula (comp. 223, XIII.),

\[
\text{V. } 0 = S\rho''\rho'\rho = x''(z'y - y'z) + y''(x'z - z'x) + z''(y'x - x'y);
\]

agreeing again with known results.

(7.) When the three lines \( \rho, \rho', \rho'' \), or \( op, op', op'' \), are not in one plane, the recent expression for \( S\rho''\rho'\rho \) gives, by 294, (3.), the volume of the parallelepiped

\* In the Lectures, the three rectangular unit-lines, \( i, j, k \), were supposed (in order to fix the conceptions, and with a reference to northern latitudes) to be directed, respectively, towards the south, the west, and the zenith; and then the contrast of the two formulæ, \( ij = +k, ji = -k \), came to be illustrated by conceiving, that we at one time turn a moveable line, which is at first directed westward, round an axis (or handle) directed towards the south, with a right-handed (or screwing) motion, through a right angle, which causes the line to take an upward position, as its final one; and that at another time we operate, in a precisely similar manner, on a line directed at first southward, with an axis directed to the west, which obliges this new line to take finally a downward (instead of, as before, an upward) direction.

\(\dagger\) Compare also 222, IV.

Hamilton's Elements of Quaternions.
(comp. 223, (9.)) of which they are edges; and this volume, thus expressed, is a positive or a negative scalar, according as the rotation round $\rho$ from $\rho'$ to $\rho''$ is itself positive or negative: that is, according as it has the same direction as that round $+x$ from $+y$ to $+z$ (or round $i$ from $j$ to $k$), or the direction opposite thereto.

(8.) It may be noticed here (comp. 223, (13.)), that if $\alpha, \beta, \gamma$ be any three vectors, then (by 294, III. and V.) we have:

$$\text{VI. } S\alpha\beta\gamma = - S\gamma\beta\alpha = \frac{1}{2}(\alpha\beta\gamma - \gamma\beta\alpha);$$
$$\text{VII. } V\alpha\beta\gamma = + V\gamma\beta\alpha = \frac{1}{2}(\alpha\beta\gamma + \gamma\beta\alpha).$$

(9.) More generally (comp. 223, (12.)), since a vector, considered as representing a right quaternion (290), is always (by 144) the opposite of its own conjugate, so that we have the important formula,*

$$\text{VIII. } K\alpha = - \alpha, \text{ and therefore IX. } K\Pi\alpha = \pm \Pi'\alpha,$$

we may write for any number of vectors, the transformations,

$$\text{X. } S\Pi\alpha = \pm S\Pi'\alpha = \frac{1}{2}(\Pi\alpha \pm \Pi'\alpha),$$
$$\text{XI. } V\Pi\alpha = \mp V\Pi'\alpha = \frac{1}{2}(\Pi\alpha \mp \Pi'\alpha),$$

upper or lower signs being taken, according as that number is even or odd: it being understood that

$$\text{XII. } \Pi'\alpha = \ldots \gamma\beta\alpha, \text{ if } \Pi\alpha = a\beta\gamma \ldots$$

(10.) The relations of rectangularity,

$$\text{XIII. } Ax. i \perp Ax. j; \ Ax. j \perp Ax. k; \ Ax. k \perp Ax. i,$$

which result at once from the definitions (181), may now be written more briefly, as follows:

$$\text{XIV. } i \perp j; \ j \perp k; \ k \perp i;$$

and similarly in other cases, where the axes, or the planes, of any two right quaternions are at right angles to each other.

(11.) But, with the notations of the Second Book, we might also have written, by 123, 181, such formulæ of complanarity as the following, $Ax. j \parallel i$, to express (comp. 225) that the axis of $j$ was a line in the plane of $i$; and it might cause some confusion, if we were now to abridge that formula to $j \parallel i$.

* If, in like manner, we interpret, on our present plan, the symbols $U\alpha, T\alpha, N\alpha$ as equivalent to $U\alpha', T\alpha', N\alpha'$, we are reconducted (compare the Notes to page 137) to the same signification of those symbols as before $(155, 185, 273)$; and it is evident that on the same plan we have now,

$$S\alpha = 0, \ V\alpha = \alpha.$$
In general, it seems convenient that we should not henceforth employ the
sign ||, except as connecting either symbols of three lines, considered still as
complanar; or else symbols of three right quaternions, considered as being col­
linear (209), because their indices (or axes) are complanar: or finally, any two
complanar quaternions (123).

(12.) On the other hand, no inconvenience will result, if we now insert
the sign of parallelism, between the symbols of two right quaternions which
are, in the former sense (123), complanar; for example, we may write, on our
present plan,

\[ \mathbf{x} \parallel \mathbf{i}, \quad \mathbf{y} \parallel \mathbf{j}, \quad \mathbf{z} \parallel \mathbf{k} \]

if \( \mathbf{x} \mathbf{y} \mathbf{z} \) be any three scalars.

296. There are a few particular but remarkable cases, of ternary and other
products of vectors, which it may be well to mention here, and of which some
may be worth a student's while to remember: especially as regards the
products of successive sides of closed polygons, inscribed in circles, or in spheres.

(1.) If \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \) be any four concircular points, we know, by the sub-
articles to 260, that their anharmonic function \((\mathbf{abcd})\), as defined in 259, (9.),
is scalar; being also positive or negative, according to a law of arrangement of
those four points, which has been already stated.

(2.) But, by that definition, and by the scalar (though negative) character
of the square of a vector (282), we have generally, for any plane or gauche
quadrilateral \( \mathbf{abcd} \), the formula:

I. \[ \varepsilon^2 (\mathbf{abcd}) = \mathbf{ab} \cdot \mathbf{bc} \cdot \mathbf{cd} \cdot \mathbf{da} = \text{the continued product of the four sides} \]
in which the coefficient \( \varepsilon^2 \) is a positive scalar, namely the product of two nega­
tive or of two positive squares, as follows:

II. \[ \varepsilon^2 = \mathbf{bc}^2 \cdot \mathbf{da}^2 = \mathbf{bc}^2 \cdot \mathbf{da}^2 > 0. \]

(3.) If then \( \mathbf{abcd} \) be a plane and inscribed quadrilateral, we have, by
260, (8.), the formula,

III. \[ \mathbf{ab} \cdot \mathbf{bc} \cdot \mathbf{cd} \cdot \mathbf{da} = \text{a positive or negative scalar,} \]
according as this quadrilateral in a circle is a crossed or an uncrossed one.

(4.) The product \( \mathbf{a} \beta \gamma \) of any three complanar vectors is a vector, because its
scalar part \( S \mathbf{a} \beta \gamma \) vanishes, by 294, (3.) and (4.); and if the factors be three
successive sides \( \mathbf{ab}, \mathbf{bc}, \mathbf{cd} \) of a quadrilateral thus inscribed in a circle, their
product has either the direction of the fourth successive side, \( \mathbf{da} \), or else the
opposite direction, or in symbols,

IV. \[ \mathbf{ab} \cdot \mathbf{bc} \cdot \mathbf{cd} : \mathbf{da} = \text{or < 0,} \]
according as the quadrilateral \( \mathbf{abcd} \) is an uncrossed or a crossed one.
(5.) By conceiving the fourth point d to approach, continuously and infinitely, to the first point a, we find that the product of the three successive sides of any plane triangle, abc, is given by an equation of the form:

\[ V \ldots AB \cdot BC \cdot CA = AT; \]

at being a line (comp. fig. 63) which touches the circumscribed circle, or (more fully) which touches the segment abc of that circle, at the point a; or represents the initial direction of motion, along the circumference, from a through b to c: while the length of this tangential product line, \( AT \), is equal to, or represents, with the usual reference to an unit of length, the product of the lengths of the three sides, of the same inscribed triangle abc.

(6.) Conversely, if this theorem respecting the product of the sides of an inscribed triangle be supposed to have been otherwise proved, and if it be remembered, then since it will give in like manner the equation,

\[ VI \ldots AC \cdot CD \cdot DA = AU, \]

if d be any fourth point, concircular with a, b, c, while au is, as in the annexed figures 63, a tangent to the new segment acd, we can recover easily the theorem (3.), respecting the product of the sides of an inscribed quadrilateral; and thence can return to the corresponding theorem (260, (8.)), respecting the anharmonic function of any such figure abcd: for we shall thus have, by V. and VI., the equation,

\[ VII \ldots AB \cdot BC \cdot CD \cdot DA = (AT \cdot AU) : (CA \cdot AC), \]

in which the divisor \( CA \cdot AC \) or \( N \cdot AC \), or \( AC^2 \), is always positive (282, (1.)), but the dividend \( AT \cdot AU \) is negative (281, (9.)) for the case of an uncrossed quadrilateral (fig. 63), being on the contrary positive for the other case of a crossed one (fig. 63, bis).

(7.) If p be any point on the circle through a given point a, which touches at a given origin o a given line \( or = \tau \), as represented in fig. 64, we shall then have by (5.) an equation of the form,

\[ VIII \ldots OA \cdot AP \cdot PO = x \cdot OT, \]

* [Or directly by Euclid \( \frac{AT}{AB} = \frac{CA}{CB} \) or \( \frac{AT}{CA} = \frac{AB}{CB} \).]
in which $x$ is some scalar coefficient, which varies with the position of $p$. Making then $oa = a$, and $op = p$, as usual, we shall have

$$IX. \ldots (p - a) \rho = -x t,$$

or

$$IX'. \ldots \rho^{-1} - a^{-1} = x t : a^2 \rho^2,$$

or

$$IX''. \ldots V \tau \rho^{-1} = V \tau a^{-1};$$

and any one of these may be considered as a form of the equation of the circle, determined by the given conditions.

(8.) Geometrically, the last formula $IX''.$ expresses, that the line $\rho^{-1} - a^{-1}$, or $R_0 - R_0$, or $a'c'$ (see again fig. 64), if $oa' = a' = R_0 = R_0$, and $op' = p'' = R . op$, is parallel to the given tangent $t$ at $o$; which agrees with fig. 58, and with Art. 260.

(9.) If $b$ be the point opposite to $o$ upon the circle, then the diameter $ob$, or $\beta$, as being $\perp t$, so that $t\beta^{-1}$ is a vector, is given by the formula,

$$X. \ldots \tau \beta^{-1} = V \tau a^{-1}; \quad \text{or} \quad X'. \ldots \beta = - t : V \tau a^{-1};$$

in which the tangent $t$ admits, as it ought to do, of being multiplied by any scalar, without the value of $\beta$ being changed.

(10.) As another verification, the last formula gives,

$$XI. \ldots \overline{ob} = T \beta = T \alpha : TV \tau a^{-1} = \overline{oa} : \sin aot.$$

(11.) If a quadrilateral $oabc$ be not inscriptible in a circle, then whether it be plane or gauche, we can always circumscribe (as in fig. 65) two circles, $oab$ and $obc$, about the two triangles, formed by drawing the diagonal $ob$; and then, on the plan of (6.), we can draw two tangents $ot$, $ou$, to the two segments $oab$, $obc$, so as to represent the two ternary products,

$$oa . ab . bo, \quad \text{and} \quad ob . bc . co;$$

after which we shall have the quaternary product,

$$XII. \ldots \overline{oa} . ab . bc . co = ot . ou : \overline{ob}^2;$$

where the divisor, $\overline{ob}^2$, or $bo . ob$, or $N . ob$, is a positive scalar, but the dividend $ot . ou$, and therefore also the quotient in the second member, or the product in the first member, is a quaternion.
(12.) The axis of this quaternion is perpendicular to the plane tou of the two tangents; and therefore to the plane itself of the quadrilateral oabc, if that be a plane figure; but if it be gauche, then the axis is normal to the circumscribed sphere at the point o: being also in all cases such, that the rotation round it, from ot to ou, is positive.

(13.) The angle of the same quaternion is the supplement of the angle tou between the two tangents above mentioned; it is therefore equal to the angle u'ot, if ou' touch the new segment ocb, or proceed in a new and opposite direction from o (see again fig. 65); it may therefore be said to be the angle between the two arcs, oab and ocb, along which a point should move, in order to go from o, on the two circumferences, to the opposite corner b of the quadrilateral oabc, through the two other corners, a and c, respectively: or the angle between the arcs ocb, oab.

(14.) These results, respecting the axis and angle of the product of the four successive sides, of any quadrilateral oabc, or abcd, apply without any modification to the anharmonic quaternion (259, (9.)) of the same quadrilateral; and although, for the case of a quadrilateral in a circle, the axis becomes indeterminate, because the quaternary product and the anharmonic function degenerate together into scalars, or because the figure may then be conceived to be inscribed in indefinitely many spheres, yet the angle may still be determined by the same rule as in the general case: this angle being $\pi$, for the inscribed and uncrossed quadrilateral (fig. 63); but $= 0$, for the inscribed and crossed one (fig. 63, bis).

(15.) For the gauche quadrilateral oabc, which may always be conceived to be inscribed in a determined sphere, we may say, by (13.), that the angle of the quaternion product, $\angle (oa \cdot ab \cdot bc \cdot cb)$, is equal to the angle of the lunule, bounded (generally) by the two arcs of small circles oab, ocb; with the same construction for the equal angle of the anharmonic,

$\angle (oabc)$, or $\angle (oa : ab \cdot bc : cb)$.

(16.) It is evident that the general principle 223, (10.), of the permissibility of cyclical permutation of quaternion factors under the sign $S$, must hold good for the case when those quaternions degenerate (294) into vectors; and it is still more obvious, that every permutation of factors is allowed, under the sign $T$: whence cyclical permutation is again allowed, under this other sign $SU$; and consequently also (comp. 196, XVI.) under the sign $\angle$. 
(17.) Hence generally, for any four vectors, we have the three equations,

\[ \text{XIII. } S\alpha\beta\gamma \delta = S\beta\gamma \delta \alpha; \quad \text{XIV. } S\alpha\beta\gamma \delta = S\beta\gamma \delta \alpha; \]

\[ \text{XV. } \angle \alpha\beta\gamma \delta = \angle \beta\gamma \delta \alpha; \]

and in particular, for the successive sides of any plane or gauche quadrilateral \(abcd\), we have the four equal angles,

\[ \text{XVI. } \angle (ab \cdot bc \cdot cd \cdot da) = \angle (bc \cdot cd \cdot da \cdot ab) = \&c.; \]

with the corresponding equality of the angles of the four anharmonics,

\[ \text{XVII. } \angle (abcd) = \angle (bcda) = \angle (cdab) = \angle (dabc); \]

or of those of the four reciprocal anharmonics (259, XVII.),

\[ \text{XVII'.. } \angle (adcb) = \angle (badc) = \angle (cbad) = \angle (dcba). \]

(18.) Interpreting now, by (13.) and (15.), these last equations, we derive from them the following theorem, for the plane, or for space:—

Let \(abcd\) be any four points, connected by four circles, each passing through three of the points: then, not only is the angle at \(a\), between the arcs \(abc, adc\), equal to the angle at \(c\), between \(cda\) and \(cba\), but also it is equal (comp. fig. 66) to the angle at \(b\), between the two other arcs \(bcd\) and \(bad\), and to the angle at \(d\), between the arcs \(dab, dcb\).

(19.) Again, let \(abcde\) be any pentagon, inscribed in a sphere; and conceive that the two diagonals \(ac, ad\) are drawn. We shall then have three equations, of the forms,

\[ \text{XVIII. } ab \cdot bc \cdot ca = at; \quad ac \cdot cd \cdot da = au; \quad ad \cdot de \cdot ea = av; \]

where \(at, au, av\) are three tangents to the sphere at \(a\), so that their product is a fourth tangent at that point. But the equations XVIII. give

\[ \text{XIX. } ab \cdot bc \cdot cd \cdot de \cdot ea = (at \cdot au \cdot av) : (ac^2 \cdot ad^2) \]

\[ = aw = a \text{ new vector, which touches the sphere at } a. \]

We have therefore this Theorem, which includes several others under it:—

"The product of the five successive sides of any (generally gauche) pentagon inscribed in a sphere, is equal to a tangential vector, drawn from the point at which the pentagon begins and ends."
(20.) Let then \( p \) be a point on the sphere which passes through \( o \), and through three given points \( A, B, C \); we shall have the equation,

\[
XX \ldots 0 = S(\mathbf{OA} \cdot \mathbf{AB} \cdot \mathbf{BC} \cdot \mathbf{CP} \cdot \mathbf{PO}) = S a(\beta - a) (\gamma - \beta) (\rho - \gamma) (-\rho) = a^2S\beta\gamma\rho + \beta^2S\gamma\rho + \gamma^2S\alpha\beta\rho - \rho^2S\alpha\beta\gamma.
\]

(21.) Comparing with 294, XIV., we see that the condition for the four co-initial vectors \( a, \beta, \gamma, \rho \) thus terminating on one spheric surface, which passes through their common origin \( o \), may be thus expressed:

\[
XXI \ldots \text{if } \rho = xa + y\beta + z\gamma, \text{ then } \rho^2 = xa^2 + y\beta^2 + z\gamma^2.
\]

(22.) If then we project (comp. 62) the variable point \( \mathbf{p} \) into points \( A', B', C' \) on the three given chords \( OA, OB, OC \), by three planes through that point \( \mathbf{p} \), respectively parallel to the planes \( BOC, COA, AOB \), we shall have the equation:

\[
XXII \ldots op^2 = oA \cdot oA' + oB \cdot oB' + oC \cdot oC'.
\]

(23.) That the equation XX. does in fact represent a spheric locus for the point \( \mathbf{p} \), is evident from its mere form (comp. 282, (10.)); and that this sphere passes through the four given points, \( o, A, B, C \), may be proved by observing that the equation is satisfied, when we change \( \rho \) to any one of the four vectors, \( 0, a, \beta, \gamma \).

(24.) Introducing an auxiliary vector, \( \mathbf{d} \), determined by the equation,

\[
XXIII \ldots S\delta\alpha\beta\gamma = a^2V^2\beta\gamma + \beta^2V^2\gamma\alpha + \gamma^2V^2\alpha\beta,
\]

or by the system of the three scalar equations (comp. 294, (25.)),

\[
XXIV \ldots a^2 = S\delta\alpha, \quad \beta^2 = S\delta\beta, \quad \gamma^2 = S\delta\gamma,
\]

or

\[
XXIV' \ldots S\delta\alpha^{-1} = S\delta\beta^{-1} = S\delta\gamma^{-1} = 1,
\]

the equation XX. of the sphere becomes simply,

\[
XXV \ldots \rho^2 = S\delta\rho, \text{ or } XXV' \ldots S\delta\rho^{-1} = 1;
\]

so that \( \mathbf{d} \) is the point of the sphere opposite to \( o \), and \( \delta \) is a diameter (comp. 282, IX'; and 196, (6.).)

(25.) The formula XXIII., which determines this diameter, may be written in this other way:

\[
XXVI \ldots S\delta\alpha\beta\gamma = V a (\beta - a) (\gamma - \beta) \gamma;
\]

or

\[
XXVI' \ldots 6 \cdot oABC \cdot oD = - V (oA \cdot AB \cdot BC \cdot CO);
\]

where the symbol \( oABC \), considered as a coefficient, is interpreted as in 294,
XLIV.; namely, as denoting the volume of the pyramid \( \text{OABC} \), which is here an inscribed one.

(26.) This result of calculation, so far as it regards the direction of the axis of the quaternion \( \text{OA} \cdot \text{AB} \cdot \text{BC} \cdot \text{CO} \), agrees with, and may be used to confirm, the theorem (12.), respecting the product of the successive sides of a gauche quadrilateral, \( \text{OABC} \); including the rule of rotation, which distinguishes that axis from its opposite.

(27.) The formula XXIII. for the diameter \( \delta \) may also be thus written:

\[
\text{XXVII. . . } \delta \cdot \text{Sa}^{-1} \beta^{-1} \gamma^{-1} = \sqrt{\left(\beta^{-1} \gamma^{-1} + \gamma^{-1} \alpha^{-1} + \alpha^{-1} \beta^{-1}\right)}
\]

and the equation XX. of the sphere may be transformed to the following:

\[
\text{XXVIII. . . } 0 = \delta \left(\beta^{-1} - \alpha^{-1}\right) \left(\gamma^{-1} - \alpha^{-1}\right) \left(\beta^{-1} - \alpha^{-1}\right);
\]

which expresses (by 294, (34.), comp. 260, (10.),) that the four reciprocal vectors,

\[
\text{XXIX. . . OA} = \alpha = \alpha^{-1}, \quad \text{OB} = \beta = \beta^{-1}, \quad \text{OC} = \gamma = \gamma^{-1}, \quad \text{OP} = \rho = \rho^{-1};
\]

are termino-complanar (64); the plane \( \text{A'B'C'P'} \), in which they all terminate, being parallel to the tangent plane to the sphere at \( \text{O} \); because the perpendicular let fall on this plane from \( \text{O} \) is

\[
\text{XXX.. . } \delta' = \delta^{-1},
\]

as appears from the three scalar equations,

\[
\text{XXXI. . . } \text{Sa} \delta = \delta \beta \gamma = \delta \gamma \beta = 1.
\]

(28.) In general, if \( \text{D} \) be the foot of the perpendicular from \( \text{O} \), on the plane \( \text{ABC} \), then

\[
\text{XXXII. . . } \delta = \text{Sa} \beta \gamma : \sqrt{(\beta \gamma + \gamma a + a \beta)};
\]

because this expression satisfies, and may be deduced from, the three equations,

\[
\text{XXXIII. . . } \text{Sa} \delta^{-1} = \delta \beta \delta^{-1} = \delta \gamma \delta^{-1} = 1.
\]

As a verification, the formula shows that the length \( \text{T}\delta \), of this perpendicular, or altitude, \( \text{OD} \), is equal to the sextuple volume of the pyramid \( \text{OABC} \), divided by the double area of the triangular base \( \text{ABC} \). (Compare 281, (4.), and 294, (3.), (33.).)

(29.) The equation XX., of the sphere \( \text{OABC} \), might have been obtained by the elimination of the vector \( \delta \), between the four scalar equations XXIV. and XXV., on the plan of 294, (27.).

Hamilton's Elements of Quaternions.
(30.) And another form of equation of the same sphere, answering to the development of XXVIII., may be obtained by the analogous elimination of the same vector \( \delta \), between the four other equations, XXIV', and XXV'.

(31.) The product of any even number of complanar vectors is generally a quaternion with an axis perpendicular to their plane; but the product of the successive sides of a hexagon ABCDEF, or any other even-sided figure, inscribed in a circle, is a scalar: because by drawing diagonals AC, AD, AE from the first (or last) point \( A \) of the polygon, we find as in (6.) that it differs only by a scalar coefficient, or divisor, from the product of an even number of tangents, at the first point.

(32.) On the other hand, the product of any odd number of complanar vectors is always a line, in the same plane; and in particular (comp. (19.)), the product of the successive sides of a pentagon, or heptagon, &c., inscribed in a circle, is equal to a tangential vector, drawn from the first point of that inscribed and odd-sided polygon: because it differs only by a scalar coefficient from the product of an odd number of such tangents.

(33.) The product of any number of lines in space is generally a quaternion (289); and if they be the successive sides of a hexagon, or other even-sided polygon, inscribed in a sphere, the axis of this quaternion (comp. (12.)) is normal to that sphere, at the initial (or final) point of the polygon.

(34.) But the product of the successive sides of a heptagon, or other odd-sided polygon in a sphere, is equal (comp. (19.)) to a vector, which touches the sphere at the initial or final point; because it bears a scalar ratio to the product of an odd number of vectors, in the tangent plane at that point.*

(35.) The equation XX., or its transformation XXVIII., may be called the condition or equation of homosphericity (comp. 260, (10.)) of the five points \( O, A, B, C, P \); and the analogous equation for the five points abcdb, with vectors \( a \beta \gamma \delta \varepsilon \) from any arbitrary origin \( O \), may be written thus:

\[
XXXIV... 0 = S (a - \beta) (\beta - \gamma) (\gamma - \delta) (\delta - \varepsilon) (\varepsilon - a);
\]
or thus

\[
XXXV... 0 = aa^2 + b\beta^2 + c\gamma^2 + d\delta^2 + e\varepsilon^2;\]

* [The inscription of polygons in a sphere is treated very fully in the "Lectures." If \( p_1, p_2, \ldots, p_n \) are the vectors from the centre to the vertices, and if \( n_1 = p_2 - p_1, n_2 = p_3 - p_2, \&c. \) denote the vector sides, then by 213 (5.) \( p = -n_1 p_1 n_1^{-1}, p_2 = -n_2 p_2 n_2^{-1} = n_2 n_1 p_1 n_1^{-1} n_2^{-1} \) and \( p_{n+1} = p = (-)^n q_1 p q_1^{-1} \).

Hence \( p q = (-)^n q_1 p_1 \); or when \( n \) is even \( p q = q_1 p_1 \) or \( q \parallel p_1 \); but when \( n \) is odd the quaternion equation \( p_1 S q + S q p_1 = 0 \) affords the conditions \( S q = 0 \) and \( S p q \parallel q = 0, \) or \( q \) is a vector at right angles to \( p_1 \). See Lecture VI., Art. 336.]

† [On change of origin XXI. may be written in the form

\[
\begin{align*}
\alpha (a - e) + b (\beta - e) + c (\gamma - e) + d (\delta - e) + e (\varepsilon - e) &= 0, \\
a (a - e)^2 + b (\beta - e)^2 + c (\gamma - e)^2 + d (\delta - e)^2 &= 0.
\end{align*}
\]

Introducing \( e \) defined by XXXIX., XXXV. and XXXVII. follow. Eliminating \( a, b, c, d, \) and \( e \) from five equations connecting the squares of the mutual distances between the points, analogous to that here given, a determinant relation is at once found.]
six times the second member of this last formula being found to be equal to
the second member of the one preceding it, if

\[ XXXVI. \quad a = BCDE, \quad b = CDEA, \quad c = DEAB, \quad d = EABC, \quad e = ABCD, \]
or more fully,

\[ XXXVII. \quad 6a = 6 (\gamma - \beta)(\delta - \beta)(\varepsilon - \beta) = S (\gamma \delta \varepsilon - \delta \varepsilon \beta + \varepsilon \beta \gamma - \beta \gamma \delta), \&c.; \]
so that, by 294, XLVIII. and XLVII., we have also (comp. 65, 70) the
equation,

\[ XXXVIII. \quad 0 = a + b + c + d + e, \]
with the relation between the coefficients,

\[ XXXIX. \quad 0 = a + b + c + d + e, \]
which allows (as above) the origin of vectors to be arbitrary.

(36.) The equation or condition XXXV. may be obtained as the result of
an elimination (294, (27.),) of a vector \( k \), and of a scalar \( g \), between five scalar
equations of the form 282, (10.), namely the five following,

\[ XL. \quad a^2 - 2Sk + g = 0, \quad \beta^2 - 2Sk + g = 0, \quad \varepsilon^2 - 2Sk + g = 0; \]
\( k \) being the vector of the centre \( k \) of the sphere \( ABCD \), of which the equation
may be written as

\[ XLI. \quad \rho^2 - 2Sk + g = 0, \]
g being some scalar constant; and on which, by the condition referred to, the
fifth point \( e \) is situated.

(37.) By treating this fifth point, or its vector \( e \), as arbitrary, we recover
the condition or equation of concircularity (3.), of the four points \( A, B, C, D \); or
the formula,

\[ XLII. \quad 0 = V (a - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - a). \]

(38.) The equation of the circle \( ABC \), and the equation of the sphere \( ABCD \),
may in general be written thus:

\[ XLIII. \quad 0 = V (a - \beta)(\beta - \gamma)(\gamma - \rho)(\rho - a); \]
\[ XLIV. \quad 0 = S (a - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \rho)(\rho - a); \]
\( \rho \) being as usual the vector of a variable point \( \rho \), on the one or the other locus.

(39.) The equations of the tangent to the circle \( ABC \), and of the tangent
plane to the sphere \( ABCD \), at the point \( A \), are respectively,

\[ XLV. \quad 0 = V (a - \beta)(\beta - \gamma)(\gamma - \rho)(\rho - a), \]
and \[ XLVI. \quad 0 = S (a - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - \rho)(\rho - a). \]
(40.) Accordingly, whether we combine the two equations XLIII. and XLV., or XLIV. and XLVI., we find in each case the equation,

\[ (\rho - a)^2 = 0, \ \text{giving} \ \rho = a, \ \text{or} \ \rho = A (20); \]
it being supposed that the three points \( A, B, C \) are not collinear, and that the four points, \( A, B, C, D \) are not coplanar.

(41.) If the centre of the sphere, \( ABCD \) be taken for the origin \( O \), so that

\[ a^2 = \beta^2 = \gamma^2 = \delta^2 = -r^2, \ \text{or} \ \|a\| = \|\beta\| = \|\gamma\| = \|\delta\| = r, \]

the positive scalar \( r \) denoting the radius, then after some reductions we obtain the transformation,

\[ V(a - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - a) = 2aS(\beta - a)(\gamma - a)(\delta - a). \]

(42.) Hence, generally, if \( k \) be, as in (36.), the centre of the sphere, we have the equation (comp. XXVI.),

\[ V(a - \beta)(\beta - \gamma)(\gamma - \delta)(\delta - a) = 2KA. ABCD. \]

(43.) We may therefore enunciate this theorem:

"The vector part of the product of four successive sides, of a gauche quadrilateral inscribed in a sphere, is equal to the diameter drawn to the initial point of the polygon, multiplied by the sextuple volume of the pyramid, which its four points determine."

(44.) In effecting the reductions (41.), the following general formulae of transformation have been employed, which may be useful on other occasions:

\[ aq + qa = 2(aSq + Sqa); \]

\[ aq = a^2Kq + 2aSqa; \]

where \( a \) may be any vector, and \( q \) may be any quaternion.

SECTION 7.

On the Fourth Proportional to Three Dпланar Vectors.

297. In general, when any four quaternions, \( q, q', q'', q''' \), satisfy the equation of quotients,

\[ q'' : q''' = q' : q, \]
or the equivalent formula,

\[ q''' = (q' : q). q'' = q'q^{-1}q'', \]

we shall say that they form a Proportion; and that the fourth, namely \( q''' \), is the Fourth Proportional to the first, second, and third quaternions, namely to
q, q', and q'', taken in this given order. This definition will include (by 288) the one which was assigned in 226, for the fourth proportional to three com-
planar vectors, a, β, γ, namely that fourth vector in the same plane, \( \delta = \beta a^{-1} \gamma \), which has been already considered; and it will enable us to interpret (comp. 
289) the symbol 

\[ \mathbf{III} \ldots \beta a^{-1} \gamma, \quad \text{when } \gamma \not\parallel a, \beta, \]
as denoting not indeed a Vector, in this new case, but at least a Quaternion, which may be called (on the present general plan) the Fourth Proportional to 
these three Diplanar Vectors, a, β, γ. Such fourth proportionals possess some 
interesting properties, especially with reference to their vector parts, which it 
will be useful briefly to consider, and to illustrate by showing their connexion 
with spherical trigonometry, and generally with spherical geometry.

(1.) Let a, β, γ be (as in 208, (1), &c.) the vectors of the corners of a 
triangle \( \triangle ABC \) on the unit-sphere, whereof the sides are a, b, c; and let us write,

\[ \begin{cases} 
    l = \cos a = S\gamma\beta^{-1} = -S\beta \gamma, \\
    m = \cos b = Sa\gamma^{-1} = -S\gamma a, \\
    n = \cos c = S\beta a^{-1} = -Sa \beta; 
\end{cases} \]

where it is understood that 

\[ \mathbf{V} \ldots a^2 = \beta^2 = \gamma^2 = -1, \quad \text{or} \quad \mathbf{VI} \ldots Ta = T\beta = T\gamma = 1; \]
it being also at first supposed, for the sake of fixing the conceptions, that 
each of these three cosines, \( l, m, n \), is greater than zero, or that each side of 
the triangle \( \triangle ABC \) is less than a quadrant.

(2.) Then, introducing three new vectors, \( \delta, \varepsilon, \zeta \), defined by the equations,

\[ \begin{cases} 
    \delta = V\beta a^{-1} \gamma = V\gamma a^{-1} \beta = m \beta + n \gamma - la, \\
    \varepsilon = V\gamma \beta^{-1} a = V a \beta^{-1} \gamma = n \gamma + la - m \beta, \\
    \zeta = V a \gamma^{-1} \beta = V \beta \gamma^{-1} a = la + m \beta - n \gamma, 
\end{cases} \]

we find that these three derived vectors have all one common length, say \( r \), 
because they have one common norm; namely, 

\[ \mathbf{VIII} \ldots N\delta = N\varepsilon = N\zeta = l^2 + m^2 + n^2 - 2lmn = r^2; \]

so that 

\[ \mathbf{IX} \ldots T\delta = T\varepsilon = T\zeta = r = \sqrt{(l^2 + m^2 + n^2 - 2lmn)}. \]

(3.) This common length, \( r \), is less than unity; for if we write, 

\[ \mathbf{X} \ldots Sa \beta \gamma = S\beta a^{-1} \gamma = \varepsilon, \]

we shall have the relation, 

\[ \mathbf{XI} \ldots \varepsilon^2 + r^2 = N\beta a^{-1} \gamma = 1; \]

and the scalar \( \varepsilon \) is different from zero, because the vectors \( a, \beta, \gamma \) are diplanar.
(4.) Dividing the three lines δ, ε, ζ by their length, r, we change them to their versors (155, 156); and so obtain a new triangle, DEF, on the unit-sphere, of which the corners are determined by the three new unit-vectors,

\[ \text{XII. } \text{OD} = U\delta = r^{-1}\delta; \quad \text{OE} = U\varepsilon = r^{-1}\varepsilon; \quad \text{OF} = U\zeta = r^{-1}\zeta. \]

(5.) The sides opposite to D, E, F, in this new or derived triangle, are bisected, as in fig. 67, by the corners A, B, C of the old or given triangle; because we have the three equations,

\[ \text{XIII. } \varepsilon + \zeta = 2\alpha; \quad \zeta + \delta = 2\beta; \quad \delta + \varepsilon = 2\gamma. \]

(6.) Denoting the halves of the new sides by \(a', b', c'\) (so that the arc \(EF = 2\alpha', \&c.)\), the equations XIII. show also, by XIV. and IX., that

the cosines of the half-sides of the new (or bisected) triangle, DEF, are therefore proportional to the cosines of the sides of the old (or bisecting) triangle ABC.

(7.) The equations (IV.) give, by 279, (1.),

\[ \text{XV. } 2l = -(\beta\gamma + \gamma\beta), \quad 2m = -(\gamma a + a\gamma), \quad 2n = -(a\beta + \beta a); \]

we have therefore, by VII., the three following equations between quaternions,

\[ \text{XVI. } a\varepsilon = \zeta a, \quad \beta\zeta = \delta\beta, \quad \gamma\delta = \gamma\varepsilon; \]

which may also be thus written,

\[ \text{XVI'. } a\varepsilon = \zeta a, \quad \beta\zeta = \delta\beta, \quad \gamma\delta = \gamma\varepsilon, \]

and express in a new way the relations of bisection (5.).

(8.) We have therefore the equations between vectors,

\[ \text{XVII. } \varepsilon = a\zeta a^{-1}, \quad \zeta = \beta\delta\beta^{-1}, \quad \delta = \gamma\varepsilon\gamma^{-1}; \]

or

\[ \text{XVII'. } \varepsilon = a\zeta a^{-1}, \quad \zeta = \beta\delta\beta^{-1}, \quad \delta = \gamma\varepsilon\gamma^{-1}. \]

(9.) Hence also, by V., or because \(a, \beta, \gamma\) are unit-vectors,

\[ \text{XVIII. } \varepsilon = -a\zeta a, \quad \zeta = -\beta\delta\beta, \quad \delta = -\gamma\varepsilon\gamma; \]

or

\[ \text{XVIII'. } \varepsilon = -a\zeta a, \quad \zeta = -\beta\delta\beta, \quad \delta = -\gamma\varepsilon\gamma. \]

(10.) In general, whatever the length of the vector \(a\) may be, the first equation XVII. expresses that the line \(\varepsilon\) is (comp. 138) the reflexion of the line \(\zeta\), with respect to that vector \(a\); because it may be put (comp. 279) under the form

\[ \text{XIX. } \zeta a^{-1} = a^{-1}\varepsilon = K\varepsilon a^{-1}, \quad \text{or} \quad \text{XIX'. } a\varepsilon = K\zeta a^{-1}. \]
Another mode of arriving at the same interpretation of the equation $\varepsilon = a\zeta a^{-1}$, is to conceive $\zeta$ decomposed into two summand vectors, $\zeta'$ and $\zeta''$, one parallel and the other perpendicular to $a$, in such a manner that

$$XX... \zeta = \zeta' + \zeta'', \quad \zeta' \parallel a, \quad \zeta'' \perp a;$$

for then we shall have, by 281, (10.), the transformations,

$$XXI... \varepsilon = a\zeta'a^{-1} + a\zeta''a^{-1} = \zeta'aa^{-1} - \zeta''aa^{-1} = \zeta' - \zeta'';$$

the parallel part of $\zeta$ being thus preserved, but the perpendicular part being reversed, by the operation $a(\quad)a^{-1}$.

Or we may return from $\varepsilon = a\zeta a^{-1}$ to the form $\varepsilon a = a\zeta$, that is, to the first equation XVI.; and then this equation between quaternions will show, as suggested in (7.), that whatever may be the length of $a$, we must have,

$$XXII... T\varepsilon = T\zeta, \quad \Delta x. \varepsilon a = \Delta x. \zeta, \quad \angle \varepsilon a = \angle a\zeta;$$

so that the two lines $\varepsilon$, $\zeta$ are equally long, and the rotation from $\varepsilon$ to $a$ is equal to that from $a$ to $\zeta$; these two rotations being similarly directed, and in one common plane.

We may also write the equations XVII, XVII'. under the forms,

$$XXIII... \varepsilon = a^{-1}\zeta a, \quad \&c.; \quad XXIII'... \zeta = a^{-1}\varepsilon a, \quad \&c.$$

Substituting this last expression for $\zeta$ in the second equation XVII', we derive this new equation,

$$XXIV... \delta = \beta a^{-1}\varepsilon a\beta^{-1}; \quad \text{or} \quad XXIV'... \varepsilon = a\beta^{-1}\delta\beta a^{-1};$$

that is, more briefly,

$$XXV... \delta = q\varepsilon q^{-1}, \quad \text{and} \quad XXV'... \varepsilon = q^{-1}\delta q, \quad \text{if} \quad XXVI... q = \beta a^{-1}.$$

An expression of this form, namely one with such a symbol as

$$XXVII... q(\quad)q^{-1}$$

for an operator, occurred before, in 179, (1.), and in 191 (5.); and was seen to indicate a conical rotation of the axis of the operand quaternion (of which the symbol is to be conceived as being written within the parentheses) round the axis of $q$, through an angle $= 2\angle q$, without any change of the angle, or of the tensor, of that operand; so that a vector must remain a vector, after any operation of

* It was remarked in 291, that this characteristic $\Delta x.$ can be dispensed with, because it admits of being replaced by $UV$; but there may still be a convenience in employing it occasionally.
this sort, as being still a right-angled quaternion (290); or (comp. 223, (10.))
because

$$\text{XXVIII. } Sqq'q^{-1} = Sq'q Sq = S \rho = 0.$$  

(16.) If then we conceive two opposite points, $p'$ and $p$, to be determined
on the unit-sphere, by the conditions of being respectively the positive poles
of the two opposite arcs, $ab$ and $ba$, so that

$$\text{XXIX. } op' = Ax. \beta a^{-1} = Ax. q, \quad \text{and} \quad op = p' o = Ax. a \beta^{-1} = Ax. q^{-1},$$

we can infer from XXIX. that the line $od$ may be derived from the line $oe$, by a
conical rotation round the line $op'$ as an axis, through an angle equal to the double
of the angle $\angle aob$ (if o be still the centre of the sphere).

(17.) And in like manner we can infer from XXIV', that the line $oe$
admits of being derived from $od$, by an equal but opposite conical rotation,
round the line $op$ as a new positive axis, through an angle equal to twice the
angle $\angle boa$.

(18.) To illustrate these and other connected results, the annexed figure 68
is drawn; in which $p$ represents, as above, the
positive pole of the arc $ba$, and arcs are drawn
from it to $d, e$, $f$, meeting the great circle
through $a$ and $b$ in the points $r, s, t$. (The
other letters in the figure are not, for the
moment, required, but their significations will
soon be explained.)

(19.) This being understood, we see, first,
that because the arcs $ef$ and $fd$ are bisected (5.)
at $a$ and $b$, the three arcual perpendiculars, $es,
ft, dr$, let fall from $e, f, d$, on the great circle
through $a$ and $b$, are equally long; and that
therefore the point $p$ is the interior pole of the small circle $def'$, if $f'$ be the
point diametrically opposite to $f$: so that a conical rotation round this pole $p$, or
round the axis $op$, would in fact bring the point $d$, or the line $od$, to the position $e$, or $oe$, which is one part of the theorem (17.).

(20.) Again, the quantity of this conical rotation, is evidently measured by
the arc $rs$ of the great circle with $p$ for pole; but the bisections above mentioned
give (comp. 165) the two arcual equations,

$$\text{XXX. } \angle RB = \angle BT, \quad \angle TA = \angle AS; \quad \text{whence} \quad \text{XXXI. } \angle RS = 2 \angle BA,$$

and the other part of the same theorem (17.) is proved.
(21.) The point \( f \) may be said to be the reflexion, on the sphere, of the point \( d \), with respect to the point \( b \), which bisects the interval between them; and thus we may say that two successive reflexions of an arbitrary point upon a sphere (as here from \( d \) to \( f \), and then from \( f \) to \( e \)), with respect to two given points (\( b \) and \( a \)) of a given great circle, are jointly equivalent to one conical rotation, round the pole (\( p \)) of that great circle; or to the description of an arc of a small circle, round that pole, or parallel to that great circle: and that the angular quantity (\( dpe \)) of this rotation is double of that represented by the arc (\( ba \)) connecting the two given points; or is the double of the angle (\( bpa \)), which that given arc subtends, at the same pole (\( p \)).

(22.) There is, as we see, no difficulty in geometrically proving this theorem of rotation: but it is remarkable how simply quaternions express it: namely by the formula,

$$XXXII. \quad a \cdot \beta^{-1} \beta \cdot a^{-1} = a \beta^{-1} \cdot \beta a^{-1},$$

in which \( a, \beta, \rho \) may denote any three vectors; and which, as we see by the points, involves essentially the associative principle of multiplication.

(23.) Instead of conceiving that the point \( d \), or the line \( od \), has been reflected into the position \( f \), or \( of \), with respect to the point \( b \), or to the line \( ob \), with a similar successive reflexion from \( f \) to \( e \), we may conceive that a point has moved along a small semicircle, with \( b \) for pole, from \( d \) to \( f \), as indicated in fig. 69, and then along another small semicircle, with \( a \) for pole, from \( f \) to \( e \); and we see that the result, or effect, of these two successive and semicircular motions is equivalent to a motion along an arc \( de \) of a third small circle, which is parallel (as before) to the great circle through \( b \) and \( a \), and has a projection \( rs \) thereon, which (still as before) is double of the given arc \( ba \).

(24.) And instead of thus conceiving two successive arcual motions of a point \( d \) upon a sphere, or two successive conical rotations of a radius \( od \), considered as compounding themselves into one resultant motion of that point, or rotation of that radius, we may conceive an analogous composition of two successive rotations of a solid body (or rigid system), round axes passing through a point \( o \), which is fixed in space (and in the body): and so obtain a theorem respecting such rotation, which easily suggests itself from what precedes, and on which we may perhaps return.

(25.) But to draw some additional consequences from the equations VII., &c., and from the recent fig. 68, especially as regards the Construction of the
Fourth Proportional to three diplanar vectors, let us first remark, generally, that when we have (as in 62) a linear equation, of the form
\[ a\alpha + b\beta + c\gamma + d\delta = 0, \]
connecting four co-initial vectors \( \alpha \ldots \delta \), whereof no three are complanar, then this fifth vector.
\[ \varepsilon = a\alpha + b\beta = -c\gamma - d\delta, \]
is evidently complanar (22) with \( \alpha, \beta \), and also with \( \gamma, \delta \) (comp. 294, (6.) ); it is therefore part of the indefinite line of intersection of the plane \( \alpha\beta\gamma \), \( \gamma\delta\alpha \), of these two pairs of vectors.

(26.) And if we divide this fifth vector \( \varepsilon \) by the two (generally unequal) scalars,
\[ a + b, \quad \text{and} \quad -c - d, \]
the two (generally unequal) vectors,
\[ (a\alpha + b\beta) : (a + b), \quad \text{and} \quad (c\gamma + d\delta) : (c + d), \]
which are obtained as the quotients of these two divisions, are (comp. 25, 64) the vectors of two (generally distinct) points of intersection, of lines with planes, namely the two following:
\[ \mathbf{AB} \cdot \mathbf{OCD}, \quad \text{and} \quad \mathbf{CD} \cdot \mathbf{OAB}. \]

(27.) When the two lines, \( \mathbf{AB} \) and \( \mathbf{CD} \), happen to intersect each other, the two last-mentioned points coincide; and thus we recover, in a new way, the condition (63), for the complanarity of the four points \( \mathbf{O, A, B, C} \), or for the termino-complanarity of the four vectors \( \alpha, \beta, \gamma, \delta \); namely the equation
\[ a + b + c + d = 0, \]
which may be compared with 294, XLV., and L.

(28.) Resuming now the recent equations VII., and introducing the new vector,
\[ \mathbf{XXXIII} \ldots \lambda = la - m\beta = \frac{1}{2} (\varepsilon - \delta), \]
which gives,
\[ \mathbf{XXXIV} \ldots S\gamma \lambda = 0, \quad \text{and} \quad \mathbf{XXXV} \ldots T\lambda = \sqrt{(r^2 - n^2)} = r \sin \epsilon'. \]
we see that the two arcs \( \mathbf{BA, DE} \), prolonged, meet in a point \( \mathbf{L} \) (comp. fig. 68), for which \( \mathbf{OL} = U\lambda \), and which is distant by a quadrant from \( \mathbf{C} \): a result which may be confirmed by elementary considerations, because (by a well-known theorem respecting transversal arcs) the common bisector \( \mathbf{BA} \) of the two sides, \( \mathbf{DE} \) and \( \mathbf{EP} \), must meet the third side in a point \( \mathbf{L} \), for which
\[ \sin DL = \sin EL. \]
Art. 297.] CONSTRUCTION OF A FOURTH PROPORTIONAL.

(29.) To prove by quaternions this last equality of sines, and to assign their common value, we have only to observe that by XXXIII.,

\[ \text{XXXVI.} \quad V\delta \lambda = V\epsilon \lambda = \frac{1}{2} V\delta \epsilon; \]

in which, \( T\delta \lambda = T\epsilon \lambda = r^2 \sin \epsilon \), and \( TV\delta \epsilon = r^2 \sin 2\epsilon \);

the sines in question are therefore (by 204, XIX.),

\[ \text{XXXVI'} \quad TVU\delta \lambda = TVU\epsilon \lambda = \frac{1}{2} r^2 \sin 2\epsilon; \]

\[ r^2 \sin \epsilon = \cos \epsilon. \]

(30.) On similar principles, we may interpret the two vector-equations,

\[ \text{XXXVII} \quad V/3X = lYfia, \quad VaX = mVb\alpha, \]

in which \( \text{XXXVIII} \quad TA: TV\beta \alpha = r \sin \epsilon; \quad \sin \epsilon = \tan \epsilon; \quad \tan \epsilon, \)

an equivalent to the trigonometric equations,

\[ \text{XXXIX} \quad \tan \frac{cd}{ab} = \frac{\cos bc \cos ac}{\sin bl}; \quad \frac{\cos ac}{\sin al}. \]

(31.) Accordingly, if we let fall the perpendicular \( cq \) on \( ab \) (see again fig. 68), so that \( q \) bisects \( rs \), and if we determine two new points \( m, n \) by the areal equations,

\[ \text{XL} \quad rLM = aAB = nQR, \quad nLN = nCD, \]

the arcs \( mr, nd \) will be quadrants; and because the angle at \( r \) is right by construction (18.), \( m \) is the pole of \( dr \), and \( dm \) is a quadrant; whence \( d \) is the pole of \( mn \) and the angle \( lnm \) is right: conceiving then that the arcs \( ca \) and \( cb \) are drawn, we have three triangles \( [bcq, acq, and lmn] \), right-angled at \( q \) and \( n \), which show, by elementary principles, that the three trigonometric quotients in XXXIX. have in fact a common value, namely \( \cos q \), or \( \cos l \).

(32.) To prove this last result by quaternions, and without employing the auxiliary points \( m, n, q, r \), we have the transformations,

\[ \text{XLI} \quad \cos l = SU \quad \frac{\sqrt{\beta a}}{\sqrt{\delta \epsilon}} = \frac{\sqrt{\beta a}}{\gamma \lambda} = T \frac{\lambda}{\sqrt{\beta a} \gamma \lambda} = T \frac{\lambda}{\sqrt{\beta a} \gamma \lambda}; \]

because

\[ \text{XLII} \quad \delta = \nu \gamma - \lambda, \quad \epsilon = \nu \gamma + \lambda, \quad V\delta \epsilon = 2n\gamma \lambda, \quad UV\epsilon = U\gamma \lambda, \]

and

\[ \text{XLIII} \quad S \frac{\beta a}{\gamma \lambda} = S \frac{\beta a \gamma \lambda}{(\gamma \lambda)^2} = - S \beta a \gamma \lambda^{-1} = - S \beta a \gamma \lambda^{-1} = 1; \]

it being remembered that \( \lambda \perp \gamma \), whence

\[ V\gamma \lambda = \gamma \lambda = - \lambda \gamma, \quad (\gamma \lambda)^2 = - \gamma^2 \lambda^2 = \lambda^2, \quad S \gamma \lambda^{-1} = 0. \]
(33.) At the same time we see that if \( p \) be (as before) the positive pole of \( ba \), and if \( k, k' \) be the negative and positive poles of \( de \), while \( l' \) is the negative (as \( l \) is the positive) pole of \( cq \), whereby all the letters in fig. 68 have their significations determined, we may write,

\[
\text{XLIV} \ldots \text{op} = UV\beta a; \quad \text{ok'} = \gamma U\lambda; \quad \text{ok} = -\gamma U\lambda; \quad \text{ol'} = -U\lambda;
\]

while

\[
\text{ol} = +U\lambda, \text{ as before.}
\]

(34.) Writing also,

\[
\text{XLV} \ldots k = -\gamma \lambda, \quad \text{or} \quad \lambda = \gamma k, \quad \text{and} \quad \mu = \beta a^{-1} \lambda,
\]

so that

\[
\text{XLV'} \ldots \text{ok} = U\kappa, \quad \text{and} \quad \text{om} = U\mu,
\]

we have

\[
\text{XLVI} \ldots \beta a^{-1} \cdot \gamma = \mu \lambda^{-1} \cdot \lambda k^{-1} = \mu k^{-1};
\]

this fourth proportional, to the three equally long but diplanar vectors, \( a, \beta, \gamma \), is therefore a versor, of which the representative arc (162) is \( km \), and the representative angle (174) is \( kdm \), or \( l'dr \), or \( edp \); and we may write for this versor, or quaternion, the expression:

\[
\text{XLVII} \ldots \beta a^{-1} \gamma = \cos l'dr + od \cdot \sin l'dr.
\]

(35.) The double of this representative angle is the sum of the two base-angles of the isosceles triangle \( dpe \); and because the two other triangles, \( epf' \), \( f'pd \), are also isosceles (19.), the lune \( ff' \) shows that this sum is what remains, when we subtract the vertical angle \( f \), of the triangle \( def \), from the sum of the supplements of the two base-angles \( d \) and \( e \) of that triangle; or when we subtract the sum of the three angles of the same triangle from four right angles. We have therefore this very simple expression for the Angle of the Fourth Proportional:

\[
\text{XLVIII} \ldots \angle \beta a^{-1} \gamma = l'dr = \pi - \frac{1}{2} (d + e + f).
\]

(36.) Or, if we introduce the area, or the spherical excess, say \( \Sigma \), of the triangle \( def \), writing thus

\[
\text{XLIX} \ldots \Sigma = d + e + f - \pi,
\]

we have these other expressions:

\[
\text{L} \ldots \angle \beta a^{-1} \gamma = \frac{1}{2} \pi - \frac{1}{2} \Sigma; \quad \text{II} \ldots \beta a^{-1} \gamma = \sin \frac{1}{2} \Sigma + r^{-1} \delta \cos \frac{1}{2} \Sigma;
\]

because

\[
\text{od} = U\delta = r^{-1} \delta, \quad \text{by XII.}
\]

* [Since \( \beta a^{-1} \gamma \cdot \gamma \cdot (\beta a^{-1} \gamma)^{-1} = \beta a^{-1} \gamma \cdot (\beta a^{-1})^{-1} = \gamma' \) suppose, \( c \) is brought to a point \( c' \) by a conical rotation round \( od \) or round \( or' \) where \( r' \) is the opposite of \( r \) (XXIX.). Hence \( c \) and \( c' \) are the points of intersection of small circles whose poles are \( d \) and \( r' \), and \( c' \) is the reflexion of \( c \) to the great circle \( rd \). This shows that the angle of the quaternion \( \beta a^{-1} \gamma \) is \( 
\]
(37.) Having thus expressed $\beta a^{-1}\gamma$, we require no new appeal to the figure, in order to express this other fourth proportional, $\gamma a^{-1}\beta$, which is the negative of its conjugate, or has an opposite scalar, but an equal vector part (comp. 204, (1.), and 295, (9.)): the geometrical difference being merely this, that because the rotation round $a$ from $\beta$ to $\gamma$ has been supposed to be negative, the rotation round $a$ from $\gamma$ to $\beta$ must be, on the contrary, positive.

(38.) We may thus write, at once,

$$LII. \ldots \gamma a^{-1}\beta = -K\beta a^{-1}\gamma = -\sin \frac{1}{2}\Sigma + r^{-1}\delta \cos \frac{1}{2}\Sigma;$$

and we have, for the angle of this new fourth proportional, to the same three vectors $a, \beta, \gamma$, of which the second and third have merely changed places with each other, the formula:

$$LIII. \ldots \angle \gamma a^{-1}\beta = r\delta = \frac{1}{2}(D + E + F) = \frac{1}{2}\pi + \frac{1}{2}\Sigma.$$

(39.) But the common vector part of these two fourth proportionals is $\delta$, by VII.; we have therefore, by XI.,

$$LIV. \ldots r = \cos \frac{1}{2}\Sigma; \quad e = \pm \sin \frac{1}{2}\Sigma;$$

the upper sign being taken, when the rotation round $a$ from $\beta$ to $\gamma$ is negative, as above supposed.

(40.) It follows by (6.) that when the sides $2a', 2b', 2c'$, of a spherical triangle $DEF$, of which the area is $\Sigma$, are bisected by the corners $A, B, C$ of another spherical triangle, of which the sides* are $a, b, c$, then

$$LV. \ldots \cos a : \cos a' = \cos b : \cos b' = \cos c : \cos c' = \cos \frac{1}{2}\Sigma.$$

(41.) It follows also, from what has been recently shown, that the angle $RDK$, or $MDN$, or the arc $MN$ in fig. 68, represents the semi-area of the bisected triangle $DEF$; whence, by the right-angled triangle $LMN$, we can infer that the sine of this semi-area is equal to the sine of a side of the bisecting triangle $ABC$, multiplied into the sine of the perpendicular, let fall upon that side from the opposite corner of the latter triangle; because we have

$$LVI. \ldots \sin \frac{1}{2}\Sigma = \sin MN = \sin LM \cdot \sin L = \sin AB \cdot \sin CQ.$$

(42.) The same conclusion can be drawn immediately, by quaternions, from the expression,

$$LVII. \ldots \sin \frac{1}{2}\Sigma = e = Sa\beta\gamma = S(V\beta a \cdot \gamma^{-1}) = TV\beta a \cdot SU(V\beta a : \gamma);$$

in which one factor is the sine of $AB$, and the other factor is the cosine of $CP$, or the sine of $CQ$.

* These sides $abc$, of the bisecting triangle $ABC$, have been hitherto supposed for simplicity (1.) to be each less than a quadrant, but it will be found that the formula LV. holds good, without any such restriction.
(43.) Under the same conditions, since

\[ a = U(e + \xi) = \frac{1}{2} e^{-1}(e + \xi), \quad \&c., \]

we may write also,

\[ \sin \frac{1}{2} \Sigma = SU(e + \xi)(e + \xi) = S\xi \xi; \]

in which, by IV. and XII.,

\[ \cos \frac{1}{2} \Sigma = r = (r^2 - rS(e\xi + \xi\xi + \xi\xi)) : 4lnn; \]

(44.) Hence also, by LIV.,

\[ \tan \frac{1}{2} \Sigma = \frac{e}{r} = \frac{S\xi \xi}{r^2 - rS(e\xi + \xi\xi + \xi\xi)} = \frac{SU\xi \xi}{1 - SUe\xi - SU\xi \xi - SU\xi}; \]

and under this last form, we have a general expression for the tangent of half the spherical opening at \( o \), of any triangular pyramid \( o\text{def} \), whatever the lengths \( T\xi, T\xi, T\xi \) of the edges at \( o \) may be.

(45.) As a verification, we have

\[ (4lnn)^2 = -\frac{1}{2} (e + \xi)^2 (\xi + \xi)^2 (\xi + \xi)^2 = 2 (r^2 - S\xi \xi) (r^2 - S\xi \xi) (r^2 - S\xi \xi); \]

but the elimination of \( \frac{1}{2} \Sigma \) between LIX. LXI. gives

\[ (4lnn)^2 = (S\xi \xi)^2 + (r^2 - r(S\xi \xi + S\xi \xi + S\xi \xi))^2; \]

we ought then to find that

\[ (S\xi \xi)^2 = r^2 - r(S\xi \xi + S\xi \xi + S\xi \xi) - 2S\xi \xi S\xi \xi S\xi \xi, \]

if \( \xi^2 = \xi^2 = -r^2 \); and in fact this equality results immediately from the general formula 294, LIII.

(46.) Under the same condition, respecting the equal lengths of \( \xi, \xi, \xi \), we have also the formula,

\[ -V(\xi + \xi)(\xi + \xi) = 2 \xi (r^2 - S\xi \xi - S\xi \xi - S\xi \xi) = 8lnn\xi; \]

whence other verifications may be derived.

(47.) If \( \sigma \) denote the area* of the bisecting triangle \( \Delta ABC \), the general principle LXII. enables us to infer that

\[ \tan \frac{\sigma}{2} = \frac{S\alpha \beta \gamma}{1 - S\beta \gamma - S\gamma \alpha - S\alpha \beta} = \frac{e}{1 + b + m + n} \]

\[ \sin c \sin \frac{\rho}{2} = \frac{1 + \cos \alpha + \cos \beta + \cos \gamma}{1 + \cos \alpha + \cos \beta + \cos \gamma}; \]

* The reader will observe that the more usual symbol \( \Sigma \), for this area of \( ABC \), is here employed 6.1 to denote the area of the inscribed triangle \( DEF \).
if $p$ denote the perpendicular $cq$ from $c$ on $AB$, so that

$$e = \sin c \sin p = \sin b \sin c \sin \lambda = &c. \text{ (comp. 210, (21.)).}$$

(48.) But, by (IX.) and (XI.),

LXVIII. \[e^2 + (1 + l + m + n)^2 = 2 (1 + l) (1 + m) (1 + n)\]

$$= \left(4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}\right)^2;$$

hence the cosine and sine of the new semi-area are,

LXIX. \[\cos \frac{\sigma}{2} = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}};\]

LXX. \[\sin \frac{\sigma}{2} = \frac{\sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}}{\cos \frac{c}{2}} = &c.\]

(49.) Returning to the bisected triangle, DEF, the last formula gives,

LXXI. \[\sin \frac{\Sigma}{2} = \frac{\sin a' \sin b' \sin \Gamma}{\cos \frac{c'}{2}} = \sin p' \sin c \sec c',\]

if $p'$ denote the perpendicular from $\Gamma$ on the bisecting arc $AB$, or $FT$ in fig. 68; but $\cos \frac{\Sigma}{2} = \cos c sec c'$, by LV.; hence

LXXII. \[\tan \frac{\Sigma}{2} = \sin p' \tan c = \sin FT \tan AB.\]

Accordingly, in fig. 68, we have, by spherical trigonometry,

$$\sin FT = \sin ES = \sin LE \sin L = \cos LN \sin MN \cosec LM = \tan MN \cot AB.\]

(50.) The arc $MN$, which thus represents in quantity the semi-area of $DEF$, has its pole at the point $D$, and may be considered as the representative arc (162) of a certain new quaternion $Q$, or of its versor, of which the axis is the radius on, or $UD$; and this new quaternion may be thus expressed:

LXXIII. \[Q = \delta \gamma a \beta = - \delta^2 + \delta S a \beta \gamma = r^2 + c \delta;\]

its tensor and versor being, respectively,

LXXIV. \[TQ = r = \cos \frac{\Sigma}{2}; \quad LXXV. \quad UQ = \cos \frac{\Sigma}{2} + OD \cdot \sin \frac{\Sigma}{2}.\]

(51.) An important transformation of this last versor may be obtained as follows:

LXXVI. \[UQ = U (\delta \gamma^{-1} \cdot a \zeta^{-1} \cdot \zeta \beta^{-1}) = (\delta \epsilon^{-1}) \delta (\zeta \epsilon^{-1}) \delta (\zeta \delta^{-1}) \delta;\]

so that

LXXVII. \[\frac{\Sigma}{2} = \angle Q = \angle \delta \gamma a \beta = \angle (\delta \epsilon^{-1}) \delta (\zeta \epsilon^{-1}) \delta (\zeta \delta^{-1}) \delta;\]
these powers of quaternions, with exponents each = \( \frac{1}{2} \), being interpreted as square roots (199, (1.)), or as equivalent to the symbols \( \sqrt{S} \), &c.

(52.) The conjugate (or reciprocal) versor, \( UQ^{-1} \), which has \( N \) for its representative arc, may be deduced from \( UQ \) by simply interchanging \( \beta \) and \( \gamma \), or \( \varepsilon \) and \( \zeta \); the corresponding quaternion is,

\[
LXXVIII. \quad Q' = KQ = \delta \beta a \gamma = r^2 - e \delta;
\]

and we have

\[
LXXIX. \quad UQ' = \cos \frac{1}{2} \Sigma - od \cdot \sin \frac{1}{2} \Sigma = (\delta \zeta^{-1})^\frac{1}{2} (\zeta \varepsilon^{-1})^\frac{1}{2} (\varepsilon \delta^{-1})^\frac{1}{2};
\]

the rotation round \( d \), from \( e \) to \( r \), being still supposed to be negative.

(53.) Let \( h \) be any other point upon the sphere, and let \( oh = \eta \); also let \( \Sigma' \) be the area of the new spherical triangle, \( \Delta FH \); then the same reasoning shows that

\[
LXXX. \quad \cos \frac{1}{2} \Sigma' + od \cdot \sin \frac{1}{2} \Sigma' = (\delta \zeta^{-1})^\frac{1}{2} (\zeta \eta^{-1})^\frac{1}{2} (\eta \delta^{-1})^\frac{1}{2},
\]

if the rotation round \( d \) from \( r \) to \( h \) be negative; and therefore, by multiplication of the two co-axal versors, \( LXXVI. \) and \( LXXX. \), we have by \( LXXV. \), the analogous formula:

\[
LXXXI. \quad \cos \frac{1}{2} (\Sigma + \Sigma') + od \cdot \sin \frac{1}{2} (\Sigma + \Sigma') = (\delta \varepsilon^{-1})^\frac{1}{2} (\varepsilon \zeta^{-1})^\frac{1}{2} (\zeta \eta^{-1})^\frac{1}{2} (\eta \delta^{-1})^\frac{1}{2};
\]

where \( \Sigma + \Sigma' \) denotes the area of the spherical quadrilateral, \( \Delta FH \).

(54.) It is easy to extend this result to the area of any spherical polygon, or to the spherical opening (44.) of any pyramid; and we may even conceive an extension of it, as a limit, to the area of any closed curve upon the sphere, considered as decomposed into an indefinitely number of indefinitely small triangles, with some common vertex, such as the point \( d \), on the spherical surface, and with indefinitely small arcs \( EF, FH, \ldots \) of the curve, for their respective bases: or to the spherical opening of any cone, expressed thus as the Angle of a Quaternion, which is the limit* of the product of indefinitely many factors, each equal to the square-root of a quaternion, which differs indefinitely little from unity.

* This Limit is closely analogous to a definite integral, of the ordinary kind; or rather, we may say that it is a Definite Integral, but one of a new kind, which could not easily have been introduced without Quaternions. In fact, if we did not employ the non-commutative property (168) of quaternion multiplication, the products here considered would evidently become each equal to unity: so that they would furnish no expressions for spherical or other areas, and in short, it would be useless to speak of them. On the contrary, when that property or principle of multiplication is introduced, these expressions of product-form are found, as above, to have extremely useful significations in spherical geometry; and it will be seen that they suggest and embody a remarkable theorem, respecting the resultant of rotations of a system, round any number of successive axes, all passing through one fixed point, but in other respects succeeding each other with any gradual or sudden changes.
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(55.) To assist the recollection of this result, it may be stated as follows (comp. 180, (3.) for the definition of an arcual sum):—

"The Arcual Sum of the Halves of the Successive Sides of any Spherical Polygon, is equal to an arc of a Great Circle, which has the Initial (or Final) Point of the Polygon for its Pole, and represents the Semi-area of the Figure"; it being understood that this resultant arc is reversed in direction, when the half-sides are (arcually) added in an opposite order.

(56.) As regards the order thus referred to, it may be observed that in the arcual addition, which corresponds to the quaternion multiplication in LXXVI., we conceive a point to move, first, from \( b \) to \( f \), through half the arc \( df \); which half-side of the triangle \( def \) answers to the right-hand factor, or square-root \((\mathbb{S}^{-1})^3\). We then conceive the same point to move next from \( f \) to \( a \), through half the arc \( fe \), which answers to the factor placed immediately to the left of the former; having thus moved, on the whole, so far, through the resultant arc \( ba \) (as a transvector, 180, (3.)), or through any equal arc (163), such as \( ml \) in fig. 68. And finally, we conceive a motion through half the arc \( ed \), or through any arc equal to that half, such as the arc \( ln \) in the same figure, to correspond to the extreme left-hand factor in the formula; the final resultant (or total transvector arc), which answers to the product of the three square roots, as arranged in the formula, being thus represented by the final arc \( mn \), which has the point \( d \) for its positive pole, and the half-area, \( \frac{1}{2} \mathbb{S} \), for the angle (51.) of the quaternion (or versor) product which it represents.

(57.) Now the direction of positive rotation on the sphere has been supposed to be that round \( d \), from \( f \) to \( e \); and therefore along the perimeter, in the order \( dfe \), as seen* from any point of the surface within the triangle: that is, in the order in which the successive sides \( df, fe, ed \) have been taken, before adding (or compounding) their halves. And accordingly, in the conjugate (or reciprocal) formula LXXIX., we took the opposite order, \( def \), in proceeding as usual from right-hand to left-hand factors, whereof the former are supposed to be multiplied by† the latter; while the result was, as we saw in (52.), a new

* In this and other cases of the sort, the spectator is imagined to stand on the point of the sphere, round which the rotation on the surface is conceived to be performed; his body being outside the sphere. And similarly when we say, for example, that the rotation round the line, or radius, \( oa \), from the line on to the line \( oc \), is negative (or left-handed), as in the recent figures, we mean that such would appear to be the direction of that rotation, to a person standing thus with his feet on \( a \), and with his body in the direction of \( oa \) prolonged: or else standing on the centre (or origin) \( o \), with his head at the point \( a \). Compare 174, II.; 177; and the second note to page 152.
† Compare the Notes to pages 147, 159.

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versor, in the expression for which, the area $\Sigma$ of the triangle was simply changed to its own negative.

(58.) To give an example of the reduction of the area to zero, we have only to conceive that the three points $d$, $e$, $f$ are co-arucal (165), or situated on one great circle; or that the three lines $\delta$, $\varepsilon$, $\zeta$ are complanar. For this case, by the laws* of complanar quaternions, we have the formula,

$$LXXXII. \ldots (\delta e^{-1})^{\frac{1}{2}} (\varepsilon \zeta^{-1})^{\frac{1}{2}} (\zeta \delta^{-1})^{\frac{1}{2}} = 1,$$

thus $\cos \frac{1}{2} \Sigma = 1$, and $\Sigma = 0$.

(59.) Again, in (53.) let the point $h$ be co-arucal with $d$ and $f$, or let $SS\varepsilon \eta = 0$; then, because

$$LXXXII'. \ldots (\delta n^{-1})^{\frac{1}{2}} (\eta \varepsilon^{-1})^{\frac{1}{2}} = (\zeta \delta^{-1})^{\frac{1}{2}},$$

the product of four factors $LXXXI.$ reduces itself to the product of three factors $LXXVI.$; the geometrical reason being evidently that in this case the added area $\Sigma'$ vanishes; so that the quadrilateral $DEFH$ has only the same area as the triangle $DEF$.

(60.) But this added area (53.) may even have a negative† effect, as for example when the new point $h$ falls on the old side $de$. Accordingly, if we write

$$LXXXIII. \ldots Q_1 = (\varepsilon \zeta^{-1})^{\frac{1}{2}} (\varepsilon \eta^{-1})^{\frac{1}{2}} (\eta \varepsilon^{-1})^{\frac{1}{2}},$$

and denote the product $LXXXI.$ of four square-roots by $Q_3$, we shall have the transformation,

$$LXXXIV. \ldots Q_2 = (\delta e^{-1})^{\frac{1}{2}} Q_1 (\varepsilon \delta^{-1})^{\frac{1}{2}},$$

which shows (comp. (15.)) that in this case the angle of the quaternary product $Q_2$ is that of the ternary product $Q_1$, or the half-area of the triangle $DEF$ (= $DEF - DHF$), although the axis of $Q_2$ is transferred from the position of the axis of $Q_1$, by a rotation round the pole of the arc $ed$, which brings it from $oe$ to $od$.

(61.) From this example, it may be considered to be sufficiently evident, how the formula $LXXXI.$ may be applied and extended, so as to represent (comp. (54.)) the area of any closed figure on the sphere, with any assumed point

* Compare the Second Chapter of the Second Book.
† In some investigations respecting areas on a sphere, it may be convenient to distinguish (comp. (28.), (63.)) between the two symbols $DEF$ and $DFE$, and to consider them as denoting two opposite triangles, of which the sum is zero. But for the present, we are content to express this distinction, by means of the two conjugate quaternion products (61.) and (52.).
Art. 297.]

**AREA OF POLYGON OR CURVE ON SPHERE.**

D on the surface as a sort of spherical origin; even when this auxiliary point is not situated on the perimeter, but is either external or internal thereto.

(62.) A new quaternion \( Q_0 \), with the same axis \( \od \) as the quaternion \( Q \) of (50.), but with a double angle, and with a tensor equal to unity, may be formed by simply squaring the versor \( UQ \); and although this squaring cannot be effected by removing the fractional exponents,* in the formula LXXXVI., yet it can easily be accomplished in other ways. For example we have, by LXXXIII. LXXXIV., and by VII. IX. X., the transformations†:

\[
\text{LXXXV... } Q_0 = UQ^2 = r^2 (\delta \gamma \alpha \beta)^2 = - \delta^2 \cdot \gamma \alpha \beta \delta \cdot \delta \gamma \alpha \beta \\
= - (\gamma \alpha \beta)^2 = - (e - \delta)^2 = r^2 - e^2 + 2e\delta;
\]

and in fact, because \( \delta = r \cdot \od \), by XII., the trigonometric values LIV. for \( r \) and \( e \) enable us to write this last result under the form,

\[
\text{LXXXVI... } Q_0 = - (\gamma \alpha \beta)^2 = \cos \Sigma + \od \cdot \sin \Sigma.
\]

(63.) To show its geometrical signification, let us conceive that ABC and LMN have the same meanings in the new fig. 70, as in fig. 68; and that \( A_1B_1M_1 \) are three new points, determined by the three arcual equations (163),

\[
\text{LXXXVII. } \od AC = \od CA_1, \od BC = \od CB_1, \od MN = \od NM_1;
\]

which easily conduct to this fourth equation of the same kind,

\[
\text{LXXXVII'. } \od LM_1 = \od B_1A_1.
\]

This new arc \( LM_1 \) represents thus (comp. 167, and fig. 43) the product \( a \gamma^{1}, \gamma \beta^{1} = \gamma \alpha^{-1} \cdot \beta \gamma^{1} \); while the old arc \( ML \), or its equal \( BA \) (31.), represents \( a \beta^{-1} \); whence the arc \( MM_1 \), which has its pole at \( D \), and is numerically equal to the whole area \( \Sigma \) of \( DEF \) (because \( MN \) was seen to be equal (50.) to half that area), represents the product \( \gamma \alpha^{-1} \beta \gamma^{1} \cdot a \beta^{-1} \), or \(- (\gamma \alpha \beta)^2 \), or \( Q_0 \). The formula LXXXVI. has therefore been interpreted, and may be said to have been proved anew, by these simple geometrical considerations.

(64.) We see, at the same time, how to interpret the symbol,

\[
\text{LXXXVIII... } Q_0 = \frac{\gamma \beta}{\alpha \gamma \beta};
\]

namely as denoting a versor, of which the axis is directed to, or from, the

---

* Compare the Note to (54.).
† The equation \( \delta \gamma \alpha \beta = \gamma \alpha \beta \delta \) is not valid generally; but we have here \( \delta = - V \gamma \alpha \beta \); and in general, \( q = pq \), if \( p \parallel Vq \).
corner \( \mathbf{d} \) of a certain auxiliary spherical triangle \( \mathbf{DEF} \), whereof the sides respectively opposite to \( \mathbf{d} \), \( \mathbf{e} \), \( \mathbf{f} \), are bisected (5.) by the given points \( \mathbf{A} \), \( \mathbf{B} \), \( \mathbf{C} \), according as the rotation round \( \mathbf{A} \) from \( \mathbf{B} \) to \( \mathbf{C} \) is negative or positive; and of which the angle represents, or is numerically equal to, the area \( \Sigma \) of that auxiliary triangle, at least if we still suppose, as we have hitherto for simplicity done (1.), that the sides of the given triangle \( \mathbf{ABC} \) are each less than a quadrant.

298. The case when the sides of the given triangle are all greater, instead of being all less, than quadrants, may deserve next to be (although more briefly) considered; the case when they are all equal to quadrants, being reserved for a short subsequent Article: and other cases being easily referred to these, by limits, or by passing from a given line to its opposite.

(1.) Supposing now that

\[
\begin{align*}
\text{I.} & \quad l < 0, \quad m < 0, \quad n < 0, \\
\text{II.} & \quad a > \frac{\pi}{2}, \quad b > \frac{\pi}{2}, \quad c > \frac{\pi}{2} 
\end{align*}
\]

we may still retain the recent equations IV. to XI.; XIII.; and XV. to XXVI., of 297; but we must change the sign of the radical, \( r \), in the equations XII. and XIV., and also the signs of the versors, \( \mathbf{U}_\delta \), \( \mathbf{U}_\varepsilon \), \( \mathbf{U}_\zeta \) in XII., if we desire that the sides of the auxiliary triangle, \( \mathbf{DEF} \), may still be bisected (as in figures 67, 68) by the corners of the given triangle, \( \mathbf{ABC} \), of which the sides \( a, b, c \) are now each greater than a quadrant. Thus, \( r \) being still the common tensor of \( \delta, \varepsilon, \zeta \), and therefore being still supposed to be itself > 0, we must write now, under these new conditions I. or II., the new equations,

\[
\begin{align*}
\text{III.} & \quad \mathbf{o} = -\mathbf{U}_\delta = -r^{-1}\delta; \quad \mathbf{o} = -\mathbf{U}_\varepsilon = -r^{-1}\varepsilon; \quad \mathbf{o} = -\mathbf{U}_\zeta = -r^{-1}\zeta; \\
\text{IV.} & \quad \cos a = -r \cos a', \quad \cos b = -r \cos b', \quad \cos c = -r \cos c'.
\end{align*}
\]

(2.) The equations IV. and VIII. of 297 still holding good, we may now write,

\[
\begin{align*}
\text{V.} & \quad \pm 2r \cos a' \cos b' \cos c' = \cos a'^2 + \cos b'^2 + \cos c'^2 - 1,
\end{align*}
\]

according as we adopt positive values (297), or negative values (298), for the cosines \( l, m, n \) of the sides of the bisecting triangle; the value of \( r \) being still supposed to be positive.

(3.) It is not difficult to prove (comp. 297, LIV., LXIX.), that

\[
\begin{align*}
\text{VI.} & \quad r = \pm \cos \frac{1}{2}\Sigma, \quad \text{according as} \quad l > 0, \quad &c., \quad \text{or} \quad l < 0, \quad &c.;
\end{align*}
\]

the recent formula V. may therefore be written unambiguously as follows:

\[
\begin{align*}
\text{VII.} & \quad 2 \cos a' \cos b' \cos c' \cos \frac{1}{2}\Sigma = \cos a'^2 + \cos b'^2 + \cos c'^2 - 1;
\end{align*}
\]

and the formula 297, L.V. continues to hold good.
(4.) In like manner, we may write, without an ambiguous sign (comp. 297, L. I.), the following expression for the fourth proportional $\beta a^{-1} \gamma$ to three unit-vectors $a$, $\beta$, $\gamma$, the rotation round the first from the second to the third being negative:

\[ \text{VIII. } \beta a^{-1} \gamma = \sin \frac{1}{2} \Sigma + od \cdot \cos \frac{1}{2} \Sigma; \]

where the scalar part changes sign, when the rotation is reversed.

(5.) It is, however, to be observed, that although this formula VIII holds good, not only in the cases of the last article and of the present, but also in that which has been reserved for the next, namely when $l = 0$, &c.; yet because, in the present case (298) we have the area $\Sigma > \pi$, the radius $od$ is no longer the (positive) axis $U \delta$ of the fourth proportional $\beta a^{-1} \gamma$; nor is $\frac{1}{2} \pi - \frac{1}{2} \Sigma$ any longer, as in 297, L., the (positive) angle of that versor. On the contrary we have now, for this axis and angle, the expressions:

\[ \text{IX. } A_0 \cdot \beta a^{-1} \gamma = do = - od; \quad \text{X. } \angle \beta a^{-1} \gamma = \frac{1}{2} (\Sigma - \pi). \]

(6.) To illustrate these results by a construction, we may remark that if, in fig. 67, the bisecting arcs $bc$, $ca$, $ab$ be supposed each greater than a quadrant, and if we proceed to form from it a new figure, analogous to 68, the perpendicular $cq$ will also exceed a quadrant, and the poles $p$ and $k$ will fall between the points $c$ and $q$; also $m$ and $r$ will fall on the arcs $lq$ and $ql'$ prolonged: and although the arc $km$, or the angle $kdm$, or $l'dr$, or $edp$, may still be considered, as in 297, (34.), to represent the versor $\beta a^{-1} \gamma$, yet the corresponding rotation round the point $d$ is now of a negative character.

(7.) And as regards the quantity of this rotation, or the magnitude of the angle at $d$, it is again, as in fig. 68, a base-angle of one of three isosceles triangles, with $p$ for their common vertex; but we have now, as in fig. 71, a new arrangement, in virtue of which this angle is to be found by halving what remains, when the sum of the supplements of the angles at $d$ and $e$, in the triangle $def$, is subtracted from the angle at $f$, instead of our subtracting (as in 297, (35.)) the latter angle from the former sum; it is therefore now, in agreement with the recent expression $X.$,

\[ \text{XI. } \angle \beta a^{-1} \gamma = \frac{1}{2} (d + e + f) - \pi. \]

(8.) The negative of the conjugate of the formula VIII. gives,

\[ \text{XII. } \gamma a^{-1} \beta = - \sin \frac{1}{2} \Sigma + od \cdot \cos \frac{1}{2} \Sigma; \]
and by taking the negative of the square of this equation, we are conducted to the following:

$$\frac{\gamma}{a} \frac{\beta}{\gamma} \frac{a}{\beta} = - (\gamma a^{-1} \beta)^2 = \cos \Sigma + \od \sin \Sigma;$$

a result which had only been proved before (comp. 297, (62.), (64.) ) for the case \( \Sigma < \pi \); and in which it is still supposed that the rotation round \( a \) from \( \beta \) to \( \gamma \) is negative.

(9.) With the same direction of rotation, we have also the conjugate or reciprocal formula,

$$\frac{\beta}{a} \frac{\gamma}{\beta} \frac{a}{\gamma} = - (\beta a^{-1} \gamma)^2 = \cos \Sigma - \od \sin \Sigma.$$

(10.) If it happened that only one side, as \( AB \), of the given triangle \( ABC \), was greater, while each of the two others was less than a quadrant, or that we had \( l > 0, m > 0, \) but \( n < 0 \); and if we wished to represent the fourth proportional to \( a, \beta, \gamma \) by means of the foregoing constructions: we should only have to introduce the point \( c' \) opposite to \( c \), or to change \( \gamma \) to \( \gamma' = - \gamma \); for thus the new triangle \( ABC' \) would have each side greater than a quadrant, and so would fall under the case of the present Article; after employing the construction for which, we should only have to change the resulting versor to its negative.

(11.) And in like manner, if we had \( l \) and \( m \) negative, but \( n \) positive, we might again substitute for \( c \) its opposite point \( c' \), and so fall back on the construction of Art. 297: and similarly in other cases.

(12.) In general, if we begin with the equations 297, XII., attributing any arbitrary (but positive) value to the common tensor, \( r \), of the three co-initial vectors \( \delta, \varepsilon, \zeta \), of which the versors, or the unit-vectors \( U\delta, \&c. \), terminate at the corners of a given or assumed triangle \( DEF \), with sides \( = 2a', 2b', 2c' \), we may then suppose (comp. fig. 67) that another triangle \( ABC \), with sides denoted by \( a, b, c \), and with their cosines denoted by \( l, m, n \), is derived from this one, by the condition of bisecting its sides; and therefore by the equations (comp. 297, LVIII.),

$$\frac{a}{a} = U(\varepsilon + \zeta), \quad \frac{\beta}{\beta} = U(\zeta + \delta), \quad \frac{\gamma}{\gamma} = U(\delta + \varepsilon),$$

with the relations 297, IV. V. VI., as before; or by these other equations (comp. 297, XIII. XIV.),

$$\frac{\varepsilon + \zeta}{2ra \cos a'}, \quad \frac{\zeta + \delta}{2r\beta \cos b'}, \quad \frac{\delta + \varepsilon}{2r\gamma \cos c'}.$$
(13.) When this simple construction is adopted, we have at once (comp. 297, LX.), by merely taking scalars of products of vectors, and without any reference to areas (compare however 297, LXIX., and 298, VII.), the equations,

\[
\begin{align*}
\text{XVII.} \quad & 4 \cos a \cos b' \cos c' = 4 \cos b \cos c' \cos a' = 4 \cos c \cos a' \cos b' \\
& = - r^2 S (\zeta + \delta) (\delta + \varepsilon) = \&c. = 1 + \cos 2a' + \cos 2b' + \cos 2c' ;
\end{align*}
\]

or

\[
\begin{align*}
\text{XVIII.} \quad & \frac{\cos a}{\cos a'} = \frac{\cos b}{\cos b'} = \frac{\cos c}{\cos c'} = \frac{\cos a'^2 + \cos b'^2 + \cos c'^2 - 1}{2 \cos a' \cos b' \cos c'} ;
\end{align*}
\]

which can indeed be otherwise deduced, by the known formulæ of spherical trigonometry.

(14.) We see, then, that according as the sum of the squares of the cosines of the half-sides, of a given or assumed spherical triangle, DEF, is greater than unity, or equal to unity, or less than unity, the sides of the inscribed and bisecting triangle, ABC, are together less than quadrants, or together equal to quadrants, or together greater than quadrants.

(15.) Conversely, if the sides of a given spherical triangle ABC be thus all less, or all greater than quadrants, a triangle DEF, but only one* such triangle, can be escribed to it, so as to have its sides bisected, as above: the simplest process being to let fall a perpendicular, such as cq in fig. 68, from c on AB, &c.; and then to draw new arcs, through C, &c., perpendicular to these perpendiculars, and therefore coinciding in position with the sought sides DE, &c., of DEF.

(16.) The trigonometrical results of recent sub-articles, especially as regards the area† of a spherical triangle, are probably all well known, as certainly some of them are; but they are here brought forward only in connexion with quaternion formulæ; and as one of that class, which is not irrelevant to the present subject, and includes the formula 294, LIII., the following may be mentioned, wherein a, β, γ denote any three vectors, but the order of the factors is important:

\[
\text{XIX.} \quad (a \beta \gamma)^2 = 2a^2 \beta^2 \gamma^2 + a^2 (\beta \gamma)^2 + \beta^2 (a \gamma)^2 + \gamma^2 (a \beta)^2 - 4a \gamma S a \beta S \beta \gamma .
\]

* In the next Article, we shall consider a case of indeterminateness, or of the existence of indefinitely many escribed triangles DEF: namely, when the sides of ABC are all equal to quadrants.

† This opportunity may be taken of referring to an interesting Note, to pages 96, 97 of Luby's Trigonometry (Dublin, 1882); in which an elegant construction, connected with the area of a spherical triangle, is acknowledged as having been mentioned to Dr. Luby, by a since deceased and lamented friend, the Rev. William Digby Sadleir, F.T.C.D. A construction nearly the same, described in the sub-articles to 297, was suggested to the present writer by quaternions, several years ago.

‡ [Using the relation \( V\alpha \beta S \gamma S a = a^2 \beta \gamma S \beta \gamma + a^2 \gamma \beta \alpha S \beta + a^2 \gamma a \alpha S \beta S \beta \gamma \), this easily follows on squaring \((V + S) a \beta \gamma\); or multiply XIX. by \( \beta^2 \) and put \( a \beta = r, \beta \gamma = p, \) and \( \beta^2 \alpha = np. \) ]
(17.) And if, as in 297, (1.), &c., we suppose that \(u, \beta, \gamma\) are three unit-vectors, \(oa, ob, oc\), and denote, as in 297, (47.), by \(\sigma\) the area of the triangle \(abc\), the principle expressed by the recent formula XIII. may be stated under this apparently different, but essentially equivalent form:

\[
\frac{a + \beta}{\beta + \gamma} \cdot \frac{\gamma + a}{a + \beta} \cdot \frac{\beta + \gamma}{\gamma + a} = \cos \sigma + a \sin \sigma;
\]

which admits of several verifications.

(18.) We may, for instance, transform it as follows (comp. 297, LXVII.):

\[
\begin{align*}
XXI. & \quad \frac{a + \beta}{\beta + \gamma} \cdot \frac{\gamma + a}{a + \beta} \cdot \frac{\beta + \gamma}{\gamma + a} = -2a + 2a(1 + l + m + n) \frac{1 + a \tan \frac{\sigma}{2}}{1 - a \tan \frac{\sigma}{2}} = \frac{\cos \frac{\sigma}{2} + a \sin \frac{\sigma}{2}}{\cos \frac{\sigma}{2} - a \sin \frac{\sigma}{2}} \\
& \quad = \left(\cos \frac{\sigma}{2} + a \sin \frac{\sigma}{2}\right)^2 = \cos \sigma + a \sin \sigma, \text{ as above.}^*
\end{align*}
\]

(19.) This seems to be a natural place for observing (comp. (16.),), that if \(a, \beta, \gamma, \delta\) be any four vectors, the lately cited equation 294, LIIL, and the square of the equation 294, XV., with \(\delta\) written in it instead of \(\rho\), conduct easily to the following very general and symmetric formula:

\[
XXII. \quad a^2\beta^2\gamma^2\delta^2 + (S\beta\gamma S\delta)^2 + (S\gamma a S\beta \delta)^2 + (S a S \beta S \gamma \delta)^2
\]

\[
+ 2a^2S\beta\gamma S\beta S\gamma S\delta + 2\gamma^2 S\beta a S\gamma S\delta + 2\gamma S a S \beta \delta S \gamma S a + 2\delta^2 S a S \beta S \gamma S \delta
\]

\[
= 2S\gamma a S \beta S \delta S \gamma \delta + 2S a S \beta S \gamma S \delta S a + 2S \beta S \gamma S a S \delta S \beta \delta
\]

\[
+ \beta^2 \gamma^2 (Sa \delta)^2 + \gamma^2 a^2 (S \beta \delta)^2 + a^2 \beta^2 (S \gamma \delta)^2
\]

\[
+ \alpha^2 \delta^2 (S \beta \gamma)^2 + \beta^2 \delta^2 (S \gamma a)^2 + \gamma^2 \delta^2 (S a \beta)^2.\]

* [Since \(U(\beta + \gamma)\) bisects the angle between \(\beta\) and \(\gamma\),

\[
\left(\begin{array}{c}
\beta \\
\gamma
\end{array}\right) \frac{1}{\gamma} = U \begin{array}{c}
\frac{\beta + \gamma}{\gamma} \\
\frac{\gamma + a}{a + \beta}
\end{array}; \text{ and therefore}
\]

\[
\left(\begin{array}{c}
\frac{\beta}{\gamma} \\
\frac{\gamma}{a + \beta}
\end{array}\right) = \frac{U(a + \beta)}{\beta} \cdot \frac{\beta}{\beta + \gamma} \cdot \frac{a + \beta}{a + \beta + a + \gamma + \beta}.
\]

This is a direct transformation from 297, LXXVI. to XX.]

† [This may perhaps be more rapidly derived by operating on \(aa + b\beta + c\gamma + d\delta = 0\) by \(S\alpha, S\beta, S\gamma, \text{ and } S\delta, \text{ and eliminating } a, b, c, \text{ and } d \text{ in the form of a determinant from the four results of operation.}]

**
(20.) If then we take any spherical quadrilateral $ABCD$, and write

$XXIII. \ldots \ell = \cos AD = -SU\alpha, \quad \ell' = \cos BD = -SU\beta, \quad \ell'' = \cos CD = -SU\gamma,$

treating $\alpha, \beta, \gamma$ as the unit-vectors of the points $A, B, C,$ and $l, m, n$ as the cosines of the arcs $BC, CA, AB$, as in 297, (1.), we have the equation,

$XXIV. \ldots 1 + \ell l' + m^2 + n^2 + \ell' l'' + 2mn + 2n'n'' + 2ml'm'' + 2lmn$

$= 2mn + 2n' + 2m' + 2l' + l'' + m'' + n'';$

which can be confirmed by elementary considerations,* but is here given merely as an interpretation of the quaternion formula $XXII$.  

(21.) In squaring the lately cited equation 294, XV., we have used the two following formulae of transformation (comp. 204, XXII., and 210, XVII.), in which $\alpha, \beta, \gamma$ may be any three vectors, and which are often found to be useful:

$XXV. \ldots (\ell V\beta) = (S\ell \alpha)^2 - \ell \alpha^2; \quad XXVII. \ldots S(\ell V\beta \gamma) = (\ell V\gamma a)^2 - (\ell V\gamma a)^2.$

299. The two cases, for which the three sides $a, b, c,$ of the given triangle $ABC$, are all less, or all greater, than quadrants, having been considered in the two foregoing Articles, with a reduction, in 298, (10.) and (11.), of certain other cases to these, it only remains to consider that third principal case, for which the sides of that given triangle are all equal to quadrants: or to inquire what is, on our general principles, the Fourth Proportional to Three Rectangular Vectors. And we shall find, not only that this fourth proportional is not itself a Vector, but that it does not even contain any vector part (292) different from zero: although, as being found to be equal to a Scalar, it is still included (131, 276) in the general conception of a Quaternion.

(1.) In fact, if we suppose, in 297, (1.), that

$I.\ldots I = 0, \quad m = 0, \quad n = 0, \quad \text{or that} \quad II.\ldots a = b = c = \frac{\pi}{2},$

or

$III.\ldots S\ell \gamma = S\ell a = S\ell b = 0, \quad \text{while} \quad IV.\ldots T\ell = T\beta = T\gamma = 1,$

the formulae 297, VII. give,

$V.\ldots \delta = 0, \quad \varepsilon = 0, \quad \zeta = 0;$

but these are the vector parts of the three pairs of fourth proportions to the

* A formula equivalent to this last equation of seventeen terms, connecting the six cosines of the arcc which join, two by two, the corners of a spherical quadrilateral $ABCD$, is given at page 407 of Carnot's Géométrie de Position (Paris, 1803).
three rectangular unit-lines, \(a, \beta, \gamma\), taken in all possible orders; and the same evanescence of vector parts must evidently take place, if the three given lines be only at right angles to each other, without being equally long.

(2.) Continuing, however, for simplicity, to suppose that they are unit lines, and that the rotation round \(a\) from \(\beta\) to \(\gamma\) is negative, as before, we see that we have now \(r = 0\), and \(e = 1\), in 297, (3); and that thus the six fourth proportionals reduce themselves to their scalar parts, namely (here) to positive or negative unity. In this manner we find, under the supposed conditions, the values:

\[
\begin{align*}
\text{VI} & \quad \beta a^{-1} \gamma = \gamma \beta^{-1} a = a \gamma^{-1} \beta = +1; \\
\text{VI'} & \quad \gamma a^{-1} \beta = a \beta^{-1} \gamma = \beta \gamma^{-1} a = -1.
\end{align*}
\]

(3.) For example (comp. 295) we have, by the laws (182) of \(i, j, k\), the values,

\[
\begin{align*}
\text{VII} & \quad ij^{-1} k = jk^{-1} i = ki^{-1} j = +1; \\
\text{VII'} & \quad kj^{-1} i = ik^{-1} j = ji^{-1} k = -1.
\end{align*}
\]

In fact, the two fourth proportionals, \(ij^{-1} k\) and \(kj^{-1} i\), are respectively equal to the two ternary products, \(-ijk\) and \(-kji\), and therefore to \(+1\) and \(-1\), by the laws included in the Fundamental Formula \(A\) (183).

(4.) To connect this important result with the constructions of the two last Articles, we may observe that when we seek, on the general plan of 298, (15.), to escribe a spherical triangle, \(DEF\), to a given tri-quadrantal (or tri-rectangular) triangle, \(ABC\), as for instance to the triangle \(ijk\) (or \(jik\)) of 181, in such a manner that the sides of the new triangle shall be bisected by the corners of the old, the problem is found to admit of indefinitely many solutions. Any point \(P\) may be assumed, in the interior of the given triangle \(ABC\); and then, if its reflexions \(D, E, F\) be taken, with respect to the three sides, \(a, b, c\), so that (comp. fig. 72) the arcs \(PD, PE, PF\) are perpendicularly bisected by those three sides, the three other arcs \(EF, FD, DE\) will be bisected by the points \(A, B, C\), as required: because the arcs \(AE, AF\) have each the same length as \(AP\), and the angles subtended at \(A\) by \(PE\) and \(PF\) are together equal to two right angles, \&c.

(5.) The positions of the auxiliary points, \(D, E, F\), are therefore, in the present case, indeterminate, or variable; but the sum of the angles at those three points is constant, and equal to four right angles; because, by the six isosceles triangles on \(PD, PE, PF\) as bases, that sum of the three angles \(D, E, F\) is equal to the sum of the angles subtended by the sides of the given triangle \(ABC\), at the assumed interior point \(P\). The spherical
excess of the triangle DEF is therefore equal to two right angles, and its area \( \Sigma = \pi \); as may be otherwise seen from the same figure 72, and might have been inferred from the formula 297, LV., or LVI.

(6.) The radius \( \text{on} \), in the formula 297, XLVII., for the fourth proportional \( \beta a^{-1} \gamma \), becomes therefore, in the present case, indeterminate; but because the angle \( L'DR \), or \( \frac{1}{2}(\pi - \Sigma) \), in the same equation, vanishes, the formula becomes simply \( \beta a^{-1} \gamma = 1 \), as in the recent equations VI.; and similarly in other examples, of the class here considered.

(7.) The conclusion, that the Fourth Proportional to Three Rectangular Lines is a Scalar, may in several other ways be deduced, from the principles of the present Book. For example, with the recent suppositions, we may write,

\[
\begin{align*}
\text{VIII} \ldots \beta a^{-1} &= - \gamma, \quad \gamma \beta^{-1} = - a, \quad a \gamma^{-1} = - \beta; \\
\text{VIII'} \ldots \gamma a^{-1} &= + \beta, \quad a \beta^{-1} = + \gamma, \quad \beta \gamma^{-1} = + a;
\end{align*}
\]

the three fourth proportionals VI. are therefore equal, respectively, to \(-\gamma^2\)\(, -a^2\), \(-\beta^2\), and consequently to \(+1\); while the corresponding expressions VI'. are equal to \(+\beta^2\), \(+\gamma^2\), \(+a^2\), and therefore to \(-1\).

(8.) Or (comp. (3.)) we may write generally the transformation (comp. 282, XXI.),

\[
\begin{align*}
\text{IX} \ldots \beta a^{-1} \gamma &= a^{-2} \beta a \gamma, \quad \text{if} \quad a^{-2} = 1 : a^2,
\end{align*}
\]

in which the factor \(a^{-2}\) is always a scalar, whatever vector \(a\) may be; while the vector part of the ternary product \(\beta a \gamma\) vanishes, by 294, III., when the recent conditions of rectangularity III. are satisfied.

(9.) Conversely, this ternary product \(\beta a \gamma\), and this fourth proportional \(\beta a^{-1} \gamma\), can never reduce themselves to scalars, unless the three vectors \(a, \beta, \gamma\) (supposed to be all actual (Art. 1)) are perpendicular each to each.

SECTION 8.

On an equivalent Interpretation of the Fourth Proportional to Three Diplanar Vectors, deduced from the Principles of the Second Book.

300. In the foregoing section, we naturally employed the results of preceding sections of the present Book, to assist ourselves in attaching a definite signification to the Fourth Proportional (297) to Three Diplanar Vectors; and thus, in order to interpret the symbol \(\beta a^{-1} \gamma\), we availed ourselves of the interpretations previously obtained, in this Third Book, of \(a^{-1}\) as a line, and of

\[\text{3 C 2}\]
$a\beta, a\beta\gamma$ as quaternions. But it may be interesting, and not un instructive, to inquire how the equivalent symbol,

$$I. \ldots (\beta : a) \cdot \gamma, \quad \text{or} \quad \frac{\beta}{a} \gamma, \quad \text{with} \gamma \not\parallel a, \beta,$$

might have been interpreted, on the principles of the Second Book, without at first assuming as known, or even seeking to discover, any interpretation of the three lately mentioned symbols,

$$II. \ldots \alpha^{-1}, \quad a\beta, \quad a\beta\gamma.$$

It will be found that the inquiry conduits to an expression of the form,

$$III. \ldots (\beta : a) \cdot \gamma = \delta + eu;$$

where $\delta$ is the same vector, and $e$ is the same scalar, as in the recent sub-articles to 297; while $u$ is employed as a temporary symbol, to denote a certain Fourth Proportional to Three Rectangular Unit Lines, namely, to the three lines $oa, ol',$ and $op$ in fig. 68; so that, with reference to the construction represented by that figure, we should be led, by the principles of the Second Book, to write the equation:

$$IV. \ldots (ob : oa) \cdot oc = od \cdot \cos \frac{1}{2} \Sigma + (ol' : oo) \cdot op \cdot \sin \frac{1}{2} \Sigma.$$

And when we proceed to consider what signification should be attached, on the principles of the same Second Book, to that particular fourth proportional, which is here the coefficient of $\sin \frac{1}{2} \Sigma$, and has been provisionally denoted by $u$, we find that although it may be regarded as being in one sense a Line, or at least homogeneous with a line, yet it must not be equated to any Vector: being rather analogous, in Geometry, to the Scalar Unit of Algebra, so that it may be naturally and conveniently denoted by the usual symbol 1, or $+1$, or be equated to Positive Unity. But when we thus write $u = 1$, the last term of the formula III. or IV., of the present Article, becomes simply $e$, or $\sin \frac{1}{2} \Sigma$; and while this term (or part) of the result comes to be considered as a species of Geometrical Scalar, the complete Expression for the General Fourth Proportional to Three Diplanar Vectors takes the Form of a Geometrical Quaternion: and thus the formula 297, XLVII., or 298, VIII., is reproduced, at least if we substitute in it, for the present, $(\beta : a) \cdot \gamma$ for $\beta\alpha^{-1}\gamma$, to avoid the necessity of interpreting here the recent symbols II.

* [*In the abstract published in the Proceedings (Royal Irish Academy, November 11th, 1844), the words 'South, West, Up' were used at first, instead of the symbols $i, j, k$; and the sought fourth proportional to $ijk$, which is here denoted by $u$, was called provisionally, 'Forward.'*—Preface to Lectures, p. (54).*]
(1.) The construction of fig. 68 being retained, but no principles peculiar to the Third Book being employed, we may write, with the same significations of c, p, &c., as before,

V. \( \frac{ob}{oa} = \frac{oa}{oc} = \cos c + \left( \frac{ol'}{oq} \right) \sin c; \)

VI. \( oc = oq \cdot \cos p + op \cdot \sin p. \)

(2.) Admitting then, as is natural, for the purposes of the sought interpretation, that distributive property which has been proved (212) to hold good for the multiplication of quaternions (as it does for multiplication in algebra); and writing for abridgment,

VII. \( u = \left( \frac{ol'}{oq} \right) \cdot op; \)

we have the quadrinomial expression:

VIII. \( \left( \frac{ob}{oa} \right) \cdot oc = \frac{ol'}{oq} \cdot \sin c \cos p + oq \cdot \cos c \cos p + op \cdot \cos c \sin p + u \cdot \sin c \sin p; \)

in which it may be observed that the sum of the squares of the four coefficients of the three rectangular unit-vectors, oq, ol', op, and of their fourth proportional, u, is equal to unity.

(3.) But the coefficient of this fourth proportional, which may be regarded as a species of fourth unit, is

IX. \( \sin c \sin p = \sin mn = \sin \frac{1}{2} \Sigma = \delta; \)

we must therefore expect to find that the three other coefficients in VIII., when divided by \( \cos \frac{1}{2} \Sigma, \) or by \( r, \) give quotients which are the cosines of the arcual distances of some point \( x \) upon the unit-sphere, from the three points \( l', q, p; \) or that a point \( x \) can be assigned, for which

X. \( \sin c \cos p = r \cos l'x; \quad \cos c \cos p = r \cos qx; \quad \cos c \sin p = r \cos px. \)

(4.) Accordingly it is found that these three last equations are satisfied, when we substitute \( d \) for \( x; \) and therefore that we have the transformation,

XI. \( ol' \cdot \sin c \cos p + oq \cdot \cos c \cos p + op \cdot \cos c \sin p = od \cdot \cos \frac{1}{2} \Sigma = \delta, \)

whence follow the equations IV. and III.; and it only remains to study and interpret the fourth unit, \( u, \) which enters as a factor into the remaining part of the quadrinomial expression VIII., without employing any principles except those of the Second Book; and therefore without using the Interpretations 278, 284, of \( \beta a, \) &c.
301. In general, when two sets of three vectors, \(a, \beta, \gamma\), and \(a', \beta', \gamma'\), are connected by the relation,

\[
\frac{\beta}{a} \frac{\gamma}{\beta'} = 1, \quad \text{or} \quad \frac{\beta}{a} \frac{\gamma}{\gamma'} = \frac{\beta'}{\alpha'},
\]

it is natural to write this other equation,

\[
III. \frac{\beta}{a} \frac{\gamma}{\gamma'} = \frac{\beta'}{\alpha'},
\]

and to say that these two fourth proportionals (297), to \(a, \beta, \gamma\), and to \(a', \beta', \gamma'\), are equal to each other: whatever the full signification of each of these two last symbols III., supposed for the moment to be not yet fully known, may be afterwards found to be. In short, we may propose to make it a condition of the sought Interpretation, on the principles of the Second Book, of the phrase,

"Fourth Proportional to three Vectors,"

and of either of the two equivalent Symbols 300, I., that the recent Equation III. shall follow from I. or II.; just as, at the commencement of that Second Book, and before concluding (112) that the general Geometric Quotient \(\beta : a\) of any two lines in space is a Quaternion, we made it a condition (103) of the interpretation of such a quotient, that the equation \((\beta : a).a = \beta\) should be satisfied.

302. There are however two tests (comp. 287), to which the recent equation III. must be submitted, before its final adoption; in order that we may be sure of its consistency, 1st, with the previous interpretation (226) of a Fourth Proportional to Three Complanar Vectors, as a Line in their common plane; and 2nd, with the general principle of all mathematical language (105), that things equal to the same thing, are to be considered as equal to each other. And it is found, on trial, that both these tests are borne: so that they form no objection to our adopting the equation 301, III., as true by definition, whenever the preceding equation II., or I., is satisfied.

(1.) It may happen that the first member of that equation III. is equal to a line \(\delta\), as in 226; namely, when \(a, \beta, \gamma\) are complanar. In this case, we have by II. the equation,

\[
IV. \frac{\delta}{\gamma'} = \frac{\delta}{\gamma} \frac{\gamma}{\gamma'} = \frac{\beta'}{\alpha'}, \quad \text{or} \quad IV'. \frac{\beta'}{\alpha'} \gamma' = \delta = \frac{\beta}{a} \gamma;
\]

so that \(a', \beta', \gamma'\) are also complanar (among themselves), and the line \(\delta\) is their fourth proportional likewise: and the equation III. is satisfied, both
members being symbols for one common line, $\delta$, which is in general situated in the intersection of the two planes, $a\beta\gamma$ and $a'\beta'\gamma'$; although those planes may happen to coincide, without disturbing the truth of the equation.

(2.) Again, for the more general case of diplanarity of $a, \beta, \gamma$, we may conceive that the equation* II. co-exists with this other of the same form,

$$\frac{\beta}{a} \gamma' = \frac{\beta''}{a''} ;$$

which gives

$$\frac{\beta}{a} \gamma = \frac{\beta''}{a''} \gamma'' ,$$

if the definition 301 be adopted. If then that definition be consistent with general principles of equality, we ought to find, by III. and VI., that this third equation between two fourth proportionals holds good:

$$\frac{\beta'}{a'} \gamma' = \frac{\beta''}{a''} \gamma'' ;$$

or that

$$\frac{\beta'}{a'} \gamma' = \frac{\beta''}{a''} \gamma'' ,$$

when the equations II. and V. are satisfied. And accordingly, those two equations give, by the general principles of the Second Book, respecting quaternions considered as quotients of vectors, the transformation,

$$\frac{\beta'}{a'} \gamma' = \frac{\beta}{a} \gamma \cdot \frac{\gamma'}{\gamma''} = \frac{\beta}{a} \gamma = \frac{\beta''}{a''} ,$$

as required.

303. It is then permitted to interpret the equation 301, III., on the principles of the Second Book, as being simply a transformation (as it is in algebra) of the immediately preceding equation II., or I.; and therefore to write, generally,

$$I. \ldots q\gamma = q'\gamma' , \quad \text{if} \quad II. \ldots q (\gamma : \gamma') = q' ;$$

where $\gamma, \gamma'$ are any two vectors, and $q, q'$ are any two quaternions, which satisfy this last condition. Now, if $v$ and $v'$ be any two right quaternions, we have (by 193, comp. 283) the equation,

$$III. \ldots Iv : Iv' = v : v' = vv' ;$$

or

$$IV. \ldots v^{-1} (Iv : Iv') = v^{-1} ;$$

whence

$$V. \ldots v^{-1} . Iv = v^{-1} . Iv' ,$$

by the principle which has just been enunciated. It follows, then, that "if a right Line ($Iv$) be multiplied by the Reciprocal ($v^{-1}$) of the Right Quaternion ($v$), of which it is the Index, the Product ($v^{-1} . Iv$) is independent of the Length, and of

---

* In this and other cases of reference, the numeral cited is always supposed to be the one which (with the same number) has last occurred before, although perhaps it may have been in connexion with a shortly preceding Article. Compare 217, (1.).
the Direction, of the Line thus operated on” ; or, in other words, that this Product has one common Value, for all possible Lines (a) in Space: which common or constant value may be regarded as a kind of new Geometrical Unit, and is equal to what we have lately denoted, in 300, III., and VII., by the temporary symbol u; because, in the last cited formula, the line or is the index of the right quotient oo : or'. Retaining, then, for the moment, this symbol, u, we have, for every line a in space, considered as the index of a right quaternion, v, the four equations:

VI. . . v⁻¹a = u; VII. . . a = vu; VIII. . . v = a : u; IX. . . v⁻¹ = u : a;

in which it is understood that a = Iv, and the three last are here regarded as being merely transformations of the first, which is deduced and interpreted as above. And hence it is easy to infer, that for any given system of three rectangular lines a, β, γ, we have the general expression:

X... (β : a).γ = xu, if a ⊥ β, β ⊥ γ, γ ⊥ a;

where the scalar co-efficient, x, of the new unit, u, is determined by the equation,

XI. . . x = ± (Tβ : Ta).Ty, according as XII... Uγ = ± Ax.(a : β).

This coefficient x is therefore always equal, in magnitude (or absolute quantity), to the fourth proportional to the lengths of the three given lines aβγ; but it is positively or negatively taken, according as the rotation round the third line γ, from the second line β, to the first line a, is itself positive or negative; or in other words, according as the rotation round the first line, from the second to the third, is on the contrary negative or positive (compare 294, (3)).

(1.) In illustration of the constancy of that fourth proportional which has been, for the present, denoted by u, while the system of the three rectangular unit-lines from which it is conceived to be derived is in any manner turned about, we may observe that the three equations, or proportions,

XIII... u : γ = β : a; γ : a = a : − γ; β : − γ = γ : β,

conduct immediately to this fourth equation of the same kind,

XIV... u : a = γ : β, or* u = (γ : β).a;

if we admit that this new quantity, or symbol, u, is to be operated on at all,

* In equations of this form, the parentheses may be omitted, though for greater clearness they are here retained.
or combined with other symbols, according to the general rules of vectors and quaternions.

(2.) It is, then, permitted to change the three letters \(a, \beta, \gamma\), by a cyclical permutation, to the three other letters, \(\beta, \gamma, a\) (considered again as representing unit-lines), without altering the value of the fourth proportional, \(u\); or in other words, it is allowed to make the system of the three rectangular lines revolve, through the third part of four right angles, round the interior and co-initial diagonal of the unit-cube, of which they are three co-initial edges.

(3.) And it is still more evident, that no such change of value will take place, if we merely cause the system of the two first lines to revolve, through any angle, in its own plane, round the third line as an axis; since thus we shall merely substitute, for the factor \(\beta : a\), another factor equal thereto. But by combining these two last modes of rotation, we can represent any rotation whatever, round an origin supposed to be fixed.

(4.) And as regards the scalar ratio of any one fourth proportional, such as \(\beta' : a' : \gamma'\), to any other, of the kind here considered, such as \(\beta : a, \gamma, u\), it is sufficient to suggest that, without any real change in the former, we are allowed to suppose it to be so prepared, that we shall have

\[
XV. \ldots a' = a; \quad \beta' = \beta; \quad \gamma' = x\gamma;
\]

\(x\) being some scalar coefficient, and representing the ratio required.

304. In the more general case, when the three given lines are not rectangular, nor unit-lines, we may on similar principles determine their fourth proportional, without referring to fig. 68 [p. 360], as follows. Without any real loss of generality, we may suppose that the planes of \(a, \beta\), and \(a, \gamma\) are perpendicular to each other; since this comes merely to substituting, if necessary, for the quotient \(\beta : a\), another quotient equal thereto. Having thus

I. \(\Delta x . (\beta : a) \perp \Delta x . (\gamma : a)\), let II. \(\beta = \beta' + \beta'', \quad \gamma = \gamma' + \gamma''\),

where \(\beta'\) and \(\gamma'\) are parallel to \(a\), but \(\beta''\) and \(\gamma''\) are perpendicular to it, and to each other; so that, by 203, I. and II., we shall have the expressions,

III. \(\beta' = S_{\beta} \frac{\beta}{\alpha} . a, \quad \gamma' = S_{\gamma} \frac{\gamma}{\alpha} . a\),

and

IV. \(\beta'' = V_{\beta} \frac{\beta}{\alpha} . a, \quad \gamma'' = V_{\gamma} \frac{\gamma}{\alpha} . a\).

Hamilton’s Elements of Quaternions,
We may then deduce, by the distributive principle (300, (2.)), the transformations,

\[ V \ldots \frac{\beta}{a} \cdot \gamma = \left( \frac{\beta'}{a} + \frac{\beta''}{a} \right) (\gamma' + \gamma'') \]

\[ = \frac{\beta'}{a} \gamma' + \frac{\beta'}{a} \gamma'' + \frac{\beta''}{a} \gamma' + \frac{\beta''}{a} \gamma'' = 3 + uv; \]

where

\[ V \ldots \delta = \beta S \frac{\gamma}{a} + \gamma'' S \frac{\beta}{a} = \gamma S \frac{\beta}{a} + \beta'' S \frac{\gamma}{a}, \quad \text{and} \quad V \ldots xu = \frac{\beta''}{a} \gamma''. \]

The latter part, \( xu \), is what we have called (300) the (geometrically) scalar part, of the sought fourth proportional; while the former part \( \delta \) may (still) be called its vector part: and we see that this part is represented by a line, which is at once in the two planes, of \( \beta \), \( \gamma'' \), and of \( \gamma, \beta'' \); or in two planes which may be generally constructed as follows, without now assuming that the planes \( a\beta \) and \( a\gamma \) are rectangular, as in I. Let \( \gamma' \) be the projection of the line \( \gamma \) on the plane of \( a, \beta \), and operate on this projection by the quotient \( \beta : a \) as a multiplier; the plane which is drawn through the line \( \beta : a \cdot \gamma' \) so obtained, at right angles to the plane \( a\beta \), is one locus for the sought line \( \delta \): and the plane through \( \gamma \), which is perpendicular to the plane \( \gamma' \gamma'' \), is another locus for that line. And as regards the length of this line, or vector part \( \delta \), and the magnitude (or quantity) of the scalar part \( xu \), it is easy to prove that

\[ V \ldots t \delta = t \cos \psi, \quad \text{and} \quad V \ldots x = \pm t \sin \psi, \]

where

\[ X \ldots t = T\beta : Ta, \quad T\gamma, \quad \text{and} \quad XI \ldots \sin \psi = \sin c \sin p, \]

if \( c \) denote the angle between the two given lines \( a, \beta \), and \( p \) the inclination of the third given line \( \gamma \) to their plane: the sign of the scalar coefficient, \( x \), being positive or negative, according as the rotation round \( a \) from \( \beta \) to \( \gamma \) is negative or positive.

(1.) Comparing the recent construction with fig. 68, we see that when the condition I. is satisfied, the four unit-lines \( U\gamma, Ua, U\beta, Ud \) take the directions of the four radii \( oc, op, or, od \), which terminate at the four corners of what may be called a tri-rectangular quadrilateral \( cqr \) on the sphere.

(2.) It may be remarked that the area of this quadrilateral is exactly equal to half the area \( \Sigma \) of the triangle \( def \); which may be inferred, either from the circumstance that its spherical excess (over four right angles) is constructed by the angle \( mdn \); or from the triangles \( dbr \) and \( eas \) being together equal.
to the triangle \( \triangle ABF \), so that the area of \( \text{desr} \) is \( \Sigma \), and therefore that of \( \text{cqrd} \) is \( \frac{1}{2} \Sigma \), as before.

(3.) The two sides \( \text{cq}, \text{qr} \) of this quadrilateral, which are remote from the obtuse angle at \( D \), being still called \( p \) and \( c \), and the side \( \text{cd} \) which is opposite to \( c \) being still denoted by \( c' \), let the side \( \text{dr} \) which is opposite to \( p \) be now called \( p' \); also let the diagonals \( \text{cr}, \text{qd} \) be denoted by \( d \) and \( d' \); and let \( s \) denote the spherical excess \( \text{cdr} - \frac{1}{2} \pi \), or the area of the quadrilateral. We shall then have the relations,

\[
\begin{align*}
\cos d &= \cos p \cos c; & \cos d' &= \cos p \cos c'; \\
\tan c' &= \cos p \tan c; & \tan p' &= \cos c \tan p; \\
\cos s &= \cos p \sec p' = \cos c \sec c' = \cos d \sec d';
\end{align*}
\]

of which some have virtually occurred before, and all are easily proved by right-angled triangles, arcs being when necessary prolonged.

(4.) If we take now two points, \( A \) and \( B \), on the side \( \text{qr} \), which satisfy the areal equation (comp. 297, XL., and fig. 68),

\[
\begin{align*}
\cos d &= \cos p \cos c; & \cos d' &= \cos p \cos c'; \\
\tan c' &= \cos p \tan c; & \tan p' &= \cos c \tan p; \\
\cos s &= \cos p \sec p' = \cos c \sec c' = \cos d \sec d';
\end{align*}
\]

and if we then join \( AC \), and let fall on this new arc the perpendiculars \( BB', DD' \); it is easy to prove that the projection \( B'D' \) of the side \( BD \) on the arc \( AC \) is equal to that arc, and that the angle \( DBB' \) is right; so that we have the two new equations,

\[
\begin{align*}
\cos d &= \cos p \cos c; & \cos d' &= \cos p \cos c'; \\
\tan c' &= \cos p \tan c; & \tan p' &= \cos c \tan p; \\
\cos s &= \cos p \sec p' = \cos c \sec c' = \cos d \sec d';
\end{align*}
\]

and the new quadrilateral \( BB'D'D \) is also tri-rectangular.

(5.) Hence the point \( D \) may be derived from the three points \( ABC \), by any two of the following conditions: 1st, the equality XIII. of the arcs \( AB, QR \); IIInd, the corresponding equality XIV. of the arcs \( AC, B'D' \); IIIInd, the tri-rectangular character of the quadrilateral \( \text{cqrd} \); IVth, the corresponding character of \( BB'D'D \).

(6.) In other words, this derived point \( D \) is the common intersection of the four perpendiculars, to the four arcs \( AB, AC, \text{cq}, BB' \), erected at the four points \( R, D', C, B \); \( \text{cq}, BB' \) being still the perpendiculars from \( C \) and \( B \), on \( AB \) and \( AC \); and \( R \) and \( D' \) being deduced from \( Q \) and \( B' \), by equal arcs, as above.

305. These consequences of the construction employed in 297, &c., are here mentioned merely in connexion with that theory of fourth proportions to vectors, which they have thus served to illustrate; but they are perhaps
numerous and interesting enough, to justify us in suggesting the name, “Spherical Parallelogram,”* for the quadrilateral $cabc$, or $bacd$, in fig. 68 (or 67), p. 360; and in proposing to say that $d$ is the Fourth Point, which completes such a parallelogram, when the three points $c, a, b$, or $b, a, c$, are given upon the sphere, as first, second, and third. It must however be carefully observed, that the analogy to the plane is here thus far imperfect, that in the general case, when the three given points are not co-arcual, but on the contrary are corners of a spherical triangle $abc$, then if we take $c, d, b$, or $b, d, c$, for the three first points of a new spherical parallelogram, of the kind here considered, the new fourth point, say $A_1$, will not coincide with the old second point $a$; although it will very nearly do so, if the sides of the triangle $abc$ be small: the deviation $AA_1$ being in fact found to be small of the third order, if those sides of the given triangle be supposed to be small of the first order; and being always directed towards the foot of the perpendicular, let fall from $a$ on $bc$.

(1.) To investigate the law of this deviation, let $\beta, \gamma$ be still any two given unit-vectors, $ob, oc$, making with each other an angle equal to $a$, of which the cosine is $l$; and let $p$ or $or$ be any third vector. Then, if we write,

$$I. \quad \rho_1 = \phi (\rho) = \frac{1}{2}N\rho \left( \frac{\beta}{\rho} \gamma + \frac{\gamma}{\rho} \beta \right), \quad oq = Up, \quad oq_1 = Up_1,$$

the new or derived vector, $\phi \rho$ or $\rho_1$, or $or_1$, will be the common vector part of the two fourth proportionals, to $\rho, \beta, \gamma$, and to $\rho, \gamma, \beta$, multiplied by the square of the length of $\rho$; and $bacq_1$ will be what we have lately called a spherical parallelogram. We shall also have the transformation (compare 297, (2.)),

$$II. \quad \rho_1 = \phi \rho = \beta \rho \frac{\rho}{\gamma} + \gamma \rho \frac{\rho}{\beta} - \rho \rho \frac{\gamma}{\beta};$$

and the distributive symbol of operation $\phi$ will be such that

$$III. \quad \phi \rho \parallel \beta, \gamma, \quad \phi^2 \rho = \rho, \quad \rho \parallel \beta, \gamma; \quad \phi \rho = \rho \rho \rho \rho \rho,$$

but

$$IV. \quad \phi \rho = - l\rho, \quad \rho \parallel Ax. (\gamma : \beta).$$

(2.) This being understood, let

$$V. \quad \rho = \rho' + \rho''; \quad \phi \rho' = \rho'_1; \quad \rho' \parallel \beta, \gamma; \quad \rho'' \parallel Ax. (\gamma : \beta);$$

* By the same analogy, the quadrilateral $cabc$, in fig. 68, may be called a Spherical Rectangle.

† [In fact $\phi \beta = \gamma$ and $\phi \gamma = \beta$. So $\phi (y \beta + z \gamma) = z \beta + y \gamma.$]
so that \( p' \), or \( op' \), is the projection of \( p \) on the plane of \( \beta \gamma \); and \( p'' \), or \( op'' \), is the part (or component) of \( p \), which is perpendicular to that plane. Then we shall have an indefinite series of derived vectors, \( \rho_1, \rho_2, \rho_3, \ldots \) or rather two such series, succeeding each other alternately, as follows:

\[
\begin{align*}
\rho_1 &= \phi \rho = \rho' - l \rho''; \\
\rho_2 &= \phi^2 \rho = \rho' + l^2 \rho''; \\
\rho_3 &= \phi^3 \rho = \rho' - l^3 \rho''; \\
&
\end{align*}
\]

the two series of derived points, \( p_1, p_2, p_3, p_4, \ldots \) being thus ranged, alternately, on the two perpendiculars, \( pp' \) and \( p_1p'_1 \), which are let fall from the points \( p \) and \( p_1 \), on the given plane \( \beta \gamma \); and the intervals, \( pp_2, p_1p_3, p_2p_4, \ldots \) forming a geometrical progression, in which each is equal to the one before it, multiplied by the constant factor \( -l \), or by the negative of the cosine of the given angle \( \beta \gamma \).

(3.) If then this angle be still supposed to be distinct from 0 and \( \pi \), and also in general from the intermediate value \( \frac{1}{2} \pi \), we shall have the two limiting values,

\[
\begin{align*}
\rho_{2n} &= \rho', \\
\rho_{2n+1} &= \rho'', 
\end{align*}
\]

or in words, the derived points \( p_2, p_4, \ldots \) of even orders, tend to the point \( p' \), and the other derived points, \( p_1, p_3, \ldots \) of odd orders, tend to the other point \( p'' \), as limiting positions; these two limit points being the feet of the two (rectilinear) perpendiculars, let fall (as above) from \( p \) and \( p' \) on the plane \( \beta \gamma \).

(4.) But even the first deviation \( pp_2 \) is small of the third order, if the length \( T_\rho \) of the line \( \rho \) be considered as neither large nor small, and if the sides of the spherical triangle \( \beta \gamma \) be small of the first order. For we have by VI. the following expressions for that deviation,

\[
\begin{align*}
VIII \ldots pp_2 &= \rho_2 - \rho = (l^2 - 1) \rho'' = -\sin^2 a \cdot \sin p_a \cdot T_\rho \cdot U \rho'';
\end{align*}
\]

where \( p_a \) denotes the inclination of the line \( \rho \) to the plane \( \beta \gamma \); or the arcual perpendicular from the point \( q \) on the side \( \beta \gamma \), or \( a \), of the triangle. The statements lately made (305) are therefore proved to have been correct.

(5.) And if we now resume and extend the spherical construction, and conceive that \( \lambda_1 \) is deduced from \( \beta \lambda_2 \gamma \), as \( A_1 \) was from \( \beta \gamma \), or \( D \) from \( B \gamma C \); while \( A_2 \) may be supposed to be deduced by the same rule from \( \beta \gamma \), and \( \lambda_2 \) from \( \beta \lambda_2 \gamma \), \&c., through an indefinite series of spherical parallelograms, in which the fourth point of any one is treated as the second point of the next, while the first and third points remain constant: we see that the points \( A_1, A_2, \ldots \) are all situated on the arcual perpendicular let fall from \( \lambda \) on \( \beta \gamma \); and that in like manner the points \( \lambda_1, \lambda_2, \ldots \) are all situated on that other arcual perpendicular,
which is let fall from \( d \) on \( bc \). We see also that the ultimate positions, \( a' \) and \( d' \), coincide precisely with the feet of those two perpendiculars: a remarkable theorem, which it would perhaps be difficult to prove, by any other method than that of the Quaternions, at least with calculations so simple as those which have been employed above.

(6.) It may be remarked that the construction of fig. 68 might have been otherwise suggested (comp. 223, IV.), by the principles of the Second Book, if we had sought to assign the fourth proportional (297) to three right quaternions; for example, to three right versors, \( v, v', v'' \), whereof the unit lines \( a, \beta, \gamma \) should be supposed to be the axes. For the result would be in general a quaternion \( v'v''e \), with \( e \) for its scalar part, and with \( e \) for the index of its right part: \( e \) and \( e \) denoting the same scalar, and the same vector, as in the subarticles to 297.

306. Quaternions may also be employed to furnish a new construction, which shall complete (comp. 305, (5.)) the graphical determination of the two series of derived points,

\[ I \ldots D, A_3, D_1, A_5, D_2, \&c., \]

when the three points \( A, B, C \) are given upon the unit-sphere; and thus shall render visible (so to speak), with the help of a new figure, the tendencies of those derived points to approach, alternately and indefinitely, to the feet, say \( d' \) and \( a' \), of the two arcual perpendiculars let fall from the two opposite corners, \( d \) and \( a \), of the first spherical parallelogram, \( BACD \), on its given diagonal \( BC \); which diagonal (as we have seen) is common to all the successive parallelograms.

(1.) The given triangle \( ABC \) being supposed for simplicity to have its sides \( abc \) less than quadrants, as in 297, so that their cosines \( lmn \) are positive, let \( A', B', C' \) be the feet of the perpendiculars let fall on these three sides from the points \( A, B, C \); also let \( M \) and \( N \) be two auxiliary points, determined by the equations,

\[ II \ldots \angle BM = \angle MC, \quad \angle AM = \angle MN; \]

so that the arcs \( AN \) and \( BC \) bisect each other in \( M \). Let fall from \( N \) a perpendicular \( ND' \) on \( BC \), so that

\[ III \ldots \angle BD' = \angle A'C; \]

and let \( b'', c'' \) be two other auxiliary points, on the sides \( b \) and \( c \), or on those sides prolonged, which satisfy these two other equations,

\[ IV \ldots \angle b'B'' = \angle AC, \quad \angle c'C'' = \angle AB. \]
(2.) Then the perpendiculars to these last sides, CA and AB, erected at these last points, H'' and C'', will intersect each other in the point D, which completes (305) the spherical parallelogram BACD; and the foot of the perpendicular from this point N, on the third side BC of the given triangle, will coincide (comp. 305, (2.)) with the foot N' of the perpendicular on the same side from N; so that this last perpendicular ND' is one locus of the point D.

(3.) To obtain another locus for that point, adapted to our present purpose, let E denote now* that new point in which the two diagonals, AD and BC, intersect each other; then because (comp. 297, (2.)) we have the expression,

\[ V \ldots \text{od} = u (m \beta + n \gamma - la), \]

we may write (comp. 297, (25.), and (30.)),

\[ VI \ldots \text{oe} = u (m \beta + n \gamma), \] whence \[ VII \ldots \sin BE : \sin EC = n : m = \cos BA' : \cos A'C; \]

the diagonal AD thus dividing the arc BC into segments, of which the sines are proportional to the cosines of the adjacent sides of the given triangle, or to the cosines of their projections BA' and A'C on BC; so that the greater segment is adjacent to the lesser side, and the middle point M of BC (1.) lies between the points A' and E.

(4.) The intersection E is therefore a known point, and the great circle through A and E is a second known locus for D; which point may therefore be found, as the intersection of the arc AE prolonged, with the perpendicular ND' from N (1.). And because E lies (3.) beyond the middle point M of BC, with respect to the foot A' of the perpendicular on BC from A, but (as it is easy to prove) not so far beyond M as the point D', or in other words falls between M and D' (when the arc BC is, as above supposed, less than a quadrant), the prolonged arc AE cuts ND' between N and D'; or in other words, the perpendicular distance of the sought fourth point D, from the given diagonal BC of the parallelogram, is less than the distance of the given second point A, from the same given diagonal. (Compare the annexed fig. 73.)

---

* It will be observed that M, N, E have not here the same significations as in fig. 68; and that the present letters c' and c'' correspond to a and b in that figure.
(5.) Proceeding next (305) to derive a new point \( A_1 \) from \( b, d, c \), as \( d \) has been derived from \( b, a, c \), we see that we have only to determine a new* auxiliary point \( r \), by the equation,

\[
\text{VIII... } \tan \alpha = \tan \beta ;
\]

and then to draw \( df \), and prolong it till it meets \( AA' \) in the required point \( A_1 \), which will thus complete the second parallelogram, \( bdcA_1 \), with \( bc \) (as before) for a given diagonal.

(6.) In like manner, to complete (comp. 305, (5.)), the third parallelogram, \( bA_2Cd_1 \), with the same given diagonal \( bc \), we have only to draw the arc \( A_2E \), and prolong it till it cuts \( ND' \) in \( D_1 \); after which we should find the point \( A_2 \) of a fourth successive parallelogram \( bD_1cA_2 \), by drawing \( D_1F \), and so on for ever.

(7.) The constant and indefinite tendency, of the derived points \( D, D_1 \ldots \) to the limit-point \( D' \), and of the other (or alternate) derived points \( A_1, A_2 \ldots \) to the other limit-point \( A' \), becomes therefore evident from this new construction; the final (or limiting) results of which, we may express by these two equations (comp. again 305, (5.)),

\[
\text{IX... } D_n = D'; \quad A_n = A'.
\]

(8.) But the smallness (305) of the first deviation \( AA_1 \), when the sides of the given triangle \( abc \) are small, becomes at the same time evident, by means of the same construction, with the help of the formula VII.; which shows that the interval† \( \alpha \), or the equal interval \( mf \) (5), is small of the third order, when the sides of the given triangle are supposed to be small of the first order: agreeing thus with the equation 305, VIII.

(9.) The theory of such spherical parallelograms admits of some interesting applications, especially in connexion with spherical conies; on which however we cannot enter here, beyond the mere enunciation of a Theorem,‡ of which (comp. 271) the proof by quaternions is easy:

* This new point, and the intersection of the perpendiculars of the given triangle, are evidently not the same in the new figure 73, as the points denoted by the same letters, \( p \) and \( p' \), in the former figure 68; although the four points \( a, b, c, d \) are conceived to bear to each other the same relations in the two figures, and indeed in fig. 67 also; \( bacd \) being, in that figure also, what we have proposed to call a spherical parallelogram. Compare the Note to (3.).

† The formula VII. gives easily the relation,

\[
\text{VII... } \tan \alpha = \tan \beta \left( \tan \frac{a}{2} \right)^2;
\]

hence the interval \( \alpha \) is small of the third order, in the case (8.) here supposed; and generally, if \( a < \frac{\pi}{2} \), as in (1.), while \( b \) and \( c \) are unequal, the formula shows that this interval \( \alpha \) is less than \( \beta \), or than \( \mu \), so that \( \mu \) falls between \( \alpha \) and \( \beta \), as in (4.).

‡ This Theorem was communicated to the Royal Irish Academy in June, 1845, as a consequence of the principles of Quaternions. See the Proceedings of that date (Vol. III., page 109).
"If KLMN be any spherical quadrilateral, and I any point on the sphere; if also we complete the spherical parallelograms,

X...KILA, LIMB, MUNC, NIKD,

and determine the poles E and F of the diagonals KM and LN of the quadrilateral: then these two poles are the foci* of a spherical conic, inscribed in the derived quadrilateral ABCD, or touching its four sides."†

(10.) Hence, in a notation‡ elsewhere proposed, we shall have, under these conditions of construction, the formula:

XI...EF (...) ABCD; or XI'...EF (...) BCD; &c.

(11.) Before closing this article and section, it seems not irrelevant to remark, that the projection γ' of the unit-vector γ, on the plane of α and β, is given by the formula,

XII...γ' = \( \frac{a \sin a \cos b + \beta \sin b \cos a}{\sin c} \);

and that therefore the point P, in which (see again fig. 73) the three arcual perpendiculars of the triangle ABC intersect, is on the vector,

XIII...p = α tan α + β tan β + γ tan γ.

(12.) It may be added, as regards the construction in 305, (2.), that the right lines,

XIV...PP₁, P₁P₂, P₂P₃, P₃P₄, ...

however far their series may be continued, intersect the given plane boc, alternately, in two points s and t, of which the vectors are,

XV...os = \( \frac{\rho'_1 + l\rho'}{1 + l} \), \( \text{ot} = \frac{\rho' + l\rho'_1}{1 + l} \);

and which thus become two fixed points in the plane, when the position of the point P in space is given, or assumed.

---

* In the language of modern geometry, the conic in question may be said to touch eight given arcs; four real, namely the sides AB, BC, CD, DA; and four imaginary, namely two from each of the focal points, E and F.
† [Take \( q = \lambda r^{-1} \kappa, r = \mu r^{-1} \lambda, s = \nu r^{-1} \mu, \text{ and } t = \tau r^{-1} \nu \); then \( s q = -\nu r^{-1} \kappa = \kappa t, \nu q = \mu r, s r = \nu \lambda, \text{ and } t q = \lambda \nu \). On reference to fig. 60, p. 304, there is no difficulty in seeing that a conic having the given foci may be drawn to touch the four sides, produced when necessary.]
‡ Compare the second Note to page 310.
On a Third Method of interpreting a Product or Function of Vectors as a Quaternion; and on the Consistency of the Results of the Interpretation so obtained, with those which have been deduced from the two preceding Methods of the present Book.

307. The Conception of the Fourth Proportional to three Rectangular Unit-Lines, as being itself a species of Fourth Unit in Geometry, is eminently characteristic of the present Calculus; and offers a Third Method of interpreting a Product of two Vectors as a Quaternion: which is however found to be consistent, in all its results, with the two former methods (278, 284) of the present Book; and admits of being easily extended to products of three or more lines in space, and generally to Functions of Vectors (289). In fact we have only to conceive* that each proposed vector, \(a\), is divided by the new or fourth unit, \(u\), above alluded to; and that the quotient so obtained, which is always (by 303, VIII.) the right quaternion \(q^a\), whereof the vector \(a\) is the index, is substituted for that vector: the resulting quaternion being finally, if we think it convenient, multiplied into the same fourth unit. For in this way we shall merely reproduce the process of 284, or 289, although now as a consequence of a different train of thought, or of a distinct but Consistent Interpretation: which thus conducts, by a new Method, to the same Rules of Calculation as before.

(1.) The equation of the unit-sphere, \(p^2 + 1 = 0\) (282, XIV.), may thus be conceived to be an abridgment of the following fuller equation:

\[
I \cdot \left(\frac{p}{u}\right)^2 = -1;
\]

* It was in a somewhat analogous way that Des Cartes showed, in his Geometria (Schooten’s Edition, Amsterdam, 1659), that all products and powers of lines, considered relatively to their lengths alone, and without any reference to their directions, could be interpreted as lines, by the suitable introduction of a line taken for unity, however high the dimension of the product or power might be. Thus (at page 3 of the cited work) the following remark occurs:—

“Ubique notandum est, quod per \(a^2\) vel \(b^3\), similiterve, communiter, non nisi lineas omnino simplices conecipiam, licet illas, ut nominibus in Algebra usitatis utar, Quadrata aut Cubos, &c. appellem.”

But it was much more difficult to accomplish the corresponding multiplication of directed lines in space; on account of the non-existence of any such line, which is symmetrically related to all other lines, or common to all possible planes (comp. the Note to page 258). The Unit of Vector-Multiplication cannot properly be itself a Vector, if the conception of the Symmetry of Space is to be retained, and duly combined with the other elements of the question. This difficulty however disappears, at least in theory, when we come to consider that new Unit, of a scalar kind (300), which has been above denoted by the temporary symbol \(u\), and has been obtained, in the foregoing section, as a certain Fourth Proportional to Three Rectangular Unit-Lines, such as the three co-initial edges, \(ab\), \(ac\), \(ad\) of...
the quotient \( p : u \) being considered as equal (by 303) to the right quaternion, \( I^2p \), which must here be a right versor (154), because its square is negative unity.

2. The equation of the ellipsoid,

\[ T(p + p\kappa) = \kappa^2 - t^2 (282, XIX.) \]

may be supposed, in like manner, to be abridged from this other equation:

\[ \Pi \ldots T\left(\frac{t}{u} p + \frac{p}{u} \kappa\right) = \left(\frac{\kappa}{u}\right)^2 - \left(\frac{t}{u}\right)^2; \]

and similarly in other cases.

3. We might also write these equations, of the sphere and ellipsoid, under these other, but connected forms:

\[ \text{III.} \ldots \frac{p}{u} p = -u; \quad \text{IV.} \ldots T\left(\frac{t}{u} p + \frac{p}{u} \kappa\right) = \frac{\kappa}{u} \kappa - \frac{t}{u} t; \]

with interpretations which easily offer themselves, on the principles of the foregoing section.

4. It is, however, to be distinctly understood, that we do not propose to adopt this Form of Notation, in the practice of the present Calculus: and that we merely suggest it, in passing, as one which may serve to throw some additional light on the Conception, introduced in this Third Book, of a Product of two Vectors as a Quaternion.

5. In general, the Notation of Products, which has been employed throughout the greater part of the present Book and Chapter, appears to be
much more convenient, for actual use in calculation, than any Notation of Quotients: either such as has been just now suggested for the sake of illustration, or such as was employed in the Second Book, in connexion with that First Conception of a Quaternion (112), to which that Book mainly related, as the Quotient of two Vectors (or of two directed lines in space). The notations of the two Books are, however, intimately connected, and the former was judged to be an useful preparation for the latter, even as regarded the quotient-forms of many of the expressions used: while the Characteristics of Operation, such as

\[ S, V, T, U, K, N, \]

are employed according to exactly the same laws in both. In short, a reader of the Second Book has nothing to unlearn in the Third; although he may be supposed to have become prepared for the use of somewhat shorter and more convenient processes, than those before employed.

SECTION 10.

On the Interpretation of a Power of a Vector as a Quaternion.

308. The only symbols, of the kinds mentioned in 277, which we have not yet interpreted, are the cube \( a^3 \), and the general power \( a^t \), of an arbitrary vector base, \( a \), with an arbitrary scalar exponent, \( t \); for we have already assigned interpretations (282, 1.), (14.), and 299, (8.)) for the particular symbols \( a^2, a^1, a^{-1} \), \( a^{-2} \), which are included in this last form. And we shall preserve those particular interpretations if we now define, in full consistency with the principles of the present and preceding Books, that this Power \( a^t \) is generally a Quaternion, which may be decomposed into two factors, of the tensor and versor kinds, as follows:

\[ I \ldots a^t = Ta^t \cdot Ua^t; \]

\( Ta^t \) denoting the arithmetical value of the \( t^{th} \) power of the positive number \( Ta \), which represents (as usual) the length of the base-line \( a \); and \( Ua^t \) denoting a versor, which causes any line \( p \), perpendicular to that line \( a \), to revolve round it as an axis, through \( t \) right angles, or quadrants, and in a positive or negative direction, according as the scalar exponent, \( t \), is itself a positive or negative number (comp. 234, (5.)).
(1.) As regards the omission of parentheses in the formula I., we may observe that the recent definition, or interpretation, of the symbol \( a^t \), enables us to write (comp. 237, II. III.),

\[
\text{II.} \quad T(a^t) = (Ta)^t = Ta^t; \quad \text{III.} \quad U(a^t) = (Ua)^t = Ua^t.
\]

(2.) The axis and angle of the power \( a^t \), considered as a quaternion, are generally determined by the two following formulæ:

\[
\text{IV.} \quad \text{Ax. } a^t = \pm Ua; \quad \text{V.} \quad \angle a^t = 2n\pi \pm \frac{1}{2}t\pi;
\]

the signs accompanying each other, and the (positive or negative or null) integer, \( n \), being so chosen as to bring the angle within the usual limits, 0 and \( \pi \).

(3.) In general (comp. 235), we may speak of the (positive or negative) product, \( \frac{1}{2}t\pi \), as being the amplitude of the same power, with reference to the line \( a \) as an axis of rotation; and may write accordingly,

\[
\text{VI.} \quad \text{am. } a^t = \frac{1}{2}t\pi.
\]

(4.) We may write also (comp. 234, VII. VIII.),

\[
\text{VII.} \quad Ua^t = \cos \frac{t\pi}{2} + Ua \cdot \sin \frac{t\pi}{2}; \quad \text{or briefly, VIII.} \quad Ua^t = \cos \frac{t\pi}{2}.
\]

(5.) In particular,

\[
\text{IX.} \quad Ua^{2n} = \cos n\pi = \pm 1; \quad \text{IX'.} \quad Ua^{2n+1} = \pm Ua;
\]

upper or lower signs being taken, according as the number \( n \) (supposed to be whole) is even or odd. For example, we have thus the cubes,

\[
\text{X.} \quad Ua^3 = - Ua; \quad \text{X'.} \quad a^3 = - aNa.
\]

(6.) The conjugate and norm of the power \( a^t \) may be thus expressed (it being remembered that to turn a line \( \perp a \) through \( -\frac{1}{2}t\pi \) round \( +a \), is equivalent to turning that line through \( +\frac{1}{2}t\pi \) round \( -a \)):

\[
\text{XI.} \quad Ka^t = Ta^t \cdot Ua^t = (-a)^t; \quad \text{XII.} \quad Na^t = Ta^{2t};
\]

parentheses being unnecessary, because (by 295, VIII.) \( K = -a \).

(7.) The scalar, vector, and reciprocal of the same power are given by the formulæ:

\[
\text{XIII.} \quad S. a^t = Ta^t \cdot \cos \frac{t\pi}{2}; \quad \text{XIV.} \quad V. a^t = Ta^t \cdot Ua \cdot \sin \frac{t\pi}{2};
\]

\[
\text{XV.} \quad 1 : a^t = Ta^{-t}. Ua^{-t} = a^{-t} = Ka^t : Na^t \quad \text{(comp. 190, (3.).)}
\]
(8.) If we decompose any vector \( \rho \) into parts \( \rho' \) and \( \rho'' \), which are respectively parallel and perpendicular to \( a \), we have the general transformation:*  

\[
\text{XVI. } a' p a^{-t} = a^t (\rho' + \rho'') a^{-t} = \rho' + U a^t \rho'',
\]

= the new vector obtained by causing \( \rho \) to revolve conically through an angular quantity expressed by \( t \pi \), round the line \( a \) as an axis (comp. 297, (15.)).

(9.) More generally (comp. 191, (5.)), if \( q \) be any quaternion, and if  

\[
\text{XVII. } a' q a^{-t} = q',
\]

the new quaternion \( q' \) is formed from \( q \) by such a conical rotation of its own axis \( Ax.q \), through \( t \pi \), round \( a \), without any change of its angle \( \angle q \), or of its tensor \( Tq \).

(10.) Treating \( ijk \) as three rectangular unit-lines (295), the symbol, or expression,  

\[
\text{XVIII. } \rho = r k^{-s} j x^{s} k^{-t}, \quad \text{or} \quad \text{XIX. } \rho = r k^{t} j x^{-s} k^{-s},
\]
in which  

\[
\text{XX. } r \geq 0, \ s \geq 0, \ s \geq 1, \ t \geq 0, \ t \geq 2,
\]

may represent any vector; the length or tensor of this line \( \rho \) being \( r \); its inclination* to \( k \) being \( s \pi \); and the angle through which the variable plane \( kp \) may be conceived to have revolved, from the initial position \( ki \), with an initial direction towards the position \( kj \), being \( t \pi \).

(11.) In accomplishing the transformation XVI., and in passing from the expression XVIII. to the less symmetric but equivalent expression XIX., we employ the principle that  

\[
\text{XXI. } j k^{s} = S^{-1} 0 = - K (j k^{s}) = j^{s} k;
\]

which easily admits of extension, and may be confirmed by such transformations as VII. or VIII.

(12.) It is scarcely necessary to remark, that the definition or interpretation I., of the power \( a^t \) of any vector \( a \), gives (as in algebra) the exponential property,  

\[
\text{XXII. } a^t a^t = a^{2t},
\]

whatever scalars may be denoted by \( s \) and \( t \); and similarly when there are more than two factors of this form.

* Compare the shortly following sub-article (11.).

† If we conceive (compare the first Note to page 345) that the two lines \( i \) and \( j \) are directed respectively towards the south and west points of the horizon, while the third line \( k \) is directed towards the zenith, then \( s \pi \) is the zenith-distance of \( \rho \); and \( t \pi \) is the azimuth of the same line, measured from south to west, and thence (if necessary) through north and east, to south again.
(13.) As verifications of the expression XVIII., considered as representing a vector, we may observe that it gives,

$$\text{XXIII... } \rho = -K\rho; \quad \text{and} \quad \text{XXIV... } \rho^2 = -r^2.$$ 

(14.) More generally, it will be found that if $u^*$ be any scalar, we have the eminently simple transformation:

$$\text{XXV... } \rho^u = (r k^j s k^i k^j k^i)^u = r^u k^j s k^i k^j k^i.$$ 

In fact, the two last expressions denote generally two equal quaternions, because they have, 1st, equal tensors, each $= ru$; 2nd, equal angles, each $= \angle (k^n)$; and 3rd, equal (or coincident) axes, each formed from $\pm k$ by one common system of two successive rotations, one through $s\pi$ round $j$, and the other through $t\pi$ round $k$.

309. Any quaternion, $q$, which is not simply a scalar, may be brought to the form $a^t$ by a suitable choice of the base, $a$, and of the exponent, $t$; which latter may moreover be supposed to fall between the limits 0 and 2; since for this purpose we have only to write,

$$I... t = \frac{2 \angle q}{\pi}; \quad II... Ta = T_q^\frac{1}{t}; \quad III... Ua = Ax \cdot q;$$

and thus the general dependence of a Quaternion, on a Scalar and a Vector Element, presents itself in a new way (comp. 17, 207, 292). When the proposed quaternion is a versor, $Tq = 1$, we have thus $Ta = 1$; or in other words, the base $a$, of the equivalent power $a^t$, is an unit-line. Conversely, every versor may be considered as a power of an unit-line, with a scalar exponent, $t$, which may be supposed to be in general positive, and less than two; so that we may write generally,

$$IV... Uq = a^t, \quad \text{with} \quad V... a = Ax \cdot q = T^{-1}1,$$

and

$$VI... t > 0, \quad t < 2;$$

although if this versor degenerate into 1 or $-1$, the exponent $t$ becomes 0 or 2, and the base $a$ has an indeterminate or arbitrary direction. And from such transformations of versors new methods may be deduced, for treating questions of spherical trigonometry and generally of spherical geometry.

* The employment of this letter $u$, to denote what we called, in the two preceding sections, a fourth unit, &c., was stated to be a merely temporary one. In general, we shall henceforth simply equate that scalar unit to the number one; and denote it (when necessary to be denoted at all) by the usual symbol, 1, for that number.
(1.) Conceive that \(p, q, r\), in fig. 46 [p. 153] are replaced by \(a, b, c\), with unit-vectors, \(a, \beta, \gamma\) as usual; and let \(x, y, z\) be three scalars between 0 and 2, determined by the three equations,

\[
\begin{align*}
\text{VII.} \quad x\pi &= 2\alpha, \quad y\pi = 2\beta, \quad z\pi = 2\gamma;
\end{align*}
\]

where \(\alpha, \beta, \gamma\) denote the angles of the spherical triangle. The three versors, indicated by the three arrows in the upper part of the figure, come then to be thus denoted:

\[
\begin{align*}
\text{VIII.} \quad q &= a^x; \quad q' = \beta^y; \quad q'q = \gamma^{x+y};
\end{align*}
\]

so that we have the equation,

\[
\begin{align*}
\text{IX.} \quad \beta^x a^x &= \gamma^{x+y}; \quad \text{or} \quad X. \quad \gamma^x \beta^x a^x &= -1;
\end{align*}
\]

from which last, by easy divisions and multiplications, these two others immediately follow:

\[
\begin{align*}
\text{X.} \quad a^x \gamma^x \beta^y &= -1; \quad \text{X.} \quad \beta^y a^x \gamma^x &= -1;
\end{align*}
\]

the rotation round \(a\) from \(\beta\) to \(\gamma\) being again supposed to be negative.

(2.) In \(X\), we may write (by 308, VIII.),

\[
\begin{align*}
\text{XI.} \quad a^x &= \cos \alpha; \quad \beta^y = \cos \beta; \quad \gamma^x = \cos \gamma;
\end{align*}
\]

and then the formula becomes, for any spherical triangle, in which the order of rotation is as above:

\[
\begin{align*}
\text{XII.} \quad \cos \gamma \cos \beta \cos \alpha &= -1;
\end{align*}
\]

or (comp. IX.),

\[
\begin{align*}
\text{XIII.} \quad -\cos c + \gamma \sin c &= \left(\cos b + \beta \sin b\right)\left(\cos a + \alpha \sin a\right).
\end{align*}
\]

(3.) Taking the scalars on both sides of this last equation, and remembering that \(\frac{\beta a}{\gamma} = -\cos c\), we thus immediately derive one form of the fundamental equation of spherical trigonometry; namely, the equation,

\[
\begin{align*}
\text{XIV.} \quad \cos c + \cos \alpha \cos \beta &= \cos c \sin \alpha \sin \beta.
\end{align*}
\]

(4.) Taking the vectors, we have this other formula:

\[
\begin{align*}
\text{XV.} \quad \gamma \sin c &= \alpha \sin \alpha \cos b + \beta \sin b \cos a + \gamma \beta a \sin \alpha \sin b;
\end{align*}
\]

which is easily seen to agree with 306, XII., and may also be usefully compared with the equation 210, XXXVII.

(5.) The result XV. may be enunciated in the form of a Theorem, as follows:

"If there be any spherical triangle \(\alpha \beta \gamma\), and three lines be drawn from the centre \(O\) of the sphere, one towards the point \(\alpha\), with a length = \(\sin \alpha \cos \beta\); another
towards the point $b$, with a length $= \sin b \cos a$; and the third perpendicular to the plane $aob$, and towards the same side of it as the point $c$, with a length $= \sin c \sin a \sin b$; and if, with these three lines as edges, we construct a parallelepiped: the intermediate diagonal from $o$ will be directed towards $c$, and will have a length $= \sin c$.

(6.) Dividing both members of the same equation XV. by $\rho$, and taking scalars, we find that if $p$ be any fourth point on the sphere, and $q$ the foot of the perpendicular let fall from this point on the arc $ab$, this perpendicular $pq$ being considered as positive when $c$ and $p$ are situated at one common side of that arc (or in one common hemisphere, of the two into which the great circle through $a$ and $b$ divides the spheric surface), we have then,

$$\text{XVI. } \sin c \cos p c = \sin a \cos b \cos p a + \sin b \cos a \cos p b + \sin a \sin b \sin p q,$$

a formula which might have been derived from the equation 210, XXXVIII., by first cyclically changing $abc$ to $bcab$, and then passing from the former triangle to its polar or supplementary; and from which many less general equations may be deduced, by assigning particular positions to $p$.

(7.) For example, if we conceive the point $p$ to be the centre of the circumscribed small circle $abc$, and denote by $R$ the arcual radius of that circle, and by $s$ the semisum of the three angles, so that $2s = a + b + c = \pi + \sigma$, if $\sigma$ again denote, as in 297, (47.), the area* of the triangle $abc$, whence

$$\text{XVII. } p a = p b = p c = R, \text{ and } \sin p q = \sin R \sin (s - c),$$

the formula XVI. gives easily,

$$\text{XVIII. } 2 \cot R \sin \frac{\sigma}{2} = \sin a \sin b \sin c,$$

a relation between radius and area, which agrees with known results, and from which we may, by 297, LXX., &c., deduce the known equation:

$$\text{XIX. } e \tan R = 4 \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2};$$

in which we have still, as in 297, (47.), &c.,

$$\text{XX. } e = (8\sigma3\gamma) \sin a \sin b \sin c = &c.$$

(8.) In like manner we might have supposed, in the corresponding general equation 210, XXXVIII., that $p$ was placed at the centre of the inscribed

---

* Compare the Note to the cited sub-article.
small circle, and that the areal radius of that circle was \( r \), the semisum of the sides being \( s \); and thus should have with ease deduced this other known relation, which is a sort of polar reciprocal of XVIII.

\[
XXI. \ldots 2 \tan r \cdot \sin s = e.
\]

But these results are mentioned here, only to exemplify the fertility of the formula, to which the present calculus conducts, and from which the theorem in (5.) was early seen to be a consequence.

(9.) We might develop the ternary product in the equation XII., as we developed the binary product XIII.; compare scalar and vector parts; and operate on the latter, by the symbol \( S \cdot p' \). New general theorems, or at least new general forms, would thus arise, of which it may be sufficient in this place to have merely suggested the investigation.

(10.) As regards the order of rotation (1.) (2.), it is clear, from a mere inspection of the formula XV., that the rotation round \( \gamma \) from \( \beta \) to \( \alpha \), or that round \( c \) from \( b \) to \( a \), must be positive, when that equation XV. holds good; at least if the angles \( \alpha, \beta, \gamma \) of the triangle \( \triangle abc \) be (as usual) treated as positive: because the rotation round the line \( \overline{V\beta a} \) from \( \beta \) to \( a \) is always positive (by 281, (3.).)

(11.) If, then, for any given spherical triangle, \( \triangle abc \), with angles still supposed to be positive, the rotation round \( c \) from \( b \) to \( a \) should happen to be (on the contrary) negative, we should be obliged to modify the formula XV.; which could be done, for example, so as to restore its correctness, by interchanging \( a \) with \( \beta \), and at the same time \( a \) with \( b \).

(12.) There is, however, a sense in which the formula might be considered as still remaining true, without any change in the mode of writing it; namely, if we were to interpret the symbols, \( \alpha, \beta, \gamma \) as denoting negative angles, for the case last supposed (11.). Accordingly, if we take the reciprocal of the equation X., we get this other equation,

\[
XXII. \ldots a^x \beta^y \gamma^z = -1;
\]

where \( x, y, z \) are positive, as before, and therefore the new exponents, \(-x, -y, -z\), are negative, if the rotation round \( \alpha \) from \( \beta \) to \( \gamma \) be itself negative, as in (1.).

(13.) On the whole, then, if \( a, \beta, \gamma \) be any given system of three co-initial and diplanar unit-lines, \( \overline{oa}, \overline{ob}, \overline{oc} \), we can always assign a system of three scalars, \( x, y, z \), which shall satisfy the exponential equation X., and shall have relations of the form VII. to the spherical angles \( \alpha, \beta, \gamma \); but these three scalars, if determined so as to fall between the limits \( \pm 2 \), will be all positive, or all negative,
according as the rotation round $\alpha$ from $\beta$ to $\gamma$ is negative, as in (1.), or positive, as in (11.).

(14.) As regards the limits just mentioned, or the inequalities,

$$XXIII...x < 2, \ y < 2, \ z < 2; \ x > -2, \ y > -2, \ z > -2,$$
they are introduced with a view to render the problem of finding the exponents $xyz$ in the formula $X.$ determinate; for since we have, by 308,

$$XXIV...\alpha^t = \beta^t = \gamma^t = +1,$$
we might otherwise add any multiple (positive or negative) of the number four, to the value of the exponent of any unit-line, and the value of the resulting power would not be altered.

(15.) If we admitted exponents $= \pm 2$, we might render the problem of satisfying the equation $X.$ indeterminate in another way; for it would then be sufficient to suppose that any one of the three exponents was thus equal to $+2$, or $-2$, and that the two others were each $= 0$; or else that all three were of the form $\pm 2$.

(16.) When it was lately said (13.), that the exponents, $x, y, z$, in the formula $X.$, if limited as above, would have one common sign, the case was tacitly excluded, for which those exponents, or some of them, when multiplied each by a quadrant, give angles not equal to those of the spherical triangle $abc$, whether positively or negatively taken; but equal to the supplements of those angles, or to the negatives of those supplements.

(17.) In fact, it is evident (because $\alpha^2 = \beta^2 = \gamma^2 = -1$), that the equation $X.$, or the reciprocal equation $XXII.$, if it be satisfied by any one system of values of $xyz$, will still be satisfied, when we divide or multiply any two of the three exponential factors, by the squares of the two unit-vectors, of which those factors are supposed to be powers: or in other words, if we subtract or add the number two, in each of two exponents.

(18.) We may, for example, derive from $XXII.$ this other equation:

$$XXV...a^{z}b^{z}y^z = -1; \ \text{or} \ XXVI...a^{2z}b^{2z}y^z = \gamma^{2z};$$
which, when the rotation is as supposed in (1.), so that $xyz$ are positive, may be interpreted as follows.

(19.) Conceive a lune $cc'$, with points $a$ and $b$ on its two bounding semi-circles, and with a negative rotation round $\alpha$ from $b$ to $c$; or, what comes to the same thing, with a positive rotation round $\alpha$ from $b$ to $c'$. Then, on the plan illustrated by figures 45 and 46, the supplements $\pi - \alpha, \pi - \beta$, of the
angles \( \alpha \) and \( \beta \) in the triangle \( \triangle ABC \), or the angles at the \textit{same points} \( A \) and \( B \) in the \textit{co-lunar} triangle \( \triangle ABC' \), will \textit{represent} two \textit{versors}, a \textit{multiplier}, and a \textit{multiplicand}, which are precisely those denoted, in XXVI., by the \textit{two factors}, \( \alpha^{2-\alpha} \) and \( \beta^{2-\beta} \); and the \textit{product} of these \textit{two factors}, \textit{taken in this order}, is that \textit{third versor}, which has its \textit{axis} directed to \( C' \), and is \textit{represented}, on the same \textit{general plan} (177), by the \textit{external angle of the lune, at that point,} \( C' \); which, in \textit{quantity}, is equal to the \textit{external angle of the same lune at} \( C \), or to the \textit{angle} \( \pi - c \). This \textit{product} is therefore equal to that \textit{power of the unit-line} \( OC' \), or \( -\gamma \), which has its \textit{exponent} \( \frac{2}{\pi} (\pi - c) = 2 - z \); we have therefore, by this \textit{construction}, the equation,

\[
XXVII. \ldots \alpha^{2-\alpha} \beta^{2-\beta} = (-\gamma)^{2-z};
\]

which (by 308, (6.) ) agrees with the recent formula XXVI.

310. The \textit{equation},

\[
I. \ldots \gamma \beta^n \alpha^m a^n b^m = -1,
\]

which results from 309, (1.), and in which \( \alpha, \beta, \gamma \) are the \textit{unit-vectors} \( OA, OB, OC \) of any three points on the unit-sphere; while the three \textit{scalars} \( \alpha, \beta, c \), in the \textit{exponents} of the \textit{three factors}, represent generally the \textit{angular quantities} of \textit{rotation}, round those three \textit{unit-lines}, or \textit{radii}, \( \alpha, \beta, \gamma \), from the \textit{plane} \( \triangle AOC \) to the \textit{plane} \( \triangle AOB \), from \( \triangle BOC \) to \( \triangle BOA \), and from \( \triangle COB \) to \( \triangle COA \), and are \textit{positive or negative} according as these \textit{rotations} of \textit{planes} are themselves \textit{positive or negative}: must be regarded as an important \textit{formula}, in the \textit{applications} of the \textit{present Calculus}. It \textit{includes}, for example, the \textit{whole} \textit{doctrine of Spherical Triangles}; not merely because it \textit{conducts}, as we have seen (309, (3.)), to \textit{one form} of the \textit{fundamental scalar equation of spherical trigonometry}, namely to the equation,

\[
\Pi. \ldots \cos c + \cos \alpha \cos b = \cos c \sin \alpha \sin b;
\]

but also because it \textit{gives} a \textit{vector equation} (309, (4.)), which \textit{serves} to \textit{connect} the \textit{angles}, or the \textit{rotations}, \( \alpha, \beta, c \), with the \textit{directions*} of the radii, \( \alpha, \beta, \gamma \), or \( OA, OB, OC \), for \textit{any system} of \textit{three diverging right lines} from one \textit{origin}. It

---

* This may be considered to be another instance of that habitual \textit{reference} to \textit{direction}, as \textit{distinguished} from mere \textit{quantity} (or \textit{magnitude}), although \textit{combined} therewith, which \textit{pervades} the \textit{present Calculus}, and is \textit{eminently characteristic} of it; whereas Des Cartes, on the contrary, had \textit{aimed} to \textit{reduce} all \textit{problems} of \textit{geometry} to the \textit{determination} of the \textit{lengths} of \textit{right lines}; although (as all who use his \textit{co-ordinates} are of course \textit{well aware}) a \textit{certain} \textit{reference} to \textit{direction} is \textit{even} in \textit{his} \textit{theory inevitable}, in \textit{connexion} with the \textit{interpretation} of \textit{negative roots} (by \textit{him} called \textit{inverse} or false roots)
may, therefore, be not improper to make here a few additional remarks, respecting the nature, evidence, and extension of the recent formula I.

(1.) Multiplying both members of the equation I., by the inverse exponential $\gamma^{-\frac{a}{\pi}}$, we have the transformation (comp. 309, (1.)):

$$\text{III.} \quad \beta^\pi \alpha^\pi = -\gamma^\pi = \gamma$$

(2.) Again, multiplying both members of I. into $a^\pi$, we obtain this other formula:

$$\text{IV.} \quad \gamma^\pi \beta^\pi = -a^\pi = a^\pi$$

(3.) Multiplying this last equation IV. by $a^\pi$, and the equation III. into $\gamma^\pi$, we derive these other forms:

$$\text{V.} \quad a^\pi \gamma^\pi \beta^\pi = -1; \quad \text{VI.} \quad \beta^\pi a^\pi \gamma^\pi = -1;$$

so that cyclical permutation of the letters, $a, \beta, \gamma$, and $\lambda, \mu, \nu$, is allowed in the equation I.; as indeed was to be expected, from the nature of the theorem which that equation expresses.

(4.) From either V. or VI. we can deduce the formula:

$$\text{VII.} \quad a^\pi \gamma^\pi \beta^\pi = -1; \quad \text{VIII.} \quad a^\pi \beta^\pi \gamma^\pi = -1;$$

by comparing which with III. and IV., we see that cyclical permutation of letters is permitted, in these equations also.

(5.) Taking the reciprocal (or conjugate) of the equation I., we obtain (compare 309, XXII.) this other equation:

$$\text{IX.} \quad a^\pi \beta^\pi \gamma^\pi = +1;$$

of equations. Thus in the first sentence of Schooten's recently cited translation (1659) of the Geometry of Des Cartes, we find it said: "Omnia Geometriae Problemeta facili ad hujusmodi terminos reduci possunt, ut deinde ad illorum constructionem, opus tantum sit rectarum Quarundam longitudinum cognoscere."

The very different view of geometry, to which the present writer has been led, makes it the more proper to express here the profound admiration with which he regards the cited Treatise of Des Cartes, containing as it does the germs of so large a portion of all that has since been done in mathematical science, even as concerns imaginary roots of equations, considered as marks of geometrical impossibility.

* For the distinction between multiplying a quaternion into and by a factor, see the Notes to pages 147, 159.
in which cyclic permutation of letters is again allowed, and from which (or from III.) we can at once derive the formula,

$$X \cdot \frac{2a}{\pi} \cdot \frac{2b}{\pi} \cdot \frac{2c}{\pi} = - \gamma \frac{\pi}{\pi}.$$

(6.) The equation $X$ may also be thus written (comp. 309, XXVII.):

$$\begin{align*}
\text{(XI.)} \quad \alpha \frac{2(\pi - a)}{\pi} \beta \frac{2(\pi - b)}{\pi} \gamma \frac{2(\pi - c)}{\pi} &= (- \gamma) \frac{2(\pi - c)}{\pi}.
\end{align*}$$

(7.) And all the foregoing equations may be interpreted (comp. 309, (19.)), and at the same time proved, by a reference to that general construction (177) for the multiplication of versors, which the figures 45 and 46 were designed to illustrate; if we bear in mind that a power $a^t$ of an unit-line $a$, with a scalar exponent, $t$, is (by 308, 309) a versor, which has the effect of turning a line $\perp a$, through $t$ right angles, round $a$ as an axis of rotation.

(8.) The principle expressed by the equation I., from which all the subsequent equations have been deduced, may be stated in the following manner, if we adopt the definition proposed in an earlier part of this work (180, (4.)), for the spherical sum of two angles on a spheric surface:

"For any spherical triangle, the Spherical Sum of the three angles, if taken in a suitable Order, is equal to Two Right Angles."

(9.) In fact, when the rotation round $a$ from $b$ to $c$ is negative, if we spherically add the angle $b$ to the angle $a$, the spherical sum so obtained is (by the definition referred to) equal to the external angle at $c$; if then we add to this sum, or supplement of $c$, the angle $c$ itself, we get a final or total sum, which is exactly equal to $\pi$; addition of spherical angles at one vertex, and therefore in one plane, being accomplished in the usual manner; but the spherical summation of angles with different vertices being performed according to those new rules, which were deduced in the Ninth Section of Book II., Chapter I.; and were connected (180, (5.)) with the conception of angular transvection, or of the composition of angular motions, in different and successive planes.

(10.) Without pretending to attach importance to the following notation, we may just propose it in passing, as one which may serve to recall and represent the conception here referred to. Using a plus in parentheses, as a symbol or characteristic of such spherical addition of angles, the formula I. may be abridged as follows:

$$\begin{align*}
\text{(XII.)} \quad c (+) b (+) a = \pi;
\end{align*}$$
the symbol of an added angle being written to the left of the symbol of the angle to which it is added (comp. 264, (4.)); because such addition corresponds (as above) to a multiplication of versors, and we have agreed to write the symbol of the multiplier to the left* of the symbol of the multiplicand, in every multiplication of quaternions.

311. There is, however, another view of the important equation 310, I., according to which it is connected rather with addition of arcs (180, (3.)), than with addition of angles (180, (4.)); and may be interpreted, and proved anew, with the help of the supplementary or polar triangle, $A'b'c'$, as follows.

(1.) The rotation round $A$ from $b$ to $c$ being still supposed to be negative, let $A'$, $b'$, $c'$ be (as in 175) the positive poles of the sides $bc$, $ca$, $ab$; and let $a'$, $b'$, $c'$ be their unit-vectors. Then, because the rotation round $A$ from $c'$ to $b'$ is positive (by 180, (2.)), and is in quantity the supplement of the spherical angle $A$, the product $c'b'$ will be (by 281, (2.), (3.)) a versor, of which $a$ is the axis, and $A$ the angle; with similar results for the two other products, $a'c'$, $b'a'$.

(2.) If then we write (comp. 291),

$$I. \quad a' = UV \beta \gamma, \quad \beta' = UV \gamma a, \quad \gamma' = UV \alpha \beta,$$

supposing that

$$II. \quad Ta = T\beta = T\gamma = 1, \quad \text{and} \quad III. \quad S\alpha \beta \gamma > 0$$

we shall have (comp. again 180, (2.)),

$$IV. \quad a = UV \gamma' \beta', \quad \beta = UV a' \gamma', \quad \gamma = UV \beta' a', \dagger$$

and

$$V. \quad A = \angle \gamma' \beta', \quad B = \angle a' \gamma', \quad C = \angle \beta' a';$$

whence (by 308 or 309) we have the following exponential expressions for these three last products of unit-lines.

$$VI. \quad \gamma' \beta' = a^\alpha; \quad a' \gamma' = \beta^\alpha; \quad \beta' a' = \gamma^\alpha.$$

(3.) Multiplying these three expressions, in an inverted order, we have, therefore, the new product:

$$VII. \quad \gamma^\alpha \beta^\alpha a^\alpha = \beta' a'. a' \gamma'. \gamma' \beta' = \gamma' \beta^2 a^2 = 1;$$

and the equation 310, I. is in this way proved anew.

* Compare the Note to page 147.
† [Here $UV \beta' \gamma' = UVV \gamma a \alpha \beta = U(-a\alpha \beta \gamma) = -aUSa \beta \gamma.]$
(4.) And because, instead of VI., we might have written,

\[ \frac{2a}{2b} = -\frac{\gamma'}{\beta'}; \quad \beta = -\frac{a'}{\gamma}; \quad \frac{2a}{2b} = -\frac{\beta'}{a'}, \]

we see that the equation to be proved may be reduced to the form of the identity

\[ \frac{\beta'}{a'} \frac{a'}{\gamma'} \frac{\gamma'}{\beta'} = +1; \]

and may be interpreted as expressing, what is evident, that if a point be supposed to move first along the side \( b'c' \), of the polar triangle \( \Delta'b'c' \), from \( b' \) to \( c' \); then along the successive side \( c'a' \), from \( c' \) to \( a' \); and finally along the remaining side \( a'b' \), from \( a' \) to \( b' \), it will thus have returned to the position from which it set out, or will on the whole have not changed place at all.

(5.) In this view, then, we perform what we have elsewhere called an addition of arcs (instead of angles as in 310); and in a notation already used (264, (4.)), we may express the result by the formula,

\[ \ldots \wedge a'b' + \wedge c'a' + \wedge b'c' = 0; \]

each of the two left-handed symbols denoting an arc, which is conceived to be added (as a successive vector-arc, 180, (3.)), to the arc whose symbol immediately follows it, or is written next it, but towards the right-hand.

(6.) The expressions VI. or VIII., for the exponential factors in 310, I., show in a new way the necessity of attending to the order of those factors, in that formula: for if we should invert that order, without altering (as in 310, VIII.) the exponents, we may now see that we should obtain this new product:

\[ \ldots a'^2 b'^2 = -\frac{\gamma'}{\beta'} \frac{a'}{\gamma} \frac{\beta'}{a'} = + (\gamma' \beta' a')^2; \]

which, on account of the diplanarity of the lines \( a', \beta', \gamma' \), is not equal to negative unity, but to a certain other versor; the properties of which may be inferred from what was shown in 297, (64.), and in 298, (8.), but upon which we cannot here delay.

312. In general (comp. 221), an equation, such as

\[ \ldots q' = q, \]

between two quaternions, includes a system of four* scalar equations, such as the following:

\[ \ldots Sq' = Sq; \quad Sqq' = Sqq; \quad Sbq' = Sbq; \quad S\gamma q' = S\gamma q; \]

* The propriety, which such results as this establish, for the use of the name, Quaternions, as applied to this whole Calculus, on account of its essential connexion with the number Four, does not require to be again insisted on.
where \(a, \beta, \gamma\) may be any three actual and diplanar vectors: and conversely, if \(a, \beta, \gamma\) be any three such vectors, then the four scalar equations II. reproduce, and are sufficiently replaced by, the one quaternion equation I. But an equation between two vectors is equivalent only to a system of three scalar equations, such as the three last equations II.; for example, in 294, (12.), the one vector equation XXII. is equivalent to the three scalar equations XXI., under the immediately preceding condition of diplanarity XX. In like manner, an equation between two versors of quaternions, such as the equation

\[ \text{III. } Uq' = Uq, \]

includes generally a system of three, but of not more than three, scalar equations; because the versor \(Uq\) depends generally (comp. 157) on a system of three scalars, namely the two which determine its axis \(Ax\), \(q\), and the one which determines its angle \(\angle q\); or because the versor equation III. requires to be combined with the tensor equation,

\[ \text{IV. } Tq' = Tq, \]

in order to reproduce the quaternion equation I. Now the recent equation, 310, I., is evidently of this versor-form III., if \(a, \beta, \gamma\) be still supposed to be unit-lines. If then we met that equation, or if one of its form had occurred to us, without any knowledge of its geometrical signification, we might propose to resolve it, with respect to the three scalars \(a, b, c\), treated as three unknown quantities. The few following remarks, on the problem thus proposed, may be not out of place, nor uninstructive, here.

(1.) Writing for abridgment,

\[ \text{V. } \cot a = t, \quad \cot b = u, \quad \cot c = v, \]

and

\[ \text{VI. } s = - \cosec a \cosec b \cosec c, \]

the equation to be resolved becomes (by 308, VII., or 309, XII.),

\[ \text{VII. } (v + \gamma)(u + \beta)(t + a) = s; \]

in which the tensors on both sides are already equal, because

\[ \text{VIII. } s^3 = (v^2 + 1)(u^2 + 1)(t^2 + 1). \]

* An equation, \(U\rho' = U\rho\), or \(UVq' = UVq\), between two versors of vectors (156), or between the axes of two quaternions (291), is equivalent only to a system of two scalar equations; because the direction of an axis, or of a vector, depends on a system of two angular elements (111).
(2.) Multiplying the equation VII. by \( t + a \), and into \( t - a \), and dividing the result by \( t^2 + 1 \), we have this new equation of the same form, but differing by cyclical permutation (comp. 310, (3.)):

\[
IX. \ldots (t + a)(v + \gamma)(u + \beta) = s;
\]
and in like manner, \n
\[
X. \ldots (u + \beta)(t + a)(v + \gamma) = s.
\]

(3.) Taking the half difference of the two last equations, and observing that (by 279, IV., and 294, II.)

\[
\begin{align*}
\langle \frac{1}{2}(\beta a \gamma - a \gamma \beta) &= V. \beta V a \gamma = \gamma S a \beta - a S \beta \gamma, \\
\langle \frac{1}{2}(\beta a - a \beta) &= V. \beta a, \quad \langle \frac{1}{2}(\beta \gamma - \gamma \beta) = V. \beta \gamma,
\end{align*}
\]

we arrive at this new equation, of vector form:

\[
XII. \ldots 0 = vV. \beta a + tV. \beta \gamma + \gamma S a \beta - a S \beta \gamma;
\]
which is equivalent only to a system of two scalar equations, because it gives \( 0 = 0 \), when operated on by \( S. \beta \) (comp. 294, (9.)).

(4.) It enables us, however, to determine the two scalars, \( t \) and \( v \); for if we operate on it by \( S. a \), we get (comp. 298, XXVI.),

\[
XIII. \ldots t S a \beta \gamma = a^2 S \beta \gamma - S \beta a S a \gamma = S (V. \beta a . V a \gamma);
\]
and if we operate on the same equation XII. by \( S. \gamma \), we get in like manner,

\[
XIV. \ldots v S a \beta \gamma = \gamma^2 S a \beta - S a \gamma S \gamma \beta = S (V a \gamma . V \gamma \beta).
\]

(5.) Processes quite similar give the analogous result,

\[
XV. \ldots u S a \beta \gamma = \beta^2 S \gamma a - S \gamma \beta S \beta a = S (V \gamma \beta . V \beta a);
\]
and thus the problem is resolved, in the sense that expressions have been found for the three sought scalars, \( t, u, v \), or for the cotangents \( V. \) of the three sought angles \( A, B, C \); whence the fourth scalar, \( s \), in the quaternion equation VII., can easily be deduced, as follows.

(6.) Since (by 294, (6.), changing \( \delta \) to \( a \), and afterwards cyclically permuting) we have, for any three vectors \( a, \beta, \gamma \), the general transformations,

\[
\begin{align*}
\langle a S a \beta \gamma &= V (V. \beta a . V a \gamma), \\
\langle \beta S a \beta \gamma &= V (V \gamma \beta . V \beta a), \\
\langle \gamma S a \gamma &= V (V a \gamma . V \gamma \beta),
\end{align*}
\]
the expressions XIII. XV. XIV. give,

\[ \begin{align*}
(t + \alpha) \beta \gamma &= \sqrt{\beta a} \cdot V\beta a; \\
(u + \beta) \beta \gamma &= \sqrt{\gamma \beta} \cdot V\beta a; \\
(v + \gamma) \beta \gamma &= V\alpha \gamma \cdot V\gamma \beta;
\end{align*} \]

whence, by VII.,

\[ (S\beta \gamma)^3 = (V\gamma \beta)^2 (V\beta a)^2 (V\alpha \gamma)^2; \]

and thus the remaining scalar, \( s \), is also entirely determined.

(7.) And the equation VIII. may be verified, by observing that the expressions XVII. give,

\[ (t^2 + 1) (S\beta \gamma)^2 = (V\gamma \beta)^2 (V\beta a)^2; \]

\[ (u^2 + 1) (S\beta \gamma)^2 = (V\gamma \beta)^2 (V\beta a)^2; \]

\[ (v^2 + 1) (S\beta \gamma)^2 = (V\alpha \gamma)^2 (V\gamma \beta)^2. \]

(8.) The equations XIII. XIV. XV. XVI. give, by elimination of \( S\beta \gamma \), these new expressions:

\[ \begin{align*}
(\ell + 1) (S\beta \gamma)^3 &= (V\gamma \beta)^2 (V\beta a)^2; \\
(u^2 + 1) (S\beta \gamma)^3 &= (V\gamma \beta)^2 (V\beta a)^2; \\
(v^2 + 1) (S\beta \gamma)^3 &= (V\alpha \gamma)^2 (V\gamma \beta)^2.
\end{align*} \]

by comparing which with the formula 281, XXVIII., after suppressing (291) the characteristic I., we find that the three scalars, \( t, u, v \), are either 1st, the cotangents of the angles opposite to the sides \( a, b, c \), of the spherical triangle in which the three given unit-lines \( a, \beta, \gamma \) terminate, or IIInd, the negatives of those cotangents, the angles themselves of that triangle being as usual supposed to be positive (309, (10.)), according as the rotation round \( a \) from \( \beta \) to \( \gamma \) is negative or positive; that is (294, (3.)), according as \( S\beta \gamma > 0 \) or \( < 0 \); or finally, by XVIII., according as the fourth scalar, \( s \), is negative or positive, because the second member of that equation XVIII. is always negative, as being the product of three squares of vectors (282, 292).

(9.) In the 1st case, which is that of 309, (1.), we see then anew, by V. and VI., that we are permitted to interpret the scalars \( \lambda, \mu, \nu \), in the exponential formula 310, I., as equal to the angles of the spherical triangle (8.), which are usually denoted by the same letters. But we see also, that we may add any even multiples of \( \pi \) to those three angles, without disturbing the exponential equation; or any one even, and two odd multiples of \( \pi \), in any order, so as to preserve a positive product of cosecants, because \( s \) is, for this case, negative in VI., by (8.).
(10.) In the IIInd case, which is that of 309, (11.), we may, for similar reasons, interpret the scalars \(a, b, c\) in the formula 310, I., as equal to the negatives of the angles of the triangle; and as thus having, what VI. now requires, because \(s\) is now positive (8.), a negative product of cosecants, while their cotangents have the values required. But we may also add, as in (9.), any multiples of \(\pi\), to the scalars thus found for the formula, provided that the number of the odd multiples, so added, is itself even (0 or 2).

(11.) The conclusions of 309, or 310, respecting the interpretation of the exponential formula, are therefore confirmed, and might have been anticipated, by the present new analysis: in conducting which it is evident that we have been dealing with real scalars, and with real vectors, only.

(12.) If this last restriction were removed, and imaginary values admitted, in the solution of the quaternion equation VII., we might have begun by operating, as in II., on that equation, by the four characteristics,

\[
XXI. \ldots S, S \cdot a, S \cdot \beta, \text{ and } S \cdot \gamma;
\]

which would have given, with the significations 297, (1.), (3.), of \(i, m, n, \) and \(e\), and therefore with the following relation between those four scalar data,

\[
XXII. \ldots e^2 = 1 - t^2 - m^2 - n^2 + 2imn,
\]

a system of four scalar equations, involving the four sought scalars, \(s, t, u, v\); from which it might have been required to deduce the (real or imaginary) values of those four scalars, by the ordinary processes of algebra.

(13.) The four scalar equations, so obtained, are the following:

\[
XXIII. \quad \begin{cases} 
0 = e + t + mu + nv - tuv + s; \\
0 = et + mtu + ntv + uv - l; \\
0 = -eu + Itu + tv + nuv + m - 2ln; \\
0 = ev + tu + Itv + muv - n;
\end{cases}
\]

eliminating \(uv\) and \(u\) between the three last of which, we find, with the help of XXII., the determinant,

\[
XXIV. \quad 0 = \begin{vmatrix} 
1, m, ntv + et - l \\
m, t, ltv + ev - n \\
n, ll - e, tv + m - 2ln
\end{vmatrix} = e (t^2 + 1)(ev - n + lm);
\]

and analogous eliminations give,

\[
XXV. \quad 0 = e (t^2 + 1) (eu - m + nl),
\]

and \(XXVI. \quad 0 = (t^2 + 1) \{e^2 uv - (m - nl) (n - im) + (1 - t^2) (et - l + mn)\}.
\)
(14.) Rejecting then the factor \( t^2 + 1 \) we find, as the only real solution of
the problem (12.), the following system of values:

\[
XXVII. \quad et = l - mn; \quad ev = m - nl; \quad ev = n - im;
\]

and

\[
XXVIII. \quad e^2 s = -(1 - t^2)(1 - m^2)(1 - n^2);
\]

which correspond precisely to those otherwise found before, in (4.), (5.), (6.),
and might therefore serve to reproduce the interpretation of the exponential
formula (310).

(15.) But on the purely algebraic side, it is found, by a similar analysis,
that the four equations XXIII. are satisfied also by a system of four imaginary
solutions, represented by the following formulse:

\[
XXIX. \begin{cases}
  t^2 + 1 = 0; \\
  v^2 + 1 = 0; \\
  s = tuv - lt - mu - nv - e = 0;
\end{cases}
\]

which it may be sufficient to have mentioned in passing, since they do not
appear to have any such geometrical interest, as to deserve to be dwelt on here;
though, as regards the consistency of the different processes employed, it may
be remembered that in passing (2.) from the equation VII. to IX., after
certain preliminary multiplications, we divided by \( t^2 + 1 \), as we were entitled
to do, when seeking only for real solutions, because \( t \) was supposed to be a
scalar.

(16.) This seems to be a natural occasion for remarking that the following
general transformation exists, whatever three vectors may be denoted by \( \alpha, \beta, \gamma \):

\[
XXX. \quad S(V\beta\gamma . V\gamma\alpha . V\alpha\beta) = -(S\alpha\beta\gamma)^2;
\]

which proves in a new way (comp. 180), that the rotation round the line \( V\beta\gamma \),
from \( V\gamma\alpha \) to \( V\alpha\beta \), is always positive; or is directed in the same sense (281, (3.)),
as the rotation round \( V\alpha\beta \) from \( a \) to \( \beta \), &c.

(17.) In like manner we have generally,

\[
XXXI. \quad S(V\alpha\beta . V\gamma\alpha . V\beta\gamma) = + (S\alpha\beta\gamma)^2;
\]

and

\[
XXXII. \quad S(V\gamma\beta . V\alpha\gamma . V\beta\alpha) = + (S\alpha\beta\gamma)^2;
\]

so that the rotation round \( V\gamma\beta \) from \( V\gamma\alpha \) to \( V\beta\alpha \) is negative, whatever arrangement
the three diplanar vectors \( \alpha, \beta, \gamma \) may have among themselves.

(18.) If then \( \Lambda'' \), \( \beta'' \), \( \gamma'' \) be the negative poles of the three successive sides,
\( bc, ca, ab \), of any spherical triangle, the rotation round \( \Lambda'' \) from \( \beta'' \) to \( \gamma'' \) is
negative: which is entirely consistent with the opposite result (180), respecting
the system of the three positive poles \( A', B', C' \).

(19.) A quantitative interpretation of the equation XXX. may also be easily
assigned: for we may infer from it (by 281, (4.), and 294, (3.) ) that if \( OABC \)
be any pyramid, and if normals \( OA', OB', OC' \) to the three faces \( BOC, COA, AOB \) have
their lengths numerically equal to the areas of those faces (as bearing the same
ratios to units, &c.), then (with a similar reference to units) the volume of the
new pyramid, \( OA'B'C' \), will be three quarters of the square of the volume of the old
pyramid, \( OABC \).

313. But an allusion was made, in 310, to an extension of the exponential
formula which has lately been under discussion; and in fact, that formula
admits of being easily extended, from triangles to polygons upon the sphere:
for we may write, generally,

\[
I. \ldots a_n^{-\frac{2a_n}{n}} a_{n-1}^{-\frac{2a_{n-1}}{n}} \ldots a_2^{-\frac{2a_2}{n}} a_1^{-\frac{2a_1}{n}} = (-1)^n,
\]

if \( A_1A_2 \ldots A_{n-1}A_n \) be any spherical polygon, and if the scalars \( A_1, A_2, \ldots \) in the
exponents denote the positive or negative angles of that polygon, considered as
the rotations \( A_nA_1A_2, A_1A_2A_3, \ldots \) namely those from \( A_1A_n \) to \( A_1A_2, \&c. \); while \( n \)
is any positive whole number* \( n > 2 \).

(1.) One mode of proving this extended formula is the following. Let
\( O\gamma = \gamma \) be the unit-vector of an arbitrary point \( c \) on the spheric surface; and
conceive that arcs of great circles are drawn from this point \( c \) to the \( n \) suc­
cessive corners of the polygon. We shall thus have a system of \( n \) spherical
triangles, and each angle of the polygon will (generally) be decomposed into
two (positive or negative) partial angles, which may be thus denoted:

\[
II. \ldots CA_1A_2 = A'_1, \quad CA_2A_3 = A'_2, \ldots ;
\]

\[
III. \ldots A_nA_1C = A''_1, \quad A_1A_2C = A''_2, \ldots ;
\]

so that, with attention to signs of angles in the additions,

\[
IV. \ldots A_1 = A'_1 + A''_1, \quad A_2 = A'_2 + A''_2, \&c.
\]

Also let

\[
V. \ldots A_2CA_1 = c_1, \quad A_3CA_2 = c_2, \&c. ;
\]

and therefore

\[
VI. \ldots c_1 + c_2 + \ldots + c_n = \text{an even multiple of } \pi,
\]

which reduces itself to \( 2\pi \) in the simple case of a polygon with no re-entrant
angles, and with the point \( c \) in its interior.

* The formula admits of interpretation, even for the case \( n = 2 \).
Then, for the triangle $\Delta CA_1A_2$, of which the angles are $c_1, \alpha', \alpha''_2$, we have, by 310, III., the equation,

$$\frac{2\alpha''_2}{2a_2} \frac{2\alpha'}{2a_1} = -\gamma$$

and in like manner, for the triangle $\Delta CA_2A_3$, we have

$$\frac{2\alpha''_3}{2a_3} \frac{2\alpha'}{2a_1} = -\gamma$$

But, when we multiply VII. by VIII., we obtain, by IV., the product,

$$\frac{2\alpha''_3}{2a_3} \frac{2\alpha''_2}{2a_2} \frac{2\alpha'}{2a_1} \frac{2(c_1 + c_2)}{2(c_1 + c_2)} = +\gamma$$

and so proceeding, we have at last, by VI., a product of the form,

$$a_1^\pi a_2^\pi \ldots a_n^\pi = (-1)^n$$

which reduces itself to I., when it is multiplied by $a^\pi$, and into $a^\pi$ (comp. 310, (3.)). The theorem is therefore proved.

(3.) In words (comp. 310, (8.)), "the spherical sum of the successive angles of any spherical polygon, if taken in a suitable order, is equal to a multiple of two right angles, which is odd or even, according as the number of the sides (or corners) of the polygon is itself odd or even": the definition formerly given (180, (4.)), of a Spherical Sum of Angles, being of course retained. And the reasoning may be briefly stated thus. When an arbitrary point $c$ is taken on the spherical surface, as in (1.), the spherical sum of the two partial angles, at the ends of any one side, is the supplement of the angle which that side subtends, at the point $c$; but the sum of all such subtended angles is either four right angles, or some whole multiple thereof: therefore the sum of their supplements can differ only by some such multiple from $n\pi$, if $n$ be the number of the sides.

(4.) Whatever that number may be, if we denote by $p_n$ the exponential product in the formula I., we have for every vector $\rho$, and for every quaternion $q$, the equations:

$$p_n \rho p_n^{-1} = \rho; \quad p_n q p_n^{-1} = q;$$

whereof the former may (by 308, (8.)) be thus interpreted:

"If any line $op$, drawn from the centre $o$ of a sphere, be made to revolve conically round any $n$ radii, $OA_1, \ldots OA_n$, as $n$ successive axes of rotation, through
angles equal respectively to the doubles of the angles of the spherical polygon \( A_1 \ldots A_n \), the line will be brought back to its initial position, by the composition of these \( n \) rotations."

(5.) Another way of proving the extended formula I., for any spherical polygon, is analogous to that which was employed in 311 for the case of a triangle on a sphere, and may be stated as follows. Let \( \lambda'_1, \lambda'_2, \ldots \lambda'_n \) be the positive poles of the arcs \( A_1A_2, A_2A_3, \ldots A_nA_1 \); and let \( a'_1, a'_2, \ldots a'_n \) be the unit-vectors of those \( n \) poles. Then the point \( A_1 \) is the positive pole of the new arc \( \lambda'_1\lambda'_n \), and the angle \( \lambda_1 \) at the polygon at that point is measured by the supplement of that arc; with similar results for other corners of the polygon. Thus we have the system of expressions (comp. 311, VI.):

\[
\frac{2\lambda_1}{a_1} = a'_1a'_n; \quad \frac{2\lambda_n}{a_n} = a'_n a'_{n-1};
\]

so that the product of powers in I. is equal to the following product of \( n \) squares of unit-lines, and therefore to the \( n^\text{th} \) power of negative unity.

\[
\frac{a''_n a'_{n-1} a'_{n-2} \ldots a'_1 a'_n}{a_1 a_2 \ldots a_n} = (-1)^n;
\]

and thus the extended theorem is proved anew.

(6.) This latter process may be translated into another theorem of rotation, on which it is possible that we may briefly return,\(^*\) in the Second and last Chapter of this Third Book, but upon which we cannot here delay.

(7.) It may be remarked however here (comp. 309, XII.), that the extended exponential formula I. may be thus written:

\[
\text{XV.} \quad ca_n S \lambda_n \cdot ca_{n-1} S \lambda_{n-1} \ldots ca_2 S \lambda_2 \cdot ca_1 S \lambda_1 = (-1)^n.
\]

(8.) For example, if \( abcd \) be any spherical quadrilateral, of which the angles (suitably measured) are denoted by \( \alpha, \beta, \gamma, \delta \), so that \( \alpha \) represents the positive or negative rotation from \( ad \) to \( ab \), &c., while \( a, \beta, \gamma, \delta \) are the unit vectors of its corners, then

\[
\text{XVI.} \quad \cos c \sin d \cdot 
\]

and therefore, by taking scalars on both sides, and changing signs,

\[
\text{XVIII.} \quad - c \cos d + \sin c \sin d \cos cd = - c \cos b \cos a + \sin b \sin a \cos ba;
\]

* Compare 297, (24.).
in fact, each member of this last formula is equal (by 309, XIV.) to the cosine of the angle \( \text{ARB} \) or \( \text{CED} \), if the opposite sides \( \text{AD} \), \( \text{BC} \) of the quadrilateral intersect in \( \text{E} \).

(10.) Let \( \rho = \text{op} \) be the unit vector of any fifth point, \( p \), upon the spheric surface; then operating by \( S \cdot \rho \) on XVII., we obtain this other general formula,

\[
0 = \sin A \cos B \cos AP + \sin B \cos A \cos BP + \sin A \sin B \sin AB \sin PQ
\]
\[
+ \sin C \cos D \cos CP + \sin D \cos C \cos DP + \sin C \sin D \sin CD \sin PR;
\]

in which the sines of the sides \( \text{AB}, \text{CD} \) are treated as always positive; but the sines of the perpendiculars \( \text{PQ} \) and \( \text{PR} \), on those two sides, are regarded as positive or negative, according as the rotations round \( p \), from \( \text{A} \) to \( \text{B} \) and from \( \text{C} \) to \( \text{D} \), are negative or positive: and hence, by assigning particular positions to \( p \), several other but less general equations of spherical tetragonometry can be derived.

(11.) For example, if we place \( p \) at the intersection, say \( f \), of the opposite sides \( \text{AB}, \text{CD} \), the two last perpendiculars will vanish, and two of the six terms will disappear, from the general formula XIX.; and a similar reduction to four terms will occur, if we make the arbitrary point \( p \) the pole of a side, or of a diagonal.

314. The definition of the power \( a' \), which was assigned in 308, enables us to form some useful expressions, by quaternions, for circular, elliptic, and spiral loci, in a given plane, or in space, a few of which may be mentioned here.

(1.) Let \( a \) be any given unit-vector \( \text{oa} \), and \( \beta \) any other given line \( \text{ob} \), perpendicular to it; then, by the definition (308), if we write,

\[
\text{I... op} = p = a'\beta, \quad Ta = 1, \quad Sa\beta = 0,
\]

the locus of the point \( p \) will be the circumference of a circle, with \( o \) for centre, and \( \text{ob} \) for radius, and in a plane perpendicular to \( \text{oa} \).

(2.) If we retain the condition \( Ta = 1 \), but not the condition \( Sa\beta = 0 \), then the product \( a'\beta \) will be in general a quaternion, and not merely a vector; but if we take its vector-part (292), we can form this new vector-expression,

\[
\text{II... op} = p = V. a'\beta = \beta \cos x + \gamma \sin x,
\]

where

\[
\text{III... } 2x = t\pi, \quad \text{and IV... } \gamma = oc = V_{a'\beta};
\]

and now the locus of \( p \) is a plane ellipse, with its centre at \( o \), and with \( \text{ob} \) and \( \text{oc} \) for its major and minor semiaxes: while the angular quantity, \( x \), is what is often called the eccentric anomaly.

Hamilton’s Elements of Quaternions.
(3.) If we write, under the same conditions (2.),

\[ V \ldots \mathbf{o} \mathbf{b}' = \mathbf{b}' = V \mathbf{b} a : a = a' \gamma, \quad \text{and} \quad VI. \ldots \mathbf{a} \mathbf{p}' = \mathbf{p}' = V \mathbf{p} a : a = a' \gamma, \]

so that \( b' \) and \( p' \) are the projections (203) of \( b \) and \( p \) on a plane drawn through \( o \), at right angles to the unit-line \( o a \), we have then, by II., the equation,

\[ VII. \ldots \mathbf{p}' = \mathbf{b}' \cos \gamma + \gamma \sin \gamma = a' \beta'; \]

so that the **locus of this projected point** \( p' \) is a circle, with \( o b' \) and \( o c \) for two rectangular radii.

(4.) Under the same conditions, the **elliptic locus** (2.), of the point \( p \) itself, is the section of the right cylinder (compare 203, (5.)),

\[ VIII. \ldots TVa \beta = TVa \beta = T \gamma, \]

made by the plane,

\[ IX. \ldots 0 = S \gamma \beta \rho, \quad \text{or} \quad IX'. \ldots \beta S \alpha \beta = S \alpha \beta S \beta \rho \quad \text{comp. 298, XXVI.}; \]

as a confirmation of which last form we have, by II. and IV.,

\[ X. \ldots S \alpha \beta = S \alpha \beta \cos \gamma, \quad S \beta \rho = \beta \cos \gamma. \]

(5.) If we retain the condition \( S \alpha \beta = 0 \) (1.), but not now the condition \( T a = 1 \), we may again write the equation I. for \( p \); but the **locus of** \( p \) will now be a **logarithmic spiral**, with \( o \) for its pole, in the plane perpendicular to \( o a \); because **equal angular motions**, of the **turning line** \( o \mathbf{p} \), correspond now to **equal multiplications** of the length of that line \( p \).

(6.) For example, when the scalar exponent \( t \) is increased by 4, so that the **revolving unit line**, 

\[ XI. \ldots U \mathbf{r} = U \alpha U \beta \]

returns (comp. 309, XXIV.) to the direction which it had before the increase of \( t \) was made, the **length** \( T \rho \) of the **turning line** \( \rho \) itself, or of the **radius vector** of the locus, is **multiplied** by \( T a^4 \); which constant and positive scalar is not now equal to unity.

(7.) If we reject both the conditions (1.),

\[ Ta = 1, \quad \text{and} \quad S \alpha \beta = 0, \]

so that the **line** \( a \), or the **base** of the **power** \( a^i \), is now **neither an unit-line**, nor **perpendicular** to \( \beta \), namely to the line on which that power operates, as a factor, we must again take **vector parts**, but we have now this **new expression**:

\[ XII. \ldots \mathbf{o} \mathbf{p} = \rho = V. \alpha \beta = a \beta (\beta \cos \gamma + \gamma \sin \gamma); \]

in which we have written, for abridgment,

\[ XIII. \ldots a = Ta, \quad \gamma = V (Ua \beta). \]
(8.) In this more complex case, the locus of \( p \) is still a plane curve, and may be said to be now an elliptic* logarithmic spiral; for if we suppress the scalar factor, \( a^t \), we fall back on the form II., and have again an ellipse as the locus: but when we take account of that factor, we find (comp. (2.)) that equal increments of eccentric anomaly \( (x) \), in the auxiliary ellipse so determined, correspond to equal multiplications of the length \( (T_p) \), of the vector of the new spiral.

(9.) We may also project \( b \) and \( p \), as in (3.), into points \( b' \) and \( p' \), on the plane through \( o \) perpendicular to \( oA \), which plane still contains the extremity \( c \) of the auxiliary vector \( \gamma \); and then, since it is easily proved that \( \gamma = Ua \cdot \beta' \), the equation of the projected spiral becomes (with \( Ta > or < 1 \)),

\[
\text{XIV.}\ldots p' = a^t (\beta' \cos x + \gamma \sin x) = a^t \beta';
\]

so that we are brought back to the case (5.), and the projected curve is seen to be a logarithmic spiral, of the known and ordinary kind.

(10.) Several spirals of double curvature are easily represented, on the same general plan, by merely introducing a vector-term proportional to \( t \), combined or not with a constant vector-term, in each of the expressions above given, for the variable vector \( \rho \). For example, the equation,

\[
\text{XV.}\ldots \rho = cta + a^t \beta, \quad \text{with} \quad Ta = 1, \quad \text{and} \quad Sa\beta = 0,
\]

while \( c \) is any constant scalar different from zero, represents a helix, on the right circular cylinder VIII.

(11.) And if we introduce a new and variable scalar, \( u \), as a factor in the right-hand term, and so write,

\[
\text{XVI.}\ldots \rho = cta + ua^t \beta,
\]

we shall have an expression for a variable vector \( \rho \), considered as depending on two variable scalars \( (t \) and \( u) \), which thus becomes (99) the expression for a vector of a surface: namely of that important Screw Surface, which is the locus of the perpendiculars, let fall from the various points of a given helix, on the axis of the cylinder of revolution, on which that helix, or spiral curve, is traced.

* The usual logarithmic spiral might perhaps be called, by contrast to this one, a circular logarithmic spiral. Compare the following sub-article (9.), respecting the projection of what is here called an elliptic logarithmic spiral.
315. Without at present pursuing farther the study of these loci by quaternions, it may be remarked that the definition (308) of the power $a^t$, especially for the case when $T_a = 1$, combined with the laws (182) of $i, j, k$, and with the identification (295) of those three important right versors with their own indices, enables us to establish the following among other transformations, which will be found useful on several occasions.

(1.) Let $a$ be any unit-vector, and let $t$ be any scalar; then,

I. $S . a^t = S . a^t$;  
II. $S . a^{t-1} = S . a^{t+1} = - S . a^{t+1}$;

III. $a^t = S . a^t + aS . a^{t-1}$;  
IV. $a^t = S . a^t - aS . a^{t-1}$;

V. $(S . a^t)^2 + (S . a^{t-1})^2 = a^t a^{t-1} = 1$.

(2.) Let $a$ and $i$ be any two unit-vectors, and let $t$ be still any scalar; then

VI. $S . a^t = S . i^t$;  
VII. $a^t = aS . a^{t-1}$;

VIII. $aV . a^t = a^2 S . a^{t-1} = S . a^{t+1}$.

(3.) Hence, by the laws of $i, j, k$,

IX. $iV . i^t = jV . j^t = kV . k^t = S . a^{t+1}$.

(4.) We have also, by the same principles and laws,

X. $iV . j^t = V . k^t$;  
$J . V . k^t = V . i^t$;  
$kV . i^t = V . j^t$;

XI. $jV . i^t = - V . k^t$;  
$kV . j^t = - V . i^t$;  
$iV . k^t = - V . j^t$.

(5.) The expression 308, (10.), for an arbitrary vector $\rho$, may be put under the following form:

XII. $\rho = r V . k^{2t+1} + r k^{2t} V . i^{2t} *$

(6.) And it may be expanded as follows:

XIII. $\rho = r \{(i \cos t\pi + j \sin t\pi) \sin s\pi + k \cos s\pi\}$.

(7.) We shall return, briefly, in the Second Chapter of this Book [337], on some of these last expressions, in connexion with differentials and derivatives of powers of vectors; but, for the purposes of the present section, they may suffice.

* [Since $j^{2t} = j^{2t+1} + S k^{2t}$ this follows at once from $\rho = r k^{2t} j^{2t} k^{t-1}$; remembering that $jk^{t-1} = k^{t-1} j = k^{t+1}$.]
SECTION 11.

On Powers and Logarithms of Diplanar Quaternions; with some Additional Formulae.

316. We shall conclude the present Chapter with a short Supplementary Section, in which the recent definition (308) of a power of a vector, with a scalar exponent, shall be extended so as to include the general case, of a Power of a Quaternion, with a Quaternion Exponent, even when the two quaternions so combined are diplanar: and a connected definition shall be given (consistent with the less general one of the same kind, which was assigned in the Second Chapter of the Second Book), for the Logarithm of a Quaternion in an arbitrary Plane:* together with a few additional Formulae, which could not be so conveniently introduced before.

(1.) We propose, then, to write, generally,

\[ \varepsilon^\eta = 1 + \frac{\eta}{1} + \frac{\eta^2}{1 \cdot 2} + \frac{\eta^3}{1 \cdot 2 \cdot 3} + \&c. \]

\( q \) being any quaternion, and \( \varepsilon \) being the real and known base of the natural (or Napierian) system of logarithms, of real and positive scalars: so that (as usual),

\[ \varepsilon = \varepsilon^1 = 1 + \frac{1}{1} + \frac{1^2}{1 \cdot 2} + \&c. = 2.71828 \ldots \]

(Compare 240, (1.) and (2.).)

(2.) We shall also write, for any quaternion \( q \), the following expression for what we shall call its principal logarithm, or simply its Logarithm:

\[ \log q = \log q + \angle q \cdot UV q; \]

and thus shall have (comp. 243) the equation,

\[ \varepsilon^{\log q} = q. \]

(3.) When \( q \) is any actual quaternion (144), which does not degenerate (131) into a negative scalar, the formula III. assigns a definite value for the logarithm, \( \log q \); which is such (comp. again 243) that

\[ \log q = \log q; \quad \log q = \angle q \cdot UV q; \]

\[ \log q = \log q; \quad \log q = \log q; \]

* The quaternions considered, in the Chapter referred to, were all supposed to be in the plane of the right versor \( i \). But see the Second Note to page 277.
the scalar part of the logarithm being thus the (natural) logarithm of the tensor; and the vector part of the same logarithm $lq$ being constructed by a line in the direction of the axis $\Delta x \cdot q$, of which the length bears, to the assumed unit of length, the same ratio as that which the angle $\angle q$ bears, to the usual unit of angle (comp. 241, (2.), (4.)).

(4.) If it were merely required to satisfy the equation,

$$\text{IX. } q^e = q,$$

in which $q$ is supposed to be a given and actual quaternion, which is not equal to any negative scalar (3.), we might do this by writing (compare again 243),

$$\text{X. } q' = (\log q)_n = lq + 2n\pi U V q,$$

where $n$ is any whole number, positive or negative or null; and in this view, what we have called the logarithm, $lq$, of the quaternion $q$, is only what may be considered as the simplest solution of the exponential equation IX., and may, as such, be thus denoted:

$$\text{XI. } lq = (\log q).$$

(5.) The excepted case (3.), where $q$ is a negative scalar, becomes on this plan a case of indetermination, but not of impossibility: since we have, for example, by the definition III., the following expression for the logarithm of negative unity,

$$\text{XII. } 1(-1) = \pi \sqrt{-1};$$

which in its form agrees with old and well-known results, but is here interpreted as signifying any unit-vector, of which the length bears to the unit of length the ratio of $\pi$ to $1$ (comp. 243, VII.).

(6.) We propose also to write, generally, for any two quaternions, $q$ and $q'$, even if diplanar, the following expression (comp. 243, (4.)) for what may be called the principal value of the power, or simply the Power, in which the former quaternion $q$ is the base, while the latter quaternion $q'$ is the exponent:

$$\text{XIII. } q^{q'} = e^{q q};$$

and thus this quaternion power receives, in general, with the help of the definitions I. and III., a perfectly definite signification.

(7.) When the base, $q$, becomes a vector, $p$, its angle becomes a right angle; the definition III. gives therefore, for this case,

$$\text{XIV. } l p = l T p + \frac{\pi}{2} U p;$$

and this is the quaternion which is to be multiplied by $q'$, in the expression,

$$\text{XV. } p^{q'} = e^{q p}.$$
(8.) When, for the same vector-base, the exponent \( q' \) becomes a scalar, \( t \), the last formula becomes:

\[
\text{XVI.} \quad \rho' = e^{\rho t} = T_{\rho} e^{x \cdot T_{\rho}}, \quad \text{if} \quad 2x = t\pi; 
\]

and because, by I., the relation \((U_{\rho})^2 = -1\) gives,

\[
\text{XVII.} \quad e^{x \cdot U_{\rho}} = \cos x + U_{\rho} \sin x, \quad \text{or briefly,} \quad \text{XVII'.} \quad e^{x \cdot U_{\rho}} = \cos x, 
\]

we see that the former definition, 308, I., of the power \( a' \), is in this way reproduced, as one which is included in the more general definition XII., of the power \( q' \); for we may write, by the last mentioned definition,

\[
\text{XVIII.} \quad (U_{\rho})^t = e^{x \cdot U_{\rho}} = \cos \frac{t\pi}{2} \quad \text{(comp. 234, VIII.)} 
\]

with the recent values XVI. and XVII., of \( x \) and \( e^{x \cdot U_{\rho}} \).

(9.) In the present theory of diplanar quaternions, we cannot expect to find that the sum of the logarithms of any two proposed factors, shall be generally equal to the logarithm of the product; but for the simpler and earlier case of complanar quaternions, that algebraic property may be considered to exist, with due modifications for multiplicity of value.*

(10.) The definition III. enables us, however, to establish generally the very simple formula (comp. 243, II. III.):

\[
\text{XIX.} \quad \log q = (T_{q} \cdot U_{q}) = lT_{q} + lU_{q}; 
\]

in which (comp. (3.),

\[
\text{XX.} \quad lU_{q} = \angle q \cdot U_{q} = Vl_{q}; \quad \text{XXI.} \quad lT_{l}U_{q} = \angle q; \quad \text{XXII.} \quad lU_{l}U_{q} = U_{q}. 
\]

(11.) We have also generally, by XIII., for any scalar exponent, \( t \), and any quaternion base, \( q \), the power,

\[
\text{XXIII.} \quad q^{t} = e^{t_{q}} = (T_{q})^{t} \cdot (\cos t \angle q + U_{q} \cdot \sin t \angle q); 
\]

or briefly,

\[
\text{XXIII'.} \quad q^{t} = T_{q} \cdot \cos t \angle q, \quad \text{if} \quad v = U_{q}; 
\]

in which the parentheses about \( T_{q} \) may be omitted, because

\[
\text{XXIV.} \quad T_{q} = (T_{q})^{t} = T_{q}^{t} \quad \text{(comp. 237, II.).} 
\]

* In 243, (3), it might have been observed, that every value of each member of the formula IX., there given, is one of the values of the other member; and a similar remark applies to the formulæ I. and II. of 239.
(12.) When the base and exponent of a power are two rectangular vectors, \( \rho \) and \( \rho' \), then, whatever their lengths may be, the product \( \rho' \rho \) is, by XIV., a vector; but \( e^a \) is always a versor,

\[
XXV. \ldots e^a = \cos Ta + Ua \sin Ta, \text{ if } a \text{ be any vector;}
\]

we have therefore,

\[
XXVI. \ldots T. \rho'^a = 1, \text{ if } S. \rho'^a = 0;
\]

or in words, the power \( \rho'^a \) is a versor, under this condition of rectangularity.

(13.) For example (comp. 242, (7.),* and the shortly following formula XXVIII.),

\[
XXVII. \ldots i^j = e^{ji} = -k; \quad j^i = e^{ji} = +k;
\]

and generally if the base be an unit-line, and the exponent a line of any length, but perpendicular to the base, the axis of the power is a line perpendicular to both; unless the direction of that axis becomes indeterminate, by the power reducing itself to a scalar, which in certain cases may happen.

(14.) Thus whatever scalar \( c \) may be, we may write,

\[
XXVIII. \ldots i^c = e^{ci} = e^{i\phi} = \cos \frac{c\pi}{2} - k \sin \frac{c\pi}{2};
\]

this power, then, is a versor (12.), and its axis is generally the line \( \pm k \); but in the case when \( c \) is any whole and even number, this versor degenerates into positive or negative unity (153), and the axis becomes indeterminate (131).

(15.) If, for any real quaternion \( q \), we write again,

\[
XXIX. \ldots UVq = v, \text{ and therefore XXX... } vq = \bar{q}v, \text{ and XXXI... } v^3 = -1,
\]

the process of 239 will hold good, when we change \( i \) to \( v \); the series, denoted in I. by \( u^q \), is therefore always at last convergent,† however great (but finite) the tensor \( Tq \) may be; and in like manner the two following other series, derived from it, which represent (comp. 242, (3.)) what we shall call, generally, by analogy to known expressions, the cosine and sine of the quaternion \( q \), are always ultimately convergent:

\[
XXXII. \ldots \cos q = \frac{1}{2} (e^{q} + e^{-q}) = 1 - \frac{q^2}{1.2} + \frac{q^4}{1.2.3.4} - \&c.;
\]

\[
XXXIII. \ldots \sin q = \frac{1}{2v} (e^{q} - e^{-q}) = \frac{q}{1} - \frac{q^2}{1.2.3} + \frac{q^4}{1.2.3.4.5} - \&c.
\]

* In the theory of complanar quaternions, it was found convenient to admit a certain multiplicity of value for a power, when the exponent was not a whole number; and therefore a notation for the principal value of a power was employed, with which the conventions of the present section enable us now to dispense.

† In fact, it can be proved that this final convergence exists, even when the quaternion is imaginary, or when it is replaced by a biquaternion (214, (8.)); but we have no occasion here to consider any but real quaternions.
(16.) We shall also define that the secant, cosecant, tangent, and cotangent of a quaternion, supposed still to be real, are the functions:

\[
XXXIV. \quad \sec q = \frac{2}{e^{q} + e^{-q}}; \quad \cosec q = \frac{2v}{e^{q} - e^{-q}};
\]

\[
XXXV. \quad \tan q = \frac{v (e^{q} - e^{-q})}{e^{q} + e^{-q}}; \quad \cot q = \frac{v (e^{q} + e^{-q})}{e^{q} - e^{-q}};
\]

and thus shall have the usual relations, \( \sec q = 1 : \cos q, \&c. \)

(17.) We shall also have,

\[
XXXVI. \quad e^{q} = \cos q + v \sin q, \quad e^{-q} = \cos q - v \sin q;
\]

and therefore, as in trigonometry (comp. 315, (1.)),

\[
XXXVII. \quad (\cos q)^{2} + (\sin q)^{2} = e^{q} e^{-q} = e^{0} = 1,
\]

whatever quaternion \( q \) may be.

(18.) And all the formulae of trigonometry, for cosines and sines of sums of two or more arcs, \&c., will thus hold good for quaternions also, provided that the quaternions to be combined are in any common plane; for example,

\[
XXXVIII. \quad \cos (q' + q) = \cos q' \cos q - \sin q' \sin q, \quad \text{if} \quad q' \parallel q.
\]

(19.) This condition of complanarity is here a necessary one; because (comp. (9.)) it is necessary for the establishment of the exponential relation between sums and powers.

(20.) Thus, we may indeed write,

\[
XXXIX. \quad e^{q' + q} = e^{q'} e^{q}, \quad \text{if} \quad q' \parallel q;
\]

but, in general, the developments of these two expressions give the difference,

\[
XL. \quad e^{q' + q} - e^{q'} e^{q} = \frac{qq' - q'q}{2} + \text{terms of third and higher dimensions};
\]

and

\[
XLI. \quad \frac{1}{2} (qq' - q'q) = V(Vq \cdot Vq'),
\]

an expression which does not vanish, when the quaternions \( q \) and \( q' \) are diplanar.

(21.) A few supplementary formulæ, connected with the present Chapter, may be appended here, as was mentioned at the commencement of this Article (316). And first it may be remarked, as connected with the theory of powers of vectors, that if \( a, b, c \) be any three unit-lines, \( OA, OB, OC \), and if \( \sigma \) denote the
area of the spherical triangle \(abc\), then the formula 298, XX. may be thus written:

\[
\frac{a + \beta}{\beta + \gamma}, \frac{\gamma + a}{\beta + \gamma}, \frac{\beta + \gamma}{a + \beta}, \frac{\gamma + a}{a + \beta} = \frac{2\pi}{\nu};
\]

the exponent being here a scalar.

(22.) The immediately preceding formula, 298, XIX., gives for any three vectors, the relation:

\[
(Ua\beta\gamma)^2 + (U\beta\gamma)^2 + (Ua\gamma)^2 + 4Ua\gamma \cdot SUa\beta \cdot SU\beta\gamma = -2;
\]

for example, if \(a, \beta, \gamma\) be made equal to \(i, j, k\), the first member of this equation becomes, \(1 - 1 - 1 + 0 = -2\).

(23.) The following is a much more complex identity, involving as it does not only three arbitrary vectors \(a, \beta, \gamma\), but also four arbitrary scalars, \(a, b, c, r\); but it has some geometrical applications, and a student would find it a good exercise in transformations, to investigate a proof of it for himself. To abridge notation, the three vectors \(a, \beta, \gamma\), and the three scalars \(a, b, c, r\), are considered as each composing a cycle, with respect to which are formed sums \(\Sigma\), and products \(\Pi\), on a plan which may be thus exemplified:

\[
\Sigma aV\beta\gamma = aV\beta\gamma + bV\gamma a + cV\alpha\beta;
\]

This being understood, the formula to be proved is the following:

\[
(\Sigma a\beta\gamma)^2 + (\Sigma aV\beta\gamma)^2 + r^2(\Sigma V\beta\gamma)^2 - r^2(\Sigma a(\beta - \gamma))^2 + 2\Pi(r^2 + S\beta\gamma + bc) - 2\Pi a^2 + \Sigma(r^2 + a^2 + a^2)(V\beta\gamma)^2 + 2bc(r^2 + S\beta\gamma) - r^2(\beta - \gamma)^2;
\]

the sign of summation in the last line governing all that follows it.

(24.) For example, by making the four scalars \(a, b, c, r\) each = 0, this formula gives, for any three vectors \(a, \beta, \gamma\), the relation,

\[
(\Sigma a\beta\gamma)^2 + 2\Pi S\beta\gamma = 2\Pi a^2 + \Sigma a^2(V\beta\gamma)^2;
\]

which agrees with the very useful equation 294, LIII., because

\[
(\Sigma a\beta\gamma)^2 = a^2((S\beta\gamma)^2 - \beta^2\gamma^2) = (aS\beta\gamma)^2 - \Pi a^2.
\]

(25.) Let \(a, \beta, \gamma\) be the vectors of three points \(a, b, c\), which are exterior to a given sphere, of which the radius is \(r\), and the equation is,

\[
\rho^2 + r^2 = 0 \quad (\text{comp. 282, XIII.});
\]
and let \( a, b, c \) denote the lengths of the tangents to that sphere, which are drawn from those three points respectively. We shall then have the relations:

\[
XLIX. \quad a^2 + a^2 = \beta^2 + \delta^2 = \gamma^2 + \epsilon^2 = -r^2;
\]

thus \( r^2 + a^2 = -a^2 \), &c., and the second member of the formula XLV. vanishes; the first member of that formula is therefore also equal to zero, for these significations of the letters: and thus a theorem is obtained, which is found to be extremely useful, in the investigation by quaternions of the system of the eight (real or imaginary) small circles, which touch a given set of three small circles on a sphere.

(26.) We cannot enter upon that investigation here; but may remark that because the vector \( \rho \) of the foot \( \rho \), of the perpendicular or let fall the origin \( o \) on the right line \( AB \), is given by the expression,

\[
L. \quad \rho = aS \frac{\beta}{\beta - a} + \beta S \frac{a}{a - \beta} = \frac{V\beta a}{a - \beta},
\]

as may be proved in various ways, the condition of contact of that right line \( AB \) with the sphere XLVIII. is expressed by the equation,

\[
LI. \quad T\beta a = rT (a - \beta); \quad \text{or} \quad LII. \quad (V\beta a)^2 = r^2 (a - \beta)^2;
\]

or by another easy transformation, with the help of XLIX.,

\[
LIII. \quad (r^2 + Sa\beta)^2 = (r^2 + a^2) (r^2 + \beta^2) = a^2 b^2.
\]

(27.) This last equation evidently admits of decomposition into two factors, representing two alternative conditions, namely,

\[
LIV. \quad r^2 + Sa\beta - ab = 0; \quad LV. \quad r^2 + Sa\beta + ab = 0;
\]

and if we still consider the tangents \( a \) and \( b \) (25.) as positive, it is easy to prove, in several different ways, that the first or the second factor is to be selected, according as the point \( r \), at which the line \( AB \) touches the sphere, does or does not fall between the points \( A \) and \( B \); or in other words, according as the length of that line is equal to the sum, or to the difference, of those two tangents.

(28.) In fact we have, for the first case,

\[
LVI. \quad T(\beta - a) = b + a, \quad \text{or} \quad 0 = (\beta - a)^2 + (b + a)^2 = -2 (r^2 + Sa\beta - ab),
\]

in virtue of the relations XLIX.; but, for the second case,

\[
LVII. \quad T(\beta - a) = \pm (b - a), \quad \text{or} \quad 0 = (\beta - a)^2 + (b - a)^2 = -2 (r^2 + Sa\beta + ab);
\]

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and it may be remarked, that we might in this way have been led to find the system of the two conditions (27.) and thence the equation LIII., or its transformations, LII. and LI.

(29.) We may conceive a cone of tangents from A, circumscribing the sphere XLVIII., and touching it along a small circle, of which the plane, or the polar plane of the point A, is easily found to have for its equation,

LVIII. \( S \rho + r^2 = 0 \) (comp. 294, (28.), and 215, (10.));

and in like manner the equation,

LIX. \( S \beta \rho + r^2 = 0 \),

represents the polar plane of the point B, which plane cuts the sphere in a second small circle: and these two circles touch each other, when either of the two conditions (27.) is satisfied; such contact being external for the case LIV., but internal for the case LV.

(30.) The condition of contact (26.), of the line and sphere, might have been otherwise found, as the condition of equality of roots in the quadratic equation (comp. 216, (2.)),

LX. \( 0 = (x \alpha + y \beta)^2 + (x + y)^2 r^2 \),

or

LXI. \( 0 = x^2 (r^2 + a^2) + 2xy (r^2 + S \alpha \beta) + y^2 (r^2 + \beta^2) \);

the contact being thus considered here as a case of coincidence of intersections.

(31.) The equation of conjugation (comp. 215, (13.)), which expresses that each of the two points A and B is in the polar plane of the other, is (with the present notations),

LXII. \( r^2 + S \alpha \beta = 0 \);

the equal but opposite roots of LXI., which then exist if the line cuts the sphere, answering here to the well-known harmonic division of the secant line AB (comp. 215, (16.)), which thus connects two conjugate points.

(32.) In like manner, from the quadratic equation 216, III., we get this analogous equation,

LXIII. \( S \frac{\lambda}{a} S \frac{\mu}{a} - S \left( \frac{\lambda}{\beta} \cdot \frac{\mu}{\beta} \right) = 1 \),

connecting the vectors \( \lambda, \mu \) of any two points L, M, which are conjugate relatively to the ellipsoid 216, II.; and if we place the point L, on the surface, the
equation LXIII. will represent the tangent plane at that point \( l \), considered as the locus of the conjugate point \( m \); whence it is easy to deduce the normal, at any point of the ellipsoid. But all researches respecting normals to surfaces can be better conducted, in connexion with the Differential Calculus of Quaternions, to which we shall next proceed.

(33.) It may however be added here, as regards Powers of Quaternions with scalar exponents (11.), that the symbol \( q'rq^{-1} \) represents a quaternion formed from \( r \), by a conical rotation of its axis round that of \( q \), through an angle \( = 2t_1 q \); and that both members of the equation,

\[ LXIV \ldots (qrq^{-1})' = qr'q^{-1}, \]

are symbols of one common quaternion.

[Some care must be taken in the interpretation of the expressions \( q^{a''} r' \) and \( (q^{a'})^{a''} \). By the definition XIII.,

\[ q^{a''} r' = \mathcal{e} q^{a''} q' q = \mathcal{e} q^{a''} q' q = \mathcal{e} q^{a''} q' q = q^{a''} q'. \]

This is quite consistent with the rule that in an operating product the factor to the right operates first on the operand. If the expression \( l q r' \) had been interpreted as equal to \( l q' r' \) instead of \( r' l q' \), then indeed the equality \( (q^{a'})^{a''} = q^{a''} q' r' = \mathcal{e} q' q'' \) would have held good, but the general rule would have been disobeyed.]
CHAPTER II.

ON DIFFERENTIALS AND DEVELOPMENTS OF FUNCTIONS OF QUATERNIONS; AND ON SOME APPLICATIONS OF QUATERNIONS, TO GEOMETRICAL AND PHYSICAL QUESTIONS.

SECTION 1.

On the Definition of Simultaneous Differentials.

317. In the foregoing Chapter of the present Book, and in several parts of the Book preceding it, we have taken occasion to exhibit, as we went along, a considerable variety of Examples, of the Geometrical Application of Quaternions: but these have been given, chiefly as assisting to impress on the reader the meanings of new notations, or of new combinations of symbols, when such presented themselves in turn to our notice. In this concluding Chapter, we desire to offer a few additional examples, of the same geometrical kind, but dealing, more freely than before, with tangents and normals to curves and surfaces; and to give at least some specimens, of the application of quaternions to Physical Inquiries. But it seems necessary that we should first establish here some Principles, and some Notations, respecting Differentials of Quaternions, and of their Functions, generally.

318. The usual definitions, of differential coefficients, and of derived functions, are found to be inapplicable generally to the present Calculus, on account of the (generally) non-commutative character of quaternion-multiplication (168, 191). It becomes, therefore, necessary to have recourse to a new Definition of Differentials, which yet ought to be so framed, as to be consistent with, and to include, the usual Rules of Differentiation: because scalars (131), as well as vectors (292), have been seen to be included, under the general Conception of Quaternions.
319. In seeking for such a new definition, it is natural to go back to the first principles of the whole subject of Differentials: and to consider how the great Inventor of Fluxions might be supposed to have dealt with the question, if he had been deprived of that powerful resource of common calculation, which is supplied by the commutative property of algebraic multiplication; or by the familiar equation,

\[ xy = yx, \]

considered as a general one, or as subsisting for every pair of factors, \( x \) and \( y \); while limits should still be allowed, but infinitesimals be still excluded: and indeed the fluxions themselves should be regarded as generally finite,* according to what seems to have been the ultimate view of Newton.

320. The answer to this question, which a study of the Principia appears to suggest, is contained in the following Definition, which we believe to be a perfectly general one, as regards the older Calculus, and which we propose to adopt for Quaternions:—

"Simultaneous Differentials (or Corresponding Fluxions) are Limits of Equimultiples† of Simultaneous and Decreasing Differences."

And conversely, whenever any simultaneous differences, of any system of variables, all tend to vanish together, according to any law, or system of laws; then, if any equimultiples of those decreasing differences all tend together to any system of finite limits, those Limits are said to be Simultaneous Differentials of the related Variables of the System; and are denoted, as such, by prefixing the letter \( d \), as a characteristic of differentiation, to the Symbol of each such variable.

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* Compare the remarks annexed to the Second Lemma of the Second Book of the Principia (Third Edition, London, 1726); and especially the following passage (page 244):

"Neque enim spectatur in hoc Lemmate magnitudo momentorum, sed prima nascentinm proportio. Eodem recidit si loco momentorum usurpentur vel velocitates incrementorum et decrementorum (quas etiam motus, mutationes et fluxiones quantitatum nominare licet) vel finitae quaevis quantitates velocietatis hisce proportionales."

† As regards the notion of multiplying such differences, or generally any quantities which all diminish together, in order to render their ultimate relations more evident, it may be suggested by various parts of the Principia of Sir Isaac Newton; but especially by the First Section of the First Book. See for example the Seventh Lemma (p. 31), under which such expressions as the following occur: "intelligantur semper \( AB \) et \( AD \) ad puncta longinqua \( b \) et \( d \) produci," .... "ideoque recte semper finites \( AB, AD, \ldots \)" The direction, "ad puncta longinqua produci," is repeated in connexion with the Eighth and Ninth Lemmas of the same Book and Section; while under the former of those two Lemmas we meet the expression, "triangula semper finitas," applied to the magnified representations of three triangles, which all diminish indefinitely together: and under the latter Lemma the words occur, "manente longitudinim \( AE \)," where \( AE \) is a finite and constant line, obtained by a constantly increasing multiplication of a constantly diminishing line \( AE \) (page 33 of the edition cited).
321. More fully and symbolically, let

\[ g, r, s, \ldots \]

denote any system of connected variables (quaternions or others); and let

\[ \Delta g, \Delta r, \Delta s, \ldots \]

denote, as usual, a system of their connected (or simultaneous) differences; in such a manner that the sums,

\[ g + \Delta g, \ r + \Delta r, \ s + \Delta s, \ldots \]

shall be a new system of variables, satisfying the same laws of connexion, whatever they may be, as those which are satisfied by the old system \( g, r, s, \ldots \). Then, in returning gradually from the new system to the old one, or in proceeding gradually from the old to the new, the simultaneous differences \( \Delta g, \Delta r, \Delta s, \ldots \) can all be made (in general) to approach together to zero, since it is evident that they may all vanish together. But if, while the differences themselves are thus supposed to decrease indefinitely together, we multiply them all by some one common but increasing number, \( n \), the system of their equimultiples,

\[ n\Delta g, \ n\Delta r, \ n\Delta s, \ldots \]

may tend to become equal to some determined system of finite limits. And when this happens, as in all ordinary cases it may be made to do, by a suitable adjustment of the increase of \( n \) to the decrease of \( \Delta g, \Delta r, \Delta s, \ldots \), the limits thus obtained are said to be simultaneous differentials of the related variables, \( g, r, s \); and are denoted, as such, by the symbols,

\[ dq, \ dr, \ ds, \ldots \]

SECTION 2.

Elementary Illustrations of the Definition, from Algebra and Geometry.

322. To leave no possible doubt, or obscurity, on the import of the foregoing Definition, we shall here apply it to determine the differential of a square, in algebra, and that of a rectangle, in geometry; in doing which we shall show, that while for such cases the old rules are reproduced, the differentials treated of need not be small; and that it would be a vitiation, and not a

\[ \ast \text{ A quaternion may be said to decrease, when its tensor decreases; and to decrease indefinitely, when that tensor tends to zero.} \]
correction, of the results, if any additional terms were introduced into their
expressions, for the purpose of rendering all the differentials equal to the cor-
responding differences: though some of them may be assumed to be so, namely,
in the first Example, one, and in the second Example, two.

(1.) In Algebra, then, let us consider the equation,

\[ y = x^n, \]

which gives,

\[ y + \Delta y = (x + \Delta x)^n, \]

and therefore, as usual,*

\[ \Delta y = 2x\Delta x + \Delta x^n; \]

or what comes to the same thing,

\[ n\Delta y = 2nx\Delta x + n^{-1}(n\Delta x)^2, \]

where \( n \) is an arbitrary multiplier, which may be supposed, for simplicity, to
be a positive whole number.

(2.) Conceive now that while the differences \( \Delta x \) and \( \Delta y \), remaining always
connected with each other and with \( x \) by the equation III., decrease, and
tend together to zero, the number \( n \) increases, in the transformed equation IV.,
and tends to infinity, in such a manner that the product, or multiple, \( n\Delta x \), tends
to some finite limit \( a \); which may happen, for example, by our obliging \( \Delta x \) to
satisfy always the condition,

\[ \Delta x = n^{-1}a, \text{ or } n\Delta x = a, \]

after a previous selection of some given and finite value for \( a \).

(3.) We shall then have, with this last condition V., the following ex-
pression by IV., for the equimultiple \( n\Delta y \), of the other difference, \( \Delta y \):

\[ n\Delta y = 2xa + a^n = b + n^{-1}a^n, \text{ if } b = 2xa. \]

But because \( a \), and therefore \( a^n \), is given and finite, (2.), while the number \( n \)
increases indefinitely, the term \( n^{-1}a^n \), in this expression VI. for \( n\Delta y \), indefi-
nitely tends to zero, and its limit is rigorously null. Hence the two finite
quantities, \( a \) and \( b \) (since \( x \) is supposed to be finite), are two simultaneous limits,
to which, under the supposed conditions, the two equimultiples, \( n\Delta x \) and \( n\Delta y \),

* We write here, as is common, \( \Delta x^2 \) to denote \( (\Delta x)^2 \); while \( \Delta \cdot x^2 \) would be written, on the same
known plan, for \( \Delta (x^2) \), or \( \Delta y \). In like manner we shall write \( dx^2 \), as usual, for \( (dx)^2 \); and shall
denote \( d(x^2) \) by \( d \cdot x^2 \). Compare the notations \( S^2, S \cdot q^2, \text{ and } V^2, V \cdot q^2 \), in 199 and 204.

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tend;* they are, therefore, by the definition (320), simultaneous differentials of \( x \) and \( y \): and we may write accordingly (321),

\[
\text{VII.} \quad \frac{dx}{a} = \frac{dy}{b} = \frac{2a}{2x}
\]

or, as usual, after elimination of \( a \),

\[
\text{VIII.} \quad \frac{dy}{b} = \frac{dx}{a} = \frac{2x}{2}
\]

(4.) And it would not improve, but vitiate, according to the adopted definition (320), this usual expression for the differential of the square of a variable \( x \) in algebra, if we were to add to it the term \( dx^2 \), in imitation of the formula III. for the difference \( \Delta x^2 \). For this would come to supposing that, for a given and finite value, \( a \), of \( dx \), or of \( n\Delta x \), the term \( n^{-1}a^2 \), or \( n^{-1}dx^2 \), in the expression VI. for \( n\Delta y \), could fail to tend to zero, while the number, \( n \), by which the square of \( dx \) is divided, increases without limit, or tends (as above) to infinity.

(5.) As an arithmetical example, let there be the given values,

\[
\text{IX.} \quad x = 2, \quad y = x^2 = 4, \quad dx = 1000;
\]

and let it be required to compute, as a consequence of the definition (320), the arithmetical value of the simultaneous differential, \( dy \). We have now the following equimultiples of simultaneous differences,

\[
\text{X.} \quad n\Delta x = dx = 1000; \quad n\Delta y = 4000 + 100000n^{-1};
\]

but the limit of the \( n^{th} \) part of a million (or of any greater, but given and finite number) is exactly zero, if \( n \) increase without limit; the required value of \( dy \) is, therefore, rigorously, in this example,

\[
\text{XI.} \quad dy = 4000.
\]

(6.) And we see that these two simultaneous differentials,

\[
\text{XII.} \quad dx = 1000, \quad dy = 4000,\]

are not, in this example, even approximately equal to the two simultaneous differences,

\[
\text{XIII.} \quad \Delta x = dx = 1000, \quad \Delta y = 1002^2 - 2^2 = 1004000,
\]

which answer to the value \( n = 1 \); although, no doubt, from the very conception

* In this case, indeed, the multiple \( n\Delta x \) has by V. a constant value, namely \( a \); but it is found convenient to extend the use of the word, limit, so as to include the case of constants: or to say, generally, that a constant is its own limit.
of simultaneous differentials, as embodied in the definition (320), they must admit of having such equisubmultiples of themselves taken,

\[ \text{XIV. } n^{-1}dx \text{ and } n^{-1}dy, \]

as to be nearly equal, for large values of the number \( n \), to some system of simultaneous and decreasing differences,

\[ \text{XV. } \Delta x \text{ and } \Delta y; \]

and more and more nearly equal to such a system, even in the way of ratio, as they all become smaller and smaller together, and tend together to vanish.

(7.) For example, while the differentials themselves retain the constant values XII., their millionth parts are, respectively,

\[ \text{XVI. } n^{-1}dx = 0.001, \text{ and } n^{-1}dy = 0.004, \text{ if } n = 1000000; \]

and the same value of the number \( n \) gives, by X., the equally rigorous values of two simultaneous differences, as follows,

\[ \text{XVII. } \Delta x = 0.001, \text{ and } \Delta y = 0.004001; \]

so that these values of the decreasing differences XV. may already be considered to be nearly equal to the two equisubmultiples, XIV. or XVI., of the two simultaneous differentials, XII. And it is evident that this approximation would be improved, by taking higher values of the number, \( n \), without the rigorous and constant values XII., of \( dx \) and \( dy \), being at all affected thereby.

(8.) It is, however, evident also, that after assuming \( y = x^2 \), and \( x = 2 \), as in IX., we might have assumed any other finite value for the differential \( dx \), instead of the value 1000; and should then have deduced a different (but still finite) value for the other differential, \( dy \), and not the formerly deduced value, 4000; but there would always exist, in this example, or for this form of the function, \( y \), and for this value of the variable, \( x \), the rigorous relation between the two simultaneous differentials, \( dx \) and \( dy \),

\[ \text{XVIII. } dy = 4dx, \]

which is obviously a case of the equation VIII., and can be proved by similar reasonings.

323. Proceeding to the promised Example from Geometry (322), we shall again see that differences and differentials are not in general to be confounded with each other, and that the latter (like the former) need not be small. But we shall also see that the differentials (like the differences), which enter into a
statement of relation, or into the enunciation of a proposition, respecting quantities which vary together, according to any law or laws, need not even be homogeneous among themselves: it being sufficient that each separately should be homogeneous with the variable to which it corresponds, and of which it is the differential, as line of line, or area of area. It will also be seen that the definition (320) enables us to construct the differential of a rectangle, as the sum of two other (finite) rectangles, without any reference to units of length, or of area, and without even the thought of employing any numerical calculation whatever.

(1.) Let, then, as in the annexed figure 74, ABCD be any given rectangle, and let BE and DG be any arbitrary but given and finite increments of its sides, AB and AD. Complete the increased rectangle GAEF, or briefly AF, which will thus exceed the given rectangle AC, or CA, by the sum of the three partial rectangles, CE, CF, CG; or by what we may call the gnomon,* CBEFGDC. On the diagonal CF take a point I, so that the line CI may be any arbitrarily selected submultiple of that diagonal; and draw through I, as in the figure, lines HM, KL, parallel to the sides AD, AB; and therefore intercepting, on the sides AB, AD prolonged, equisubmultiples BH, DK of the two given increments, BE, DG, of those two given sides.

(2.) Conceive now that, in this construction, the point I approaches to C, or that we take a series of new points I, on the given diagonal CF, nearer and nearer to the given point C, by taking the line CI successively a smaller and smaller part of that diagonal. Then the two new linear intervals, BH, DK, and the new gnomon, CBHKDC, or the sum of the three new partial rectangles, CH, CI, CK, will all indefinitely decrease, and will tend to vanish together: remaining, however, always a system of three simultaneous differences (or increments), of the two given sides, AB, AD, and of the given area, or rectangle, AC.

(3.) But the given increments, BE and DG, of the two given sides, are always (by the construction) equimultiples of the two first of the three new and decreasing differences; they may, therefore, by the definition (320), be arbitrarily taken as two simultaneous differentials of the two sides, AB and AD, provided that we then treat, as the corresponding or simultaneous differential of the rectangle AC, the

* The word, gnomon, is here used with a slightly more extended signification, than in the Second Book of Euclid.
limit of the equimultiple of the new gnomon (2.), or of the decreasing difference between the two rectangles, AC and AI, whereof the first is given.

(4.) We are then, first, to increase this new gnomon, or the difference of AC, AI, or the sum (2.) of the three partial rectangles, CH, CI, CK, in the ratio of BE to BH, or of DG to DK; and secondly, to seek the limit of the area so increased. For this last limit will, by the definition (320), be exactly and rigorously equal to the sought differential of the rectangle AC; if the given and finite increments, BE and DG, be assumed (as by (3.) they may) to be the differentials of the sides, AB, AD.

(5.) Now when we thus increase the two new partial rectangles, CH and CK, we get precisely the two old partial rectangles, CE and CG; which, as being given and constant, must be considered to be their own limits.* But when we increase, in the same ratio, the other new partial rectangle CI, we do not recover the old partial rectangle CF, corresponding to it; but obtain the new rectangle CL, or the equal rectangle CM, which is not constant, but diminishes indefinitely as the point I approaches to C; in such a manner that the limit of the area, of this new rectangle CL or CM, is rigorously null.

(6.) If, then, the given increments, BE, DG, be still assumed to be the differentials of the given sides, AB, AD (an assumption which has been seen to be permitted), the differential of the given area, or rectangle, AC, is proved (not assumed) to be, as a necessary consequence of the definition (320), exactly and rigorously equal to the sum of the two partial rectangles CE and CG; because such is the limit (5.) of the multiple of the new gnomon (2.), in the construction.

(7.) And if any one were to suppose that he could improve this known value for the differential of a rectangle, by adding to it the rectangle CF, as a new term, or part, so as to make it equal to the old or given gnomon (1.), he would (the definition being granted) commit a geometrical error, equivalent to that of supposing that the two similar rectangles CI and CF, bear to each other the simple ratio, instead of bearing (as they do) the duplicate ratio, of their homologous sides.

* Compare the note to page 434.
SECTION 3.

On some general Consequences of the Definition.

324. Let there be any proposed equation of the form,

\[ I \ldots Q = F(q, r, \ldots); \]

and let \( dq, dr, \ldots \) be any assumed (but generally finite) and simultaneous differentials of the variables, \( q, r, \ldots \) whether scalars, or vectors, or quaternions, on which \( Q \) is supposed to depend, by the equation \( I \). Then the corresponding (or simultaneous) differential of their function, \( Q \), is equal (by the definition 320, compare 321) to the following limit:

\[ \Pi \ldots dQ = \lim_{n \to \infty} n \left\{ F(q + n^{-1} dq, r + n^{-1} dr, \ldots) - F(q, r, \ldots) \right\}; \]

where \( n \) is any whole number (or other positive* scalar) which, as the formula expresses, is conceived to become indefinitely greater and greater, and so to tend to infinity. And if, in particular, we consider the function \( Q \) as involving only one variable \( q \), so that

\[ III \ldots Q = f(q) = f_q, \]

then

\[ IV \ldots dQ = df_q = \lim_{n \to \infty} n \left\{ f(q + n^{-1} dq) - f_q \right\}; \]

a formula for the differential of a single explicit function of a single variable, which agrees perfectly with those given, near the end of the First Book, for the differentials of a vector, and of a scalar, considered each as a function (100) of a single scalar variable, \( t \): but which is now extended, as a consequence of the general definition (320), to the case when the connected variables, \( q, Q \), and their differentials, \( dq, dQ \), are quaternions: with an analogous application, of the still more general Formula of Differentiation \( II. \), to Functions of several Quaternions.

(1.) As an example of the use of the formula \( IV. \), let the function of \( q \) be its square, so that

\[ V \ldots Q = f_q = q^2. \]

Then, by the formula,

\[ VI \ldots dQ = df_q = \lim_{n \to \infty} n \left\{ (q + n^{-1} dq)^2 - q^2 \right\} = \lim_{n \to \infty} (q \cdot dq + dq \cdot q + n^{-1} dq^2), \]

* Except in some rare cases of discontinuity, not at present under our consideration, this scalar \( n \) may as well be conceived to tend to negative infinity.
where \( dq^2 \) signifies* the square of \( dq \); that is,

\[
\text{VII} \ldots d \cdot q^2 = q \cdot dq + dq \cdot q;
\]
or without the points† between \( q \) and \( dq \),

\[
\text{VII'} \ldots d \cdot q^2 = q dq + dq q;
\]
an expression for the differential of the square of a quaternion, which does not in general admit of any further reduction: because \( q \) and \( dq \) are not generally commutative, as factors in multiplication. When, however, it happens, as in algebra, that \( q \cdot dq = dq \cdot q \), by the two quaternions \( q \) and \( dq \) being complanar, the expression VII. then evidently reproduces the usual form, 322, VIII., or becomes,

\[
\text{VIII} \ldots d \cdot q^2 = 2q dq, \quad \text{if} \quad dq \parallel q \quad (123).
\]

(2.) As another example, let the function be the reciprocal,

\[
\text{IX} \ldots Q = f q = q^{-1}.
\]

Then, because

\[
\text{X} \ldots f (q + n^{-1} dq) - f q = (q + n^{-1} dq)^{-1} - q^{-1}
\]

\[
= (q + n^{-1} dq)^{-1} \left[ q - (q + n^{-1} dq) \right] q^{-1}
\]

\[
= - n^{-1} (q + n^{-1} dq)^{-1} \cdot dq \cdot q^{-1},
\]
of which, when multiplied by \( n \), the limit is \(- q^{-1} dq \cdot q^{-1} \), we have the following expression for the differential of the reciprocal of a quaternion,

\[
\text{XI} \ldots d \cdot q^{-1} = - q^{-1} \cdot dq \cdot q^{-1};
\]
or without the points‡ in the second member, \( dq \) being treated (as in VII'.) as a whole symbol,

\[
\text{XI'} \ldots d \cdot q^{-1} = - q^{-1} dq \cdot q^{-1};
\]
an expression which does not generally admit of being any farther reduced, but becomes, as in the ordinary calculus,

\[
\text{XII} \ldots d \cdot q^{-1} = - q^{-2} dq, \quad \text{if} \quad dq \parallel q,
\]
that is, for the case of complanarity, of the quaternion and its differential.§

* Compare the note to page 433.
† The point between \( d \) and \( q^2 \), in the first member of VII., is indispensable, to distinguish the differential of the square from the square of the differential. But just as this latter square is denoted briefly by \( dq^2 \), so the products, \( q \cdot dq \) and \( dq \cdot q \), may be written as \( q dq \) and \( dq q \); the symbol, \( dq \), being thus treated as a whole one, or as if it were a single letter. Yet, for greater clearness of expression, we shall retain the point between \( q \) and \( dq \), in several (though not in all) of the subsequent formulae, leaving it to the student to omit it, at his pleasure.
‡ Compare the note immediately preceding.
§ [See 329 (4.) for a result including XI.]
325. Other Examples of Quaternion Differentiation will be given in the following section; but the two foregoing may serve sufficiently to exhibit the nature of the operation, and to show the analogy of its results to those of the older calculus, while exemplifying also the distinction which generally exists between them. And we shall here proceed to explain a notation, which (at least in the statement of the present theory of differentials) appears to possess some advantages; and will enable us to offer a still more brief symbolical definition, of the differential of a function \( f(q) \), than before.

(1.) We have defined (320, 324), that if \( dq \) be called the differential of a (quaternion or other) variable, \( q \), then the limit of the multiple,

\[
\lim_{n \to \infty} n \left[ f(q + n^{-1} dq) - f(q) \right],
\]

of an indefinitely decreasing difference of the function, \( f(q) \), of that (single) variable \( q \), when taken relatively to an indefinite increase of the multiplying number, \( n \), is the corresponding or simultaneous differential of that function, and is denoted, as such, by the symbol \( df(q) \).

(2.) But before we thus pass to the limit, relatively to \( n \), and while that multiplier, \( n \), is still considered and treated as finite, the multiple is evidently a function of that number, \( n \), as well as of the two independent variables, \( q \) and \( dq \). And we propose to denote (at least for the present) this new function of the three variables,

\[
\lim_{n \to \infty} n \left[ f(q + n^{-1} dq) - f(q) \right]
\]

in such a manner as to write, for any two variables, \( q \) and \( q' \), and any number, \( n \), the equation,

\[
\frac{f_n(q, dq)}{n} = f(q + n^{-1} q') - f(q);
\]

which may obviously be also written thus,

\[
f(q + n^{-1} q') = f(q) + n^{-1} f_n(q, dq),
\]

and is here regarded as rigorously exact, in virtue of the definitions, and without anything whatever being neglected, as small.
Abts. 325, 326.] DISTRIBUTIVE PROPERTY. 441

(3.) For example, it appears from the little calculation in 324, (1.), that,

\[ f_n(q, q') = qq' + q'q + n^{-1}q^n, \quad \text{if } fq = q^n; \]

and from 324, (2.), that,

\[ f_n(q, q') = -(q + n^{-1}q')^{-1}q'q^{-1}, \quad \text{if } fq = q^{-1}. \]

(4.) And the definition of \( dfq \) may now be briefly thus expressed:

\[ \text{VIII.} \quad dfq = f_\infty (q, dq); \]

or, if the sub-index \( \infty \) be understood, we may write, still more simply,

\[ \text{IX.} \quad dfq = f(q, dq); \]

this last expression, \( f(q, dq) \), or \( f(q, q') \), denoting thus a function of two independent variables \( q \) and \( q' \), of which the form is derived* or deduced (comp. (2.)), from the given or proposed form of the function \( f_q \) of a single variable, \( q \), according to a law which it is one of the main objects of the Differential Calculus (at least as regards Quaternions) to study.

326. One of the most important general properties, of the functions of this class \( f(q, q') \), is that they are all distributive with respect to the second independent variable, \( cf \), which is introduced in the foregoing process of what we have called derivatioh,† from some given function \( f_q \), of a single variable, \( q \): a theorem which may be proved as follows, whether the two independent variables be, or be not, quaternions.

(1.) Let \( q'' \) be any third independent variable, and let \( n \) be any number; then the formula 325, V. gives the three following equations, resulting from the law of derivation of \( f_n(q, q') \) from \( f_q \):

\[ \text{I.} \quad f(q + n^{-1}q'') = f_q + n^{-1}f_n(q, q''); \]
\[ \text{II.} \quad f(q + n^{-1}q'' + n^{-1}q') = f(q + n^{-1}q'') + n^{-1}f_n(q + n^{-1}q'', q'); \]
\[ \text{III.} \quad f(q + n^{-1}q' + n^{-1}q'') = f_q + n^{-1}f_n(q, q' + q''); \]

* It was remarked, or hinted, in 318, that the usual definition of a derived function, namely, that given by Lagrange in the Calcul des Fonctions, cannot be taken as a foundation for a differential calculus of quaternions: although such derived functions of scalars present themselves occasionally in the applications of that calculus, as in 100, (3.) and (4.), and in some analogous but more general cases, which will be noticed soon. The present Law of Derivation is of an entirely different kind, since it conducts, as we see, from a given function of one variable, to a derived function of two variables, which are in general independent of each other. The function \( f_n(q, q') \), of the three variables, \( n, q, q' \), may also be called a derived function, since it is deduced, by the fixed law IV., from the same given function \( f_q \), although it has in general a less simple form than its own limit, \( f_\infty (q, q') \), or \( f(q, q') \).

† Compare the note immediately preceding.

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by comparing which we see at once that

\[ IV. \quad f_n(q, q' + q'') = f_n(q + n^{-1}q'', q') + f_n(q, q''), \]

the form of the original function, \( f_q \), and the values of the four variables, \( q, q', q'', \) and \( n \), remaining altogether arbitrary: except that \( n \) is supposed to be a number, or at least a scalar, while \( q, q', q'' \) may (or may not) be quaternions.

(2.) For example, if we take the particular function \( f_q = q^2 \), which gives the form 325, VI. of the derived function \( f_n(q, q') \), we have

\[ V. \quad f_n(q, q'') = qq'' + q'q + n^{-1}q''^2; \]

\[ VI. \quad f_n(q, q' + q'') = q(q' + q'') + (q' + q'') q + n^{-1}(q' + q'')^2; \]

and therefore

\[ VII. \quad f_n(q, q' + q'') - f_n(q, q'') = qq' + q'q + n^{-1}(q'^2 + q''^2 + q'q'') \]

\[ = (q + n^{-1}q''') q' + q' (q + n^{-1}q'') + n^{-1}q'^2 \]

\[ = f_n(q + n^{-1}q'', q'), \]

as required by the formula IV.

(3.) Admitting then that formula as proved, for all values of the number \( n \), we have only to conceive that number (or scalar) to tend to infinity, in order to deduce this limiting form of the equation:

\[ VIII. \quad f_n(q, q' + q'') = f_n(q, q') + f_n(q, q''); \]

or simply, with the abridged notation of 325, (4.),

\[ IX. \quad f(q, q' + q'') = f(q, q') + f(q, q''); \]

which contains the expression of the functional property, above asserted to exist.

(4.) For example, by what has been already shown (comp. 325, (3.) and (4.)),

\[ X. \quad \text{if } f_q = q^2, \text{ then } f(q, q') = qq' + q'q; \]

and \( XI. \quad \text{if } f_q = q^{-1}, \text{ then } f(q, q') = -q^{-1}q'q^{-1}; \)

in each of which instances we see that the derived function \( f(q, q') \) is distributive relatively to \( q' \), although it is only in the first of them that it happens to be distributive with respect to \( q \) also.

(5.) It follows at once from the formula IX. that we have generally*

\[ XII. \quad f(q, 0) = 0; \]

* We abstract here from some exceptional cases of discontinuity, &c.
and it is not difficult to prove, as a result including this, that

\[ f(q, xq') = xf(q, q'), \text{ if } x \text{ be any scalar.} \]

(6.) As a confirmation of this last result, we may observe that the definition of \( f(q, q') \) may be expressed by the following formula (comp. 324, IV., and 325, IX.):

\[ f(q, q') = \lim_{n \to \infty} n\{f(q + n^{-1}q') - q\}; \]

we have therefore, if \( x \) be any finite scalar, and \( m = x^{-1}n \),

\[ f(q, xq') = x \lim_{m \to \infty} m\{f(q + m^{-1}q') - q\}; \]

a transformation which gives the recent property XIII., since it is evident that the letter \( m \) may be written instead of \( n \), in the formula of definition XIV.

327. Resuming then the general expression 325, IX., or writing anew,

\[ dfq = f(q, dq), \]

we see (by 326, IX.) that this derived function, \( dfq \), of \( q \) and \( dq \), is always (as in the examples 324, VII. and XI.) distributive with respect to that differential \( dq \), considered as an independent variable, whatever the form of the given function \( f(q) \) may be. We see also (by 326, XIII.), that if the differential \( dq \) of the variable, \( q \), be multiplied by any scalar, \( x \), the differential \( dfq \), of the function \( f(q) \), comes to be multiplied, at the same time, by the same scalar, or that

\[ f(q, xdq) = xf(q, dq), \text{ if } x \text{ be any scalar.} \]

And in fact it is evident, from the very conception and definition (320) of simultaneous differentials, that every system of such differentials must admit of being all changed together to any system of equimultiples, or equisubmultiples, of themselves, without ceasing to be simultaneous differentials: or more generally, that it is permitted to multiply all the differentials of a system, by any common scalar.

(1.) It follows that the quotient,

\[ dfq : dq = f(q, dq) : dq, \]

of the two simultaneous differentials, \( dfq \) and \( dq \), does not change when the differential \( dq \) is thus multiplied by any scalar; and consequently that this quotient \( III \) is independent of the tensor \( Tdq \), although it is not generally independent of the versor \( Udq \), if \( q \) and \( dq \) be quaternions: except that it remains
in general unchanged, when we merely change that versor to its own opposite (or negative), or to $-Udq$, because this comes to multiplying $dq$ by $-1$, which is a scalar.

(2.) For example, the quotient,

$$d \cdot g^2 : dq = g + dq \cdot g \cdot dq^{-1} = g + Udq \cdot g \cdot Udq^{-1},$$

in which $dq^{-1}$ and $Udq^{-1}$ denote the reciprocals of $dq$ and $Udq$, is very far from being independent of $dq$, or at least of $Udq$; since it represents, as we see, the sum of the given quaternion $g$, and of a certain other quaternion, which latter, in its geometrical interpretation (comp. 191, (5.)), may be considered as being derived from $g$, by a conical rotation of $Ax \cdot g$ round $Ax \cdot dq$, through an angle $= 2\angle dq$: so that both the axis and the quantity of this rotation depend on the versor $Udq$, and vary with that versor.

(3.) In general, we may, if we please, say that the quotient III. is a Differential Quotient; but we ought not to call it a Differential Coefficient (comp. 318), because $d/q$ does not generally admit of decomposition into two factors, whereof one shall be the differential $dq$, and the other a function of $g$ alone.

(4.) And for the same reason, we ought not to call that Quotient a Derived Function (comp. again 318), unless in so speaking we understand a Function of Two* independent Variables, namely of $q$ and $Udq$, as before.

(5.) When, however, a quaternion, $g$, is considered as a function of a scalar variable, $t$, so that we have an equation of the form,

$$q = ft,$$

it is then permitted (comp. 100, (3.) and (4.)) to write,

$$dq : dt = d/ft : dt = \lim_{h \to 0} \frac{n}{dt} \left\{ f\left( t + \frac{dt}{n} \right) - ft \right\}$$

$$= \lim_{h \to 0} \frac{n}{h} \left\{ f(t + h) - ft \right\}$$

$$= f' = Dq ft = Dq f;$$

and to call this limit, as usual, a derived function of $t$, because it is (in fact) a function of that scalar variable, $t$, alone, and is independent of the scalar differential, $dt$.

* Compare the note to 325, (4.).
(6.) We may also write, under these circumstances, the differential equation,

\[ \text{VII. } \frac{dq}{dt} = D_g \cdot dt, \quad \text{or} \quad \text{VIII. } \frac{df}{dt} = f't \cdot dt, \]

and may call the derived quaternion, \(D_g\), or \(f't\), as usual, a differential coefficient in this formula, because the scalar differential, \(dt\), is (in fact) multiplied by it, in the expression thus found for the quaternion differential, \(dq\) or \(df\).

(7.) But as regards the logic of the question (comp. again 100, (3.) ), it is important to remember that we regard this derived function, or differential coefficient,

\[ \text{IX. } f't, \quad \text{or} \quad D_t f, \quad \text{or} \quad D_g, \]

as being an actual quotient VI., obtained by dividing an actual quaternion,

\[ \text{X. } \frac{df}{dt}, \quad \text{or} \quad \frac{dq}{dt}, \]

by an actual scalar, \(dt\), of which the value is altogether arbitrary, and may (if we choose) be supposed to be large (comp. 322); while the dividend quaternion \(X\) depends, for its value, on the values of the two independent scalars, \(t\) and \(dt\), and on the form of the function \(ft\), according to the law which is expressed by the general formula 324, IV., for the differentiation of explicit functions of any single variable.

328. It is easy to conceive that similar remarks apply to quaternion functions of more variables than one; and that when the differential of such a function is expressed (comp. 324, II.) under the form,

\[ \text{I. } dQ = dF(q, r, s, \ldots) = F(q, r, s, \ldots dq, dr, ds, \ldots), \]

the new function \(F\) is always distributive, with respect to each separately of the differentials \(dq, dr, ds, \ldots\); being also homogeneous of the first dimension (comp. 327), with respect to all those differentials, considered as a system; in such a manner that, whatever may be the form of the given quaternion function, \(Q\), or \(F\), the derived* function \(F\); or the third member of the formula I., must possess this general functional property (comp. 326, XIII., and 327, II.),

\[ \text{II. } F(q, r, s, \ldots x dq, xdr, xds \ldots) = xF(q, r, s, \ldots dq, dr, ds, \ldots), \]

where \(x\) may be any scalar: so that products, as well as squares, of the differentials \(dq, dr, \&c.,\) of \(q, r, \&c.\) considered as so many variables on which \(Q\) depends, are excluded from the expanded expression of the differential \(dQ\) of the function \(Q\).

* Compare the note last referred to.
(1.) For example, if the function to be differentiated be a product of two quaternions,

\[ Q = F(q, r) = qr, \]

then it is easily found from the general formula 324, II., that \((\text{because the limit of } n^{-1}. dq . dr \text{ is null, when the number } n \text{ increases without limit})\) the differential of the function is,

\[ dQ = d. qr = dF(q, r) = F(q, r, dq, dr) = q . dr + dq . r; \]

with analogous results, for differentials of products of more than two quaternions.

(2.) Again, if we take this other function,

\[ Q = F(q, r) = q^{-1}r, \]

then, applying the same general formula 324, II., and observing that we have, for all values of the number (or other scalar), \(n\), and of the four quaternions, \(q, r, q', r'\), the identical transformation (comp. 324, (2.)),

\[ n^{-1}(q + n^{-1}q')^{-1}(r + n^{-1}r') = q^{-1}r' - (q + n^{-1}q')^{-1} q'q^{-1}(r + n^{-1}r'), \]

we find, as the required limit, when \(n\) tends to infinity, the following differential of the function:

\[ dQ = d . q^{-1}r = dF(q, r) = F(q, r, dq, dr) = q^{-1} . dr - q^{-1} . dq . q^{-1}r; \]

which is again, like the expression IV., distributive with respect to each of the differentials \(dq, dr\), of the variables \(q, r\), and does not involve the product of those two differentials: although these two differential expressions, IV. and VII., are both entirely rigorous, and are not in any way dependent on any supposition that the tensors of \(dq\) and \(dr\) are small (comp. again 322).

329. In thus differentiating a function of more variables than one, we are led to consider what may be called Partial Differentials of Functions of two or more Quaternions; which may be thus denoted,

\[ d_q Q, d_r Q, d_s Q, \ldots \]

if \(Q\) be a function, as above, of \(q, r, s, \ldots\) which is here supposed to be differentiated with respect to each variable separately, as if the others were constant. And then, if \(dQ\) denote, as before, what may be called, by contrast, the Total Differential of the function \(Q\), we shall have the General Formula,

\[ dQ = d_q Q + d_r Q + d_s Q + \ldots; \]
or, briefly and symbolically,

$$III...d = dq + dr + ds + \ldots,$$

if $q, r, s, \ldots$ denote the quaternion variables on which the quaternion function depends, of which the total differential is to be taken; whether those variables be all independent, or be connected with each other, by any relation or relations.

(1.) For example (comp. 328, (1.)),

$$IV...\text{if } Q = qr, \text{ then } dqQ = dq \cdot r, \text{ and } drQ = q \cdot dr;$$

and the sum of these two partial differentials of $Q$ makes up its total differential $dQ$, as otherwise found above.

(2.) Again (comp. 328, (2.)),

$$V...\text{if } Q = q^2r, \text{ then } dqQ = -q^{-1}dq \cdot q^{-1}r; \text{ and } drQ = q^{-1}dr;$$

and $dqQ + drQ =$ the same $dQ$ as that which was otherwise found before, for this form of the function $Q$.

(3.) To exemplify the possibility of a relation existing between the variables $q$ and $r$, let those variables be now supposed equal to each other in $V.;$ we shall then have $Q = 1, dQ = 0;$ and accordingly we have here $dqQ = -q^{-1}dq = -drQ.$

(4.) Again, in $IV.,$ let $qr = c =$ any constant quaternion; we shall then again have $0 = dQ = dqQ + drQ;$ and may infer that

$$VI...dr = -q^{-1}dq \cdot r, \text{ if } qr = c = \text{const.;}$$

a result which evidently agrees with, and includes, the expression 324, XI., for the differential of a reciprocal.

(5.) A quaternion, $q$, may happen to be expressed as a function of two or more scalar variables, $t, u, \ldots$; and then it will have, as such, by the present Article, its partial differentials, $dtq, duq, \&c.$ But because, by 327, VII., we may in this case write,

$$VII...dq = D_{dq} \cdot dt, \text{ duq} = D_{duq} \cdot du, \ldots$$

where the coefficients are independent of the differentials (as in the ordinary calculus), we shall have (by II.) an expression for the total differential $dq$, of the form,

$$VIII...dq = dq + duq + \ldots = D_{dq} \cdot dt + D_{duq} \cdot du + \ldots;$$
and may at pleasure say, under the conditions here supposed, that the derived quaternions,

\[ \text{IX.} \quad D_q q, \quad D_w q, \ldots \]

are either the Partial Derivatives, or the Partial Differential Coefficients, of the Quaternion Function,

\[ \text{X.} \quad q = F(t, u, \ldots) ; \]

with analogous remarks for the case, when the quaternion, \( q \), degenerates (comp. 289) into a vector, \( \rho \).

330. In general, it may be considered as evident, from the definition in 320, that the differential of a constant is zero; so that if \( Q \) be changed to any constant quaternion, \( c \), in the equation 324, I., then \( dQ \) is to be replaced by 0, in the differentiated equation, 324, II. And if there be given any system of equations, connecting the quaternion variables, \( q, r, s, \ldots \) we may treat the corresponding system of differentiated equations, as holding good, for the system of simultaneous differentials, \( dq, dr, ds, \ldots \); and may therefore, legitimately in theory, whenever in practice it shall be found to be possible, eliminate any one or more of those differentials, between the equations of this system.

(1.) As an example, let there be the two equations,

\[ \text{I.} \quad q r = c, \quad \text{and} \quad \text{II.} \quad s = r^2, \]

where \( c \) denotes a constant quaternion. Then (comp. 328, (1.), and 324, (1.)) we have the two differentiated equations corresponding,

\[ \text{III.} \quad q \cdot dr + dq \cdot r = 0; \quad \text{IV.} \quad ds = r \cdot dr + dr \cdot r; \]

in which the points* might be omitted. The former gives,

\[ \text{V.} \quad dr = -q^{-1} dq \cdot r, \text{as in 329, VI.}; \]

and when we substitute this value in the latter, we thereby eliminate the differential \( dr \), and obtain this new differential equation,

\[ \text{VI.} \quad ds = -rg^{-1} \cdot dq \cdot r - q^{-1} \cdot dq \cdot r^2. \]

(2.) The equation I. gives also the expression,

\[ \text{VII.} \quad r = q^{-1} c; \]

the equation II. gives therefore this other expression,

\[ \text{VIII.} \quad s = (q^{-1} c)^2 = q^{-1} c q^{-1} c, \]

* Compare the second note to 324, (1.).
by elimination before differentiation. And if, in the formula VI., we substitute the expressions VII. and VIII. for \( r \) and \( s \), we get this other differential equation,

\[ \text{IX. } d \cdot (q^{-1}c)^2 = -q^{-1}cq^{-1} \cdot dq \cdot q^{-1}c - q^{-1} \cdot dq \cdot q^{-1}cq^{-1}c; \]

which might have been otherwise obtained (comp. again 324, (1.) and (2.)), under the form,

\[ \text{X. } d \cdot (q^{-1}c)^2 = q^{-1}c \cdot d(q^{-1}c) + d(q^{-1}c) \cdot q^{-1}c. \]

331. No special rules are required, for the differentiation of functions of functions of quaternions; but it may be instructive to show, briefly, how the consideration of such differentiation conducts (comp. 326) to a general property of functions of the class \( f(q, q') \); and how that property can be otherwise established.

(1.) Let \( f, \phi, \) and \( \psi \) denote any functional operators, such that

\[ \text{I. } \psi q = \phi(fq); \]

\[ \text{II. } r = \phi q, \text{ and III. } s = \phi r, \text{ we have IV. } s = \psi q; \]

whence \( \text{V. } ds = d\psi q = d\phi r. \)

That is, we may (as usual) differentiate the compound function, \( \phi(fq) \), as if \( fq \) were an independent variable, \( r \); and then, in the expression so found, replace the differential \( dfq \) by its value, obtained by differentiating the simple function, \( fq \). For this comes virtually to the elimination of the differential \( dr \), or of the symbol \( dfq \), in a way which we have seen to be permitted (330):

(2.) But, by the definitions of \( dfq \) and \( f_n(q, q') \), we saw (325, VIII. IX.) that the differential \( dfq \) might generally be denoted by \( f_n(q, dq) \), or briefly by \( f(q, dq) \); whence \( d\phi r \) and \( d\psi q \) may also, by an extension of the same notation, be represented by the analogous symbols, \( \phi_n(r, dr) \) and \( \psi_n(q, dq) \), or simply by \( \phi(r, dr) \) and \( \psi(q, dq) \).

(3.) We ought, therefore, to find that

\[ \text{VI. } \psi_n(q, dq) = \phi_n(fq, f_n(q, dq)), \text{ if } \psi q = \phi(fq); \]

or briefly that

\[ \text{VII. } \psi(q, q') = \phi(fq, f(q, q')), \text{ if } \psi q = \phi f q, \]

for any two quaternions, \( q, q' \), and any two functions, \( f, \phi \); provided that the functions \( f_n(q, q'), \phi_n(q, q'), \psi_n(q, q') \) are deduced (or derived) from the functions \( fq, \phi q, \psi q \), according to the law expressed by the formula 325, IV.;
and that then the limits to which these derived functions \(f_n(q, q')\), \\
&c. tend, when the number \(n\) tends to infinity, are denoted by these other functional \nsymbols, \(f(q, q')\), \\
&c.

(4.) To prove this otherwise, or to establish this general property VII., of \nfunctions of this class \(f(q, q')\), without any use of differentials, we may observe \nthat the general and rigorous transformation 325, V., of the formula 325, IV. \nbased on the functions \(f_n(q, q')\) are defined, gives for all values of \(n\) the \nequation:

\[
VIII. \ \ \ \ \ \ \ \ \ \phi f(q + n^{-1}g') = \phi (f\phi + n^{-1}f_n(q, q')) = \phi f\phi + n^{-1}\phi f_n(f\phi, f_n(q, q'));
\]

but also, by the same general transformation,

\[
IX. \ \ \ \ \ \ \ \ \ \psi (q + n^{-1}g') = \psi q + n^{-1}\psi q_n(q, q');
\]

hence generally, for all values of the number \(n\), as well as for all values of the \ntwo independent quaternions, \(q, q'\), and for all forms of the two functions, \(f, \phi\), we \nmay write,

\[
X. \ \ \ \ \ \ \ \ \ \psi_n(q, q') = \phi_n(f\phi, f_n(q, q')), \ \ \ \ \ \ \ \ \ \text{if} \ \ \ \psi q = \psi f q;
\]

an equation of which the limiting form, for \(n = \infty\), is (with the notations used) \nthe equation VII. which was to be proved.

(5.) It is scarcely worth while to verify the general formula X., by any \nparticular example: yet, merely as an exercise, it may be remarked that if \nwe take the forms,

\[
XI. \ \ \ \ \ \ \ \ \ \phi q = q^2, \ \ \ \ \ \ \ \ \ \psi q = q^4,
\]

of which the two first give, by 325, VI., the common derived form,

\[
XII. \ \ \ \ \ \ \ \ \ \phi q = q^2, \ \ \ \ \ \ \ \ \ \psi q = q^4,
\]

the formula X. becomes,

\[
XIII. \ \ \ \ \ \ \ \ \ \psi_n(q, q') = \phi_n(q^2, gq' + q'g + n^{-1}g'^2)
= q^2(qg' + q'g + n^{-1}g'^2) + (qq' + q'q + n^{-1}g'^2) g^2 + n^{-1}(qg' + q'g + n^{-1}g'^2)^2;
\]

which agrees with the value deduced immediately from the function \(\psi q\) or \(q^4\), \nbased on the definition 325, IV., namely,

\[
XIV. \ \ \ \ \ \ \ \ \ \psi_n(q, q') = n[(q + n^{-1}g')^4 - q^4] = n[(q^2 + n^{-1}(g^2 + q'q + n^{-1}g'^2))^2 - (q^2)^2].
\]
(6.) In general, the theorem, or rule, for differentiating as in (1.) a function of a function, of a quaternion or other variable, may be briefly and symbolically expressed by the formula,

$$ XV. \ldots d(qf)q = dq(fq) ; $$

and if we did not otherwise know it, a proof of its correctness would be supplied, by the recent proof of the correctness of the equivalent formula VII.

SECTION 4.

Examples of Quaternion Differentiation.

332. It will now be easy and useful to give a short collection of Examples of Differentiation of Quaternion Functions and Equations, additional to and inclusive of those which have incidentally occurred already, in treating of the principles of the subject.

(1.) If $c$ be any constant quaternion (as in 330), then

$$
I. \ldots dc = 0 ; \quad II. \ldots d (fq + c) = d/fq ; \\
III. \ldots d . cfq = cd/q ; \quad IV. \ldots d (fq . c) = d/fq . c .
$$

(2.) In general,

$$ V. \ldots d (fq + pq + \ldots) = dfq + d/pq + \ldots ; \quad VI. \ldots d \Sigma = \Sigma d , $$

if $\Sigma$ be used as a mark of summation.

(3.) Also,

$$ VII. \ldots d (fq . pq) = dfq . pq + pq . dq ; $$

and similarly for a product of more functions than two: the rule being simply, to differentiate each factor separately, in its own place, or without disturbing the order of the factors (comp. 318, 319); and then to add together the partial results (comp. 329).

(4.) In particular, if $m$ be any positive whole number,

$$ VIII. \ldots d . q^m = q^{m-1}dq + q^{m-2}dq . q + qdq . q^{m-3} + dq . q^{m-1} , $$

and because we have seen (324, (2.) ) that

$$ IX. \ldots d . q^{-1} = - q^{-1} . dq . q^{-1} , $$
we have this analogous expression for the differential of a power of a quaternion, with a negative but whole exponent,

\[ X \ldots d \cdot q^{-m} = -q^{-m}d \cdot q^m \cdot q^{-m} \]

\[ = -q^{-1}dq \cdot q^m - q^2dq \cdot q^{-m} - \ldots - q^{-m}d \cdot q^m \cdot q^{-m}dq \cdot q^m \cdot q^{-1}. \]

(5.) To differentiate a square root, we are to resolve the linear equation,

\[ XI \ldots q^3 \cdot d \cdot q^4 + d \cdot q^4 \cdot q^3 = dq; \text{ or } XI' \ldots rr' + r'r = q', \]

if we write, for abridgment,

\[ XII \ldots r = q^3, \quad q' = dq, \quad r' = d \cdot q^4 = dr. \]

(6.) Writing also, for this purpose,

\[ XIII \ldots s = Kr = K \cdot q^3, \]

whence (by 190, 196) it will follow that

\[ XIV \ldots rs = Nr = Tr^2 = Tq, \quad \text{and } XV \ldots r + s = 2Sr = 2S \cdot q^3, \]

the product and sum of these two conjugate quaternions, \( r \) and \( s \), being thus scalars (140, 145), we have, by XI',

\[ XVI \ldots r^{-1}q's = r's + sr'; \]

whence, by addition,

\[ XVII \ldots q' + r^{-1}q's = (r + s)r' + r'(r + s) = 2r'(r + s); \]

and finally,

\[ XVIII \ldots r' = \frac{q' + r^{-1}q's}{2(r + s)}, \text{ or } XIX \ldots d \cdot q^4 = \frac{dq + q^{-4}dq \cdot K \cdot q^4}{4S \cdot q^3}; \]

an expression for the differential of the square-root of a quaternion, which will be found to admit of many transformations, not needful to be considered here.

(7.) In the three last sub-articles, as in the three preceding them, it has been supposed, for the sake of generality, that \( q \) and \( dq \) are two diplanar

* Although such solution of a linear equation, or equation of the first degree, in quaternions, is easily enough accomplished in the present instance, yet in general the problem presents difficulties, without the consideration of which the theory of differentiation of implicit functions of quaternions would be entirely incomplete. But a general method, for the solution of all such equations, will be sketched in a subsequent Section.
quaternions; but if in any application they happen, on the contrary, to be complanar, the expressions are then simplified, and take usual, or algebraic forms, as follows:

\[
\begin{align*}
XX. \quad & d \cdot q^{m} = mq^{m-1}dq; \\
XXI. \quad & d \cdot q^{m} = -mq^{m-1}dq; \\
XXII. \quad & d \cdot q^{i} = \frac{1}{2}q^{-1}dq, \quad \text{if} \quad XXIII. \quad dq|||q(123); \\
\end{align*}
\]

because, when \( q' \) is complanar with \( q \), and therefore with \( q^{i} \), or with \( r \), in the expression XVIII., the numerator of that expression may be written as \( r^{-1}q' (r + s) \).

(8.) More generally, if \( x \) be any scalar exponent, we may write, as in the ordinary calculus, but still under the condition of complanarity XXIII.,

\[
\begin{align*}
XXIV. \quad & d \cdot q^{x} = xq^{x-1}dq; \quad \text{or} \quad XXV. \quad q \cdot d \cdot q^{x} = xq^{x}dq. \\
\end{align*}
\]

333. The functions of quaternions, which have been lately differentiated, may be said to be of algebraic form; the following are a few examples of differentials of what may be called, by contrast, transcendental functions of quaternions: the condition of complanarity \( dq|||q \) being however here supposed to be satisfied, in order that the expressions may not become too complex. In fact, with this simplification, they will be found to assume, for the most part, the known and usual forms, of the ordinary differential calculus.

(1.) Admitting the definitions in 316, and supposing throughout that \( dq|||q \), we have the usual expressions for the differentials of \( \varepsilon^{q} \) and \( lg \), namely,

\[
\begin{align*}
I. \quad & d \cdot \varepsilon^{q} = \varepsilon d\varepsilon; \\
II. \quad & d lg = q^{-1}dq. \\
\end{align*}
\]

(2.) We have also, by the same system of definitions (316),

\[
\begin{align*}
III. \quad & d \sin g = \cos g dq; \\
IV. \quad & d \cos g = -\sin g dq; \quad & \text{&c.} \\
\end{align*}
\]

(3.) Also, if \( r \) and \( dr \) be complanar with \( q \) and \( dq \), then, by 316,

\[
\begin{align*}
IV'. \quad & d \cdot q^{r} = d \cdot \varepsilon^{rq} = q^{r}d\varepsilon \cdot rq = q^{r}(lgdr + q^{-1}rdq); \\
\end{align*}
\]

or in the notation of partial differentials (329),

\[
\begin{align*}
V. \quad & d_{q} \cdot q^{r} = rq^{-1}dq, \quad \text{and} \quad VI. \quad d_{r} \cdot q^{r} = q^{r}lgdr. \\
\end{align*}
\]

(4.) In particular, if the base \( q \) be a given or constant vector, \( a \), and if the exponent \( r \) be a variable scalar, \( t \), then (by the value 316, XIV. of \( lp \)) the recent formula IV. becomes,

\[
\begin{align*}
VII. \quad & d \cdot a^{t} = (1T\alpha + \frac{\pi}{2} U\alpha)a^{t}dt. \\
\end{align*}
\]
(5.) If then the base $a$ be a given unit line, so that $1T_a = 0$, and $U_a = a$, we may write simply,

$$
\text{VIII.} \ldots \text{d} \cdot a^t = \frac{\pi}{2} a^{t+1} \text{d}t, \quad \text{if} \quad \text{d}a = 0, \quad \text{and} \quad T_a = 1.
$$

(6.) This useful formula, for the differential of a power of a constant unit line, with a variable scalar exponent, may be obtained more rapidly from the equation 308, VII., which gives,

$$
\text{IX.} \ldots a^t = \cos \frac{t\pi}{2} + a \sin \frac{t\pi}{2}, \quad \text{if} \quad T_a = 1;
$$

since it is evident that the differential of this expression is equal to the expression itself multiplied by $\frac{1}{2}\pi \text{ad} \text{t}$, because $a^t = -1$.

(7.) The formula VIII. admits also of a simple geometrical interpretation, connected with the rotation through $t$ right angles, in a plane perpendicular to $a$, of which rotation, or version, the power $a^t$, or the versor $Ua^t$, is considered (308) to be the instrument, or agent, or operator (comp. 293).  

334. Besides algebraical and transcendental forms, there are other results of operation on a quaternion, $q$, or on a function thereof, which may be regarded as forming a new class (or kind) of functions, arising out of the principles and rules of the Quaternion Calculus itself: namely those which we have denoted in former Chapters by the symbols,

$$
\text{I.} \ldots Kq, Sg, Vq, Ng, Tg, Ug,
$$

or by symbols formed through combinations of the same signs of operation, such as

$$
\text{II.} \ldots SUq, VUq, UUq, \text{&c.}
$$

And it is essential that we should know how to differentiate expressions of these forms, which can be done in the following manner, with the help of the principles of the present and former Chapters, and without now assuming the complanarity, $dq \parallel q$.

(1.) In general, let $f$ represent, for a moment, any distributive symbol, so that for any two quaternions, $q$ and $q'$, we shall have the equation,

$$
\text{III.} \ldots f(q + q') = f(q) + f(q');
$$

* Compare the first note to page 135.
and therefore also* (comp. 326, (5.)),

IV. . . \( f(xq) = x\zeta q \), if \( x \) be any scalar.

(2.) Then, with the notation 325, IV., we shall have

\[ \text{V. . . } f_n(q, q') = n\{f(q + n^{-1}q') - f\zeta q\} = f\zeta q'; \]

and therefore, by 325, VIII., for any such function \( f\zeta q \), we shall have the differential expression,

\[ \text{VI. . . } df\zeta q = f\zeta dq. \]

(3.) But \( S, V, K \) have been seen to be distributive symbols (197, 207); we can therefore infer at once that

\[ \text{VII. . . } dKq = Kdq; \quad \text{VIII. . . } dSq = Sdq; \quad \text{IX. . . } dVq = Vdq; \]

or in words, that the differentials of the conjugate, the scalar, and the vector of a quaternion are, respectively, the conjugate, the scalar, and the vector of the differential of that quaternion.

(4.) To find the differential of the norm, \( Nq \), or to deduce an expression for \( dNq \), we have (by VII. and 145) the equation,

\[ \text{X. . . } dNq = dq \cdot qKq = dq \cdot Kq' + q \cdot Kdq; \]

but

\[ qKq' = K \cdot q'Kq, \text{ by 145, and 192, II.}; \]

and

\[ (1 + K) \cdot q'Kq = 2S \cdot q'Kq = 2S(Kq \cdot q'), \text{ by 196, II., and 198, I.}; \]

therefore

\[ \text{XI. . . } dNq = 2S(Kq \cdot dq). \]

(5.) Or we might have deduced this expression XI. for \( dNq \), more immediately, by the general formula 324, IV., from the earlier expression 200, VII., or 210, XX., for the norm of a sum, under the form,

\[ \text{XI'. . . } dNq = \lim_{n \to \infty} n\{N(q + n^{-1}dq) - Nq\} \]

\[ = \lim_{n \to \infty} \{2S(Kq \cdot dq) + n^{-1}Ndq\} \]

\[ = 2S(Kq \cdot dq), \]

as before.

* In quaternions the equation III. is not a necessary consequence of IV., although the latter is so of the former; for example, the equation IV., but not the equation III., will be satisfied, if we assume \( f\zeta q = qeq^{-1}d\zeta q \), where \( e \) and \( e' \) are any two constant quaternions, which do not degenerate into scalars.
(6.) The tensor, \( T_q \), is the square-root (190) of the norm, \( N_q \); and because \( T_q \) and \( N_q \) are scalars, the formula 332, XXII., may be applied; which gives, for the differential of the tensor of a quaternion, the expression (comp. 158),

\[
XII. \quad dT_q = \frac{dN_q}{2T_q} = S(KU_q \cdot dq) = S \frac{dq}{U_q},
\]

a result which is more easily remembered, under the form,

\[
XIII. \quad \frac{dT_q}{T_q} = S \frac{dq}{q}.
\]

(7.) The versor \( U_q \) is equal (by 188) to the quotient, \( q : T_q \), of the quaternion \( q \) divided by its tensor \( T_q \); hence the differential of the versor is,

\[
XIV. \quad dU_q = \frac{dq}{T_q} = \left( \frac{dq}{q} - S \frac{dq}{q} \right) \frac{q}{T_q} = v \frac{dq}{q} \cdot U_q;
\]

whence follows at once this formula, analogous to XIII., and like it easily remembered,

\[
XV. \quad \frac{dU_q}{U_q} = v \frac{dq}{q}.
\]

(8.) We might also have observed that because (by 188) we have generally \( q = T_q \cdot U_q \), therefore (by 332, (3.)) we have also,

\[
XVI. \quad dq = dT_q \cdot U_q + T_q \cdot dU_q,
\]

and

\[
XVII. \quad \frac{dq}{q} = \frac{dT_q}{T_q} + \frac{dU_q}{U_q};
\]

if then we have in any manner established the equation XIII., we can immediately deduce XV.; and conversely, the former equation would follow at once from the latter.

(9.) It may be considered as remarkable, that we should thus have generally, or for any two quaternions, \( q \) and \( dq \), the formula:*

\[
XVIII. \quad S(dU_q : U_q) = 0; \quad \text{or} \quad XVIII'. \quad dU_q : U_q = S^{-1}0;
\]

* When the connexion of the theory of normals to surfaces, with the differential calculus of quaternions, shall have been (even briefly) explained in a subsequent Section, the student will perhaps be able to perceive, in this formula XVIII., a recognition, though not a very direct one, of the geometrical principle, that the radii of a sphere are its normals.
but this vector character of the quotient \( dUg: Ug \) can easily be confirmed, as follows. Taking the conjugate of that quotient, we have, by VII. (comp. 192, II.; 158; and 324, XI.),

\[
XIX. \ldots K (dUg \cdot Ug^{-1}) = KUg^{-1} \cdot dKUg = Ug \cdot d(Ug^{-1}) = -dUg \cdot Ug^{-1};
\]

whence

\[
XX. \ldots (1 + K) (dUg \cdot Ug^{-1}) = 0;
\]

which agrees (by 196, II.) with XVIII.

(10.) The scalar character of the tensor, \( Tg \), enables us always to write, as in the ordinary calculus,

\[
XXI. \ldots dTg = dTg : Tg;
\]

but \( lTg = Slq \), by 316, V.; the recent formula XIII. may therefore, by VIII., be thus written,

\[
XXII. \ldots SDLg = dSLg = dTg : dq = S(dq : g); \text{ or } XXII' \ldots dlq - q^{-1}dq = S^{-0}.
\]

(11.) When \( dq \parallel q \), this last difference vanishes, by 333, II.; and the equation XV. takes the form,

\[
XXIII. \ldots dlUq = Vdlq = dVlg.
\]

And in fact we have generally, \( lUq = Vlg \), by 316, XX., although the differentials of these two equal expressions do not separately coincide with the members of the recent formula XV., when \( q \) and \( dq \) are diplanar. We may however write generally (comp. XXII.),

\[
XXIV. \ldots dlUq - dUq : Uq = V(dlq - dq : g) = dlq - dq : g.
\]

335. We have now differentiated the six simple functions 334, I., which are formed by the operation of the six characteristics,

\[ K, S, V, N, T, U; \]

and as regards the differentiation of the compound functions 334, II., which are formed by combinations of those former operations, it is easy on the same principles to determine them, as may be seen in the few following examples.

(1.) The axis \( Ax. q \) of a quaternion has been seen (291) to admit of being represented by the combination \( UVg \); the differential of this axis may therefore, by 334, IX. and XIV., be thus expressed:
The differential of the axis is therefore, generally, a line perpendicular to that axis, or situated in the plane of the quaternion; but it vanishes, when the plane (and therefore the axis) of that quaternion is constant; or when the quaternion and its differential are complanar.

(2.) Hence,

\[ dUg = 0, \text{ if } dVg \parallel q; \]

and conversely this complanarity \( IV \). may be expressed by the equation \( III \).

(3.) It is easy to prove, on similar principles, that

\[ IV. \quad dVUg = V(V_{dq} \cdot Uq); \]

and

\[ V. \quad dSUq = S(V_{dq} \cdot Uq). \]

(4.) But in general, for any two quaternions, \( q \) and \( q' \), we have (comp. 223, (5.)) the transformations,

\[ VII. \quad S(V' \cdot q) = S(V' \cdot Vq) = S \cdot q' \cdot Vq; \]

and when we thus suppress the characteristic \( V \) before \( dq : q \), and insert it before \( Uq \), under the sign \( S \) in the last expression \( VI. \), we may replace the new factor \( VUq \) by \( TVUq \cdot UVUq \) (188), or by \( TVUq \cdot Uq \) (274, XIII.), or by \( TVUq \cdot Uq \) (204, V.), where the scalar factor \( TVUq \) may be taken outside (by 196, VIII.); also for \( q : Uq \) we may substitute \( 1 : (UVq \cdot q) \), or \( 1 : qUVq \), because \( UVq \parallel q \); the formula \( VI. \) may therefore be thus written,

\[ VIII. \quad dSUq = -S \frac{dq}{q' \cdot Uq} \cdot TVUq. \]

(5.) Now it may be remembered, that among the earliest connexions of quaternions with trigonometry, the following formulæ occurred (196, XVI., and 204, XIX.),

\[ IX. \quad SUq = \cos \angle q, \quad TVUq = \sin \angle q; \]

we had also, in 316, these expressions for the angle of a quaternion,

\[ X. \quad \angle q = TVlq = TlUq; \]
we may therefore establish the following expression for the differential of the angle of a quaternion,

\[ \text{XI. } \frac{dq}{qUVq} = dTVlq = dTVUq = S \frac{dq}{qUVq}. \]

(6.) The following is another way of arriving at the same result, through the differentiation of the sine instead of the cosine of the angle, or through the calculation of \( dTVUq \), instead of \( dSUq \). For this purpose, it is only necessary to remark that we have, by 334, XII. XIV., and by some easy transformations of the kind lately employed in (4.), the formula,

\[ \text{XII. } dTVUq = S \frac{VdUq}{UVUq} = S \frac{dUq}{UVq} = S \left( \frac{V dq}{q} \frac{Uq}{UVq} \right) = S \frac{dq}{qUVq}.SUq; \]

dividing which by \( SUq \), and attending to IX. and X., we arrive again at the expression XI., for the differential of the angle of a quaternion.

(7.) Eliminating \( Sdq:qUVq \) between VIII. and XII., we obtain the differential equation,

\[ \text{XIII. } (SUq)^2 + (TVUq)^2 = \text{const.}; \]

of which, on account of the scalar character of the differentiated variables, the integral is evidently of the form,

\[ \text{XIV. } (SUq)^2 + (TVUq)^2 = \text{const.}; \]

and accordingly we saw, in 204, XX., that the sum in the first member of this equation is constantly equal to positive unity.

(8.) The formula XI. may also be thus written,

\[ \text{XV. } \frac{dq}{qUVq} = S(Vdq.q):UVq; \]

with the verification, that when we suppose \( dq \parallel q \), as in IV., and therefore \( dUVq = 0 \) by III., the expression under the sign \( S \) becomes the differential of the quotient, \( Vq:UVq \), and therefore, by 316, VI., of the angle \( q \) itself.

336. An important application of the foregoing principles and rules consists in the differentiation of scalar functions of vectors, when those functions are defined and expressed according to the laws and notations of quaternions. It will be found, in fact, that such differentiations play a very extensive part, in the applications of quaternions to geometry; but, for the moment, we shall treat them here, as merely exercises of calculation. The following are a few examples.
(1.) Let $\rho$ denote, in these sub-articles, a variable vector; and let the following equation be proposed,

\[ r^2 + \rho^2 = 0, \quad \text{in which } \nabla r = 0, \]

so that $r$ is a (generally variable) scalar. Differentiating, and observing that, by 279, III., $\rho \rho' + \rho' \rho = 2S\rho \rho'$, if $\rho'$ be any second vector, such as we suppose $d\rho$ to be, we have, by 322, VIII., and 324, VII., the equation,

\[ rdr + S\rho d\rho = 0; \quad \text{or } \quad dr = -r^{-1}S\rho d\rho = rS\rho^{-1}d\rho. \]

In fact, if $r$ be supposed positive, it is here, by 282, II., the tensor of $\rho$; so that this last expression for $dr$ is included in the general formula, 334, XIII.

(2.) If this tensor, $r$, be constant, the differential equation II. becomes simply,

\[ S\rho d\rho = 0, \quad \text{if } -\rho^2 = \text{const.}, \quad \text{or if } dT\rho = 0. \]

(3.) Again, let the proposed equation be (comp. 282, XIX.),

\[ r^2 = T(\rho + \rho \kappa), \quad \text{with } d\iota = 0, \quad d\kappa = 0, \]

so that $\iota$ and $\kappa$ are here two constant vectors. Then, squaring and differentiating, we have (by 334, XI., because $\kappa'\rho = \rho'\kappa$, &c.),

\[ 2r^3dr = \frac{1}{2}dN(\iota + \rho \kappa) = S(\rho \iota + \rho \kappa)(d\rho + d\rho \kappa) = (i^2 + k^2) S\rho d\rho + 2S\kappa \rho d\rho; \]

or more briefly,

\[ 2r^3dr = S\nu d\rho, \]

if $\nu$ be an auxiliary vector, determined by the equation,

\[ r^4 \nu = (i^2 + k^2) \rho + 2S\kappa \rho \iota; \]

which admits of several transformations.

(4.) For example we may write, by 295, VII.,

\[ r^4 \nu = (i^2 + k^2) \rho + k\rho \iota + i\rho \kappa = \iota(\rho + \rho \kappa) + k(\rho \iota + \rho \kappa); \]

or, by 294, III., and 282, XII.,

\[ r^4 \nu = (i^2 + k^2) \rho + 2(\kappa S\rho - \rho S\kappa + iS\kappa) = (i - k)^2 \rho + 2(iS\kappa + kS\iota); \quad \&c. \]
(5.) The equation $V$ gives (comp. 190, $V$), when squared without differentiation,

$$XI. \ldots \rho^4 = N(\rho + \rho k) = (\rho + \rho k)(\rho + \rho k)$$

$$= (\rho^2 + k^2)(\rho^2 + \rho k \rho + \rho k \rho)$$

$$= (\rho^2 + k^2)(\rho^2 + 2S\rho k \rho)$$

$$= (\rho - k)^2 |\rho|^2 + 4S|\rho| S\rho k \rho = \xi_\rho,$$

by transformations of the same kind as before; we have therefore, by the recent expressions for $\rho^4$, the following remarkably simple relation between the two variable vectors, $\rho$ and $\nu$,

$$XII. \ldots S\rho \nu = 1; \text{ or } XII'. \ldots S\rho \nu = 1.$$

(6.) When the scalar, $r$, is constant, we have, by VII., the differential equation,

$$XIII. \ldots \rho v_{\rho \nu} = 0; \text{ whence also } XIV. \ldots \rho v_{\rho \nu} = 0, \text{ by XII.}$$

a relation of reciprocity thus existing, between the two vectors $\rho$ and $\nu$, of which the geometrical signification will soon be seen.

(7.) Meanwhile, supposing $r$ again to vary, we see that the last expression VI. for $2r^3 dr$ may be otherwise obtained, by taking half the differential of either of the two last expanded expressions XI. for $r^4$; it being remembered, in all these little calculations, that cyclical permutation of factors, under the sign $S$, is permitted (223, (10.)), even if those factors be quaternions, and whatever their number may be: and that if they be vectors, and if their number be odd, it is then permitted, under the sign $V$, to invert their order (295, (9.)), and so to write, for instance, $V_{\rho k} \rho$ instead of $V_{\rho k} \rho$, in the formula VIII.

(8.) As another example of a scalar function of a vector, let $p$ denote the proximity (or nearness) of a variable point $p$ to the origin $o$; so that

$$XV. \ldots p = (-p^2)^{-1} = Tp^{-1}, \text{ or } XV. \ldots p^2 + p^2 = 0.$$  

Then,

$$XVI. \ldots dp = S\rho dp, \text{ if } XVII. \ldots v = p^2 \rho = p^2 U \rho;$$

$v$ being here a new auxiliary vector, distinct from the one lately considered (VIII.), and having (as we see) the same versor (or the same direction) as the vector $\rho$ itself, but having its tensor equal to the square of the proximity of $p$ to $o$; or equal to the inverse square of the distance, of one of those two points from the other.

337. On the other hand, we have often occasion, in the applications, to consider vectors as functions of scalars, as in 99, but now with forms arising
out of operations on quaternions, and therefore such as had not been considered in the First Book. And whenever we have thus an expression such as either of the two following,

I. \( \rho = \phi(t) \), or II. \( \rho = \phi(s, t) \),

for the variable vector of a curve, or of a surface (comp. again 99), \( s \) and \( t \) being two variable scalars, and \( \phi(t) \) and \( \phi(s, t) \) denoting any functions of vector form, whereof the latter is here supposed to be entirely independent* of the former, we may then employ (comp. 100, (4.) and (9.) and the more recent sub-articles, 327, (5.), (6.), and 329, (5.) ) the notations of derivatives, total or partial; and so may write, as the differentiated equations, resulting from the forms I. and II. respectively, the following:

III. \( \frac{d\rho}{dt} = \phi'(t) \cdot dt = \rho' \cdot dt = D_{t\rho} \cdot dt \);

IV. \( \frac{d\rho}{dt} = \phi'(s, t) \cdot ds + \phi'(s, t) \cdot dt = D_{s\rho} \cdot ds + D_{t\rho} \cdot dt \);

of which the geometrical significations have been already partially seen, in the sub-articles to 100, and will soon be more fully developed.

(1.) Thus, for the circular locus, 314, (1.), for which

V. \( \rho = a^\pi \beta \), \( T \alpha = 1 \), \( S a \beta = 0 \),

we have, by 333, VIII., the following derived vector,

VI. \( \rho' = D_{t\rho} = \frac{\pi}{2} a^\pi \beta = \frac{\pi}{2} a \rho \).

(2.) And for the elliptic locus, 314, (2.), for which

VII. \( \rho = V. a^\pi \beta \), \( T \alpha = 1 \), but not \( S a \beta = 0 \),

we have, in like manner, this other derived vector,

VIII. \( \rho' = D_{t\rho} = \frac{\pi}{2} V. a^\pi \beta \).

(3.) As an example of a vector-function of more scalars than one, let us resume the expression (308, XVII.),

IX. \( \rho = r k^4 j^4 k^{-4} j^{-4} k^4 \);  

* We are therefore not employing here the temporary notation of some recent Articles, according to which we should have had, \( d\phi \rho = \phi(q, dq) \).
in which we shall now suppose that the tensor $r$ is given, so that $\rho$ is the variable vector of a point upon a given spheric surface, of which the radius is $r$, and the centre is at the origin; while $s$ and $t$ are two independent scalar variables, with respect to which the two partial derivatives of the vector $\rho$ are to be determined.

(4.) The derivation relatively to $t$ is easy; for, since $ijk$ are vector-units (295), and since we have generally, by 333, VIII.,

$$\mathbf{X} \ldots \; d \cdot a^{x} = \frac{\pi}{2} a^{x+1} dx,$$

and therefore

$$\mathbf{XI} \ldots \; D_{t} \cdot a^{x} = \frac{\pi}{2} a^{x+1} D_{t} a^{x},$$

if $Ta = 1$, and if $x$ be any scalar function of $t$, we may write, at once, by 279, IV.,

$$\mathbf{XII} \ldots \; D_{t} \rho = \frac{\pi}{2} (k \rho - \rho k) = \pi \nabla k \rho;$$

and we see that

$$\mathbf{XIII} \ldots \; \nabla_{t} D_{t} \rho = 0,$$

a result which was to be expected, on account of the equation,

$$\mathbf{XIV} \ldots \; \rho^{x} + r^{x} = 0,$$

which follows, by 308, XXIV., from the recent expression IX. for $\rho$.

(5.) To form an expression of about the same degree of simplicity, for the other partial derivative of $\rho$, we may observe that $j^{x+1}k^{x}$ is equal to its own vector part (its scalar vanishing); hence

$$\mathbf{XV} \ldots \; D_{s} \rho = \pi k^{x} j^{x} \rho; \quad \text{or} \quad \mathbf{XVI} \ldots \; D_{s} \rho = \pi k^{x} j^{x} \rho = \pi k^{x} j^{x} \rho;$$

by the transformation 308, (11.). And because the scalar of $k^{x}j^{x}$ is zero, we have thus the equation,

$$\mathbf{XVII} \ldots \; \nabla_{s} D_{t} \rho = 0,$$

which is analogous to XIII., and might have been otherwise obtained, by taking the derivative of XIV. with respect to the variable scalar $s$.

(6.) The partial derivative $D_{s} \rho$ must be a vector; hence, by XV. or XVI., $\rho$ must be perpendicular to the vector $k^{x}j^{x}$, or $k^{x}j^{x}$; a result which, under the last form, is easily confirmed by the expression 315, XII. for $\rho$. In fact that expression gives, by 315, (3.) and (4.), and by the recent values

* [Thus $D_{s}(j^{x}j^{x}) = \frac{\pi}{2} (j^{x+1}j^{x} + j^{x+1}j^{x}) = \pi j^{x+1}j^{x}$, and therefore $D_{s} \rho = \pi r k^{x}j^{x}k^{x}j^{x}k^{x}$ which is equivalent to XV.]
XII. XVI., these other forms for the two partial derivatives of $\rho$, which have been above considered:

$$\text{XVIII.} \quad D_\rho = \pi r k^2 V. j^2; \quad \text{XIX.} \quad D_\phi = \pi r (k^2 V. f^2 s^2 - V. k^2);$$

which might have been immediately obtained, by partial derivations, from the expression 315, XII. itself, and of which both are vector-forms.

(7.) And hence, or immediately by differentiating the expanded expression 315, XIII., we obtain these new forms for the partial derivatives of $\rho$:

$$\text{XX.} \quad D_\rho = \pi r (j \cos t\pi - i \sin t\pi) \sin s\pi;$$
$$\text{XXI.} \quad D_\phi = \pi r [(i \cos t\pi + j \sin t\pi) \cos s\pi - k \sin s\pi].$$

(8.) We may add that not only is the variable vector $\rho$ perpendicular to each of the two derived vectors, $D_\rho$ and $D_\phi$, but also they are perpendicular to each other; for we may write, by XII. and XVI.,

$$\text{XXII.} \quad S (L_\rho \cdot D_\phi) = \pi r S. k^2 ij = 0,$$

and the same conclusion may be drawn from the expressions XX. and XXI.

(9.) A vector may be considered as a function of three independent scalar variables, such as $r$, $s$, $t$; or rather it must be so considered, if it is to admit of being the vector of an arbitrary point of space: and then it will have a total differential (329) of the trinomial form,

$$\text{XXIII.} \quad d\rho = d_r \rho + d_s \rho + d_t \rho = D_\rho . dr + D_\rho . ds + D_\rho . dt;$$

and will thus have three* partial derivatives.

(10.) For example, when $\rho$ has the expression IX., we have this third partial derivative,

$$\text{XXIV.} \quad D_r \rho = r^{-1} \rho = U_\rho,$$

which may also be thus more fully written (comp. again 315, XIII.),

$$\text{XXV.} \quad D_r \rho = k^2 j^2 k^2 j^2 k^2 = (i \cos t\pi + j \sin t\pi) \sin s\pi + k \cos s\pi;$$

and we see that the three derived vectors,

$$\text{XXVI.} \quad D_r \rho, D_\phi \rho, D_\phi \rho,$$

compose here a rectangular system.

* That is to say, three of the first order; for we shall soon have occasion to consider successives differentials, of functions of one or more variables, and so shall be conducted to the consideration of orders of differentials and derivatives, higher than the first.
SECTION 5.

On Successive Differentials, and Developments, of Functions of Quaternions.

338. There will now be no difficulty in the successive differentiation, total or partial, of functions of one or more quaternions; and such differentiation will be found to be useful, as in the ordinary calculus, in connexion with developments of functions: besides that it is necessary for many of those geometrical and physical applications of differentials of quaternions, on which we have not entered yet. A few examples of successive differentiation may serve to show, more easily than any general precepts, the nature and effects of the operation; and we shall begin, for simplicity, with explicit functions of one quaternion variable.

(1.) Take then the square, \( q^2 \), of a quaternion, as a function \( f_q \), which is to be twice differentiated. We saw, in 324, VII., that a first differentiation gave the equation,

\[
I. \quad df_q = d \cdot q^2 = q \cdot dq + dq \cdot q;
\]

but we are now to differentiate again, in order to form the second differential \( d^2 f_q \) of the function \( q^2 \), treating the differential of the variable \( q \) as still equal to \( dq \), and in general writing \( ddq = d^2 q \), where \( d^2 q \) is a new arbitrary quaternion, of which the tensor, \( Td^2 q \), need not be small (comp. 322). And thus we get, in general, this twice differentiated expression, or differential of the second order,

\[
II. \quad d^2 f_q = d^2 \cdot q^2 = q \cdot d^2 q + 2dq \cdot q.
\]

(2.) The second differential of the reciprocal of a quaternion is generally (comp. 324, XL),

\[
III. \quad d^2 \cdot q^{-1} = 2(q^{-1}dq)q^{-1} - q^{-1}dq^2q^{-1}.
\]

(3.) If \( p \) be a variable vector, then (comp. 336, (1.)) we have, for the first and second differentials of its square, the expressions:

\[
IV. \quad d \cdot p^2 = 2Spdp; \quad V. \quad d^2 \cdot p^2 = 2Spd^2p + 2dp^2.
\]

(4.) If \( f_\rho \) be any other scalar function of a variable vector \( \rho \), and if (comp. again the sub-articles to 336) its first differential be put under the form,

\[
VI. \quad df_\rho = 2Svdp, \text{ when } v \text{ is another variable vector,}
\]
then the second differential of the same function may be expressed as follows:

\[ \text{VII. } d^2f = 2Svd^2v + 2Svd\rho ; \]

in which we have written, briefly, Sdv\rho, instead of S(dv \cdot d\rho).

(5.) The following very simple equation will be found useful, in the theory of motions, performed under the influence of central forces:

\[ \text{VIII. } d\mathbf{V}d\rho = \mathbf{V}d\rho^2 ; \text{ because } d\mathbf{V}d\rho^2 = 0. \]

(6.) As an example of the second differential of a quaternion, considered as a function of a scalar variable (comp. 333, VIII., and 337, (1.)), the following may be assigned, in which \( a \) denotes a given unit line, so that \( a^2 = -1 \), \( da = 0 \), but \( x \) is a variable scalar:

\[ \text{IX. } d^2 a^x = d\left(\frac{\pi}{2} a^{x+1}dx \right) = \frac{\pi}{2} a^{x+1}dx - \left(\frac{\pi}{2}\right)a^x dx. \]

(7.) The second differential of the product of any two functions of a quaternion \( q \) may be expressed as follows (comp. II.):

\[ \text{X. } d^2(fq \cdot \phi q) = d^2fq \cdot \phi q + 2d\phi q \cdot d^2\phi q. \]

339. The second differential, \( d^2q \), of the variable quaternion \( q \), enters generally (as has been seen) into the expression of the second differential \( d^2fq \), of the function \( fq \), as a new and arbitrary quaternion: but, for that very reason, it is permitted, and it is frequently found to be convenient, to assume that this second differential \( d^2q \) is equal to zero: or, what comes to the same thing, that the first differential \( dq \) is constant. And when we make this new supposition,

\[ \text{I. } dq = \text{constant}, \text{ or } \text{I'. } d^2q = 0, \]

the expressions for \( d^2fq \) become of course more simple, as in the following examples.

(1.) With this last supposition, I. or I', we have the following second differentials, of the square and the reciprocal of a quaternion:

\[ \text{II. } d^2q^2 = 2dq^2; \text{ III. } d^2q^{-1} = 2(q^{-1}dq)^2q^{-1} = 2q^{-1}dq\cdot q^{-1}. \]

(2.) Again, if we suppose that \( c_0, c_1, c_2 \) are any three constant quaternions, and take the function,

\[ \text{IV. } fq = c_0q, c_1q, c_2q, \]
we find, under the same condition I. or I', that its first and second differentials are,

\[ V \ldots d_1 f = c_0 dq \cdot c_0 q c_2 + c_0 q c_0 dq \cdot c_2; \quad VI \ldots d_2 f = 2c_0 dq \cdot c_1 dq \cdot c_2; \]

in writing which, the points* may be omitted.

(3.) The first differential, \( dq \), remaining still entirely arbitrary (comp. 322, (8.), and 325, (2.)), so that no supposition is made that its tensor \( Tdq \) is small, although we now suppose this differential \( dq \) to be constant (I.) we have rigorously,

\[ VII \ldots (q + dq)^2 = q^2 + d q^2 + \frac{1}{2} d^2 q^2; \]

an equation which may be also written thus,

\[ VIII \ldots (q + dq)^2 = (1 + d + \frac{1}{3} d^2) q^2. \]

(4.) And in like manner we shall have, more generally, under the same condition of constancy of \( dq \), the equation,

\[ IX \ldots f(q + dq) = (1 + d + \frac{1}{2} d^2) f(q); \]

if the function \( f(q) \) be the sum of any number of monomes, each separately of the form IV., and therefore each rational, integral, and homogeneous of the second dimension, with respect to the variable quaternion, \( q \); or of such monomes, combined with others of the first dimension, and with constant terms: that is, if \( a_0, b_0, b_1, b'_0, b'_1, \ldots \) and \( c_0, c_1, c_2, c'_0, c'_1, c'_2, \ldots \) be any constant quaternions, and

\[ X \ldots f(q) = a_0 + \Sigma b_0 q b_1 + \Sigma c_0 q c_0 q c_2. \]

340. It is easy to carry on the operation of differentiating, to the third and higher orders; remembering only that if, in any former stage, we have denoted the first differentials of \( q, dq, \ldots \) by \( dq, d^2 q, \ldots \) we then continue so to denote them, in every subsequent stage of the successive differentiation: and that if we find it convenient to treat any one differential as constant, we must then treat all its successive differentials as vanishing. A few examples may be given, chiefly with a view to the extension of the recent formula 339, IX., for the function \( f(q + dq) \) of a sum, of any two quaternions, \( q \) and \( dq \), to polynomial forms, of dimensions higher than the second.

* Compare the second Note to page 439.
(1.) The third differential of a square is generally (comp. 338, II.),

\[ d^3 q^2 = q \cdot d^3 q + d^3 q \cdot q + 3 (dq \cdot d^2 q + d^2 q \cdot dq). \]

(2.) More generally, the third differential of a product of two quaternion functions (comp. 338, X.) may be thus expressed:

\[ d^3 (fq \cdot \phi q) = d^3 fq \cdot \phi q + 3 d^2 fq \cdot d\phi q + 3 dfq \cdot d^2 \phi q + dfq \cdot d^3 \phi q. \]

(3.) More generally still, the \( n \)th differential of a product is, as in the ordinary calculus,

\[ d^n (fq \cdot \phi q) = d^n fq \cdot \phi q + n d^{n-1} fq \cdot d\phi q + n_s d^{s-1} fq \cdot d^s \phi q + \ldots + fq \cdot d^n \phi q, \]

if

\[ n_s = \frac{n(n-1)}{2}, \quad n_s = \frac{n(n-1)(n-2)}{2 \cdot 3}, \quad \text{&c.} \]

the only thing peculiar to quaternions being, that we are obliged to retain (generally) the order of the factors, in each term of this expansion III.

(4.) Hence, in particular, denoting briefly the function \( fq \) by \( r \), and changing \( \phi q \) to \( q \),

\[ d^n rq = d^n r \cdot q + nd^{n-1} r \cdot dq, \quad \text{if} \quad d^2 q = 0. \]

(5.) Hence also, under this condition that \( dq \) is constant, if \( c \) be any other constant quaternion, we have the transformation,

\[ (1 + d + \frac{1}{3} d^2 + \frac{1}{2 \cdot 3} d^3 + \ldots + \frac{1}{2 \cdot 3 \ldots n} d^n) \cdot r c = (1 + d + \frac{1}{3} d^2 + \frac{1}{2 \cdot 3} d^3 + \ldots + \frac{1}{2 \cdot 3 \ldots (n-1)} d^{n-1}) r \cdot (q + dq) c, \quad \text{if} \quad d^n r = 0. \]

(6.) Hence, by 339, (4.), it is easy to infer that if we interpret the symbol \( \varepsilon^d \) by the equation (comp. 316, I.),

\[ \varepsilon^d = 1 + d + \frac{1}{3} d^2 + \frac{1}{2 \cdot 3} d^3 + \&c., \]

that is, if we interpret this other symbol \( \varepsilon^d f q \), as concisely denoting the series which is formed from \( f q \), by operating on it with this symbiotic development; and if the function \( f q \), thus operated on, be any finite polynome, involving (like the expression 339, X.) no fractional nor negative exponents; we may then write, as an extension of a recent equation (339, IX.) the formula:

\[ \varepsilon^d f q = f (q + dq), \quad \text{if} \quad d^2 q = 0; \]
which is here a perfectly rigorous one, \textit{all} the terms of this expansion for a function of a sum of two quaternions, $q$ and $dq$, becoming separately equal to zero, as soon as the symbolic exponent of $d$ becomes greater than the dimension of the polynome.

(7.) We shall soon [342] see that there is a sense, in which this exponential transformation VII. may be extended, to other functional forms which are not composed as above: and that thus an analogue of Taylor’s Theorem can be established for Quaternions. Meanwhile it may be observed that by changing $dq$ to $\Delta q$, in the finite expansion obtained as above, we may write the formula as follows:

\[\text{VIII. } \varepsilon^m f/q = f(q + \Delta q) = (1 + \Delta) f/q, \quad \text{or briefly, IX. } \varepsilon^m = 1 + \Delta;\]

which last symbolical equation may be operated on, or transformed, as in the usual calculus of differences and differentials. For instance, it being understood that we treat $\Delta^2 q$ as well as $d^2 q$ as vanishing, we have thus (for any positive and whole exponent $m$), the two following transformations of IX.,

\[\text{X. } \Delta^m = (\varepsilon^m - 1)^m, \quad \text{and XI. } d^m = (\log (1 + \Delta))^m;\]

the results of operating, with the symbols thus equated, on any polynomial function $f/q$, of the kind above described, being always finite expansions, which are rigorously equal to each other.

341. Let $Fx$ and $\phi x$ be any two functions of a scalar variable, of which both vanish with that variable; so that they satisfy the two conditions,

\[\text{I. } F0 = 0, \quad \phi 0 = 0.\]

Then the \textit{three simultaneous values},

\[\text{II. } x, Fx, \phi x,\]

of the variable and the two functions, are at the same time (comp. 320, 321) \textit{three simultaneous differences}, as compared with this other system of three simultaneous values,

\[\text{III. } 0, F0, \phi 0.\]

If, then, any \textit{equimultiples},

\[\text{IV. } nx, nFx, n\phi x,\]

of the three values II., can be made, by any suitable increase of the number, $n$, combined with a decrease of the variable, $x$, to tend together to any system of
limits, those limits must (by the definition in 320, compare again 321) admit of being considered as a system of simultaneous differentials,

\[ V \ldots dx, \; dF, \; d\phi, \]

answering to the system of initial values III.; and must be proportional to the ultimate values of the connected system of derivatives,

\[ VI. \ldots 1, \; F', \; \phi', \] when \( x \) tends to zero.

We may therefore write, as expressions for those ultimate values of the two last derived functions,

\[ VII. \ldots F'0 = \lim_{n \to \infty} nF', \; \phi'0 = \lim_{n \to \infty} n\phi', \] if \( F0 = \phi0 = 0. \]

And even if these last values vanish, or if the two new conditions

\[ VIII. \ldots F'0 = 0, \; \phi'0 = 0, \]

are satisfied, so that \( x, \; F', \) and \( \phi' \) are now (comp. II.) a new system of simultaneous differences, we may still establish the following equation of limits of quotients, which is independent of these last conditions VIII.,

\[ IX. \ldots \lim_{x \to 0} (F : \phi) = \lim_{x \to 0} (F' : \phi'), \] if \( F0 = \phi0 = 0; \]

it being understood that, in certain cases, these two quotients may both vanish with \( x; \) or may tend together to infinity, when \( x \) tends, as before, to zero.

(1.) This theorem is so important, that it will not be useless to confirm it by a geometrical illustration, which may at the same time serve for a geometrical proof; at least for the extensive case where both the functions \( f \) and \( \phi \) are of scalar forms, and consequently may be represented, or constructed, by the corresponding ordinates, \( XY \) and \( XZ \) (or ordinates answering to one common abscissa \( OX \)), of two curves \( OY \) and \( Oz \), which are in one plane, and set out from (or pass through) one common origin \( O \), as in the annexed figure 75.

We shall afterwards see that the result, so obtained, can be extended to quaternion functions.

(2.) Suppose then, first, that the ordinates of these two curves are proportional, or that they bear to each other one fixed and constant ratio; so that the equation,

\[ X \ldots XY : XZ = xy : xz, \]

is satisfied for every pair of abscissa, \( OX \) and \( Ox \), however great or small the corresponding ordinates may be. Prolonging then (if necessary) the chord
of the first curve, to meet the axis of abscissae in some point \( t \), and so to determine a subsecant \( tX \), we see at once (by similar triangles) that the corresponding chord \( Zs \) of the second curve will meet the same axis in the same point, \( t \); and therefore that it will determine (rigorously) the same subsecant, \( tX \).

(3.) Hence, if the point \( x \) be conceived to approach to \( X \), so that the secant \( Yyt \) of the first curve tends to coincide with the tangent \( YT \) to that curve at the point \( Y \), the secant \( Zzt \) of the second curve must tend to coincide with the line \( ZT \), which line therefore must be the tangent to that second curve: or in other words, corresponding subtangents coincide, and of course are equal, under the supposed condition \( X \), of a constant proportionality of ordinates.

(4.) Suppose next that corresponding ordinates only tend to bear a given or constant ratio to each other; or that their (now) variable ratio tends to a given or fixed limit, when the common abscissa is indefinitely diminished, or when the point \( X \) tends to \( O \); and let \( T \) be still the variable point in which the tangent to the first curve at \( T \) meets the axis, so that the line \( TX \) is still the first subtangent. Then the corresponding tangent to the second curve at \( Z \) will not in general pass through the point \( T \), but will meet the axis in some different point \( U \). But the ratio of the two corresponding subtangents, \( TX \) and \( UX \), which had been a ratio of equality, when the condition of proportionality \( X \) was satisfied rigorously, will now at least tend to such a ratio; so that we shall have, under this new condition, of tendency to proportionality of ordinates, the limiting equation,

\[ \lim (TX : UX) = 1; \]

whence the equation \( IX \) results, under the geometrical form,

\[ \lim (\tan XTY : \tan XUZ) = \lim (XY : XZ). \]

(5.) We might also have observed that, when the proportion \( X \) is rigorous, corresponding areas* (such as \( xXYy \) and \( xXZz \)) of the two curves are then exactly in the given ratio of the ordinates; so that this other equation, or proportion,

\[ \lim (OXYyO : OXZzO) = XY : XZ, \]

* Compare the Fourth Lemma of the First Book of the Principia; and see especially its Corollary, in which the reasoning of the present sub-article is virtually anticipated.
is then also rigorous. Hence if we only suppose, as in (4.), that the ordinates
tend to some fixed limiting ratio, the areas must tend to the same; so that if
the second member of the equation IX. have any definite value, as a limit, the
first member must have the same; whereas the recent proof, by subtangents,
served rather to show that if the first (or left hand) limit in IX. existed, then
the second limit in that equation existed also, and was equal to the first.

(6.) If the function $F_x$ be a quaternion, we may (by 221) express it as
follows,

$$XIV. \quad F_x = W + iX + jY + kZ,$$

where $W$, $X$, $Y$, $Z$ are four scalar functions of $x$, of which each separately can
be constructed, as the ordinate of a plane curve; and the recent geometrical*
reasoning will thus apply to each of them, and therefore to their linear combi­
nation $F_x$: which quaternion function reduces itself to a vector function of $x$,
when $W = 0$.

(7.) And if $\phi x$ were another quaternion or vector function, we might first
substitute it for $F_x$, and then eliminate the scalar function $\phi x$; so that a
limiting equation of the form IX. may thus be proved to hold good, when
both the functions compared are vectors, or quaternions, supposed still to vanish
with $x$.

(8.) The general considerations, however, on which the equation IX., was
lately established, appear to be more simple and direct; and it is evident
that they give, in like manner, this other but analogous equation, in which
$F^{"}x$ and $\phi^{"}x$ are second derivatives, and the conditions VIII. are now supposed
to be satisfied:

$$XV. \quad \lim_{x \to 0} F^{"}x : \phi^{"}x = \lim_{x \to 0} (F^{"}x : \phi^{"}x), \quad \text{if } F'0 = 0, \phi'0 = 0.$$

* Instead of the equation IX., it has become usual, in modern works on the Differential Calculus,
to give one of the following form (deduced from principles of Lagrange):

$$\frac{F(x)}{\phi(x)} = \frac{F'(\delta x)}{\phi'(\delta x)}, \quad \text{if } F(0) = \phi(0) = 0;$$

$\delta$ denoting some proper fraction, or quantity between 0 and 1. And a geometrical illustration, which
is also a geometrical proof, when the functions $F_x$ and $\phi x$ can be constructed (or conceived to be con­
bructed) as the ordinates of two plane curves, is sometimes derived from the axiom (or geometrical
intuition), that the chord of any finite and plane arc must be parallel to the tangent, drawn at some
point of that finite arc. But this parallelism no longer exists, in general, when the curve is one of
double curvature; and accordingly the equation in this note is not generally true, when the functions
are quaternions; or even when one of them is a quaternion, or a vector.
And so we might proceed, as long as successive derivatives, of higher orders, continue to vanish together.

(9.) Hence, in particular, if we take this scalar form,

\[ \phi x = \frac{x^m}{2 \cdot 3 \ldots m}, \]

which evidently gives the values,

\[ \phi 0 = 0, \quad \phi' 0 = 0, \quad \phi'' 0 = 0, \ldots \phi^{(m-1)} 0 = 0, \quad \phi^{(m)} 0 = 1, \]

and if we suppose that the function \( Fx \) is such that

\[ F 0 = 0, \quad F' 0 = 0, \quad F'' 0 = 0, \ldots \quad F^{(m-1)} 0 = 0, \]

while \( F^{(m)} 0 \) has any finite value, we may then establish this limiting equation:

\[ \lim (Fx : \phi x) = F^{(m)} 0; \]

in which the function \( Fx \), and the value \( F^{(m)} 0 \), are here supposed to be generally quaternions; although they may happen, in particular cases, to reduce themselves (292) to vectors, or to scalars.

342. It will now be easy to extend the Exponential Transformation 340, VII.; and to show that there is a sense in which that very important Formula,

\[ e^{d\phi x} = f(q + dq), \] \[ \text{if } d^2 q = 0, \]

which is, in fact, a known* mode of expressing the Series or Theorem of Taylor, holds good for Quaternion Functions generally, and not merely for those functions of finite and polynomial form, with positive and whole exponents, for which it was lately deduced, in 340, (6.). For let \( \phi q \) and \( f(q + dq) \) denote any two states, or values, of which neither is infinite, of any function of a quaternion; and of the \( m \) first differentials,

\[ d\phi q, \quad d^2 \phi q, \ldots \quad d^m \phi q, \]

let it be supposed that no one is infinite, and that the last of them is different from zero; while all that precede it, and the functions \( \phi q \) and \( f(q + dq) \) themselves, may or may not happen to vanish. Let the first \( m \) terms, of the

\[ \sum_{n=0}^{m} \frac{d^n \phi q}{n!}, \]

where \( d \) denotes the value which the function \( u \) receives, when the variable \( x \) receives the arbitrary increment \( dx \) (l'accroissement quelconque \( dx \)).

* Lacroix, for instance, in page 168 of the First Volume of his larger Treatise on the Differential and Integral Calculus (Paris, 1810), presents the Theorem of Taylor under the form,

\[ u' = u + \frac{d \phi q}{1} + \frac{d^2 \phi q}{1.2} + \frac{d^3 \phi q}{1.2.3} + \frac{d^4 \phi q}{1.2.3.4} + \&c.; \]

where \( u' \) denotes the value which the function \( u \) receives, when the variable \( x \) receives the arbitrary increment \( dx \) (l'accroissement quelconque \( dx \)).
exponential development of the symbol \((\mathbf{e}^d - 1)\mathbf{f}_q\), be denoted briefly by \(g_1, g_2, \ldots g_m\); and let \(r_m\) denote what may be called the remainder of the series, or the correction which must be conceived to be added to the sum of these \(m\) terms, in order to produce the exact value of the difference,

\[
\text{III. } \Delta \mathbf{f}_q = \mathbf{f}(q + \Delta q) - \mathbf{f}_q = \mathbf{f}(q + dq) - \mathbf{f}_q;
\]
in such a manner that we shall have rigorously, by the notations employed, the equation,

\[
\text{IV. } \mathbf{f}(q + dq) = \mathbf{f}_q + g_1 + g_2 + \ldots + g_m + r_m, \quad \text{where } g_m = \frac{d^{m+1}\mathbf{f}_q}{2 \cdot 3 \cdots m};
\]
this term \(g_m\) being different from zero, but no one of the terms being infinite, by what has been above supposed. Then we shall prove, as a Theorem, that

\[
\text{V. } \lim (\mathbf{T}_m : \mathbf{T}_q) = 0, \quad \text{if } \lim \mathbf{T}_{dq} = 0;
\]
or in words, that the tensor of the remainder may be made to bear as small a ratio as we please, to the tensor of the last term retained, by diminishing the tensor, without changing the versor, of the differential (or difference) \(dq\). And this very general result, which will soon be seen to extend to functions of several quaternions, is in the present Calculus that analogue of Taylor’s theorem to which we lately alluded (in 340, (7.)); and it may be called, for the sake of reference, “Taylor’s Theorem adapted to Quaternions.”

(1.) Writing

\[
\text{VI. } F_x = \mathbf{f}(q + xdq) - \mathbf{f}_q - xd\mathbf{f}_q - \frac{x^2}{2} d^2\mathbf{f}_q - \ldots - \frac{x^{m-1}}{2 \cdot 3 \cdots (m-1)} d^{m-1}\mathbf{f}_q,
\]
we shall have the following successive derivatives with respect to \(x\),

\[
\begin{align*}
F_x &= d\mathbf{f}(q + xdq) - d\mathbf{f}_q - xd^2\mathbf{f}_q - \ldots - \frac{x^{m-2}}{2 \cdot 3 \cdots (m-2)} d^{m-2}\mathbf{f}_q; \\
F''_x &= d^2\mathbf{f}(q + xdq) - d^2\mathbf{f}_q - \ldots - \frac{x^{m-3}}{2 \cdot 3 \cdots (m-3)} d^{m-3}\mathbf{f}_q; \\
F^{(m-1)}_x &= d^{m-1}\mathbf{f}(q + xdq) - d^{m-1}\mathbf{f}_q; \quad \text{and finally,} \\
F^{(m)}_x &= d^m\mathbf{f}(q + xdq);
\end{align*}
\]
because, by 327, VI., and 324, IV.,

\[
\text{VIII. } D\mathbf{f}(q + xdq) = \lim_{n \to \infty} n \{f(q + xdq + n^3dq) - f(q + xdq)\} = d\mathbf{f}(q + xdq),
\]
and in like manner,

\[
\text{IX. } D^2\mathbf{f}(q + xdq) = d^2\mathbf{f}(q + xdq), \quad \&c.;
\]
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the mark of derivation D referring to the scalar variable x, while d operates
on q alone, and not here on x, nor on dq.

(2.) We have therefore, by VI. and VII., the values,

\[ X \ldots F_0 = 0, \quad F'0 = 0, \quad F''0 = 0, \ldots F^{(m-1)}0 = 0, \quad F^{(m)}0 = q^m f_q; \]

whence, by 341, XIX., we have this limiting equation,

\[ \lim_{x \to 0} \left( \frac{F_x}{x} \right) = \frac{d^m f_q}{2.3 \ldots m}; \]

or

\[ \lim_{x \to 0} \left( F_x : \psi x \right) = 1, \quad \psi x = \left( \frac{d^m d^m f_q}{2.3 \ldots m} \right). \]

(3.) But these two functions, \( F_x \) and \( \psi x \), are formed by IV. from \( q_m + r_m \) and \( q_m \), by changing \( dq \) to \( x dq \); and instead of thus multiplying \( dq \) by a
decreasing scalar, \( x \), we may diminish its tensor \( Tdq \), without changing its
versor \( V dq \). We may therefore say that, when this is done, the quotient
\( (q_m + r_m) : q_m \) tends to unity, or this other quotient \( r_m : q_m \) to zero, as its limit;
or in other words, the limiting equation V. holds good.

(4.) As an example, let the function \( f_q \) be the reciprocal, \( q^{-1} \); then (comp.
339, III.) its \( m \)th differential is (for \( dq = \text{const.} \)),

\[ \frac{d^m f_q}{dq} = d. q^{-1} = 2 \cdot 3 \ldots m \cdot q^{-1} (-r)^m, \quad \text{if} \quad r = dq \cdot q^{-1}; \]

and it is easy to prove, without differentials, that

\[ (g + r) q^{-1} = q^{-1} (1 + r)^{-1} = q^{-1} \{ 1 - r + r^2 - \ldots + (-r)^m + (-r)^{m+1} (1 + r)^{-1} \}; \]

we have therefore here

\[ q_m = q^{-1} (-r)^m, \quad r_m = - q_m r (1 + r)^{-1}, \quad T (r_m : q_m) = Tr \cdot T (1 + r)^{-1}; \]

and this last tensor indefinitely diminishes with \( Tdq \), the quaternion \( q \) being
supposed to have some given value different from zero.

(5.) In general, if we establish the following equation,

\[ f'(g + n^{-1} dq) = f_q + n^{-1} df_q + \frac{n^2}{2} d^2 f_q + \ldots + \frac{n^{1-m}}{2 \cdot 3 \ldots (m-1)} d^{m-1} f_q \]

\[ + \frac{n^{-m}}{2 \cdot 3 \ldots m} f_n (q, dq), \]

as a definitional extension of the equation 325, V.; and if we suppose that
neither the function \( f_q \) itself, nor any one of its differentials as far as \( d^{m-1} f_q \)
is infinite; the result contained in the limiting equation XI. may then be
expressed by the formula,

\[ f_x^{(m)} (q, dq) = d^m f_q; \]
which for the particular value \( m = 1 \), if we suppress the upper index, coincides with the form 325, VIII. of the definition \( dfx \), but for higher values of \( m \) contains a theorem: namely (when \( d^m f g \) is supposed neither to vanish, nor to become infinite), what we have called Taylor’s Theorem adapted to Quaternions.

343. That very important theorem may be applied to cases, in which a quaternion (as in 327, (5.)), or a vector (as in 337), is expressed as a function of a scalar; also to transcendental forms (333), whenever the differentiations can be effected; and to those new forms (334), which result from the peculiar operations of the present Calculus itself. A few such applications may here be given.

(1.) Taking first this transcendental and quaternion function of a variable scalar,

\[ \ldots q = \frac{df}{dx} = \alpha^t, \quad \text{with} \quad Ta = 1, \quad da = 0, \quad dt = \text{const.}, \]

we have, by 333, VIII., the general term,

\[ \Pi \ldots q_m = \frac{d^m \alpha^t}{2.3\ldots m} = \frac{\alpha^t}{2.3\ldots m} \left(\frac{\pi a^t}{2}\right)^m = \frac{\alpha^t (xa)^m}{2.3\ldots m}, \quad \text{if} \quad 2\pi = \pi dt; \]

dividing then \( \alpha^t \cdot \alpha^t \) by \( \alpha^t \), we obtain an infinite series, which is found to be correct, and convergent; namely (comp. 308, (4.)),

\[ \Pi \ldots \alpha dt = 1 + xa + \frac{(xa)^2}{2} + \ldots + \frac{(xa)^m}{2.3\ldots m} + \ldots = \epsilon xa = \cos \frac{\pi dt}{2} + a \sin \frac{\pi dt}{2}. \]

(2.) Correct and finite expansions, for \( S (q + dq) \), \( V (q + dq) \), \( K (q + dq) \), and \( N (q + dq) \), are obtained when we operate with \( \epsilon^a \cdot \alpha^t \) on \( Sg, Vg, Kg, \) and \( Ng \); for example (\( dq \) being still constant), the third and higher differentials of \( Ng \) vanish by 334, XI., and we have

\[ \Pi \ldots \epsilon^a Ng = (1 + d + \frac{1}{2} d^2) Ng = Ng + 2S(Kg \cdot dq) + Ndq = N (q + dq); \]

an expression for the norm of a sum, which agrees with 210, XX., and with 200, VII.

(3.) To develop, on like principles, the tensor and versor of a sum, let us again write \( r \) for \( dq : q \), and denote the scalar and vector parts of this quotient by \( s \) and \( v \); so that, by 334, XIII. and XV.,

\[ \Pi \ldots s = Sr = S \frac{dq}{q} = \frac{dTq}{Tq}; \quad VI \ldots v = Vr = V \frac{dq}{q} = \frac{dUq}{Uq}. \]

(4.) Then writing also, for abridgment, as in a known notation of factorials,

\[ \Pi \ldots [-1]^m = (-1) (-2) (-3) \ldots (-m), \]
we shall have, by 342, XIII., \(dq\) being still treated as constant, the equation,

\[VIII... d^m(s + v) = d^{m-1} = [-1]^{m+1} = [-1] (s + v)^{m+1},\]

of which it is easy to separate the scalar and vector parts; for example,

\[IX... ds = - s \cdot (s + v) = - (s^2 + v^2); \quad dv = - V \cdot (s + v) = - 2sv.\]

(5.) We have also, by V. and VI.,

\[X... \frac{d^m Tq}{Tq} = (s + d) \frac{d^{m-1} Tq}{Tq} = ... = (s + d)^m 1;\]

\[XI... \frac{d^m Uq}{Uq} = (v + d) \frac{d^{m-1} Uq}{Uq} = ... = (v + d)^m 1;\]

the notation being such that we have, for instance, by IX.,

\[XII... (s + d)^1 = s; \quad (s + d)^2 = (s + d) s = s^2 + ds = s^2;\]

\[XIII... (v + d)^1 = v; \quad (v + d)^2 = (v + d) v = v^2 + dv = v^2 - 2sv.\]

(6.) The exponential formula 342, I., gives, therefore,

\[XIV... T(q + dq) = e^{s+1} Tq;\]

\[XV... U(q + dq) = e^{v+1} Uq;\]

or, dividing and substituting,

\[XVI... T(1 + s + v) = e^{s+1} 1; \quad XVII... U(1 + s + v) = e^{v+1} 1;\]

\(s\) and \(v\) being here a scalar and a vector, which are entirely independent of each other; but of which, in the applications, the tensors must not be taken too large, in order that the series may converge.

(7.) The symbolical expressions, XVI. and XVII., for those two series, may be developed by (4.) and (5.); thus, if we only write down the terms which do not exceed the second dimension, with respect to \(s\) and \(v\), we have by XII. and XIII. the development,

\[XVIII... T(1 + s + v) = 1 + s - \frac{1}{2} v^2 + ...;\]

\[XIX... U(1 + s + v) = 1 + v + \left(\frac{1}{2} v^2 - sv\right) + ...;\]

of which accordingly the product is 1 + \(s + v\), to the same order of approximation.

(8.) A function of a sum of two quaternions can sometimes be developed, without differentials, by processes of a more algebraical character; and when
this happens, we may compare the result with the form given by Taylor's Series, as adapted to quaternions in 342, and so deduce the values of the successive differentials of the function; for example, we can infer the expression 342, XIII. for $d^n q^{-1}$, from the series 342, XIV., for the reciprocal of a sum.

(9.) And not only may we verify the recent developments, XVIII. and XIX., by comparing them with the more algebraical forms,*

\[ XX. . . T (1 + s + v) = (1 + s + v)^{3/2} (1 + s - v)^{3/2}, \]

\[ XXI. . . U (1 + s + v) = (1 + s + v)^{3/2} (1 + s - v)^{-3/2}, \]

but also, if the first of these, for example (when expanded by ordinary processes, which are in this case applicable), have given us, without differentials,

\[ XXII. . . T (q + q') = (1 + s - \frac{1}{2}v^2 \ldots) T_q, \]

where $s = Sqq^{-1}$, and $v = \nabla q'q^{-1}$, we can then infer the values of the first and second differentials of the tensor of a quaternion, as follows:

\[ XXIII. . . dT_q = S \frac{dq}{q}. T_q; \quad d^2 T_q = - \left( \frac{\nabla dq}{q} \right)^2 T_q; \]

whereof the first agrees with 334, XII. or XIII., and the second can be deduced from it, under the form,

\[ XXIV. . . d^2 T_q = d \left( S \frac{dq}{q}. T_q \right) = \left( \left( S \frac{dq}{q} \right)^2 - S \left( \frac{dq}{q} \right)^2 \right) T_q. \]

(10.) In general, if we can only develop a function $f(q + q')$ as far as the term or terms which are of the first dimension relatively to $q'$, we shall still obtain thus an expression for the first differential $df_q$, by merely writing $dq$ in the place of $q'$. But we have not chosen (comp. 100, (14.)) to regard this property of the differential of a function as the fundamental one, or to adopt it as the definition of $df_q$; because we have not chosen to postulate the general possibility of such developments of functions of quaternion sums, of which in fact it is in many cases difficult to discover the laws, or even to prove the existence, except in some such way as that above explained.

(11.) This opportunity may be taken to observe, that (with recent notations) we have, by VIII., the symbolical expression,

\[ XXV. . . \varepsilon^{\pi + q} l = 1 + s + v; \quad \text{or} \quad XXVI. . . \varepsilon^{\pi + d} l = 1 + r. \]

* [These are equivalent to the transformations

\[ \nabla q \sqrt{K_q} = \frac{T_q}{\sqrt{K_q}} \quad \text{and} \quad \sqrt{q} \sqrt{K_q} = T_q \]

\[ \text{and} \quad \frac{\nabla q}{\sqrt{K_q}} \sqrt{q} \sqrt{K_q} = T_q = U_q. ]

† [In fact by VIII., $dr = - r^2$ and $(r + d)^2 . l = (r + d) . r = r^2 - r^2 = 0.$]
344. Successive differentials are also connected with successive differences, by
laws which it is easy to investigate, and on which only a few words need here
be said.

(1.) We can easily prove, from the definition 324, IV. of \( \frac{df_q}{d} \), that if \( dq \)
be constant,

\[
\frac{d^2f_q}{d} = \lim_{n \to \infty} n^2 \left( f(g + 2n^{-1} dq) - 2f(g + n^{-1} dq) + f_q \right);
\]

with analogous expressions for differentials of higher orders.

(2.) Hence we may say (comp. 340, X.) that the successive differentials,

\[
\frac{d^2f_q}{d}, \frac{d^3f_q}{d}, \frac{d^4f_q}{d}, \ldots \text{ for } d^2q = 0,
\]

are limits to which the following multiples of successive differences,

\[
\Delta \frac{d^2f_q}{d}, \Delta \frac{d^3f_q}{d}, \Delta \frac{d^4f_q}{d}, \ldots \text{ for } \Delta^2q = 0,
\]

all simultaneously tend, when the multiple \( n\Delta q \) is either constantly equal to
\( dq \), or at least tends to become equal thereto, while the number \( n \) increases
indefinitely.

(3.) And hence we might prove, in a new way, that if the function

\( f(q + dq) \)
can be developed, in a series proceeding according to ascending and
whole dimensions with respect to \( dq \), the parts of this series, which are of those
successive dimensions, must follow the law expressed by Taylor's Theorem*
adapted to Quaternions (342).

345. It is easy to conceive that the foregoing results may be extended
(comp. 338), to the successive differentiations of functions of several quater-
nions; and that thus there arises, in each such case, a system of successive
differentials, total and partial: as also a system of partial derivatives, of orders
higher than the first, when a quaternion, or a vector, is regarded (comp. 337)
as a function of several scalars.

(1.) The general expression for the second total differential,

\[
\frac{d^2Q}{d} = \frac{d^2F(q, r, \ldots)}{d},
\]

involves \( d^2q, d^2r, \ldots \); but it is often convenient to suppose that all these second
differentials vanish, or that the first differentials \( dq, dr, \ldots \) are constant; and
then \( d^mQ \), or \( d^mF(q, r, \ldots) \), becomes a rational, integral, and homogeneous
function of the \( m \)th dimension, of those first differentials \( dq, dr, \ldots \), which
may (comp. 329, III.) be thus denoted,

\[
\frac{d^mQ}{d} = (d_q + d_r + \ldots)^m Q; \text{ or briefly, } \frac{d^mQ}{d} = (d_q + d_r + \ldots)^m,
\]

* Some remarks on the adaptation and proof of this important theorem will be found in the
Lectures, pages 589, &c.
in developing which symbolical power, the multinomial theorem of algebra may be employed: because we have generally, for quaternions as in the ordinary calculus,

\[ \text{IV... } d_v d_q = d_q d_r. \]

(2.) For example, if we denote \( dq \) and \( dr \) by \( q' \) and \( r' \), and suppose

\[ \text{V... } Q = qr', \quad \text{VI... } d_v Q = q'r' \; ; \quad \text{VII... } d_r Q = r'qr + rqr'; \]

and

\[ \text{VIII... } d_v d_r Q = d_q d_r Q = r'q'r + rq'r'. \]

And in general, each of the two equated symbols IV. gives, by its operation on \( F(q, r) \), the limit of this other function, or product (comp. 344, I.),

\[ \text{IX... } mn' \{ F(q + n^{-1} dq, r + n'^{-1} dr) - F(q, r + n'^{-1} dr) - F(q + n^{-1} dq, r) + F(q, r) \}; \]

in which the numbers \( n \) and \( n' \) are supposed to tend to infinity.

(3.) We may also write, for functions of several quaternions,

\[ \text{X... } Q + \Delta Q = F(q + dq, r + dr, \ldots) = e^{d_q + d_r + \cdots} F(q, r) ; \]

or briefly,

\[ \text{XI... } 1 + \Delta = e^{d_q + d_r + \cdots} ; \]

with interpretations and transformations analogous to those which have occurred already, for functions of a single quaternion.

(4.) Finally, as an example of successive and partial derivation, if we resume the vector expression 308, XVIII. (comp. 315, XII. and XIII.), namely,

\[ \text{XII... } \rho = rk^t j^2 k^t \]

which has been seen to be capable of representing the vector of any point of space, we may observe that it gives, without trigonometry, by the principle mentioned in 308, (11.), and by the sub-articles to 315, not only the form

\[ \text{XIII... } \rho = rk^t j^2 k^t, \]

but also, if \( a \) be any vector unit,

\[ \text{XIV... } \rho = r k^{t+1} j^{2s} k^{-t} = r k^t(kS \cdot a^{2s} + iS \cdot a^{2s-1}) \cdot k^{-t} ; \]

whence

\[ \text{XV... } \rho = r V \cdot k^{2s+1} + r k^{2t} V \cdot j^{2s}, \] as in 315, XII.

(5.) We have therefore the following new expressions (compare the sub-articles to 337), for the two partial derivatives of the first order, of this variable vector \( \rho \), taken with respect to \( s \) and \( t \):

\[ \text{XVI... } D_s \rho = \pi r k^t j^2 k^t = - \pi \rho k^t jk^t, \]

with the verification, that

\[ \text{XVII... } d_s \rho = \pi r^2 \cdot h^t k^s j^2 k^t \cdot h^t j^2 j^2 j^{-1} k^t = \pi r^2 h^t j k^t ; \quad \text{and} \]

\[ \text{XVIII... } d_r \rho = \pi r^2 \cdot h^t k^s j^2 k^t \cdot h^t j^2 j^2 j^{-1} k^t = \pi r^2 h^t j k^t ; \quad \text{and} \]
PARTIAL DIFFERENTIALS AND DERIVATIVES.

\[ X \text{VIII.} \quad D_t \rho = \pi r k^2 V \cdot j^{28} = \pi r k^2 j S \cdot a^{28-1} = r^{-1} \rho D_s \rho \cdot S \cdot a^{28-1} \]

whence

\[ X \text{IX.} \quad \rho D_t \rho = - r D_s \rho \cdot S \cdot a^{28-1}, \quad \text{and} \quad X \text{X.} \quad D_s \rho \cdot D_t \rho = \pi^2 r^2 S \cdot a^{28-1}; \]

while

\[ X \text{XI.} \quad D_r \rho = r^{-1} \rho = k^2 k^2 j^{-1} k^2 t, \quad \text{as in 337, XXV}; \]

so that we have the following ternary product of these derived vectors of the first order,

\[ X \text{XII.} \quad D_r \rho \cdot D_s \rho \cdot D_t \rho = \pi^2 r^2 S \cdot a^{28-1} = \pi^2 D_s S \cdot a^{28}; \]

the scalar character of which product depends (comp. 299, (9.)) on the circumstance, that the vectors thus multiplied compose (337, (10.)) a rectangular system.

(6.) It is easy then to infer, for the six partial derivatives of \( \rho \), of the second order, taken with respect to the same three scalar variables, \( r, s, t \), the expressions:

\[ X \text{XIII.} \quad D_r^2 \rho = 0; \quad D_r D_s \rho = D_r D_t \rho = r^{-1} D_s \rho; \quad D_r D_t \rho = D_t D_r \rho = r^{-1} D_t \rho; \]

\[ X \text{XIV.} \quad D_s^2 \rho = - \pi^2 \rho; \quad D_s D_t \rho = D_t D_s \rho = \pi^2 r k^2 V \cdot j^{28+1}; \quad D_t^2 \rho = - \pi^2 r k^2 V \cdot j^{28}. \]

(7.) The three partial differentials of the first order, of the same variable vector \( \rho \), are the following:

\[ X \text{XV.} \quad d_r \rho = r^{-1} \rho dr; \quad d_s \rho = D_s \rho \cdot ds; \quad d_t \rho = D_t \rho \cdot dt; \]

with the products,

\[ X \text{XVI.} \quad d_r \rho \cdot d_s \rho = - \pi r^2 \rho dS \cdot a^{28} \cdot dt; \]

\[ X \text{XVII.} \quad d_r \rho \cdot d_t \rho \cdot d_s \rho = \pi r^2 dr \cdot dS \cdot a^{28} \cdot dt. \]

(8.) These differential vectors, \( d_r \rho, d_s \rho, d_t \rho \), are (in the present theory) generally finite; \( d_r \rho \), like \( D_r \rho \), being a line in the direction of \( \rho \), or of the radius of this sphere round the origin, at least if \( dr \), like \( r \), be positive; while \( d_s \rho \), like \( D_s \rho \), is (comp. 100, (9.)) a tangent to the meridian of that spheric surface, for which \( r \) and \( t \) are constant; but \( d_t \rho \), like \( D_t \rho \), is on the contrary a tangent to the small circle (or parallel), on the same sphere, for which \( r \) and \( s \) are constant.

(9.) Treating only the radius \( r \) as constant, and writing \( \rho = or \), if we pass from the point \( p \), or \( (s, t) \), to another point \( q \), or \( (s + \Delta s, t) \), on the same meridian, the chord \( pq \) is represented by the finite difference, \( \Delta \rho \); and in like
manner, if we pass from \( r \) to a point \( r \), or \((s, t + \Delta t)\), on the same parallel, the new chord \( rr \) is represented by the other partial and finite difference, \( \Delta t \); while the point \((s + \Delta s, t + \Delta t)\) may be denoted by \( s \).

(10.) If now the two points \( q \) and \( r \) be conceived to approach to \( r \), and to come to be very near it, the chords \( qr \) and \( rr \) will very nearly coincide with the two corresponding arcs of meridian and parallel; or with the tangents to the same two circles at \( r \), so drawn as to have the lengths of those two arcs: or finally with the differential and tangential vectors, \( ds \) and \( dp \), if we suppose (as we may, comp. 322) that the two arbitrary and scalar differentials, \( ds \) and \( dt \), are so assumed as to be constantly equal to the two differences, \( \Delta s \) and \( \Delta t \), and consequently to diminish with them.

(11.) Whether the differentials \( ds \) and \( dt \) be large or small, the product \( ds \cdot dp \), like the product \( Ds \cdot Dp \), represents rigorously a normal vector (as in XXVI. and XX.); of which the length bears to the unit of length (comp. 281) the same ratio, as that which the rectangle under the two perpendicular tangents, \( ds \) and \( dp \), to the sphere, bears to the unit of area. Hence, with the recent suppositions (10.), we may regard this product \( ds \cdot dp \) as representing, with a continually and indefinitely increasing accuracy, even in the way of ratio, what we may call the directed element of spheric surface, \( pqrs \), considered as thus represented (or constructed) by a normal at \( r \); and the tensor of the same product, namely (by XXVI.),

\[
XXVIII. \quad T(ds \cdot dp) = - \pi r^2 dS \cdot a^{z2} \cdot dt,
\]

in which the negative sign is retained, because \( S \cdot a^{z2} \) decreases from +1 to -1, while \( s \) increases from 0 to 1, is an expression on the same plan for what we may call by contrast the undirected element of spheric area, or that element considered with reference merely to quantity, and not with reference to direction.

(12.) Integrating, then, this last differential expression XXVIII., from \( t = 0 \) to \( t = 2 \), and from \( s = s_0 \) to \( s = s_1 \), that is, taking the limit of the sum of all the elements \( pqrs \) between these bounding values, we find the following equation:

\[
XXIX. \quad \text{Area of Spheric Zone} = 2\pi r^2 S(a^{z2} - a^{z2}) ;
\]

whence

\[
XXX. \quad \text{Area of Spheric Cap (s)} = 2\pi r^2 (1 - S \cdot a^{z2}) = 4\pi r^2 (T \cdot a)^2 ;
\]

and finally,

\[
XXXI. \quad \text{Area of Sphere} = 4\pi r^2 , \text{as usual.}
\]
(13.) In like manner the expression XXVII., with its sign changed (on account of the decrease of \( S \cdot a^2 \), as in (11.)), represents the element of volume; and thus, by integrating from \( r = r_0 \) to \( r = r_1 \), from \( s = 0 \) to \( s = 1 \), and from \( t = 0 \) to \( t = 2 \), we obtain anew the known values:

\[
XXXII. \ldots \text{Volume of Spheric Shell} = \frac{4\pi}{3}(r_1^3 - r_0^3);
\]

and

\[
XXXIII. \ldots \text{Volume of Sphere} (r) = \frac{4\pi r^3}{3}, \text{as usual.}
\]

(14.) These are however only specimens of what may be called Scalar Integration, although connected with quaternion forms; and it will be more characteristic of the present Calculus, if we apply it briefly to take the Vector Integral, or the limit of the vector-sum of the directed elements (11.) of a portion of a spheric surface: a problem which corresponds, in hydrostatics, to calculating the resultant of the pressures on that surface, each pressure having a normal direction, and a quantity proportional to the element of area.

(15.) For this purpose, we may employ the expression XXVI. with its sign changed, in order to denote an inward normal, or a pressure acting from without; and if we then substitute for \( p \) its value XV, and observe that

\[
XXXIV. \ldots \int_0^2 k^{2+t} \, dt = 0, \text{ because } k^2 = -1,*
\]

and remember that \( \nabla \cdot k^{2s+1} = kS \cdot a^{2s} \), we easily deduce the expressions:

\[
XXXV. \ldots \text{Sum of Directed Elements of Elementary Zone} = \pi r^2 k \cdot (S \cdot a^{2s})^2;
\]

\[
XXXVI. \ldots \text{Sum of Directed Elements of Spheric Cap (s)} = -\pi r^2 k(1 - (S \cdot a^{2s})^2)
\]

\[= \pi r^2 k(\nabla \cdot a^{2s})^2 = \pi^{-1} k(D\rho)^2 = \pi k(\nabla k\rho)^2.\]

(16.) But the radius of the plane and circular base, of the spheric segment corresponding, is TVk\(\rho \), so that its area is in quantity = \(-\pi(\nabla k\rho)^2\); and the common direction of all its inward normals is that of \(+ k \); hence, if we still represent the directed elements by normals thus drawn inwards, we have this new expression:

\[
XXXVII. \ldots \text{Sum of Directed Elements of Circular Base} = -\pi k(\nabla k\rho)^2;
\]

comparing which with XXXVI., we arrive at the formula,

\[
\text{Since } d(k^{2s+1}) = \frac{1}{2} k^{2s}d(2t - 1) \text{ by } 333 (5.), \text{ the integral } \int_0^1 k^{2s} \, dt = \frac{1}{2} (k^{2s-1} - k^{-1}).
\]

* [Since \( d(k^{2s+1}) = \frac{1}{2} k^{2s}d(2t - 1) \) by 333 (5.), the integral \( \int_0^1 k^{2s} \, dt = \frac{1}{2} (k^{2s-1} - k^{-1}). \)]
XXXVIII. . . Sum of Directed Elements of Spheric Segment = Zero;
a result which may be greatly extended, and which evidently answers to a
known case of equilibrium in hydrostatics.

(17.) These few examples may serve to show already, that Differentials of
Quaternions (or of Vectors) may be applied to various geometrical and physical
questions; and that, when so applied, it is permitted to treat them as small, if
any convenience be gained thereby; as in cases of integration there always is.
But we must now pass to an important investigation of another kind, with
which differentials will be found to have only a sort of indirect or suggestive
connexion.

SECTION 6.

On the Differentiation of Implicit Functions of Quaternions;
and on the General Inversion of a Linear Function, of a
Vector or a Quaternion: with some connected Investigations.

346. We saw, when differentiating the square-root of a quaternion (332,
(5.) and (6.),) that it was necessary for that purpose to resolve a linear
equation, or an equation of the first degree; namely the equation,

\[ rr' + q' = q', \]

in which \( r \) and \( q' \) represented two given quaternions, \( q^4 \) and \( dq \), while \( r' \)
represented a sought quaternion, namely \( dr \) or \( d q^4 \). And generally, from
the linear or distributive form (327), of the quaternion differential

\[ dQ = d\mathbf{q} = f(q, dq), \]

of any given and explicit function \( f(q) \), when considered as depending on the
differential \( dq \) of the quaternion variable \( q \), we see that the return from the
former differential to the latter, that is from \( dQ \) to \( dq \), or the differentiation of
the inverse or implicit function \( f^{-1}Q \), requires for its accomplishment the
Solution of an Equation of the First Degree: or what may be called the
Inversion of a Linear Function of a Quaternion. We are therefore led to con-
sider here that general Problem; to which accordingly, and to investigations
connected with which, we shall devote the present Section, dismissing however
now the special consideration of the Differentials above mentioned, or treating

† Compare the Note to page 462.
them only as Quaternions, sought or given, of which the relations to each other are to be studied.

347. Whatever the particular form of the given linear or distributive function, $f_q$, may be, we can always decompose it as follows:

I. \[ f_q = f(Sq + Vq) = fSq + fVq = Sq \cdot f1 + fVq; \]

taking then separately scalars and vectors, or operating with $S$ and $V$ on the proposed linear equation,

II. \[ f_q = r, \]

where $r$ is a given quaternion, and $q$ a sought one, we can in general eliminate $Sq$, and so reduce the problem to the solution of a linear and vector equation, of the form,

III. \[ \phi \rho = \sigma; \]

where $\sigma$ is a given vector, but $\rho (= Vq)$ is a sought one, and $\phi$ is used as the characteristic of a given linear and vector function of a vector, which function we shall throughout suppose to be a real one, or to involve no imaginary constants in its composition. But, to every such function $\phi \rho$, there always corresponds what may be called a conjugate linear and vector function $\phi' \rho$, connected with it by the following Equation of Conjugation,

IV. \[ S\lambda \phi \rho = S\rho \phi' \lambda; \]

where $\lambda$ and $\rho$ are any two vectors. Assuming then, as we may, that $\mu$ and $\nu$ are two auxiliary vectors, so chosen as to satisfy the equation,

V. \[ YfXV = \tau, \]

and therefore also,

VI. \[ S\lambda \sigma = S\lambda \mu \nu, \quad S\mu \sigma = 0, \quad S\nu \sigma = 0, \]

where $\lambda$ is a third auxiliary and arbitrary vector, we may (comp. 312) replace the one vector equation III. by the three scalar equations,

VII. \[ S\rho \phi' \lambda = S\lambda \mu \nu, \quad S\rho \phi' \mu = 0, \quad S\rho \phi' \nu = 0. \]

And these give, by principles with which the reader is supposed to be already familiar,* the expression,

VIII. \[ m\rho = \psi \sigma, \quad \text{or IX.} \quad \rho = \phi^{-1} \sigma = m^{-1} \psi \sigma, \]

* A student might find it useful, at this stage, to read again the Sixth Section of the preceding Chapter; or at least the early sub-articles to Art. 294, a familiar acquaintance with which is presumed in the present Section.
if \( m \) be a scalar-constant, and \( \psi \) an auxiliary linear and vector function, of which the value and the form are determined by the two following equations:

\[
X \ldots mS\lambda\mu\nu = S(\phi'\lambda \cdot \phi'\mu \cdot \phi'\nu); \\
XI. \ldots \psi(\nabla\mu\nu) = \nabla(\phi'\mu \cdot \phi'\nu);
\]

or briefly,

\[
X' \ldots mS\lambda\mu\nu = S \cdot \phi'\lambda\phi'\mu\phi'\nu; \\
XI' \ldots \psi\nabla\mu\nu = \nabla \cdot \phi'\mu\phi'\nu.
\]

And thus the proposed \textit{Problem of Inversion}, of the linear and vector function \( \phi \), may be considered to be, in all its generality, \textit{resolved}; because it is always possible so to prepare the second members of the equations \( X \) and \( XI \), that they shall take the forms indicated in the first members of those equations.*

(1.) For example, if we \textit{assume} any three diplanar vectors \( a, a', a'' \), and \textit{deduce} from them three other vectors \( \beta_0, \beta'_0, \beta''_0 \), by the equations,

\[
XII. \ldots \beta_0 Saa'a'' = Va'a'', \quad \beta'_0 Saa'a'' = Va'a', \quad \beta''_0 Saa'a'' = Vaa',
\]

then \textit{any} vector \( \rho \) may, by 294, \( XV. \), be expressed as follows:

\[
XIII. \ldots \rho = \beta_0 S\alpha \rho + \beta'_0 S\alpha' \rho + \beta''_0 S\alpha'' \rho;
\]

if then we write,

\[
XIV. \ldots \beta = \phi\beta_0, \quad \beta' = \phi\beta'_0, \quad \beta'' = \phi\beta''_0, \dagger
\]

we shall have the following \textit{General Expression}, or \textit{Standard Trinomial Form}, for a \textit{Linear and Vector Function of a Vector},

\[
XV. \ldots \phi\rho = \beta S\alpha \rho + \beta' S\alpha' \rho + \beta'' S\alpha'' \rho;
\]

containing, as we see, \textit{three vector constants}, \( \beta, \beta', \beta'' \), or \textit{nine scalar constants}, such as

\[
XVI. \ldots S\alpha \beta, \quad S\alpha' \beta, \quad S\alpha'' \beta; \quad S\alpha \beta', \quad S\alpha' \beta', \quad S\alpha'' \beta'; \quad S\alpha \beta'', \quad S\alpha' \beta'', \quad S\alpha'' \beta'';
\]

which may (and generally will) all \textit{vary}, in passing from \textit{one} linear and \textit{vector function} \( \phi\rho \) to \textit{another} such function; but which are \textit{all} supposed to be \textit{real}, and \textit{given}, for each \textit{particular form} of that function.

* [For a more elementary solution of the problem of Inversion, see sub-art. (4.).]

† [The equations XIV. lead to a useful expression for a linear vector function in terms of three diplanar vectors \( \beta_0, \beta'_0, \) and \( \beta''_0 \), and the derived vectors \( \beta, \beta', \) and \( \beta'' \).]
(2.) Passing to what we have called the conjugate linear function \( \phi' \), the form XV. gives, by IV., the expression,

\[
\text{XVII. } \phi' = aS\beta' + a'S\beta' + a''S\beta' ;
\]

but

\[
V. (aS\beta + a'S\beta') = Vaa'S. \beta'v - S\beta v
\]

therefore the transformation XI. succeeds, and gives,

\[
\text{XVIII. } \psi' = V\beta'\beta''S\alpha' + V\alpha''S\beta' + Vaa'S\beta'\beta' ;
\]

as an expression for the auxiliary function \( \psi \); of which the conjugate may be thus written,

\[
\text{XIX. } \psi' = V\beta'\beta''S\alpha' + V\alpha''S\beta' + Vaa'S\beta'\beta' ;
\]

so that \( \phi \) is changed to \( \psi \), when \( \phi \) is changed to \( \phi' \), by interchanging each of the three alphas with the corresponding beta.

(3.) If we write, as in this whole investigation we propose to do,

\[
\text{XX. } \lambda' = V\mu' = Vv' = V\lambda',
\]

the formulae XI. and X. become,

\[
\text{XXI. } \psi'\lambda' = V. \phi'\mu'\nu', \quad \text{and} \quad \text{XXII. } mS\lambda\lambda' = S. \phi'\lambda'\psi' ;
\]

with the same sort of abridgment of notation as in XI'.; and because the coefficient of \( Saa'\alpha'' \) in this last expression XXII. is by XVII. XVIII.,

\[
S\beta'\lambda\beta''\beta'\lambda' + S\beta'\lambda\beta''\beta'\lambda' + S\beta'\lambda\beta''\beta'\lambda' = S\beta'\beta''\beta \lambda',
\]

the division by \( S\lambda' \), or by \( S\lambda\mu' \), succeeds, and we find the expression,

\[
\text{XXIII. } m = Saa'\alpha''S\beta''\beta' ;
\]

which may also be thus written,

\[
\text{XXIII' } m = S\beta'\beta''S\alpha' ;
\]

so that \( m \) does not change when we pass from \( \phi \) to \( \phi' \), on which account we may write also,

\[
\text{XXIV. } mS\lambda\lambda' = S. \phi'\lambda'\psi' \quad \text{or} \quad \text{XXIV' } mS\lambda\mu' = S. \phi'\mu'\nu',
\]

because, by (2.), we can deduce from XI. the conjugate expression,

\[
\text{XXV. } \psi' = V. \phi'\mu'\nu' .
\]
(4.) We ought then to find that the linear equation,

$$XXVI. \ldots \sigma = \phi \rho = \beta Sa^\rho + \beta' Sa'^\rho + \beta" Sa''^\rho,$$

has its solution expressed (comp. VIII.) by the formula,

$$XXVII. \ldots pSa' a'' S\beta'' \beta' \beta = \nabla a' a'' S\beta'' \beta' \sigma + \nabla a'' a S\beta'' \beta \sigma + \nabla a' S\beta' \beta \sigma;$$

and accordingly, if we operate on the expression XXVI. for \(\sigma\) with the three symbols,

$$XXVIII. \ldots S \cdot \beta'' \beta', \ S \cdot \beta \beta'', \ S \cdot \beta' \beta,$$

we obtain the three scalar equations,

$$XXIX. \ldots S \beta'' \beta' \sigma = S \beta'' \beta' \beta Sa^\rho, \ \&c.,$$

from which the equation XXVII. follows immediately, without any introduction of the auxiliary vectors \(\lambda, \mu, \nu\), although these are useful in the theory generally.

(5.) Conversely, if the equation XXVII. were given, and the value of \(\sigma\) sought, we might operate with the three symbols,

$$XXX. \ldots S \cdot a, \ S \cdot \beta, \ S \cdot \gamma,$$

and so obtain the three scalar equations XXIX., from which the expression XXVI. for \(\sigma\) would follow.

(6.) It will be found a useful check on formulae of this sort, to consider each beta, in what we have called the Standard Form (1.) of \(\phi \rho\), as being of the first dimension; for then we may say that \(\phi\) and \(\phi'\) are also of the first dimension, but \(\psi\) and \(\psi'\) of the second, and \(m\) of the third; and every formula, into which these symbols enter, will thus be homogeneous: \(a, a', a''\), and \(\lambda, \mu, \nu, \rho\), being not counted, in this mode of estimating dimensions, but \(\sigma\) being treated as of the first dimension, when it is taken as representing \(\phi \rho\).*

(7.) And although the trinomial form XV. has been seen to be sufficiently general, yet if we choose to take the more expanded form,

$$XXXI. \ldots \phi \rho = \Sigma \beta Sa^\rho, \text{ which gives } XXXII. \ldots \phi' \rho = \Sigma a S\beta^\rho,$$

any number of terms of \(\phi \rho\), such as \(\beta Sa^\rho, \beta' Sa'^\rho, \&c.,\) being now included in the sum \(\Sigma\), there is no difficulty in proving that the equations VIII. and IX. are satisfied, when we write,

---

* [Compare the first Note to Art. 350.]
XXXIII. \( \psi \rho = \Sigma V a a' \beta' \beta \rho \), with XXXIV. \( \psi' \rho = \Sigma V \beta' \beta' S a' a \rho \), and

XXXV. \( m = \Sigma S a a' a'' \beta'' \beta' \beta = \Sigma \beta' \beta' \beta'' S a' a' a. \)

(8.) The important property (2.), that the auxiliary function \( \psi \) is changed to its own conjugate \( \psi' \), when \( \phi \) is changed to \( \phi' \), may be proved without any reference to the form \( \Sigma \beta S a \rho \) of \( \phi \rho \), by means of the definitions IV. and XI., of \( \phi' \) and \( \psi \), as follows. Whatever four vectors \( \mu, \nu, \mu_1, \) and \( \nu_1 \) may be, if we write

XXXVI. \( \lambda_1 = \nu \mu_1 \nu_1 \), and XXXVII. \( \psi' \psi' \nu = \nu \phi \mu \phi \nu \),

adopting here this last equation as a definition of the function \( \psi' \), we may proceed to prove that it is conjugate to \( \psi' \), by observing that we have the transformations,

XXXVIII. \( S \lambda'_1 \psi' \lambda' = S (\nu \mu_1 \nu_1 \cdot \nu \cdot \phi' \mu \phi' \nu) = S \cdot \mu_1 \nu (\nu \cdot \nu \cdot \phi \mu \phi \nu) = S \mu_1 \phi' \mu \cdot S \nu_1 \phi' \mu \cdot S \nu \phi' \mu_1 \cdot S \nu \phi \mu \phi' \nu_1 = S . \mu (\nu \nu \cdot \nu \cdot \phi' \mu_1 \cdot \nu_1) = S \lambda' S \lambda' \),

which establish the relation in question, between \( \psi \) and \( \psi' \).

(9.) And the not less important property (3.), that \( m \) remains unchanged when we pass from \( \phi \) to \( \phi' \), may in like manner be proved, without reference to the form XV. or XXXI. of \( \phi \rho \), by observing that we have by XXXVII., &c, the transformations,

XXXIX. \( S . \phi \lambda' \psi \phi' \mu \phi' \nu = S . \phi \lambda' \psi' \phi' \mu \phi' \nu = S \lambda' S \lambda' \phi \phi \nu = m S \lambda' \lambda = m S \lambda \nu \),

because the equations III. and VIII. give,

XL. \( \psi \phi \rho = m \rho \), whatever vector \( \rho \) may be;

so that the value of this scalar constant \( m \) may now be derived from the original linear function \( \phi \), exactly as it was in X. or X'. from the conjugate function \( \phi' \).

348. It is found, then, that the linear and vector equation,

I. \( \phi \rho = \sigma \), gives II. \( m \rho = \psi' \sigma \),

as its formula of solution; with the general method, above explained, of deducing \( m \) and \( \psi \) from \( \phi \). We have therefore the two identities,

III. \( m \sigma = \phi \psi' \sigma \), \( m \rho = \psi' \phi \rho \);

or briefly and symbolically,

III'. \( m = \phi = \psi' \phi \);
with which, by what has been shown, we may connect these \textit{conjugate equations},

\[ III' \ldots m = \phi' \psi' = \psi' \phi'. \]

Changing then successively \( \mu \) and \( \nu \) to \( \psi' \mu \) and \( \psi' \nu \), in the equation of definition of the auxiliary function \( \psi \), or in the formula,

\[ \psi \nabla \mu \nu = \nabla. \phi' \mu \phi' \nu, \]

we get these two other equations,

\[ \text{IV.} \ldots - \psi \nabla. \nu \psi' \mu = m \nabla. \mu \phi' \nu; \quad \text{V.} \ldots \psi \nabla. \psi' \mu \nu = m^2 \nabla \mu \nu; \]

in the former of which the \textit{points} may be omitted, while in each of them \textit{accented} may be exchanged with unaccented symbols of operation: and we see that the \textit{law of homogeneity} (347, (6.) ) is preserved. And many other transformations of the same sort may be made, of which the following are a few examples.

1. Operating on \( \text{V.} \) by \( \psi' \), or by \( m^2 \phi' \), we get this new formula,

\[ \text{VI.} \ldots \nabla. \psi' \mu \psi' \nu = m \nabla. \phi' \nu; \]

comparing which with the lately cited definition of \( \psi \), we see that we may change \( \phi \) to \( \psi' \), if we at the same time change \( \mu \) to \( m^2 \); \( \phi' \) being then changed to \( \psi' \), and \( \psi' \) to \( m \phi' \).

2. For example, we may thus pass from \( \text{IV.} \) and \( \text{V.} \) to the formulæ,

\[ \text{VII.} \ldots - \psi \nabla. \nu \psi' \mu = \nabla. \mu \psi' \nu, \quad \text{VIII.} \ldots \phi \nabla. \phi' \mu \psi' \nu = m \nabla \mu \nu; \]

in which we see that the lately cited \textit{law of homogeneity} is still observed.

3. The equation \( \text{VII.} \) might have been otherwise obtained, by interchanging \( \mu \) and \( \nu \) in \( \text{IV.} \), and operating with \( - m^2 \phi' \), or with \( - \psi' \); and the formula \( \text{VIII.} \) may be at once deduced from the equation of definition of \( \psi \), by operating on it with \( \phi \). In fact, our \textit{rule of inversion}, of the \textit{linear function} \( \phi \), may be said to be contained in the formula,

\[ \text{IX.} \ldots \phi^{-1} \nabla \mu \nu = m \phi^{-1} \nabla. \phi' \mu \phi' \nu; \]

where \( m \) is a scalar constant, as above.

4. By similar operations and substitutions,

\[ \text{X.} \ldots \phi^2 \nabla. \phi' \mu \phi' \nu = m \phi \nabla \mu \nu = \nabla. \psi' \mu \psi' \nu; \]

\[ \text{XI.} \ldots m \phi \nabla. \phi' \mu \phi' \nu = m^2 \nabla \mu \nu = \psi \nabla. \psi' \mu \psi' \nu; \]

\[ \text{XII.} \ldots \phi' \mu \phi' \nu = m^2 \phi \nabla \mu \nu = \psi \nabla. \psi' \mu \psi' \nu; \]

\[ \text{XIII.} \ldots \phi^{-2} \nabla. \phi' \mu \phi' \nu = \psi \nabla. \phi' \mu \phi' \nu = \psi^2 \nabla \mu \nu; \quad \&c. \]
(5.) But we have also,

\[ \text{XIV.} \ldots S \cdot \lambda \phi^2 \phi' = S \cdot \phi \phi' \lambda = S \cdot \rho \phi^2 \lambda, \]

so that the second functions \( \phi^2 \) and \( \phi'^2 \) are conjugate (compare 347, IV.); hence, by XIII., \( \psi^2 \) is formed from \( \phi^2 \), as \( \psi \) from \( \phi \); and generally it will be found, that if \( n \) be any whole number, and if we change \( \phi \) to \( \phi^n \), we change at the same time \( \phi' \) to \( \phi'^n \), \( \psi \) to \( \psi^n \), \( \psi' \) to \( \psi'^n \), and \( m \) to \( m^n \).

(6.) It may also be remarked that the changes (1.) conduct to the equation,

\[ \text{XV.} \ldots (S \cdot \phi \lambda \mu \phi \nu)^2 = S \lambda \mu \nu S \cdot \psi \lambda \mu \psi \nu; \]

and to many other analogous formulae.

349. The expressions,

\[ \lambda' \phi \lambda + \mu' \phi \mu + \nu' \phi \nu, \quad \lambda' \psi \lambda + \mu' \psi \mu + \nu' \psi \nu \]

with the significations 347, XX. of \( \lambda', \mu', \nu' \), and others of the same type, are easily proved to vanish when \( \lambda, \mu, \nu \) are complanar,* and therefore to be divisible by \( S \lambda \mu \nu \), since each such expression involves each of the three auxiliary vectors \( \lambda, \mu, \nu \) in the first degree only; the quotients of such divisions being therefore certain constant quaternions, independent of \( \lambda, \mu, \nu \), and depending only on the particular form of \( \phi \), or on the (scalar or vector; but real) constants, which enter into the composition of that given function. Writing, then,

\[ \text{I.} \ldots q_1 = (\lambda' \phi \lambda + \mu' \phi \mu + \nu' \phi \nu) : S \lambda \mu \nu, \]

and

\[ \text{II.} \ldots q_2 = (\lambda' \psi \lambda + \mu' \psi \mu + \nu' \psi \nu) : S \lambda \mu \nu, \]

we shall find it useful to consider separately the scalar and vector parts of these two quaternion constants, \( q_1 \) and \( q_2 \); which constants are, respectively, of the first and second dimensions, in a sense lately explained.†

(1.) Since \( V \lambda' \phi \lambda = \mu S \nu \phi \lambda - \nu S \lambda \phi' \mu, \) &c., it follows that the vector parts of \( q_1 \) and \( q_2 \) change signs, when \( \phi \) is changed to \( \phi' \), and therefore \( \psi \) to \( \psi' \). On the other hand, we may change the arbitrary vectors \( \lambda, \mu, \nu \) to \( \lambda', \mu', \nu' \), if we at the same time change \( \lambda' \) to \( V \mu \nu' \), or to \( - S \lambda \mu \nu, \) &c., and \( S \lambda \mu \nu \), or \( S \lambda \lambda' \), to \( - (S \lambda \mu \nu)^2 \); dividing then by \( - S \lambda \mu \nu \), we find these new expressions,

\[ \text{III.} \ldots q_1 = (\lambda' \phi \lambda' + \mu \phi \mu' + \nu \phi \nu') : S \lambda \mu \nu, \]

\[ \text{IV.} \ldots q_2 = (\lambda' \psi \lambda' + \mu \psi \mu' + \nu \psi \nu') : S \lambda \mu \nu; \]

* [By putting \( \nu = x \lambda + y \mu \).]
† [It may be instructive to the student to reduce these quaternion constants by replacing \( \lambda, \mu, \) and \( \nu \) by \( x' i + y' j + z' k, \) \( x' i + y' j + z' k, \) and \( x'' i + y'' j + z'' k \).]

3 R 2
operating on which by $S$, we return to the scalars of the expressions I. and II., with $\phi$ and $\psi$ changed to $\phi'$ and $\psi'$.

(2.) Hence the conjugate quaternion constants, $Kq_1$ and $Kq_2$, are obtained by passing to the conjugate linear functions; and thus we may write,

\[
V \ldots Kq_1 = (\lambda'\phi'\lambda + \mu'\phi'\mu + \nu'\phi'\nu) : S\lambda\nu;
\]

\[
VI \ldots Kq_2 = (\lambda'\psi'\lambda + \mu'\psi'\mu + \nu'\psi'\nu) : S\lambda\nu;
\]

or, interchanging $\lambda$ with $\lambda'$, &c., in the dividends,

\[
VII \ldots Kq_1 = (\lambda\phi'\lambda' + \mu\phi'\mu' + \nu\phi'\nu') : S\lambda\nu;
\]

\[
VIII \ldots Kq_2 = (\lambda\psi'\lambda' + \mu\psi'\mu' + \nu\psi'\nu') : S\lambda\nu;
\]

where $\lambda' = V\mu\nu$, &c., as before.

(3.) Operating with $V \cdot \rho$ on $Vq_2$, and observing that

\[
V \cdot \rho V\lambda'\phi\lambda = \phi'(\lambda S\lambda'\rho) - \lambda'S\lambda\phi'\rho, \quad \text{&c.,}
\]

while

\[
\phi'(\lambda S\lambda'\rho + \mu S\mu'\rho + \nu S\nu'\rho) = \rho S\lambda\nu;
\]

and

\[
\lambda'S\lambda\phi'\rho + \mu'S\mu\phi'\rho + \nu'S\nu\phi'\rho = \rho' S\lambda\nu;
\]

with similar transformations for $V \cdot \rho Vq_2$, we find that

\[
IX \ldots V \cdot \rho Vq_1 = \phi\rho - \phi'\rho;
\]

and

\[
X \ldots V \cdot \rho Vq_2 = \psi\rho - \psi'\rho.
\]

(4.) Accordingly, since

\[
S\rho(\phi\rho - \phi'\rho) = - S\rho(\phi\rho - \phi'\rho) = 0,
\]

the vector $\phi\rho - \phi'\rho$, if it do not vanish, must be a line perpendicular to $\rho$, and therefore of the form,

\[
XI \ldots \phi\rho - \phi'\rho = 2\gamma\rho,
\]

in which $\gamma$ is some constant vector;* so that we may write,

\[
XII \ldots \phi\rho = \phi\rho + V\gamma\rho, \quad \phi'\rho = \phi\rho - V\gamma\rho,
\]

where the function $\phi\rho$ is its own conjugate, or is the common self-conjugate part of $\phi\rho$ and $\phi'\rho$; namely the part,

\[
XIII \ldots \phi\rho = \frac{1}{2}(\phi\rho + \phi'\rho).
\]

And we see that, with this signification of $\gamma$,

\[
XIV \ldots V(\lambda'\phi\lambda + \mu'\phi\mu + \nu'\phi\nu) = - 2\gamma S\lambda\nu, \quad \text{or} \quad XIV' \ldots Vq_1 = - 2\gamma;
\]

* [This vector $\gamma$ has been called the spin-vector of the function $\phi$.]
while we have, in like manner,

\[ XV. \ldots V(\lambda' \psi \lambda + \mu' \psi \mu + v' \psi v) = -2\delta \lambda \mu \nu, \quad \text{or} \quad XV. \ldots Vg_\gamma = -2\delta, \]

if

\[ XVI. \ldots \psi_\rho - \psi'_\rho = 2Vд_\rho. \]

As a confirmation, the part \( \phi_\alpha \) of \( \phi \) has by (1.) no effect in \( Vg_\gamma \); and if we change \( \phi_\lambda \) to \( V\gamma_\lambda \), &c., in the first member of XIV., we have thus,

\[ \lambda \delta \lambda' + \mu \delta \mu' + v \delta v' = -\gamma \delta (\lambda' + \mu' + v') = \lambda \delta \lambda \mu \nu - 3\gamma \delta \lambda \mu \nu. \]

(5.) Since \( \lambda \delta \lambda' = \phi V\lambda \phi' \lambda' \), &c., by 348, VII., while we may write, by (1.), (2.), and (4.),

\[ XVII. \ldots V(\alpha \phi \alpha' + \mu \phi \mu + v \phi v) = -2\delta \lambda \mu \nu, \]

\[ XVIII. \ldots V(\alpha \phi \phi' \alpha + \mu \phi \mu' + v \phi v') = -2\delta \lambda \mu \nu, \]

or

\[ XIX. \ldots V(\alpha \phi' \alpha + \mu \phi' \mu + v \phi' v) = +2\delta \lambda \mu \nu, \]

and

\[ XX. \ldots V(\alpha \phi' \lambda + \mu \phi' \mu + v \phi' v) = +2\delta \lambda \mu \nu, \]

we have this relation between the two new vector constants,

\[ XXI. \ldots \delta = -\phi \gamma = -\phi' \gamma = -\phi_\alpha \gamma; \]

for \( \phi, \phi' \), and \( \phi_\alpha \) have all the same effect, on this particular vector, \( \gamma \).

(6.) We may add that the vector constant \( \gamma \) is of the first dimension, and that \( \delta \) is of the second dimension, with respect to the betas of the standard form; in fact, with that form, 347, XV., of \( \phi_\rho \), we have the expressions,

\[ XXII. \ldots \gamma = \frac{1}{2}V(\beta a + \beta' a' + \beta'' a''), \]

and

\[ XXIII. \ldots \delta = \frac{1}{2}V(Vb' b'' . Va'a'' + Vb'' b . Va'a + Vb' b' . Va'a). \]

(7.) If we denote by \( \psi_\phi \) and \( m_\phi \), what \( \psi \) and \( m \) become when \( \phi \) is changed to \( \phi_\alpha \), we easily find that

\[ XXIV. \ldots \psi_\rho = \psi_\phi_\rho - \gamma \delta \gamma \rho + V\delta_\rho; \quad XXV. \ldots \psi'_\rho = \psi_\phi_\rho - \gamma \delta \gamma \rho - V\delta_\rho; \]

so that the self-conjugate part of \( \psi_\rho \) contains a term, \( -\gamma \delta \gamma \rho \), which involves the vector \( \gamma \), but only in the second degree; and in like manner,

\[ XXVI. \ldots m = m_\phi + S\gamma \delta = m_\phi - S\gamma \phi \gamma; \]

\( \gamma \) again entering only in an even degree, because \( m \) remains unchanged, when we pass from \( \phi \) to \( \phi' \), or from \( \gamma \) to \( -\gamma \).

* [Expand \( Vm_\nu = V(\phi_\phi \mu - V\gamma \mu)(\phi_\phi \nu - V\gamma \nu) \).]
(8.) It is evident that we have the relations,

\[ XXVII. \ldots m_0 = \phi_0 \psi_0 = \psi \phi_0; \]

and that, in a sense already explained, \( \phi_0, \psi_0, \) and \( m_0 \) are of the first, second, and third dimensions, respectively.

350. After thus considering the vector parts of the two quaternion constants, \( q_1 \) and \( q_2 \), we proceed to consider their scalar parts; which will introduce two new scalar constants, \( m'' \) and \( m' \), and will lead to the employment of two new conjugate auxiliary functions, \( \chi \phi \) and \( \chi' \rho \); whence also will result the establishment of a certain Symbolic and Cubic Equation,

\[ I. . . 0 = m - m'\phi + m''\phi^2 - \phi^3; \]

which is satisfied by the Linear Symbol of Operation, \( \phi \), and is of great importance in this whole Theory of Linear Functions.*

(1.) Writing, then,

\[ II. . . m'' = S q_1, \quad \text{and} \quad III. . . m' = S q_2, \]

we see first that neither of these two new constants changes value, when we pass from \( \phi \) to \( \phi' \), or from \( \gamma \) to \( -\gamma \); because, in such a passage, it has been seen that we only change \( q_1 \) and \( q_2 \) to \( K q_1 \) and \( K q_2 \). Accordingly, if we denote by \( m''_0 \) and \( m''_0 \) what \( m' \) and \( m'' \) become, when \( \phi \) is changed to \( \phi_0 \), we easily find the expressions,

\[ IV. . . m'' = m''_0; \quad \text{and} \quad V. . . m' = m'_0 - \gamma^2. \]

* [Or directly, without introducing \( \chi \) or \( \chi' \), for an arbitrary vector \( A \) the relation

\[ \phi^3 A - m_1 \phi^2 A + m_2 \phi A - m_3 A = 0, \]

will generally exist. This may be briefly written in the form,

\[ \phi^3 - m_1 \phi^2 A + m_2 \phi A - m_3 A = 0, \]

where the coefficients \( m \) can only depend on \( \phi \) and \( \lambda \). Operating on this by \( \phi \) and \( \phi^2 \),

\[ \phi^4 A - m_1 \phi^3 A + m_2 \phi^2 A - m_3 A = 0 \]

and

\[ \phi^5 A - m_1 \phi^4 A + m_2 \phi^3 A - m_3 A = 0. \]

But an arbitrary vector \( \rho \) may be expressed in the form

\[ xA + y \phi A + z \phi^2 A, \]

and hence from the three equations, on multiplying by \( x, y, \) and \( z, \) and adding, the equation

\[ \phi^3 \rho - m_1 \phi^2 \rho + m_2 \phi \rho - m_3 \rho = 0 \]

results. This must be identical with the equation found by treating \( \rho \) directly, in the same manner as \( A \) has been treated, and therefore the coefficients \( m \) must be independent of \( A \). The suffixes here printed serve to indicate the dimensions of the \( m. \) See 347 (6.).]
(2.) It may be noted that $m''$, or $m''_0$, is of the first dimension, but that $m'$ and $m'_0$ are of the second, with respect to the standard form of $\phi$; and accordingly, with that form we have,

$$\text{VI. } m'' = S(a\beta + a'\beta' + a''\beta'');$$

and

$$\text{VII. } m' = S(Va'\alpha'' \cdot V\beta''\beta' + V\alpha''\alpha \cdot V\beta\beta'' + V\alpha\alpha' \cdot V\beta\beta).$$

(3.) If we introduce two new linear functions, $\chi_\rho$ and $\chi'\rho$, such that

$$\text{VIII. } \chi\nabla \mu \nu = V(\mu\phi'\nu - \nu\phi'\mu),$$

and

$$\text{IX. } \chi'\nabla \mu \nu = V(\mu\phi\nu - \nu\phi\mu),$$

it is easily proved that these functions are conjugate to each other, and that each is of the first dimension; in fact, with the standard form of $\phi_\rho$, we have the expressions,

$$\text{X. } \chi_\rho = V(a\nabla_\beta_\rho + a'\nabla_\beta'\rho + a''\nabla_\beta''\rho),$$

$$\text{XI. } \chi'\rho = V(\beta\nabla_\alpha_\rho + \beta'\nabla_\alpha'\rho + \beta''\nabla_\alpha''\rho);$$

and $S \cdot \lambda \nabla_\beta_\rho = S \cdot \rho \nabla_\alpha \lambda$, &c. Also, if $\chi_0$ be formed from $\phi_0$, as $\chi$ from $\phi$, it will be found that

$$\text{XII. } \chi_0 = \chi_\rho - \nabla_\gamma \rho, \text{ and } \text{XIII. } \chi'\rho = \chi_\rho + \nabla_\gamma \rho;$$

where $\chi_0$ is of the first dimension.

(4.) Since

$$S\lambda \chi'\lambda' = S \cdot \lambda(\mu\phi'\nu - \nu\phi'\mu) = S(\mu'\phi'\mu + \nu'\phi'\nu),$$

the expression II. gives, by 349, V., the equation,

$$\text{XIV. } m'' S\lambda \chi' = S \cdot \lambda(\phi + \chi)\lambda',$$

$\lambda$ and $\lambda'$ being two arbitrary and independent vectors; which can only be, by our having the functional relation,

$$\text{XV. } \phi_\rho + \chi_\rho = m'' \rho;$$

or briefly and symbolically,

$$\text{XVI. } \chi + \phi = m''.$$
(5.) The formula XVI. gives,

\[ \chi' = \phi'' - \phi' = \phi \chi; \]

and accordingly the identity of \( \chi' \phi \) and \( \phi \chi \) may easily be otherwise proved, by changing \( \mu \) and \( \nu \) to \( \psi \mu \) and \( \psi \nu \) in the definition VIII. of \( \chi \), and remembering that

\[ \nabla \cdot \psi \mu \psi \nu = m \phi \nabla \mu \nu, \quad \phi' \psi' = m, \quad \text{and} \quad \nabla \psi' \nu = -\phi \nabla \nu \phi' \mu; \]

for thus we have,

\[ \chi' \phi \mu \nu = \nabla (\mu \psi' \nu - \nu \psi' \mu) = \phi \nabla (\mu \phi' \nu - \nu \phi' \mu) = \phi \chi \nabla \mu \nu, \]

as required.

(6.) Since, then,

\[ S \cdot \lambda \phi \chi \lambda' = S \cdot \lambda (\mu \psi' \nu - \nu \psi' \mu) = S (\mu \psi' \mu + \nu \psi' \nu), \]

the value III. of \( m' \) gives, by 349, VI., the equation,

\[ \lambda' \cdot S \lambda \lambda' = S \cdot \lambda (\psi + \phi \chi) \lambda', \]

\( \lambda \) and \( \lambda' \) being independent vectors; hence,

\[ \psi \rho + \phi \chi \rho = m', \rho, \]

or briefly,

\[ \psi + \phi \chi = m'. \]

And in fact, with the standard form of \( \phi \rho \), we have

\[ \chi' \phi \rho = \chi \phi \rho = \nabla (\beta' \beta' \beta' \cdot \nabla \rho \frac{a}{a'} + \nabla \beta' \beta' \cdot \nabla \rho \frac{a}{a'} + \nabla \beta' \beta' \cdot \nabla \rho \frac{a}{a'}); \]

which verifies the equation XX., when it is combined with the value VII. of \( m' \), and with the expression 347, XVIII. for \( \psi \rho \).

(7.) Eliminating the symbol \( \chi \), between the two equations XVI. and XXI., and remembering that \( \phi \psi = \psi \phi = m \), we find the symbolic expression,

\[ \chi' \cdot m \phi^{-1} = \psi = m' - m'' \phi + \phi'; \]

and thus the symbolic and cubic equation I. is proved.

(8.) And because the coefficients, \( m, m', m'' \), of that equation, have been seen to remain unaltered, in the passage from \( \phi \) to \( \phi' \), we may write also this conjugate equation,

\[ 0 = m - m' \phi' + m'' \phi'^2 - \phi'^3. \]

* [This may also be proved thus:—If \( \rho \) and \( \sigma \) are arbitrary vectors, by 348 (5.),

\[ S \rho (\phi^3 - m'' \phi'^3 + m' \phi' - m) = S \sigma (\phi^3 - m'' \phi'^3 + m' \phi' - m) \rho, \]

and therefore vanishes. This requires \( \sigma (\phi^3 - m'' \phi'^3 + m' \phi' - m) = 0 \), as \( \rho \) is arbitrary.]
(9.) Multiplying symbolically the equation I. by $-m^{-1}\psi^2$, and reducing by $\psi\phi = m$, we eliminate the symbol $\phi$, and obtain this cubic in $\psi$,

$$XXV. \ldots 0 = m^2 - mm'\psi + m'\psi^2 - \psi^3;$$

in which $\psi'$ may be substituted for $\psi$.

(10.) In general, it may be remarked, that when we change $\phi$ to $\psi$, and therefore $\psi$ to $m\phi$, as before, we change not only $m$ to $m^2$, but also $m'$ to $mm''$, and $m''$ to $m'$; while $\chi$ is at the same time changed to $\phi\chi$, or to $\chi\phi$, and the quaternion $q_1$ is changed to $q_2$. Accordingly, we may thus pass from the relation XVI. to XXI.; and conversely, from the latter to the former.

(11.) And if the two new auxiliary functions, $\chi$ and $\chi'$, be considered as defined by the equations VIII. and IX., their conjugate relation (3.) to each other may be proved, without any reference to the standard form of $\phi\rho$, by reasonings similar to those which were employed in 347, (8.), to establish the corresponding conjugation of the functions $\psi$ and $\psi'$.

(12.) It may be added that the relations between $\phi$, $\phi'$, $\chi$, $\chi'$, and $m''$ give the following additional transformations, which are occasionally useful:

$$XXVI. \ldots \phi'V\mu\nu = V(\mu\chi\nu + \nu\phi\mu) = -V(\nu\chi\mu + \mu\phi\nu);$$

$$XXVII. \ldots \phi V\mu\nu = V(\mu\chi'\nu + \nu\phi'\mu) = -V(\nu\chi'\mu + \mu\phi'\nu);$$

with others on which we cannot here delay.*

351. The cubic in $\phi$ may be thus written:

$$I. \ldots 0 = m\rho - m'\phi\rho + m''\phi^2\rho - \phi^3\rho;$$

where $\rho$ is an arbitrary vector. If then it happen that for some particular but actual vector, $\rho$, the linear function $\phi\rho$ vanishes, so that $\phi\rho = 0$, $\phi^2\rho = 0$, $\phi^3\rho = 0$, &c., the constant $m$ must be zero; or in symbols,

$$II. \ldots \text{if } \phi\rho = 0, \text{ and } T\rho > 0, \text{ then } m = 0.$$  

Hence, by the expression 347, XXIII. for $m$, when the standard form for $\phi\rho$ is adopted, we must have either

$$III. \ldots S\alpha'\alpha'' = 0, \text{ or else } IV. \ldots S\beta'\beta'\beta = 0;$$

so that, in each case, that generally trinomial form, 347, XV., must admit of

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* [Without introducing $\chi$, since for any three vectors $m''S\lambda\mu\nu = S\lambda(\phi V\mu\nu + V\mu\phi\nu + V\phi\mu\nu)$, it follows, as $\lambda$ is arbitrary, that $m''V\mu\nu = \phi' V\mu\nu + V\mu\phi'\nu + V\phi'\mu\nu$. This is equivalent to XXVI.]
being reduced to a binomial. Conversely, when we have thus a function of the particular form,

V. \( \phi \rho = \beta S \alpha \rho + \beta' S \alpha' \rho \),

we have then,

VI. \( \phi V \alpha \alpha' = 0 \),

so that if \( \alpha \) and \( \alpha' \) be actual and non-parallel lines, the real and actual vector \( V \alpha \alpha' \) will be a value of \( \rho \), which will satisfy the equation \( \phi \rho = 0 \); but no other real and actual value of \( \rho \), except \( \rho = \mathcal{V} \alpha \alpha' \), will satisfy that equation, if \( \beta \) and \( \beta' \) be actual, and non-parallel. In this case V., the operation \( \phi \) reduces every other vector to the fixed plane of \( \beta, \beta' \), which plane is therefore the locus of \( \phi \rho \); and since we have also,

VII. \( \phi' \rho = aS \beta \rho + a' S \beta' \rho \),

we see that the locus of the functionally conjugate vector, \( \phi' \rho \), is another fixed plane, namely that of \( a, a' \). Also, the normal to the latter plane is the line which is destroyed by the former operation, namely by \( \phi \); while the normal to the former plane is in like manner the line, which is annihilated by the latter operation, \( \phi' \), since we have

VIII. \( \phi' V \beta \beta' = 0 \),

but not \( \phi' \rho = 0 \), for any actual \( \rho \), in any direction except that of \( V \beta \beta' \), or its opposite, which may however, for the present purpose, be regarded as the same.*

In this case we have also monomial forms for \( \psi \rho \) and \( \psi' \rho \), namely

IX. \( \psi \rho = V \alpha \alpha' S \beta' \beta \rho \), and X. \( \psi' \rho = V \beta \beta' S \alpha' \alpha \rho \);

so that the operation \( \psi \) destroys every line in the first fixed plane (of \( \beta, \beta' \)), and the conjugate operation \( \psi' \) annihilates every line in the second fixed plane (of \( a, a' \)). On the other hand, the operation \( \psi \) reduces every line, which is out of the first plane, to the fixed direction of the normal to the second plane; and the operation \( \psi' \) reduces every line which is out of the second plane, to that other fixed direction, which is normal to the first plane. And thus it comes to pass, that whether we operate first with \( \psi \), and then with \( \phi \); or first with \( \phi \), and then with \( \psi \); or first with \( \psi' \) and then with \( \phi' \); or first with \( \phi' \), and then with \( \psi' \); in all these cases, we arrive at last at a null line, in conformity with the symbolic equations,

XI. \( \psi \psi = \psi \phi = \phi' \psi' = \psi' \phi' = m = 0 \),

which belong to the case here considered.

* Accordingly, in the present investigation, whenever we shall speak of a "fixed direction," or the "direction of a given line," &c., we are always to be understood as meaning, "or the opposite of that direction."
(1.) Without recurring to the standard form of $\phi \rho$, the equation 348, VI., namely $V. \psi \mu \psi' v = m \phi V \mu v$, and the analogous equation $V. \psi \mu \psi v = m \phi' V \mu v$, might have enabled us to foresee that $\psi \rho$ and $\psi \rho$, if they do not both constantly vanish, must (if $m = 0$) have each a fixed direction; and therefore that each must be expressible by a monome, as above: the fixed direction of $\psi \rho$ being that of a line which is annihilated by the operation $\phi$, and similarly for $\psi' \rho$ and $\phi'$.

(2.) And because, by 347, XI. and XXV., we have

$$\psi V \mu v = V. \phi' \mu \phi' v, \quad \text{and} \quad V. \phi \mu v = V. \phi \mu \phi v,$$

so that the line $\phi' \mu$, if actual, is perpendicular to $\psi V \mu v$, and the line $\phi \mu$ perpendicular to $\psi' V \mu v$, we see that each of the two lines, $\phi' \rho$ and $\phi \rho$, must have (in the present case) a plane locus; whence the binomial forms of the two conjugate vector functions, $\phi \rho$ and $\phi' \rho$, might have been foreseen: $\psi \rho$ and $\psi' \rho$ being here supposed to be actual vectors.

(3.) The relations of rectangularity, of the two fixed lines (or directions), to the two fixed planes, might also have been thus deduced, through the two conjugate binomial forms, V. and VII., without the previous establishment of the more general trinomial (or standard) form of $\phi \rho$.

(4.) The existence of a plane locus for $\phi \rho$, and of another for $\phi' \rho$, for the case when $m = 0$, might also have been foreseen from the equations,

$$S. \phi \lambda \phi' \mu \phi v = S. \phi' \lambda \mu \phi' v = m S \lambda \mu v ;$$

and the same equations might have enabled us to foresee, that the scalar constant $m$ must be zero, if for any one actual vector, such as $\lambda$, either $\phi \lambda$ or $\phi' \lambda$ becomes null.

(5.) And the reducibility of the trinomial to the binomial form, when this last condition is satisfied, might have been anticipated, without any reference to the composition of the constant $m$, from the simple consideration (comp. 294, (10.)), that no actual vector $\rho$ can be perpendicular, at once, to three diplanar lines.

352. It may happen, that besides the recent reduction (351) of the linear function $\phi \rho$ to a binomial form, when the relation

$$I. \ldots \; m = 0$$

exists between the constants of that function, in which case the symbolic and cubic equation 350, I. reduces itself to the form,

$$II. \ldots \phi^3 - m'' \phi^2 + m' \phi = 0,$$
thus losing its absolute term, or having one root equal to zero, this equation may undergo a further reduction, by two of its roots becoming equal to each other; namely either by our having

\[ \text{III. } m' = 0, \quad \text{and} \quad \text{IV. } \phi^2(\phi - m'') = 0; \]

or in another way, by the existence of these other equations,

\[ \text{V. } m''^2 - 4m' = 0, \quad \text{and} \quad \text{VI. } \phi(\phi - \frac{1}{2}m'')^2 = 0. \]

In each of these two cases, we shall find that certain new geometrical relations arise, which it may be interesting briefly to investigate; and of which the principal is the mutual rectangularity of two fixed planes, which are the loci (comp. 351) of certain derived, and functionally conjugate vectors: namely, in the case III. IV., the loci of \( \phi \rho \) and \( \phi' \rho \); and in the case V. VI., the loci of \( \Phi \rho \) and \( \Phi' \rho \), if

\[ \text{VII. } \Phi = \phi - \frac{1}{3}m'', \quad \text{and} \quad \text{VIII. } \Phi' = \phi' - \frac{1}{3}m'', \]

so that, in this last case, the symbol \( \Phi \) satisfies this new cubic,

\[ \text{IX. } 0 = \Phi^3(\Phi + \frac{1}{3}m''); \]

while \( \Phi' \) satisfies at the same time a cubic equation with the same coefficients (comp. 350, (8.)), namely

\[ \text{X. } 0 = \Phi'^2(\Phi' + \frac{1}{3}m''). \]

(1.) We saw in 351, (1.), (2.), that when \( m = 0 \) the line \( \psi \rho \) has generally a fixed direction, to which that of the line \( \phi \rho \) is perpendicular; and that in like manner the line \( \psi \rho \) has then another fixed direction, to which \( \phi' \rho \) is perpendicular. If then the plane loci of \( \phi \rho \) and \( \phi' \rho \) be at right angles to each other, we must also have the fixed lines \( \psi \lambda \) and \( \psi \mu \) rectangular, or

\[ \text{XI. } 0 = S \cdot \psi \lambda \psi \mu = S \lambda \psi \mu, \]

independently of the directions of \( \lambda \) and \( \mu \); whence

\[ \text{XII. } 0 = \psi^2 \mu, \quad \text{or} \quad \text{XIII. } \psi^2 = 0, \]

since \( \mu \) is an arbitrary vector.

(2.) Now in general, by the functional relation 350, XXI. combined with \( \psi \phi = m \), we have the transformation,

\[ \text{XIV. } \psi^2 = \psi (m' - \phi \chi) = m' \psi - m \chi; \]
if then $m = 0$, as in I., the symbol $\psi$ must satisfy the depressed or quadratic equation,

$$\text{XV. } 0 = m'\psi - \psi^2;$$

which is accordingly a factor of the cubic equation,

$$\text{XVI. } 0 = m'\psi^2 - \psi^3,$$

where to the general equation 350, XXV. is reduced, by this supposition of $m$ vanishing.

(3.) If then we have not only $m = 0$, as in I., but also $m' = 0$, as in III., the condition XIII. is satisfied, by XV.; and the two planes, above referred to, are generally rectangular.

(4.) We might indeed propose to satisfy that condition XIII., by supposing that we had always,

$$\text{XVII. } \psi = 0, \text{ that is, } \text{XVII'. } \psi\rho = 0,$$

for every direction of $\rho$; but in this case, the quaternion constant $q_3$ would vanish (by 349, II.); and therefore the constant $m'$, as being its scalar part (by 350, III.), would still be equal to zero.

(5.) The particular supposition XVII. would however alter completely the geometrical character of the question; for it would imply (comp. 351, (2.)) that the directions of the lines $f\rho$ and $f'\rho$ (when not evanescent) are fixed, instead of those lines having only certain planes for their loci, as before.

(6.) On the side of calculation, we should thus have, for the two conjugate functions, $f\rho$ and $f'\rho$, monomial expressions of the forms,

$$\text{XVIII. } f\rho = \beta S\alpha\rho, \quad f'\rho = aS\beta\rho;$$

whence, by 347, XVIII., and 350, VII., we should recover the equations, $\psi\rho = 0$ and $m' = 0$.

(7.) We should have also, in this particular case,

$$\text{XIX. } f\rho = 0, \text{ if } \rho \perp \alpha, \quad \text{and XX. } f'\rho = 0, \text{ if } \rho \perp \beta;$$

so that $f\rho$ now vanishes, if $\rho$ be any line in the fixed plane perpendicular to $\alpha$; and in like manner $f'\rho$ is a null line, if $\rho$ be in that other fixed plane, which is at right angles to the other given line, $\beta$.

(8.) These two planes, or their normals $\alpha$ and $\beta$, or the fixed directions of the two lines $f'\rho$ and $f\rho$, will be rectangular (comp. (1.)), if we have this new equation,

$$\text{XXI. } \phi^2 = 0, \quad \text{or XXI'. } \phi^2\rho = 0,$$
for every direction of $\rho$; and accordingly the expression XVIII. gives

$$\phi^2 \rho = S a \beta \cdot \phi \rho = 0, \quad \text{if} \quad \beta \perp a, \quad \text{and reciprocally.}$$

(9.) Without expressly introducing $a$ and $\beta$, the equation 350, XXIII. shows that when $\psi = 0$, and therefore also $m' = 0$, as in (4.), the symbol $\phi$ satisfies (comp. (2.)) the new quadratic or depressed equation,

$$0 = \phi^2 - m'' \phi;$$

which is accordingly a factor of the cubic IV., but to which that cubic is not reducible, unless we have thus $\psi = 0$, as well as $m' = 0$.

(10.) The condition, then, of the existence and rectangularity of the two planes (7.), for which we have respectively $\phi \rho = 0$ and $\phi' \rho = 0$, without $\phi \rho$ generally vanishing (a case which it would be useless to consider), is that the four following equations should subsist:

$$m = 0, \quad m' = 0, \quad m'' = 0, \quad \text{and} \quad \psi = 0;$$

or that the cubic IV., and its quadratic factor XXII., should reduce themselves to the very simple forms,

$$\phi^3 = 0, \quad \text{and} \quad \phi^2 = 0;$$

the cubic in $\phi$ having thus its three roots equal, and null, and $\psi \rho$ vanishing.

(11.) We may also observe that as, when even one root of the general cubic 350, I. is zero, that is when $m = 0$, the vector equation

$$\phi \rho = 0$$

was seen (in 351) to be satisfied by one real direction of $\rho$, so when we have also $m' = 0$, or when the cubic in $\phi$ has two null roots, or takes the form IV., then the two vector equations,

$$\phi \rho = 0, \quad \psi \rho = 0,$$

are satisfied by one common direction of the real and actual line $\rho$; because we have, by 350, XVII. and XX., the general relation,

$$\psi \rho = m' \rho - \chi \phi \rho.$$

(12.) And because, by 350, XV., we have also the relation $\chi \rho = m'' \rho - \phi \rho$, it follows that when the three roots of the cubic all vanish, or when the three
scalar equations XXIII. are satisfied, then the three vector equations,

\[ \phi \rho = 0, \chi \rho = 0, \chi \rho = 0, \]

have a common (real and actual) vector root; or are all satisfied by one common direction of \( \rho \).

(13.) Since \( m'' - \phi = \chi \), the cubic IV. may be written under any one of the following forms,

\[ 0 = \phi^2 \chi = \phi \chi \phi = \chi \phi^2 = \phi \cdot \chi \chi = \&c., \]

in which accented may be substituted for unaccented symbols: and its geometrical signification may be illustrated by a reference to certain fixed lines, and fixed planes, as follows.

(14.) Suppose first that \( m \) and \( m' \) both vanish, but that \( m'' \) is different from zero, so that the cubic in \( \phi \) is reducible to the form IV., but not to the form XXIV.; and that the operation \( \psi \), which is here equivalent to \( -\phi \chi \), or to \( -\chi \phi \), does not annihilate every vector \( \rho \), so that (comp. (4.) (5.) (6.) \( \phi \rho \) and \( \phi' \rho \) have not the directions of two fixed lines, but have only (comp. (1.) and (3.)) two fixed and rectangular planes, \( \Pi \) and \( \Pi' \), as their loci; and let the normals to these two planes be denoted by \( \lambda \) and \( \lambda' \), so that these two rectangular lines, \( \lambda \) and \( \lambda' \), are situated respectively in the planes \( \Pi \) and \( \Pi' \).

(15.) Then it is easily shown (comp. 351) that the operation \( \phi \) destroys the line \( \lambda' \) itself, while it reduces* every other line (that is, every line which is not of the form \( a \lambda' \), with \( V x = 0 \)) to the plane \( \Pi \perp \lambda \); and that it reduces every line in that plane to a fixed direction, \( \mu \), in the same plane, which is thus the common direction of all the lines \( \phi^2 \rho \), whatever the direction of \( \rho \) may be. And the symbolical equation, \( \chi \cdot \phi^2 = 0 \), expresses that this fixed direction \( \mu \) may also be denoted by \( \chi^{-1} 0 \); or that we have the equation,

\[ 0 = \chi \mu = m'' \mu - \phi \mu, \quad \text{if} \quad \mu = \phi^2 \rho, \]

which can accordingly be otherwise proved: with similar results for the conjugate symbols, \( \phi' \) and \( \chi' \).

(16.) For example, we may represent the conditions of the present case by the following system of equations (comp. 351, V. VII. IX. X., and 350, VI. VII. X. XI.) :

\[
\left\{ \begin{array}{l}
\phi \rho = \beta S \alpha \rho + \beta' S \alpha' \rho, \\
\phi' \rho = \alpha S \beta \rho + \alpha' S \beta' \rho,
\end{array} \right.
\]

\[ 0 = m' = S (V a \alpha' . V \beta' \beta) = S a \beta S \alpha' \beta' - S a \beta' S \alpha' \beta, \]

\[ m'' = S a \beta + S a' \beta'; \]

---

* We propose to include the case where an operation of this sort destroys a line, or reduces it to zero, under the case when the same operation reduces a line to a fixed direction, or to a fixed plane.
XXXII... \[
\begin{align*}
\chi' \rho &= V(\alpha V\beta \rho + a'V\beta' \rho) = m'' \rho - \phi \rho, \\
\chi'' \rho &= V(\beta V\alpha \rho + \beta'V\alpha' \rho) = m'' \rho - \phi' \rho, \\
- \psi \rho &= \phi \chi \rho = \chi \phi \rho = V\alpha a'S\beta' \rho, \\
- \psi' \rho &= \phi' \chi' \rho = \chi' \phi' \rho = V\beta'\beta S\alpha \rho, \\
\end{align*}
\]
and may then write (not here supposing \(\lambda' = V\mu \nu, \&c.\)),
XXXIII... \[
\begin{align*}
\lambda &= V\beta \beta', \quad \lambda' = V\alpha a', \quad S\lambda \lambda' = 0, \\
\mu &= \phi \beta \parallel \phi' \beta', \quad \mu' = \phi' \alpha \parallel \phi' \alpha, \quad S\lambda \mu = S\lambda' \mu' = 0;
\end{align*}
\]
after which we easily find that
XXXIV... \[
\begin{align*}
\phi \lambda &= 0, \quad \phi^2 \rho \parallel \mu, \quad \phi \mu = m'' \mu, \quad \chi \mu = 0; \\
\phi' \lambda &= 0, \quad \phi' \rho \parallel \mu', \quad \phi' \mu' = m'' \mu', \quad \chi' \mu' = 0.
\end{align*}
\]
(17.) Since we have thus \(\chi' \mu' = 0\), where \(\mu'\) is a line in the fixed direction of \(\phi'' \rho\), we have also the equation,
XXXV... \(0 = S\rho \chi' \mu' = S\mu' \chi \rho, \) or \(\chi \rho \perp \mu'\);
the locus of \(\chi \rho\) is therefore a plane perpendicular to the line \(\mu'\); and in like manner, \(\mu\) is the normal to a plane, which is the locus of the line \(\chi' \rho\). And the symbolical equations, \(\phi \cdot \phi \chi = 0, \phi^2 \cdot \chi = 0\), may be interpreted as expressing, that the operation \(\phi\) reduces every line in this new plane of \(\chi \rho\) to the fixed direction of \(\phi' = 0\), or of \(\lambda'\); and that the operation \(\phi^2\) destroys every line in this plane \(\perp \mu'\); with analogous results, when accented are interchanged with unaccented symbols. Accordingly we see, by XXXII., that \(\phi \chi \rho\) has the fixed direction of \(V\alpha a'\), or of \(\lambda'\); and that \(\phi' \cdot \phi \chi \rho = 0\), because \(\phi' \lambda' = 0\).
(18.) We see also, that the operation \(\phi \chi\), or \(\chi \phi\), destroys every line in the plane \(\Pi\), to which the operation \(\phi\) reduces every line; and that thus the symbolical equations, \(\phi \chi \cdot \phi = 0, \chi \phi \cdot \phi = 0\), may be interpreted.
(19.) As a verification, it may be remarked that the fixed direction \(\lambda'\), of \(\phi \chi \rho\) or \(\chi \phi \rho\), ought to be that of the line of intersection of the two fixed planes of \(\phi \rho\) and \(\chi \rho\); and accordingly it is perpendicular by XXXIII. to their two normals, \(\lambda\) and \(\mu'\): with similar remarks respecting the fixed direction \(\lambda\), of \(\phi' \chi' \rho\) or \(\chi' \phi' \rho\), which is perpendicular to \(\lambda'\) and to \(\mu\).
(20.) Let us next suppose, that besides \(m = 0\), and \(m' = 0\), we have \(\psi = 0\), but that \(m''\) is still different from zero. In this case, it has been seen (6.) that the expression for \(\phi \rho\) reduces itself to the monomial form, \(\beta S a \rho\); and therefore that the operation \(\phi\) destroys every line in a fixed plane \(\perp \alpha\), while it reduces every other line to a fixed direction \(\parallel \beta\), which is not contained in that plane, because we have not now \(S a \beta = 0\).
(21.) In this case we have by (16.), equating $a'$ or $\beta'$ to 0, the expressions,

$\phi \rho = \beta \omega \phi \rho, \quad \phi' \rho = a \beta \omega \rho, \quad m'' = \omega \beta \omega' \geq 0,$

$\chi \rho = \omega \alpha \alpha \omega \rho' = (m'' - \phi) \rho, \quad \chi' \rho = \omega \beta \omega \alpha \rho = (m'' - \phi') \rho,$

so that the equations XVIII. are reproduced; and the depressed cubic, or the quadratic XXII. in $\phi$, may be written under the very simple form,

XXXVII. . . $0 = \phi \chi = \chi \phi.$

(22.) Accordingly (comp. (5.) and (7.)), the operation $\phi$ here reduces an arbitrary line to the fixed direction of $\beta$, while $\chi$ destroys every line in that direction; and conversely, the operation $\chi$ reduces an arbitrary line to the fixed plane perpendicular to $\alpha$, and $\phi$ destroys every line in that fixed plane. But because we do not here suppose that $m'' = 0$, the fixed direction of $\phi \rho$ is not contained in the fixed plane of $\chi \rho$; and (comp. (8.) and (10.)) the directions of $\phi \rho$ and $\phi' \rho$ are not rectangular to each other.

(23.) On the other hand, if we suppose that the three roots of the cubic in $\phi$ vanish, or that we have $m = 0$, $m' = 0$, and $m'' = 0$, as in XXIII., but that the equation $x \rho = 0$ is not satisfied for all directions of $\rho$, then the binomial forms XXXI. of $\phi \rho$ and $\phi' \rho$ reappear, but with these two equations of condition between their vector constants, whereof only one had occurred before:

XXXVIII. . . $0 = S \alpha \beta S \alpha' \beta' - S \alpha \beta' S \alpha' \beta, \quad 0 = S \alpha \beta + S \alpha' \beta'.$

(24.) We have also now the expressions,

XXXIX. . . $\chi \rho = - \phi \rho, \quad \chi' \rho = - \phi' \rho ;$

and the cubic in $\phi$ becomes simply $\phi^3 = 0$, as in XXIV.; but it is important to observe that we have not here (comp. (9.)) the depressed or quadratic equation $\phi^2 = 0$, since we have now on the contrary the two conjugate expressions,

XL. . . $\phi^2 \rho = \psi \rho = \omega \alpha \alpha \beta' \beta' \phi \rho, \quad \phi^2 \rho = \psi' \rho = \omega \beta \omega' \alpha \alpha \beta \phi \rho,$

which do not generally vanish. And the equation $\phi^3 = 0$ is now interpreted, by observing that $\phi^3$ here reduces every line to the fixed direction of $\phi' \alpha 0$; while $\phi$ reduces an arbitrary vector to that fixed plane, all lines in which are destroyed by $\phi^2$.

(25.) In this last case (23.), in which all the roots of the cubic in $\phi$ are equal, and are null, the theorem (12.), of the existence of a common vector root
of the three equations XXVIII., may be verified by observing that we have now,

\[ \text{XLII. } \phi \nu = 0, \ \phi' \nu = 0, \ \nu \mu = 0, \ \sigma \mu \mu' = 0, \]

the third of which would not have here held good, unless we had supposed \( m'' = 0 \).

(26.) This last condition allows us to write, by (16.),

\[ \text{XLIII. } \nu = \nu \mu \nu', \text{ so that } \sigma \mu \nu = 0, \ \sigma \mu' \nu = 0, \]

this new vector \( \nu \) will be a line in the intersection of those two rectangular planes, which were lately seen (14.) to be the loci of the lines \( \phi \rho \) and \( \phi' \rho \), and are now (comp. (17.)) the loci of \( \chi \rho \) and \( \chi' \rho \); and the three lines \( \mu, \mu', \nu \) (or \( \lambda', \lambda, \nu \)) will compose a rectangular system.

(27.) In general, it is easy to prove that the expressions,

\[ \text{XLIV. } \begin{cases} \beta = a \beta_1 + b \beta_1' & \beta' = a' \beta_1 + b' \beta_1' \\ a_1 = aa + a'a' & a_1' = ba + b'a' \end{cases} \]

in which \( a, \beta, a', \beta' \) may be any four vectors, and \( a, b, a', b' \) may be any four scalars, conduct to the following transformations (in which \( \rho \) may be any vector):

\[ \text{XLV. } \sigma a_1 \beta_1 + \sigma a_1' \beta_1' = \sigma a \beta + \sigma a' \beta' ; \]

\[ \text{XLVI. } \beta \sigma a_1 \rho + \beta' \sigma a_1' \rho = \beta \sigma \rho + \beta' \sigma \rho ; \]

\[ \text{XLVII. } \nu a_1 a' \cdot \nu \beta_1 \beta_1' = \nu a \cdot \nu \beta \beta' ; \]

so that the scalar, \( \sigma a \beta + \sigma a' \beta' \); the vector, \( \beta \sigma a \rho + \beta' \sigma a' \rho \); and the quaternion,\(^*\) \( \nu a \cdot \nu \beta \beta' \), remain unaltered in value, when we pass from a given system of four vectors \( a \beta a' \beta' \), to another system of four vectors \( a_1 \beta_1 a_1' \beta_1' \), by expressions of the forms XLIV.

(28.) With the help of this general principle (27.), and of the remarks in (26.), it may be shown, without difficulty, that in the case (23.) the vector

\[ * \text{ We have, in these transformations, examples of what may be called Quaternion Invariants.} \]
constants of the binomial expression \( \beta S\alpha p + \beta' S\alpha' p \) for \( \phi p \) may, without any real loss of generality, be supposed subject to the four following conditions,

\[
\text{XLVIII. } 0 = S\alpha \beta = S\alpha' \beta = S\beta \beta' = S\alpha' \beta';
\]

which evidently conduct to these other expressions,

\[
\text{XLIX. } \phi^3 p = \beta S\alpha \beta' S\alpha' p, \quad \phi^3 p = 0;
\]

and thus put in evidence, in a very simple manner, the general non-depression of the cubic \( \phi^3 = 0 \), to the quadratic, \( \phi^2 = 0 \).

(29.) The case, or sub-case, when we have not only \( m = 0, m' = 0, m'' = 0 \), but also \( \psi = 0 \), and therefore \( \phi^2 = 0 \), as a depressed form of \( \phi^3 = 0 \), by the linear function \( \phi p \) reducing itself to the monomial \( \beta S\alpha p \), with the relation \( S\alpha \beta = 0 \) between its constants, has been already considered (in (10.)); and thus the consequences of the supposition III., that there are (at least) two equal but null roots of the cubic in \( \phi \), have been perhaps sufficiently discussed.

(30.) As regards the other principal case of equal roots, of the cubic equation in \( \phi \), namely that in which the vector constants are connected by the relation V., or by the equation of condition,

\[
\text{L. } 0 = m''^2 - 4m' = (S\alpha \beta + S\alpha' \beta')^2 - 4S(Vaa' V\beta' \beta)
\]

\[
= (S\alpha \beta - S\alpha' \beta')^2 + 4S\alpha \beta' S\alpha' \beta,
\]

it may suffice to remark that it conducts, by VI., or by VII. and IX., to the symbolical equation,

\[
\text{LI. } 0 = \phi \Phi^2, \quad \text{if } \Phi = \phi - \frac{1}{2} m'';
\]

and that thus its interpretation is precisely similar to that of the analogous equation,

\[
\chi \phi^3 = 0, \quad \text{where } \chi = m'' - \phi, \quad \text{XXIX.}
\]
as given in (14.), and in the following sub-articles.*

* [The following résumé of the special cases discussed in recent articles may not be superfluous:—
Assuming arbitrarily any three constant and diplanar vectors \( \beta, \beta', \text{ and } \beta'' \), any linear vector function \( \phi p \) may be resolved along these three vectors; thus \( \phi p = \chi \beta + x' \beta' + x'' \beta'' \). In this expression \( x, x' \), and \( x'' \) are linear and scalar functions of \( p \), and may consequently be replaced by \( S\alpha p \), \( S\alpha' p \), and \( S\alpha'' p \). Hence the trinomial form \( \phi p = \beta S\alpha p + \beta' S\alpha' p + \beta'' S\alpha'' p \) is established, and the function \( \phi \) is made to depend upon the three vectors \( \alpha, \alpha', \text{ and } \alpha'' \). When these are given, \( \phi \) is determined; and conversely, when \( \phi \) is given, the three vectors \( \alpha, \alpha', \text{ and } \alpha'' \) are determinate, retaining always the same set of vectors of reference \( \beta, \beta' \), and \( \beta'' \). Special cases will arise when special relations connect \( \alpha, \alpha', \) and \( \alpha'' \).
If \( \phi p = 0 \) for a particular value of \( p \), \( S\alpha p = S\alpha' p = S\alpha'' p = 0 \) are necessary consequences, and]
353. When we have \( m = 0 \), but not \( m' = 0 \), nor \( m'' = 4m' \), the three roots of the cubic in \( \phi \) are all unequal, while one of them is still null, as before; and the two roots of the quadratic and scalar equation, with real coefficients (347),

\[
I... 0 = c^2 + m''c + m',
\]

which is formed from the cubic by changing \( \phi \) to \( -c \), and then dividing by \( c \), are also necessarily unequal, whether they be real or imaginary. We shall find that when these two scalar roots, \( c_1, c_2 \), are real, there are then two real directions, \( \rho_1 \) and \( \rho_2 \), in that fixed plane \( \Pi \) which is the locus (351, 352) of the line \( \phi \rho \), possessing the property that for each of them the homogeneous and vector equation of the second degree,

\[
II... V\phi\rho = 0, \quad \text{or} \quad \phi \parallel \rho,
\]

is satisfied, without \( \rho \) vanishing; namely by our having, for the first of these two directions, the equation

\[
III... \phi \rho_1 = -c_1\rho_1, \quad \text{or} \quad \phi_1\rho_1 = 0, \quad \text{if} \quad \phi_1 = \phi + c_1;
\]

and for the second of them the analogous equation,

\[
IV... \phi \rho_2 = -c_2\rho_2, \quad \text{or} \quad \phi_2\rho_2 = 0, \quad \text{if} \quad \phi_2 = \phi + c_2:
\]

but that no other direction of the real and actual vector \( \rho \), satisfies the equation

Therefore \( a, a', \) and \( a'' \) (if actual) must be complanar. But if \( a, a', \) and \( a'' \) are complanar, the trinomial form reduces on rearrangement to the binomial form

\[
\phi \rho = (\beta + a\beta') Sa\rho + (a' + a\beta') Sa'\rho,
\]

provided \( a \) and \( a' \) are the scalars determined by the relation of complanarity \( a'' = a a + a'a \). Conversely, if the trinomial reduces to a binomial form, the three vectors \( a, a', \) and \( a'' \) (if actual) must be complanar.

Further reduction to the monomial form will not be possible unless these three vectors are parallel. In general, also, as \( \psi \rho = Va'a'S\beta'\beta'\rho + Va''a'SS\beta'\beta'\rho + Va'a'S\beta'\beta'\rho + Va'a'S\beta'\beta'\rho, \psi \rho \) will not vanish identically, or the equation \( \psi = 0 \) will not be true, unless the vectors are parallel. This easily follows on replacing \( \rho \) successively by \( \beta, \beta', \) and \( \beta'' \).

Remark that, when \( \phi \) is expressible in a binomial form, it reduces those vectors which it does not annul to a fixed plane, we may assume a plane containing a pair of arbitrarily chosen vectors \( \beta \) and \( \beta' \), and consider all those functions \( \phi \) which reduce vectors to this particular plane. Just as in the case of the trinomial form, these functions \( \phi \) may be expressed by the type \( \phi \rho = BS\alpha \rho + \beta'S\alpha'\rho \), and they depend on and may be determined by the vectors \( \alpha \) and \( \alpha' \) if the vectors \( \beta \) and \( \beta' \) are preserved unchanged.

A second root of the cubic will vanish if \( m' = SVa'a'V\beta' \) is equal to zero. This may happen in two ways—(1) when \( Vaa' = 0 \), in which case the binomial is reducible to the monomial form, and \( \psi \rho \) will vanish for all values of \( \rho \), or \( \psi = 0 \); (2) when \( Vaa' \) is actual and perpendicular to \( V\beta' \), that is, when the plane of \( a \) and \( a' \) is at right angles to that of \( \beta \) and \( \beta' \). In this latter case, the assumptions
V., except that third which has already been considered (351), as satisfying the linear and vector equation,

$$V\ldots \phi p = 0, \quad \text{with } T_p > 0.$$

It will also be shown that these two directions, $\rho_1$, $\rho_2$, are not only real, but rectangular, to each other and to the third direction $\rho$, when the linear function $\phi p$ is self-conjugate (349, (4.)), or when the condition

$$VI\ldots \phi'\rho = \phi p, \quad \text{or} \quad VI'\ldots S\lambda\phi p = S\rho\phi\lambda,$$

is satisfied by the given form of $\phi$, or by the constants which enter into the composition of that linear symbol; but that when this condition of self-conjugation is not satisfied, the roots of the quadratic $L$ may happen to be imaginary: and that in this case there exists no real direction of $\rho$, for which the vector equation $II.$ of the second degree is satisfied, by actual values of $\rho$, except that one direction which has been seen before to satisfy the linear equation $V$.

(1.) The most obvious mode of seeking to satisfy $II.$, otherwise than through $V.$, is to assume an expression of the form, $\rho = x\beta + x'\beta'$, and to seek thereby to satisfy the equation, $$(\phi + c)\rho = 0,$$ with $\phi p = \beta S\rho + \beta' S\rho'$, by satisfying separately the two scalar equations,

$$VII\ldots 0 = x(c + Sa\beta) + x' Sa\beta', \quad 0 = x'(c + S\rho'\beta') + x Sa'\beta,$$

$$\alpha = \alpha + \beta V\beta'$$
and $\alpha' = \alpha' + \beta' V\beta'$ are legitimate when $\alpha$, $\alpha'$, $\beta$, and $\beta'$ are scalars, while $\beta'$ is some vector in the plane of $\beta$ and $\beta'$, and not, as before, diplanar to them. Replacing $\alpha$ and $\alpha'$, the new binomial form, $\phi p = (\alpha + \beta') S\rho + (\beta + \beta') S\rho'$. $\rho$ is obtained, and $\phi p = (\alpha' + \alpha') \phi\rho$. Again, a third root will vanish if $m'' = Sa\beta + Sa\beta' = S(\beta + \beta')\beta'' = 0$, or if $\beta'' \parallel V(\beta + \beta')V\beta'$. Examining separately the case in which the symbolic equation of the binomial is depressed to a quadratic, it is seen at once that it must be of the form $\phi^2 + x\phi = 0$. It cannot be of the form $\phi^2 + x\phi + y = 0$, for $\phi p$, $\phi p'$, &c., are in the plane of $\beta$ and $\beta'$, and $\rho$ is not generally in that plane. On calculation of $\phi\rho$, it is found that

$$\phi^2 p + x\phi p = \beta (Sa\beta Sa + Sa\beta' Sa' + x Sa) + \beta' (Sa\beta Sa + Sa\beta' Sa' + x Sa')$$
and if this vanishes for all values of $\rho$,

$$x = - Sa\beta = - Sa\beta', \quad \text{and} \quad Sa\beta' = Sa\beta = 0.$$

The second pair of equations is satisfied by assuming $\alpha = V\tau\beta'$ and $\alpha' = V\tau\beta$, and then from the first pair $x = - Sr\beta'\beta = - S\tau\beta'. \beta$. Hence, it is easy to see that the general solutions are

$$a = a\phi\beta' - x V\beta', \quad \text{and} \quad a' = a\phi\beta + x V\beta,$$
and that $V\alpha = - x \left( a + a' + x V\beta' \right)$. From these $x = - 3m''$, and $x^2 = + m' = \frac{1}{2} m'^2$. If $x$ vanishes, the function becomes monomial.

Of course when $m$ is zero, the usual solution $m\rho = \psi\sigma$ of the equation $\phi p = \sigma$ is nugatory. In this case, since $\phi^2 p - m'\phi^2 p + m'\phi p = 0$, or $\phi^2 - m'\phi + m'\phi = 0$, the solution is $m\rho = m'\sigma - \phi\sigma + \phi^{-1}0$, and it is indeterminate; if in addition $m' = 0$, the solution is $m'\rho = \sigma + \phi^{-1}0$.]
which give, by elimination of $x' : x$, the following quadratic in $c$,

\[ (c + S\alpha \beta) (c + S\alpha' \beta') = S\alpha \beta S\alpha' \beta', \]

which is easily seen to be only another form of I. Denoting then, as above, by $c_1$ and $c_2$, the roots of that quadratic I., supposed for the present to be real, we have these two real directions for $\rho$, in the plane $II$ of $\beta, \beta'$:

IX. $\rho_1 = \beta(c_1 + S\alpha' \beta') - \beta'S\alpha' \beta = c_1\beta + V\alpha' V\beta'\beta'$;

X. $\rho_2 = \beta(c_2 + S\alpha' \beta') - \beta'S\alpha' \beta = c_2\beta + V\alpha' V\beta'\beta'$;

which satisfy the equations III. and IV. In fact, the expression IX. gives

$\phi_{\rho_1} = c_1\phi\beta + m'\beta = -c_1\rho_1$, or $\phi_{\rho_1} = 0$,

because we may write it thus,

XI. $\rho_1 = (m'' + c_1)\beta - \phi\beta = -c_2\beta - \phi\beta = -\phi_1\beta = -\phi\beta - m'c_1\beta$;

and in like manner, the expression X. may be thus written,

XII. $\rho_2 = (m'' + c_2)\beta - \phi\beta = -c_1\beta - \phi\beta = -\phi_1\beta = -\phi\beta - m'c_2\beta$;

and gives,

$\phi_{\rho_2} = c_2\phi\beta + m'\beta = -c_2\rho_2$, or $\phi_{\rho_2} = 0$.

(2.) We may also write,

XIII. $\rho_1' = \beta'(c_1 + S\alpha\beta) - \beta S\alpha\beta' = c_1\beta' + V\alpha V\beta'\beta'$;

XIV. $\rho_2' = \beta'(c_2 + S\alpha\beta) - \beta S\alpha\beta' = c_2\beta' + V\alpha V\beta'\beta'$;

and shall then have the equations,

XV. $\phi_1\rho_1' = 0$, $\phi_2\rho_2' = 0$;

but the directions of $\rho_1'$ and $\rho_2'$ will be the same by VIII. as those of $\rho_1$ and $\rho_2$, and so will furnish no new solution of the problem just resolved.

(3.) Since we have thus,

XVI. $\phi_2\beta' \parallel \phi_2\beta \parallel \rho_1 \parallel \phi_1^{-0}$, and XVI. $\phi_1\beta' \parallel \phi_1\beta \parallel \rho_2 \parallel \phi_2^{-0}$,

it follows that the operation $\phi_2$ reduces every line in the fixed plane of $\phi\rho$ to the fixed direction of $\phi_1^{-0}$; and that, in like manner, the operation $\phi_1$ reduces every line, in the same fixed plane of $\phi\rho$, to the other fixed direction of $\phi_2^{-0}$.

(4.) Hence we may write the symbolic equations,

XVII. $\phi_1 \cdot \phi_2 \cdot \phi = 0$, $\phi_2 \cdot \phi_1 \cdot \phi = 0$,

in which the points may be omitted; and in fact we have the transformations,

XVIII. $\phi_1\phi_2 = \phi_2\phi_1 = (\phi + c_1)(\phi + c_2) = \phi^2 - m''\phi + m' = \psi$,

so that

$\phi_1\phi_2 \cdot \phi = \phi_2\phi_1 \cdot \phi = \psi \phi = m = 0$. 
(5.) If we propose to form $\psi_1$ from $\phi_1$, by the same general rule (347, XI.) by which $\psi$ is formed from $\phi$, we have

\[ XIX. \quad \psi_1 V \mu \nu = V. \phi' \mu \phi' \nu = V. (\phi' \mu + c_1 \mu) (\phi' \nu + c_1 \nu), \]

and therefore, by the definition 350, VIII. of $\chi$,

\[ XX. \quad \psi \rho = \psi \rho + c_1 \chi \rho + c_1^2 \rho, \quad \text{or} \quad XXI. \quad \psi_1 = \psi + c_1 \chi + c_1^2; \]

and in like manner,

\[ XXII. \quad \psi_2 = \psi + c_2 \chi + c_2^2, \]

even if $m$ be different from zero, and if $c_1$, $c_2$ be arbitrary scalars.

(6.) Accordingly, without assuming that $m$ vanishes, if we operate on $\psi \rho$ with $\phi_1$, or symbolically multiply the expression XXI. for $\psi_1$ by $\phi_1$, we get the symbolic product,

\[ XXIII. \quad \phi_1 \psi_1 = (\phi + c_1)(\psi + c_1 \chi + c_1^2) \]

\[ = \phi \psi + c_1(\phi \chi + \psi) + c_1^2 (\phi + \chi) + c_1^3 \]

\[ = m + c_1 m' + c_1^2 m'' + c_1^3 = m, \]

where $m_1$ is what the scalar $m$ becomes, when $\phi$ is changed to $\phi_1$, or is such that

\[ XXIV. \quad m_1 S \lambda \mu \nu = S. \phi' \lambda \phi' \mu \phi' \nu = S. (\phi' \lambda + c_1 \lambda) (\phi' \mu + c_1 \mu) (\phi' \nu + c_1 \nu); \]

as appears by the definitions of $\phi'$, $\psi$, $\chi$, $m$, $m'$, $m''$, and by the relations between those symbols which have been established in recent Articles, or in the sub-articles appended to them.

(7.) Supposing now again that $m = 0$, and that $c_1$, $c_2$ are the roots of the quadratic I. in $c$, we have by XXIII.,

\[ XXV. \quad \phi_1 \psi_1 = m_1 = 0; \quad \text{and in like manner} \quad XXVI. \quad \phi_2 \psi_2 = m_2 = 0, \]

if $m_2$ be formed from $m_1$, by changing $c_1$ to $c_2$.

(8.) Comparing XXV. with XXVII., we may be led to suspect the existence of an intimate connexion existing between $\psi_1$ and $\phi_1$, since each reduces an arbitrary vector to the fixed direction of $\phi_1^{-1}0$, or of $\rho_1$; and in fact these two operations are identical, because, by XXI., and by the known relations between the symbols, we have the transformations,

\[ XXVII. \quad \psi_1 = \psi + c_1 \chi + c_1^2 = (m' - m'' \phi + \phi^2) + c_1(m'' - \phi) + c_1^3 \]

\[ = \phi^2 - (m'' + c_1) \phi = \phi^2 + c_1 \phi = \phi \phi_2; \]

and similarly

\[ XXVIII. \quad \psi_2 = \phi^2 + c_1 \phi = \phi \phi_1; \]

while $\psi = \phi_1 \phi_2$, as before.
(9.) We have thus the new symbolic equation,

\[ \phi \phi_1 \phi_2 = 0, \]

in which the three symbolic factors, \( \phi, \phi_1, \phi_2 \) may be in any manner grouped and transposed, so that it includes the two equations XVII.; and in which the subject of operation is an arbitrary vector \( \rho \). Its interpretation has been already partly given; but we may add, that while \( \phi \) reduces every vector to the fixed plane \( \Pi \), \( \phi_1 \) reduces every line to another fixed plane, \( \Pi_1 \), and \( \phi_2 \) reduces to a third plane, \( \Pi_2 \); thus \( \phi_1 \phi_2 \), or \( \phi_2 \phi_1 \), while it destroys two lines \( \rho_1, \rho_2 \), and therefore every line in the plane \( \Pi \), reduces an arbitrary line to the fixed direction of the intersection of the two planes \( \Pi_1 \Pi_2 \), which intersection must thus have the direction of \( \phi^{-1}0 \); and in like manner, the fixed direction \( \rho_1 \) of \( \phi_1^{-1}0 \), as being that to which an arbitrary vector is reduced (3.) by the compound operation \( \phi_2 \phi_1 \), or \( \phi_1 \phi_2 \), must be that of the intersection of the planes \( \Pi_1 \Pi_2 \); and \( \rho_2 \), or \( \phi_2^{-1}0 \), has the direction of the intersection of \( \Pi_1 \Pi_2 \); while on the other hand \( \phi \phi_2 \) destroys every line in \( \Pi_1 \), and \( \phi \phi_1 \) every line in \( \Pi_2 \); so that these three planes, with their three lines of intersection, are the chief elements in the geometrical interpretation of the equation \( \phi \phi_1 \phi_2 = 0 \).

(10.) The conjugate equation,

\[ \phi' \phi'_1 \phi'_2 = 0, \]

may be interpreted in a similar way, and so conducts to the consideration of a conjugate system of planes and lines; namely the planes \( \Pi'_1, \Pi'_1, \Pi'_2 \), which are the loci of \( \phi' \rho, \phi'_1 \rho, \phi'_2 \rho \), while the operations \( \phi' \phi'_2, \phi'_2 \phi'_1, \) and \( \phi' \phi'_1 \) destroy all lines in these three planes respectively, and reduce arbitrary lines to the fixed directions of the intersections, \( \Pi'_1 \Pi'_2, \Pi'_2 \Pi'_1, \Pi'_1 \Pi'_1 \), which are also those of \( \phi'^{-1}0, \phi'^{-1}0, \phi'^{-1}0 \).

(11.) It is important to observe that these three last lines are the normals to the three first planes, \( \Pi, \Pi', \Pi'' \); and that, in like manner, the three former lines are perpendicular to the three latter planes. To prove this, it is sufficient to observe that

\[ S \phi' \rho = S \phi' \rho' = 0, \quad \text{if} \quad \phi' \rho' = 0, \quad \text{or that} \quad \phi \rho \perp \phi'^{-1}0; \]

and similarly, \( \phi' \rho \perp \phi'^{-1}0 \), &c.*

* [More symmetrically, without assuming one root to be zero, if \( \phi \) satisfies the symbolical cubic \( (\phi + c_1) (\phi + c_2) (\phi + c_3) = 0 \), it is easy to show that \( \rho_1 \), the result of operating by \( (\phi + c_2) (\phi + c_3) \) on any vector \( \rho \), is parallel to a fixed direction. For a second arbitrary vector \( \sigma \) may be expressed in the form \( x \phi' \rho + y \phi' \rho + z \rho \), and so \( (\phi + c_1) (\phi + c_2) (\phi + c_3) \sigma = x \phi' \rho_1 + y \phi' \rho_1 + z \rho_1 = (x \phi_1^2 - y \phi_1 + z) \rho_1 \) (since by the symbolical cubic \( (\phi + c_1) \rho_1 = 0 \) is likewise \( \| \rho_1 \)). Thus the operators \( (\phi + c_2) (\phi + c_3), (\phi + c_3) (\phi + c_1), \)
(12.) Instead of eliminating \( z' : z \) between the two equations VII., we might have eliminated \( c \); which would have given this other quadratic,

\[ XXXII. \ldots 0 = x'^2 Sa' \beta + xx' (Sa' \beta' - Sa \beta) - x'^2 Sa \beta' ; \]

also, if \( x'_1 : x_1 \) and \( x'_2 : x_2 \) be the two values of \( x' : x \), then

\[ XXXIII. \ldots \rho_1 || x_1 \beta + x'_1 \beta', \quad \rho_2 || x_2 \beta + x'_2 \beta', \]

and

\[ XXXIV. \ldots x_1 x_2 : (x_1 x'_2 + x_2 x'_1) : x'_1 x'_2 = -Sa_\beta' : (Sa \beta - Sa' \beta') : Sa \beta' ; \]

hence the condition of rectangularity of the two lines \( \rho_1, \rho_2 \), or \( \phi_1 \perp 0, \phi_2 \perp 0 \), is expressed by the equation,

\[ XXXV. \ldots 0 = -\beta'^2 Sa \beta' + \beta \beta' (Sa \beta - Sa' \beta') + \beta'^2 Sa \beta' = S \cdot \beta \beta' V (\beta \alpha + \beta' \alpha') ; \]

and consequently it is satisfied, if the given function \( \phi \) be self-conjugate (VI.), because we have then the relation,

\[ XXXVI. \ldots V \beta \alpha + V \beta' \alpha' = 0 ; \]

in fact the binomial form of \( \phi \) gives (comp. 349, XXII.),

\[ XXXVII. \ldots \phi' \rho - \phi \rho = (a S \beta \rho - S a \rho) + (a' S \beta' \rho - S a' \rho) = V \cdot \rho V (\beta \alpha + \beta' \alpha') , \]

which cannot vanish independently of \( \rho \), unless the constants satisfy the condition XXXVI.

(13.) With this condition then, of self-conjugation of \( \phi \), we have the relation of rectangularity,

\[ XXXVIII. \ldots S_{\rho_1 \rho_2} = 0, \]  

or \( \phi_1 \perp 0 \perp \phi_2 \perp 0 \); at least if these directions \( \rho_1 \) and \( \rho_2 \) be real, which they can easily be proved to be, as follows. The condition XXXVI. gives,

\[ XXXIX. \ldots 0 = S \cdot a a' V (\beta \alpha + \beta' \alpha') = a'^2 Sa \beta' + S a' (Sa' \beta' - Sa \beta) - a'^2 Sa \beta' ; \]

and \((\phi + c_1) (\phi + c_2)\) reduce any vector to lines parallel respectively to three fixed directions \( \rho_1, \rho_2, \) and \( \rho_3 \). Further, by the property of the conjugate function \( \phi' \), \((\phi + c_1) \rho \) is a general expression for a vector perpendicular to \( \rho_1 \). In the same way \((\phi' + c_2) (\phi' + c_3) \rho \) is perpendicular to \( \rho_2 \) and also to \( \rho_3 \) and parallel to a fixed direction \( \rho'_1 \) which satisfies \((\phi' + c_1) \rho'_1 = 0 \); and \( \rho'_2 \) and \( \rho'_3 \) similarly found and satisfying \((\phi' + c_2) \rho'_2 \) and \((\phi' + c_3) \rho'_3 = 0 \) are at right angles respectively to the planes of \( \rho_2, \rho_1, \) and of \( \rho_1, \rho_2 \). Taking unit vectors through a common origin and parallel to these fixed vectors, \( U_{\rho_1}, U_{\rho_2}, \) and \( U_{\rho_3} \) determine a triangle on the unit sphere and \( U_{\rho'_1}, U_{\rho'_2}, \) and \( U_{\rho'_3} \) are the vectors to the vertices of the supplemental triangle. Again if \( \gamma \) is the spin-vector defined in 349 (4.), \( U (\phi + c_1) \gamma \) or its equal \( U (\phi' + c_1) \gamma \) terminates at the pole of the great circle through \( U_{\rho_1} \) and \( U_{\rho'_1} \), and the point determined by \( UV \gamma \phi \gamma \) is the common orthocentre of the two triangles. When the function is self-conjugate, the two supplemental triangles coincide, and consequently the solutions of \( V \rho \phi \phi = 0 \) are mutually perpendicular (16.).

* [In general by 349 (4.), \( 2 S \gamma \rho_1 \rho_2 = S (\phi - \phi') \rho_1 \rho_2 = (c_2 - c_1) S \rho_1 \rho_2 \). So if \( \rho_1 \) is perpendicular to \( \rho_2, \gamma \), if it does not vanish, lies in their plane. Conversely, if \( \gamma \) lies in the plane of \( \rho_1 \) and \( \rho_2 \), either \( S \rho_1 \rho_2 = 0 \), or \( c_1 = c_2 \).]
hence
\[(a^2Sαβ - a'^2Sαβ')^2 = (Sαα')^2 (Sαβ - Sα'β')^2,\]
\[a^2a'^2(m''^2 - 4m') = a^2a'^2 [(Sαβ - Sα'β')^2 + 4Sαβ'Sα'β'] \]
\[= (a^2a'^2 - (Sαα')^2) (Sαβ - Sα'β')^2 + (a^2Sαβ + a'^2Sαβ')^2 > 0, \]
and
\[XL. \ldots (Sαβ - Sα'β')^2 + 4Sαβ'Sα'β' = m''^2 - 4m' > 0; \]
so that each of the two quadratics, I. (or VIII.), and XXXII., has real and unequal roots: a conclusion which may also be otherwise derived, from the expressions \[β = aa + ba', \quad β' = ba + a'a', \]
which the condition allows us to substitute for \(β\) and \(β'\).

(14.) The same condition XXXVI. shows that the four vectors \(aβa'β'\) are coplanar, or that we have the relations,
\[XL. \ldots Sαββ' = 0, \quad Sα'ββ' = 0, \quad V(Vaα'.Vβ'β) = 0; \]
hence \(Vaα'\), or \(φ^{-1}0\) is now normal to the plane \(Π\); and therefore by (13.), when the function \(φ\) is self-conjugate (VI.), the three directions,
\[XLII. \ldots ρ, \quad ρ_1, \quad ρ_2, \quad φ^{-1}0, \quad φ_1^{-1}0, \quad φ_2^{-1}0, \]
compose a real and rectangular system.

(15.) In the present series of sub-articles (to 353), we suppose that the three roots of the cubic in \(φ\) are all unequal, the cases of equal roots (with \(m = 0\)) having been discussed in a preceding series (352); but it may be remarked, in passing, that when a self-conjugate function \(φρ\) is reducible to the monomial form \(βSαρ\), we must have the relation \(Vβa = 0\); and that thus the line \(β\), to the fixed direction of which (comp. 352, (5.) and (6.)) the operation \(φ\) then reduces an arbitrary vector, is perpendicular to the fixed plane (352, (7.)), every line in which is destroyed by that operation \(φ\).

(16.) In general, if \(φ\) be thus self-conjugate, it is evident that the three planes \(Π', \quad Π_1', \quad Π_2'\), which are (comp. (10.)) the loci of \(φ'ρ\), \(φ''ρ\), \(φ'ρ'\), coincide with the planes \(Π, \quad Π_1, \quad Π_2\), which are the loci of \(φρ\), \(φρ_1\), \(φρ_2\).

(17.) When \(φ\) is not self-conjugate, so that \(φρ\) and \(φ'ρ\) are not generally equal, it has been remarked that the scalar quadratic I., and therefore also the symbolical cubic in \(φ\), may have imaginary roots; and that, in this case, the vector equation II. of the second degree cannot be satisfied by any real direction of \(ρ\), except that one which satisfies the linear equation V., or causes \(φρ\) itself to vanish, while \(ρ\) remains real and actual. As an example of such imaginary scalars, as roots of I., and of what may be called imaginary directions, or
imaginary vectors (comp. 214, (4.),) which correspond to those scalars, and are
themselves imaginary roots of II., we may take the very simple expressions
(comp. 349, XII.),

\[ \phi \rho = V\gamma \rho, \quad \phi' \rho = -V\gamma \rho ; \]
in which \( \gamma \) denotes some real and given vector, and which evidently do not
satisfy the condition VI., the function \( \phi \) being here the negative of its own
conjugate, so that its self-conjugate part \( \phi_0 \) is zero (comp. 349, XIII.). We
have thus,

\[ m_0 = 0, \quad m'_0 = 0, \quad m''_0 = 0, \quad \phi_0 = 0, \quad \psi_0 = 0, \quad \chi_0 = 0, \]
and consequently, by the sub-articles to 349 and 350,

\[ m = 0, \quad m' = -\gamma^2, \quad m'' = 0, \quad \psi \rho = -\gamma S\gamma \rho, \quad \chi \rho = -V\gamma \rho ; \]
the quadratic I., and its roots \( c_1, c_2 \), become therefore,

\[ c^2 - \gamma^2 = 0, \quad c_1 = +\sqrt{-1}.T\gamma, \quad c_2 = -\sqrt{-1}.T\gamma, \]
where \( \sqrt{-1} \) is the imaginary of algebra (comp. 214, (3.)); thus by XX. or
XXI., and XXII. we have now

\[ \phi_1 \sigma = -\gamma S\gamma \sigma - c_1 V\gamma \sigma + c_1^2 \sigma = (\gamma - c_1)V\gamma \sigma, \quad \phi_2 \sigma = (\gamma - c_2)V\gamma \sigma; \]

and hence

\[ S\gamma \phi_1 \sigma = 0, \quad V\gamma \phi_1 \sigma = \gamma \phi_1 \sigma, \quad &e., \]
and

\[ \phi_1 \psi_1 \sigma = (\phi + c_1)\psi_1 \sigma = (\gamma + c_1)(\gamma - c_1)V\gamma \sigma = (\gamma^2 - c_1^2)V\gamma \sigma = 0, \]
and in like manner

\[ \phi_2 \psi_2 \sigma = 0 ; \]
if then we take an arbitrary vector \( \sigma \), and derive (or rather conceive as derived)
from it two (imaginary) vectors \( \rho_1 \) and \( \rho_2 \) by the (imaginary) operations \( \psi_1 \) and
\( \psi_2 \), we shall have (comp. III. and IV.) the equations,

\[ \rho_1 = \psi_1 \sigma, \quad \phi_1 \rho_1 = 0, \quad \phi_1 \rho_1 = -c_1 \rho_1, \quad V\rho_1 \phi_1 \rho_1 = 0, \]
and

\[ \rho_2 = \psi_2 \sigma, \quad \phi_2 \rho_2 = 0, \quad \phi_2 \rho_2 = -c_2 \rho_2, \quad V\rho_2 \phi_2 \rho_2 = 0, \]
as ones which are at least symbolically true. We find then that the two imaginary
directions, \( \rho_1 \) and \( \rho_2 \), satisfy (at least in a symbolical sense, or as far as
calculation is concerned) the vector equation II., or that \( \rho_1 \) and \( \rho_2 \) are two
imaginary vector roots of \( V\rho \phi \rho = 0 \); but that, because the scalar quadratic I.
has here imaginary roots, this vector equation II. has (as above stated) no real
vector root \( \rho \), except one in the direction of the given and real vector \( \gamma \), which
satisfies the linear equation \( V\gamma \), or gives \( \phi \rho = 0 \).
(18.) This particular example might have been more simply treated, by a
less general method, as follows. We wish to satisfy the equation,

\[ \mathbf{L}\ldots 0 = \mathbf{V} \cdot \rho \mathbf{V} \mathbf{\gamma} \rho = \rho \mathbf{S} \mathbf{\gamma} \rho - \rho^2 \mathbf{\gamma} ; \]

which gives, when we operate on it by \( \mathbf{V} \cdot \mathbf{\gamma} \) and \( \mathbf{V} \cdot \rho \), these others,

\[ \mathbf{LII} \ldots 0 = \mathbf{V} \mathbf{\gamma} \rho \cdot \mathbf{S} \mathbf{\gamma} \rho, \quad 0 = \rho^2 \mathbf{V} \mathbf{\gamma} \rho ; \]

if then we wish to avoid supposing \( \phi \rho = \mathbf{V} \mathbf{\gamma} \rho = 0 \), we must seek to satisfy the
two scalar equations,

\[ \mathbf{LIII} \ldots \mathbf{S} \mathbf{\gamma} \rho = 0, \quad \rho^2 = 0 ; \]

and conversely, if we can satisfy these by any (real or imaginary) \( \rho \), we shall have satisfied (really or symbolically) the vector equation \( \mathbf{L} \). Now the first
equation \( \mathbf{LIII} \) is satisfied, when we assume the expression,

\[ \mathbf{LIV} \ldots \rho = (c + \gamma) \mathbf{V} \mathbf{\gamma} \sigma = \mathbf{V} \mathbf{\gamma} \sigma \cdot (c - \gamma) , \]

where \( \sigma \) is an arbitrary vector, and \( c \) is any scalar, or symbol subject to the
laws of scalars; and this expression \( \mathbf{LIV} \) for \( \rho \), with its transformation just
assigned, gives

\[ \mathbf{LV} \ldots \rho^2 = (c^2 - \gamma^2) (V \mathbf{\gamma} \sigma)^2 = 0, \quad \text{if} \quad c^2 - \gamma^2 = 0 ; \]

the quadratic \( \mathbf{XLVI} \) is therefore reproduced, and we have the same imaginary
roots, and imaginary directions, as before.

(19.) Geometrically, the imaginary character of the recent problem, of
satisfying the equation \( \mathbf{V} \cdot \rho \mathbf{V} \mathbf{\gamma} \rho = 0 \) by any direction of \( \rho \) except that of the
given line \( \gamma \), is apparent from the circumstance that \( \phi \rho \), or \( \mathbf{V} \mathbf{\gamma} \rho \), is here a
vector perpendicular to \( \rho \), if both be actual lines; and that therefore the one
cannot be also parallel to the other, so long as both are real.*

354. In the three preceding Articles, and in the sub-articles annexed, we
have supposed throughout that the absolute term of the cubic in \( \phi \) is wanting,
or that the condition \( m = 0 \) is satisfied; in which case we have seen (351)

* Accordingly the two imaginary directions, above found for \( \rho \), are easily seen to be those which
in modern geometry are called the directions of lines drawn in a given plane (perpendicular here to the
given line \( \gamma \)), to the circular points at infinity: of which supposed directions the imaginary character
may be said to be precisely this, that each is (in the given plane) its own perpendicular.

[As additional examples:—
If \( \phi \rho = \mathbf{q} \rho \mathbf{q}^{-1} \), it is obvious that \( \phi^* = \phi^{-1} \). This shows that the cubic of \( \phi \) is reciprocal, and it
may easily be reduced to \( (\phi - 1) (\phi^2 - 2 \cos 2u \phi + 1) = 0 \) if \( u = \angle q \). The real direction is \( \mathbf{V} \mathbf{q} \), and
the imaginary directions are the lines to the circular points at infinity in the plane perpendicular to
\( \mathbf{V} \mathbf{q} \). Again, if \( \phi \) changes \( a \) into \( b \), \( b \) into \( \gamma \), and \( \gamma \) into \( a \), the cubic is \( \phi^3 - 1 = 0 \). The directions
are \( a + \omega b + \omega^2 \gamma \), where \( \omega \) is an algebraic cube root of unity.]
that it is always possible to satisfy the linear equation \( \phi \rho = 0 \), by at least one real and actual value of \( \rho \) (with an arbitrary scalar coefficient); or by at least one real direction. It will be easy now to show, that although conversely (comp. 351, (4.)) the function \( \phi \rho \) cannot vanish for any actual vector \( \rho \), unless we have thus \( m = 0 \), yet there is always at least one real direction for which the vector equation of the second degree,

\[
I. \ldots V\rho \phi \rho = 0,
\]

which has already been considered (353) in combination with the condition \( m = 0 \), is satisfied; and that if the function \( \phi \) be a self-conjugate one, then this equation \( I. \) is always satisfied by at least three real and rectangular directions, but not generally by more directions than three; although, in this case of self-conjugation, namely when

\[
\text{II.} \ldots \phi' \rho = \phi \rho, \text{ or } \text{II'.} \ldots S\lambda \phi \rho = S\rho \phi \lambda,
\]

for all values of the vectors \( \rho \) and \( \lambda \), the equation \( I. \) may happen to become true, for one real direction of \( \rho \), and for every direction perpendicular thereto: or even for all possible directions, according to the particular system of constants, which enter into the composition of the function \( \phi \rho \). We shall show also that the scalar (or algebraic) and cubic equation,

\[
\text{III.} \ldots 0 = m + m' c + m'' c^2 + c^3,
\]

which is formed from the symbolic and cubic equation 350, \( I. \), by changing \( \phi \) to \( -c \), enters importantly into this whole theory; and that if it have one real and two imaginary roots, the quadratic and vector equation \( I. \) is satisfied by only one real direction of \( \rho \); but that it may then be said (comp. 353, (17.)) to be satisfied also by two imaginary directions, or to have two imaginary and vector roots: so that this equation \( I. \) may be said to represent generally a system of three right lines, whereof one at least must be real. For the case II., the scalar roots of III. will be proved to be always real; so that if \( m_0, m', \) and \( m'' \) be formed (as in sub-articles to 349 and 350) from the self-conjugate part \( \phi \rho \) of any linear and vector function \( \phi \rho \), as \( m, m', \) and \( m'' \) are formed from that function \( \phi \rho \) itself, then the new cubic,

\[
\text{IV.} \ldots 0 = m_0 + m' c + m'' c^2 + c^3,
\]

which thus results, can never have imaginary roots.

(1.) If we write,

\[
V. \ldots \Phi \rho = \phi \rho + cp, \quad \Phi' \rho = \phi' \rho + cp, \quad \text{or briefly, } \quad V'. \ldots \Phi = \phi + c, \quad \Phi' = \phi' + c,
\]

where \( c \) is an arbitrary scalar, and if we denote by \( \Psi, \Psi', \) and \( M \) what \( \psi, \Psi', \)
and \( m \), become, by this change of \( \Phi \) to \( \phi + c \) or \( \Phi \), the calculations in 353, (5.), (6.), show that we have the expressions,

\[
\text{VI. } \Psi = \psi + cx + c^2, \quad \Psi' = \psi' + c\psi' + c^2,
\]

and

\[
\text{VII. } M = m + m'c + m''c^2 + c^3,
\]

with

\[
\text{VIII. } M = \Phi \Psi = \Psi \Phi = \Phi' \Psi' = \Psi' \Phi'.
\]

(2.) Hence it may be inferred that the functions \( \psi, \psi' \), and the constants \( m', m'' \) become,

\[
\text{IX. } x = D_e \psi = \chi + 2c, \quad x' = D_e \psi' = \chi' + 2c,
\]

\[
\text{X. } \left\{ \begin{array}{l}
M' = D_e M = m' + 2m''c + 3c^2, \\
M'' = \frac{1}{3} D_e^2 M = m'' + 3c;
\end{array} \right.
\]

with the verifications,

\[
\text{XI. } \phi + x = \phi' + x' = M', \quad \phi x + \psi = \phi' x' + \psi = M',
\]

as we had, by the sub-articles to 350,

\[
\phi + \chi = \phi' + \chi' = m'', \quad \phi \chi + \psi = \phi' \chi' + \psi' = m'.
\]

(3.) The new linear symbol \( \Phi \) must satisfy the new cubic,

\[
\text{XII. } 0 = M - M' \Phi + M'' \Phi^2 - \Phi^3;
\]

which accordingly can be at once derived from the old cubic 350, I., under the form,

\[
\text{XIII. } 0 = m + m'(c - \Phi) + m''(c - \Phi)^2 + (c - \Phi)^3.
\]

(4.) Now it is always possible to satisfy the condition,

\[
\text{XIV. } M = 0,
\]

by substituting for \( c \) a real root of the scalar cubic III.; and thereby to reduce the new symbolical cubic XII. to the form,

\[
\text{XV. } 0 = \Phi^3 - M'' \Phi^2 + M \Phi;
\]

which is precisely similar to the form,

\[
0 = \phi^3 - m'' \phi^2 + m' \phi,
\]

and conducts to analogous consequences, which need not here be developed in detail, since they can easily be supplied by anyone who will take the trouble to read again the few recent series of sub-articles.

(5.) For example, unless it happen that \( \Psi \rho \) constantly vanishes, in which case \( M' = 0 \), and \( \Phi \rho \) (if not identically null) takes a monomial form, which is
reduced to zero (comp. 352, (7.)) for every direction of $\rho$ in a given plane, the operation $\Psi$ reduces (comp. 351) an arbitrary vector to a given direction; and the operation $\Phi$ destroys every line in that direction: so that, in every case, there is at least one real way of satisfying the vector equation $\Phi \rho = 0$, and therefore also (as above asserted) the equation I., without causing $\rho$ itself to vanish.

(6.) And since that equation I. may be thus written,

$$\text{XVI. } \nabla \rho \Phi = 0, \text{ or } \Phi \rho \parallel \rho,$$

we see that it can be satisfied without $\Phi \rho$ vanishing, if this new scalar and quadratic equation,

$$\text{XVII. } 0 = C^2 + M'C + M', \quad \text{comp. 353, I.,}$$

have real and unequal roots, $C_1, C_2$; for if we then write,

$$\text{XVIII. } \Phi_1 = \Phi + C_1, \quad \Phi_2 = \Phi + C_2,$$

the line $\Phi \rho$ will generally have for its locus a given plane, and there will be two real and distinct directions $\rho_1$ and $\rho_2$ in that plane, for one of which $\Phi_1 \rho_1 = 0$, while $\Phi_2 \rho_2 = 0$ for the other, so that each satisfies XVI., or I.; and these are precisely the fixed directions of $\Psi_1 \rho$ and $\Psi_2 \rho$, if $\Psi_1$ and $\Psi_2$ be formed from $\Psi$ by changing $\Phi$ to $\Phi_1$ and $\Phi_2$ respectively.

(7.) Cases of equal and of imaginary roots need not be dwelt on here; but it may be remarked in passing, that if the function $\phi \rho$ have the particular form ($g$ being any scalar constant),

$$\text{XIX. } \phi = g \rho, \text{ then XX. } (g - \phi)^3 = 0, \text{ and XXI. } \mathcal{M} = (g + c)^3;$$

the cubic XIV. or III. having thus all its roots equal, and the equation I. being satisfied by every direction of $\rho$, in this particular case.

(8.) The general existence of a real and rectangular system of three directions satisfying I., when the condition II. is satisfied, may be proved as in 353, (14.); and it is unnecessary to dwell on the case where, by two roots of the cubic becoming equal, all lines in a given plane, and also the normal to that plane, are vector roots of I., with the same condition II.

(9.) And because the quadratic, $0 = c^2 + m'c + n'$ (353, I.), has been proved to have always real roots (353, (13.)) when $\phi' \rho = \phi \rho$, the analogous quadratic XVII. must likewise then have real roots, $C_1, C_2$; whence it immediately follows (comp. XII. and XIII.), that (under the same condition of self-conjugation) the cubic III. has three real roots, $c, c + C_1, c + C_2$; and therefore that (as above stated) the other cubic IV., which is formed
from the self-conjugate part \( \phi_0 \) of the general linear and vector function \( \phi \), and which may on that account be thus denoted,

\[
\text{XXII. } M_0 = 0, \text{ has its roots always real.}
\]

(10.) If we denote in like manner by \( \Phi_0 \) the symbol \( \phi_0 + c \), the equation

\[
m = m_0 - S\gamma \Phi_0 \gamma
\]

(349, XXVI., comp. 349, XXI.) becomes,

\[
\text{XXIII. } M = M_0 - S\gamma \Phi_0 \gamma;
\]

whence, by comparing powers of \( c \), we recover the relations,

\[
m' = m'_0 - \gamma^3, \text{ and } m'' = m''_0, \text{ as in 350, (1.).}\]

(11.) On a similar plan, the equation \( m \phi \nu \mu \nu = \nu. \psi \mu \nu \psi \nu \) becomes,

\[
\text{XXIV. } M \phi \nu \mu \nu = \nu. \psi \mu \nu \psi \nu, \quad \text{comp. 348, (1.),}
\]

in which \( \mu \) and \( \nu \) are arbitrary vectors, and \( c \) is an arbitrary scalar; or more fully,

\[
\text{XXV. } (m + m'c + m''c^2 + c^3)(\phi' + c)\nu \mu \nu = \nu.(\psi \mu + c\chi \mu + c^2 \mu)(\psi \nu + c\chi \nu + c^2 \nu);
\]

whence follow these new equations,

\[
\text{XXVI. } (m + m'c)\nu \mu \nu = \nu(\psi \mu \cdot \chi \nu - \psi \nu \cdot \chi \mu),
\]

\[
\text{XXVII. } (m' + m''c)\nu \mu \nu = \nu(\mu \psi \nu - \nu \psi \mu + \chi \mu \cdot \chi \nu),
\]

\[
\text{XXVIII. } (m'' + c)\nu \mu \nu = \nu(\mu \chi \nu - \nu \chi \mu),
\]

which can all be otherwise proved, and from the last of which (by changing \( \phi \) to \( \psi \), &c.) we can infer this other of the same kind,

\[
\text{XXIX. } (m' + c)\nu \mu \nu = \nu(\mu \phi \nu \cdot v - \nu \phi \chi \mu).
\]

(12.) As an example of the existence of a real and rectangular system of three directions (8.), represented jointly by an equation of the form I., and of a system of three real roots of the scalar cubic III., when the condition II. is satisfied, let us take the form

\[
\text{XXX. } \phi \rho = g \rho + \nu \lambda \rho \mu = \phi' \rho,
\]

\( g \) being here any real and given scalar, and \( \lambda, \mu \) any real and non-parallel

---

\* [If

\[
\phi_1 = \phi_0 \rho_1 + \nu \gamma \rho_1 = -c_1 \rho_1, \quad \text{then } \rho_1 = -\phi_0 (\phi_0 + c_1)^{-1} \nu \gamma \rho_1
\]

\[
(m_0 + m'c_1 + m''c_1^2 + c_1^3)\rho_1 = -\nu(\phi_0 + c_1)\gamma(\phi_0 + c_1)\rho_1 = \nu(\phi_0 + c_1)\gamma \nu \gamma \rho_1
\]

\[
= \rho_1 \nu \gamma (\phi_0 + c_1)\gamma - \gamma \nu \gamma \rho_1 (\phi_0 + c_1)\gamma = \rho_1 \nu \gamma (\phi_0 + c_1)\gamma.
\]

From this, \( c_1 \) is a root of

\[
(m_0 - \gamma \phi \gamma) + (m'_0 - \gamma^3) + m''c^2 + c^3 = 0,
\]

and this cubic must be identical with \( m + m'c + m''c^2 + c^3 = 0 \), as they have three roots common.]
given vectors; to which form, indeed, we shall soon find that every self-conjugate function \( \phi \rho \) can be brought. We have now (after some reductions),

\[
XXXI. \quad \psi \rho = V \lambda \mu S \lambda \mu - V \lambda \mu S \lambda \rho \mu - g(\lambda S \mu \rho + \mu S \lambda \rho) + g \rho,
\]

\[
XXXII. \quad \chi \rho = - (\lambda S \mu \rho + \mu S \lambda \rho) + 2g \rho,
\]
and

\[
XXXIII. \quad m = (g - S \lambda \mu)(g^2 - \lambda^2 \mu^2), \quad m' = - \lambda^2 \mu^2 - 2gS \lambda \mu + 3g^2,
\]

\[
m'' = - S \lambda \mu + 3g;
\]

where the part of \( \psi \rho \) which is independent of \( g \) may be put under several other forms, such as the following,

\[
XXXIV. \quad Y (X \mu S X \mu - X \mu S X \mu) = X \mu S X \mu - X \mu S X \mu
\]

\[
= X (\mu S X \mu + X \mu S \mu) = X (\mu S X \rho - X \mu S \rho) = \text{&c.};
\]

and \( \Phi, \Psi, \chi, M, M', M'' \) may be formed from \( \phi, \psi, \chi, m, m', m'' \), by simply changing \( \phi \) to \( \phi + g \). The equation \( M = 0 \) has therefore here three real and unequal roots, namely the three following

\[
XXXV. \quad c = - g + S \lambda \mu, \quad c + C_1 = - g + T \lambda \mu, \quad c + C_2 = - g - T \lambda \mu;
\]

and the corresponding forms of \( \Psi \rho \) are found to be,

\[
XXXVI. \quad \psi \rho = V \lambda \mu S \lambda \mu, \quad \psi_1 \rho = - (T \lambda \mu + \mu T \lambda) S \rho (T \lambda \mu + \mu T \lambda),
\]

\[
\psi_2 \rho = - (T \lambda \mu - \mu T \lambda) S \rho (T \lambda \mu - \mu T \lambda).
\]

Thus \( \Psi \rho, \Psi_1 \rho, \) and \( \Psi_2 \rho \) have in fact the three fixed and rectangular directions of \( \nabla \lambda \mu, \) \( T \lambda \mu + \mu T \lambda, \) and \( T \lambda \mu - \mu T \lambda, \) namely of the normal to the given plane of \( \lambda, \mu, \) and the bisectors of the angles made by those two given lines; and these are accordingly the only directions which satisfy the vector equation of the second degree,

\[
XXXVII. \quad (V \rho \phi \rho = V. \rho V \lambda \rho \mu = V \rho \lambda \mu S \rho + V \rho \mu S \lambda \rho = 0;
\]

so that this last equation represents (as was expected) a system of three right lines, in these three respective directions.

(13.) In general, if \( c_1, c_2, c_3 \) denote the three roots (real or imaginary) of the cubic equation \( M = 0 \), and if we write,

\[
XXXVIII. \quad \Phi_1 = \phi + c_1, \quad \Phi_2 = \phi + c_2, \quad \Phi_3 = \phi + c_3,
\]

the corresponding values of \( \Psi \) will be (comp. VI.),

\[
XXXIX. \quad \Psi_1 = \psi + c_1 \chi + c_1^2, \quad \Psi_2 = \psi + c_2 \chi + c_2^2, \quad \Psi_3 = \psi + c_3 \chi + c_3^2;
\]

\textit{Hamilton's Elements of Quaternions}.
also we have the relations,
\[
\begin{align*}
&c_1 + c_2 + c_3 = -m'' = -\phi - \chi, \\
&c_2c_3 + c_3c_1 + c_1c_2 = +m = \phi\chi + \psi, \\
&c_1c_2c_3 = -m = -\phi\psi;
\end{align*}
\]
whence it is easy to infer the expressions,
\[
\begin{align*}
\Phi_1 &= (c_2 - c_3)^{-1}(\Psi_3 - \Psi_2), \\
\Phi_2 &= (c_3 - c_1)^{-1}(\Psi_1 - \Psi_3), \\
\Phi_3 &= (c_1 - c_2)^{-1}(\Psi_2 - \Psi_1);
\end{align*}
\]
which enable us to express the functions \(\Phi_1\rho\), \(\Phi_2\rho\), \(\Phi_3\rho\) as binomials (comp. 351, &c.), when \(\Psi_1\rho\), \(\Psi_2\rho\), \(\Psi_3\rho\) have been expressed as monomes, and to assign the planes (real or imaginary), which are the loci of the lines \(\Phi_1\rho\), \(\Phi_2\rho\), \(\Phi_3\rho\).

(14.) Accordingly, the three operations, \(\Phi\), \(\Phi_1\), \(\Phi_2\), by which lines in the three lately determined directions (12.) are destroyed, or reduced to zero, and which at first present themselves under the forms,
\[
\begin{align*}
\Phi_1 &= \lambda\Psi\rho + \mu\Psi\lambda\rho, \\
\Phi_2 &= \Psi\lambda\rho\mu + \rho\lambda\mu, \\
\Phi_3 &= \Psi\mu\rho\lambda - \rho\lambda\mu,
\end{align*}
\]
are found to admit of the transformations,
\[
\begin{align*}
\Phi_1 &= \Psi\rho\lambda - \Psi\lambda\rho, \\
\Phi_2 &= \Psi\rho\lambda + \rho\lambda\mu, \\
\Phi_3 &= \Psi\mu\lambda - \Psi\lambda\mu;
\end{align*}
\]
where \(\Psi\), \(\Psi_1\), \(\Psi_2\) have the recent forms XXXVI, and the loci of \(\Phi_1\rho\), \(\Phi_2\rho\), \(\Phi_3\rho\) compose a system of three rectangular planes.

(15.) In general, the relations (13.) give also (comp. 353, (9.),
\[
\begin{align*}
\Psi_1 &= \Phi_1\Phi_2, \\
\Psi_2 &= \Phi_2\Phi_3, \\
\Psi_3 &= \Phi_3\Phi_1,
\end{align*}
\]
and
\[
\begin{align*}
\Psi_1\Psi_2 &= \Phi_1\Psi_2 = \Phi_2\Psi_3 = \Phi_3\Phi_1 = 0,
\end{align*}
\]
whence also,
\[
\begin{align*}
\Psi_1\Psi_2\Psi_3 &= \Psi_2\Psi_3\Psi_1 = \Psi_3\Psi_1\Psi_2 = 0,
\end{align*}
\]
the symbols (in any one system of this sort) admitting of being transposed and grouped at pleasure; if then the roots of \(M = 0\) be real and unequal, there arises a system of three real and distinct planes, which are connected with the interpretation of the symbolical equation, \(\Phi_1\Phi_2\Phi_3 = 0\), exactly as the three planes in 353, (9.) were connected with the analogous equation \(\phi\phi_1\phi_2 = 0\).

(16.) And when the cubic has two imaginary roots, it may then be said that there is one real plane (such as the plane \(\perp\gamma\) in 353, (18.), (19.),) containing the two imaginary directions which then satisfy the equation I.; and two imaginary planes, which respectively contain those two directions, and intersect each other in one real line (such as the line \(\gamma\) in the example cited), namely the one real vector root of the same equation I.
355. Some additional light may be thrown upon that vector equation of the second degree, by considering the system of the two scalar equations,

I. \( S\lambda p\phi = 0 \), and II. \( S\lambda \rho = 0 \),

and investigating the condition of the reality of the two directions, \( \rho_1 \) and \( \rho_2 \), by which they are generally satisfied, and for each of which the plane of \( \rho \) and \( \phi \rho \) contains generally the given line \( \lambda \) in I., or is normal to the plane locus II. of \( \rho \). We shall find that these two directions are always real and rectangular (except that they may become indeterminate), when the linear function \( \phi \) is its own conjugate; and that then, if \( \lambda \) be a root \( \rho_0 \) of the vector equation,

III. \( V\rho\phi = 0 \),

which has been already otherwise discussed, the lines \( \rho_1 \) and \( \rho_2 \) are also roots of that equation; the general existence (354) of a system of three real and rectangular directions, which satisfy this equation III. when \( \phi\rho = \phi \rho \), being thus proved anew: whence also will follow a new proof of the reality of the scalar roots of the cubic \( M = 0 \), for this case of self-conjugation of \( \phi \); and therefore of the necessary reality of the roots of that other cubic, \( M_0 = 0 \), which is formed (354, IV. or XXII.) from the self-conjugate part \( \phi_0 \) of the general linear and vector function \( \phi \), as \( M = 0 \) was formed from \( \phi \).

(1.) Let \( \lambda, \mu, \nu \) be a system of three rectangular vector units, following in all respects the laws (182, 183), of the symbols \( i, j, k \). Writing then,

IV. \( \rho = y\mu + z\nu, \) and therefore, \( \lambda \rho = y\nu - z\mu, \quad \phi \rho = y\phi \mu + z\phi \nu, \)

the equation II. is satisfied, and I. becomes,

V. \( 0 = y^2S\nu\phi \mu + yz(S\nu\phi \nu - S\mu \phi \mu) - z^2S\mu \phi \nu; \)

the roots of which quadratic will be real and unequal, if

VI. \( (S\nu\phi \nu - S\mu \phi \mu)^2 + 4S\mu \phi \nu S\nu \phi \mu > 0; \)

* Geometrically, the equation I. represents a cone of the second order, with \( \lambda \) for one side, and with the three lines \( \rho \) which satisfy III. for three other sides; and II. represents a plane through the vertex, perpendicular to the side \( \lambda \). The two directions sought are thus the two sides, in which this plane cuts the cone. [The general equation of a quadric may be written in the form \( S\phi \rho = 1 \) where the function \( \phi \) is self-conjugate. The cone, through its intersection with a concentric sphere, is \( S\phi(\phi + r^2)\rho = 0 \) if \( r \) is the radius of the sphere. If this touches the plane \( S\lambda \rho = 0 \), it is geometrically evident that the edge of contact is a principal axis of the plane section of the quadric as it passes through the points of contact of the concentric sections of the quadric and the sphere. The condition for contact is \( \lambda \parallel (\phi + r^2)\rho, \) or \( S\lambda \phi \rho \rho = 0, \) coupled with \( S\lambda \rho = 0 \). The directions of the principal axes thus determined are always real whether the plane cuts the quadric in a real curve or not.]
and the corresponding directions of $\rho$ will be rectangular, if

$$\text{VII. } 0 = S(y_1\mu + z_1v) (y_2\mu + z_2v) = -(y_1y_2 + z_1z_2);$$

that is, if

$$\text{VIII. } S\mu\phi = S\mu\phi,$$

at least for this particular pair of vectors, $\mu$ and $v$.

(2.) Introducing now the expression, $\phi\rho = \phi_0\rho + V\gamma\rho$ (349, XII.), the conditions VI. and VIII. take the forms,

$$\text{IX. } (S\mu\phi_0v - S\mu\phi_0)^2 + 4S(\mu\phi_0v)^2 > 4(S\gamma\mu\nu)^2, \text{ and X. } S\gamma\mu\nu = 0;$$

which are both satisfied generally when $\gamma = 0$, or $\phi = \phi' = \phi_0$; the only exception being, that the quadratic $V$. may happen to become an identity, by all its coefficients vanishing: but the opposite inequality (to VI. and IX.) can never hold good, that is to say, the roots of that quadratic can never be imaginary, when $\phi$ is thus self-conjugate.

(3.) On the other hand, when $\gamma$ is actual, or $\phi'\rho$ not generally $= \phi\rho$, the condition X. of rectangularity can only accidentally be satisfied, namely by the given or fixed line $\gamma$ happening to be in the assumed plane of $\mu$, $v$; and when the two directions of $\rho$ are thus not rectangular, or when the scalar $S\gamma\mu\nu$ does not vanish, we have only to suppose that the square of this scalar becomes large enough, in order to render (by IX.) those directions coincident, or imaginary.

(4.) When $\phi' = \phi$, or $\gamma = 0$, we may take $\mu$ and $v$ for the two rectangular directions of $\rho$, or may reduce the quadratic to the very simple form $yz = 0$; but, for this purpose, we must establish the relations,

$$\text{XI. } S\mu\phi = S\mu\phi = 0.$$

(5.) And if, at the same time, $\lambda$ satisfies the equation III., so that $\phi\lambda || \lambda$, we shall have these other scalar equations,

$$\text{XII. } 0 = S\mu\phi\lambda = S\nu\phi\lambda = S\lambda\phi\mu = S\lambda\phi\nu;$$

whence

$$\phi\mu || V\nu\lambda || \mu, \text{ and } \phi\nu || V\lambda\mu || v,$$

or,

$$\text{XIII. } 0 = V\lambda\phi\lambda = V\mu\phi\mu = V\nu\phi\nu;$$

$\lambda, \mu, \nu$ thus forming (as above stated) a system, of three real and rectangular roots, of that vector equation III.

(6.) But in general, if III. be satisfied by even two real and distinct directions of $\rho$, the scalar and cubic equation $M = 0$ can have no imaginary
root; for if those two directions give two unequal but real and scalar values, $c_1$ and $c_2$, for the quotient $-\varphi \rho : \rho$, then $c_1$ and $c_2$ are two real roots of the cubic, of which therefore the third root is also real; and if, on the other hand, the two directions $\rho_1$ and $\rho_2$ give one common real and scalar value, such as $c_1$, for that quotient, then $\varphi \rho = -c_1 \rho$, or $\varphi \rho = (\phi + c_1) \rho = 0$, for every line in the plane of $\rho_1$, $\rho_2$; so that $\varphi \rho$ must be of the form, $-c_1 \rho + \beta \rho_1 \rho_2 \rho$, and the cubic will have at least two equal roots, since it will take the form,

$$XIV\ldots 0 = (c - c_1)^2 (c - c_1 + \delta \rho_1 \rho_2 \beta),$$

as is easily shown from principles and formulae already established.

(7.) It is then proved anew, that the equation $M = 0$ has all its roots real, if $\varphi' \rho = \varphi \rho$; and therefore that the equation $M_0 = 0$ (as above stated) can never have an imaginary root.

(8.) And we see, at the same time, how the scalar cubic $M = 0$ might have been deduced from the symbolical cubic 350, I., or from the equation 351, I., as the condition for the vector equation $\text{III.}$ being satisfied by any actual $\rho$; namely by observing that if $\varphi \rho = -c \rho$, then $\varphi^2 \rho = c^2 \rho$, $\varphi^3 \rho = -c^3 \rho$, &c., and therefore $M \rho = 0$, in which $\rho$, by supposition, is different from zero.

(9.) Finally, as regards the case* of indetermination, above alluded to, when the quadratic $V.$ fails to assign any definite values to $y : z$, or any definite directions in the given plane to $\rho$, this case is evidently distinguished by the condition,

$$XV\ldots \delta \mu \varphi \mu = \delta \nu \varphi \nu,$$

in combination with the equations $\text{XI.}$

356. The existence of the Symbolic and Cubic Equation (350), which is satisfied by the linear and vector symbol $\varphi$, suggests a Theorem† of Geometrical Deformation, which may be thus enunciated:—

"If, by any given Mode, or Law, of Linear Derivation, of the kind above denoted by the symbol $\varphi$, we pass from any assumed Vector $\rho$ to a Series of Successively Derived Vectors, $\rho_1$, $\rho_2$, $\rho_3$, . . . or $\varphi^1 \rho$, $\varphi^2 \rho$, $\varphi^3 \rho$, . . .; and if, by constructing a Parallelepiped, we decompose any Line of this Series, such as $\rho_3$, into three partial or component lines, $m \rho$, $-m' \rho_1$, $m'' \rho_2$, in the Directions of the three

* It will be found that this case corresponds to the circular sections of a surface of the second order; while the less particular case in which $\varphi' \rho = \varphi \rho$, but not $\delta \mu \varphi \mu = \delta \nu \varphi \nu$, so that the two directions of $\rho$ are determined, real, and rectangular, corresponds to the axes of a non-circular section of such a surface.

† This theorem was stated, nearly in the same way, in page 568 of the Lectures; and the problem of inversion of a linear and vector function was treated, in the few preceding pages (559, &c.), though with somewhat less of completeness and perhaps of simplicity than in the present Section, and with a
which precede it, as here of \( \rho, \rho_1, \rho_2 \); then the Three Scalar Coefficients, \( m, -m', m'' \), or the Three Ratios which these three Components of the Fourth Line \( \rho_4 \) bear to the Three Preceding Lines of the Series, will depend only on the given Mode or Law of Derivation, and will be entirely independent of the assumed Length and Direction of the Initial Vector."

(1.) As an Example of such successive Derivation, let us take the law,

\[
\rho_1 = \phi_1 = -\sqrt{\beta}\rho_{1}\gamma, \quad \rho_2 = \phi_2 = -\sqrt{\beta}\rho_{1}\gamma, \quad \&c.,
\]

which answers to the construction in 305, (1.), \&c., when we suppose that \( \beta, \gamma \) are unit-lines. Treating them at first as any two given vectors, our general method conducts to the equation,

\[
\rho_3 = m\rho - m'\rho_1 + m''\rho_2,
\]

with the following values of the coefficients,

\[
\begin{align*}
\text{III.} & \quad m = -\beta^2\gamma^2\beta\gamma, \quad m' = -\beta^2\gamma^2, \quad m'' = \beta\gamma;
\end{align*}
\]

as may be seen, without any new calculation, by merely changing \( g, \lambda, \mu \), in 354, XXXIII., to \( 0, \beta, \) and \(-\gamma\).

(2.) Supposing next, for comparison with 305, that

\[
\begin{align*}
\text{IV.} & \quad \beta^2 = \gamma^2 = -1, \quad \text{and} \quad S\beta\gamma = -l,
\end{align*}
\]

so that \( \beta, \gamma \) are unit lines, and \( l \) is the cosine of their inclination to each other, the values III. become,

\[
\begin{align*}
\text{V.} & \quad m = l, \quad m' = -1, \quad m'' = -l;
\end{align*}
\]

slightly different notation. The general form of such a function which was there adopted may now be thus expressed:

\[
\phi_\rho = 2\beta S\rho_\rho + V\rho_\rho, \quad \& \rho \text{ being a given quaternion;}
\]

the resulting value of \( m \) was found to be (page 561),

\[
m = 2\beta Sa'a'S\beta'\beta + 2S(rVaa'.\beta'\beta) + Sr2S\beta_r - 2S\beta r + Sr^2;
\]

and the auxiliary function which we now denote by \( \psi \) was,

\[
im\phi^{-1}\psi = \psi = 2Vaa'S\beta'\beta + 2V.aV(\beta'\beta.r) + (VrSr - VrSr);
\]

where the sum of the two last terms of \( \psi \) might have been written as \( \sigma S\rho_\rho - r\sigma r \). A student might find it an useful exercise, to prove the correctness of these expressions by the principles of the present Section. One way of doing so would be, to treat \( 2\beta S\rho_\rho \) and \( r \) as respectively equal to \( \phi_\rho + V\rho_\rho \) and \( e + \epsilon \); which would transform \( m \) and \( \psi_\rho \), as above written, into the following,

\[
M_0 - S(\gamma + \epsilon) (\phi_0 + e) (\gamma + \epsilon), \quad \text{and} \quad \psi_0 = -S(\gamma + e) S(\gamma + \epsilon) + V(\phi_0 + e) (\gamma + \epsilon);
\]

that is, into the new values which the \( M \) and \( \psi \) of the Section assume, when \( \phi_\rho \) takes the new value,

\[
\phi_\rho = (\phi_0 + e)\rho + V(\gamma + \epsilon)\rho.
\]
and the equation II., connecting four successive lines of the series, takes the form,

\[ \rho_5 = l_5 + \rho_4 - l_3 \rho_2 \quad \text{or} \quad \rho_6 = \rho_5 - l_4 (\rho_3 - \rho) ; \]

a result which agrees with 305, (2.), since we there found that if \( \rho = \text{or} \), &c., the interval \( \rho_1 \rho_5 \) was \( -l \times \rho_2 \).

(3.) And as regards the inversion of a linear and vector function (347), or the return from any one line \( \rho_1 \) of such a series to the line \( \rho \) which precedes it, our general method gives, for the example I., by 354, (12.),

\[ \psi_1 = \frac{1}{2} \beta (\beta \gamma \rho_1 + \rho_1 \beta \gamma) \gamma, \]

and

\[ \rho = \phi^{-1} \rho_1 = m^{-1} \psi_1 = -\frac{\beta \rho_1 \beta^{-1} + \gamma \rho \gamma^{-1}}{\beta \gamma + \gamma \beta}; \]

a result which it is easy to verify and to interpret, on principles already explained.

357. We are now prepared to assign some new and general Forms, to which the Linear and Vector Function (with real constants) of a variable vector can be brought, without assuming its self-conjugation; one of the simplest of which forms is the following,

\[ \phi_\rho = V_{q_0} \rho + V_{\lambda \rho \mu} \],

\( q_0 \) being here a real and constant quaternion, and \( \lambda, \mu \) two real and constant vectors, which can all be definitely assigned, when the particular form of \( \phi \) is given; except that \( \lambda \) and \( \mu \) may be interchanged (by 295, VII.), and that either may be multiplied by any scalar, if the other be divided by the same. It will follow that the scalar, quadratic, and homogeneous function of a vector, denoted by \( S_{\rho \phi \rho} \), can always be thus expressed:

\[ \Pi \ldots S_{\rho \phi \rho} = g \rho^2 + S_{\lambda \rho \mu} \rho; \]
or thus,

\[ \Pi' \ldots S_{\rho \phi \rho} = g' \rho^2 + 2S_{\lambda \rho \mu}, \text{ if } g' = g - S_{\lambda \mu}; \]
a general and (as above remarked) definite transformation, which is found to be one of great utility in the theory of Surfaces* of the Second Order.

(1.) Attending first to the case of self-conjugate functions \( \phi_\rho \), from which we can pass to the general case by merely adding the term \( V_{\gamma \rho} \), and

---

* In the theory of such surfaces, the two constant and real vectors, \( \lambda \) and \( \mu \), have the directions of what are called the cyclic normals.
supposing (in virtue of what precedes) that \(a_1a_2a_3\) are three real and rectangular vector-units, and \(c_1c_2c_3\) three real scalars (the roots of the cubic \(M_0 = 0\)), such that
\[
\begin{align*}
\phi_1a_1 &= (\phi_0 + c_1)a_1 = 0, & \phi_2a_2 &= (\phi_0 + c_2)a_2 = 0, & \phi_3a_3 &= (\phi_0 + c_3)a_3 = 0,
\end{align*}
\]
we may write
\[
\begin{align*}
IV. & \quad \rho = -(a_1Sa_1\rho + a_2Sa_2\rho + a_3Sa_3\rho), \\
V. & \quad \phi_0\rho = c_1a_1Sa_1\rho + c_2a_2Sa_2\rho + c_3a_3Sa_3\rho;
\end{align*}
\]
and therefore
\[
\begin{align*}
IV. & \quad \phi_0\rho = c_1a_1Sa_1\rho + c_2a_2Sa_2\rho + c_3a_3Sa_3\rho; \\
V. & \quad \phi_1\rho = (c_2 - c_1)a_2Sa_2\rho + (c_3 - c_1)a_3Sa_3\rho, \\
VI. & \quad \phi_2\rho = (c_3 - c_2)a_3Sa_3\rho + (c_1 - c_2)a_1Sa_1\rho, \\
\end{align*}
\]
the binomial forms of \(\phi_1, \phi_2, \phi_3\) being thus put in evidence.

(2.) We have thus the general but scalar expressions:
\[
\begin{align*}
VII. & \quad -\rho^2 = (Sa_1\rho)^2 + (Sa_2\rho)^2 + (Sa_3\rho)^2; \\
VIII. & \quad S\rho\phi\rho = S\rho\phi\rho = c_1(Sa_1\rho)^2 + c_2(Sa_2\rho)^2 + c_3(Sa_3\rho)^2 \\
& \quad = -c_1\rho^2 + (c_2 - c_1)(Sa_2\rho)^2 + (c_3 - c_1)(Sa_3\rho)^2 \\
& \quad = -c_2\rho^2 + (c_3 - c_1)(Sa_3\rho)^2 + (c_3 - c_2)(Sa_3\rho)^2 \\
& \quad = -c_3\rho^2 - (c_2 - c_1)(Sa_1\rho)^2 - (c_3 - c_2)(Sa_3\rho)^2;
\end{align*}
\]
in which it is in general permitted to assume that
\[
\begin{align*}
IX. & \quad c_1 < c_2 < c_3, \text{ or that } XI. & \quad c_2 - c_1 = 2e^2, & c_3 - c_2 = 2e'^2,
\end{align*}
\]
e and \(e'\) being real scalars, and the numerical coefficients being introduced for a motive of convenience which will presently appear.

(3.) Comparing the last but one of the expressions VIII. with IX., we see that we may bring \(S\rho\phi\rho\) to the proposed form IX., by assuming,
\[
\begin{align*}
XI. & \quad \lambda = ea_1 + e'a_3, & \mu = -ea_1 + e'a_3, & g = S\lambda \mu - c_3 = -\frac{1}{2}(c_1 + c_3),
\end{align*}
\]
because \(S\lambda \mu = e^2 - e'^2 = c_2 - \frac{1}{2}(c_1 + c_3).

(4.) But in general (comp. 349, (4.)) we cannot have, for all values of \(\rho\),
\[
\begin{align*}
XII. & \quad S\rho\phi\rho = S\rho\phi\rho, \text{ unless } XIII. & \quad \phi_0\rho = \phi_0\rho,
\end{align*}
\]
that is, unless the self-conjugate parts of \(\phi\) and \(\phi'\) be equal; we can therefore infer from IX. that \(\phi_0\rho = \rho + V\lambda \rho \mu\), because \(V\lambda \rho \mu = V\mu \rho \lambda\) is its own conjugate; and thus the transformation I. is proved to be possible, and real.
(5.) Accordingly, with the values XI. of \( \lambda, \mu, g \), the expression,

\[ \phi \rho = g \rho + V \lambda \rho \mu = \rho (g - S \lambda \mu) + \lambda S \mu \rho + \mu S \lambda \rho, \]

becomes,

\[ \phi \rho = -c_2 \rho + (e' a_3 + e a_1) S(e'a_3 - e a_1) \rho + (e' a_3 - e a_1) S(e'a_3 + e a_1) \rho \]

\[ = -c_2 \rho - 2 e^2 a_3 Sa_1 \rho + 2 e^2 o_3 Sa_3 \rho; \]

which agrees, by XI., with VI.

(6.) Conversely if \( g, \lambda, \) and \( \mu \) be constants such that \( \phi \rho = g \rho + V \lambda \rho \mu \), then \( \phi \rho V \lambda \mu = g' V \lambda \mu \), where \( g' = g - S \lambda \mu \), as before; hence \( g' \) must be one of the three roots \( c_1, c_2, c_3 \) of the cubic \( M_0 = 0 \), and the normal to the plane of \( \lambda, \mu \) must have one of the three directions of \( a_1, a_2, a_3 \); if then we assume, on trial, that this plane is that of \( a_1, a_3 \), and write accordingly,

\[ \lambda = a a_1 + a' a_3, \quad \mu = b a_1 + b' a_3, \quad \phi \rho = \lambda S \mu \rho + \mu S \lambda \rho, \]

we are, by VI., to seek for scalars \( aa' bb' \) which shall satisfy the three conditions,

\[ 2ab = c_1 - c_2, \quad 2a'b' = c_3 - c_2, \quad ab' + ba' = 0; \]

but these give

\[ (2ab')^2 = (2ba')^2 = (c_3 - c_2) (c_2 - c_1), \]

so that if the transformation is to be a real one, we must suppose that \( c_2 - c_1 \) and \( c_3 - c_2 \) are either both positive, as in IX., or else both negative; or in other words, we must so arrange the three real roots of the cubic, that \( c_2 \) may be (algebraically) intermediate in value between the other two. Adopting then the order IX., with the values X., we satisfy the conditions XVII. by supposing that

\[ a' = b' = e', \quad a = -b = e; \]

and are thus led back from XVI. to the expressions XI., as the only real ones for \( \lambda, \mu, \) and \( g \) which render possible the transformations I. and II.; except that \( \lambda \) and \( \mu \) may be interchanged, &c., as before.

(7.) We see, however, that in an imaginary sense there exist two other solutions of the problem, to transform \( \phi \rho \) and \( S \phi \phi \rho \) as above; for if we retain the order IX., and equate \( g' \) in II'. to either \( -c_1 \) or \( -c_3 \), we may in each case conceive the corresponding sum of two squares in VIII. as being the product of two imaginary but linear factors; the planes of the two imaginary pairs of vectors which result being real, and perpendicular respectively to \( a_i \) and \( a_3 \).

Hamilton's Elements of Quaternions.
(8.) And if the real expression XIV. for \( \phi_\rho \) be given, and it be required to pass from it to the expression V., with the order of inequality IX., the investigation in 354, (12.) enables us at once to establish the formulæ:

\[
\begin{align*}
XX. & \quad c_1 = -g - T\lambda \mu, \quad c_2 = -g + S\lambda \mu, \quad c_3 = -g + T\lambda \mu; \\
XXI. & \quad a_1 = U(\lambda T\mu - \mu T\lambda), \quad a_2 = UV\lambda \mu, \quad a_3 = U(\lambda T\mu + \mu T\lambda);
\end{align*}
\]

in which however it is permitted to change the sign of any one of the three vector units. Accordingly the expressions XI. give,

\[
\begin{align*}
T\lambda \mu + S\lambda \mu &= 2e^2 = c_3 - c_1, \quad T\lambda \mu - S\lambda \mu = 2e^2 = c_3 - c_1, \quad S\lambda \mu = g + c_2; \\
T\lambda &= T\mu, \quad \lambda - \mu = 2e_1 a_1, \quad V\lambda \mu = -2e' a_1 a_1 = -2e_1 a_2, \quad \lambda + \mu = 2e' a_3.
\end{align*}
\]

(9.) We have also the two identical transformations,

\[
\begin{align*}
XXII. & \quad S\lambda \rho \mu \rho = \rho^2 T\lambda \mu + ((S\lambda \rho \mu)^2 + (S\lambda \rho T\mu + S\mu \rho T\lambda)^2) (T\lambda \mu - S\lambda \mu)^{-1}, \\
XXIII. & \quad S\lambda \rho \mu \rho = -\rho^2 T\lambda \mu - ((S\lambda \rho \mu)^2 + (S\lambda \rho T\mu - S\mu \rho T\lambda)^2) (T\lambda \mu + S\lambda \mu)^{-1},
\end{align*}
\]

which hold good for any three vectors, \( \lambda, \mu, \rho \), and may (among other ways) be deduced, through the expressions XX. and XXI., from II. and VIII.

(10.) Finally, as regards the expressions VI. for \( \phi_\rho, \) &c., if we denote the corresponding forms of \( \psi_\rho \) by \( \phi_\rho, \) &c., we have (comp. 354, (15.) ) these other expressions, which are as usual (comp. 351, &c.) of monomial form:

\[
\begin{align*}
XXIV. & \quad \psi_\rho = \phi_\rho = (c_2 - c_1) (c_1 - c_3) a_2 S\lambda \rho; \\
XXV. & \quad \psi_\rho = \phi_\rho = (c_3 - c_2) (c_2 - c_1) a_3 S\lambda \rho.
\end{align*}
\]

and which verify the relations 354, XLI., and several other parts of the whole foregoing theory.

358. The general linear and vector function \( \phi_\rho \) of a vector has been seen (347, (1.)) to contain, at least implicitly, nine scalar constants; and accordingly the expression 357, I. involves that number, namely four in the term \( V q_\rho \), on account of the constant quaternion \( q_\rho \), and five in the other term \( V\lambda \rho \mu \), each of the two unit-vectors, \( U\lambda \) and \( U\mu \), counting as two scalars, and the tensor \( T\lambda \mu \) as one more. But a self-conjugate linear and vector function, or the self-conjugate part \( \phi_\rho \) of the general function \( \phi_\rho \), involves only six scalar constants; either because three disappear with the term \( V\gamma \rho \) of \( \phi_\rho \); or because the
condition of self-conjugation, $\Sigma V\beta a = 2\gamma = 0$ (comp. 349, XXII. and 353, XXXVI.), which arises when we take for $\phi\rho$ the form $\Sigma \beta S a\rho$ (347, XXXI.), is equivalent to a system of three scalar equations, connecting the nine constants. And for the same reason the general quadratic but scalar function, $S\rho\phi\rho$, involves in like manner only six scalar constants. Accordingly there enter only six such constants into the expressions 357, II., V., VIII., XIV.; $c_1, c_2, c_3$, for instance, being three such, and the rectangular unit system $a_1, a_2, a_3$, answering to three others. The following other general transformations of $S\rho\phi\rho$ and $\phi\rho$, although not quite so simple as 357, II. and XIV., involve the same number (six) of scalar constants, and deserve to be briefly considered: namely the forms,

I. $S\rho\phi\rho = a(\nabla a\rho)^2 + b(\Sigma \beta\rho)^2$;

II. $\phi\rho = -a\nabla a\rho + b\beta S\beta\rho$;

in which $a, b$ are two real scalars, and $a, \beta$ are two real unit-vectors. We shall merely set down the leading formulæ, leaving the reader to supply the analysis, which at this stage he cannot find difficult.

1. In accomplishing the reduction of the expressions,

$$S\rho\phi\rho = c_1(Sa_1\rho)^2 + c_2(Sa_2\rho)^2 + c_3(Sa_3\rho)^2,$$

$$\phi\rho = c_1a_1Sa_1\rho + c_2a_2Sa_2\rho + c_3a_3Sa_3\rho,$$

to these new forms I. and II., it is found that, if the result is to be a real one, $-a$ must be that root of the scalar cubic $M_\alpha = 0$, the reciprocal of which is algebraically intermediate, between the reciprocals of the other two. It is therefore convenient here to assume this new condition, respecting the order of the inequalities,

$$\text{III. } c_1^{-1} > c_2^{-1} > c_3^{-1};$$

which will indeed coincide with the arrangement 357, IX., if the three roots $c_1, c_2, c_3$ be all positive, but will be incompatible with it in every other case.

2. This being laid down (or even, if we choose, the opposite order being taken), the (real) values of $a, b, a, \beta$ may be thus expressed:

IV. $a = -c_2$, $b = c_1 - c_2 + c_3$;

V. $a = xa_1 + za_3$, $\beta = x'a_1 + z'a_3$;

in which

VI. $x^2 = \frac{c_1^{-1} - c_2^{-1}}{c_1^{-1} - c_3^{-1}}$, $z^2 = \frac{c_2^{-1} - c_3^{-1}}{c_1^{-1} - c_3^{-1}}$.
VII. \( \frac{c_2 x}{x'} = \frac{c_3 z}{z'} = b(x x' + z z') = - b S a \beta = (s a y) b' \)

VIII. \( b'^2 = c_1 c_2 - c_3 b = c_1^2 x^2 + c_3^2 z^2 \); IX. \( x^2 + y^2 = x'^2 + y'^2 = 1 \);

X. \( b x' z' = c_3 x z \);

XI. \( c_1 x^2 + c_3 z^2 = b^{-1} b'^2 = b (S a \beta)^2 \), \( c_1 c_3 = - a b (S a \beta)^2 \);

XII. \( b' \beta = - b \beta S a \beta = c_1 x a_1 + c_3 z a_3 \); &c.

(3.) And there result the transformations:

XIII. \( \phi_1 \rho = (c_1 - c_3) a_1 S a_1 \rho + (c_3 - c_2) a_3 S a_3 \rho = - c_2 (x a_1 + z a_3) S (x a_1 + z a_3) \rho + \frac{c_2}{c_1 c_3} (x c_1 a_1 + z c_3 a_3) S (x c_1 a_1 + z c_3 a_3) \rho \);

XIV. \( \phi_2 \rho = c_1 a_1 S a_1 \rho + c_3 a_3 S a_3 \rho + c_3 S a_3 \rho = c_2 (x a_1 + z a_3) S (x a_1 + z a_3) \rho + \frac{c_2}{c_1 c_3} (x c_1 a_1 + z c_3 a_3) S (x c_1 a_1 + z c_3 a_3) \rho \);

XV. \( S \phi_1 \rho = - c_2 (V a_1 + z a_3) \rho^2 + \frac{c_2}{c_1 c_3} (S (x c_1 a_1 + z c_3 a_3) \rho^2 \)

which last, if \( c_1 c_3 \) be positive, gives this other real form,

XVI. \( S \phi_2 \rho = - c_2 (V a_1 + z a_3) \rho^2 + \frac{c_2}{c_1 c_3} (S (x c_1 a_1 + z c_3 a_3) \rho^2 \)

\( x^2 \) and \( z^2 \) being determined by the expressions VI.

(4.) Those expressions allow us to change the sign of \( z : x \), and thereby to determine a second pair of real unit lines, \( a' \) and \( \beta' \), which may be substituted for \( a \) and \( \beta \) in the forms I. and II.; the order of inequalities III. (or the opposite order), and the values IV. of \( a \) and \( b \), remaining unchanged. We have therefore the double transformations:

XVII. \( S \phi_1 \rho = - c_2 (V a_1 + z a_3) \rho^2 + (c_1 - c_2 + c_3) (S \beta \rho)^2 = - c_2 (V a' \rho)^2 + (c_1 - c_2 + c_3) (S \beta' \rho)^2 \);

XVIII. \( \phi_2 \rho = c_1 a_1 V a_1 + (c_1 - c_2 + c_3) S \beta S \beta \rho = c_2 a' V a' \rho + (c_1 - c_2 + c_3) \beta' S \beta' \rho \).

(5.) If either of the two connected forms I. and II. had been given, we might have proposed to deduce from it the values of \( c_1 c_2 c_3 \), and of \( a_1 a_2 a_3 \), by the general method of this Section. We should thus have had the cubic,

XIX. \( 0 = M_0 = (c + a) \{ c^2 + (a - b) c - a b (S a \beta)^2 \} \);
and because the quadratic \((c + a)^{-1} M_0 = 0\) may be thus written,

\[
XX. \ldots (c^{-1} + a^{-1})^2 \left(S \alpha \beta \right)^2 - (c^{-1} + a^{-1}) (a^{-1} S \cdot (a \beta)^2 + b^{-1}) + a^{-2} (V a \beta)^2 = 0,
\]

it gives two real values of \(c^{-1} + a^{-1}\), one positive and the other negative; if then we arrange the reciprocals of the three roots of \(M_0 = 0\) in the order III., we have the expressions,

\[
XXI. \ldots \begin{cases} 
  c_1 = \frac{1}{2} (b - a) + \frac{1}{2} a b \sqrt{(a^{-2} + 2 a^{-1} b^{-1} S \cdot (a \beta)^2 + b^{-2})}; & c_2 = - a; \\
  c_2 = \frac{1}{2} (b - a) - \frac{1}{2} a b \sqrt{(a^{-2} + 2 a^{-1} b^{-1} S \cdot (a \beta)^2 + b^{-2})}; & k_3;
\end{cases}
\]

the signs of the radical being determined by the condition that \((c_1 - c_3) : a b (S a \beta)^2 = c_1^{-1} - c_3^{-1} > 0\). Accordingly these expressions for the roots agree evidently with the former results, IV. and XI., because \(S \cdot (a \beta)^2 = 2 (S a \beta)^2 - 1\).

(6.) The roots \(c_1, c_2, c_3\) being thus known, the same general method gives for the directions of \(a_1, a_2, a_3\) the versors of the following expressions (or of their negatives):

\[
XXII. \ldots \begin{cases} 
  \psi_{1} = ac_1^{-1} (c_0 a + b \beta S a \beta) S (c_0 a + b \beta S a \beta) \rho; \\
  \psi_{2} = ab V a \beta S \beta a \rho; \\
  \psi_{3} = ac_2^{-1} (c_1 a + b \beta S a \beta) S (c_1 a + b \beta S a \beta) \rho;
\end{cases}
\]

of which the monomial forms may again be noted, and which give,

\[
XXII. \ldots a_1 = \pm \Omega (c_0 a + b \beta S a \beta), \quad a_2 = \pm \Omega V a \beta a, \quad a_3 = \pm \Omega (c_0 a + b \beta S a \beta).
\]

(7.) Accordingly the expressions in (2.) give (if we suppose \(a_0 a_1 = + a_2\)),

\[
XXIII. \ldots c_0 a + b \beta S a \beta = (c_3 - c_1) x a_1, \quad V a \beta = (x^2 - x z^2) a_2, \quad c_0 a + b \beta S a \beta = (c_1 - c_3) z a_3;
\]

and as an additional verification of the consistency of the various parts of this whole theory, it may be observed (comp. 357, XXIV.), that

\[
XXIV. \ldots - a c_2^{-1} (c_0 a + b \beta S a \beta)^2 = (c_0 - c_1) (c_1 - c_3), \quad a b (V a \beta)^2 \\
= (c_2 - c_0) (c_2 - c_1), \quad - a c_1^{-1} (c_1 a + b \beta S a \beta)^2 = (c_1 - c_3) (c_2 - c_1).
\]

(8.) As regards the second transformations, XVII. and XVIII., it is easy to prove that we may write,

\[
XXV. \ldots (c_3 - c_1) a' = b \beta a \beta - a a, \quad (c_3 - c_1) \beta' = a a \beta a - b \beta,
\]
XXVI. \( (c_3 - c_1)^2 = (b \beta a \beta - aa)^2 = (a a \beta a - b \beta)^2; \)

so that we have the following equation,

XXVII. \( (a (V a \rho)^2 + b (S \beta \rho)^2) \left( a^2 + 2ab S (a \beta)^2 + b^2 \right) = a (V (b \beta a \beta - aa) \rho)^2 + b (S (a a \beta a - b \beta) \rho)^2; \)

which is true for any vector \( \rho \), any two unit lines \( a, \beta \), and any two scalars \( a, b. \)

(9.) Accordingly it is evident from (4.), that \( a_1, a_3 \) must be the bisectors of the angles made by \( a, a' \), and also of those made by \( \beta, \beta' \); and the expressions XXV. may be thus written (because \( b - a = c_1 + c_3 \),

XXVIII. \( (c_3 - c_1) a' = (c_3 + c_1) a + 2b \beta S a \beta, \quad (c_1 - c_3) \beta' = (c_1 + c_3) \beta - 2a a S a \beta; \)

whence, by XXIII., we may write,

XXIX. \( a + a' = 2x a_1, \quad a - a' = 2z a_3; \)

so that \( a_1 \) bisects the internal angle, and \( a_3 \) the external angle, of the lines \( a, a' \).

(10.) At the same time we have these other expressions,

XXX. \( (c_1 - c_3) (\beta + \beta') = 2(c_1 \beta - a a S a \beta), \quad (c_1 - c_3) (\beta - \beta') = 2(c_3 \beta - a a S a \beta); \)

which can easily be reduced to the simple forms,

XXXI. \( \beta + \beta' = 2x' a_1, \quad \beta - \beta' = 2z' a_3, \)

with the recent meanings of the coefficients \( x' \) and \( z' \).

(11.) And although, for the sake of obtaining real transformations, we have supposed (comp. III.) that

XXXII. \( (c_1^{-1} - c_2^{-1}) (c_2^{-1} - c_3^{-1}) > 0, \)

because the assumed relation \( a = x a_1 + z a_3 \) between the three unit vectors \( a a_1 a_3 \), whereof the two latter are rectangular, gives \( x^2 + z^2 = 1 \), as in IX., so that each of the two expressions VI. involves the other, and their comparison gives the ratio,

XXXIII. \( x^2 : z^2 = (c_1^{-1} - c_2^{-1}) : (c_2^{-1} - c_3^{-1}), \)

yet we see that, without this inequality XXXII. existing, the foregoing transformations hold good in an imaginary (or merely symbolical) sense: so that we may say, in general, that the functions \( S_{\rho \phi \rho} \) and \( \phi_{\rho \phi} \) can be brought
to the \textit{forms} I. and II. in \textit{six} distinct \textit{ways}, whereof \textit{two} are \textit{real}, and the \textit{four} others are \textit{imaginary}.

(12.) It may be added that the first equation XXII. admits of being replaced by the following,

\[ XXXIV. \ldots \psi_1 \rho = -bc_i^{-1}(c_i \beta - aaSa\beta) S(c_i \beta - aaSa\beta) \rho, \]

with a corresponding form for \( \psi_2 \rho \); and that thus, instead of XXII., we are at liberty to write the expressions,

\[ XXXV. \ldots a_1 = U(c_i \beta - aaSa\beta), \quad a_2 = UVa\beta, \quad a_3 = U(c_i \beta - aaSa\beta), \]

for the rectangular unit system, deduced from I. or II.

359. If we call, as we naturally may, the \textit{expressions}

\[ \text{I. } \ldots \phi_1 \rho = c_1a_1Sa_1 \rho + c_2a_2Sa_2 \rho + c_3a_3Sa_3 \rho, \]
\[ \text{II. } \ldots S_\rho \phi_\rho = c_1(Sa_1 \rho)^2 + c_2(Sa_2 \rho)^2 + c_3(Sa_3 \rho)^2, \]

the \textit{Rectangular Transformations} of the \textit{Functions} \( \phi_1 \rho \) and \( S_\rho \phi_\rho \), then by another geometrical analogy, which will be seen when we come to speak briefly of the theory of \textit{Surfaces of the Second Order}, we may call the \textit{expressions},

\[ \text{III. } \ldots \phi_3 \rho = g\rho + V\lambda \rho \mu, \]
\[ \text{IV. } \ldots S_\rho \phi_\rho = g\rho^2 + S\lambda \rho \mu \rho, \]

the \textit{Cyclic* Transformations} of the same two functions; and may say that the two other and more recent \textit{expressions},

\[ \text{V. } \ldots \phi_5 \rho = -aV \rho \beta S_\beta \rho, \]
\[ \text{VI. } \ldots S_\rho \phi_\rho = a(V \rho \beta)^2 + b(S_\beta \rho)^2, \]

are \textit{Focal† Transformations} of the same. We have already shown (357) how to exchange \textit{rectangular forms} with \textit{cyclic ones}; and also (358) how to pass from \textit{rectangular expressions} to \textit{focal ones}, and reciprocally: but it may be worth while to consider briefly the mutual relations which exist, between \textit{cyclic} and \textit{focal expressions}, and the modes of passing from either to the other.

* Compare the Note to Art. 357.

† It will be found that the \textit{two real vectors} \( a, a' \), of 358, are the \textit{two real focal lines} of the \textit{real or imaginary cone}, which is \textit{asymptotic} to the \textit{surface of the second order}, \( S_\rho \rho = \text{const.} \)
(1.) To pass from IV. to VI., or from the cyclic to the focal form, we may first accomplish the rectangular transformation II., with the values 357, XX., and XXI., of \( c_1, c_2, c_3 \), and of \( a_1, a_2, a_3 \), the order of inequality being assumed to be

\[
VII. \ldots c_3 > c_2 > c_1,
\]
as in 357, IX.;

and then shall have (comp. 358, XV.) the following expressions:

\[
VIII. \ldots 4S\rho \phi \rho = \left\{ S \cdot \rho \left( c_1^3 (U\lambda - U\mu) + c_3^3 (U\lambda + U\mu) \right) \right\}^2 - \left\{ V \cdot \rho \left( c_1^3 (U\lambda + U\mu) + c_3^3 (U\lambda - U\mu) \right) \right\}^2;
\]

\[
VIII'. \ldots 4S\rho \phi \rho = -\left\{ S \cdot \rho \left( (-c_1)^3 (U\lambda - U\mu) + (-c_3)^3 (U\lambda + U\mu) \right) \right\}^2 + \left\{ V \cdot \rho \left( (-c_1)^3 (U\lambda + U\mu) + (-c_3)^3 (U\lambda - U\mu) \right) \right\}^2;
\]

\[
IX. \ldots (c_3 - c_1)^2 S\rho \phi \rho = \left\{ V \cdot \rho \left( c_3^3 V\lambda \mu + (-c_2)^3 (\lambda T\mu + \mu T\lambda) \right) \right\}^2 + \left\{ S \cdot \rho \left( (-c_2)^3 V\lambda \mu - c_3^3 (\lambda T\mu + \mu T\lambda) \right) \right\}^2;
\]

\[
X. \ldots (c_3 - c_1)^2 S\rho \phi \rho = -\left\{ V \cdot \rho \left( (-c_3)^3 V\lambda \mu + c_2^3 (\lambda T\mu - \mu T\lambda) \right) \right\}^2 - \left\{ S \cdot \rho \left( -c_2^3 V\lambda \mu + (-c_3)^3 (\lambda T\mu - \mu T\lambda) \right) \right\}^2;
\]
in which it is to be remembered that (by 357, XX.),

\[
XI. \ldots c_1 = -g - T\lambda \mu, \quad c_2 = -g + S\lambda \mu, \quad c_3 = -g + T\lambda \mu;
\]

and of which all are symbolically true, or give (as in IV.) the real value \( gp^2 + S\lambda \rho \mu \rho \) for \( S\rho \phi \rho \), if \( g, \lambda, \mu, \rho \) be real. And in this symbolical sense, although they have been written down as four, they only count as three distinct focal transformations, of a given and real cyclic form; because the expression VIII'. is an immediate consequence of VIII.; and other formulæ IX'. and X'. might in like manner be at once derived from IX. and X.

(2.) But if we wish to confine ourselves to real focal forms, there are then four cases to be considered, in each of which some one of the four equations VIII. VIII'. IX. X. is to be adopted, to the exclusion of the other three. Thus, if

\[
XII. \ldots c_3 > c_2 > c_1 > 0, \quad \text{and therefore} \quad c_1^{-1} > c_2^{-1} > c_3^{-1} > 0,
\]

the form VIII. is the only real one. If

\[
XIII. \ldots c_3 > c_2 > 0 > c_1, \quad c_2^{-1} > c_3^{-1} > 0 > c_1^{-1}, \quad \text{then X. is the real form.}
\]

If

\[
XIV. \ldots c_3 > 0 > c_4 > c_1, \quad c_3^{-1} > 0 > c_4^{-1} > c_1^{-1}, \quad \text{the only real form is IX.}
\]
Finally if $0 > c_5 > c_3 > c_1$, $0 > c_1^{-1} > c_2^{-1} > c_3^{-1}$,
that is, if all the roots of the cubic $M_0 = 0$ be negative, then VIII'. is the form
to be adopted, under the same condition of reality.

(3.) When all the roots $c$ are positive, or in the case when VIII. is the
real focal form, the unit lines $a$, $\beta$ in VI. may be thus expressed:

$$
\begin{align*}
\alpha &= \frac{1}{3} \left( \frac{c_3}{c_2} \right) (U\lambda - U\mu) + \frac{1}{3} \left( \frac{c_1}{c_2} \right) (U\lambda + U\mu); \\
\beta &= \frac{1}{3} \left( \frac{c_3}{b} \right) (U\lambda - U\mu) + \frac{1}{3} \left( \frac{c_1}{b} \right) (U\lambda + U\mu);
\end{align*}
$$

with $b = c_1 - c_2 + c_3$ as before (358, IV.).

(4.) In the same case VIII., the expressions for $4S\phi\phi\rho$ may be written
(comp. 358, XVI.) under either of these two other real forms:

$$
\begin{align*}
\text{XVII.} & \quad 4S\phi\phi\rho = N \{(c_3 + c_1) \rho \cdot U\lambda + (c_3 \hat{-} c_1) U\mu \cdot \rho\}; \\
\text{XVII'.} & \quad 4S\phi\phi\rho = N \{(c_3 + c_1) U\lambda \cdot \rho + (c_3 \hat{-} c_1) \rho \cdot U\mu\};
\end{align*}
$$

so that if we write, for abridgment,

$$
\text{XVIII.} \quad \iota = \frac{1}{3} (c_3 + c_1) U\lambda, \quad \kappa_0 = \frac{1}{3} (c_3 \hat{-} c_1) U\mu,
$$

we shall have, briefly,

$$
\text{XIX.} \quad S\phi\phi\rho = N (\iota \rho + \mu \kappa_0) = N (\rho \iota_0 + \kappa_0 \rho).
$$

(5.) Or we may make

$$
\text{XX.} \quad \iota = \frac{1}{3} (c_3 \hat{-} c_1)^2 U\lambda, \quad \kappa = \frac{1}{3} (c_3 \hat{-} c_1)^2 U\mu, \quad \text{whence } \kappa^2 - \iota^2 = c_1 c_3 \hat{-};
$$

and shall then have the transformation,

$$
\text{XXI.} \quad S\phi\phi\rho = N \frac{\iota \rho + \rho \kappa}{\kappa^2 - \iota^2},
$$

which may be compared with the equation 281, XXIX. of the ellipsoid, and
for the reality of which form, or of its two vector constants, $\iota$, $\kappa$, it is necessary
that the roots $c$ of the cubic should all be positive as above.

(6.) It was lately shown (in 358, (8.), &c.) how to pass from a given and
real focal form to a second of the same kind, with its new real unit lines $a'$, $\beta'$
in the same plane as the two old or given lines, $a$, $\beta$; but we have not yet
shown how to pass from a focal form to a cyclic one, although the converse passage has been recently discussed. Let us then now suppose that the form VI. is real and given, or that the two scalar constants \(a, b\), and the two unit vectors \(a, \beta\), have real and given values; and let us seek to reduce this expression VI. to the earlier form IV.

(7.) We might, for this purpose, begin by assuming that

\[
XXII. . . c_1^{-1} > c_2^{-1} > c_3^{-1}, \text{ as in 358, III. ;}
\]

which would give the expressions 358, XXI. and XXII., for \(c_1c_2c_3\) and \(a_1a_2a_3\), and so would supply the rectangular transformation, from which we could pass, as before, to the cyclic one.

(8.) But to vary a little the analysis, let us now suppose that the given focal form is some one of the four following (comp. (1.)):

\[
XXIII. . . S_{\rho \phi \rho} = (S_{\beta_{0}\rho})^2 - (V_{a_{0}\rho})^2; \quad XXIII'. . . S_{\rho \phi \rho} = (V_{a_{0}\rho})^2 - (S_{\beta_{0}\rho})^2;
\]

\[
XXIV. . . S_{\rho \phi \rho} = (S_{\beta_{0}\rho})^2 + (V_{a_{0}\rho})^2; \quad XXIV'. . . S_{\rho \phi \rho} = -(V_{a_{0}\rho})^2 - (S_{\beta_{0}\rho})^2;
\]

in each of which \(a_{0}\) and \(\beta_{0}\) are conceived to be given and real vectors, but not generally unit lines; and which are in fact the four cases included under the general form, \(a(V_{a_{0}\rho})^2 + b(S_{\beta_{0}\rho})^2\), according as the scalars \(a\) and \(b\) are positive or negative. It will be sufficient to consider the two cases, XXIII. and XXIV., from which the two others will follow at once.

(9.) For the case XXIII. we easily derive the real cyclic transformation,

\[
XXV. . . S_{\rho \phi \rho} = (S_{\beta_{0}\rho})^2 - (S_{a_{0}\rho})^2 + a_{0}^2 \rho^2
\]

\[
= S(\beta_{0} + a_{0})\rho \cdot S(\beta_{0} - a_{0})\rho + a_{0}^2 \rho^2
\]

\[
= g\rho^2 + S\lambda \mu \rho^2 = (g - S\lambda\mu)\rho^2 + 2S\lambda\mu S\rho^2,
\]

where

\[
XXVI. . . \lambda = \beta_{0} + a_{0}, \quad \mu = \frac{1}{2}(\beta_{0} - a_{0}), \quad g = \frac{1}{2}(a_{0}^2 + \beta_{0}^2);
\]

and the equations 357, (9.) enable us to pass thence to the two imaginary cyclic forms.

(10.) For example, if the proposed function be (comp. XIX.),

\[
XXVII. . . S_{\rho \phi \rho} = N(t_{0}\rho + \rho \kappa_{0}) = (S(t_{0} + \kappa_{0})\rho)^2 - (V(t_{0} - \kappa_{0})\rho)^2;
\]

we may write

\[
a_{0} = t_{0} - \kappa_{0}, \quad \beta_{0} = t_{0} + \kappa_{0}, \quad \lambda = 2t_{0}, \quad \mu = \kappa_{0}, \quad g = t_{0}^2 + \kappa_{0}^2.
\]
and the required transformation is (comp. 336, XI.),

\[ XXVIII. \ldots N(t_0\rho + \rho \kappa) = (t_0^2 + \kappa_0^2)\rho^2 + 2S_{t_0}\rho\kappa_0\rho. \]

(11.) To treat the case XXIV. by our general method, we may omit for simplicity the subindices \( \circ \), and write simply (comp. V. and VI.) the expressions,

\[ XXIX. \ldots \phi_\rho = -aV\alpha + \beta S\beta_\rho, \quad \text{and} \quad XXX. \ldots S_\rho \phi_\rho = (V\alpha)^2 + (S\beta_\rho)^2; \]

in which, however, it is to be observed that \( a \) and \( \beta \), though real vectors, are not now unit lines (8.). Hence, because \(-aV\alpha = aS\alpha - a^2\rho\), we easily form the expressions:

\[ XXXI. \ldots m = a^2(Sa\beta)^2, \quad m' = a^2(a^2 - \beta^2) - (Sa\beta)^2, \quad m'' = \beta^2 - 2a^2; \]

\[ XXXII. \ldots \psi_\rho = V\alpha\beta S\alpha\beta\rho - a^2(aV\alpha + \beta V\beta_\rho) + a^4\rho \]

and therefore

\[ XXXIII. \ldots M = (c - a^2)(c^2 + (\beta^2 - a^2)c) - (Sa\beta)^2, \]

\[ XXXIV. \ldots \psi_\rho = V\alpha\beta S\alpha\beta + (\beta^2 - a^2)(c\rho - aS\alpha_\rho) - c(aS\alpha + \beta S\beta_\rho) + c^2\rho \]

(12.) Introducing then a real and positive scalar constant, \( r \), such that

\[ XXXV. \ldots r^4 = (a^2 - \beta^2)^2 + 4(Sa\beta)^2 = (a^2 + \beta^2)^3 + 4(Va\beta)^2 \]

\[ = a^4 + (a\beta)^2 + (\beta a)^2 + \beta^4 = a^4 + 2S(\beta a)^2 + \beta^4 \]

\[ = a^2(a^2 + \beta a^2) = \beta^2(\beta^3 + a\beta a)^2 = \&c., \]

in which (by 199, \&c.),

\[ S(a\beta)^2 = (Sa\beta)^2 + (Va\beta)^2 = 2(Sa\beta)^2 - a^2\beta^3 = 2(Va\beta)^2 + a^2\beta^2, \]

the roots of \( M = 0 \) admit of being expressed as follows:

\[ XXXVI. \ldots c_1 = \frac{1}{3}(a^2 - \beta^2 + r^2), \quad c_2 = a^2, \quad c_3 = \frac{1}{3}(a^2 - \beta^2 - r^2); \]

and when they are thus arranged, we have the inequalities,

\[ XXXVII. \ldots c_1 > 0 > c_2 > c_3, \quad c_1^{-1} > 0 > c_2^{-1} > c_3^{-1}. \]
(13.) The corresponding forms of \( \Psi \rho \) are the three monomial expressions,

\[
\begin{align*}
\Psi_1 & = c_1^{-1}(ac_3 + \beta Sa\beta) \cdot S(ac_3 + \beta Sa\beta) \rho, \\
\Psi_3 & = \sqrt{a} \beta S\beta a \rho,
\end{align*}
\]

which may be variously transformed and verified, and give the three following rectangular vector units,

\[
\begin{align*}
a_1 & = U(ac_3 + \beta Sa\beta), \\
a_2 & = UVa\beta, \\
a_3 & = U(ac_1 + \beta Sa\beta);
\end{align*}
\]
in connexion with which it is easy to prove that

\[
\begin{align*}
T(ac_3 + \beta Sa\beta) & = (c_3 - c_2) (c_1 - c_3) (c_1 - c_2) = r (c_1 - c_2) (c_3 - c_2), \\
TVa\beta & = (c_1 - c_2) (c_3 - c_2), \\
T(ac_1 + \beta Sa\beta) & = c_1 (c_3 - c_2) (c_1 - c_2) = r (c_2 - c_3) (c_1 - c_2);
\end{align*}
\]

the radicals being all real, by XXXVII.

(14.) We have thus, for the given focal form XXX., the rectangular transformation,

\[
\begin{align*}
S_\rho & = (Vap)^2 + (S\beta \rho)^2 \\
& = \frac{c_1(S(ac_3 + \beta Sa\beta) \rho)^2}{c_1(c_3 - c_2)} + \frac{c_3(S(ac_3 + \beta Sa\beta) \rho)^2}{c_1(c_3 - c_2)} + \frac{c_2(S(ac_3 + \beta Sa\beta) \rho)^2}{c_1(c_3 - c_2)},
\end{align*}
\]
or briefly,

\[
\begin{align*}
S_\rho & = (Vap)^2 + (S\beta \rho)^2 = c_1(S . \rho U(ac_3 + \beta Sa\beta) \rho)^2 \\
& + a^2(S . \rho UVa\beta)^2 + c_3(S . \rho U(ac_1 + \beta Sa\beta)^2;
\end{align*}
\]
in which the first term is positive, but the two others are negative, and \( c_1, c_3 \) are the roots of the quadratic,

\[
0 = c^2 + (\beta^2 - a^2)c - (Sa\beta)^2.
\]

(15.) We have also the parallelisms,

\[
\begin{align*}
ac_3 + \beta Sa\beta & \parallel \beta c_1 - aSa\beta, \\
ac_1 + \beta Sa\beta & \parallel \beta c_3 - aSa\beta,
\end{align*}
\]

because

\[c_1c_3 = -(Sa\beta)^2;
\]

and may therefore write,

\[
\begin{align*}
S_\rho & = (Vap)^2 + (S\beta \rho)^2 = c_1(S . \rho U(\beta c_1 - aSa\beta))^2 \\
& + a^2(S . \rho UVa\beta)^2 + c_3(S . \rho U(\beta c_3 - aSa\beta))^2;
\end{align*}
\]
while

\[ T(\beta \theta_1 - aS\alpha \beta_1) = r_c \left( \theta_1 - \theta_2 \right), \quad T(\beta \theta_3 - aS\alpha \beta_3) = r \left( - \theta_3 \right) \left( \theta_1 - \theta_2 \right), \]

and \( r = (\theta_1 - \theta_2)^2 \), with real radicals as before.

(16.) Multiplying then by \( r^2(TV\alpha \beta)^2 \), or by \((\theta_1 - \theta_2)(\theta_1 - \theta_3)(\theta_3 - \theta_2)\), we obtain this new equation,

\[
(c_1 - c_3) \left\{ (TV\alpha \beta)^2 \left( (V\alpha \rho)^2 + (S\beta \rho)^2 \right) - a^2(S\alpha \beta \rho) \right\} = (c_3 - a^2) \left( c_1 S\beta \rho - aS\alpha \beta \right)^2 - (c_1 - a^2) \left( c_3 S\beta \rho - aS\alpha \beta \right)^2;
\]

which is only another way of expressing the same rectangular transformation as before, but has the advantage of being freed from divisors.

(17.) Developing the second member of XLVII., and dividing by \( c_1 - c_3 \), we obtain this new transformation:

\[
(c_1 - c_3) \left\{ (TV\alpha \beta)^2 \left( (V\alpha \rho)^2 + (S\beta \rho)^2 \right) - a^2(S\alpha \beta \rho) \right\} = a^2(S\alpha \beta \rho) \left( V\alpha \rho \right)^2 - a^2(S\alpha \beta \rho) \left( V\alpha \rho \right)^2 + 2a^2S\alpha \beta \rho V\alpha \rho S\beta \rho + C(S\beta \rho)^2;
\]

in which we have written for abridgment,

\[
C = c_1 c_3 - a^2(c_1 + c_3).
\]

(18.) The expressions XXXVI. for \( c_1, c_3 \) give thus,

\[
0 = - a^2 - (V\alpha \beta)^2;
\]

and accordingly, when this value is substituted for \( C \) in XLVIII., that equation becomes an identity, or holds good for all values of the three vectors, \( a, \beta, \rho \); as may be proved* in various ways.

(19.) Admitting this result, we see that for the mere establishment of the equation XLVII., it is not necessary that \( c_1 \) and \( c_3 \) should be roots of the particular quadratic XLIII. It is sufficient, for this purpose, that they should be roots of any quadratic,

\[
c^2 + Ac + B = 0, \quad \text{with the relation} \quad Aa^2 + B + a^4 + (V\alpha \beta)^2 = 0,
\]

between its coefficients. But when we combine with this the condition of rectangularity, \( a, \beta \perp a_1 \), or

\[
0 = S \left( c_1 \beta - aS\alpha \beta \right) \left( c_2 \beta - aS\alpha \beta \right) = A(S\alpha \beta)^2 + B\beta^2 + a^2(S\alpha \beta)^2;
\]

* Many such proofs, or verifications, as the one here alluded to, are purposely left, at this stage, as exercises, to the student.
we obtain thus a second relation, which gives definitely, for the two coefficients, the values,

\[ A = \beta^2 - \alpha^2, \quad B = -(Sa\beta)^2; \]

and so conducts, in a new way, to the equation XLIII.

(20.) In this manner, then, we might have been led to perceive the truth of the rectangular transformation XLVIII., with the quadratic equation XLIII. of which \( c_1 \) and \( c_3 \) are roots, without having previously found the cubic XXXIII., of which the quadratic is a factor, and of which the other root is \( c_2 = \alpha^2 \). But if we had not employed the general method of the present Section, which conducted us to form first that cubic equation, there would have been nothing to suggest the particular form XLVII., which could thus have only been by some sort of chance arrived at.

(21.) The values of \( a_1a_2a_3 \) give also (comp. 357, VII.),

\[ -\rho^2 = (S\cdot \rho U(\beta c_1 - aSa\beta))^2 + (S\cdot \rho UVa\beta)^2 + (S\cdot \rho U(\beta c_3 - aSa\beta))^2; \]

that is, by XL. and XLVI.,

\[ \rho (c_1 - c_3) \equiv (\rho(Va\beta)^2 - (Sa\beta\rho)^2) = c_3(c_1 - c_3) (c_3S\beta\rho - Sa\beta S\alpha)^2; \]

and accordingly the values XXXVI. of \( c_1, c_3 \) enable us to express each member of this last equation under the common form, \(-c_1c_3(c_1 - c_3)(aS\beta\rho - S\alpha S\beta)^2\).

(22.) Comparing the recent inequalities \( c_1 > c_3 > c_2 \) (XXXVII.) with the arrangement 357, IX., we see, by 357, (6.), that for the real cyclic transformation (6.) at present sought, the plane of \( \lambda, \mu \) is to be perpendicular to \( a_3 \) (and not to \( a_2 \), as in 357, (3.), &c.). We are therefore to eliminate \((cS\beta\rho - Sa\beta S\alpha)^2\) between the equations XLVII. and LVI., which gives (after a few reductions) the real transformation:

\[ \rho (c_1 - aSa\beta + c_3Va\beta) \equiv \rho (c_1 - aSa\beta - c_3Va\beta); \]

which is of the kind required.

(23.) Accordingly it will be found that the following equation,

\[ ((Sa\beta)^2 - c_1\beta^2) (Va\beta)^2 + (c - a^2) (c(S\beta\rho)^2 - \rho^2S(\alpha^2)\beta) = (cS\beta\rho - Sa\beta S\alpha)^2 - c(Sa\beta\rho)^2; \]
is an identity, or that it holds good for all values of the scalar \( c \), and of the vectors \( a, \beta, \rho \); since, by addition of \( c(Va\beta)^2\rho^2 \) on both sides, it takes this obviously identical form,

\[
LIX \ldots ((Sa\beta)^2 - c\beta^2)(S\rho)^2 + c(c - a^2)(S\beta\rho)^2 = (cS\beta\rho - Sa\beta S\rho)^2 - c(aS(\beta\rho - (\beta S\rho)^2;
\]

so that if \( c_i \) be either root of the quadratic XLIII., or if \( c_i(c_i - a^2) = (Sa\beta)^2 - c_i\beta^2 \), the transformation LVII. is at least symbolically valid: but we must take, as above, the positive root of that quadratic for \( c_i \), if we wish that transformation to be a real one, as regards the constants which it employs. And if we had happened (comp. (20.)) to perceive this identity LIX., and to see its transformation LVIII., we might have been in that way led to form the quadratic XLIII., without having previously formed the cubic XXXIII..

(24.) Already, then, we see how to obtain one of the two imaginary cyclic transformations of the given focal form XXX., namely by changing \( c_i \) to \( c_3 \) in LVII.; and the other imaginary transformation is had, on principles before explained, by eliminating \( (Sa\beta\rho)^2 \) between XLVII. and LVI.; a process which easily conducts to the equation,

\[
LX. \ldots (V\rho)^2 + (S\beta\rho)^2 + a^2\rho^2 = \left(c_1 - c_3\right)^{-1}(c_1^{-1}(cS\beta\rho - Sa\beta S\rho)^2 - c_3^{-1}(c_3S\beta\rho - Sa\beta S\rho)^2),
\]

where the second member is the sum of two squares (\( c_1 \) being > 0, but \( c_3 < 0 \)), as the second expression LVII. would also become, if \( c_i \) were replaced by \( c_3 \). Accordingly, each member of LX. is equal to \( (Sa\beta)^2 + (S\beta\rho)^2 \), if \( c_1, c_3 \) be the roots of any quadratic LI., with only the one condition,

\[
LXI. \ldots c_1c_3 = B = -(Sa\beta)^2;
\]

which however, when combined with the condition of rectangularity LI., suffices to give also \( A = \beta^2 - a^2 \), as in LIV., and so to lead us back to the quadratic XLIII., which had been deduced by the general method, as a factor of the cubic equation XXXIII.

(25.) And since the values XXXVI. of \( c_i, c_3 \) reduce, as above, the second member of LX. to the simple form \( (Sa\beta)^2 + (S\beta\rho)^2 \), we may thus, or even without employing the roots \( c_1, c_3 \) at all, deduce the following expression for the last imaginary cyclic transformation:

\[
LXII. \ldots S\rho\phi\rho = (V\rho)^2 + (S\beta\rho)^2 = -a^2\rho^2 + S(a + \sqrt{1}\beta)^2 S(a - \sqrt{1}\beta),
\]
where $\sqrt{-1}$ is the imaginary of algebra (comp. 214, (6.)); while the real scalar $r^4$ of XXXV. may at the same time receive the connected imaginary form,

$$LXIII. \ldots r^4 = (a^2 - \beta^2)^2 + 4(Sa\beta)^2 = (a + \sqrt{-1}\beta)^2 (a - \sqrt{-1}\beta)^2.$$  

(26.) Finally, as regards the passage from the given form XXX., to a second real focal form (comp. 358, (4.)), or the transformation,

$$LXIV. \ldots (V\alpha)^2 + (S\beta\rho)^2 = (V\alpha')^2 + (S\beta'\rho)^2,$$

in which $\alpha'$ and $\beta'$ are real vectors, distinct from $\pm a$ and $\pm \beta$, but in the same plane with them, it may be sufficient (comp. 358, (8.)), to write down the formulæ:

$$LXV. \ldots r^2\alpha' = -(a^2 + \beta a\beta), \quad r^2\beta' = -(\beta^2 + a\beta a),$$

with the same real value of $r^2$ as before; so that (by XXXV., &c.) we have the relations,

$$LXVI. \ldots T\alpha' = Ta, \quad T\beta' = T\beta, \quad Sa\beta' = Sa\beta;$$

$$LXVII. \ldots \begin{cases} r^2(a + a') = a(r^2 - a^2 + \beta^2) - 2\beta Sa\beta = -2(\alpha c_1 + \beta Sa\beta) \parallel a, \\ r^2(a - a') = a(r^2 + a^2 - \beta^2) + 2\beta Sa\beta = 2(\alpha c_1 + \beta Sa\beta) \parallel a; \end{cases}$$

$$LXVIII. \ldots \begin{cases} r^2(\beta + \beta') = \beta(r^2 + a^2 + \beta^2) - 2a\beta Sa\beta = 2(\beta c_1 - a\beta Sa\beta) \parallel a, \\ r^2(\beta - \beta') = \beta(r^2 - a^2 + \beta^2) + 2a\beta Sa\beta = -2(\beta c_1 - a\beta Sa\beta) \parallel a. \end{cases}$$

(27.) We have then the identity,

$$LXIX. \ldots (V(a^2 + \beta a\beta)\rho)^2 + (S(\beta^2 + a\beta a)\rho)^2$$

$$= (a^4 + 2S(a\beta)^2 + \beta^4)((V\alpha)^2 + (S\beta\rho)^2);$$

with which may be combined this other of the same kind,

$$LXX. \ldots -(V(a^2 - \beta a\beta)\rho)^2 + (S(\beta^2 - a\beta a)\rho)^2$$

$$= (a^4 - 2S(a\beta)^2 + \beta^4)(-(V\alpha)^2 + (S\beta\rho)^2),$$

which enables us to pass from the focal form XXIII., to a second real focal form, with its two new lines in the same plane as the two old ones: and it may be noted that we can pass from LXIX. to LXX., by changing $a$ to $a\sqrt{-1}$.
360. Besides the rectangular, cyclic, and focal transformations of $S\rho\phi\pi$, which have been already considered, there are others, although perhaps of less importance: but we shall here mention only two of them, as specimens, whereof one may be called the Bifocal, and the other the Mixed Transformation.

(1.) The two lines $u$, $a'$, of 359, LXV., being called focal lines,* an expression which shall introduce them both may be called on that account a bifocal transformation.

(2.) Retaining then the value 359, XXXV. of $r^4$, and introducing a new auxiliary constant $e$, which shall satisfy the equation,

$$I. \ldots \beta^2 - a^2 = r^2e,$$

and therefore

$$II. \ldots 4(Sa\beta)^2 = r^4(1 - e^2),$$

so that

$$III. \ldots 4e^2(Sa\beta)^2 = (1 - e^2)(\beta^2 - a^2)^2,$$

the first equation 359, LXV. gives,

$$IV. \ldots r^2(ea - a') = 2\beta Sa\beta,$$

$$V. \ldots r^2(eSa\rho - Sa'\rho) = 2Sa\beta S\beta\rho;$$

and therefore, with the form 359, XXX. of $S\rho\phi\pi$,

$$VI. \ldots (1 - e^2)S\rho\phi\pi = (1 - e^2)((V\rho)^2 + (S\beta\rho)^2)$$

$$= (1 - e^2)(V\rho)^2 + (eSa\rho - Sa'\rho)^2$$

$$= (e^2 - 1)a^2\rho^2 + (Sa\rho)^2 - 2eSa\rhoSa'\rho + (Sa'\rho)^2;$$

in which $a^2 = a'^2$, by 359, LXVI., so that $a$ and $a'$ may be considered to enter symmetrically into this last transformation, which is of the bifocal kind above mentioned.

(3.) For the same reason, the expression last found for $S\rho\phi\pi$ involves again (comp. 358) six scalar constants; namely, $e$, $Ta (= Ta')$, and the four involved in the two unit lines, $U_a$, $U_a'$.

(4.) In all the foregoing transformations, the scalar and quadratic function $S\rho\phi\pi$ has been evidently homogeneous, or has been seen to involve no terms below the second degree in $\rho$. We may however also employ this apparently heterogeneous or mixed form,

$$VII. \ldots S\rho\phi\pi = g' (\rho - e)^2 + 2S\lambda (\rho - \zeta) S\mu (\rho - \zeta) + e;$$

* Compare the Note to Art. 359 [p. 535].
in which $g', \lambda, \mu$ have the same significations as in 357, but $e, \varepsilon, \zeta$ are three new constants, subject to the two conditions of homogeneity,

$$\text{VIII. } g' \varepsilon + \lambda S \mu \zeta + \mu S \lambda \zeta = 0,$$

and

$$\text{IX. } g' \varepsilon^2 + 2S \lambda \zeta S \mu \zeta + e = 0,$$

in order that the expression VII. may admit of reduction to the form,

$$\text{X. } S \rho \phi \rho = g' \rho^2 + 2S \lambda \rho S \mu \rho, \text{ as in 357, IV'}.$$

(5.) Other general homogeneous transformations of $S \rho \phi \rho$, which are themselves real, although connected with imaginary* cyclic forms (comp. 357, (7.)), because a sum of two squares of linear and scalar functions is, in an imaginary sense, a product of two such functions, are the two following (comp. 357, (9.)):

$$\text{XI. } S \rho \phi \rho = g \rho^2 + S \lambda \rho \mu \rho = g_1 \rho^2 + (S \lambda_1 \rho)^2 + (S \mu_1 \rho)^2;$$

$$\text{XII. } S \rho \phi \rho = g_2 \rho^2 + S \lambda \rho \mu \rho = g_3 \rho^2 - (S \lambda_3 \rho)^2 - (S \mu_3 \rho)^2;$$

in which (comp. 357, (2.) and (8.)),

$$\text{XIII. } g_1 = g + T \lambda \mu = - e_1; \quad g_2 = g - T \lambda \mu = - e_3,$$

$$\text{XIV. } \lambda_1 = \nabla \lambda \mu (T \lambda \mu - S \lambda \mu)^\frac{1}{2}, \quad \mu_1 = (\lambda T \mu + \mu T \lambda) (T \lambda \mu - S \lambda \mu)^\frac{1}{2},$$

and

$$\text{XV. } \lambda_3 = \nabla \lambda \mu (T \lambda \mu + S \lambda \mu)^\frac{1}{2}, \quad \mu_3 = (\lambda T \mu - \mu T \lambda) (T \lambda \mu + S \lambda \mu)^\frac{1}{2};$$

so that $g_1, \lambda_1, \mu_1$, and $g_2, \lambda_3, \mu_3$ are real, if $g, \lambda, \mu$ be such.

(6.) We have therefore the two new mixed transformations following:

$$\text{XVI. } S \rho \phi \rho = g_1 (\rho - \varepsilon_1)^2 + (S \lambda_1 (\rho - \zeta_1))^2 + (S \mu_1 (\rho - \zeta_1))^2 + e_1;$$

$$\text{XVII. } S \rho \phi \rho = g_2 (\rho - \varepsilon_3)^2 - (S \lambda_3 (\rho - \zeta_3))^2 - (S \mu_3 (\rho - \zeta_3))^2 + e_3;$$

with these two new pairs of equations, as conditions of homogeneity,

$$\text{XVIII. } g_1 \varepsilon_1 + \lambda_1 S \zeta_1 \lambda_1 + \mu_1 S \zeta_1 \mu_1 = 0,$$

$$\text{XIX. } g_2 \varepsilon_3 + (S \zeta_3 \lambda_1)^2 + (S \zeta_1 \mu_3)^2 + e_1 = 0,$$

and

$$\text{XX. } g_2 \varepsilon_3 - \lambda_3 S \zeta_3 \lambda_3 - \mu_3 S \zeta_3 \mu_3 = 0,$$

$$\text{XXI. } g_2 \varepsilon_3 - (S \zeta_3 \lambda_3)^2 - (S \mu_3 \zeta_3)^2 + e_3 = 0.$$

* $\lambda_1 \pm \sqrt{-1} \mu_1$, and $\lambda_3 \pm \sqrt{-1} \mu_3$, may here be said to be two pairs of imaginary cyclic normals, of that real surface of the second order, of which the equation is, as before, $S \rho \phi \rho = \text{const.}$ Compare the Notes to pages 527, 534.
361. We saw, in the sub-articles to 336, that the differential, $df\rho$, of a scalar function of a vector, may in general be expressed under the form,

$$I. \ldots df\rho = nSv dp,$$

where $v$ is a derived vector function, of the same variable vector $\rho$, and $n$ is a scalar coefficient. And we now propose to show, that if

$$II. \ldots f\rho = S\phi \rho d\rho,$$

$\phi \rho$ still denoting the linear and vector function which has been considered in the present Section, and of which $\phi_\rho$ is still the self-conjugate part, we shall have the equation $I.$ with the values,

$$III. \ldots n = 2, \; v = \phi_\rho;$$

so that the part $\phi_\rho$ may thus be deduced from $\phi \rho$ by operating with $\frac{1}{2} dS \cdot \rho$, and seeking the coefficient of $d\rho$ under the sign $S.$ in the result: while there exist certain general relations of reciprocity (comp. 336, (6.)), between the two vectors $\rho$ and $v$, which are in this way connected, as linear functions of each other.

(1.) We have here, by the supposed linear form of $\phi \rho$, the differential equation (comp. 334, VI.),

$$IV. \ldots d\phi \rho = \phi d\rho;$$

also

$$S(d\rho \cdot \phi \rho) = S(\phi_\rho \cdot d\rho), \; \text{and} \; S(\rho \cdot \phi d\rho) = S(\phi' \rho \cdot d\rho);$$

hence, by 349, XIII., we have, as asserted,

$$V. \ldots dS\phi \rho \rho = S(\phi \rho + \phi' \rho) d\rho = 2S \cdot \phi \rho d\rho.$$

(2.) As an example of the employment of this formula, in the deduction of $\phi_\rho$ from $\phi \rho$, let us take the expression,

$$VI. \ldots \phi \rho = \Sigma \beta S a \rho,$$

which gives,

$$VII. \ldots f\rho = S\rho \phi \rho = \Sigma Sa \rho S \beta \rho,$$

and therefore

$$VIII. \ldots df\rho = \Sigma S(\beta Sa \rho + aS \beta \rho) d\rho.$$

Comparing this with the general formula,

$$IX. \ldots \frac{1}{2} df\rho = Sv d\rho = S \cdot \phi_\rho d\rho,$$
we find that the form VI. of \( \phi \rho \) has for its self-conjugate part,

\[ X. \ldots v = \phi \rho = \frac{1}{2} \Sigma (bS_\alpha \rho + aS_\beta \rho); \]

and in fact we saw (347, XXXII.) that this form gives, as its conjugate, the expression,

\[ XI. \ldots \phi' \rho = \Sigma aS_\beta \rho. \]

(3.) Supposing now, for simplicity, that the function \( \phi \) is given, or made, self-conjugate, by taking (if necessary) the semisum of itself and its own conjugate function, we may write \( \phi \) instead of \( \phi_0 \), and shall thus have, simply,

\[ XII. \ldots v = \phi \rho, \quad XIII. \ldots f_\rho = S_\nu \rho, \quad XIV. \ldots df_\rho = 2S_\nu d_\rho; \]

whence also (comp. 348, I. II.),

\[ XV. \ldots \rho = \phi^{-1} \nu = m^{-1} \psi \nu, \quad \text{and} \quad XVI. \ldots S_\nu d_\rho = S_\rho d_\nu. \]

(4.) Writing, then,

\[ XVII. \ldots F_\nu = S_\nu \phi^{-1} \nu = m^{-1} S_\nu \psi \nu, \]

we shall have the equations,

\[ XVIII. \ldots F_\nu = f_\rho, \quad XIX. \ldots dF_\nu = 2S_\nu d_\nu = 2S \cdot \phi^{-1} \nu d_\nu; \]

so that \( \rho \) may be deduced from \( F_\nu \), as \( \nu \) was deduced from \( f_\rho \); and generally, as above stated, there exists a perfect reciprocity of relations, between the vectors \( \rho \) and \( \nu \), and also between their scalar functions, \( f_\rho \) and \( F_\nu \).

(5.) As regards the deduction, or derivation, of \( \nu \) from \( f_\rho \), and of \( \rho \) from \( F_\nu \), it may occasionally be convenient to denote it thus: *

\[ XX. \ldots \nu = \frac{1}{2} (S \cdot d_\rho)^{-1} d_\rho; \quad XXI. \ldots \rho = \frac{1}{2} (S \cdot d_\nu)^{-1} d_\nu; \]

* [Hamilton suggested the notation \( \{ = (S \cdot d_\rho)^{-1} d \) in page 291 of a paper published in the "Proceedings of the Royal Irish Academy," vol. iii. On the same page he introduced the "more general characteristic of operation,"

\[ \frac{d}{d \rho} + \frac{d}{d \nu} + \frac{k}{d \nu} \cdot \alpha, \]

in which \( x, y, \) and \( z \) are ordinary rectangular coordinates," while \( i, j, \) and \( k \) are unit vectors parallel to the coordinate axes. More recently \( \nu \) has been printed \( v \), and in accordance with the notation for partial differentiation used in the "Elements" \( \nu = iD_x + jD_y + kD_z \). Now if \( \rho = ix + jy + kz \), for any system of rectangular axes,

\[ df_\rho = (dxD_x + dyD_y + dzD_z)f_\rho = - Sd_\nu v \cdot f_\rho. \]

Comparing this with \( df_\rho = nS_\nu d_\rho \), it is evident, as \( d_\rho \) may have any direction whatever, that the equation \( n = - v \cdot f_\rho \) must be true. Hence it may be inferred that \( v \) is independent of any particular set of coordinate axes.]
in fact, these last may be considered as only symbolical transformations of the expressions,

\[ \text{XXII. } df_p = 2S(d_p \cdot v), \quad dF_v = 2S(d_v \cdot \rho), \]

which follow immediately from XIV. and XIX.

(6.) As an example of the passage from an expression such as \( f_p \), to an equal expression of the reciprocal form \( F_v \), let us resume the cyclic form 357, II., writing thus,

\[ \text{XXIII. } f_p = S\varphi\rho = g\rho^2 + S\lambda\mu\rho, \]

and supposing that \( g, \lambda, \) and \( \mu \) are real. Here, by what has been already shown (in subArticles to 354 and 357), if \( \varphi\rho \) be supposed self-conjugate, as in (3.), we have,

\[ \text{XXIV. } v = \varphi\rho = g\rho + V\lambda\mu\rho; \]

\[ \text{XXV. } m = (g - S\lambda\mu) (g^2 - \lambda^2\mu^2) = -c_i c_\alpha c_\beta; \]

\[ \text{XXVI. } \psi = \nabla\varphi\mu S\lambda\mu - \nabla\lambda S\mu\nu - g(\lambda S\nu\mu + \mu S\lambda\nu) + g^2;^* \]

and therefore

\[ \text{XXVII. } mF_v = S\nu\psi \]

\[ = S\lambda\nu\mu S\lambda\mu + (S\lambda\nu\mu)^2 - 2gS\lambda\nu S\mu\nu + g^2;^* \]

\[ = (g^2 - \lambda^2\mu^2)\nu^2 + \lambda^2(S\mu\nu)^2 + \mu^2(S\lambda\nu)^2 - 2gS\lambda\nu S\mu\nu; \]

which last, when compared with 360, VI., is seen to be what we have called a bifocal form: its focal lines \( a, a' \) (360, (1.)) having here the directions of \( \lambda, \mu \), that is of what may be called the cyclic lines \( f \) of the form XXIII. The cyclic and bifocal transformations are therefore reciprocals of each other.

(7.) As another example of this reciprocal relation between cyclic and focal lines, in the passage from \( f_p \) to \( F_v \), or conversely from the latter to the former, let us now begin with the focal form,

\[ \text{XXVIII. } f_p = S\rho\varphi\rho = (V\varphi\rho)^2 + (S\beta\rho)^2, \]

\[ 359, \text{XXX.} \]

* [Since \( \nu = g\rho + \lambda\mu - S\lambda\rho\mu \), it follows that \( \lambda\mu = g\lambda\mu + \lambda^2\mu^2 - \lambda\mu S\lambda\mu = (\lambda^2\mu^2 - g^2)\rho + (g - \lambda\mu)S\lambda\rho\mu + g\nu. \)

From this \( S\lambda\nu\mu = (g - S\lambda\mu)S\lambda\rho\mu, \)

and, on substitution, equation XXVI. may at once be found, remembering that \( \psi = m\varphi^{-1}. \)]

† They are in fact (compare the Note to page 527) the cyclic normals, or the normals to the cyclic planes, of that surface of the second order, which has for its equation \( f_p = \text{const.} \); while they are, as above, the focal lines of that other or reciprocal surface, of which \( \nu \) is the variable vector, and the equation is \( F_v = \text{const.} \).
in which \(a\) and \(\beta\) are supposed to be given and real vectors. We have now, by 359, (11.),

\[
\begin{align*}
\nu &= \phi_0 = -aV\alpha + \beta S\beta_0, \\
\psi &= V\alpha\beta S\alpha_0 + a(\alpha^2 - \beta^2)S\alpha,
\end{align*}
\]

and therefore,

\[
\begin{align*}

mF\nu &= a^2(Sa\beta)^2F\nu = S\nu\phi_0 \\
&= SavSvS\alpha\beta + (\alpha^2 - \beta^2)(Sav)^2 \\
&= -\nu^2(Sa\beta)^2 + Sav((\alpha^2 - \beta^2)S\alpha\nu + 2Sa\beta S\beta\nu) \\
&= -\nu^2(Sa\beta)^2 + Sav\alpha^2 + \beta a\beta)v,
\end{align*}
\]

an expression which is of cyclic form; one cyclic line of \(F\nu\) being the given focal line \(a\) of \(\phi_0\); and the other cyclic line of \(F\nu\) having the direction of \(\pm(\alpha^2 + \beta a\beta)\), and consequently (by 359, LXV.) of \(\pm\alpha'\), where \(\alpha'\) is the second real and focal line of \(\phi_0\).

(8.) And to verify the equation XVIII., or to show by an example that the two functions \(\phi_0\) and \(F\nu\) are equal in value, although they are (generally) different in form, it is sufficient to substitute in XXX. the value XXIX. of \(\nu\); which, after a few reductions, will exhibit the asserted equality.

362. It is often convenient to introduce a certain scalar and symmetric function of two independent vectors, \(\rho\) and \(\rho'\), which is linear with respect to each of them, and is deduced from the linear and self-conjugate vector function \(\phi_0\), of a single vector \(\rho\), as follows:

\[
I. \quad f(\rho, \rho') = f(\rho', \rho) = S\rho'\phi_0 = S\rho\phi\rho'.
\]

With this notation, we have

\[
\begin{align*}
II. \quad f(\rho + \rho') &= f(\rho) + 2f(\rho, \rho') + f\rho' \\
III. \quad f(\rho, \rho' + \rho'') &= f(\rho, \rho') + f(\rho, \rho'') \\
IV. \quad f(\rho, \rho) &= f\rho \\
V. \quad d\rho = 2f(\rho, d\rho) \\
VI. \quad f(x\rho, y\rho') &= xyf(\rho, \rho'), \quad \text{if} \quad Vx = Vy = 0
\end{align*}
\]

and as a verification,

\[
VII. \quad f(x\rho) = x^2f\rho,
\]

a result which might have been obtained, without introducing this new function I.
(1.) It appears to be unnecessary, at this stage, to write down proofs of the foregoing consequences, II. to VI., of the definition I.; but it may be worth remarking, that we here depart a little, in the formula V., from a notation (325) which was used in some early Articles of the present Chapter, although avowedly only as a temporary one, and adopted merely for convenience of exposition of the principles of Quaternion Differentials.

(2.) In that provisional notation (comp. 325, IX.) we should have had, for the differentiation of the recent function $f_p$ (361, II.), the formulae,

$$df_p = f'(p, dp), \quad f'(p, \rho') = 2S\rho'\phi_p;$$

the numerical coefficient being thus transferred from one of them to the other, as compared with the recent equations, I. and V. But there is a convenience now in adopting these last equations V. and I., namely,

$$df_p = 2f'(p, dp), \quad f'(p, \rho') = S\rho'\phi_p;$$

because this function $S\rho'\phi_p$, or $S\phi_p$, occurs frequently in the applications of quaternions to surfaces of the second order, and not always with the coefficient 2.

(3.) Retaining then the recent notations, and treating $dp$ as constant, or $d^2p$ as null, successive differentiation of $f_p$ gives, by IV. and V., the formulæ,

$$d^2f_p = 2f'(p, dp); \quad d^3f_p = 0; \quad \text{&c.};$$

so that the theorem 342, I. is here verified, under the form,

$$\text{IX. } \epsilon^2f_p = (1 + d + \frac{1}{2!}d^3)f_p = f_p + 2f(p, dp) + fdp;$$

or briefly,

$$\text{X. } \epsilon^2f_p = f(p + dp),$$

an equation which by II. is rigorously exact (comp. 339, (4.)), without any supposition whatever being made, respecting any smallness of the tensor, $Tdp$.

363. Linear and vector functions of vectors, such as those considered in the present Section, although not generally satisfying the condition of self-conjugation, present themselves generally in the differentiation of non-linear but vector functions of vectors. In fact, if we denote for the moment such a non-linear function by $\omega(p)$, or simply by $\omega p$, the general distributive property (326) of differential expressions allows us to write,

$$\text{I. } d\omega(p) = \phi(dp), \quad \text{or briefly, } \text{I'. } d\omega p = \phi dp;$$

where $\phi$ has all the properties hitherto employed, including that of not being
generally self-conjugate, as has been just observed. There is, however, as we shall soon see, an extensive and important case, in which the property of self-conjugation exists, for such a function $\phi$; namely when the differentiated function, $\omega \rho$, is itself the result $\nu$ of the differentiation of a scalar function $f_\rho$ of the variable vector $\rho$, although not necessarily a function of the second dimension, such as has been recently considered (361); or more fully, when it is the coefficient of $d\rho$, under the sign $S_\lambda$, in the differential (361, I.) of that scalar function $f_\rho$, whether it be multiplied or not by any scalar constant (such as $n$, in the formula last referred to). And generally (comp. 346), the inversion of the linear and vector function $\phi$ in I. corresponds to the differentiation of the inverse (or implicit) function $\omega^{-1}$; in such a manner that the equation I. or I' may be written under this other form,

$$II. \ldots d\omega^{-1} \sigma = \phi^{-1} d\sigma = \nu^{-1} \psi d\sigma, \quad \text{if} \quad \sigma = \omega \rho.$$ 

(1.) As a very simple example of a non-linear but vector function, let us take the form,

$$III. \ldots \sigma = \omega(\rho) = \rho \alpha \rho, \quad \text{where} \quad \alpha \text{ is a constant vector.}$$

This gives, if $d\rho = \rho$,

$$IV. \ldots \phi \rho' = \phi d\rho = d\omega \rho = \rho' \alpha \rho + \rho \alpha \rho' = 2V \rho \alpha \rho';$$

$$V. \ldots S \lambda \phi \rho' = 2S \lambda \rho \alpha \rho' = S \rho' \phi' \lambda;$$

$$VI. \ldots \phi' \lambda = 2V \lambda \rho \alpha = 2V \rho \alpha \lambda, \quad \phi' \rho' = 2V \alpha \rho \rho';$$

so that $\phi \rho'$ and $\phi' \rho'$ are unequal, and the linear function $\phi \rho'$ is not self-conjugate.

(2.) To find its self-conjugate part $\phi \rho'$, by the method of Art. 361, we are to form the scalar expression,

$$VII. \ldots \frac{1}{2} f\rho' = \frac{1}{2} S \rho' \phi \rho' = \rho'^2 S \rho;$$

of which the differential, taken with respect to $\rho'$, is

$$VIII. \ldots \frac{1}{2} d f\rho' = S. \phi \rho' d\rho' = 2S \rho \alpha \rho' d\rho', \quad \text{giving} \quad IX. \ldots \phi \rho' = 2\rho' S \rho;$$

and accordingly this is equal to the semisum of the two expressions, IV. and VI., for $\phi \rho'$ and its conjugate.

(3.) On the other hand, as an example of the self-conjugation of the linear and vector function,

$$X. \ldots d\nu = d\omega \rho = \phi d\rho, \quad \text{when} \quad X'. \ldots d f\rho = 2S \nu d\rho = 2S. \omega \rho d\rho;$$
even if the scalar function $f_p$ be of a higher dimension than the second, let this last function have the form,

\[ f_p = Sqpq'pq''p, \quad q, q', q'' \] being three constant quaternions.

Here

\[
XII. \quad v = \omega \rho = \frac{1}{\mu} V(qpq'pq'' + q'pq''pq + q''pqpq');
\]

\[
XIII. \quad dv = \phi d\rho = \phi\rho' = \frac{1}{\mu} V(qpq'pq'' + q'pq''pq) + \frac{1}{\mu} V(q'pq''pq + q''pqpq')
\]

and

\[
XIV. \quad S\lambda \phi \rho' = \frac{1}{\mu} S(qpq''\phi + \rho'q\phi) + &c. = S\rho \phi \lambda;
\]

so that $\phi' = \phi$, as asserted.

(4.) In general, if $\delta$ be used as a second and independent symbol of differentiation, we may write (comp. 345, IV.),

\[
XV. \quad \delta df_q = d\delta f_q,
\]

where $f_q$ may denote any function of a quaternion; in fact, each member is, by the principles of the present Chapter (comp. 344, I., and 345, IX.), an expression for the limit,

\[
XVI. \quad \lim_{n \to \infty} \frac{f(q + n^{-1}dq + n^{-1}\delta q) - f(q + n^{-1}dq) - f(q + n^{-1}\delta q) + f_q}{n^{-1}}.
\]

(5.) As another statement of the same theorem, we may remark that a first differentiation of $f_q$, with each symbol separately taken, gives results of the forms,

\[
XVII. \quad d_f q = f(q, dq), \quad \delta f q = f(q, \delta q);
\]

and then the assertion is, that if we differentiate the first of these with $\delta$, and the second with $d$, operating only on $q$ with each, and not on $dq$ nor on $\delta q$, we obtain equal results, of these other forms,

\[
XVIII. \quad \delta d f q = f(q, dq, \delta q) = f(q, \delta q, dq) = d\delta f q.
\]

For example, if

\[
XIX. \quad f_q = qeo, \quad \text{where } e \text{ is a constant quaternion},
\]

* We may also say that each of the two symbols $XV.$ represents the coefficient of $x^y^{'}$, in the development of $f(q + xdq + y\delta q)$ according to ascending powers of $x$ and $y$, when such development is possible.
the common value of these last expressions is,

\[ \delta dfq = d\delta fq = \delta q \cdot c \cdot dq + dq \cdot c \cdot \delta q. \]

(6.) Writing then, by X.,

\[ \delta \omega \rho = \phi \delta \rho, \quad \text{with} \quad d\omega \rho = \phi d\rho, \quad \text{as before}, \]

we have the general equation,

\[ S(\delta \rho \cdot \phi \delta \rho) = S(\delta \rho \cdot \phi d\rho), \]

in which \( \delta \rho \) and \( \delta \rho \) may represent any two vectors; the linear and vector function, \( \phi \), which is thus derived from a scalar function \( f\rho \) by differentiation, is therefore (as above asserted and exemplified) always self-conjugate.*

(7.) The equation XXIII. may be thus briefly written,

\[ Sd\rho \delta \nu = S\delta \rho d\nu; \]

and it will be found to be virtually equivalent to the following system of three known equations, in the calculus of partial differential coefficients,

\[ \frac{\partial f}{\partial y} = \frac{\partial L}{\partial y'}, \quad \frac{\partial D}{\partial y} = D' \frac{\partial L}{\partial y}, \quad \frac{\partial D}{\partial z} = D' \frac{\partial L}{\partial z}. \]

* [If \( n \) defined by the equation \( Afp = nSrAp \) (361, 1.)

is not a constant scalar but a function of \( \rho \), the function \( \phi \) generally ceases to be self-conjugate. For example, comparing \( Afp = 2SvAp = 2F(\rho)S\mu d\rho \),

since \( d\rho \) is arbitrary, \( \nu = \mu F(\rho) \). Differentiating this again

\[ d\nu = \phi d\rho = \mu \partial F + \partial \mu \cdot F = \mu S\lambda d\rho + \theta d\rho \cdot F, \]

if

\[ dF = S\lambda d\rho, \quad \text{and} \quad \partial \mu = \theta d\rho. \]

Here again, as \( d\rho \) is arbitrary,

\[ \theta(\rho) = (\phi(\rho) - \mu S\lambda(\rho))F^{-1}, \]

and the conjugate of \( \theta \) is

\[ \theta'(\rho) = (\phi(\rho) - \lambda S\mu(\rho))F^{-1}. \]

Hence the spin-vector of \( \theta \) is

\[ \frac{1}{2} \nabla \lambda F^{-1}, \quad \text{or} \quad \frac{1}{2} \nabla \nu F^{-2}. \]

This vanishes only when \( F \) is some function of \( f(\rho) \), or a constant as may be easily verified, and in this case \( \theta \) is self-conjugate.]
364. At the commencement of the present Section, we reduced (347) the problem of the inversion (346) of a linear (or distributive) quaternion function of a quaternion, to the corresponding problem for vectors; and, under this reduced or simplified form, have resolved it. Yet it may be interesting, and it will now be easy, to resume the linear and quaternion equation,

I. \(fq = r\), with II. \(f(q + q') = fq + f(q')\),

and to assign a quaternion expression for the solution of that equation, or for the inverse quaternion function,

III. \(q = f^{-1}r\),

with the aid of notations already employed, and of results already established.

(1.) The conjugate of the linear and quaternion function \(fq\) being defined (comp. 347, IV.) by the equation,

IV. \(Spfq = Sqf'p\),

in which \(p\) and \(q\) are arbitrary quaternions, if we set out (comp. 347, XXXI.) with the form,

V. \(fq = tqs + t'qs' + \ldots = \Sigma tqs\),

in which \(s, s', \ldots\) and \(t, t', \ldots\) are arbitrary but constant quaternions, and which is more than sufficiently general, we shall have (comp. 347, XXXII.) the conjugate form,

VI. \(f'p = spt + s'pt' + \ldots = \Sigma spt\);

whence

VII. \(f1 = \Sigma ts\), and VIII. \(f'1 = \Sigma st\);

it is then possible, for each given particular form of the linear function \(fq\), to assign one scalar constant \(e\), and two vector constants, \(e, e'\), such that

IX. \(fl = e + e',\quad f'1 = e + e'\).
and then we shall have the general transformations (comp. 347, I.):

\[ X. \ldots S f q = S \cdot f q' = e S q + S e' q ; \]

\[ XI. \ldots V f q = e S q + V \cdot f V q = e S q + f V q ; \]

and

\[ XII. \ldots f q = (e + e) S q + S e' q + f V q ; \]

in which \( S e' q = S \cdot e' V q \), and \( f V q \) or \( V f V q \) is a linear and vector function of \( V q \), of the kind already considered in this Section; being also such that, with the form \( V \) of \( f q \), we have

\[ XIII. \ldots f q = \Sigma V \rho \phi . \]

(2.) As regards the number of independent and scalar constants which enter, at least implicitly, into the composition of the quaternion function \( f q \), it may in various ways be shown to be sixteen; and accordingly, in the expression \( XII. \), the scalar \( e \) is one; the two vectors, \( e \) and \( e' \), count each as three; and the linear and vector function, \( f V q \), counts as nine (comp. 347, (1.)).

(3.) Since we already know (347, &c.) how to invert a function of this last kind \( f \), we may in general write,

\[ XIV. \ldots r = S r + V r = S r + \phi \rho , \quad \text{where} \quad XV. \ldots \rho = \phi^{-1} V r = m^{\psi} V r ; \]

the scalar constant, \( m \), and the auxiliary linear and vector function, \( \psi \), being deduced from the function \( f \) by methods already explained. It is required then to express \( q \), or \( S q \) and \( V q \), in terms of \( r \), or of \( S r \) and \( \rho \), so as to satisfy the linear equation,

\[ XVI. \ldots (e + e) S q + S e' q + f V q = S r + \phi \rho ; \]

the constants \( e \), \( e' \), and the form of \( \phi \), being given.

---

* [By a method analogous to that of the Note on page 507, if any three diplanar vectors \( \beta_1, \beta_2, \beta_3 \) are chosen, any quaternion function \( f q \) may have its vector part resolved along these three vectors, so that \( f q = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + x_4 \), in which the coefficients \( x \) are scalar functions of \( q \), and are moreover linear if \( f q \) is linear in \( q \). So for a linear function,

\[ f q = \beta_1 S p_1 q + \beta_2 S p_2 q + \beta_3 S p_3 q + S p_4 , \]

and in this expression \( p_1, p_2, p_3, \) and \( p_4 \) are four constant quaternions involving sixteen scalar constants and determining the function \( f \). Denoting \( Sp \) by \( a \), and \( V p \) by \( a \), on rearrangement,

\[ f q = (\beta_1 S a_1 + \beta_2 S a_2 + \beta_3 S a_3) V q + (a_1 \beta_1 + a_2 \beta_2 + a_3 \beta_3 + a_4) S q + S a_4 V q , \]

and this is manifestly of the type,

\[ f q = (e + e) S q + S e' q + f V q . \)
(4.) Assuming for this purpose the expression,

\[ X \text{VII. } q = q' + \rho, \]

in which \( q' \) is a new sought quaternion, we have the new equation,

\[ X \text{VIII. } f q' = S r + \phi \rho - f \rho = S(r - \varepsilon \rho); \]

whence

\[ X \text{IX. } q' = S(r - \varepsilon \rho) \cdot f^{-1} \]

and

\[ X \text{X. } q = \rho + S(r - \varepsilon \rho) \cdot f^{-1}; \]

in which \( \rho \) is (by supposition) a known vector, and \( S(r - \varepsilon \rho) \) is a known scalar; so that it only remains to determine the unknown but constant quaternion, \( f^{-1} \), or to resolve the particular equation,

\[ X \text{XI. } f q_0 = 1, \quad \text{in which } X \text{XII. } q_0 = c + \gamma = f^{-1}, \]

c being a new and sought scalar constant, and \( \gamma \) being a new and sought vector constant.

(5.) Taking scalar and vector parts, the quaternion equation \( X \text{XI.} \) breaks up into the two following (comp. \( X. \) and \( X \text{I.} \)):

\[ X \text{XIII. } 1 = S f(c + \gamma) = ec + S \varepsilon \gamma; \quad X \text{XIV. } 0 = \nabla f(c + \gamma) = \varepsilon c + \phi \gamma; \]

which give the required values of \( c \) and \( \gamma \), namely,

\[ X \text{XXV. } c = (e - S \varepsilon \phi^{-1} \varepsilon)^{-1}, \quad \text{and } X \text{XXVI. } \gamma = -c \phi^{-1} \varepsilon; \]

whence

\[ X \text{XXVII. } f^{-1} = \frac{1 - \phi^{-1} \varepsilon}{e - S \varepsilon \phi^{-1} \varepsilon}; \]

and accordingly we have, by \( X \text{II.} \), the equation,

\[ X \text{XXVIII. } f(1 - \phi^{-1} \varepsilon) = e - S \varepsilon \phi^{-1} \varepsilon = V^{-1} 0. \]

(6.) The problem of quaternion inversion is therefore reduced anew to that of vector inversion, and solved thereby; but we can now advance some steps further, in the elimination of inverse operations, and in the substitution for them of direct ones. Thus, if we observe that \( \phi^{-1} = m^{-1} \psi \), as before, and write for abridgment,

\[ X \text{XXIX. } n = me - S \varepsilon \psi \varepsilon = f(m - \psi \varepsilon), \]
so that \( n \) is a new and known scalar constant, we shall have, by XV. XX. XXVII. XXIX.,

\[
XXX \ldots mp = \psi Vr; \quad XXXI. \ldots n^{-1}f = m - \psi \epsilon;
\]

and

\[
XXXII. \ldots mnq = n\psi Vr + (mSr - S\epsilon\psi Vr) \cdot (m - \psi \epsilon),
\]

an expression from which all inverse operations have disappeared, but which still admits of being simplified, through a division by \( m \), as follows.

(7.) Substituting (by XXIX.), in the term \( n\psi Vr \) of XXXII., the value \( me - S\epsilon\psi \epsilon \) for \( n \), and changing (by XXX.) \( \psi Vr \) to \( mp \), in the terms which are not obviously divisible by \( m \), such a division gives,

\[
XXXIII. \ldots nq = (m - \psi \epsilon)Sr + e\psi Vr - S\epsilon\psi Vr + \sigma,
\]

where

\[
XXXIV. \ldots \sigma = -pS\epsilon\psi \epsilon + \psi S\epsilon \rho = V. \epsilon'Vp^2V.
\]

But (by 348, VII., interchanging accents) we have the transformation,

\[
XXXV. \ldots Vp^2V = -\epsilon'Vp^2V = -\epsilon'V\epsilon Vr,
\]

because \( \phi \rho = Vr \), by XIV. or XV.; everything inverse therefore again disappears with this new elimination of the auxiliary vector \( \rho \), and we have this final expression,

\[
XXXVI. \ldots nq = n^{-1}f^{-1}r = (me - S\epsilon\psi \epsilon) \cdot f^{-1}r
\]

\[
= (m - \psi \epsilon)Sr + e\psi Vr - S\epsilon\psi Vr - V\epsilon' \phi' V\epsilon Vr,
\]

in which each symbol of operation governs all that follows it, except where a point indicates the contrary, and which it appears to be impossible further to reduce, as the formula of solution of the linear equation I., with the form XII. of the quaternion function, \( fg \).

* [The following solution is possibly more direct. Equating the scalar and vector parts of

the two equations

\[
fq = (e + \epsilon) Sq + S\epsilon Vq + \phi Vq = r = Sr + Vr,
\]

\[
eSq + S\epsilon Vq = Sr, \quad \text{and} \quad eSq + \phi Vq = Vr
\]

are found. Operating on the second equation by \( \phi^{-1} \), and replacing \( Vq \) in the first, \( Sq \) is seen to be given by

\[
(e - S\epsilon\phi^{-1} \epsilon) Sq = Sr - S\epsilon\phi^{-1} Vr.
\]

Now

\[
q = Sq + Vq = (1 - \phi^{-1} \epsilon) Sq + \phi^{-1} Vr,
\]
(8.) Such having been the analysis of the problem, the synthesis, by which an \textit{a posteriori proof} of the correctness of the resulting formula is to be given, may be simplified by using the scalar value XXIX. of $f(m - \psi \varepsilon)$; and it is sufficient to show (denoting $Vr$ by $\omega$), that for every vector $\omega$ the following equation holds good, with the same form XII. of $f$:

$$XXXVII. \ldots f(\psi \omega - S \varepsilon \psi \omega) - f'V \varepsilon \phi' \phi V \varepsilon \omega = (m - S \varepsilon \psi \omega) \cdot \omega.$$

(9.) Accordingly, that form of $f$ gives, with the help of the principle employed in XXXV.,

$$XXXVIII. \begin{cases} 
\phi \psi \omega = \phi(S \varepsilon \psi \omega + m \omega), \\
-f \varepsilon \psi \omega = -(e + \varepsilon) \varepsilon \psi \omega, \\
-f \varepsilon \phi' \phi V \varepsilon \omega = - \phi \varepsilon \phi' \phi V \varepsilon \omega = V(V \varepsilon \omega \cdot \psi' \varepsilon) = e \varepsilon \psi \omega - \omega \varepsilon \psi \varepsilon,
\end{cases}$$

because $S \omega \psi \varepsilon' = S \varepsilon \psi \omega$, &c.; and thus the equation XXXVI. is proved, by actually operating with $f$.

(10.) As an example, if we take the particular form,

$$XXXIX. \ldots r = f q = pq + q p,$$

in which

$$XL. \ldots p = a + a = a \text{ given quaternion},$$

we have then,

$$XLI. \ldots f 1 = f' 1 = 2 p, \quad e = 2 a, \quad \varepsilon = \varepsilon' = 2 a, \quad \phi \rho = 2 a \rho;$$

whence by the theory of linear and vector functions,

$$XLII. \ldots \phi' \rho = 2 a \rho, \quad \psi \rho = 4 a^2 \rho, \quad m = 8 a^3,$$

and therefore,

$$XLIII. \ldots \psi \varepsilon = 8 a^2 a, \quad m - \psi \varepsilon = 8 a^2 (a - a), \quad n = 16 a^2 (a^2 - a^2);$$

so that, dividing by $8a$, the formula XXXVI. becomes,

$$XLIV. \ldots 2a(a^2 - a^2)q = a(a - a)Sr + a^2 Vr - aS \cdot a Vr - aV \cdot aVr,$$

or

$$XLV. \ldots 2a(a + a)q = aSr + (a + a) Vr - Sar,$$

so on substituting the value of $S \phi$ just found,

$$q = \frac{(1 - \phi^{-1} \varepsilon) (Sr - S \varepsilon \phi^{-1} Vr)}{\varepsilon - S \varepsilon \phi^{-1} \varepsilon} + \phi^{-1} Vr.$$

It only remains to replace $\phi^{-1}$ by $w^{-1} \psi$ in order to recover XXXVI.}
or

\[ \text{XLVI.} \quad 2pqSp = S \cdot rKp + pVr = rSp + V(Vp \cdot Vr), \]

or

\[ \text{XLVII.} \quad 4pqSp = 2rSp + (pr - rp) = pr + rKp; \]

or finally,

\[ \text{XLVIII.} \quad q = f^{-1}r = \frac{r + p^{-1}rKp}{4Sp} = \frac{r + Kp \cdot rp^{-1}}{4Sp} \]

Accordingly,

\[ \text{XLIX.} \quad (pr + rKp) + (rp + Kp \cdot r) = 2r(p + Kp) = 4rSp. \]

(11.) In so simple an example as the last, we may with advantage avail ourselves of special methods; for instance (comp. 346), we may use that which was employed in 332, (6.), to differentiate the square root of a quaternion, and which conducted there more rapidly to a formula (332, XIX.) agreeing with the recent XLVIII.

(12.) We might also have observed, in the same case XXXIX., that

\[ \text{L.} \quad pr - rp = p^2q - qp^2 = 2V(V(p^2) \cdot Vq) = 4Sp \cdot V(Vp \cdot Vq) = 2Sp \cdot (pq - qp); \]

whence \( pq - qp \), and therefore \( pq \) and \( qp \), can be at once deduced, with the same resulting value for \( q \), or for \( f^{-1}r \), as before; and generally it is possible to differentiate, on a similar plan, the \( n^{th} \) root of a quaternion.

365. We shall conclude this Section on Linear Functions, of the kinds above considered, by proving the general existence of a Symbolic and Biquadratic Equation, of the form,

\[ \text{I.} \quad 0 = n - n'f + n''f^2 - n'''f^3 + f^4, \]

which is thus satisfied by the Symbol \( f \) of Linear and Quaternion Operation on a Quaternion, as the Symbolic and Cubic Equation,

\[ \text{I'.} \quad 0 = m - m'\phi + m''\phi^2 - m'''\phi^3, \]

was satisfied by the symbol \( \phi \) of linear and vector operation on a vector; the four coefficients, \( n, n', n'', n''' \), being four scalar constants, deduced from the function \( f \) in this extended or quaternion theory, as the three scalar coefficients \( m, m', m'' \) were constants deduced from \( \phi \), in the former or vector theory. And at the same time we shall see that there exists a System of Three Auxiliary Functions, \( F, G, H \), of the Linear and Quaternion kind, analogous to the two vector functions, \( \psi \) and \( \chi \), which have been so useful in the foregoing theory of vectors,
and like them connected with each other, and with the given quaternion function \( f \), by several simple and useful relations.*

(1.) The formula of solution, 364, XXXVII., of the linear and quaternion equation \( fq = r \), being denoted briefly as follows,

\[
\text{II. } \ldots \quad nq = ny^{-1}r = Fr,
\]

so that (comp. 348, III.') we may write, briefly and symbolically,

\[
\text{III. } \ldots \quad fF = Ef = n,
\]

it may next be proposed to examine the changes which the scalar \( n \) and the function \( Fr \) undergo, when \( Fr \) is changed to \( Fr + cr \), or \( f \) to \( f + c \), where \( c \) is any scalar constant; that is, by 364, XII., when \( e \) is changed to \( e + c \), and \( \phi \) to \( \phi + c \); \( \phi', \psi \), and \( m \) being at the same time changed, according to the laws of the earlier theory.

(2.) Writing, then,

\[
\text{IV. } \ldots \quad f'e = f + c, \quad e_c = e + c, \quad \phi_c = \phi + c, \quad \phi'_c = \phi' + c,
\]

and

\[
\text{V. } \ldots \quad \psi_c = \psi + c\chi + c^2, \quad m_c = m + m'c + m''c^2 + c^3,
\]

we may represent the new form of the equation 364, XXXVII. as follows:

\[
\text{VI. } \ldots \quad n_c f^{-1}r = F_c r, \quad \text{or} \quad \text{VII. } \ldots \quad f_c F_c = n_c;
\]

where

\[
\text{VIII. } \quad F_c r = (m_c - \psi_c e)S_r + e_c \psi_c V_r - S e' \psi_c V_r - V e' \phi'_c V e V r,
\]

* [That a linear quaternion function satisfies a symbolic quartic may be established as follows:

On inquiry whether it is possible to determine a scalar \( c \) and a quaternion \( q \) so that \( fq + cq = 0 \), the two equations

\[
(e + c)S q + S e' V q = 0, \quad \text{and} \quad e S q + (\phi + c)V q = 0,
\]

are found by equating to zero the scalar and vector parts. Hence from the second equation \( V q = - (\phi + c)^{-1} e S q \), and, on substitution in the first, it appears that \( c \) must satisfy the relation \( e + c - S e' (\phi + c)^{-1} e = 0 \). It may be shown without difficulty, as in the text, that this leads to a quartic equation in \( e \).

If \( c_n \) is any root of this quartic, and if \( a_n = - (\phi + c_n)^{-1} e \), the quaternion \( q_n = 1 + a_n \) will satisfy \( (f + c_n)q_n = 0 \). Corresponding to the four values of \( c_n \) are four quaternions, and in terms of these any arbitrary quaternion may in general be expressed.

Assuming

\[
q = x_1 q_1 + x_2 q_2 + x_3 q_3 + x_4 q_4,
\]

and deriving from these the equations

\[
V q = \Sigma x_n a_n, \quad \text{and} \quad S q = \Sigma x_n,
\]

and again from these the equation

\[
V q - a_1 S q = x_2(a_2 - a_1) + x_3(a_3 - a_1) + x_4(a_4 - a_1),
\]

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and

\[ \mathbf{X} \ldots F'_e = F + cG + c^2H + c^3, \]

and

\[ \mathbf{XI} \ldots n_e = n + n'e + n''e^2 + n'''e^3; \]

where \( F, G, H \) are three functional symbols such that

\[
\begin{align*}
F_r &= (m - \phi \epsilon)S_r + e\phi V_r - S\epsilon\psi V_r - V\epsilon'\phi'\psi' V_r - V\epsilon'\psi' V_r; \\
G_r &= (m' - \chi \epsilon)S_r + (e\chi + \psi) V_r - S\epsilon\chi V_r - V\epsilon'\psi V_r; \\
H_r &= (m'' - \epsilon)S_r + (e + \chi) V_r - S\epsilon'\chi V_r;
\end{align*}
\]

and \( n, n', n'', n''' \) are four scalar constants, namely,

\[
\begin{align*}
n &= em - S\epsilon\psi\epsilon \quad \text{(as in 364, XXIX.)}; \\
n' &= m + em' - S\epsilon\chi\epsilon; \\
n'' &= m' + em'' - S\epsilon'\chi; \\
n''' &= m'' + e.
\end{align*}
\]

The scalar \( x_4 \) is given, on operating by \( SV(a_2 - a_1) (a_3 - a_1) \), by

\[
x_4S(a_2a_3a_1 - a_1a_2 + a_3a_1 - a_1a_3) = SV(a_2a_1 + a_3a_1 + a_1a_2) - S_2S_1a_2a_3,
\]

and the values of the other scalars may be written down from symmetry (comp. p. 48). In general \( x_1, x_2, x_3, \) and \( x_4 \) are uniquely determinate provided the four vectors \( a_n \) do not terminate on a common plane. As \( e \) varies, the curve traced out by \( p = - (\phi + c)^{-1}e \) is a twisted cubic and upon this curve the vectors \( a_n \) terminate, and consequently their four extremities do not lie on a plane.

To verify that \( p = - (\phi + c)^{-1}e \) is a twisted cubic, the equation

\[
S\lambda(\phi + c)^{-1}e = 1, \quad \text{or} \quad S\lambda\psi\epsilon = m,
\]

is found determining the values of \( e \) for the points in which the curve cuts the arbitrary plane \( S\lambda p + 1 = 0 \). As this is a cubic equation in \( e \), the curve cuts the plane in but three points.

In general then

\[
q = x_1q_1 + x_2q_2 + x_3q_3 + x_4q_4.
\]

Operating on this by \( f + c_1 \), and

\[
(f + c_1)q = x_2(c_1 - c_0)q_2 + x_3(c_1 - c_0)q_3 + x_4(c_1 - c_0)q_4,
\]

from which \( q_1 \) has disappeared. Similarly operating by \( f + c_2 \) destroys the term in \( q_2 \), and finally

\[
(f + c_4) (f + c_3) (f + c_2) (f + c_1) q = 0,
\]

which is equivalent to 1.]
(4.) Developing then the symbolical equation VII., with the help of X.
and XI., and comparing powers of c, we obtain these new symbolical equa-
tions (comp. 350, XVI. XXI. XXIII.):

\[
\begin{align*}
H &= n''' - f; \\
G &= n'' - fH = n'' - n'''f + f^2; \\
F &= n' - fG = n' - n''f + n'''f^2 - f^3;
\end{align*}
\]

and finally,

\[
xV. \ldots n = Ff = n'f - n''f^2 + n'''f^3 - f^4,
\]

which is only another way of writing the symbolic and biquadratic equation I.

(5.) Other functional relations exist, between these various symbols of
operation, which we cannot here delay to develop; but we may remark that,
as in the theory of linear and vector functions, these usually introduce a
mixture of functions with their conjugates (comp. 347, XI., &c.).

(6.) This seems however to be a proper place for observing, that if we
write, as temporary notations, for any four quaternions, p, q, r, s, the equations,

\[
\begin{align*}
\text{XVI.} \ldots [pq] &= pq - qp; \quad \text{XVII.} \ldots (pqr) = S.p[qr] \\
\text{XVIII.} \ldots [pqr] &= (pqr) + [rq]S_p + [pr]S_q + [qp]S_r; \\
\text{XIX.} \ldots (pqrs) &= S.p[qrs],
\end{align*}
\]

so that [pq] is a vector, (pqr) and (pqrs) are scalars, and [pqr] is a qua-
ternion, we shall have, in the first place, the relations:

\[
\begin{align*}
\text{XX.} \ldots [pq] &= -[qp], \quad [pp] = 0; \\
\text{XXI.} \ldots (pqr) &= - (qpr) = (grp) = \&c., \quad (ppr) = 0; \\
\text{XXII.} \ldots [pqr] &= - [qpr] = [grp] = \&c., \quad [ppr] = 0;
\end{align*}
\]

and

\[
\begin{align*}
\text{XXIII.} \ldots (pqrs) &= - (qprs) = (qrs) = - (qsp) = \&c., \quad (pprs) = 0.
\end{align*}
\]

(7.) In the next place, if \( t \) be any fifth quaternion, the quaternion equation,

\[
\text{XXIV.} \ldots 0 = p(qrst) + q(rstp) + r(stpq) + s(tpqr) + t(pqrs),
\]
which may also be thus written,

\[ q(prst) = p(qrst) + r(pqst) + s(prqt) + t(prsq), \]

and which is analogous to the vector equation,

\[ 0 = aS\beta S\delta - \beta S\gamma S\alpha + \gamma S\delta S\alpha - S\alpha S\beta S\gamma, \]

or to the continually† occurring transformation (comp. 294, XIV.),

\[ \delta S\alpha S\beta S\gamma = aS\delta S\beta S\gamma + \beta S\alpha S\delta S\gamma + \gamma S\alpha S\beta S\delta, \]

is satisfied generally, because it is satisfied for the four distinct suppositions,

\[ q = p, \quad q = r, \quad q = s, \quad q = t. \]

(8.) In the third place, we have this other general quaternion equation,

\[ q(prst) = [rst]Spq - [stp]Srq + [tpr]Ssq - [prs]Stq, \]

which is analogous to this other‡ useful vector formula (comp. 294, XV.),

\[ \delta S\alpha S\beta S\gamma = V\beta S\gamma S\delta + V\gamma S\alpha S\delta + V\alpha S\beta S\gamma; \]

because the equation XXIX. gives true results, when it is operated on by the four distinct symbols (comp. 312),

\[ XXXI. \quad S.p, \quad S.r, \quad S.s, \quad S.t. \]

* [Or again as a determinant

\[
\begin{vmatrix}
 p & q & r & s & t \\
 x_1 & x_2 & x_3 & x_4 & x_5 \\
 y_1 & y_2 & y_3 & y_4 & y_5 \\
 z_1 & z_2 & z_3 & z_4 & z_5 \\
 w_1 & w_2 & w_3 & w_4 & w_5 \\
\end{vmatrix} = 0,
\]

if \[ p = w_1 + ix_1 + jy_1 + kz_1, \&c.]

† The equations XXVII. and XXX., which had been proved under slightly different forms in the sub-articles to 294, have been in fact freely employed as transformations in the course of the present Chapter, and are supposed to be familiar to the student. Compare the Note to page 485.

‡ Compare the Note immediately preceding.
(9.) Assuming then any four quaternions, \( p, r, s, t \), which are not connected by the relation,

\[ (prst) = 0, \]

and deducing from them four others, \( p', r', s', t' \), by the equations,

\[
\begin{align*}
\{ p'(prst) &= f[rst], \quad r'(prst) = -f[stp], \\
\quad s'(prst) &= f[tpr], \quad t'(prst) = -f[prs],
\end{align*}
\]

in which \( f \) is still supposed to be a symbol of linear and quaternion operation on a quaternion, the formula XXIX. allows us to write generally, as an expression for the function \( fq \), which may here be denoted by \( q' \) (because \( r \) is now otherwise used):

\[ q' = fq = p'Spq + r'Srq + s'Ssq + t'Stg; \]

and its sixteen scalar constants (comp. 364, (2.)) are now those which are involved in its four quaternion constants, \( p', r', s', t' \).

(10.) Operating on this last equation with the four symbols,

\[
\begin{align*}
S[r's't'], \quad S[s't'p'], \quad S[t'p'r'], \quad S[p'r's'],
\end{align*}
\]

we obtain the four following results:

\[
\begin{align*}
(g'q's't') &= (p'r's't')Spq; \quad (g's't'p') = (r's't'p')Srq; \\
(g't'p'r') &= (s't'p'r')Ssq; \quad (g'p'r's') = (t'p'r's')Stg;
\end{align*}
\]

and when the values thus found for the four scalars,

\[ Spq, \quad Srq, \quad Ssq, \quad Stg, \]

are substituted in the formula XXIX., we have the following new formula of quaternion inversion:

\[ (p'r's't') (prst)q = (p'r's't') (prst)f^{-1}q' \]

\[ = [rst] (g'q's't') + [spt] (g's't'p') + [tpr] (g't'p'r') + [prs] (g'p'r's'); \]

which shows, in a new way, how to resolve a linear equation in quaternions, when put under what we may call (comp. 347, (1.)) the Standard Quadrinomial Form, XXXIV.
(11.) Accordingly, if we operate on the formula XXXVIII. with \( f \), attending to the equations XXXIII., and dividing by \((prst)\), we get this new equation

\[
XXXIX. \ldots (p'r's't')(f)q = p'(q'r's't') - r'(q's't'p' + s'(q't'p'r') - t'(q'p'r's');
\]

whence

\[
fq = q', \text{ by XXV.}
\]

(12.) It has been remarked (9.), that \( p, r, s, t \), in recent formulæ, may be any four quaternions, which do not satisfy the equation XXXII.; we may therefore assume,

\[
XL. \ldots p = 1, \quad r = i, \quad s = j, \quad t = k,
\]

with the laws of 182, &c., for the symbols \( i, j, k \), because those laws give here,

\[
XL\text{I.} \ldots (ijk) = -2;
\]

and then it will be found that the equations XXXIII. give simply,

\[
XL\text{II.} \ldots p' = f1, \quad r' = -fi, \quad s' = -fj, \quad t' = -fk;
\]

so that the standard quadrinomial form XXXIV. becomes, with this selection of \( prst \),

\[
XL\text{III.} \ldots fq = f1. Sq - fi. Siq - fj. Sjq - fk. Skq,
\]

and admits of an immediate verification, because any quaternion, \( q \), may be expressed (comp. 221) by the quadrinomial,

\[
XL\text{IV.} \ldots q = Sq - isiq - jsiq - kskq.
\]

(13.) Conversely, if we set out with the expression,

\[
XL\text{V.} \ldots q = w + ix + jy + kz, \quad \text{221, III.,}
\]

which gives,

\[
XL\text{VI.} \ldots fq = wfi + xfi + yfi + zf'k,
\]

or briefly,

\[
XL\text{VII.} \ldots e = aw + bx + cy + dz,
\]
the letters abode being here used to denote five known quaternions, while wxyz are four sought scalars, the problem of quaternion inversion comes to be that of the separate determination (comp. 312) of these four scalars, so as to satisfy the one equation XLVII.; and it is resolved (comp. XXV.) by the system of the four following formulæ:

\[
\begin{align*}
    XLVIII. \quad & w(abcd) = (abcd) ; \\
    & x(abcd) = (aecd) ; \\
    & y(abcd) = (abed) ; \\
    & z(abcd) = (abce) ;
\end{align*}
\]

the notations (6.) being retained.

(14.) Finally it may be shown, as follows, that the biquadratic equation I., for linear functions of quaternions, includes* the cubic I', or 350, I., for vectors. Suppose, for this purpose, that the linear and quaternion function, \( fq \), reduces itself to the last term of the general expression 364, XII., or becomes,

\[
XLIX. \quad fg = \phi Vq, \quad \text{so that} \quad L. \quad e = 0, \quad \epsilon = \epsilon' = 0, \quad f1 = f'1 = 0;
\]

the coefficients \( n, n', n'', n''' \) take then, by XIII., the values,

\[
LI. \quad n = 0, \quad n' = m, \quad n'' = m', \quad n''' = m'';
\]

and the biquadratic I. becomes,

\[
LII. \quad 0 = (-m + m'f - m''f^2 + f^3)f.
\]

But \( fg \) is now a vector, by XLIX., and it may be any vector, \( \rho \); also the operation \( f \) is now equivalent to that denoted by \( \phi \), when the subject of the

* In like manner it may be said, that the cubic equation includes a quadratic one, when we confine ourselves to the consideration of vectors in one plane; for which case \( m = 0 \), and also \( \phi \rho = 0 \), if \( \rho \) be a line in the given plane: for we have then \( \phi X = m' - \psi = m' \), or

\[
\phi^2 - m''\phi + m' = 0,
\]

with this understanding as to the operand. In fact, the cubic gives here (because \( m = 0 \)),

\[
(\phi^2 - m''\phi + m')\phi \rho = 0;
\]

and therefore

\[
(\phi^2 - m''\phi + m')\sigma = 0;
\]

if \( \sigma \) be already the result of an operation with \( \phi \), on any vector \( \rho \); that is if it be, as above supposed, a line in the given plane.
operation is a vector; we may therefore, in the case here considered, write this last equation LII. under the form,

\[ LIII. \ldots 0 = (-m + m'\phi - m''\phi^2 + \phi^3)\rho, \]

which agrees with 351, I., and reproduces the symbolical cubic, when the symbol of the operand (\(\rho\)) is suppressed.*

*A few additional remarks may be made concerning the solutions of \(Vq^{-1}fq = 0\), and of \(Vq^{-1}f'q = 0\), and the relations connecting them.

It is easy to see, in various ways, that \(f\) and its conjugate \(f'\) satisfy the same symbolical biquadratic. If, for instance, \(q\) and \(q'\) are any arbitrary quaternions

\[ Sq(f'^4 - n''f'^3 + n''f'^2 - n'f' + n)q' = Sq'(f^4 - n''f^3 + n''f^2 - n'f + n)q = 0 \]

by I., and therefore as the quaternions are arbitrary,

\[ (f'^4 - n''f'^3 + n''f'^2 - n'f' + n)q' = 0. \]

Again, the same property follows from the equation

\[ e + e = S\epsilon((\phi + e)^{-1}e = S\epsilon(\phi' + e)^{-1}e'. \]

(See the Note to page 561.)

Now if, as in the Note just cited, \(q_1, q_2, q_3,\) and \(q_4\) are the solutions of \(Vq^{-1}fq = 0\), and \(q'_1, q'_2, q'_3,\) and \(q'_4\) are those of \(Vq^{-1}f'q = 0,\)

\[ cSqq'q'2 = -Sqfq'q'2 = -Sqq'f'q'2 = c_2Sqq'q'2, \]

as the roots \(e\) are the same for \(f\) and for its conjugate \(f'\). Hence if \(e_1\) is not equal to \(e_3,\) it is necessary to have \(Sqq'q'2 = 0\); and, in general, \(Sqq'q' = 0,\) where \(n\) is different from \(n'.\)

If then \(q_1 = 1 + a_1,\) and \(q'_1 = 1 + a'_1, \&c.,\)

\[ Sq_1q'_2 = 1 + a_1a'_2 = 0. \]

Interpreted geometrically this property shows that if vectors are drawn through the origin equal to \(a_1, a_2, a_3, a_4,\) and to \(a'_1, a'_2, a'_3,\) and \(a'_4,\) they will terminate on the polar plane of \(a_1\) with respect to the unit sphere \(p^2 + 1 = 0.\) In other words, the tetrahedron determined by the extremities of \(a_1, a_2, a_3,\) and \(a_4,\) is the polar reciprocal of that determined by \(a'_1, a'_2, a'_3,\) and \(a'_4,\) in the particular case in which \(f\) is self-conjugate, \(1 + Saa_2 = 0,\) and the tetrahedron is self-reciprocal with respect to the unit sphere; or, without reference to a sphere, the tetrahedron may be said to be orthocentric as the perpendiculars \((- a_1^{-1}, \&c.)\) from the origin on the faces pass through the corresponding vertices.

Hence, any quaternion \(q\) may be expressed in the form (compare again the note to page 561)

\[ q = q_1 \frac{Sqq'q_1}{Sqq'q_1} + q_2 \frac{Sqq'q_2}{Sqq'q_2} + q_3 \frac{Sqq'q_3}{Sqq'q_3} + q_4 \frac{Sqq'q_4}{Sqq'q_4}, \]

and

\[ fq = -c_1q_1 \frac{Sqq'q_1}{Sqq'q_1} - c_2q_2 \frac{Sqq'q_2}{Sqq'q_2} - c_3q_3 \frac{Sqq'q_3}{Sqq'q_3} - c_4q_4 \frac{Sqq'q_4}{Sqq'q_4}, \]

may be regarded as a canonical form of a function \(f.\)

It is easy to see from the properties of the reciprocal tetrahedra that the vector

\[ a'_1 = -\frac{V(a_2a_4 + a_4a_2 + a_3a_5)}{Saa_3a_5}, \]

being the negative of the reciprocal of the vector perpendicular on the plane through the extremities of \(a'_2, a'_3,\) and \(a'_4,\)]
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