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The Twenty-Seven Lines upon the Cubic Surface
THE TWENTY-SEVEN LINES
UPON
THE CUBIC SURFACE

by

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NOTE

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A. H.

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## CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>HISTORICAL SUMMARY</strong></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td><strong>INTRODUCTION</strong></td>
<td>8</td>
</tr>
<tr>
<td>I</td>
<td><strong>Preliminary Theorems</strong></td>
<td>10</td>
</tr>
<tr>
<td>II</td>
<td><strong>The Double Six Configuration. Auxiliary Theorems</strong></td>
<td>13</td>
</tr>
<tr>
<td>III</td>
<td><strong>The Trihedral Pair Configuration</strong></td>
<td>26</td>
</tr>
<tr>
<td>IV</td>
<td><strong>Analytical Investigation of the Twenty-seven Lines and Forty-five Triple Tangent Planes for the General Equation of the Cubic Surface</strong></td>
<td>43</td>
</tr>
<tr>
<td>V</td>
<td><strong>The Construction of a Model of a Double Six</strong></td>
<td>54</td>
</tr>
<tr>
<td>VI</td>
<td><strong>The Construction of the Configurations of the Straight Lines upon the Twenty-one Types of the Cubic Surface</strong></td>
<td>58</td>
</tr>
<tr>
<td>VII</td>
<td><strong>On some Configurations associated with the Configurations of the Lines upon the Cubic Surface</strong></td>
<td>83</td>
</tr>
<tr>
<td></td>
<td><strong>BIBLIOGRAPHY</strong></td>
<td>96</td>
</tr>
<tr>
<td></td>
<td><strong>INTERSECTION TABLE</strong></td>
<td>To face p. 24</td>
</tr>
<tr>
<td></td>
<td><strong>PLATES 1—13</strong></td>
<td>At End</td>
</tr>
</tbody>
</table>
HISTORICAL SUMMARY

While it is doubtless true that the classification of cubic surfaces is complete, the number of papers dealing with these surfaces which continue to appear from year to year furnish abundant proof of the fact that they still possess much the same fascination as they did in the days of the discovery of the twenty-seven lines upon the cubic surface. The literature of the subject is very extensive. In a bibliography on curves and surfaces compiled by J. E. Hill, of Columbia University, New York, the section on cubic surfaces contained two hundred and five titles*. The Royal Society of London Catalogue of Scientific Papers, 1800–1900, volume for Pure Mathematics (1908), contains very many more.

The first paper that deals specifically with the cubic surface was by L. Mossbrugger, “Untersuchungen über die geometrische Bedeutung der constanten Coeﬃcienten in den allgemeinen Gleichungen der Flächen des zweiten und dritten Grades,” which appeared in the first volume of the Archiv der Mathematik und Physik, 1841.

The theory of straight lines upon a cubic surface was first studied in a correspondence between the British mathematicians Salmon and Cayley; and the results were published, Camb. and Dublin Math. Journal, Vol. iv. (1849), pp. 118–132 (Cayley), pp. 252–260 (Salmon). The observation that a deﬁnite number of straight lines must lie on the surface is initially due to Cayley, whereas the determination of that number was first made by Salmon†.

The basis for a purely geometric theory of cubic surfaces was laid by Steiner‡ in a short but extremely fruitful and suggestive memoir. This paper contained many theorems, given either wholly without

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‡ “Ueber die Flächen dritten Grades,” read to the Berlin Academy, 31st January, 1856; Crelle’s Journ., Vol. liii.
HISTORICAL SUMMARY

proof, or with at most the barest indication of the method of derivation—a habit of "ce célèbre sphinx;" as he has been styled by Cremona.

On account of what Cayley described as the "complicated and many-sided symmetry" among the relations between the twenty-seven lines upon the cubic surface, great difficulty was at first experienced in obtaining any adequate conception of the complete configuration. The notation first given by Cayley was obtained by starting from some arrangement that was not unique, but one of a system of several like arrangements; but it was so complicated as scarcely to be considered as at all putting in evidence the relations of the lines and triple tangent planes. Hart gave a very elegant and symmetrical notation for the lines and planes, an account of which is to be found in the original paper by Salmon*, who also gave a notation of limited usefulness. Schläfli† it was who invented the notation which may be called epoch-making—that of the double six‡; and this notation has remained unimproved upon up to the present time. This notation is one out of a possible thirty-six of like character among the twenty-seven lines. More recently, Taylor§ has devised a notation for the lines independent of any particular initial choice; but this cannot be regarded as an improvement upon the notation devised by Schläfli.

The foundations for subsequent analytic investigations concerning the twenty-seven lines were laid, as shown, by Cayley and Salmon. Indeed Sylvester‖ once remarked, in his characteristically florid style: "Surely with as good reason as had Archimedes to have the cylinder, cone and sphere engraved on his tombstone might our distinguished countrymen leave testamentary directions for the cubic eikosioheptagram to be engraved on theirs."

The first significant papers on cubic surfaces from the synthetic standpoint, following Steiner's memoir above mentioned, were by Cremona and Rudolf Sturm. These were two of the four papers submitted in competition for the prize offered by Steiner through the Royal Academy of Sciences of Berlin in 1864, which was divided between Cremona and Sturm on Leibniz Day, 1866. The beauty and simplicity of many of the methods employed in these papers eminently justified Steiner's original remark: "Es ist daraus zu sehen, dass diese

‡ For the history of the double six theorem, see infra, § 6.
Flächen fortan fast eben so leicht und einlässlich zu behandeln sind, als bisher die Fläche zweiten Grades.” Cremona’s “Mémoire de géométrie pure sur les surfaces du troisième ordre” is found in Crelle’s Journal*. Sturm’s memoir was subsequently expanded into a treatise†.

Schläfli (l.c.) first considered a division of the general surface of the third order into species, in regard to the reality of the twenty-seven lines. He contented himself with a mere survey of the problem. This was in 1858. In 1862, F. August‡ published a more elaborate investigation of the subject. In 1863 appeared a valuable memoir by Schlafli§ treating the subject in great detail. He makes there, as the title of the paper indicates, a division of the surface into types—depending upon the nature of the singularities. His classification was adopted by Cayley in his “Memoir on Cubic Surfaces”‖.

If Cayley and Salmon had wished to follow Sylvester’s advice and to insert a clause in their wills, directing that a figure of the eikosiheptagram be engraved upon their monuments, they would have had no certainty of the correct fulfilment of their directions until the year 1869, when Christian Wiener made a model of a cubic surface showing twenty-seven real lines lying upon it¶. This achievement of Wiener’s, Sylvester** once remarked, is one of the discoveries “which must for ever make 1869 stand out in the Fasti of Science.” In his address to the Mathematical and Physical Section of the British Association, H. J. S. Smith†† states that “a model showing the distribution in space of the lines themselves, unaccompanied by the surface on which they lie, has been constructed by Professor Henrici”; but Henrici does not seem to have published any paper on the subject.

In 1872, Clebsch and Klein, at Göttingen University, considered the question as to the shape of surfaces of the third order. Clebsch

* Vol. lxxviii. (1868), pp. 1–133.
† Synthetische Untersuchungen über Flächen dritter Ordnung, B. G. Teubner, Leipzig, 1867.
‡ Disquisitiones de superficiebus tertii ordinis, Dissert. inaup. Berolini, 1862.
first constructed a model of the diagonal surface with twenty-seven real lines. "Instigated by this investigation of Clebsch," says Klein, "I turned to the general problem of determining all possible forms of cubic surfaces. I established the fact that by the principle of continuity all forms of real surfaces of the third order can be derived from the particular surface having four conical points*." Klein’s method established completeness of enumeration—the consideration of fundamental importance. Klein exhibited a complete set of models of cubic surfaces at the World’s Exposition in Chicago in 1894, including Clebsch’s symmetrical model of the diagonal surface and Klein’s model of the cubic surface having four real conical points. Models of the typical cases of all the principal forms of cubic surfaces have been constructed by Rodenberg† for Brill’s collection; and these plaster models may now be purchased. Blythe has constructed models of certain types of cubic surfaces, and illustrated in some detail the character of the changes that take place under certain conditions‡. The list of those who have written on the mechanical construction of the configurations of the lines upon a cubic surface and the general collocation of the lines upon the surface includes the names of Cayley, Frost, Zeuthen, De Vries, Taylor and Blythe§.

The configuration of the twenty-seven lines is not only of great interest per se, but also because of its close association with, and relation to, other remarkable configurations. It was also in the year 1869—the year over which Sylvester waxed dithyrambic—that Geiser‖ showed that the projection of a cubic surface from a point upon it, on a plane of projection parallel to the tangent plane at that point, is a quartic curve; and that every quartic curve can be generated in this way. He showed the mutual interdependence of the configurations of the twenty-eight bitangents to a plane quartic curve and the twenty-seven lines upon a cubic surface, and the method of derivation of either configuration from the other. By making use of

* Lectures on Mathematics, Evanston Colloquium, 1894, Macmillan and Co.


§ Cf. infra, §§ 18, 21.

Geiser's results, Zeuthen* obtained a new demonstration of the theorems of Schläfi† upon the reality of the straight lines and triple tangent planes of a cubic surface. He proves the reality of all the twenty-eight bitangents to the quartic in the case when the curve consists of four separate closed portions. In the next year, he points out the important connection between Klein's researches on cubic surfaces (i.e.) and his own researches on plane quartic curves. If a surface with four conic nodes be chosen, the resulting quartic has four double points. By the principle of continuity, the four ovals of the quartic are readily obtained; and this, as Zeuthen showed, corresponds to Klein's derivation of the diagonal surface from the cubic surface having four conic nodes‡. Timerding has shown that it is feasible to derive the properties of the plane quartic curve and its bitangents from the known properties of the cubic surface and its straight lines, and *vice versa§.

In 1877, Cremona‖ was first to show that the Pascalian configuration might be derived from the configuration of the twenty-one lines upon the surface of the third degree with one conical point (Species II in Cayley's enumeration) by projection from the conical point. Mention should also be made here of the elaborate paper of Bertini¶.


that there are only two ways in which a Steiner system of bitangents arises from lines of the cubic surface, and the new theorem that the six transversals from any point of the cubic surface, one to each of the opposite line pairs of a double six, lie on a quadric cone.

The theory of varieties of the third order, that is to say, curved geometric forms of three dimensions contained in a space of four dimensions, has been the subject of a profound memoir by Corrado Segre*. The depth and fecundity of this paper is evinced by the fact that a large proportion of the propositions upon the plane quartic and its bitangents, Pascal's theorem, the cubic surface and its twenty-seven lines, Kummer's surface and its configuration of sixteen singular points and planes, and on the connection between these figures, are derivable from propositions relating to Segre's cubic variety, and the figure of six points or spaces from which it springs. Other investigators into the properties of this beautiful and important locus in space of four dimensions and some of its consequences are Castelnuovo and Richmond†.

The problem of the twenty-seven lines is full of interest from the group theoretic standpoint. In 1869, Camille Jordan‡ first proved that the group of the problem of the trisection of hyperelliptic functions of the first order is isomorphic with the group of the equation of the twenty-seventh degree, on which the twenty-seven lines of the general surface of the third degree depend. In 1887, Klein§ sketched the effective reduction of the one problem to the other. In 1887-1890, Maschke in a series of papers set up the complete form-system of a quaternary group of 51840 substitutions||. In 1893, Burkhardt¶, on the basis of Klein's paper above mentioned, these papers of Maschke, and a paper by Witting**, carried out the work sketched by Klein—the reduction of the one problem to the other.

Since Jordan's paper appeared in 1869, many writers have studied the Galois group of the equation of the twenty-seven lines. Dickson* has led in this investigation, publishing a number of papers on the subject. Among those who have treated the problem of the twenty-seven lines as a problem in substitution groups or Galoisian groups may be mentioned Kühnen†, Weber‡, Pascale§, and Kasner‖. This last paper is in close contact with the investigations of Moore¶ and Slaught** on the cross-ratio group of Cremona transformations.

† Ueber die Galois'sche Gruppe der Gleichung 27. Grades, von welcher die Geraden auf der allgemeinen Fläche dritter Ordnung abhängen, Diss. Marburg. 1888.
INTRODUCTION

The problem of the twenty-seven lines upon the cubic surface is of such scope and extent, and is allied to so many other problems of importance, that to give a résumé of all that has been done upon the subject would enlarge the present memoir into an extensive book. It has not proved feasible to attempt to cover even the geometrical phases of the problem, in their extension in particular to the cognate problem of the forty-five triple tangent planes, although the two subjects go hand in hand. In this memoir, however, is given a general survey of the problem of the twenty-seven lines, from the geometric standpoint, with special attention to salient features: the concept of trihedral pairs, the configuration of the double six; the solution of the problem of constructing models of the double six configuration and of the configurations of the straight lines upon the twenty-one types of the cubic surface, the derivation of the Pascalian configuration from that of the lines upon the cubic surface with one conical point, and certain allied problems. Certain of the results have been published, or presented before the American Mathematical Society, the North Carolina Academy of Science, and the Elisha Mitchell Scientific Society.

In §§ 1–4 are given certain preliminary theorems concerning the existence and number of the twenty-seven lines and forty-five planes for the general cubic surface, and remarks upon the first notation employed. In §§ 5–7 are given an account of Schläfli’s notation, a history of the double six theorem and an analytic proof; in § 8 follow certain interesting results on the anharmonic ratios of the configurations. In § 9 appear two conditions that five lines lie upon the cubic surface, and in § 10 is the description of the formation, and the tabulation, of the thirty-six double sixes. In § 11 occur certain auxiliary theorems for special features of the general configuration of the twenty-seven lines.
In § 12 are given the definition and number of trihedral pairs, and in § 13 the actual formation of the tables of the 120 forms. In § 14 these are grouped together in such a way (sets of three) as to determine in forty ways all the twenty-seven lines. In § 15 is found a formulation of conclusions.

In § 16 is given the discussion of a particular form of the general equation of the cubic surface, together with the determination and tabulation of the forty-five triple tangent planes. In § 17 is found the formulation of the analytic expression of geometrical results.

In §§ 18–19 the methods for the construction of a model of a double six are discussed; and a practical method is there given in detail.

In §§ 20–44 the general problem of constructing thread or wire models of the configurations of the straight lines upon all twenty-one types of the cubic surface is fully considered. The complete data for constructing all these models is furnished. Drawings of the configurations of the lines, displaying their collocation with respect to each other and to the fundamental tetrahedron, have been made to scale; and these serve to illustrate the text.

In § 45 is given a discussion of the derivation of the Brianchon configuration from two spatial point triads; and in §§ 46–47 the discussion of the derivation of the Pascalian configuration from that of the straight lines upon the second species of the cubic surface (Fowler’s enumeration), with a graphic representation of the combined configuration.

Finally, in § 48, appears a theorem on the number of cubic surfaces with one conical point passing through the lines of mutual intersection of two triheders.

There is appended a bibliography of the principal papers consulted which bear directly upon the present investigation.
CHAPTER I

PRELIMINARY THEOREMS

1. Existence of Straight Lines upon the Cubic Surface.

In order to find the conditions that any straight line, whose equations are

\[
\frac{x-x_0}{\lambda} = \frac{y-y_0}{\mu} = \frac{z-z_0}{\nu} = r;
\]

lie entirely upon a surface, we substitute

\[
x = x_0 + \lambda r, \quad y = y_0 + \mu r, \quad z = z_0 + \nu r
\]

in the equation of the surface; arrange the terms of the resulting equation according to powers of \( r \) and then set all the coefficients of \( r \) equal to zero, since the equation in \( r \) must be identically satisfied, for all values of \( r \). Since in the present case the equation of the surface is of the third degree, there result four conditions. But the equations of a straight line involve four disposable constants; and, as the number of conditions to be fulfilled is exactly equal to the number of disposable constants in the equations of the straight line, it follows that every surface of the third order must contain a finite number of straight lines, real or imaginary, lying entirely upon it.

2. Number of Straight Lines upon the Cubic Surface.

Suppose we pass a plane \( \Pi \) through a point \( P \) outside the surface and through a straight line \( l \) lying upon the surface. Then \( \Pi \) meets the surface in the line \( l \), and a conic \( C \) besides (since the curve of intersection is a degenerate cubic), i.e. meets the surface in a section having two double points. Hence, by definition, it is a double tangent plane. These double tangent planes \( \Pi \) to the cubic surface are also double tangent planes to the tangent cone, vertex \( P \). Now since to
every plane \( II \) corresponds one straight line \( l \) lying entirely on the surface, and there are twenty-seven* \((n = 3)\) double tangent planes to the tangent cone, vertex \( P \), therefore there are twenty-seven straight lines \( l \) upon the cubic surface\(†\).

### 3. Triple Tangent Planes.

By properly determining the plane passed through any straight line \( l \) upon the cubic surface, the conic \( C \) (§ 2) will degenerate into a pair of straight lines. Here the plane intersects the surface in three intersecting straight lines (a degenerate curve of the third order having three double points) and the points of intersection of the lines taken in pairs are the points of contact of the plane with the surface. Now, through each of the three lines in the plane there may be drawn, besides the given plane, four other triple tangent planes. For these twelve new planes give rise to twenty-four lines upon the surface, making up, with the former three lines, twenty-seven lines upon the surface. It is clear that there can be no lines upon the surface besides the twenty-seven. For since the three lines upon the triple tangent plane are the complete intersection of this plane with the surface, every other line upon the surface meets the triple tangent plane in a point upon one of the three lines, and must therefore lie in a plane passing through one of these lines, such plane (since it meets the surface in two lines, and therefore in a third line) being obviously a triple tangent plane. Hence the whole number of lines upon the surface is twenty-seven. Every straight line on the surface is met by ten others.

If all the twenty-seven intersect in pairs, there would be \(351\) points of intersection. But since each line is met by ten other lines, there remain sixteen lines by which it is not met. Therefore there are \(\frac{27 \times 16}{2} = 216\) pairs of lines that do not mutually intersect. Consequently there are \(135\) points of intersection. Since these \(135\) points, by threes, determine the triple tangent planes, there are forty-five triple tangent planes.

* Salmon (Geometry of Three Dimensions, 4th edition, § 286) gives

\[
\frac{n}{2} (n - 1) (n - 2) \left( n^3 - n^2 + n - 12 \right)
\]

as the number of double tangent planes, drawn through a point \( P \) to a surface of the \( n \)th degree.

4. **Salmon’s Notation for the Twenty-Seven Lines**.

Lemma. The general equation of the cubic surface may be reduced to the canonical form $uvw - \xi\eta\zeta = 0$, where $u, v, w, \xi, \eta, \zeta$ are linear polynomials.

The number of independent constants in the general equation of the third degree is $19 \left[ n(n^2 + 6n + 11) \right]$, for $n = 3$. Since the linear polynomials $u, v, w, \xi, \eta, \zeta$ contain eighteen ratios of constants and there is one other constant factor implicitly contained in one of the products $uvw, \xi\eta\zeta$, therefore the form $uvw - \xi\eta\zeta = 0$ contains nineteen constants and is one into which the general equation of a cubic surface may be thrown.

It will appear later (§ 15) from geometrical considerations that the problem to reduce the base cubic to the form $uvw - \xi\eta\zeta = 0$ is solvable in 120 different ways.

**Notation.** Consider the canonical form of the surface of the third degree $ace - bdf = 0$, where $a, b, c, d, e, f$ are linear polynomials. By inspection it is patent that this surface contains the nine lines $ab, ad, af, cb, cd, ef, eb, ed, ef$—where $ab$, for example, represents the line of intersection of the planes $a = 0, b = 0$. If we suppose $a = \mu b$ to be the equation of one of the triple tangent planes through the intersection of the planes $a$ and $b$, the plane $a = \mu b$ meets the surface in the same lines in which it meets the hyperboloid $\mu ce - df = 0$, that is, the two lines in the plane are generating lines of different species, and consequently one of them meets the pair of lines $cd$ and $ef$, and the other of them meets the pair of lines $ef$ and $ed$. Let us now denote each of the remaining eighteen lines by the three lines which it meets—the line meeting $ab$, $cd$ and $ef$ being denoted by the symbol $ab . cd . ef$. Since $\mu$ has three values, there are three lines that meet $ab$, $cd$, $ef$. Applying the same reasoning to the planes through $bc$ and $ca$, we employ the following symbolism for the twenty-seven lines:

$$ab, ad, \ldots, ef; \quad (ab . cd . ef)_i, (ad . cb . ef)_i, (af . cb . ed)_i, \quad (i = 1, 2, 3).$$

Unfortunately, the information furnished by this method as to how these suffixes are to be supplied is inadequate; certain postulates have to be made as to how the intersections occur. This notation of Salmon’s was the first given for the twenty-seven lines. It was soon superseded by a very superior one, to be explained in the next article.

CHAPTER II

THE DOUBLE SIX CONFIGURATION. AUXILIARY THEOREMS

5. The Double Six Notation.

Let us write down, in Salmon's notation, two systems of non-intersecting hues:

\begin{align*}
ab, cd, ef, (ad \cdot cf \cdot eb)_1, (ad \cdot cf \cdot eb)_2, (ad \cdot cf \cdot eb)_3, \\
cf, eb, ad, (ab \cdot cd \cdot ef)_1, (ab \cdot cd \cdot ef)_2, (ab \cdot cd \cdot ef)_3.
\end{align*}

In this scheme, it is postulated (§ 4) that each line of one system does not intersect the line of the other system which is written in the same vertical line, but does intersect the five other lines of the second.

This configuration was first actually observed by Schlafli* and was given by him the name it has since borne—a "double six." The concept of the double six lies at the very basis of the study of the lines upon a cubic surface. The notation derived therefrom is the simplest and most convenient that has yet been discovered for the twenty-seven lines and forty-five planes.

Notation. Starting with the double six, written

\begin{align*}
a_1, a_2, a_3, a_4, a_5, a_6, \\
b_1, b_2, b_3, b_4, b_5, b_6,
\end{align*}

we are enabled to express the complex and diversified symmetry of the twenty-seven lines and forty-five planes in unique and simple form.

* "An attempt to determine the twenty-seven lines upon a surface of the third order, and to divide such surfaces into species in reference to the reality of the lines upon the surface," Quart. Journ. Vol. ii. (1858), pp. 55–65, 110–120.
Returning to the double six, written in Salmon's notation, we see that the lines $ab$, $cb$, and $eb$ lie in the same plane, and are the only three of the twenty-seven lines that lie in the plane $b$. In like manner $cb$, $cd$, and $cf$ all lie in the plane $c$, and hence the line that lies in the plane of $ab$ and $eb$ is identical with the line that lies in the plane of $cd$ and $cf$, viz. the line $cb$.

In the new notation, we shall call the third line in the plane of $a_3$ and $b_3$, which intersect, the line $c_{12}$; and the triangle thus formed will be designated 12. As has been shown above, the side $c_{12}$ forms with $a_2$ and $b_1$ a triangle, designated 21. Hence we have 15 ($\equiv 6\cdot 5$) lines $c_i$, each of which intersects only those four lines $a$, $b$ the suffixes of which belong to the pair of numbers forming the suffix of $c$. For suppose $c_{12}$ should intersect any other line, say $a_3$, of the eight lines $a_3$, $a_4$, $a_5$, $a_6$; $b_3$, $b_4$, $b_5$, $b_6$. Then $c_{12}$ intersecting $a_1$, $b_1$, $a_2$ and $b_2$ already, $c_{12}a_3b_1$ and $c_{12}a_5b_3$ form two triangles; and since they have two lines in common, their planes are identical, and consequently $b_1$ intersects $b_2$, contrary to hypothesis.

Any two $c$'s, the suffixes of which have a number in common, do not intersect. For suppose $c_{12}$, $c_{13}$ intersect; they form a plane in which $a_1$ and $b_1$ lie, and therefore $a_1$ meets $b_1$, contrary to hypothesis. It may also be shown that any two $c$'s, the suffixes of which have no number in common, do intersect. These facts may be briefly put as follows:

\[
\begin{array}{ll}
c_{ij} \text{ intersects } a_i, b_j; a_j, b_i \\
c_{ij} \text{ intersects } c_{kl} \\
c_{ij} \text{ does not intersect } c_{ik} \\
c_{ij} \equiv c_{ji} \\
\Delta_{ij} \text{ is not identical with } \Delta_{ji}
\end{array}
\]

We see then that there are triangles of the form $c_{12}$, $c_{34}$, $c_{56}$ which may be briefly designated 12. 34. 56. Hence there are thirty ($\equiv 6\cdot 5$) triangles of the type 12, and fifteen of the type 12. 34. 56. The latter arises from the fact that, if we fix our attention upon 12, the other two sets may be written in only three ways.

6. History of the Theorem.

In 1858, Schläfli (l.c.) proved the double six theorem incidentally in connection with his investigations on the twenty-seven lines on the cubic surface. He enunciated the theorem in the following form:

Given five lines $a$, $b$, $c$, $d$, $e$ which meet the same straight line $X$; then may any four of the five lines be intersected by another line.
Suppose that \( A, B, C, D, E \) are the other lines intersecting \( (b, c, d, e), (c, d, e, a), (d, e, a, b), (e, a, b, c), \) and \( (a, b, c, d) \) respectively. Then \( A, B, C, D, E \) will all be met by one other straight line \( x \).

The double six in this case is written

\[
\begin{pmatrix}
  \alpha, \beta, \gamma, \delta, \epsilon, \\
  A, B, C, D, E, X
\end{pmatrix}
\]

Schläfli then proposes the question: "Is there, for this elementary theorem, a demonstration more simple than the one derived from the theory of cubic forms?"

Sylvester* states that the theorem admits of very simple geometrical proof; but he did not supply the proof. Salmon† has given a method for constructing a double six, by pure geometry; but it is not a proof of the theorem, independent of the cubic surface.

In 1868, Cayley‡ gave a proof of the theorem from purely static considerations, making use of theorems on six lines in involution. It has recently been remarked, by Mr G. T. Bennett, that this is erroneous§. Again in 1870, Cayley|| verified the theorem, using this time the six co-ordinates of a line. In 1903, Kasner¶ also gave a proof using the six co-ordinates of a line. More recently (January 13, 1910), Baker has given a direct algebraic proof of the theorem independently of the cubic surface, so formulated as to show that the theorem belongs to three dimensions only**.

In 1881, Schurtt originally gave a geometrical proof of the double six theorem, basing his proof on a poristic property of the plane cubic curve. Recently (November 21, 1910), Baker †† has given a geometrical proof of the double six theorem independently of the cubic surface, thus demonstrating the fundamentally projective character of the configuration.


Representing the double six as follows:

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1' & 2' & 3' & 4' & 5' & 6'
\end{pmatrix},
\]

it is seen that these twelve lines have the thirty intersections \( P_{ij} \), and determine thirty planes \( \Pi_{ij} \) (formed by the lines \( i \) and \( j' \)).

Using quadriplanar co-ordinates, I choose for the lines \( 1', 3', 4', 5', 6' \) the following equations:

\[
\begin{align*}
1': & \quad d\delta' Cx + d\delta' Az - (\alpha' \gamma' \delta' - \alpha \gamma \delta') w = 0, \quad y = 0, \\
3': & \quad \gamma' Dy - (\beta' \gamma' \delta' - K \beta \gamma \delta') z + \gamma' Bw = 0, \quad x = 0, \\
4': & \quad z = 0, \quad w = 0, \\
5': & \quad \delta' x - \alpha' w = 0, \quad \gamma' y - \beta' z = 0, \\
6': & \quad \delta x - \alpha w = 0, \quad \gamma y - \beta z = 0,
\end{align*}
\]

where we set

\[ A, B, C, D = (\alpha' - Ka), (\beta' - Kb), (\gamma' - K\gamma), (\delta' - K\delta) \]

respectively.

These equations have been so chosen that the five lines have a common tractor*. The condition that any five lines, \( 1, 2, 3, 4, 5 \) say, have a common tractor, where the equations of lines \( i \) and \( j \) are

\[
\begin{align*}
(\alpha' x + b_1 y + c_1 z + d_1 w = 0) \quad & \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldot
\]

* Cayley uses the word "tractor" to denote a line which meets any given lines in space.
respectively, and we understand by \((ij)\) the determinant
\[
\begin{vmatrix}
\alpha_i, \beta_i, \gamma_i, \delta_i \\
\alpha_j, \beta_j, \gamma_j, \delta_j \\
\alpha_k, \beta_k, \gamma_k, \delta_k \\
\alpha_l, \beta_l, \gamma_l, \delta_l \\
\end{vmatrix},
\]
is as follows :
\[
\Delta_5 = \begin{vmatrix}
0, (12), (13), (14), (15) \\
(21), 0, (23), (24), (25) \\
(31), (32), 0, (34), (35) \\
(41), (42), (43), 0, (45) \\
(51), (52), (53), (54), 0 \\
\end{vmatrix} \equiv 0^*.
\]
The five lines \(1', 3', 4', 5', 6'\) are co-tractorial, since the equations identically satisfy \(\Delta_5 = 0\), as may be shown on trial. Moreover these five lines do not mutually intersect, since in forming the determinants \((ij)\), no one of them is found to vanish. A difficulty arises in the event of the hyperboloid through any three of these five collinear lines touching a fourth, that is to say, that certain four of the lines might have a double tractor \(t^*\). That such is not the case will appear in the sequel.

Determining now the common tractor, \(2\), of these five lines, we find it to have the equations:
\[
\\begin{align*}
\gamma'B (\delta x - a'w) - \delta'A (\gamma'y - \beta'z) &= 0 \\
\gamma'B (\delta x - aw) - \delta'A (\gamma'y - \beta z) &= 0
\end{align*}
\]

Now, in general, four given lines have a pair of tractors. Since the five lines \(1', 3', 4', 5', 6'\) already have a single tractor \(2\), they have, in sets of four, five more tractors thus: the lines \(1, 3, 4, 5, 6\) are tractors of the sets \((3', 4', 5', 6')\), \((1', 4', 5', 6')\), \((1', 3', 5', 6')\), \((1', 3', 4', 6')\), \((1', 3', 4', 5')\) respectively.

Let us proceed to find the equations of the five lines \(1, 3, 4, 5, 6\). Recalling the values of \(A, B, C\) and \(D\) above, it is obvious by inspection that the equations of lines \(1\) and \(3\), meeting the quadruples \((3', 4', 5', 6')\) and \((1', 4', 5', 6')\) respectively, are
\[
\begin{align*}
1: & \quad x = 0, \quad w = 0, \\
3: & \quad y = 0, \quad z = 0.
\end{align*}
\]


The equations of line 4, since it meets the lines 5' and 6', are of the form
\[\begin{align*}
\frac{x}{\alpha} - \lambda \frac{y}{\beta} + \lambda \frac{z}{\gamma} - \frac{w}{\delta} &= 0, \\
\frac{x}{\alpha} - \mu \frac{y}{\beta} + \mu \frac{z}{\gamma} - \frac{w}{\delta} &= 0
\end{align*}\]

The conditions that this line meet the line 1', written in the form
\[\left(\frac{x}{\alpha} + \frac{z}{\gamma} - \frac{w}{\delta}\right) \frac{1}{\alpha \gamma} - K \left(\frac{x}{\alpha} + \frac{z}{\gamma} - \frac{w}{\delta}\right) \frac{1}{\alpha \gamma} = 0, \quad y = 0,
\]
are given by \(\lambda = 1, \quad \mu = 1\).

Then the line 4 has the equations
\[\begin{align*}
4: \quad &\left\{ \begin{array}{l}
\frac{x}{\alpha} - \frac{y}{\beta} + \frac{z}{\gamma} - \frac{w}{\delta} = 0, \\
\frac{x}{\alpha} - \frac{y}{\beta} + \frac{z}{\gamma} - \frac{w}{\delta} = 0
\end{array} \right. 
\end{align*}\]
and we see by inspection that this line meets the line 3' when we write its equations in the form
\[3': \quad \left\{ \begin{array}{l}
\left(\frac{-y}{\beta} + \frac{z}{\gamma} - \frac{w}{\delta}\right) \frac{1}{\beta \delta} - K \left(\frac{-y}{\beta} + \frac{z}{\gamma} - \frac{w}{\delta}\right) \frac{1}{\beta \delta} = 0, \\
x = 0
\end{array} \right. \]

Next, line 5, since it meets the lines 4' and 6', has equations of the form
\[\begin{align*}
z - 1\lambda w &= 0, \\
\left\{ \begin{array}{l}
\left(\frac{x}{\alpha} - \frac{w}{\delta}\right) - \mu \left(\frac{y}{\beta} - \frac{z}{\gamma}\right) = 0
\end{array} \right.
\end{align*}\]
Meeting line 3' (see first form), it is necessary to identify the equations
\[-\frac{\mu}{\beta} y + \frac{\mu}{\gamma} z - \frac{1}{\delta} w = 0,
\]
\[-\frac{1}{\delta} \left(\frac{\delta - K \delta}{\beta' - K \beta}\right) y + \frac{\gamma \beta' \delta - K \gamma / \beta \delta}{\gamma \delta / (\beta' - K \beta)} z - \frac{w}{\delta} = 0.
\]
Hence
\[M = \frac{\beta}{\delta} \left(\frac{\delta - K \delta}{\beta' - K \beta}\right) = \frac{1}{\gamma \delta} \left(\frac{\gamma \beta' \delta - K \gamma / \beta \delta}{\gamma \delta / (\beta' - K \beta)}\right),
\]
giving
\[\gamma: \gamma = \beta'; \quad \beta,
\]
for which
\[M = \frac{\gamma}{\delta} \left(\frac{\delta - K \delta}{\gamma' - K \gamma}\right),
\]
and therefore
\[\frac{\delta}{\gamma} \left(\frac{\gamma' - K \gamma}{\delta - K \delta}\right) \left(\frac{x}{\alpha} - \frac{w}{\delta}\right) - \left(\frac{y}{\beta} - \frac{z}{\gamma}\right) = 0.
\]
Applying similar reasoning to the equation
\[z - \lambda w = 0,
\]
with respect to the lines 1' and 3', we finally obtain
\[\lambda = \frac{\gamma'}{\delta}.
\]
Then the equations of line 5 are

\[
5: \left\{ \begin{align*}
\delta'z - \gamma'w & = 0 \\
\beta C (\delta x - a'w) - aD (\gamma y - \beta z) & = 0
\end{align*} \right.
\]

Determining in similar fashion the equations of line 6, we obtain

\[
6: \left\{ \begin{align*}
\delta z - \gamma w & = 0 \\
\beta' C (\delta' x - a'w) - a'D (\gamma' y - \beta' z) & = 0
\end{align*} \right.
\]

It remains to show that the five lines 1, 3, 4, 5, 6 have a common tractor (in other words, are collinear).

Writing out the various determinants \((ij)\) and substituting in the formula for \(\Delta_s\), we obtain (after reduction)

\[
\Delta_s = 0.
\]

Hence these five lines have a common tractor. They do not mutually intersect, since no \((ij) \equiv 0\).

Determining now the equations of the line called 2', which meets these five lines, we find

\[
2': \left\{ \begin{align*}
(\alpha \beta' - a' \beta) \delta \delta' Cx + (\gamma \delta' - \gamma' \delta) a' \beta' Bw = 0 \\
(\alpha \beta' - a' \beta) \gamma \gamma' Dy + (\gamma' \delta' - \gamma \delta) \beta' \beta' Ax = 0
\end{align*} \right.
\]

Hence we reach the following conclusion, which is Schläfli's theorem:

The five lines determined from five co-tractorial lines by choosing the remaining tractor in each set of four of the latter lines, are themselves co-tractorial.

In the above proof, the complete set of lines was derived from the five co-tractorial lines 1', 3', 4', 5', 6', but it is immaterial from which five of the primed or unprimed lines we start. Moreover, the relation between the sets 1', 3', 4', 5', 6' and 1, 3, 4, 5, 6 is a reversible one—the lines of one set are the tractors of the other set by fours, and vice versa.

8. Anharmonic Ratios.

Let us next find the co-ordinates of the points of intersection of the lines 2', 3', 4', 5', 6' with the line 1. Determining these in the usual way and writing down also the co-ordinates of the vertex \(C\) of the fundamental tetrahedron \(ABCD\), we tabulate them as follows:

\[
\begin{align*}
P_{12}: & \quad 0 \quad \beta \beta' (\gamma \delta - \gamma' \delta) A \\
P_{33}: & \quad 0 \quad \beta' \delta - K \beta \delta' \gamma \\
P_{44}: & \quad 0 \quad 1 \\
P_{55}: & \quad 0 \quad \beta' \\
P_{66}: & \quad 0 \quad \beta \\
C: & \quad 0 \quad 0
\end{align*}
\]

\[2-2\]
The anharmonic ratio of the four collinear points \( P_{12'}, P_{13'}, P_{25'}, P_{16'} \) is identical with the anharmonic ratio of the four parameters
\[
\gamma' \left( \alpha \beta - \alpha' \beta' \right) D
\]
\[
\beta' \left( \gamma' \delta - \gamma \delta' \right) A
\]
\[
\gamma' D
\]
\[
\beta' \gamma' \beta' \beta'.
\]
Calculating the value \( \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2} \) of the anharmonic ratio of these four parameters numbered in the order in which they are written, we find
\[
(P_{12'}, P_{13'}, P_{25'}, P_{16'}) = \frac{\beta' \delta'}{K \beta \delta} \left\{ \frac{\beta \left( \gamma' \delta - \gamma \delta' \right) A - \gamma \left( \alpha \beta - \alpha' \beta' \right) D}{\beta' \left( \gamma' \delta - \gamma \delta' \right) A - \gamma' \left( \alpha \beta - \alpha' \beta' \right) D} \right\}.
\]
Next, let us determine the co-ordinates of the points of intersection of the lines 1, 2, 3, 5, 6 with the line 4'. These follow in the table below:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>P_{24'}</td>
<td>A</td>
<td>B</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>P_{34'}</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>P_{45'}</td>
<td>a \gamma D</td>
<td>\beta \delta C</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>P_{64'}</td>
<td>a' \gamma' D</td>
<td>\beta' \delta' C</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The anharmonic ratio of the four collinear points \( P_{24'}, P_{34'}, P_{45'}, P_{64'} \) is identical with the anharmonic ratio of the four parameters
\[
B
\]
\[
A \neq 0, \frac{\beta \delta C}{\alpha \gamma D}, \frac{\beta' \delta' C}{\alpha' \gamma' D}.
\]
Calculating the value \( \frac{\mu_3 - \mu_1}{\mu_3 - \mu_2} \) of the anharmonic ratio of these four parameters numbered in the order in which they are written, we find
\[
(P_{24'}, P_{34'}, P_{45'}, P_{64'}) = \frac{\beta' \delta'}{K \beta \delta} \left\{ \frac{\beta \delta A C - \alpha \gamma B D}{\beta' \delta' A C - \alpha' \gamma' B D} \right\}.
\]
Recalling the fact that
\[
A, B, C, D \equiv (\alpha' - K \alpha), (\beta' - K \beta), (\gamma' - K \gamma), (\delta' - K \delta)
\]
respectively, it is easily verified that
\[
\frac{\beta' \delta'}{K \beta \delta} \left\{ \frac{\beta \left( \gamma' \delta - \gamma \delta' \right) A - \gamma \left( \alpha \beta - \alpha' \beta' \right) D}{\beta' \left( \gamma' \delta - \gamma \delta' \right) A - \gamma' \left( \alpha \beta - \alpha' \beta' \right) D} \right\} = \frac{\beta' \delta'}{K \beta \delta} \left\{ \frac{\beta \delta A C - \alpha \gamma B D}{\beta' \delta' A C - \alpha' \gamma' B D} \right\}.
\]
Accordingly
\[
\{P_{12'}, P_{13'}, P_{15'}, P_{16'}\} = \{P_{24'}, P_{34'}, P_{45'}, P_{64'}\},
\]
or in briefer notation
\[
(2', 3', 5', 6')_t = (2, 3, 5, 6)_t.'
This theorem may be stated as follows:

The anharmonic ratio of the points in which any four out of five co-tractorial lines cut the common tractor of all five is equal to the anharmonic ratio of the points where the fifth line is intersected by the correspondents of the first four.

Let us designate the anharmonic ratio of the four planes formed by the line \( i'_1 \) with the lines \( i_5, i_4, i_9, i_6 \) by the symbol \( (i_5, i_4, i_9, i_6)_{i'_1} \).

Recalling next the known theorem concerning the two tractors of four lines, viz. that the four points of either tractor and the four planes of the other tractor have the same anharmonic ratio, we obtain

\[ (i_5, i_4, i_9, i_6)_{i'_1} = (i_5, i_4, i_9, i_6)_{i'_1}. \]

Making use of the last theorem, we obtain

\[ (i'_1, i'_1, i'_1, i'_1)_{i'_1} = (i_5, i_4, i_9, i_6)_{i'_1}. \]

Hence we draw the conclusion:

The anharmonic ratio of the four points, on one of five co-tractorial lines, each of which is collinear with any three of the four remaining lines, is equal to the anharmonic ratio of the four planes determined by these remaining lines and their common tractor.


Given any five co-tractorial lines, these determine uniquely, as was shown in § 7, the double six configuration. Then if we consider the plane of \( ij \), it will be met by the lines \( i', j \) in points which lie on the line \( (ij) \). Since \( sO_5 = 15 \), the twelve lines of the double six together with the fifteen new lines make up twenty-seven in all, the total number upon the cubic surface*. Hence the condition \( \Delta_5 = 0 \) (§ 7), which is the condition that five lines be co-tractorial, is likewise the condition that five given lines lie on a cubic surface. The subject was first studied by Sylvester in connection with a theorem in the Lehrbuch der Statik, of Möbius (Leipzig).

If we are given five lines, defined by their six co-ordinates \( (a_1, b_1, c_1, f_1, g_1, h_1) \) \( \ldots \) \( (a_5, b_5, c_5, f_5, g_5, h_5) \), then the condition that these lines be co-tractorial is expressed by the equation

\[
\begin{vmatrix}
0, & 12, & 13, & 14, & 15 \\
21, & 0, & 23, & 24, & 25 \\
31, & 32, & 0, & 34, & 35 \\
41, & 42, & 43, & 0, & 45 \\
51, & 52, & 53, & 54, & 0
\end{vmatrix} = 0.
\]

where we set

\[ a_1 f_2 + a_2 f_1 + b_1 g_2 + b_2 g_1 + c_1 h_2 + c_2 h_1 = 12, \text{ etc.} \]

This is also the condition that these lines may lie in a cubic surface.

The agreement between this equation of condition and that of Sylvester \((\Delta_6 = 0 \text{ of } \S 7)\) inheres in the fact that Cayley's determinant of the fifth order above written is the square root of Sylvester's \(\Delta_6\).†

**10. Enumeration of the Double Sixes of a Cubic Surface.**

It has been shown (§ 3) that the configuration of twenty-seven lines contains two hundred and sixteen pairs of non-intersecting straight lines. Each pair determines a double six; but each double six contains six pairs of such mated lines. Hence the number of double sixes is \(\frac{216}{6} = 36\).

Let us next proceed to form a table of the double sixes. The original double six

\[
\begin{align*}
1, & \ 2, \ 3, \ 4, \ 5, \ 6 \\
1', & \ 2', \ 3', \ 4', \ 5', \ 6'
\end{align*}
\]

is the primitive. There is but one of this type.

Consider next the type

\[
\begin{align*}
1, & \ 1', \ 23, \ 24, \ 25, \ 26, \\
2, & \ 2', \ 13, \ 14, \ 15, \ 16.
\end{align*}
\]

The number of this type is clearly 15 \(= \binom{6}{3}\).

Of the type given by

\[
\begin{align*}
1, & \ 2, \ 3, \ 56, \ 46, \ 45, \\
23, & \ 13, \ 12, \ 4', \ 5', \ 6',
\end{align*}
\]

there are 20 \(= \binom{6}{3}\), since keeping 1, 2, 3 fixed, the remaining numerals are uniquely determined§. This concludes the enumeration, since

\[1 + 15 + 20 = 36.\]

Below is the table of all the double sixes.

---

§ In his paper “A Memoir on Cubic Surfaces,” *Philos. Trans. Royal Soc.* Vol. clxix. (1869), pp. 231–326, Cayley erroneously states that there are twenty of the type

\[
\begin{align*}
1, & \ 2, \ 3, \ 56, \ 46, \ 45, \\
23, & \ 13, \ 12, \ 4, \ 5, \ 6.
\end{align*}
\]
| 1, 2, 3, 4, 5, 6 | 1, 3, 6, 45, 25, 24 |
| 1', 2', 3', 4', 5', 6' | 36, 16, 13, 2', 4', 5' |
| 1, 1', 23, 24, 25, 26 | 1, 4, 5, 36, 26, 23 |
| 2, 2', 13, 14, 15, 16 | 45, 15, 14, 2', 3', 6' |
| 1, 1', 32, 34, 32, 56 | 1, 4, 6, 35, 25, 23 |
| 3, 3', 12, 14, 15, 16 | 46, 16, 14, 2', 3', 5' |
| 1, 1', 42, 43, 45, 46 | 1, 5, 6, 34, 24, 23 |
| 5, 5', 12, 13, 14, 16 | 34, 24, 23, 1', 5', 6' |
| 1, 1', 62, 63, 64, 65 | 2, 3, 5, 46, 16, 14 |
| 6, 6', 12, 13, 14, 15 | 35, 25, 23, 1', 4', 6' |
| 2, 2', 31, 34, 35, 36 | 1, 2, 3, 56, 46, 45 |
| 3, 3', 21, 24, 25, 26 | 23, 13, 12, 4', 5', 6' |
| 2, 2', 41, 43, 45, 46 | 1, 2, 4, 56, 36, 35 |
| 4, 4', 21, 23, 25, 26 | 24, 14, 12, 3', 5', 6' |
| 2, 2', 51, 53, 54, 56 | 1, 2, 5, 46, 36, 34 |
| 5, 5', 21, 23, 24, 26 | 25, 15, 12, 3', 4', 6' |
| 2, 2', 61, 63, 64, 65 | 1, 2, 6, 45, 35, 34 |
| 6, 6', 21, 23, 24, 25 | 26, 16, 12, 3', 4', 5' |
| 3, 3', 41, 42, 45, 46 | 2, 3, 6, 45, 15, 14 |
| 4, 4', 31, 32, 35, 36 | 36, 26, 23, 1', 4', 5' |
| 3, 3', 51, 52, 54, 56 | 2, 4, 5, 36, 16, 13 |
| 5, 5', 31, 32, 34, 36 | 45, 25, 24, 1', 3', 6' |
| 3, 3', 61, 62, 64, 65 | 2, 4, 6, 35, 15, 13 |
| 6, 6', 31, 32, 34, 35 | 46, 26, 24, 1', 3', 5' |
| 4, 4', 51, 52, 53, 56 | 2, 5, 6, 34, 14, 13 |
| 5, 5', 41, 42, 43, 46 | 56, 26, 25, 1', 3', 4' |
| 4, 4', 61, 62, 63, 65 | 3, 4, 5, 26, 16, 12 |
| 6, 6', 41, 42, 43, 45 | 45, 35, 34, 1', 2', 6' |
| 5, 5', 61, 62, 63, 64 | 3, 4, 6, 25, 15, 12 |
| 6, 6', 51, 52, 53, 54 | 46, 36, 34, 1', 2', 5' |
| 1, 3, 4, 56, 26, 25 | 3, 5, 6, 24, 14, 12 |
| 34, 14, 13, 2', 5', 6' | 56, 36, 35, 1', 2', 4' |
| 1, 3, 5, 46, 26, 24 | 4, 5, 6, 23, 13, 12 |
| 35, 15, 13, 2', 4', 6' | 56, 46, 45, 1', 2', 3' |
It is worthy of remark that the double sixes play a part in the theory of the nodes of the cubic surface. If a surface of the third order $f(x, y, z, w) = 0$ has a proper node $(x, y, z, w)$, then the six lines passing through such node and represented by the equations $D^2 f = 0$, $D^2 f = 0$ form a double six, in which each two corresponding (non-intersecting) lines of the two sextuples coincide.*

So in Cayley's enumeration†, the system of lines and planes for the second species of cubic surface is derived from that of the first species by supposing that in the double six the corresponding lines $1, 1'; 2, 2'$; etc. severally coincide (cf. § 25).

11. Auxiliary Theorems.

In addition to the double six configuration, there are very many others having interesting properties formed from certain parts of the complete configuration of the twenty-seven lines. Indeed, as Cayley has remarked, the number of such theorems might be multiplied indefinitely. It is possible to deduce a large number of theorems directly from the notation; or even, more cumbrously, from an intersection table (cf. figure). Below are given a few simple theorems.

1. Any straight line is not cut by sixteen other straight lines.

2. Any two non-intersecting straight lines, say $a_1$ and $b_1$, are met by the same five lines $c_{12}, c_{13}, c_{14}, c_{15}, c_{16}$. Of the remaining twenty straight lines, there are five which meet only $a_1$, five which meet only $b_1$, and ten which meet neither $a_1$ nor $b_1$. A set of lines such as $a_1, b_1$ is called a "double."

3. Any three non-intersecting straight lines $a_1, a_2, a_3$ are met by the same three straight lines $b_1, b_2, b_3$. There are six straight lines which meet neither of the three lines $a_1, a_2, a_3$; six which meet only two of these three, and nine which meet only one of the three. A set of lines such as $a_1, a_2, a_3$ is called a "triple."

4. Any four non-intersecting straight lines $a_1, a_2, a_3, a_4$ are met by two straight lines $b_5, b_6$. There are three straight lines which meet neither of the four, $a_1, a_2, a_3, a_4$; four which meet only three of them, six which meet only two of them, and eight which meet only one of them. A set of lines such as $a_1, a_2, a_3, a_4$ is called a "quadruple."

5 (1). Five non-intersecting straight lines such as $a_1, a_2, a_3, a_4, a_5$, which belong to a double six, are met by only one straight line, $b_6$. There is but one line, $a_6$, which fails to meet all of them.

* Schléfli, Quart. Journ. Vol. 11. (1858), p. 120.
Intersection Table.
V (2). Five non-intersecting straight lines \(a_1, a_2, a_3, a_4, c_{65}\), not belonging to a double six, are met by two straight lines \(b_5, b_6\). There is no straight line by which no one of the five given straight lines is met.

Either set of five non-intersecting straight lines is called a "quintuple."

VI. Finally, such a set as six non-intersecting straight lines \(a_1, a_2, a_3, a_4, a_5, a_6\) is called a "sextuple."

On the basis of the preceding, it is easy to determine immediately the number of doubles, triples, etc. in the configuration of the twenty-seven lines.

Number of doubles \(= \frac{27 \cdot 16}{1 \cdot 2} = 216;\)

" " triples \(= \frac{27 \cdot 16 \cdot 10}{1 \cdot 2 \cdot 3} = 720;\)

" " quadruples \(= \frac{27 \cdot 16 \cdot 10 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 1080;\)

" " quintuples \(= \frac{27 \cdot 16 \cdot 10 \cdot 6 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 648;\)

" " sextuples \(= \frac{27 \cdot 16 \cdot 10 \cdot 6 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 72.\)

A word must be said about the quintuples, which are of two types. Every quadruple \(a_1, a_2, a_3, a_4\) gives (1) one quintuple with two intersecting lines; and (2) two quintuples with one intersecting line each. That is, we have the three quintuples:

\[a_1a_2a_3a_4a_5\] with one intersector \(b_6;\)

\[a_1a_2a_3a_4a_6\] " " " " \[b_5;\]

\[a_1a_2a_3a_4c_{65}\] " " " " \[two\ \text{intersectors}\ b_5, b_6.\]

Thus the quintuples fall into two groups; and there are twice as many in one group as in the other. Since the total number is 648, it follows that there are 432 of the type having only one intersector, and 216 of the type having two intersectors. This explains the derivation of the number of sextuples, since two quintuples out of every set of three belong to a double six.

A large number of theorems upon special portions of the configuration of the twenty-seven lines is given by Steiner*, Sturm†, Taylor‡, and others; and to these the reader is referred.

CHAPTER III

THE TRIHEDRAL PAIR CONFIGURATION

12. Definition and Number of Trihedral Pairs.

Let us choose from the forty-five triangles, formed by the twenty-seven lines, two: 12, 43—having \(a_1, b_2, c_{12}; a_4, b_5, c_{45}\) for sides respectively. These triangles have no side in common, and their planes cut in another straight line \(K\), called their edge. Moreover their sides meet in pairs \(a_1, b_2; a_4, b_5; c_{12}, c_{45}\) upon the edge \(K\), in three points \(\delta\), say.

The pairs just written are sides of three other triangles \(a_1b_2c_{13}, a_4b_5c_{46}, c_{12}c_{45}c_{56}\), written in the abbreviated notation 13, 42, 12.43.56. The planes of the triangles 12, 43, 13.42.56 form a triheder \(T\). Since the joins of corresponding vertices are concurrent, it follows that the meets of corresponding sides lie by threes upon the three axes of the planes of the given triangles. In like manner, the planes of the triangles 13, 42, 12.43.56 form a triheder \(T_1\), upon whose axes their sides meet each other. The latter three triangles, like the former, have the nine lines \(a_1, b_2, c_{12}; a_4, b_5, c_{45}; c_{12}, c_{45}c_{56}\) as sides; and the planes of the two triheders \(T\) and \(T_1\) cut each other in these same lines. Two such triheders, \(T\) and \(T_1\), are called conjugate; and taken together, we shall refer to them as a trihedral pair (Fig. 1).

Otherwise phrased, any two triangle planes 12, 43 which have no line in common, determine a third plane 13.42.56, which forms with them a triheder. These in turn determine another triheder 13, 42, 12.43.56.

In order to determine the number of trihedral pairs directly, it suffices to fix our attention upon any one triangle plane such as \(a_1b_2c_{12}\). Through each of the lines \(a_1, b_3, c_{12}\) pass four triple tangent planes, besides the plane \((a_1b_2c_{12})\) in question, \(\Pi\) say. Hence there are \(45 - 13 = 32\) planes which have no line in common with \(\Pi\). Hence in order to find the number of triheders we must multiply 45 by 32, divide by 2 since the plane is considered twice in the enumeration, and further divide by 3, since it takes three planes to form a triheder. Hence
there are \( \frac{45 \cdot 32}{2 \cdot 3} = 240 \) triheders; but since these fall into pairs such that one triheder being given the other is uniquely determined, there are in all 120 trihedral pairs.

**13. Enumeration of Trihedral Pairs.**

It is of interest to form, actually, the 120 trihedral pairs in terms of the planes which constitute them. Remembering that every trihedral pair determines nine straight lines lying wholly upon the surface, we must choose for combination those triheders that fulfill this condition.

It is sufficiently obvious, in view of the unique notation already adopted, that the faces of the trihedral pair, of the type

\[
(12)(23)(31) + (13)(32)(21),
\]

intersect the surface in the nine lines \( a_1, a_2, a_3; b_1, b_2, b_3; c_{12}, c_{23}, c_{31} \).

These nine lines may be arranged in the following form:

\[
\begin{bmatrix}
  a_1 & b_2 & c_{12} \\
  a_2 & b_3 & c_{23} \\
  a_3 & b_1 & c_{31}
\end{bmatrix}
\]

where each horizontal line represents three co-planar lines, and each vertical column represents three non-intersecting lines—as indicated. Bearing in mind the order of the letters, as indicated by the diagram, we may represent all examples of this type in the abbreviated notation

\[
\begin{bmatrix}
  i & j & ij \\
  j & k & jk \\
  k & i & ki
\end{bmatrix} \quad i, j, k = 1, 2, \ldots 6 \ (i \neq j \neq k).
\]

Fixing our attention on the first column, it is obvious that the number of this type is clearly \( 6 \cdot 3 = 20 \).

Consider next the two similar types:

\[
(35)(64)(12 \cdot 34 \cdot 56) + (34)(65)(12 \cdot 35 \cdot 64); \\
(53)(46)(12 \cdot 34 \cdot 56) + (43)(56)(12 \cdot 35 \cdot 64).
\]

The faces of the first pair cut the surface in the nine lines \( a_5, a_6; b_4, b_5; c_{12}, c_{24}, c_{56}, c_{35}, c_{43} \). Also the faces of the second pair cut the surface in the nine lines \( a_4, a_5; b_3, b_6; c_{12}, c_{34}, c_{56}, c_{35}, c_{43} \).

These two sets of nine lines may be arranged in the following forms:

\[
\begin{bmatrix}
  a_3 & b_3 & c_{35} \\
  a_4 & b_4 & c_{46} \\
  c_{24} & c_{56} & c_{12}
\end{bmatrix}, \quad \begin{bmatrix}
  a_4 & b_2 & c_{24} \\
  a_5 & b_6 & c_{56} \\
  c_{46} & c_{25} & c_{12}
\end{bmatrix}.
\]
That there are forty-five of each of these types follows from the fact that, if we keep 12 fixed, for example, then there are three ways in which the c-triangles may be written:

\[
\begin{align*}
\{ & 35 \\
& 46 \\
& 34.56.12 \\
\} & \quad \{ & 36 \\
& 45 \\
& 35.46.12 \\
\} & \quad \{ & 34 \\
& 56 \\
& 36.45.12 \\
\}
\end{align*}
\]

Hence there are fifteen such sets.

All examples of this type may be represented in the abbreviated notation

\[
\begin{align*}
\left\{ & i \quad j \quad ij \\
& i' \quad j' \quad i'j' \\
& i''j' \quad i''j \\
\right\} & \quad i, j, k, l = 1, 2, \ldots 6 \ (i \neq j \neq k \neq l),
\end{align*}
\]

the affixes denoting that a different choice of numerals must be made for the letters in the second line to those in the first line.

Finally, there is a type:

\[
(14. 25 . 36) (35 . 16 . 24) (26 . 34 . 15)
\]

\[+ (14 . 35 . 26) (25 . 16 . 34) (36 . 24 . 15).\]

The faces of this pair cut the surface in the nine lines \(c_{14}, c_{15}, c_{16}, c_{24}, c_{25}, c_{26}, c_{34}, c_{35}, c_{36}.\) These nine lines may be arranged in the following form:

\[
\begin{align*}
\{ & c_{14} \quad c_{15} \quad c_{16} \\
& c_{24} \quad c_{25} \quad c_{26} \\
& c_{34} \quad c_{35} \quad c_{36} \\
\} & \quad i, j, k, l = 1, 2, \ldots 6 \ (i \neq j \neq k \neq l),
\end{align*}
\]

Obviously such a form arises from the two forms:

\[
\begin{align*}
\left\{ & 1 \quad 1 \quad 1 \\
& 2 \quad 2 \quad 2 \\
& 3 \quad 3 \quad 3 \\
\right\} & \quad \left\{ & 4 \quad 5 \quad 6 \\
& 4 \quad 5 \quad 6 \\
& 4 \quad 5 \quad 6 \\
\right\}.
\end{align*}
\]

Hence the number of such forms is \(\frac{1}{2} \binom{6}{3} = 10.\)

All examples of this type may be represented in the abbreviated notation

\[
\begin{align*}
\left\{ & ij \quad ik \quad il \\
& i'j \quad i'k \quad i'l \\
& i''j \quad i''k \quad i''l \\
\right\} & \quad i, j, k, l = 1, 2, \ldots 6 \ (i \neq j \neq k \neq l),
\end{align*}
\]

the affixes denoting that a different choice of numerals must be made for the letters for each line.

Hence we have enumerated all the different types, the total number of trihedral pairs being \(120 = 20 + 90 + 10.\)

Below are listed the 120 trihedral pairs, according to the rules just enumerated.
### TABLE I

<table>
<thead>
<tr>
<th>Form</th>
<th>Determining</th>
</tr>
</thead>
<tbody>
<tr>
<td>(12) (23) (31) + (32) (21) (13)</td>
<td>$a_1$, $a_2$, $a_3$; $b_1$, $b_2$, $b_3$; $c_{12}$, $c_{13}$, $c_{23}$</td>
</tr>
<tr>
<td>(12) (24) (41) + (42) (21) (14)</td>
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<tr>
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<tr>
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<tr>
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<tr>
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## TABLE II

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<td>(41) (32) (56.13.24) + (31) (42) (56.14.23)</td>
<td>$a_3$, $a_4$; $b_1$, $b_2$; $c_{15}$, $c_{24}$, $c_{50}$, $c_{14}$, $c_{23}$</td>
</tr>
<tr>
<td>(12) (34) (56.14.23) + (14) (32) (56.12.34)</td>
<td>$a_1$, $a_3$; $b_2$, $b_4$; $c_{14}$, $c_{23}$, $c_{50}$, $c_{12}$, $c_{34}$</td>
</tr>
<tr>
<td>(21) (43) (56.14.23) + (41) (23) (56.12.34)</td>
<td>$a_3$, $a_4$; $b_1$, $b_3$; $c_{14}$, $c_{23}$, $c_{56}$, $c_{12}$, $c_{34}$</td>
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### TABLE III

<table>
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<tr>
<th>Form</th>
<th>Determining</th>
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<tbody>
<tr>
<td>(14.25.36) (35.16.24) (26.34.15)</td>
<td>c_{14}, c_{15}, c_{16}, c_{24}, c_{25}, c_{26}, c_{34}, c_{35}, c_{36}</td>
</tr>
<tr>
<td>+ (14.35.26) (25.16.34) (36.24.15)</td>
<td></td>
</tr>
<tr>
<td>(13.25.46) (45.16.23) (26.43.15)</td>
<td>c_{13}, c_{15}, c_{16}, c_{23}, c_{25}, c_{26}, c_{43}, c_{45}, c_{46}</td>
</tr>
<tr>
<td>+ (13.45.26) (25.16.43) (46.23.15)</td>
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</tr>
<tr>
<td>(13.24.56) (54.16.23) (26.53.14)</td>
<td>c_{13}, c_{14}, c_{16}, c_{23}, c_{24}, c_{26}, c_{53}, c_{54}, c_{56}</td>
</tr>
<tr>
<td>+ (13.54.26) (24.16.53) (56.23.14)</td>
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</tr>
<tr>
<td>(13.24.65) (64.15.23) (25.63.14)</td>
<td>c_{13}, c_{14}, c_{16}, c_{23}, c_{24}, c_{25}, c_{63}, c_{64}, c_{65}</td>
</tr>
<tr>
<td>+ (13.64.25) (24.15.63) (65.23.14)</td>
<td></td>
</tr>
<tr>
<td>(12.35.46) (45.16.32) (36.42.15)</td>
<td>c_{12}, c_{15}, c_{16}, c_{32}, c_{35}, c_{36}, c_{42}, c_{45}, c_{46}</td>
</tr>
<tr>
<td>+ (12.45.36) (35.16.42) (46.32.15)</td>
<td></td>
</tr>
<tr>
<td>(12.34.56) (54.16.32) (36.52.14)</td>
<td>c_{12}, c_{14}, c_{16}, c_{32}, c_{34}, c_{36}, c_{52}, c_{54}, c_{56}</td>
</tr>
<tr>
<td>+ (12.54.36) (34.16.52) (56.32.14)</td>
<td></td>
</tr>
<tr>
<td>(12.34.65) (64.15.32) (35.62.14)</td>
<td>c_{12}, c_{14}, c_{16}, c_{32}, c_{34}, c_{36}, c_{62}, c_{64}, c_{65}</td>
</tr>
<tr>
<td>+ (12.64.35) (34.15.62) (65.32.14)</td>
<td></td>
</tr>
<tr>
<td>(12.43.56) (53.16.42) (46.52.13)</td>
<td>c_{12}, c_{13}, c_{16}, c_{42}, c_{43}, c_{46}, c_{52}, c_{53}, c_{56}</td>
</tr>
<tr>
<td>+ (12.53.46) (43.16.52) (56.42.13)</td>
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</tr>
<tr>
<td>(12.43.65) (63.15.42) (45.62.13)</td>
<td>c_{12}, c_{13}, c_{16}, c_{42}, c_{43}, c_{45}, c_{62}, c_{63}, c_{65}</td>
</tr>
<tr>
<td>+ (12.63.45) (43.15.62) (65.42.13)</td>
<td></td>
</tr>
<tr>
<td>(12.53.64) (63.14.52) (54.62.13)</td>
<td>c_{12}, c_{13}, c_{14}, c_{52}, c_{53}, c_{54}, c_{62}, c_{63}, c_{64}</td>
</tr>
<tr>
<td>+ (12.63.54) (53.14.62) (64.52.13)</td>
<td></td>
</tr>
</tbody>
</table>
14. The twenty-seven lines uniquely determined by three trihedral pairs.

It is inferable, since each trihedral pair gives nine lines, that it is always possible to place together three trihedral pairs so as to give all twenty-seven lines. By means of the above table it will be shown that such is the case.

Choosing two trihedral pairs from the first table, and one from the last table, such as
\[
\begin{align*}
\{1, 2, 12\}, \quad \{4, 5, 45\}, \quad \{14, 15, 16\}, \\
\{2, 3, 23\}, \quad \{5, 6, 56\}, \quad \{24, 25, 26\}, \\
\{3, 1, 31\}, \quad \{6, 4, 64\}, \quad \{34, 35, 36\}
\end{align*}
\]
we see that these determine all twenty-seven lines. There are ten sets of this type, exhausting the first and third tables.

The formation of the remaining triads is accomplished by means of the following consideration. Consider the trihedral pair
\[
\begin{align*}
\{3, 5, 35\}, \\
\{6, 4, 64\}, \\
\{34, 56, 12\}
\end{align*}
\]
Obviously the pairs to be associated with this pair are those that determine \(a_4, a_5; b_1, b_2\) and \(a_1, a_2 b_3, b_6\)—i.e. the trihedral pairs
\[
\begin{align*}
\{4, 1, 41\}, \quad \{1, 5, 15\}, \\
\{5, 2, 52\}, \quad \{2, 6, 26\}, \\
\{42, 51, 36\}, \quad \{16, 25, 34\}
\end{align*}
\]
An inspection of the table will reveal the truth of this statement.

It follows that, choosing any trihedral pair arbitrarily, then the two remaining pairs to be associated with this one are uniquely determined. The three triads so associated contain all the twenty lines; and it may easily be shown that it contains them twice.

Below are listed the forty triads of trihedral pairs in the abbreviated notation.
\[
\begin{array}{c|c|c}
1 & 2 & 12 \\
2 & 3 & 23 \\
3 & 1 & 31 \\
\hline
1 & 2 & 12 \\
2 & 4 & 24 \\
4 & 1 & 41 \\
\hline
1 & 2 & 12 \\
2 & 5 & 25 \\
5 & 1 & 51 \\
\hline
1 & 2 & 12 \\
2 & 6 & 26 \\
6 & 1 & 61 \\
\hline
1 & 3 & 13 \\
3 & 4 & 34 \\
4 & 1 & 41 \\
\hline
1 & 3 & 13 \\
3 & 5 & 35 \\
5 & 1 & 51 \\
\hline
1 & 3 & 13 \\
3 & 6 & 36 \\
6 & 1 & 61 \\
\hline
1 & 4 & 14 \\
4 & 5 & 45 \\
5 & 1 & 51 \\
\hline
1 & 4 & 14 \\
4 & 6 & 46 \\
6 & 1 & 61 \\
\hline
1 & 5 & 15 \\
5 & 6 & 56 \\
6 & 1 & 61 \\
\hline
3 & 5 & 35 \\
6 & 4 & 64 \\
34 & 56 & 12 \\
\end{array}
\]
\[
\begin{array}{c}
\{ 4 \ 3 \ 43 \} , \quad \{ 5 \ 6 \ 56 \} , \quad \{ 35 \ 46 \ 12 \} , \\
\quad \{ 3 \ 6 \ 36 \} , \quad \{ 4 \ 5 \ 45 \} , \quad \{ 1 \ 2 \ 5 \} , \\
\quad \{ 2 \ 4 \ 24 \} , \quad \{ 5 \ 2 \ 52 \} , \quad \{ 3 \ 1 \ 32 \} , \\
\quad \{ 6 \ 1 \ 61 \} , \quad \{ 2 \ 3 \ 23 \} , \quad \{ 1 \ 4 \ 14 \} , \\
\quad \{ 14 \ 25 \ 36 \} , \quad \{ 15 \ 26 \ 34 \} , \quad \{ 13 \ 24 \ 56 \} , \\
\end{array}
\]
<table>
<thead>
<tr>
<th></th>
<th>2 5 25</th>
<th>3 4 34</th>
<th>1 6 16</th>
</tr>
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<td>63</td>
<td>6 63 14</td>
<td>5 1 51</td>
<td>4 2 42</td>
</tr>
<tr>
<td>23</td>
<td>65 14</td>
<td>31 54 26</td>
<td>12 46 35</td>
</tr>
<tr>
<td>36</td>
<td>3 6 36</td>
<td>1 3 13</td>
<td>2 4 24</td>
</tr>
<tr>
<td>5 2 52</td>
<td>4 5 45</td>
<td>6 1 61</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>56 14</td>
<td>15 43 26</td>
<td>21 64 35</td>
</tr>
<tr>
<td>26</td>
<td>2 6 26</td>
<td>5 4 54</td>
<td>1 3 13</td>
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<tr>
<td>35</td>
<td>3 5 35</td>
<td>6 1 61</td>
<td>4 2 42</td>
</tr>
<tr>
<td>36</td>
<td>3 6 36</td>
<td>1 3 13</td>
<td>2 4 24</td>
</tr>
<tr>
<td>5 3 53</td>
<td>1 6 16</td>
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<tr>
<td>6 2 62</td>
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<td>52</td>
<td>63 14</td>
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<td>2 6 26</td>
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<td>3 5 35</td>
<td>6 1 61</td>
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<td>56 14</td>
<td>31 64 25</td>
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<td>3 6 36</td>
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<td>4 2 42</td>
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<td>4 2 42</td>
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<td>3 4 34</td>
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<td>41</td>
<td>65 23</td>
<td>12 53 46</td>
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</tr>
<tr>
<td>3 6 26</td>
<td>4 5 45</td>
<td>1 3 13</td>
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</tr>
<tr>
<td>24</td>
<td>35 15</td>
<td>13 54 26</td>
<td>21 65 34</td>
</tr>
<tr>
<td>43</td>
<td>4 3 43</td>
<td>1 6 16</td>
<td>2 5 25</td>
</tr>
<tr>
<td>6 2 62</td>
<td>5 4 54</td>
<td>3 1 31</td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>63 15</td>
<td>14 56 23</td>
<td>21 35 46</td>
</tr>
<tr>
<td>26</td>
<td>2 6 26</td>
<td>3 5 35</td>
<td>1 4 14</td>
</tr>
<tr>
<td>4 3 43</td>
<td>6 1 61</td>
<td>5 2 52</td>
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</tr>
<tr>
<td>23</td>
<td>46 15</td>
<td>31 65 24</td>
<td>12 54 36</td>
</tr>
</tbody>
</table>
15. The Cubic Surface referred to a Pair of Triheders.

It follows from the preceding discussion, in connection with § 4, that the equation of the cubic surface, when written in the form

\[ U V W - \lambda X Y Z = 0, \]

may be said to be referred to a pair of triheders. Since there are precisely one hundred and twenty trihedral pairs, it follows that the equation of our fundamental cubic surface may be reduced to the form

\[ U V W - \lambda X Y Z = 0 \]

in one hundred and twenty ways. That is to say, we have arrived at a geometrical solution of the algebraic problem: In how many ways may the general equation of the third degree \((x, y, z, w)^3 = 0\) be reduced to the form

\[ U V W - \lambda X Y Z = 0, \]

where \(U, V, W; X, Y, Z\) are linear polynomials in \(x, y, z, w\)?

\[
\begin{align*}
\begin{bmatrix} 3 & 4 & 34 \end{bmatrix}, & \begin{bmatrix} 1 & 6 & 16 \end{bmatrix}, & \begin{bmatrix} 2 & 5 & 25 \end{bmatrix} \\
\begin{bmatrix} 6 & 2 & 62 \end{bmatrix}, & \begin{bmatrix} 5 & 3 & 53 \end{bmatrix}, & \begin{bmatrix} 4 & 1 & 41 \end{bmatrix} \\
\begin{bmatrix} 32 & 64 & 15 \end{bmatrix}, & \begin{bmatrix} 13 & 56 & 24 \end{bmatrix}, & \begin{bmatrix} 21 & 45 & 36 \end{bmatrix} \\
\begin{bmatrix} 2 & 4 & 24 \end{bmatrix}, & \begin{bmatrix} 3 & 6 & 36 \end{bmatrix}, & \begin{bmatrix} 1 & 5 & 15 \end{bmatrix} \\
\begin{bmatrix} 5 & 3 & 53 \end{bmatrix}, & \begin{bmatrix} 4 & 1 & 41 \end{bmatrix}, & \begin{bmatrix} 6 & 2 & 62 \end{bmatrix} \\
\begin{bmatrix} 23 & 54 & 16 \end{bmatrix}, & \begin{bmatrix} 31 & 46 & 25 \end{bmatrix}, & \begin{bmatrix} 12 & 65 & 34 \end{bmatrix} \\
\begin{bmatrix} 3 & 5 & 35 \end{bmatrix}, & \begin{bmatrix} 1 & 4 & 14 \end{bmatrix}, & \begin{bmatrix} 2 & 6 & 26 \end{bmatrix} \\
\begin{bmatrix} 4 & 2 & 42 \end{bmatrix}, & \begin{bmatrix} 6 & 3 & 63 \end{bmatrix}, & \begin{bmatrix} 5 & 1 & 51 \end{bmatrix} \\
\begin{bmatrix} 32 & 45 & 16 \end{bmatrix}, & \begin{bmatrix} 13 & 64 & 25 \end{bmatrix}, & \begin{bmatrix} 21 & 56 & 34 \end{bmatrix} \\
\begin{bmatrix} 2 & 5 & 25 \end{bmatrix}, & \begin{bmatrix} 4 & 6 & 46 \end{bmatrix}, & \begin{bmatrix} 1 & 3 & 13 \end{bmatrix} \\
\begin{bmatrix} 3 & 4 & 34 \end{bmatrix}, & \begin{bmatrix} 5 & 1 & 51 \end{bmatrix}, & \begin{bmatrix} 6 & 2 & 62 \end{bmatrix} \\
\begin{bmatrix} 24 & 35 & 16 \end{bmatrix}, & \begin{bmatrix} 41 & 56 & 23 \end{bmatrix}, & \begin{bmatrix} 12 & 63 & 45 \end{bmatrix} \\
\begin{bmatrix} 4 & 3 & 43 \end{bmatrix}, & \begin{bmatrix} 1 & 5 & 15 \end{bmatrix}, & \begin{bmatrix} 2 & 6 & 26 \end{bmatrix} \\
\begin{bmatrix} 5 & 2 & 52 \end{bmatrix}, & \begin{bmatrix} 6 & 4 & 64 \end{bmatrix}, & \begin{bmatrix} 3 & 1 & 31 \end{bmatrix} \\
\begin{bmatrix} 42 & 53 & 16 \end{bmatrix}, & \begin{bmatrix} 14 & 65 & 23 \end{bmatrix}, & \begin{bmatrix} 21 & 36 & 45 \end{bmatrix} \\
\begin{bmatrix} 2 & 5 & 25 \end{bmatrix}, & \begin{bmatrix} 3 & 6 & 36 \end{bmatrix}, & \begin{bmatrix} 1 & 4 & 14 \end{bmatrix} \\
\begin{bmatrix} 4 & 3 & 43 \end{bmatrix}, & \begin{bmatrix} 5 & 1 & 51 \end{bmatrix}, & \begin{bmatrix} 6 & 2 & 62 \end{bmatrix} \\
\begin{bmatrix} 23 & 45 & 16 \end{bmatrix}, & \begin{bmatrix} 31 & 56 & 24 \end{bmatrix}, & \begin{bmatrix} 12 & 64 & 35 \end{bmatrix} \\
\begin{bmatrix} 3 & 4 & 34 \end{bmatrix}, & \begin{bmatrix} 1 & 5 & 15 \end{bmatrix}, & \begin{bmatrix} 2 & 6 & 26 \end{bmatrix} \\
\begin{bmatrix} 5 & 2 & 52 \end{bmatrix}, & \begin{bmatrix} 6 & 3 & 63 \end{bmatrix}, & \begin{bmatrix} 4 & 1 & 41 \end{bmatrix} \\
\begin{bmatrix} 32 & 54 & 16 \end{bmatrix}, & \begin{bmatrix} 13 & 65 & 24 \end{bmatrix}, & \begin{bmatrix} 21 & 46 & 35 \end{bmatrix}
\end{align*}
\]
CHAPTER IV

ANALYTICAL INVESTIGATION OF THE TWENTY-SEVEN LINES
AND FORTY-FIVE TRIPLE TANGENT PLANES FOR THE
GENERAL EQUATION OF THE CUBIC SURFACE


The general equation of a surface, of degree 3, class 12, may be
written in the form
\[(x, y, z, w)^3 = 0,\]
x, y, z, w representing co-ordinates referred to a fundamental tetra­
hedron \(ABCD\), where, as usual, we denote the plane of \(ABC\) by the
equation \(w = 0\), and so on*.

It has been shown (§ 15) that it is possible, and that in 120 ways,
to express the equation of the cubic surface in the canonical form
\[UVW - kXYZ = 0,\]
where \(U, V, W; X, Y, Z\) represent polynomes of the first degree in
\(x, y, z, w\); and then the equation is said to be referred to a pair of
tri­heders.

Let us choose for the equation of the cubic surface, with twenty­
seven distinct straight lines lying upon it—to be discussed in
subsequent articles—the following simple and symmetrical form†:
\[
\left(\frac{x}{x_2} + \frac{y}{y_2} + \frac{z}{z_2} + \frac{w}{w_2}\right) \left(\frac{xz}{x_1z_1} - \frac{yw}{y_1w_1}\right)
\]
\[- k \left(\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} + \frac{w}{w_1}\right) \left(\frac{xz}{x_2z_2} - \frac{yw}{y_2w_2}\right) = 0.
\]

† This equation with specialized coefficients was chosen because the equations
of a large number of the straight lines upon the surface can be determined by in­
spection. Moreover, it develops that the subsequent construction of a model of the
twenty-seven lines is quite feasible for this form of the surface. The equation was
(1870).
This equation may be thrown into the following form:

\[
ax \left[ y_1 y_2 w_1 w_2 (x_2 - k z_1) x - w_1 w_2 (x_1 y_2 z_1 - k x_2 z_1) y + y_1 y_2 w_1 w_2 (x_2 - k x_1) z - w_1 w_2 (x_1 z_1 w_2 - k x_2 z_1 w_1) w \right] - yw \left[ z_1 x_2 (x_1 y_2 z_2 - k x_2 y_1 w_1) x - x_1 x_2 z_1 (w_1 - k w_1) y + x_1 x_2 (y_2 z_1 w_2 - k y_1 z_2 w_1) z - x_1 x_2 z_1 z_2 (y_1 - k y_2) w \right] = 0.
\]

We now recognize it as being in the canonical form, referred to a pair of triheders.

By inspection of the first form of the equation written above it is manifest that the thirteen lines, given by the following equations, lie wholly upon the surface:

2': \[x = 0, \quad y = 0\] \(\ldots\) \(f)\),
1: \[x = 0, \quad w = 0\] \(\ldots\) \(g)\),
3: \[y = 0, \quad z = 0\] \(\ldots\) \(h)\),
4': \[z = 0, \quad w = 0\] \(\ldots\) \(i)\),
5: \[\frac{x}{x_1} + \frac{y}{y_1} = 0, \quad \frac{z}{z_1} + \frac{w}{w_1} = 0\] \(\ldots\) \(j)\),
6: \[\frac{x}{x_2} + \frac{y}{y_2} = 0, \quad \frac{z}{z_2} + \frac{w}{w_2} = 0\] \(\ldots\) \(k)\),
5': \[\frac{x}{x_3} + \frac{w}{w_3} = 0, \quad \frac{y}{y_3} + \frac{z}{z_3} = 0\] \(\ldots\) \(l)\),
6': \[\frac{x}{x_1} + \frac{w}{w_1} = 0, \quad \frac{y}{y_1} + \frac{z}{z_1} = 0\] \(\ldots\) \(m)\),
12: \[x = 0, \quad \left( \frac{y}{y_2} + \frac{z}{z_2} + \frac{w}{w_2} \right) \frac{1}{y_1 w_1} - k \left( \frac{y}{y_1} + \frac{z}{z_1} + \frac{w}{w_1} \right) \frac{1}{y_2 w_2} = 0\] \(\ldots\) \(n)\),
23: \[y = 0, \quad \left( \frac{x}{x_2} + \frac{z}{z_2} + \frac{w}{w_2} \right) \frac{1}{x_1 w_1} - k \left( \frac{x}{x_1} + \frac{z}{z_1} + \frac{w}{w_1} \right) \frac{1}{x_2 z_2} = 0\] \(\ldots\) \(p)\),
34: \[z = 0, \quad \left( \frac{x}{x_2} + \frac{y}{y_2} + \frac{w}{w_2} \right) \frac{1}{x_1 y_1} - k \left( \frac{x}{x_1} + \frac{y}{y_1} + \frac{w}{w_1} \right) \frac{1}{y_2 w_2} = 0\] \(\ldots\) \(q)\),
41: \[w = 0, \quad \left( \frac{x}{x_2} + \frac{y}{y_2} + \frac{z}{z_2} \right) \frac{1}{x_1 z_1} - k \left( \frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} \right) \frac{1}{x_2 z_2} = 0\] \(\ldots\) \(r)\),
56: \[\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} + \frac{w}{w_1} = 0, \quad \frac{x}{x_2} + \frac{y}{y_2} + \frac{z}{z_2} + \frac{w}{w_2} = 0\] \(\ldots\) \(s)\).
In order to reach the designation on the left, we must have recourse to the conception of the double six:

\[
\begin{align*}
1, & \quad 2, & \quad 3, & \quad 4, & \quad 5, & \quad 6, \\
1', & \quad 2', & \quad 3', & \quad 4', & \quad 5', & \quad 6',
\end{align*}
\]

in which no two lines in the same row intersect, but each line of the one row intersects all but the corresponding line of the other system. Moreover any two lines such as 1, 2' lie in a plane denoted 12'; similarly the lines 1', 2 lie in a plane denoted 1'2. These two planes meet in a line 12; and any three lines such as 12, 34, 56 meet in pairs, lying in a plane 12 . 34 . 56.

Now, considering the first eight lines written down, the following table showing intersections enables us to designate these eight lines as indicated by the notation shown at the left.

\[
\begin{array}{cccc}
& f & i & l \\
g & + & + & + & + \\
h & + & + & + & + \\
j & + & + & + & + \\
k & + & + & + & + \\
\end{array}
\]

Considering next the line \( (n) \), it appears that it lies in the same plane as lines 1 and 2' and hence must be the line 12. Similar reasoning holds for the lines \( (p), (q) \) and \( (r) \). Inspection of equations \( (f), (k) \) and \( (s) \) reveals the fact that the line \( (s) \) is none other than the line 56.

It remains to compute the equations of the remaining lines, fourteen in number, which lie upon the cubic. Let us first compute the remaining four lines of the double six, viz. 1', 2, 3' and 4. The following method immediately suggests itself.

The lines 3, 5, 6, and 12 are met by the line 2', and by a second line 1'. This line 1', as a line meeting 3, 5 and 6, will be given by equations of the form

\[
x + \frac{x_1}{y_1} y = \phi \left( \frac{w_1}{z_1} z + w \right); \quad x + \frac{x_2}{y_2} y = \phi \left( \frac{w_2}{z_2} z + w \right).
\]

Now, noting that these equations, setting therein \( x = 0 \), become

\[
\frac{z}{z_1} + \frac{w}{w_1} = \frac{x_1}{y_1 w_1} \phi y; \quad \frac{z}{z_2} + \frac{w}{w_2} = \frac{x_2}{y_2 w_2} \phi y,
\]
we see that the condition of intersection with the line 12 gives

\[ \phi = \frac{x_2 - k x_1}{w_2 - k w_1}. \]

Along with these equations may be written the resulting equation

\[ z_1 z_2 (x_2 y_1 - x_1 y_2) y = \phi y_1 y_2 (z_2 w_1 - z_1 w_2) z. \]

Carrying out the computations, similarly, for 2, 3', and 4, we may tabulate the equations of the four lines as follows:

\[
\begin{align*}
1': & \\
& \begin{align*}
x + \frac{x_1}{y_1} y &= \phi \left( \frac{w_1}{z_1} z + w \right); \\
x + \frac{x_2}{y_2} y &= \phi \left( \frac{w_2}{z_2} z + w \right); \\
z_1 z_2 (x_2 y_1 - x_1 y_2) y &= \phi y_1 y_2 (z_2 w_1 - z_1 w_2) z; \\
\phi &= \frac{x_2 - k x_1}{w_2 - k w_1}.
\end{align*}
\end{align*}
\]

\[
\begin{align*}
2: & \\
& \begin{align*}
x + \frac{x_1}{w_1} w &= \phi \left( y + \frac{y_1}{z_1} z \right); \\
x + \frac{x_2}{w_2} w &= \phi \left( y + \frac{y_2}{z_2} z \right); \\
z_1 z_2 (x_2 w_1 - x_1 w_2) w &= \phi w_1 w_2 (y_2 z_1 - y_1 z_2) z; \\
\phi &= -\frac{x_2 - k x_1}{y_2 - k y_1}.
\end{align*}
\end{align*}
\]

\[
\begin{align*}
3': & \\
& \begin{align*}
-\phi \left( \frac{y_1}{x_1} x + y \right) &= z + \frac{z_1}{w_1} w; \\
-\phi \left( \frac{y_2}{x_2} x + y \right) &= z + \frac{z_2}{w_2} w; \\
\phi w_1 w_2 (x_2 y_1 - x_1 y_2) x &= x_1 x_2 (z_2 w_1 - z_1 w_2) w; \\
\phi &= \frac{z_2 - k z_1}{y_2 - k y_1}.
\end{align*}
\end{align*}
\]

\[
\begin{align*}
4: & \\
& \begin{align*}
\phi \left( \frac{w_1}{x_1} x + w \right) &= z + \frac{z_1}{y_1} y; \\
\phi \left( \frac{w_2}{x_2} x + w \right) &= z + \frac{z_2}{y_2} y; \\
\phi y_1 y_2 (x_2 w_1 - x_1 w_2) x &= x_1 x_2 (y_2 z_1 - y_1 z_2) y; \\
\phi &= -\frac{z_2 - k z_1}{w_2 - k w_1}.
\end{align*}
\end{align*}
\]
On examination of the above equations of the seventeen lines it will appear that some of the forty-five triple tangent planes may be determined by inspection.

In the first place, the planes \( x = 0, y = 0, z = 0, w = 0 \) are the triple tangent planes 12', 32', 34', 14', since they contain the lines 1, 2', 12; 2', 3, 23; 3, 4', 34; and 4', 1, 41 respectively.

By inspection, we observe that \( \frac{x}{x_2} + \frac{w}{w_2} = 0 \) is the equation of plane 15', since in it lie both the lines 1 and 5'. In precisely similar manner we determine the equations of the planes 16', 35', 36', 52', 54', 62', 64'.

It appears, from an examination of the equations of lines 5 and 6', that the plane 56' has the form

\[
\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} + \frac{w}{w_1} = 0;
\]

and similarly for the plane 65'.

Considering now the third type, 13' for example, it is obvious that its equation is identical with the third equation written under 3' above, since it vanishes identically for \( x = 0, w = 0 \); that is

\[
13': \quad \phi w_1 w_2 (x_2 y_3 - x_3 y_2) x = x_2 x_3 (z_2 w_1 - z_1 w_2) w;
\]

and in this way we may determine also the equations of the planes 24', 31', and 42'.

Considering the fourth type, 26' say, it is obvious that the lines 2 and 5' both lie in the plane

\[
\left( \frac{x}{x_2} + \frac{w}{w_2} \right) + \phi \frac{y}{y_2} \frac{x_2}{y_2} \frac{k}{y_1} \left( \frac{y}{y_2} \frac{z}{z_2} \right) = 0.
\]

By analogous reasoning, we obtain similar equations for the planes 26', 45', 46', 51', 53', 61', 63'.

The remaining types are not discoverable by inspection, and direct calculation has to be resorted to. Take the plane 23' for example, on which lie the lines 2, 3', and 23. Any plane through 2 is of the form

\[
\left( x + \frac{x_2}{w_2} w \right) - \phi \left( y + \frac{y_2}{z_2} z \right) + \lambda \left( x + \frac{x_1}{w_1} w \right) - \phi \left( y + \frac{y_1}{z_1} z \right) = 0;
\]

\[
\phi = - \frac{(x_2 - k x_1)}{(y_2 - k y_1)}.
\]
Now a point on the line 3' is 

\[(0, -y_2 + ky_1, z_2 - kz_1, 0).\]

Hence (after substitution and reduction) \(\lambda = \frac{hx_1}{z_2}.\) Then we have

\[\left( x + \frac{x_2}{w_2} w \right) - \phi \left( y + \frac{y_2}{z_2} z \right) - \frac{hx_1}{z_2} \left\{ \left( \frac{x_2}{w_2} w \right) - \phi \left( \frac{y_2}{z_2} z \right) \right\} = 0,\]

which reduces to the form

\[w_1w_2(y_2 - ky_1)(z_2 - kz_1) x + w_1w_2(x_2 - kx_1)(z_2 - kz_1) y + w_1w_2(x_2 - kx_1)(y_2 - ky_1) z + (y_2 - ky_1)(x_2z_2w_1 - kx_1z_1w_2) w = 0.\]

In this fashion we determine, besides the plane 23', the three other planes of like form 21', 41', and 43'.

There remain fifteen equations to be determined. Consider the plane 12. 35. 46, which passes through the lines 12, 35, and 46. Now the planes 35', 3'5 intersect in the line 35. Hence any plane through 35 is of the form

\[\frac{y_1}{z_1} \left( \frac{z_2 - kz_1}{y_2 - ky_1} \right) \frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} + w = 0.\]

Since this plane passes through the line 12, if we place \(x = 0\) in it, we must identify

\[\left\{ \frac{1}{z_1} \left( \frac{z_2 - kz_1}{y_2 - ky_1} - \frac{\lambda}{y_2} \right) y + \left( \frac{z_2 - \lambda z_1}{z_1 z_2} \right) z + w = 0 \right\} \}

with

\[\left( \frac{w_2 - kw_1}{y_1y_2w_1w_2} \right) y + \left( \frac{y_2z_1w_2 - kyz_2w_1}{y_1y_2z_1w_1w_2} \right) z + \left( \frac{y_2 - ky_1}{y_1y_2w_1w_2} \right) w = 0 \]

Hence, multiplying equation (1) through by the factor \(y_1y_2w_1w_2\); and comparing coefficients, we have the two equations of condition:

\[\left( \frac{y_2 - ky_1}{y_1y_2w_1w_2} \right) \left( \frac{1}{z_1} \left( \frac{z_2 - kz_1}{y_2 - ky_1} - \frac{\lambda}{y_2} \right) \right) = \left( \frac{w_2 - kw_1}{y_1y_2w_1w_2} \right);\]

\[\left( \frac{y_2 - ky_1}{y_1y_2w_1w_2} \right) \left( \frac{z_2 - \lambda z_1}{z_1 z_2} \right) = \left( \frac{y_2z_1w_2 - kyz_2w_1}{y_1y_2z_1w_1w_2} \right).\]

From either one of these equations we derive the following value for the parameter:

\[\lambda = \frac{y_2}{z_1w_1} \left( \frac{z_2w_1 - z_1w_2}{y_2 - ky_1} \right).\]
Substituting this value of λ in the equation of the plane, we finally obtain, after reduction,

\[ y_1z_2w_1(z_2 - kw_1) x + x_1z_1z_2(w_2 - kw_1) y + x_1(y_2z_2w_2 - ky_1z_2w_1) z + x_1z_1z_2(y_2 - ky_1) w = 0 \]

as the equation of the plane 12.35.46.

Consider now the next type, of which there are two equations, 12.34.56 and 14.23.56.

It is sufficient to derive here the equation of the plane 12.34.56. Since it passes through the intersection of the two planes 12' and 21', its equation is of the form

\[
\{\lambda + z_1z_2(y_2 - ky_1)(w_2 - kw_1)\} x + z_1z_2(x_2 - kw_1)(w_2 - kw_1) y + (x_2 - kw_1)(y_2z_2w_2 - ky_1z_2w_1) z + z_1z_2(x_2 - kw_1)(y_2 - ky_1) w = 0.
\]

Moreover, the plane 12.34.56 also passes through the intersection of the two planes, 34' and 43', whose equations are as follows:

34': \[ z = 0; \]

43': \[ (z_2 - kz_2)(x_1y_2w_2 - kx_2y_1w_1) x + x_1y_2(z_2 - kz_2)(w_2 - kw_1) y + x_1y_2(y_2(z_2 - ky_2)(w_2 - kw_1) z + x_1y_2(z_2 - kz_2)(y_2 - ky_1) w = 0. \]

In order to identify this second form of the plane 12.34.56 with the form written above, it is obvious by inspection that we must multiply the equation 34' through by \(k(x_1 - ky_1)(z_2 - kz_2)/z_1w_1 - z_1w_2)\), and the equation 43' through by \(z_1z_2(x_2 - kw_1)/x_1y_2(z_2 - kz_2)\), and add the two resulting equations. Comparing coefficients of the same variable, in the two forms for the equation 12.34.56, we derive by inspection the desired equation

12.34.56: \[ z_1z_2(x_1y_2w_2 - kx_2y_1w_1) x + x_1z_1z_2(w_2 - kw_1) y + x_1y_2(z_2 - ky_2)(w_2 - kw_1) z + x_1z_1z_2(y_2 - ky_1) w = 0. \]

Similarly we may derive the four equations of the type 13.25.46; and likewise the equation of 13.24.56, which is unique.

The results of the investigation, the character of the derivation having been explained in detail, are given in the following tables.

H.
\begin{align*}
12' &\equiv \pi_1 \quad x = 0 \\
14' &\equiv \pi_2 \quad w = 0 \\
32' &\equiv \pi_3 \quad y = 0 \\
34' &\equiv \pi_4 \quad z = 0 \\
15' &\equiv \pi_5 \quad \frac{x}{x_2} + \frac{w}{w_2} = 0 \\
16' &\equiv \pi_6 \quad \frac{x}{x_1} + \frac{w}{w_1} = 0 \\
35' &\equiv \pi_7 \quad \frac{y}{y_2} + \frac{z}{z_2} = 0 \\
36' &\equiv \pi_8 \quad \frac{y}{y_1} + \frac{z}{z_1} = 0 \\
52' &\equiv \pi_9 \quad \frac{x}{x_1} + \frac{y}{y_1} = 0 \\
54' &\equiv \pi_{10} \quad \frac{z}{z_1} + \frac{w}{w_1} = 0 \\
62' &\equiv \pi_{11} \quad \frac{x}{x_2} + \frac{y}{y_2} = 0 \\
64' &\equiv \pi_{12} \quad \frac{z}{z_2} + \frac{w}{w_2} = 0 \\
56' &\equiv \pi_{13} \quad \frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} + \frac{w}{w_1} = 0 \\
65' &\equiv \pi_{14} \quad \frac{x}{x_2} + \frac{y}{y_2} + \frac{z}{z_2} + \frac{w}{w_2} = 0 \\
13' &\equiv \pi_{15} \quad w_1 w_2 (z_2 - k z_1) (x_2 y_1 - x_1 y_2) x - x_1 x_2 (y_2 - k y_1) (z_2 w_1 - z_1 w_2) w = 0 \\
24' &\equiv \pi_{16} \quad w_1 w_2 (x_2 - k x_1) (y_2 z_1 - y_1 z_2) z + z_1 z_2 (y_2 - k y_1) (x_2 w_1 - x_1 w_2) w = 0 \\
31' &\equiv \pi_{17} \quad z_1 z_2 (w_2 - k w_1) (x_2 y_1 - x_1 y_2) y - y_1 y_2 (x_2 - k x_1) (z_2 w_1 - z_1 w_2) z = 0 \\
42' &\equiv \pi_{18} \quad y_1 y_2 (z_2 - k z_1) (x_2 w_1 - x_1 w_2) x + x_1 x_2 (w_2 - k w_1) (y_2 z_1 - y_1 z_2) y = 0 \\
25' &\equiv \pi_{19} \quad \left( \frac{x}{x_2} + \frac{w}{w_2} \right) + \frac{y_2}{y_2} \left( x_2 - k x_1 \right) \left( \frac{y}{y_2} + \frac{z}{z_2} \right) = 0 \\
26' &\equiv \pi_{20} \quad \left( \frac{x}{x_1} + \frac{w}{w_1} \right) + \frac{y_1}{y_1} \left( x_2 - k x_1 \right) \left( \frac{y}{y_1} + \frac{z}{z_1} \right) = 0
\end{align*}
\begin{align*}
45' & \equiv \pi_{21} \quad \left( \frac{x}{x_2} + \frac{\nu}{\nu_2} + \frac{z_2}{w_2} \left( \frac{\nu_2 - kw_2}{z_2 - kx_2} \right) \right) \left( \frac{y}{y_2} + \frac{\nu}{\nu_2} \right) = 0 \\
46' & \equiv \pi_{22} \quad \left( \frac{x}{x_1} + \frac{\nu}{\nu_1} + \frac{z_1}{w_1} \left( \frac{\nu_2 - kw_2}{z_2 - kx_2} \right) \right) \left( \frac{y}{y_1} + \frac{\nu}{\nu_1} \right) = 0 \\
51' & \equiv \pi_{23} \quad \left( \frac{x}{x_1} + \frac{\nu}{\nu_1} + \frac{z_1}{w_1} \left( \frac{\nu_2 - kw_2}{z_2 - kx_2} \right) \right) \left( \frac{z}{z_1} + \frac{\nu}{\nu_1} \right) = 0 \\
53' & \equiv \pi_{24} \quad \left( \frac{x}{x_2} + \frac{\nu}{\nu_2} + \frac{z_2}{w_2} \left( \frac{\nu_2 - kw_2}{z_2 - kx_2} \right) \right) \left( \frac{z}{z_2} + \frac{\nu}{\nu_2} \right) = 0 \\
61' & \equiv \pi_{25} \quad \left( \frac{x}{x_2} + \frac{\nu}{\nu_2} + \frac{z_2}{w_2} \left( \frac{\nu_2 - kw_2}{z_2 - kx_2} \right) \right) \left( \frac{z}{z_2} + \frac{\nu}{\nu_2} \right) = 0 \\
63' & \equiv \pi_{26} \quad \left( \frac{x}{x_2} + \frac{\nu}{\nu_2} + \frac{z_2}{w_2} \left( \frac{\nu_2 - kw_2}{z_2 - kx_2} \right) \right) \left( \frac{z}{z_2} + \frac{\nu}{\nu_2} \right) = 0 \\
21' & \equiv \pi_{27} \quad z_2 \left( y_2 - ky_1 \right) \left( w_2 - kw_1 \right) x + z_1 z_2 \left( x_2 - kx_1 \right) \left( w_2 - kw_1 \right) y \\
& \quad + \left( x_2 - kx_1 \right) \left( y_2 z_1 w_2 - ky_1 z_2 w_1 \right) z + z_1 z_2 \left( x_2 - kx_1 \right) \left( y_2 - ky_1 \right) w = 0 \\
23' & \equiv \pi_{28} \quad w_1 w_2 \left( y_2 - ky_1 \right) \left( w_2 - kw_1 \right) x + w_1 w_2 \left( x_2 - kx_1 \right) \left( w_2 - kw_1 \right) y \\
& \quad + w_1 w_2 \left( x_2 - kx_1 \right) \left( y_2 - ky_1 \right) z + \left( y_2 - ky_1 \right) \left( w_2 z_2 w_1 - kx_1 z_1 w_2 \right) w = 0 \\
41' & \equiv \pi_{29} \quad y_1 y_2 \left( y_2 - ky_1 \right) \left( w_2 - kw_1 \right) x + \left( w_2 - kw_1 \right) \left( y_2 z_1 \left( x_2 - kx_1 \right) \left( w_2 - kw_1 \right) y \\
& \quad + \left( x_2 - kx_1 \right) \left( w_2 - kw_1 \right) z + y_1 y_2 \left( w_2 - kw_1 \right) \left( x_2 - kx_1 \right) \left( w_2 - kw_1 \right) \left( z_2 - kz_1 \right) w = 0 \\
43' & \equiv \pi_{30} \quad \left( x_2 - kx_1 \right) \left( x_1 y_2 w_2 - kx_2 y_1 w_1 \right) x + x_1 x_2 \left( x_2 - kx_1 \right) \left( w_2 - kw_1 \right) y \\
& \quad + x_1 x_2 \left( y_2 - ky_1 \right) \left( w_2 - kw_1 \right) z + x_1 x_2 \left( x_2 - kx_1 \right) \left( y_2 - ky_1 \right) w = 0 \\
12.36.45 & \equiv \pi_{31} \quad y_2 z_1 w_2 \left( x_2 - kx_1 \right) \left( w_2 - kw_1 \right) y + z_2 \left( y_2 z_1 w_2 - ky_1 z_2 w_1 \right) z + x_2 z_2 \left( y_2 - ky_1 \right) w = 0 \\
12.35.46 & \equiv \pi_{32} \quad y_1 z_2 w_1 \left( x_2 - kx_1 \right) \left( w_2 - kw_1 \right) y + z_1 \left( y_2 z_1 w_2 - ky_1 z_2 w_1 \right) z + x_1 z_2 \left( y_2 - ky_1 \right) w = 0 \\
14.25.36 & \equiv \pi_{33} \quad y_1 y_2 w_2 \left( x_2 - kx_1 \right) \left( w_2 - kw_1 \right) y + y_1 y_2 w_2 \left( x_2 - kx_1 \right) z + x_1 y_2 \left( y_2 - ky_1 \right) w = 0 \\
14.26.35 & \equiv \pi_{34} \quad y_1 y_2 w_1 \left( x_2 - kx_1 \right) \left( x_2 y_2 z_2 - kx_1 y_2 z_1 \right) y + y_1 y_2 w_1 \left( x_2 - kx_1 \right) z + x_1 y_2 \left( y_2 - ky_1 \right) w = 0 \\
15.23.46 & \equiv \pi_{35} \quad y_1 w_1 w_2 \left( x_2 - kx_1 \right) \left( x_2 y_2 z_2 - kx_1 y_2 z_1 \right) y + y_1 w_1 w_2 \left( x_2 - kx_1 \right) z + y_1 \left( x_2 z_2 w_1 - kx_1 z_1 w_2 \right) w = 0 \\
15.23.34 & \equiv \pi_{36} \quad z_1 \left( x_1 y_2 w_2 - kx_2 y_1 w_1 \right) x + x_1 x_2 z_1 \left( w_2 - kw_1 \right) y + y_1 w_1 w_2 \left( x_2 - kx_1 \right) z + x_1 x_2 \left( y_2 - ky_1 \right) w = 0 \\
16.23.45 & \equiv \pi_{37} \quad y_2 w_1 w_2 \left( x_2 - kx_1 \right) \left( x_2 y_2 z_2 - kx_1 y_2 z_1 \right) y + y_2 w_1 w_2 \left( x_2 - kx_1 \right) z + y_2 \left( x_2 z_2 w_1 - kx_1 z_1 w_2 \right) w = 0
\end{align*}
<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16.25.34$</td>
<td>$z_2 (x_1 y_2 w_2 - k x_2 y_1 w_1) x + x_1 w_2 z_2 (w_2 - k w_1) y$</td>
</tr>
<tr>
<td>$\equiv \pi_{38}$</td>
<td>$+ x_1 y_2 w_2 (x_2 - k x_1) z + x_1 x_2 z_2 (y_2 - k y_1) w = 0$</td>
</tr>
<tr>
<td>$12.34.56$</td>
<td>$z_1 z_2 (x_1 y_2 w_2 - k x_2 y_1 w_1) x + x_1 x_2 z_1 z_2 (w_2 - k w_1) y$</td>
</tr>
<tr>
<td>$\equiv \pi_{39}$</td>
<td>$+ x_1 x_2 (y_2 z_1 w_2 - k y_1 z_2 w_1) z + x_1 x_2 z_1 z_2 (y_2 - k y_1) w = 0$</td>
</tr>
<tr>
<td>$14.23.56$</td>
<td>$y_1 y_2 w_1 w_2 (z_2 - k z_1) x + w_1 w_2 (x_2 y_1 z_2 - k x_1 y_2 z_1) y$</td>
</tr>
<tr>
<td>$\equiv \pi_{40}$</td>
<td>$+ y_1 y_2 w_1 w_2 (x_2 - k x_1) z + y_1 y_2 (x_1 z_2 w_1 - k x_1 z_1 w_2) w = 0$</td>
</tr>
<tr>
<td>$13.25.46$</td>
<td>$y_1 z_2 w_1 w_2 (z_2 - k z_1) (x_1 y_2 - x_2 y_1) x$</td>
</tr>
<tr>
<td>$\equiv \pi_{41}$</td>
<td>$+ x_1 z_2 z_2 w_2 (w_2 - k w_1) (x_1 y_2 - x_2 y_1) y$</td>
</tr>
<tr>
<td>$\equiv \pi_{42}$</td>
<td>$- x_1 y_1 y_2 w_2 (x_2 - k x_1) (x_1 w_2 - z_2 w_1) z$</td>
</tr>
<tr>
<td>$13.26.45$</td>
<td>$y_2 z_1 w_1 w_2 (z_2 - k z_1) (x_1 y_2 - x_2 y_1) x$</td>
</tr>
<tr>
<td>$\equiv \pi_{43}$</td>
<td>$+ x_2 z_1 z_2 w_1 (w_2 - k w_1) (x_1 y_2 - x_2 y_1) y$</td>
</tr>
<tr>
<td>$\equiv \pi_{44}$</td>
<td>$- x_2 y_1 y_2 w_1 (x_2 - k x_1) (x_1 w_2 - z_2 w_1) z$</td>
</tr>
<tr>
<td>$15.24.36$</td>
<td>$y_1 y_2 z_1 w_2 (z_2 - k z_1) (x_1 w_2 - x_2 w_1) x$</td>
</tr>
<tr>
<td>$\equiv \pi_{45}$</td>
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</tr>
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</tr>
<tr>
<td>$\equiv \pi_{49}$</td>
<td>$+ x_1 y_2 z_1 z_2 w_1 w_2 (w_2 - k w_1) (x_1 y_2 - x_2 y_1) (y_1 z_2 - y_2 z_1) y$</td>
</tr>
<tr>
<td>$\equiv \pi_{50}$</td>
<td>$- x_1 y_2 z_1 z_2 w_1 w_2 (x_2 - k x_1) (y_1 z_2 - y_2 z_1) (x_1 w_2 - x_2 w_1) z$</td>
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<tr>
<td></td>
<td>$- x_1 y_2 z_1 z_2 w_1 w_2 (y_2 - k y_1) (x_1 w_2 - x_2 w_1) (x_1 w_2 - x_2 w_1) w = 0$</td>
</tr>
</tbody>
</table>

It is clear, from inspection of the equations of the forty-five triple tangent planes just tabulated, that a perfectly symmetrical system may be derived by setting $k = 1$. But the $k$ is retained at this time for a reason that will appear in the sequel. In fact, in the construction of the models, the circumstance that $k$ is within our choice enables us, after assigning numerical values to the other constants, to assign such a value to $k$ as will bring all the twenty-seven lines within easy reach. In a word, we use $k$, so to speak, as a lever.
17. The Analytic Expression of the Equation of a Cubic Surface in One Hundred and Twenty Forms of the type $UVW + \lambda XYZ = 0$.

By comparing the tables of the trihedral pairs (§ 13) with the table of triple tangent planes, we may determine at once the analytic expressions of the equation of the cubic surface into 120 different forms of the type

$$UVW + \lambda XYZ = 0,$$

where $U$, $V$, $W$, $X$, $Y$, $Z$ are linear polynomials in $x$, $y$, $z$ and $w$.

On reverting to the second form of the equation of the surface (§ 16), the constant $\lambda$ appears to be $-1$. It may, however, also equal $+1$, owing to the fact that the equations of the forty-five triple tangent planes have all been written with the variables arranged in the order $x, y, z, w$. For example,

$$\pi_1 \cdot \pi_{28} \cdot \pi_{17} - \pi_3 \cdot \pi_{27} \cdot \pi_{19} = 0$$

and

$$\pi_1 \cdot \pi_{16} \cdot \pi_{29} + \pi_{18} \cdot \pi_{27} \cdot \pi_2 = 0$$

are both forms for the equation.

Regarding the equations of the planes as absolutely fixed in sign, then the sign to be given to $\lambda (= \pm 1)$ can easily be determined—and indeed by inspection. It is unnecessary to list these forms.
CHAPTER V

THE CONSTRUCTION OF A MODEL OF A DOUBLE SIX


The construction of a model of a double six is a subject that has interested both Sylvester and Cayley. Sylvester* sketched a geometrical construction, not only for a double six, but also for the complete configuration of the twenty-seven lines. This same construction is also given by Salmon in his *Geometry of Three Dimensions* (4th edition, p. 500). This purely geometric construction, while it is of great theoretical interest, does not on its face afford any practical method of actually making a model of the configuration. Cayley† gave a verification of Schläfli's theorem by using his method of the six co-ordinates of a line, and thereby obtained the equational representation of the twelve lines of a double six. He chose arbitrarily the four sides 1', 2, 3', 4 of a skew quadrilateral; then selecting the co-ordinates of the lines 2', 5', 6' in the manner conditioned by the hypotheses for intersection, he determined the six co-ordinates of the five remaining lines 4', 1, 3, 5, and 6. Replacing the constants by numerical values, he obtained data for the construction of a model, but found on trial that it could not be constructed successfully with the values assigned.

Later, in 1873, Cayley‡ again treated the problem, this time approaching it through the medium of the cubic surface, the historical method of approach (cf. § 1). From a specialized form of the general equation of the cubic surface, he derived the equations of the twelve lines of the double six. Unfortunately, the numerical values he selected for the constants involved were ill chosen; and when he constructed a model of the configuration, he met with only partial success, some of the lines and intersections falling beyond convenient limits.

* *Comptes Rendus, Vol. XIX. (1861), pp. 977–980.
More recently, Blythe* has given a very elegant method of constructing a double six, by employing five pairs of points in involution on the line of intersection of two planes, making any convenient angle with each other.

19. A Practical Construction for the Model of a Double Six.

By making use of the equations of the lines used in the proof of Schläfi's Theorem (§ 7), it is possible to construct a perfect and simple model of a double six, which clearly brings out the salient points of the configuration.

The equations used in § 7, after some slight changes for the sake of symmetry, are as follows:

1: \[ x = 0, \quad w = 0, \]
2: \[ \{ z_2 \beta (w_2 x + x_2 w) - w_2 \alpha (z_2 y + y_2 z) = 0 \}, \]
3: \[ y = 0, \quad z = 0, \]
4: \[ \{ z_2 \beta (w_2 x + x_2 w) - w_2 \alpha (z_2 y + y_2 z) = 0 \}, \]
5: \[ \begin{align*}
  & w_2 z + z_2 w = 0 \\
  & y_1 \gamma (w_1 x + x_1 w) - x_1 \delta (z_1 y + y_1 z) = 0, \\
  & w_1 z + z_1 w = 0 \\
  & y_2 \gamma (w_2 x + x_2 w) - x_2 \delta (z_2 y + y_2 z) = 0, \\
  & \begin{cases}
  y = 0 \\
  \begin{cases}
  w_1 w_2 C x + w_2 w_3 A z + (x_2 z_2 w_1 - x_1 z_1 w_2) w = 0,
  \\
  x = 0, \\
  z = 0, \\
  w = 0,
  \end{cases}
  \end{cases}
\end{align*} \]

where we set

\[ A, B, C, D \equiv (x_2 - kx_1), -(y_2 - kx_1), (z_2 - kz_1), -(w_2 - kw_1) \]

respectively.

By inspection, we note that the three lines 1, 3 and 4' coincide with the edges BC, DA and AB respectively of the fundamental tetrahedron ABGD. Let us choose the constants as follows:

\[ x_1 = 3, \quad y_1 = -4, \quad z_1 = 5, \quad w_1 = -6; \]
\[ x_2 = z_2 = 1; \quad y_2 = w_2 = -1. \]

If now we make, using suitable constants, proper drawings of the planes ABD and BDC, inspection of the figure shows, as was verified by trial, that a suitable value for the constant \( k \) is \( -\frac{3}{16} \). Substituting these numerical values in the equations of the twelve lines and determining for each line (except the lines 1, 3, 3', and 4') the coordinates of the points where it meets the planes of ABD and BDC respectively, we obtain the following results in tabulated form:

<table>
<thead>
<tr>
<th>Line</th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( w )</th>
<th>Co-ordinates, for edge=100</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 is line BC</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( x=47.2, \ y=52.8 )</td>
</tr>
<tr>
<td>3 &quot; &quot; DA</td>
<td>25</td>
<td>28</td>
<td>0</td>
<td>0</td>
<td>( y=26.6, \ z=24.2, \ w=15.7 )</td>
</tr>
<tr>
<td>4' &quot; &quot; AB</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( x=43.3, \ y=86.6, \ w=43.3 )</td>
</tr>
<tr>
<td>2 meets AB</td>
<td>25</td>
<td>28</td>
<td>0</td>
<td>0</td>
<td>( y=17.3, \ z=43.3, \ w=26 )</td>
</tr>
<tr>
<td>&quot; BCD</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>( x=52.3, \ y=47.7 )</td>
</tr>
<tr>
<td>4 &quot; &quot; ABD</td>
<td>85</td>
<td>124</td>
<td>0</td>
<td>0</td>
<td>( y=3, \ z=41.8, \ w=41.8 )</td>
</tr>
<tr>
<td>&quot; BCD</td>
<td>0</td>
<td>6</td>
<td>85</td>
<td>85</td>
<td>( x=40.7, \ y=59.3 )</td>
</tr>
<tr>
<td>5 &quot; &quot; AB</td>
<td>34</td>
<td>31</td>
<td>0</td>
<td>0</td>
<td>( y=-3.8, \ z=41.1, \ w=49.3 )</td>
</tr>
<tr>
<td>&quot; BCD</td>
<td>0</td>
<td>-24</td>
<td>255</td>
<td>306</td>
<td>( x=48.5, \ y=51.5 )</td>
</tr>
<tr>
<td>1' &quot; &quot; CD</td>
<td>47</td>
<td>0</td>
<td>0</td>
<td>62</td>
<td>( x=43.1, \ y=56.9 )</td>
</tr>
<tr>
<td>&quot; AD</td>
<td>0</td>
<td>0</td>
<td>47</td>
<td>50</td>
<td>( x=143, \ y=243 )</td>
</tr>
<tr>
<td>2' &quot; &quot; BC</td>
<td>-14</td>
<td>0</td>
<td>0</td>
<td>31</td>
<td>( x=-52.3, \ y=182.3 )</td>
</tr>
<tr>
<td>&quot; AD</td>
<td>0</td>
<td>76</td>
<td>85</td>
<td>0</td>
<td>( y=47.2, \ z=52.8 )</td>
</tr>
<tr>
<td>3' &quot; &quot; CD</td>
<td>0</td>
<td>0</td>
<td>35</td>
<td>38</td>
<td>( y=48, \ w=52 )</td>
</tr>
<tr>
<td>5' &quot; &quot; BC</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( y=50, \ z=50 )</td>
</tr>
<tr>
<td>&quot; AD</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( x=50, \ w=50 )</td>
</tr>
<tr>
<td>6' &quot; &quot; BC</td>
<td>0</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>( y=44.4, \ z=55.6 )</td>
</tr>
<tr>
<td>&quot; AD</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>( x=33.3, \ y=66.7 )</td>
</tr>
</tbody>
</table>
For the actual details of the construction, I have for convenience taken \( x, y, z, w \) as the perpendicular distances of the current point from the faces of a regular tetrahedron, the edge of which is taken to be 100 units. It follows that the altitude of each triangle face = 86·6.

The outside values are given in the manner most convenient for the construction of a model or drawing. If any point lies in an edge of the fundamental tetrahedron, its two co-ordinates (finite) are in the ratio in which the edge is divided. So I have taken the sum of the two co-ordinates for a point on an edge = 100. Recalling the fact that, for an equilateral triangle, the sum of the co-ordinates for a point in the plane of the triangle is equal to the altitude of the triangle, I have so chosen the co-ordinates of a point lying in a face of the tetrahedron, that their sum is equal to 86·6; and in this case, the three co-ordinates denote the perpendicular distances from the sides of the triangle.

Some little ingenuity had to be exercised in constructing the model, especially in deciding which faces of the fundamental tetrahedron to dispense with, in order to leave the model open to view. On laying down the points, it appeared that the model might be constructed by using only the planes of \( ABD \) and \( BCD \), since the planes \( ABC \) and \( ADC \) are intersected by lines of the double six at no points not lying on the edges \( BC \) and \( DC \). This statement will be clear from an inspection of Plate 1, a perspective drawing of the configuration, made exactly according to the directions given above.
CHAPTER VI

20. Division of the Cubic Surface into Types.

Schläfi first conceived the idea of a division of the cubic surface into species in reference to the reality of the straight lines lying upon the surface. He later published an extensive memoir on the subject, which served as a basis for Cayley's exhaustive Memoir on Cubic Surfaces. The general surface of the third order falls into only five different types, in reference to the reality of the twenty-seven lines. These are shown in the following table:

<table>
<thead>
<tr>
<th>Type</th>
<th>No. Real Planes</th>
<th>No. Real Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>45</td>
<td>27</td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>13</td>
<td>3</td>
</tr>
</tbody>
</table>

This conclusion follows from the fact that every surface of the third order (real, general) may be generated by means of two triheders which present one or the other of the following three cases: (1) the

triheders are formed by six real planes; (2) one triheder is wholly real, while the other is formed by one real and two conjugate imaginary planes; (3) each triheder has one real and two conjugate imaginary planes.

The division according to the nature of the singularities, however, is the division with which we shall be concerned, in particular, in the sequel (cf. Schläfli’s second paper, i.e., and Cayley’s Memoir on Cubic Surfaces).

21. On the Construction of Models of the Twenty-seven Lines upon the Cubic Surface.

For the first time in 1869, Dr Christian Wiener* constructed a model of the twenty-seven lines upon the general cubic surface without singularities (cf. historical summary). Sylvester† and Salmon‡ had each given the same method of construction; but this was a “pure geometric” construction, without indication as to how it was to be carried out in making a model (thread, wire, plaster, or otherwise) of the configuration.

In 1882, Percival Frost§ gave a full description of the method by means of which he was enabled to make a thread model of the twenty-seven lines. Theoretically his method is based upon the analytical investigation of Schläfli||, in his original paper upon the twenty-seven lines. Frost determined the equations of the twenty-seven lines; and then, giving numerical values to the constants involved, he calculated in cartesian co-ordinates the co-ordinates of the one hundred and thirty-five points of intersection of the twenty-seven lines. Much care had to be exercised in the choice of numerical values for the constants, in order that all the lines might fall within reach; that the triangles might appear fairly spread out; and that coincidence or parallelism among the lines might be avoided. The method was tremendously laborious; and even after the model was constructed by Frost, it was imperfect or rather incomplete, several of the lines falling entirely out of reach.

Blythe¶ has given a purely geometrical method for constructing a model of the lines upon the cubic surface without singularities, and has

|| Quart. Journ., i.e.
briefly sketched the variations for the types of the surface possessed of singularities.

This method is based initially upon the fact that the general equation of the cubic surface may be put into the form

$$a\beta\gamma = K\delta S,$$

where $a, \beta, \gamma, \delta$ are of the first degree, $S$ is of the second degree, in the variables, and $K$ a constant. Taking a series of planes through some straight line lying wholly upon this surface, from the form of the equation the conclusion is drawn that the series of conic sections thus obtained cuts the line in pairs of points which are a range in involution.

It is shown (I.e.) that a cubic surface may be determined by straight lines $l, m, n; L, M, N; P, Q, R, S$, conditioned as follows:

$l, m, n$ are co-planar

$L, M, N$ ,, ,, $L, M, N$ ,, ,, $L, P, Q$ ,, ,, $L, R, S$ ,, ,, provided $M, P, R$ meet $m$, and $N, Q, S$ meet $n$: where $M, N; P, Q; R, S$ cut $L$ in points, which are a range in involution.

While these investigations are very interesting, the construction is not taken in all its generality; in a number of instances a special investigation is required, necessitating a marked modification of the process for the general case.

More recently, Blythe has considered the subject again in a brief paper, this time giving the construction for the twenty-seven lines upon only the cubic surface without singularities, again employing the notion of five pairs of points of an involution. His researches are recorded in full in his book, *On Models of Cubic Surfaces* (Cambridge University Press).

The papers of Korteweg and De Vries also deserve special mention.


In the present article, there is detailed a uniform method of representing the lines on each type of the cubic surface by means of models or perspective drawings. I have actually constructed, to scale, a
graphical representation of the lines in all twenty-one types of the cubic surface.

I have made use of the materials found in Cayley's *Collected Math. Papers*, Vol. vi. No. 412, employing the equations of the lines and triple tangent planes of the various surfaces, expressed in the canonical form. This has been done in every case save that of the surface without singularities, the most general case. The problem in this case is greatly simplified by choosing an equation for the surface with highly specialized coefficients. All these equations are expressed in quadriplanar co-ordinates. In the drawings or models, the lines appear not only in proper relation to each other, but also in correct position with respect to the edges of the fundamental tetrahedron. The notation employed here for the lines and planes, and also for each type of the cubic (with the exception of the most general type) is identical with that found in the article last referred to. Although the division depending upon the singularities gives rise to twenty-three types of the cubic surface, two of these are scrolls, in which there is no question of the twenty-seven lines.

A few words of explanation will serve to make the method clear. For each type of the cubic surface, I have chosen, by trial, such numerical values for the constants as will show the entire configuration in a comparatively limited space. In each case, I have made out a table of numerical co-ordinates for the points where each line meets two faces of the fundamental tetrahedron, and these points, of course, fix each line in position. A fact of essential simplicity is that, in almost every case, only two planes of the fundamental tetrahedron are used, these leaving the model wholly open to view.

To illustrate the process, take for example the general type of the cubic surface. The lines 1 and 2' coincide with the edges $BC$ and $CD$, respectively, of the fundamental tetrahedron $ABCD$. Consider now the line 12, which meets 1 and 2'. The line 12 lies in the plane $12.34.56$; hence to find where the line 12 meets the line 1, we make $x = 0, \, w = 0$ in the equation of the plane $12.34.56$. Similarly, to find where the line 12 meets the edge $CD \, (x = 0, \, y = 0)$, which is the line 2', make $x = 0, \, y = 0$ in the equation of the plane $12.34.56$. Planes passing through the line 12, other than the plane $12.34.56$, might have been used; and the simplest form has always been used.

In the case of a line such as 45, which meets the tetrahedron in only one edge $AB$, it remains to find where the line 45 meets the plane $BCD \, (x = 0)$. Here we consider two planes, 45' and 54', which
intersect in the line 45. Making \( x = 0 \) in the equations of each one of these planes, we find their lines of intersection with the plane \( BCD \); and these lines intersect where 45 meets the plane \( BCD \). Similar reasoning holds for a line which meets no edge of the fundamental tetrahedron.

The values given in the tables of co-ordinates have been carefully checked in every case. Using some other planes than the two used in determining the co-ordinates of the point in question, it has been verified that the co-ordinates actually do satisfy the equations of these other planes upon which the point should lie. Just as in the construction of the model of the double six, I have for convenience taken \( x, y, z, w \) as the perpendicular distances of the current point from the faces of a regular tetrahedron, whose edge is taken to be 100 units; in consequence, the altitude of each face is equal to 86.6. Accordingly, for the values on the right in the tables, the following checks were used: if the point lies on an edge, the sum of the co-ordinates equals 100; if the point is not on an edge, the sum of the co-ordinates equals 86.6.

23. Concerning the Graphical Representation of the Configurations of the Lines on the Cubic Surface.

In making accurate drawings, to scale, of the various configurations, there were a number of difficulties to be overcome. Five separate scales had to be brought into play.

First, a tetrahedron was drawn, attention being paid to the requirements of perspective. It will be recalled that a point which lies on an edge has, for co-ordinates, the distances to the extremities of the edge. Moreover, when a point lies in a face of the fundamental tetrahedron, the perpendicular distances to the sides of that triangle are the co-ordinates of the point. The tetrahedron is taken to be regular, with each edge equal to 100 units in length.

I shall describe the method for making the diagram of the configuration of the twenty-seven lines on the general surface of the third degree; but the principles employed are the same for all the graphic representations. First are made four separate scales, one for each of the edges \( AB, BC, CD, DA \), of lengths as laid down on the diagram; each scale is divided into 100 equal parts. Next, using the proper scale for each edge, I laid down the position of each point where the four edges \( AB, BC, CD, DA \), i.e. the lines 4', 1, 2' and 3 respectively, are met by the remaining lines.
The following method was employed for determining with accuracy the positions of the points, in the faces $ABD$, $BCD$, not lying on an edge of the tetrahedron. I first drew on two separate sheets, one on each sheet, two equilateral triangles, representing the two faces $ABD$, $BCD$. These triangles were constructed equal in size. Next was constructed a new scale, of length equal to the side of one of the triangles, and divided into 100 equal parts. I next located on these two sheets, to the new scale, each point, not on an edge, which lies in one of these faces. I next determine its position by lines drawn parallel to the sides of the triangle, two such parallels in each case being sufficient to fix the position of the point; and then find the lengths of the parallels according to the scale of the triangle.

Now, returning to the original perspective representation of the tetrahedron, I locate each point by like parallels, using for each parallel the scale of that edge, parallel to which the line is drawn, as originally chosen.

An immediate check on the accuracy of the construction is afforded by the fact that all the points, so constructed, that lie in any one face, are collinear. This graphical check is likewise a rough check on the accuracy of the numerical computations for the co-ordinates of the points.

24. Species I = 12.

The equation of the surface is taken to be (§ 16)

$$(x/y + z/w) - k(x/y + z/w) = 0.$$
ON THE CONSTRUCTION OF MODELS OF

essential simplicity in connection with the construction of this model is the employment of only two planes, viz. the planes of the two faces \( ABD, BCD \) of the fundamental tetrahedron \( ABCD \), whereby the model is left entirely open to view. The intersection table of the twenty-seven lines (§ 11), considered in connection with the diagram of the present configuration, shows immediately which lines do, and which lines do not, intersect. Actually, the notation attached to the lines is entirely self-explanatory on this point (Plate 2).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>Co-ordinates, for edge = 100</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( x = 54.5, y = 57.1 )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>( x = 50, y = 50 )</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>( x = 50, y = 50 )</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>( y = 55.6 )</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>( x = 45.5, y = 53.5 )</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>( x = 44.4, y = 53.8 )</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>( x = 44.5, y = 53.6 )</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>( x = 43.3, y = 53.7 )</td>
</tr>
<tr>
<td>1</td>
<td>25</td>
<td>3</td>
<td>3</td>
<td></td>
<td>( x = 43.2, y = 53.3 )</td>
</tr>
<tr>
<td>2</td>
<td>28</td>
<td>31</td>
<td>186</td>
<td></td>
<td>( x = 48.5, y = 53.2 )</td>
</tr>
<tr>
<td>4</td>
<td>28</td>
<td>32</td>
<td>150</td>
<td></td>
<td>( x = 48.5, y = 53.2 )</td>
</tr>
<tr>
<td>4</td>
<td>102</td>
<td>372</td>
<td>35</td>
<td>38</td>
<td>( x = 53.6, y = 53.6 )</td>
</tr>
<tr>
<td>12</td>
<td>76</td>
<td>35</td>
<td>38</td>
<td></td>
<td>( x = 48, y = 52 )</td>
</tr>
<tr>
<td>13 meets $BC$</td>
<td>$x$</td>
<td>$y$</td>
<td>$z$</td>
<td>$w$</td>
<td>Co-ordinates, for edge=100</td>
</tr>
<tr>
<td>--------------</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
<td>--------------------------</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$y = -143$, $z = 243$</td>
</tr>
<tr>
<td></td>
<td>-14</td>
<td>17</td>
<td>31</td>
<td></td>
<td>$x = -82\cdot3$, $w = 182\cdot3$</td>
</tr>
<tr>
<td>14</td>
<td>$BC$</td>
<td>100</td>
<td>109</td>
<td>124</td>
<td>$y = 47\cdot8$, $z = 52\cdot2$</td>
</tr>
<tr>
<td></td>
<td>$AB$</td>
<td></td>
<td></td>
<td></td>
<td>$x = 46\cdot8$, $y = 53\cdot2$</td>
</tr>
<tr>
<td>15</td>
<td>$BC$</td>
<td>20</td>
<td>17</td>
<td>17</td>
<td>$y = 54$, $z = 46$</td>
</tr>
<tr>
<td></td>
<td>$ABD$</td>
<td>17</td>
<td>6</td>
<td></td>
<td>$x = 36\cdot8$, $y = 13$, $w = 36\cdot8$</td>
</tr>
<tr>
<td>16</td>
<td>$BC$</td>
<td>25</td>
<td>34</td>
<td>34</td>
<td>$y = 42\cdot4$, $z = 57\cdot6$</td>
</tr>
<tr>
<td></td>
<td>$ABD$</td>
<td>17</td>
<td>-8</td>
<td></td>
<td>$x = 34\cdot2$, $y = -16\cdot1$, $w = 68\cdot5$</td>
</tr>
<tr>
<td>23</td>
<td>$CD$</td>
<td>47</td>
<td>50</td>
<td>62</td>
<td>$z = 48\cdot5$, $w = 51\cdot5$</td>
</tr>
<tr>
<td></td>
<td>$AD$</td>
<td></td>
<td></td>
<td></td>
<td>$x = 43\cdot1$, $y = 56\cdot9$</td>
</tr>
<tr>
<td>24</td>
<td>$CD$</td>
<td>14</td>
<td>5</td>
<td></td>
<td>$z = 73\cdot7$, $w = 26\cdot3$</td>
</tr>
<tr>
<td></td>
<td>$AB$</td>
<td>17</td>
<td>62</td>
<td></td>
<td>$x = 21\cdot5$, $y = 78\cdot5$</td>
</tr>
<tr>
<td>25</td>
<td>$CD$</td>
<td>28</td>
<td>25</td>
<td>-4</td>
<td>$z = 52\cdot8$, $w = 47\cdot2$</td>
</tr>
<tr>
<td></td>
<td>$ABD$</td>
<td>21</td>
<td>28</td>
<td></td>
<td>$x = 40\cdot4$, $y = 53\cdot9$, $w = -7\cdot7$</td>
</tr>
<tr>
<td>26</td>
<td>$CD$</td>
<td>7</td>
<td>10</td>
<td>3</td>
<td>$z = 41\cdot2$, $w = 58\cdot8$</td>
</tr>
<tr>
<td></td>
<td>$ABD$</td>
<td>14</td>
<td>14</td>
<td></td>
<td>$x = 39\cdot1$, $y = 39\cdot1$, $w = 8\cdot4$</td>
</tr>
<tr>
<td>34</td>
<td>$AD$</td>
<td>7</td>
<td>10</td>
<td>3</td>
<td>$x = 41\cdot2$, $w = 58\cdot8$</td>
</tr>
<tr>
<td></td>
<td>$AB$</td>
<td>17</td>
<td>20</td>
<td></td>
<td>$x = 45\cdot9$, $y = 54\cdot1$</td>
</tr>
<tr>
<td>35</td>
<td>$AD$</td>
<td>35</td>
<td>62</td>
<td>-9</td>
<td>$x = 36\cdot1$, $w = 63\cdot9$</td>
</tr>
<tr>
<td></td>
<td>$BCD$</td>
<td>70</td>
<td>70</td>
<td></td>
<td>$y = 46\cdot3$, $z = 46\cdot3$, $w = -6$</td>
</tr>
<tr>
<td>36</td>
<td>$AD$</td>
<td>28</td>
<td>31</td>
<td>4</td>
<td>$x = 47\cdot5$, $w = 52\cdot5$</td>
</tr>
<tr>
<td></td>
<td>$BCD$</td>
<td>28</td>
<td>35</td>
<td></td>
<td>$y = 36\cdot2$, $z = 45\cdot2$, $w = 5\cdot2$</td>
</tr>
<tr>
<td>45</td>
<td>$AB$</td>
<td>34</td>
<td>31</td>
<td></td>
<td>$x = 52\cdot3$, $y = 47\cdot7$</td>
</tr>
<tr>
<td></td>
<td>$BCD$</td>
<td>-24</td>
<td>255</td>
<td>306</td>
<td>$y = -3\cdot8$, $z = 41\cdot1$, $w = 49\cdot3$</td>
</tr>
<tr>
<td>46</td>
<td>$AB$</td>
<td>85</td>
<td>124</td>
<td>85</td>
<td>$x = 40\cdot7$, $y = 59\cdot3$</td>
</tr>
<tr>
<td></td>
<td>$BCD$</td>
<td>6</td>
<td>85</td>
<td></td>
<td>$y = 3$, $z = 41\cdot8$, $w = 41\cdot8$</td>
</tr>
<tr>
<td>56</td>
<td>$ABD$</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>$x = 43\cdot3$, $y = 86\cdot6$, $w = -43\cdot3$</td>
</tr>
<tr>
<td></td>
<td>$BCD$</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>$y = 17\cdot3$, $z = 43\cdot3$, $w = 26$</td>
</tr>
</tbody>
</table>

On the basis of the above data, in connection with the results of § 10, it is of course a mere matter of detail to construct all of the thirty-six double sixes belonging to the cubic surface.
25. **Species II = 12 - C₂**.

The equation of the surface is:

\[ w(a, b, c, f, g, h, x, y, z)^2 + 2Kxyz = 0. \]

Let us write this equation in the form

\[ w(1, 1, 1, l + \frac{1}{l}, m + \frac{1}{m}, n + \frac{1}{n})(x, y, z)^2 + \frac{a\beta\gamma\delta}{p} \text{xyz} = 0, \]

where for brevity

\[ a = mn - l, \]
\[ \beta = nl - m, \]
\[ \gamma = lm - n, \]
\[ \delta = lmn - 1, \]
\[ p = bmn. \]

If we take \( x = 0 \) as the equation of the plane [12], \( y = 0 \) as that of the plane [34], \( z = 0 \) as that of the plane [56], then the equations of the thirty distinct tangent planes to the cubic surface may be written down immediately; from these are obtained the equations of the twenty-one distinct lines upon the surface in the following forms:

(1) : \( x = 0, y + lz = 0, \)
(3) : \( y = 0, z + mx = 0, \)
(5) : \( z = 0, x + ny = 0, \)
(2) : \( x = 0, y + l^{-1}z = 0, \)
(4) : \( y = 0, z + m^{-1}x = 0, \)
(6) : \( z = 0, x + n^{-1}y = 0, \)
(45) : \( x + ny + mz = 0, x + \beta\gamma w = 0, \)
(16) : \( y + lz + nx = 0, y + \gamma\omega w = 0, \)
(23) : \( z + mx + ly = 0, z + a\beta w = 0, \)
(46) : \( x + n^{-1}y + mz = 0, x - a\delta w = 0, \)
(26) : \( y + l^{-1}z + nx = 0, y - \beta\delta w = 0, \)
(24) : \( z + m^{-1}x + ly = 0, z - \gamma\delta w = 0, \)
(35) : \( x + ny + m^{-1}z = 0, x - a\delta w = 0, \)
(15) : \( y + lz + n^{-1}x = 0, y - \beta\delta w = 0, \)
(13) : \( z + mx + l^{-1}y = 0, z - \gamma\delta w = 0, \)
(36) : \( x + n^{-1}y + m^{-1}z = 0, x + \beta\gamma w = 0, \)
(25): \( y + l^{-1}z + n^{-1}x = 0, \ y + \gamma aw = 0, \)

(14): \( z + m^{-1}x + l^{-1}y = 0, \ z + a\beta w = 0, \)

(12): \( x = 0, \ w = 0, \)

(34): \( y = 0, \ w = 0, \)

(56): \( z = 0, \ w = 0. \)

Note that \( p \) does not appear in the equations of any of the twenty-one lines.

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( w )</th>
<th>Co-ordinates, for edge = 100</th>
</tr>
</thead>
<tbody>
<tr>
<td>45 meets BC</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( y = 80, z = -180 )</td>
</tr>
<tr>
<td></td>
<td>( AC )</td>
<td>-187</td>
<td>561</td>
<td>288</td>
<td>( x = -24.4, z = 73.4, w = 37.6 )</td>
</tr>
<tr>
<td>16</td>
<td>( ACD )</td>
<td></td>
<td></td>
<td></td>
<td>( y = 60, z = 400 )</td>
</tr>
<tr>
<td></td>
<td>( ABD )</td>
<td>-44</td>
<td>33</td>
<td>48</td>
<td>( x = -103, y = 77.2, w = 112.4 )</td>
</tr>
<tr>
<td>23</td>
<td>( AB )</td>
<td>3</td>
<td>2</td>
<td></td>
<td>( y = 35.5, z = 17.7, w = 33.4 )</td>
</tr>
<tr>
<td></td>
<td>( BCD )</td>
<td>34</td>
<td>17</td>
<td>32</td>
<td>( y = -33.3, z = 133.3 )</td>
</tr>
<tr>
<td>46</td>
<td>( BC )</td>
<td>-4</td>
<td>4</td>
<td></td>
<td>( x = -27.2, z = 81.6, w = 32.2 )</td>
</tr>
<tr>
<td></td>
<td>( ACD )</td>
<td>-27</td>
<td>81</td>
<td>32</td>
<td>( x = 14.3, y = 85.7 )</td>
</tr>
<tr>
<td>26</td>
<td>( AC )</td>
<td>8</td>
<td>3</td>
<td></td>
<td>( x = 31.4, z = 15.7, w = 39.5 )</td>
</tr>
<tr>
<td></td>
<td>( BCD )</td>
<td>102</td>
<td>51</td>
<td>128</td>
<td>( x = 17.6, z = 52.9, w = 51.3 )</td>
</tr>
<tr>
<td>24</td>
<td>( AB )</td>
<td>1</td>
<td>6</td>
<td></td>
<td>( y = 133.3, z = -33.3 )</td>
</tr>
<tr>
<td></td>
<td>( ACD )</td>
<td>-11</td>
<td>33</td>
<td>32</td>
<td>( x = 57, y = 76, w = 67.6 )</td>
</tr>
<tr>
<td>35</td>
<td>( BC )</td>
<td>4</td>
<td>-1</td>
<td></td>
<td>( x = 27.2, z = 72.8 )</td>
</tr>
<tr>
<td></td>
<td>( ABD )</td>
<td>-27</td>
<td>36</td>
<td>32</td>
<td>( y = 20.3, z = 40.6, w = 25.6 )</td>
</tr>
<tr>
<td>15</td>
<td>( AC )</td>
<td>3</td>
<td>8</td>
<td></td>
<td>( x = 27.2, z = 72.8 )</td>
</tr>
<tr>
<td></td>
<td>( BCD )</td>
<td>51</td>
<td>102</td>
<td>64</td>
<td>( y = 14.3, \ y = 85.7 )</td>
</tr>
<tr>
<td>13</td>
<td>( AB )</td>
<td>6</td>
<td>1</td>
<td></td>
<td>( y = 17.5, z = 35.1, w = 34 )</td>
</tr>
<tr>
<td></td>
<td>( BCD )</td>
<td>33</td>
<td>66</td>
<td>64</td>
<td>( y = 180, z = -80 )</td>
</tr>
<tr>
<td>36</td>
<td>( BC )</td>
<td>9</td>
<td>-4</td>
<td></td>
<td>( x = 67.1, y = 50.3, w = 103.4 )</td>
</tr>
<tr>
<td></td>
<td>( ABD )</td>
<td>-748</td>
<td>561</td>
<td>1152</td>
<td>( x = 30.3, z = 14.7, w = 42.6 )</td>
</tr>
<tr>
<td>25</td>
<td>( AC )</td>
<td>3</td>
<td>2</td>
<td></td>
<td>( x = 60, z = 40 )</td>
</tr>
<tr>
<td></td>
<td>( BCD )</td>
<td>22</td>
<td>11</td>
<td>32</td>
<td>( y = 40, y = 60 )</td>
</tr>
<tr>
<td>14</td>
<td>( AB )</td>
<td>2</td>
<td>3</td>
<td></td>
<td>( y = 40, y = 60 )</td>
</tr>
<tr>
<td></td>
<td>( BCD )</td>
<td>17</td>
<td>34</td>
<td>64</td>
<td>( y = 40, y = 60 )</td>
</tr>
</tbody>
</table>
In these, let us set
\[ l = -\frac{1}{2}, \quad m = \frac{1}{3}, \quad n = \frac{3}{4}. \]

Consequently
\[ \alpha = \frac{3}{4}, \quad \beta = -\frac{17}{24}, \quad \gamma = -\frac{11}{12}, \quad \delta = -\frac{9}{8}. \]

Calculating the co-ordinates of the points where each line meets two faces of the tetrahedron \( ABCD \) (except in the case of the line 12, 34, 56, lying wholly in the face \( ABC \), and the lines 1, 2, 3, 4, 5, 6, for each one of which, since they lie by pairs in the faces \( ABD, BCD, ACD \), only one point has to be calculated), we obtain the results as tabulated above (p. 67).

It is to be observed that the system of lines and planes is at once deduced from that belonging to the general equation of the cubic surface, by supposing that in the primitive double six the corresponding lines 1 and 1', 2 and 2', etc. severally coincide.

In the present numbering, the lines 1, 2, 3, 4, 5, 6 pass through a conical node at \( D \) on the surface, i.e. they lie not only in the cubic surface, but also in a quadric cone of vertex \( D \) (Plate 3).

26. **Species III = 12 - B₃**.

The equation of the surface is
\[ 2w(x + y + z)(lx + my + nz) + 2kxyz = 0, \]
where we set
\[ \lambda, \mu, \nu = m - n, \quad n - l, \quad l - m, \quad \text{respectively}. \]

Let us take
\[ l = -\frac{1}{2}, \quad m = \frac{1}{3}, \quad n = \frac{3}{4}. \]

Consequently
\[ \lambda = -\frac{5}{12}, \quad \mu = \frac{5}{4}, \quad \nu = -\frac{5}{6}. \]

The equations of the fifteen distinct lines upon the surface take the form:

(1): \quad x = 0, \quad y + z = 0,
(2): \quad y = 0, \quad z + x = 0,
(3): \quad z = 0, \quad x + y = 0,
(4): \quad x = 0, \quad 4y + 9z = 0,
(5): \quad y = 0, \quad 3z - 2x = 0,
(6): \quad z = 0, \quad -3x + 2y = 0,
(14): \quad x = 0, \quad w = 0,
(25): \quad y = 0, \quad w = 0,
(36): \quad z = 0, \quad w = 0,
(15): \[ 2x - 3y - 3z = 0, \quad 8w - 5z = 0, \]
(16): \[ 3x - 2y - 2z = 0, \quad 12w + 5y = 0, \]
(26): \[ 3x - 2y + 3z = 0, \quad 24w + 5x = 0, \]
(24): \[ 9x + 4y + 9z = 0, \quad 8w - 5z = 0, \]
(34): \[ 4x + 4y + 9z = 0, \quad 12w + 5y = 0, \]
(35): \[ 2x + 2y - 3z = 0, \quad 24w + 5x = 0. \]

The plane of \( ABC \) \((w = 0)\) was chosen for the base of the tetrahedron, since it contains three of the lines. The lines 14, 25, 36 are the edges \( BC, AC, AB \); respectively, of the tetrahedron. The lines 1, 2, and 3 pass through the point \( D \) and bisect the angle \( D \) in the planes of \( BCD, CDA, ABD \) respectively. The data for the remaining lines are given in the following table:

<table>
<thead>
<tr>
<th>Passes through</th>
<th>( x ) ( y ) ( z ) ( w )</th>
<th>Co-ordinates, for edge=100</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 passes through</td>
<td>( D ) in plane ( BCD )</td>
<td>( x = 41.6, \quad z = 27.7, \quad w = 17.3 )</td>
</tr>
<tr>
<td>5 passes through</td>
<td>( D ) in plane ( ACD )</td>
<td>( x = 46.2, \quad y = 69.3, \quad w = -28.9 )</td>
</tr>
<tr>
<td>6 passes through</td>
<td>( D ) in plane ( ABD )</td>
<td>( x = 37.8, \quad y = 56.7, \quad w = -7.9 )</td>
</tr>
<tr>
<td>15 meets ( AB ) ( ACD )</td>
<td>3 ( y ) 2</td>
<td>( x = -138.6, \quad z = 138.6, \quad w = 86.6 )</td>
</tr>
<tr>
<td>16 meets ( AB ) ( ABD )</td>
<td>-4 ( y ) 9</td>
<td>( x = 207.8, \quad y = -207.8, \quad w = 86.6 )</td>
</tr>
<tr>
<td>24 meets ( BC ) ( ACD )</td>
<td>9 ( y ) -4</td>
<td>( x = 59.4, \quad z = 39.6, \quad w = -12.4 )</td>
</tr>
</tbody>
</table>

The system of lines and planes for this surface is deduced immediately from that belonging to Species II by supposing the
tangent cone to reduce itself to the pair of biplanes; three of the planes of Species II come to coincide with one biplane, three of them with the other. The line of intersection of the biplanes is called the edge.

In the present case, the point $D$ is a binode on the surface. The edge is not a line on the surface (Plate 4).

**27. Species IV = 12 $- 2C_2$.**

The equation of the surface is

\[ wxz + y^2 (yz + \delta w) + (a, b, c, \delta) x y z = 0, \]

where we set

\[ a + b + c + d = \delta. \]

Let us take

\[ f_1 = 1, f_2 = \frac{1}{2}, f_3 = \frac{1}{3}, f_4 = \frac{1}{4}; \gamma = - \frac{1}{3}, \delta = - \frac{2}{5}; \]

\[ d = \gamma \delta \left( \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4} \right) = \frac{4}{3}. \]

The equations of the sixteen distinct lines upon the surface take the forms:

- (0): \(x = 0, y = 0,\)
- (5): \(x = 0, 20y - 5z - 6w = 0,\)
- (1): \(x - y = 0, 2y - 5z = 0,\)
- (2): \(2x - y = 0, 4y - 5z = 0,\)
- (3): \(3x - y = 0, 6y - 5z = 0,\)
- (4): \(4x - y = 0, 8y - 5z = 0,\)
- (1'): \(x - y = 0, y - 3w = 0,\)
- (2'): \(2x - y = 0, 2y - 3w = 0,\)
- (3'): \(3x - y = 0, y - w = 0,\)
- (4'): \(4x - y = 0, 4y - 3w = 0,\)
- (12, 34'): \(4x - 6y + 5z = 0, 12x - 7y + 3w = 0,\)
- (13, 24'): \(6x - 8y + 5z = 0, 8x - 6y + 3w = 0,\)
- (14, 23'): \(8x - 10y + 5z = 0, 6x - 5y + 3w = 0,\)
- (23, 14'): \(12x - 10y + 5z = 0, 4x - 5y + 3w = 0,\)
- (24, 13'): \(16x - 12y + 5z = 0, 3x - 4y + 3w = 0,\)
- (34, 12'): \(24x - 14y + 5z = 0, 2x - 3y + 3w = 0.\)

The plane of $BCD$ ($x = 0$) was chosen for the base of the tetrahedron, since it contains two of the lines. The line (0) is the edge $CD$ of the tetrahedron. The data for the remaining lines are given in the following table:
<table>
<thead>
<tr>
<th>0 is line $CD$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$w$</th>
<th>Co-ordinates, for edge=100</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 meets $BD$</td>
<td>3</td>
<td>1</td>
<td>10</td>
<td>$y=23$, $w=77$</td>
<td></td>
</tr>
<tr>
<td>$BC$</td>
<td></td>
<td></td>
<td></td>
<td>$y=20$, $z=80$</td>
<td></td>
</tr>
<tr>
<td>1 passes through $D$</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>$x=36\cdot1$, $y=36\cdot1$, $z=14\cdot4$</td>
<td></td>
</tr>
<tr>
<td>meets $ABC$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 passes through $D$</td>
<td>5</td>
<td>10</td>
<td>8</td>
<td>$x=18\cdot8$, $y=37\cdot7$, $z=30\cdot1$</td>
<td></td>
</tr>
<tr>
<td>meets $ABC$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 passes through $D$</td>
<td>5</td>
<td>15</td>
<td>18</td>
<td>$x=11\cdot4$, $y=34\cdot2$, $z=41$</td>
<td></td>
</tr>
<tr>
<td>meets $ABC$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 passes through $D$</td>
<td>5</td>
<td>20</td>
<td>32</td>
<td>$x=7\cdot6$, $y=30\cdot4$, $z=48\cdot6$</td>
<td></td>
</tr>
<tr>
<td>meets $ABC$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1' passes through $C$</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>$x=37\cdot1$, $y=37\cdot1$, $w=12\cdot4$</td>
<td></td>
</tr>
<tr>
<td>meets $ABD$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2' passes through $C$</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>$x=20$, $y=40$, $w=26\cdot6$</td>
<td></td>
</tr>
<tr>
<td>meets $ABD$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3' passes through $C$</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>$x=12\cdot4$, $y=37\cdot1$, $w=37\cdot1$</td>
<td></td>
</tr>
<tr>
<td>meets $ABD$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4' passes through $C$</td>
<td>3</td>
<td>12</td>
<td>16</td>
<td>$x=8\cdot4$, $y=33\cdot5$, $w=44\cdot7$</td>
<td></td>
</tr>
<tr>
<td>meets $ABD$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12.3'4' meets $BCD$</td>
<td>630</td>
<td>756</td>
<td>1470</td>
<td>$y=19\cdot1$, $z=22\cdot9$, $w=44\cdot6$</td>
<td></td>
</tr>
<tr>
<td>$ABC$</td>
<td>8085</td>
<td>13860</td>
<td>10164</td>
<td>$x=21\cdot8$, $y=37\cdot4$, $z=27\cdot4$</td>
<td></td>
</tr>
<tr>
<td>13.2'4'</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$BCD$</td>
<td>5</td>
<td>8</td>
<td>10</td>
<td>$y=18\cdot8$, $z=30\cdot1$, $w=37\cdot7$</td>
<td></td>
</tr>
<tr>
<td>$ABC$</td>
<td>15</td>
<td>20</td>
<td>14</td>
<td>$x=26\cdot5$, $y=35\cdot4$, $z=24\cdot7$</td>
<td></td>
</tr>
<tr>
<td>14.2'3'</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$BCD$</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td>$y=18\cdot6$, $z=37\cdot1$, $w=30\cdot9$</td>
<td></td>
</tr>
<tr>
<td>$ABC$</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>$x=28\cdot9$, $y=34\cdot6$, $z=23\cdot1$</td>
<td></td>
</tr>
<tr>
<td>23.1'4'</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$BCD$</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td>$y=18\cdot6$, $z=37\cdot1$, $w=30\cdot9$</td>
<td></td>
</tr>
<tr>
<td>$ABD$</td>
<td>15</td>
<td>18</td>
<td>10</td>
<td>$x=30\cdot2$, $y=36\cdot3$, $w=20\cdot1$</td>
<td></td>
</tr>
<tr>
<td>24.1'3'</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$BCD$</td>
<td>15</td>
<td>36</td>
<td>20</td>
<td>$y=18\cdot3$, $z=43\cdot9$, $w=24\cdot4$</td>
<td></td>
</tr>
<tr>
<td>$ABD$</td>
<td>9</td>
<td>12</td>
<td>7</td>
<td>$x=27\cdot8$, $y=37\cdot1$, $w=21\cdot6$</td>
<td></td>
</tr>
<tr>
<td>34.1'2'</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$BCD$</td>
<td>5</td>
<td>14</td>
<td>5</td>
<td>$y=18\cdot1$, $z=50\cdot5$, $w=18\cdot1$</td>
<td></td>
</tr>
<tr>
<td>$ABD$</td>
<td>9702</td>
<td>16632</td>
<td>10164</td>
<td>$x=23$, $y=39\cdot5$, $w=24\cdot1$</td>
<td></td>
</tr>
</tbody>
</table>
ON THE CONSTRUCTION OF MODELS OF

In the present case, there are two conic nodes, at $C$ and $D$ respectively. The rays $1, 2, 3, 4$ and $1', 2', 3', 4'$ pass through the two nodes $D$ and $C$, respectively (Plate 5).


The equation of the surface is

$$wzx + (x + z)(y^2 - ax^2 - bx^2) = 0.$$ 

The equations of the ten distinct lines upon the surface are as follows:

(3): $x = 0, z = 0$,
(4): $x + z = 0, w = 0$,
(1): $x = 0, y - \sqrt{b}z = 0$,
(2): $x = 0, y + \sqrt{b}z = 0$,
(1'): $z = 0, -\sqrt{a}x + y = 0$,
(2'): $z = 0, \sqrt{a}x + y = 0$,
(11'): $-\sqrt{a}x + y - \sqrt{b}z = 0, \sqrt{ab}(x + z) + w = 0$,
(12'): $\sqrt{a}x + y - \sqrt{b}z = 0, -2\sqrt{ab}(x + z) + w = 0$,
(21'): $-\sqrt{a}x + y + \sqrt{b}z = 0, -2\sqrt{ab}(x + z) + w = 0$,
(22'): $\sqrt{a}x + y + \sqrt{b}z = 0, \sqrt{ab}(x + z) + w = 0$.

The numerical values chosen for the constants are

$$a = \frac{4}{9}, \ b = \frac{9}{25}.$$

The plane of $ABC (z = 0)$ was selected for the base of the tetrahedron, since it contains three of the lines. The line (3) is the edge $BD$, the line (4) passes through $A$ and is parallel to the edge $BC$. The data for the remaining lines are given in the table on page 73.

There is a binode at $D$, and the edge is torsal, i.e. the surface is touched along the edge by a plane (Plate 6).

29. Species $VI = 12 - B_3 - C_2$.

The equation of the surface is

$$wzx + y^2z + (a, b, c, d \overline{x}, y)^2 = 0,$$

where we set

$$(a, b, c, d \overline{x}, y)^2 \equiv -d (\theta_2x - y)(\theta_3x - y)(\theta_4x - y).$$

The numerical values assigned for the constants are as follows:

$$\theta_2 = \frac{2}{3}, \ \theta_3 = \frac{4}{5}, \ \theta_4 = \frac{3}{4}; \ d = 1.$$
The equations of the eleven distinct lines upon the surface are as follows:

$$(0) : \quad x = 0, \quad y = 0,$$
$$(1) : \quad x = 0, \quad y + z = 0,$$
$$(2) : \quad 2x - 3y = 0, \quad z = 0,$$
$$(3) : \quad 4x - 5y = 0, \quad z = 0,$$
$$(4) : \quad 3x - 4y = 0, \quad z = 0,$$
$$(2') : \quad 2x - 3y = 0, \quad 4x + 9w = 0,$$
$$(3') : \quad 4x - 5y = 0, \quad 16x + 25w = 0,$$
$$(4') : \quad 3x - 4y = 0, \quad 9x + 16w = 0,$$
$$(12 \cdot 3'4') : \quad 2x - 3y - 3z = 0, \quad 12x - 31y - 20w = 0,$$
$$(13 \cdot 2'4') : \quad 4x - 5y - 5z = 0, \quad 6x - 17y - 12w = 0,$$
$$(14 \cdot 2'3') : \quad 3x - 4y - 4z = 0, \quad 8x - 22y - 15w = 0.$$

The plane of $ABD$ ($z = 0$) was chosen for the base of the tetrahedron, since it contains three of the lines. The line (0) is the edge $CD$ of the tetrahedron; the line (1) passes through $D$ and is parallel
to the edge $BC$. The data for the remaining lines are given in the following table:

<table>
<thead>
<tr>
<th>Passes through $C$</th>
<th>Co-ordinates, for edge=100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$ in plane $ABD$</td>
<td>$x = 70.9$, $y = 47.2$, $w = -31.5$</td>
</tr>
<tr>
<td>$D$ in plane $ABD$</td>
<td>$x = 74.7$, $y = 59.7$, $w = -47.8$</td>
</tr>
<tr>
<td>$D$ in plane $ABD$</td>
<td>$x = 72.9$, $y = 54.7$, $w = -41$</td>
</tr>
<tr>
<td>$D$ in plane $ABD$</td>
<td>$x = 70.2$, $y = 46.8$, $w = -30.4$</td>
</tr>
<tr>
<td>$D$ in plane $ABD$</td>
<td>$x = 38.2$, $z = 25.5$, $w = 22.9$</td>
</tr>
<tr>
<td>$D$ in plane $ABD$</td>
<td>$x = 74.2$, $y = 59.4$, $w = -47$</td>
</tr>
<tr>
<td>$D$ in plane $ABD$</td>
<td>$x = 37.7$, $z = 30.1$, $w = 18.8$</td>
</tr>
<tr>
<td>$D$ in plane $ABD$</td>
<td>$x = 37.2$, $y = 54.9$, $w = -41.5$</td>
</tr>
<tr>
<td>$D$ in plane $ABD$</td>
<td>$x = 37.9$, $z = 38.5$, $w = 20.2$</td>
</tr>
</tbody>
</table>

Here there is a binode at $D$, a conic node at $G$. The axis joining the two nodes is a line on the surface (Plate 7).

30. **Species VII** = $12 - B_5$.

The equation of the surface is

$$wxz + y^2z + yx^2 - z^3 = 0.$$

The equations of the six distinct lines upon the surface are as follows:

- $(0)$: $x = 0$, $z = 0$,
- $(1)$: $y = 0$, $z = 0$,
- $(2')$: $x = 0$, $y + z = 0$,
- $(3')$: $x = 0$, $y - z = 0$,
- $(12')$: $x - w = 0$, $y + z = 0$,
- $(13')$: $x + w = 0$, $y - z = 0$. 
The plane of $BCD$ ($x=0$) was chosen for the base of the tetrahedron, since it contains three of the lines. The lines (0) and (1) are the edges $BD$ and $AD$ respectively; the line (2') passes through $D$ and is parallel to $BC$; the line (3') also passes through $D$ and bisects the edge $BC$. The line (12') passes through the middle point of $AD$ and is parallel to $BC$; the line (13') passes through the middle point of $BC$ and is parallel to $AD$.

There is a binode at $D$ and the edge is torsal. The tangent plane coincides with one of the biplanes; we have thus an ordinary biplane, and an oscular biplane (Plate 8).

### 31. Species VIII $= 12 - 3C_2$

The equation of the surface is

$$y^3 + y(x + z + w) + 4axzw = 0,$$

where we set

$$(m - 1)^2 = 4am.$$

Consequently

$$m_1 + m_2 = 2 + 4a, \quad m_1m_2 = 1.$$

Putting $a = \frac{1}{8}$, we obtain $m_1 = 2, \quad m_2 = \frac{1}{2}$.

The equations of the twelve distinct lines upon the surface are as follows:

1. $(7): \quad w = 0, \quad y = 0$,
2. $(8): \quad x = 0, \quad y = 0$,
3. $(9): \quad z = 0, \quad y = 0$,
4. $(7): \quad y + z + x = 0, \quad w = 0$,
5. $(8): \quad y + x + w = 0, \quad z = 0$,
6. $(9): \quad y + z + w = 0, \quad x = 0$,
7. $(1): \quad 2y = 2x = -z$,
8. $(2): \quad 2y = -x = 2z$,
9. $(3): \quad 2y = -w = 2x$,
10. $(4): \quad 2y = 2w = -x$,
11. $(5): \quad 2y = 2z = -w$,
12. $(6): \quad 2y = -z = 2w$.

The lines (7), (8) and (9) are the edges $CA$, $CD$ and $AD$, respectively. The lines (7), (8) and (9) are the lines at infinity in the planes $ABC$, $ABD$, $BCD$, respectively. The lines (1) and (2) pass through the point $D$, and are parallel to the internal bisectors of the angles $C$ and $A$, respectively, lying in the plane $ABC$. The lines (3) and (4) pass through the point $C$, and are parallel to the internal bisectors of the angles $D$ and $A$, respectively, lying in the plane $ABD$. 
The lines (5) and (6) pass through the point A, and are parallel to the internal bisectors of the angles D and C, respectively, lying in the plane BCD.

There are three conic nodes, at D, C, and A, respectively. The axes, each joining two nodes, are lines on the surface (Plate 9).

32. Species IX = 12 - 2B3.

The equation of the surface is

\[ wxz + (a, b, c, d \tilde{x}, y)^2 = 0, \]

where we set

\[ (a, b, c, d \tilde{x}, y)^2 = -d (f_1 x - y) (f_2 x - y) (f_3 x - y). \]

The equations of the seven distinct lines upon the surface are as follows:

(0): \[ x = 0, y = 0, \]
(1): \[ f_1 x - y = 0, z = 0, \]
(2): \[ f_2 x - y = 0, z = 0, \]
(3): \[ f_3 x - y = 0, z = 0, \]
(4): \[ f_1 x - y = 0, w = 0, \]
(5): \[ f_2 x - y = 0, w = 0, \]
(6): \[ f_3 x - y = 0, w = 0. \]

The following values were chosen for the constants:

\[ f_1 = 1/3, f_2 = 1, f_3 = 3. \]

The line (0) is the edge CD; the line (2) bisects the interior angle D, in the plane ABD; the line (5) bisects the interior angle C, in the plane ABC. The data for the remaining lines are given in the following table:

<table>
<thead>
<tr>
<th>Passes through</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>w</th>
<th>Plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 passes through meets AB</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>D in plane ABD</td>
<td></td>
</tr>
<tr>
<td>3 passes through meets AB</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>D in plane ABD</td>
<td></td>
</tr>
<tr>
<td>4 passes through meets AB</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>C in plane ABC</td>
<td></td>
</tr>
<tr>
<td>6 passes through meets AB</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>C in plane ABC</td>
<td></td>
</tr>
</tbody>
</table>
There are two binodes on the surface, at $D$ and $C$, respectively. The axis, joining the two binodes, is a line on the surface (Plate 10).

33. Species $X = 12 - B_\mathbf{2} - C_2$.

The equation of the surface is
\[wxz + (x + z)(y^2 - x^2) = 0.\]

The equations of the seven distinct lines upon the surface are as follows:

- (0): $x = 0, \ y = 0$,
- (3): $x = 0, \ z = 0$,
- (1): $y - z = 0, \ z = 0$,
- (2): $x + y = 0, \ z = 0$,
- (1'): $x - y = 0, \ w = 0$,
- (2'): $x + y = 0, \ w = 0$.

The lines (0) and (3) are the edges $CD$ and $BD$, respectively. The lines (1) and (2) lie in the plane $ABD$ and bisect the angle $D$ internally and externally, respectively. The lines (1') and (2') lie in the plane $ABC$ and bisect the angle $C$ internally and externally, respectively. The line (12) lies in the plane $ABC$ and bisects externally the angle $B$.

There is a binode at $D$, a conic node at $C$; and the axis, joining the two nodes, is a line on the surface (Plate 8).

34. Species $XI = 12 - B_\mathbf{6}$.

The equation of the surface is
\[wxz + y^2z + x^3 - z^3 = 0.\]

The equations of the three distinct lines upon the surface are as follows:

- (0): $x = 0, \ z = 0$,
- (1): $x = 0, \ y + z = 0$,
- (2): $x = 0, \ y - z = 0$.

The line (0) is the edge $BD$; the lines (1) and (2) lie in the plane $BCD$, and bisect, externally and internally, the angle $D$.

There is a binode at $D$, and the tangent plane coinciding with one of the biplanes is oscular. We thus have an ordinary biplane, and an oscular biplane. The edge is a line on the surface (Plate 8).
35. Species XII = 12 - U₅.

The equation of the surface is

$$w(x + y + z)^2 + xyz = 0.$$  

The equations of the six distinct lines upon the surface are as follows:

(1): $$x = 0, \quad y + z = 0,$$
(2): $$y = 0, \quad z + x = 0,$$
(3): $$z = 0, \quad \bar{z} + y = 0,$$
(1'): $$w = 0, \quad \bar{w} = 0,$$
(2'): $$y = 0, \quad w = 0,$$
(3'): $$z = 0, \quad w = 0.$$

The lines (1), (2), (3) pass through the point D, and are parallel to the lines BC, AC and AB, respectively. The lines (1'), (2'), (3') are the edges BC, AC, AB, respectively.

There is a unode at D, i.e. the quadric cone has become a coincident plane-pair. This uniplane meets the cubic surface in three lines through the unode. Here these three lines or rays are distinct (Plate 11).


The equation of the surface is

$$wxz + y^2(x + y + z) = 0.$$  

The equations of the eight distinct lines upon the surface are as follows:

(5): $$w = 0, \quad y = 0,$$
(6): $$z = 0, \quad y = 0,$$
(0): $$y = 0, \quad w = 0,$$
(1): $$x = 0, \quad y + z = 0,$$
(2): $$z = 0, \quad \bar{x} + y = 0,$$
(3): $$w = y = -z,$$
(4): $$w = y = -x,$$
(012): $$w = 0, \quad x + y + z = 0.$$

The lines (5), (6) and (0) are the edges CD, AD and AC, respectively. The lines (1) and (2) pass through the point D, and are parallel to BC and AB, respectively. The line (3) passes through the
point $A$, and meets the plane $BCD$ at the point of intersection of the internal bisector of the angle $C$ and the external bisectors of the angles $B$ and $D$. The line (4) passes through the point $C$, and meets the plane $ABD$ at the point of intersection of the internal bisector of the angle $A$ and the external bisectors of the angles $B$ and $D$. The line (012) lies at infinity in the plane $ABC$.

There is a binode at $D$, and two conic nodes at $A$ and $C$, respectively. The axes, each joining the binode with a conic node, and the axis, joining the two conic nodes, are lines on the surface (Plate 11).

37. Species XIV $= 12 - B_5 - C_2$.

The equation of the surface is

$$wxz + y^2z + yx^2 = 0.$$ 

The equations of the four distinct lines upon the surface are as follows:

(0): $x = 0, \ y = 0,$

(1): $x = 0, \ z = 0,$

(2): $z = 0, \ y = 0,$

(3): $w = 0, \ y = 0.$

The lines (0), (1), (2), (3) are the edges $CD, BD, AD, AC$, respectively.

There is a binode at $D$, a conic node at $C$. The axis and the edge are lines on the surface (Plate 11).

38. Species XV $= 12 - U_1$.

The equation of the surface is

$$wx^2 + wz^2 + y^2z = 0.$$ 

The equations of the three distinct lines upon the surface are as follows:

(1): $x = 0, \ y = 0,$

(2): $x = 0, \ z = 0,$

(3): $z = 0, \ w = 0.$

The lines (1), (2), (3) are the edges $CD, BD, AB$, respectively.

There is a unode at $D$, and two of the three rays in the uniplane $BDC$ are coincident (Plate 11).
39. Species XVI = 12 – 4C₂.

The equation of the surface is

\[ w(x^2 + y^2 + z^2) + x^2yz = 0. \]

The equations of the nine distinct lines upon the surface are as follows:

- (12): \( z = 0, w = 0, \)
- (13): \( y = 0, w = 0, \)
- (14): \( y = 0, z = 0, \)
- (23): \( x = 0, w = 0, \)
- (24): \( x = 0, z = 0, \)
- (34): \( x = 0, y = 0, \)
- (12. 34): \( x + y = 0, z + w = 0, \)
- (13. 24): \( x + z = 0, y + w = 0, \)
- (14. 23): \( x + w = 0, y + z = 0. \)

The lines (12), (13), (14), (23), (24), (34) are the edges AB, AC, AD, BC, BD, CD, respectively. The lines (12. 34), (13. 24), (14. 23) meet the pairs of lines AB, CD; BD, AC; BC, AD, respectively, at infinity.

There are four conic nodes, at A, B, C and D; and the axes, each through two nodes, are lines on the surface (Plate 11).

40. Species XVII = 12 – 2B₃ – C₂.

The equation of the surface is

\[ wxyz + x^2y + y^2 = 0. \]

The equations of the five distinct lines upon the surface are as follows:

- (0): \( x = 0, y = 0, \)
- (1): \( y = 0, z = 0, \)
- (2): \( y = 0, w = 0, \)
- (3): \( z = 0, x + y = 0, \)
- (4): \( w = 0, x + y = 0. \)

The lines (0), (1) and (2) are the edges CD, AD and AC, respectively. The lines (3) and (4) are parallel to the line AB, and pass through the points D and C, respectively.

There are two binodes, at D and C, respectively, and a conic node at A. The axis joining the two binodes, and the axes, each through the conic node and a binode, are lines on the surface (Plate 11).
41. **Species XVIII = 12 - B₄ - 2C₂.**

The equation of the surface is:

\[wxyz + y^2(x + z) = 0.\]

The equations of the five distinct lines upon the surface are as follows:

- (1): \(y = 0, x = 0,\)
- (2): \(y = 0, z = 0,\)
- (0): \(y = 0, w = 0,\)
- (3): \(x = 0, z = 0,\)
- (4): \(w = 0, x + z = 0.\)

The lines (1), (2), (0), (3) are the edges \(CD, AD, AC, BD\) respectively. The line (4) passes through \(B\) and is parallel to \(AC.\)

There is a binode at \(D,\) and two conic nodes at \(A\) and \(C,\) respectively. The axes, each through the binode and a conic node, the axis through the two conic nodes, and the edge of the binode are all lines on the surface (Plate 12).

42. **Species XIX = 12 - B₆ - C₂.**

The equation of the surface is:

\[wxz + y^2z + x^3 = 0.\]

The equations of the two distinct lines upon the surface are as follows:

- (1): \(x = 0, y = 0,\)
- (2): \(x = 0, z = 0.\)

The lines (1) and (2) are the edges \(CD\) and \(BD,\) respectively.

There is a binode at \(D,\) a conic node at \(C.\) The axis, joining the binode and the conic node, and the edge of the binode are lines on the surface (Plate 12).

43. **Species XX = 12 - U₆.**

The equation of the surface is:

\[x^3w + xx^2 + y^3 = 0.\]

The equation of the one distinct line upon the surface is

- (1): \(x = 0, y = 0.\)

The line (1) is the edge \(CD.\)

There is a unode at \(D,\) and the three rays in the uniplane are coincident. In this case, the line \(CD\) represents twenty-seven coincident straight lines (Plate 12).
44. Species XXI = 12 - 3B₃.

The equation of the surface is

\[ wxx + y^3 = 0. \]

The equations of the three distinct lines upon the surface are as follows:

(1): \[ y = 0, \quad x = 0, \]
(2): \[ y = 0, \quad z = 0, \]
(3): \[ y = 0, \quad w = 0. \]

The lines (1), (2), (3) are the edges CD, AD, AC, respectively.

There are three binodes, at D, C and A, respectively. The axes, each joining two binodes, are lines on the surface (Plate 12).
CHAPTER VII

ON SOME CONFIGURATIONS ASSOCIATED WITH THE CONFIGURATIONS OF THE LINES UPON THE CUBIC SURFACE

45. Concerning the Brianchon Configuration.

Cayley* has considered the question of deriving the Pascalian configuration, by projection, from a pair of triheders. Denote the three planes of one triheder by \( a_1, a_2, a_3 \); of the other by \( b_1, b_2, b_3 \). Considering the nine lines \( a_i b_j \) \((i, j = 1, 2, 3)\) and taking them in a particular way in six sets of three each, we may pass hyperboloids through each set of three lines. These hyperboloids intersect in four points \( O_1, O_2, O_3, O_4 \); and if we project the solid figure of the two triheders from any one of these four points upon an arbitrary plane, the resulting figure is the Pascalian configuration. The polar planes of any one of the points \( O \) with respect to each one of the triheders are identical.

There is, of course, by the principle of duality, a corresponding theorem for two point-triads in space. The proof of this theorem may be effected directly in a very simple manner, by use of the equations already derived in a former article. Choosing the equations in the manner given below, we gain the advantage of bringing the critical plane to coincide with the plane of one of the faces of the fundamental tetrahedron, thereby furnishing a basis of facile procedure. This will appear in the sequel.

Four point co-ordinates are used, an equation of the form

\[ xu_1 + yu_2 + zu_3 + wu_4 = 0 \]

being the equation of a point, the co-ordinates being the variables \( u_1, u_2, u_3, u_4 \), which represent the perpendiculars, from the four points \( A \),

84  ON SOME CONFIGURATIONS ASSOCIATED WITH THE
B, C, D of the fundamental tetrahedron ABCD, upon any plane
passing through the point in question.

Consider two point-triads in space, the one triad consisting of the
points designated L, M, N; the other consisting of the points desig­
nated P, Q, R. The equations of the points are chosen as below; the
results are precisely the same, had the equations of the points been
chosen in all their generality (Cayley’s paper, l.c.).

\[
\begin{align*}
L : & \quad u_4 = 0, \\
M : & \quad mu_1 + lu_3 + lmn u_3 + (mn - l) (nl - m) (lmn - 1) u_4 = 0, \\
N : & \quad nl u_1 + mnu_2 + u_2 - (mn - l) (nl - m) (lmn - 1) u_4 = 0,
\end{align*}
\]

and

\[
\begin{align*}
P : & \quad u_3 - (mn - l) (lmn - 1) u_4 = 0, \\
Q : & \quad u_2 - (nl - m) (lmn - 1) u_4 = 0, \\
R : & \quad u_2 + (mn - l) (nl - m) u_4 = 0.
\end{align*}
\]

Let \(v_1, v_2, v_3, v_4\) denote constant values of \(u_1, u_2, u_3, u_4\), respectively. Also set

\[
\left. \frac{\partial F}{\partial u_1} \right|_{u_1 = v_1} = \frac{\partial F}{\partial v_1},
\]

and similarly in other cases.

The initial problem is to find a plane such that its pole with
respect to the system of points, written in the symbolic form

\[
LMN = 0 \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (1),
\]

is identical with its pole with respect to the second system of three
points, written

\[
PQR = 0 \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (2),
\]

The pole

\[
\left. \frac{\partial F}{\partial v_1} u_3 + \frac{\partial F}{\partial v_2} u_5 + \frac{\partial F}{\partial v_3} u_7 + \frac{\partial F}{\partial v_4} u_4 = 0
\]

of the plane \((v_1, v_2, v_3, v_4)\) with respect to the system (1) given by the equation

\[
F(u_1, u_2, u_3, u_4) = 0
\]

has for its equation

\[
\begin{align*}
[2mn v_1 v_4 + n (F + m^2) v_2 v_4 + m (F m^2 + 1) v_3 v_4 + 2 \lambda \mu v. v_4] u_1 \\
+ [2mn v_2 v_4 + n (F + m^2) v_1 v_4 + l (m^2 + n^2) v_3 v_4 + 2 \lambda \mu v. v_4] u_2 \\
+ [2mn v_3 v_4 + m (F n^2 + 1) v_1 v_4 + l (m^2 + n^2) v_2 v_4 - 2 \lambda \mu v. v_4] u_3 \\
+ [mn v_2 v_3 + v_2^2 + v_3^2] - 3 \lambda^2 \mu v^2. v_1 v_4 + n (F + m^2) v_3 v_4 + m (F m^2 + 1) v_1 v_4 \\
+ l (m^2 + n^2) v_2 v_3 + 2 \lambda \mu v. v_2 v_4 + 2 \lambda \mu v. v_3 v_4 - 2 \lambda \mu v. v_4] u_4 = 0 \cdots (3),
\end{align*}
\]

where \(\lambda, \mu, v \equiv mn - l, nl - m, lmn - 1\), respectively.
Also, the pole
\[
\frac{\partial \Phi}{\partial v_1} u_1 + \frac{\partial \Phi}{\partial v_2} u_2 + \frac{\partial \Phi}{\partial v_3} u_3 + \frac{\partial \Phi}{\partial v_4} u_4 = 0
\]
of the plane \((v_1, v_2, v_3, v_4)\) with respect to the system (2) given by the equation
\[
\Phi(u_1, u_2, u_3, u_4) = 0
\]
has for its equation
\[
\left[ v_2 v_3 - \mu v_4 + \lambda \mu v_1 - \lambda \mu^2 v_2 \right] u_1 \\
+ \left[ v_1 v_3 - \mu v_4 + \lambda \mu v_2 - \lambda \mu^2 v_1 \right] u_2 \\
+ \left[ v_1 v_2 - \mu v_3 + \lambda \mu v_1 + \lambda \mu^2 v_2 \right] u_3 \\
+ \left[ -\mu v_1 v_2 - \mu v_3 - \lambda \mu^2 v_1 - \lambda \mu^2 v_2 \right] u_4 = 0 \ldots \ldots \ldots \ldots (4),
\]
where \(\lambda, \mu, \nu = mn - l, nl - m, lmn - 1\), respectively,
as before.

Now, it is evident by inspection that equations (3) and (4) are identical (aside from sign) if
\[
v_1 = v_2 = v_3 = 0.
\]
Accordingly the plane of the face \(ABC\) of the fundamental tetrahedron \(ABCD\) is such that its pole with respect to the point-triad (1) is coincident with its pole with respect to the point-triad (2).

Connect up next the six points \(L, M, N, P, Q, B\) by lines and planes in every possible way. Suppose the plane of \(ABC\) to be intersected by the line \(LM\) in the point \(LM\), and by the plane \(LMN\) in the line \(LMN\); and so in other cases. We obtain in this fashion a configuration in the plane of \(ABC\), consisting of the fifteen \((\equiv C_5)\) points \(LM, LN, ..., QR\), and of the twenty \((\equiv C_6)\) lines \(LMN, LMP, ..., PQR\); and which is such that through each of the points there pass four of the lines, and on each of the lines lie three of the points. Thus the lines
\[
\begin{align*}
LMN \\
LMP \\
LMQ \\
LMR
\end{align*}
\]
and the points
\[
\begin{align*}
LM \\
MN \\
NL
\end{align*}
\]
and so in other cases.

It will next be shown that six lines, denoted by \(1, 2, 3, 4, 5, 6\), may be drawn in the plane \(ABC\), conditioned as follows:
ON SOME CONFIGURATIONS ASSOCIATED WITH THE

\[(A): \begin{cases} 
\text{line (1) passes through the points } \ LP, \ MQ, \ NR, \\
\text{(2) } \ LP, \ MQ, \ NB, \\
\text{(3) } \ LP, \ MB, \ NQ, \\
\text{(4) } \ LQ, \ MP, \ NR, \\
\text{(5) } \ LQ, \ MP, \ NR, \\
\text{(6) } \ LR, \ MQ, \ NP.
\end{cases}\]

For this purpose, represent any line in the plane of \(ABC\) as the join of two points, whose equations are

\[
\begin{align*}
\lambda_1 u_2 + \mu_1 u_3 + v_1 u_4 &= 0, \\
\lambda_2 u_2 + \mu_2 u_3 + v_2 u_4 &= 0.
\end{align*}
\]

If, for example, this line meets the line \(LP\), the join of the two points, whose equations are

\[
L: \ u_4 = 0, \\
P: \ u_1 - (mn - l) (lmn - 1) u_4 = 0,
\]

we have the equation of condition

\[
\begin{vmatrix}
\lambda_1 & \mu_1 & v_1 \\
\lambda_2 & \mu_2 & v_2 \\
0 & 0 & 1 \\
1 & 0 & 0 & -(mn - l) (lmn - 1)
\end{vmatrix} = 0,
\]

or

\[
\frac{\mu_1}{\mu_2} = \frac{v_1}{v_2} = \frac{\mu_2}{\mu_1}.
\]

and hence the line in question may be written

\[
\begin{align*}
\mu_1 u_2 + v_1 u_3 &= 0, \\
\mu_2 u_2 + v_2 u_3 &= 0.
\end{align*}
\]

If further, this line meets the line \(MQ\), we have the equation of condition

\[
\begin{vmatrix}
1 & 0 & 0 \\
0 & \mu_1 & v_1 \\
m & l & lmn, \\
0 & 1 & 0
\end{vmatrix} = 0,
\]

or

\[
\frac{\mu_1}{v_1} = 1 : l,
\]

and hence the required line has for its equations

\[
\begin{align*}
1: \ u_1 &= 0, \\
u_2 + lu_3 &= 0.
\end{align*}
\]

If now we write the equation of the point \(N\) in the form

\[
N: \ (nl u_1 + mn u_2 + lmn u_3) - (lmn - 1) \{u_3 + (mn - l)(nl - m) u_4\} = 0,
\]

and note the equation of the point \(R:\)

\[
R: \ u_3 + (mn - l)(nl - m) u_4 = 0,
\]
it is evident that the equations of the line $NR$ may be written

$$NR: \begin{cases} nl u_1 + mn u_2 + lm u_3 = 0, \\ u_3 + (m n - l) (m l - m) u_4 = 0. \end{cases}$$

That the line (1) meets the line $NR$ is now obvious by inspection.

Determining, in similar fashion, the equations of the five remaining lines, and re-writing the equations of line (1), we obtain

1:
$$\begin{cases} u_1 = 0 \\ u_2 + lu_3 = 0 \end{cases},$$

2:
$$\begin{cases} u_2 = 0 \\ mu_1 + u_3 = 0 \end{cases},$$

3:
$$\begin{cases} u_3 = 0 \\ nu_1 + u_2 = 0 \end{cases},$$

4:
$$\begin{cases} u_1 = 0 \\ lu_2 + u_3 = 0 \end{cases},$$

5:
$$\begin{cases} u_2 = 0 \\ u_1 + mu_3 = 0 \end{cases},$$

6:
$$\begin{cases} u_3 = 0 \\ u_1 + nu_2 = 0 \end{cases}.$$

Now these six lines 1, 2, 3, 4, 5, 6 touch the conic given by the equation

$$l m n (u_1^2 + u_2^2 + u_3^2) + m n (l^2 + 1) u_2 u_3 + n l (m^2 + 1) u_3 u_1 + l m (n^2 + 1) u_1 u_2 = 0.$$  

This is most easily shown by putting $u_1, u_2, u_3$ in turn equal to zero in the equation last written. We obtain, respectively,

$$mn (u_2 + lu_3) (lu_2 + u_3) = 0,$$

$$nl (mu_1 + u_3) (u_1 + mu_3) = 0,$$

$$lm (nu_1 + u_2) (u_1 + nu_2) = 0.$$  

Moreover, it is clear from an inspection of the scheme (A) above, that the points $LP, LQ, LR; MP, MQ, MR; NP, NQ, NR$ are the points 14, 25, 36; 35, 16, 24; 26, 34, 15, respectively, where 14, for example, denotes the meet of the lines 1 and 4; and so in other cases.

Conversely, starting from the six lines 1, 2, 3, 4, 5, 6 touching the conic, and denoting the points 14, 25, 36; 35, 16, 24; 26, 34, 15 (which are, indeed, the vertices, and meets of opposite sides, of the hexad 162435) in the manner described above, then it is possible to complete the figure of the fifteen points $LM, LN, ... QR$, and of the twenty lines $LMN, LMP, ... PQR$, such that through each point pass
four lines, and on each line lie three points, as detailed in the fore­
going.

Of the fifteen points, nine, viz. the points \( LP, LQ, LR; MP, MQ, MR; NP, NQ, NR \) are, as appeared above, points on two of the six lines \( 1, 2, 3, 4, 5, 6 \); the remaining points are \( MN, NL, LM; QR, RP, PQ \). These are Brianchon Points:

\[
\begin{align*}
MN & \text{ of the hexad } 162435, \\
NL & \text{ of the hexad } 152634, \\
LM & \text{ of the hexad } 142536, \\
QR & \text{ of the hexad } 152436, \\
RP & \text{ of the hexad } 142635, \\
PQ & \text{ of the hexad } 162534,
\end{align*}
\]

for the point \( MN \) is the meet of the lines \( MNP, MNQ, MNR = MP, NP; MQ, NQ; MR, NR = 35, 26; 16, 34; 24, 15 \); that is, \( MN \) is the Brianchon point of the hexad \( 162435 \); and similar reasoning verifies the above statements for the remainder of the six points.

To summarize, we have two sets of three hexads, such that the Brianchon points of each set lie in linea; and the two lines so obtained, together with the eighteen lines through the six Brianchon points, form a system of twenty lines passing by fours through fifteen points (Fig. 2).

46. Derivation of the Pascalian Configuration by the Projection of the Straight Lines lying upon the Cubic Surface of the Second Species.

For the cubic surface with one conical point (§ 25), the tangent planes of the type \([12\cdot34\cdot56]\) are fifteen in number, viz.

\[
\begin{align*}
[12\cdot34\cdot56], & \quad [14\cdot23\cdot56], & \quad [15\cdot23\cdot46], \\
[12\cdot36\cdot45], & \quad [13\cdot24\cdot56], & \quad [15\cdot24\cdot36], \\
[12\cdot35\cdot46], & \quad [16\cdot23\cdot45], & \quad [13\cdot25\cdot46], \\
[16\cdot25\cdot34], & \quad [13\cdot26\cdot45], & \quad [14\cdot26\cdot35], \\
[15\cdot26\cdot34], & \quad [16\cdot24\cdot35], & \quad [14\cdot25\cdot36].
\end{align*}
\]

In terms of these fifteen planes, the equation of the surface may be written in the form

\[
UVW + kXYZ = 0
\]

in ten different ways (§ 13, third table of trihedral pairs).
Fig. 2.
Consider one of these forms, viz.

\[ [12 \cdot 34 \cdot 56] \times [15 \cdot 23 \cdot 46] \times [14 \cdot 26 \cdot 35] \]
\[ + k [12 \cdot 35 \cdot 46] \times [15 \cdot 26 \cdot 34] \times [14 \cdot 23 \cdot 56] = 0, \]

written in the symbolic notation. This conjugate trihedral pair cuts out from the cubic surface the nine lines

12, 14, 15; 23, 34, 35; 26, 46, 56.

Now let us consider the six lines

1, 2, 3, 4, 5, 6

upon the cubic surface, which all pass through one point \( O \), the vertex of the quadric cone upon which they lie. It is clear by inspection that

\[
\begin{align*}
\text{the line } 1 & \text{ meets the lines } 12, 15, 14, \\
& \quad \quad 3 \quad \quad \quad \quad \quad 34, 23, 35, \\
& \quad \quad \quad \quad \quad 6 \quad \quad \quad \quad \quad 56, 46, 26, \\
& \quad \quad \quad \quad \quad 2 \quad \quad \quad \quad \quad 12, 23, 26, \\
& \quad \quad \quad \quad \quad 4 \quad \quad \quad \quad \quad 34, 46, 14, \\
& \quad \quad \quad \quad \quad 5 \quad \quad \quad \quad \quad 56, 15, 35, \\
\end{align*}
\]

\( (A) \):

For the sake of brevity, I shall designate the six planes as follows:

\[ [12 \cdot 34 \cdot 56] \equiv a, \quad [12 \cdot 35 \cdot 46] \equiv f, \]
\[ [15 \cdot 23 \cdot 46] \equiv b, \quad [15 \cdot 26 \cdot 34] \equiv g, \]
\[ [14 \cdot 26 \cdot 35] \equiv c, \quad [14 \cdot 23 \cdot 56] \equiv h. \]

Consider now the six planes \( a, b, c, f, g, h \), and taking \( O \) as the point of projection, and an arbitrary plane of projection, the line of intersection of the planes \( a \) and \( b \) will be projected into a line \( ab \), and the point of intersection of the planes \( a, b, c \) into a point \( abc \); and so in other cases. We have thus a plane figure consisting of the fifteen lines \( ab, ac, \ldots gh \), and of the twenty points \( abc, af, \ldots gh \); and which is such that on each of the lines there lie four of the points, and through each of the points there pass three of the lines, viz. the points \( abc, af, abg, abh \) lie on the line \( ab \); and the lines \( bc, ca, ab \) meet in the point \( abc \); and so in other cases.

Moreover, from the above scheme, we see that the projections of the lines \( af, bg, ch \) meet in a point, and the like for each of the six triads of lines; that is, in the plane figure, we have six points 1, 3, 6, 2, 4, 5—each of them the intersection of three lines as shown in the following scheme:
and these six points lie in a conic (the intersection of the quadric cone by the plane of projection). It is clear that the lines $af, ag, ah; bf, bg, bh; cf, cg, ch$ are the lines $12, 34, 56; 46, 15, 23; 35, 26, 14$, respectively.

Of the fifteen lines, nine, viz. the lines $af, ag, ah; bf, bg, bh; cf, cg, ch$ are, as has been seen, lines through two of the six points $1, 3, 6, 2, 4, 5$; the remaining lines are $be, ca, ab; gh, hf, fg$. These are Pascalian lines:

- $bc$ for the hexagon $153264$,
- $ca$ „ „ „ 143562,
- $ab$ „ „ „ 123465,
- $gh$ „ „ „ 143265,
- $hf$ „ „ „ 123564,
- $fg$ „ „ „ 153462.

This appears as follows:

- line $bc$ contains points $bcf, bcg, bch$
- $= bf . cf, bg . cg, bh . ch$
  = 46 . 35, 15 . 26, 23 . 14$

that is, $bc$ is the Pascalian line of the hexagon $153264$; and similarly in other cases.

The twenty points $abc, abf, . . . fgh$ are as follows, viz. omitting the two points $abc, fgh$, the remaining eighteen points are the Pascalian points (the intersections of pairs of lines each through two of the points $1, 2, 3, 4, 5, 6$) which lie on the Pascalian lines $bc, ca, ab; gh, hf, fg$ respectively; the point $abc$ is the intersection of the Pascalian lines $bc, ca, ab$; and the point $fgh$ is the intersection of the Pascalian lines $gh, hf, fg$—the points in question being two of the points $S$ (Steiner's twenty points, each the intersection of three Pascalian lines).

In this process, we have projected only a single one of the ten possible trihedral pairs. This projection of a single trihedral pair gave six Pascalian lines and two Steiner points. Remembering that the six
points and the conic upon which they lie are fixed, in the process of projection, we reach the conclusion that the projection of all ten trihedral pairs upon the same plane give 60 Pascalian lines and 20 Steiner points, as should be the case.

It would not be difficult at this point to develop the theory so as to put in evidence the 60 Kirkman points, the twenty Salmon-Cayley lines, the fifteen Salmon points, and the fifteen Steiner lines for the plane configuration, derived by projection, from the point $O$, of corresponding elements in the spatial configuration. This was done by Cremona*, to whom the theorem is due, in his original paper on the subject. The subject has also been considered by Richmond†, who succeeded in giving a perfectly symmetrical form to the equations of the lines on this type of cubic surface, after Segre's method (cf. historical summary).

**47. On the Graphic Representation of the Projection of a Pair of Triheders into the Pascalian Configuration.**

The problem with which I have concerned myself here is: Can we represent to the eye, graphically or by means of a model, the figure arising from the projection of a pair of triheders into the Pascalian configuration?

I select the two triheders as follows:

$$(12 \cdot 34 \cdot 56) (15 \cdot 23 \cdot 46) (14 \cdot 26 \cdot 35) = 0.$$

$$(12 \cdot 35 \cdot 46) (15 \cdot 26 \cdot 34) (14 \cdot 23 \cdot 56) = 0.$$

These mutually intersect in the nine lines

$$12, 14, 15; 23, 34, 35; 26, 46, 56.$$

Using the numerical values given in § 25, these planes have the equations

$$a : \quad w = 0,$$

$$b : \quad 256x - 384y - 96z + 459w = 0,$$

$$c : \quad 96x - 64y - 256z + 153w = 0,$$

and

$$f^* : \quad 32x + 27w = 0,$$

$$g : \quad 64y - 51w = 0,$$

$$h : \quad 32z - 17w = 0.$$

* "Teoremi Stereometrici dai quali si deducono le Proprietà dell' Esagrammo di Pascal," Reale Accademia dei Lincei, Anno lxxiv. (1876-77).


‡ Cf. § 13.
Writing down the polar planes of these two triads $abc$, $fgh$, the condition for their coincidence requires their common pole $O$ to assume, as one of its four possible positions, the position of the vertex $D$ of the fundamental tetrahedron $ABCD$.

The lines through the point $O$, and conditioned by the scheme (A) of § 46, are the lines 1, 2, 3, 4, 5, 6 passing through the conical node of that article, which coincides with the vertex $D$ of the fundamental tetrahedron $ABCD$.

The nine lines 12, 14, 15; 23, 34, 35; 26, 46, 56, and also the six lines 1, 2, 3, 4, 5, 6, are laid down on the diagram, precisely as was done for the Species II (§ 25) of the cubic surface. The point $D$ is the point of projection, while the plane of projection, which is within our choice, may be taken to coincide with the plane of $ABC$, since this plane contains the three lines 12, 34, 56, but does not contain any one of the lines 1, 2, 3, 4, 5, 6.

The projection of the line 46, for example, was found by joining the meets of the lines 4 and 46 with the plane of projection (since the line 46 meets the line 4); and similarly for other cases. Only three of the Pascalian lines, viz. $bc$, $ca$, $ab$ are shown in the figure, in order to avoid too great complexity. The projection of line $bc$, for example, was found as follows: lines $bh$ and $ch$ (i.e. lines 23 and 14), lying in planes $b$ and $c$ respectively, intersect in a point $P$ say, on the line $bc$; and similarly the lines $bg$ and $cg$ (i.e. lines 15 and 26) intersect in a point $P_1$ say, on the line $bc$. The projections of the points $P$ and $P_1$ are the meets of the projections of the pairs of lines $bh$, $ch$; $bg$, $cg$ respectively. The projection of the line $bc$, then, is the join of the projections of the points $P$ and $P_1$.

The Steiner point shown in the figure, the common meet of the three Pascalian lines $bc$, $ca$, $ab$, is one of the two Steiner points yielded by the projection; the other is not shown as explained above.

One other detail was the construction of the conic section (in the diagram an hyperbola) given by the intersection of the quadric cone (containing the lines 1, 2, 3, 4, 5, 6) with the plane of projection. The six points I, II, III, IV, V, VI (the meets of the lines 1, 2, 3, 4, 5, 6, respectively, with the plane of projection) lie in this conic section. Hence it was constructed, projectively, by means of Pascal’s Theorem (Plate 13).

This graphic (or modelled) representation of a remarkable configuration and its projection relates not only to Cayley’s configuration, but also to the Species II of the cubic surface. Indeed, it may be
interpreted as a planographic representation of the projection of the lines upon a cubic surface having only one conical node, from the nodal point, upon an arbitrary plane into the projection of the "mystic hexagram."

It is perhaps worthy of remark that the six lines 1, 2, 3, 4, 5, 6 lie upon the quadric cone having $D$ for its vertex, and for its base the conic section (an hyperbola) having for its equations

$$12 (x^2 + y^2 + z^2) - 5(6yz - 8zx + 5xy) = 0, \quad w = 0.$$  

The intersections of this cone by the planes $x = 0, y = 0, z = 0$, respectively, have for their equations:

$$x = 0, \quad (2y - z)(y - 2z) = 0;$$  
$$y = 0, \quad (x + 3z)(3x + z) = 0;$$  
$$z = 0, \quad (3x + 4y)(4x + 3y) = 0.$$  

48. A Deduction from Cayley’s Theorem on the Pascalian Configuration.

The theorem of Cayley* mentioned in § 45, together with the argument in § 46, leads to the well-known conclusion:

Given any two triads of planes $a, b, c; f, g, h$; then it is possible to find four points $O_1, O_2, O_3, O_4$ such that the polar plane of any one of these points with respect to one triheder is identical with its polar plane with respect to the other triheder. Considering any one of the points, say $O_1$, then it is possible to draw six lines through $O_1$, whose positions are defined as follows:

$$\begin{align*}
\text{line 1 meets the lines } & af, bg, ch, \\
\text{"} 2 " " " & ag, bh, cf, \\
\text{"} 3 " " " & ah, bf, cg, \\
\text{"} 4 " " " & af, bh, cg, \\
\text{"} 5 " " " & ag, bf, ch, \\
\text{"} 6 " " " & ah, bg, cf.
\end{align*}$$

Then these six lines together with the nine lines $af, ag, ah; bf, bg, bh; cf, cg, ch$ determine a cubic surface upon which they lie, for which the point $O_1$ is the only conical point.

This conclusion may be more generally phrased as follows:

Through the nine lines of mutual intersection of two triheders can be drawn four cubic surfaces, each possessing only one conical point.

and having twenty-one distinct lines lying wholly upon the surface. The four points $O_1$, $O_2$, $O_3$, $O_4$ such that the plane of any one of them with respect to one triheder is identical with its polar plane with respect to the other triheder, are the conical points of the four cubic surfaces. Through each one of these points pass six lines, conditioned by the scheme (A) above, which lie not only upon a quadric cone but also upon the cubic surface through the nine lines above mentioned and for which that point is the only conical point.
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OF

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      Compare also Comptes Rendus, Vol. LXVIII. pp. 865....


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1882  "On the 27 lines, the 45 triple tangent planes, and the 36 double-sixers of a cubic surface, with a hint for the construction of models which give the position of the lines when they are all real."  By P. Frost.  *Quart. Journ.*  Vol. XVII. pp. 89–96.


“Representation on the plane of the twenty-seven lines upon a cubic surface.” By P. H. Schoute. *Amsterdam Akademie Verslagen*, Vol. i. (1893), pp. 143-....


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A Double Six.
Species $II = 12 - C_2$. 
Species $V = 12 - B_4$. 
Species VI = 12 – R3 – C2.
Species VII = 12 - B₅.

Species X = 12 - B₄ - C₃.

Species XI = 12 - B₆.
Species VIII = 12 – 3C₇.
Species IX = 12 - 2B₁.
Species XII = 12 - U₄,

Species XV = 12 - U₇,

Species XIII = 12 - B₃ - 2C₂,

Species XVI = 12 - 4C₂,

Species XIV = 12 - B₅ - C₂,

Species XVII = 12 - 2B₃ - C₂.
Species XVIII = $12 - B_4 - 2C_2$.

Species XIX = $12 - B_6 - C_2$.

Species XX = $12 - U_8$.

Species XXI = $12 - 3B_8$. 