FROM DARK MATTER TO DEFICIT ANGLES:
EFFECTIVE FIELD THEORY IN COSMOLOGY AND
ADS/CFT

by

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Abstract

The Standard Model (SM) of particle physics, despite its accurate and thoroughly tested description of nature, is an incomplete theory. Astrophysical observations indicate an abundance of matter beyond that of the SM, which currently can only be observed through indirect gravitational effects. The Cosmic Microwave Background (CMB) provides a means to constrain both the abundance of this dark matter and the presence of additional light particle species. The SM also fails to provide a full description of gravity, with many open questions as to the quantum nature of gravitational phenomena. In this thesis, we consider three distinct but complementary means of extending this modern framework.

Conventional attempts to determine the nature of dark matter are insensitive to models with mass below about a GeV. We consider newly proposed technology which would allow for the detection of dark matter as light as an MeV in mass, through the observation of single electron events in semiconductor materials with significantly lowered thresholds. We find that such detectors would be particularly sensitive to dark matter with electric and magnetic dipole moments, with a reach many orders of
ABSTRACT

magnitude beyond current bounds.

We then consider the effects of new light species on the CMB. We perform a thorough survey of natural, minimal models containing new light species and numerically calculate the precise contribution of each of these models to the CMB. We provide a map between the parameters of any particular theory and the results of observational experiments. Using this map, we present new constraints placed by the Planck experiment on the parameter space of several models containing new light species.

Finally, we study the universal behavior of long-distance gravitational interactions in AdS$_3$ from the perspective of conformal field theory (CFT). To do so, we compute the structure of Virasoro conformal blocks in a semi-classical, large central charge approximation. Using this result, we then prove the existence of large spin operators with fixed ‘anomalous dimensions’ indicative of the presence of deficit angles in AdS$_3$. As we approach the threshold for the BTZ black hole, interpreted as a CFT$_2$ scaling dimension, the twist spectrum of large spin operators becomes dense. We derive the BTZ quasi-normal modes and show that primary states above the BTZ threshold mimic a thermal background for light operators through exchange of the Virasoro identity block.

Advisor: David E. Kaplan
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Dedication

This thesis is dedicated to my parents, Kenneth and Joan Walters. You were my first teachers and my constant source of love and support. Thank you.
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Chapter 1

Introduction

The Standard Model (SM) of particle physics is a quantum field theory which accurately describes the observed short-distance interactions of matter. Over the past four decades, this SM has been experimentally tested to an unprecedented level of precision, with no significant discrepancies between its theoretical predictions and experimental results [1–3]. Despite its continued success, the SM is only an effective description of nature; it is inherently incomplete.

One obvious shortcoming of the SM is its inability to account for the abundance of additional matter observed throughout the universe [4,5]. Though the precise nature of this dark matter (DM) is unknown, it can be accurately approximated as a single particle species which only interacts gravitationally [6]. While observational constraints have been placed on various properties of DM, both its identity and its production in the early universe remain a mystery.
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The presence of DM suggests that there exists a set of particles which is currently hidden due to the weakness of its interactions with the SM. While it is entirely possible that this hidden sector is quite simple, it could instead contain a rich spectrum of phenomena waiting to be discovered. One well-motivated possibility is the addition of new species of light particles [7, 8]. Though such particles must be so weakly-coupled as to be unobservable in past collider experiments [9], they could still significantly impact the evolution of the early universe [10–12]. Searches for both DM and light species through cosmological observations thus provide an important means of extending the SM.

Another observation which is not described by the SM is the accelerating expansion of the universe [13]. The simplest explanation of this acceleration is the presence of a constant vacuum energy density, often referred to as a cosmological constant. Currently, there is no accepted mechanism for determining the value of this constant [14,15].

The standard framework for cosmology, referred to as the ΛCDM model, combines the particle content of the SM with cold DM and a cosmological constant. Despite being approximate and incomplete, this model successfully describes a variety of observations of the universe from its early history onwards [16–21].

In this thesis, we consider three particular questions currently unresolved by this standard framework: the nature of dark matter, the existence of additional species of light particles, and the universal long-distance properties of gravity. The first two
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topics consist of phenomenological studies of current or future experiments, while the third involves a more formal discussion of new theoretical results. We begin with a brief introduction to each individual subject, before continuing with more detail in the ensuing chapters.

1.1 Light Dark Matter and Direct Detection

There are a large number of well-motivated extensions to the SM which explain the origin of DM and predict additional interactions with ordinary matter \[22\]. These models have led to a set of experiments which attempt to determine the nature of DM through direct observation. The general strategy is to use underground detectors to observe collisions between DM and the atoms of some target material \[23\]. These experiments have led to significant constraints on the possible strength of DM interactions \[24, 26\].

Each experiment has a lower bound on the energy deposit required to produce an observable signal, and for the majority of experiments this threshold is near the keV scale \[27\]. However, DM with a mass below the GeV scale is kinematically unable to generate recoil energies above this threshold, leaving a large range of light DM models completely unconstrained by direct detection searches \[28\].

One possible means of extending our experimental reach is the study of low energy electron recoil events. The bandstructure of semiconductors allows for the ionization of electrons with energies near the eV scale, and recent technological advances have
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made the detection of such ionization events a realistic possibility [29,30]. In chapter 2 we consider the potential sensitivity of one proposal, referred to as CDMSLite [31], which would modify the existing Cryogenic Dark Matter Search experiment [25] to reduce the energy threshold by approximately three orders of magnitude, thereby allowing detection of single electron recoils.

To determine the observational reach of such an experiment, we consider the possible models for DM-electron interactions and compute the rate for a light DM particle to scatter off a valence electron bound to a semiconductor. We combine these rates with an estimate of the possible backgrounds at CDMSLite to determine the sensitivity of such a device to DM-electron interactions, which is found to be orders of magnitude beyond current bounds. This chapter is based on [32], which was written in collaboration with Peter Graham, Dave Kaplan, and Surjeet Rajendran.

1.2 New Light Particles in the Early Universe

One of the best probes we have to study the ΛCDM model is the Cosmic Microwave Background (CMB) [33], an approximately thermal distribution of photons emitted shortly after the Big Bang. In the earliest moments of the universe, photons were continually interacting with electrons and protons, maintaining an approximate thermal equilibrium. As the universe expanded, this system cooled, until the charged electrons and protons could bind together into neutral hydrogen. The photons quickly ceased to scatter and began to stream freely, and those photons which have freely...
CHAPTER 1. INTRODUCTION

propagated since that time compose the CMB, effectively providing a snapshot of the early universe. Since its discovery fifty years ago [34], the CMB has been repeatedly measured by many independent observational experiments [16–19], leading to a remarkably precise understanding of the early universe.

Precision measurements of the CMB allow us to determine what fraction of the early universe consisted of massive particles with negligible kinetic energy and what fraction consisted of light particles moving with a substantial velocity. Note that in this context the term 'light' refers to particles with masses approximately at or below the eV scale, whereas in the previous section it was used to describe much more massive particles.

The energy density of massive particles was largely dominated by DM, so determining this portion of the early universe particle content can be viewed as an indirect measure of the amount of DM. However, many extensions of the SM predict the existence of additional species of light particles beyond those of the SM, so CMB observations also provide a largely model-independent means of detecting new light particles [35–37].

In chapter 3, we consider the effects of new light species on the CMB. Using an effective field theory approach, we discuss the possible models of light particles which are compatible with the simplifying criteria of naturalness and minimality. We then numerically determine the contribution of each new light species to the CMB as a function of the parameters in the underlying theory. We also interpret the viable
parameter space of each model in terms of recent measurements of the CMB from the Planck satellite [16], placing additional constraints on theories using this new data. This chapter is based on [38], which was written in collaboration with Chris Brust and Dave Kaplan.

1.3 AdS/CFT and Long-Distance Interactions

The SM completely ignores the existence of gravity. From a practical standpoint, this omission is not worrisome. On long distances, the effects due to gravity are accurately described using general relativity, with no need to include any quantum corrections [39]. In fact, some of the universal quantum effects of gravity have been shown to necessarily match those of relativity at long distances [40–42]. For all short distance experiments, the gravitational effects due to individual particles are completely negligible in comparison to other interactions, and the effects due to macroscopic bodies can be accurately treated as an unchanging gravitational background through which fundamental particles travel [43,44].

From a conceptual perspective, though, the question of describing gravity as a fully quantum theory remains one of the most compelling goals of modern theoretical physics. Such a description is necessary to describe a rich spectrum of phenomena in extreme conditions, such as the formation and evaporation of black holes and the very first moments after the Big Bang. As we currently have no experimental data on the quantum structure of gravity, many basic questions about such a theory remain
unanswered and require much more theoretical progress \[45,46\].

Over the past fifteen years, a rather robust new framework for approaching this issue has been rapidly developing. Known as the AdS/CFT correspondence \[47,48\], this framework posits that quantum theories of gravity in an anti de Sitter (AdS) spacetime have a dual description in terms of a conformal field theory (CFT) in a flat spacetime with one fewer dimension. Phrased more precisely, this correspondence claims that the Hilbert space of states within a theory of AdS\(_{d+1}\) quantum gravity is identical to that of some CFT\(_d\).

This correspondence has many surprising and far-reaching consequences \[49,50\]. One of the most striking is the ability to relate the quantum properties of a full theory of gravitational interactions to a separate theory which makes no reference to gravity. Using this correspondence, any question about gravity can presumably be rephrased as a question about ordinary quantum field theory. While these results technically only apply to theories in an AdS background, this framework serves as a remarkably useful arena in which to study general properties of quantum gravity.

Though there has been a tremendous amount of progress in understanding this correspondence, there are many open questions remaining. One obvious question is whether all CFTs have a dual description in terms of higher-dimensional AdS. As a first step towards answering this question, it was recently shown \[51,52\] that all CFTs in \(d \geq 3\) have a structure consistent with long-distance locality in AdS. More specifically, the Hilbert space of every CFT\(_{\geq3}\) satisfies the principle of ‘cluster
CHAPTER 1. INTRODUCTION

decomposition', which can be interpreted in AdS as the independence of two systems at large separation. This result can be viewed both as a basic check of the generality of this correspondence and as a test of the universal nature of long-distance interactions in AdS$_{\geq 4}$.

In chapter 4, we consider the generalization of this work to AdS$_3$/CFT$_2$. The structure of 2d CFTs is qualitatively very different from that of higher dimensions [53], such that our generalization is only able to describe theories in the semi-classical limit. However, in this limit we are able to discover dramatic properties of gravity in AdS$_3$. In particular, we derive the existence of deficit angles [54] surrounding any object with mass below a particular threshold. We also show that all objects with mass above this threshold mimic the effects of a thermal background. This behavior is completely consistent with the interpretation of these massive objects as BTZ black holes [55]. This chapter is based on [56], which was written in collaboration with Liam Fitzpatrick and Jared Kaplan.
Chapter 2

Semiconductor Probes of Light

Dark Matter

2.1 Introduction

The particle nature of dark matter (DM) has been well established by astronomical and cosmological data ([5]6 and references therein). It is reasonable to expect the DM particle to carry non-gravitational interactions. These non-gravitational interactions may permit direct detection of DM, plausibly leading to a deeper understanding of its origins and the structure of particle physics. A good case can be made for weak-scale interactions between the Standard Model (SM) and DM. The possible existence of new states at the weak scale, as suggested by the hierarchy problem, could lead to such interactions. A variety of experiments are currently probing these interactions.
CHAPTER 2. SEMICONDUCTOR PROBES OF LIGHT DARK MATTER

These experiments measure the energy deposited by DM as it scatters off the atoms in a detector. Current experiments are sensitive to recoil energies $\gtrsim$ keV \cite{27}. At these recoil energies, the experiments are dominantly sensitive to the scattering of DM particles with masses larger than $\sim 1$ GeV off the atomic nucleus \cite{28}.

A lighter particle bound to the DM halo is kinematically forbidden from depositing energies greater than a keV. Owing to our ignorance of the physics responsible for DM, it is desirable to develop technological tools to explore all possible regions of the DM parameter space. The ability to detect low energy ($\sim$ eV) electron recoil events will significantly extend our reach into this parameter space, as was demonstrated recently in \cite{57}. The energy deposited by a light DM particle on the nucleus is suppressed by the nuclear mass. The difficulty of detecting such low energy nuclear recoils is further complicated by the anaemic response of the nucleus to such events. However, since the electron is light, DM can dump more energy into it. Further, energy deposition into electrons can lead to more readily identifiable events such as ionization in the detector. Technological advances in semiconductor-based DM detectors have made the detection of such ionization events a realistic possibility. Strategies to suppress backgrounds to ultra low levels similar to typical direct detection experiments also seem feasible \cite{29,30}. One possible experiment of this type is the CDMSLite proposal, which would modify existing CDMS technology to reduce the energy threshold by approximately three orders of magnitude, thereby allowing detection of single electron recoils \cite{58}.
CHAPTER 2. SEMICONDUCTOR PROBES OF LIGHT DARK MATTER

The ionization of electrons always involves transfer of momentum from DM to the atomic nucleus. The cross-section for such interactions is suppressed by a form factor for momentum transfers much bigger than the inverse Bohr radius of the concerned electron. The electrons in semiconductors and noble gas detectors have comparable Bohr radii. However, the bandstructure of the semiconductor allows for the ionization of electrons with relatively lower energy ($\sim 1$ eV) in comparison with the energy needed to ionize electrons in a noble gas detector ($\sim 10$ eV). Since the DM particle has to lose more energy in the case of a noble gas detector, the momentum transferred to the nucleus is higher, leading to a form factor suppression of the cross-section. Owing to the smaller energy transferred in the case of the semiconductor, the momentum transferred to the nucleus is also smaller, leading to an unsuppressed cross-section.

In this chapter, we argue that semiconductor detectors able to measure the production of single electron-hole pairs have the potential to detect light DM in a wide range of parameter space, orders of magnitude beyond current bounds. We also show that such semiconductor devices possess an enhanced sensitivity to light DM in comparison with noble gas detectors. We begin, in section 2.2, by considering the possible models for DM-electron interactions, either through renormalizable couplings or effective operators such as electromagnetic dipoles. We illustrate, through the aid of a simple concrete example, the ease with which these electromagnetic moments are generated for DM particles as a consequence of new states at the weak scale. In section 2.3, we compute the rate for a light DM particle to scatter off a valence electron
CHAPTER 2. SEMICONDUCTOR PROBES OF LIGHT DARK MATTER

bound to a semiconductor. In computing this rate for various operators, we focus in particular on electromagnetic moments because they are both easily generated in many models and have enhanced cross-sections due to their coupling to a long range force carrier, namely, the photon. Consequently, these operators may offer the easiest way to probe the existence of such light DM. Using an estimate of the possible backgrounds at CDMSLite, we examine the sensitivity of such a device to DM-electron interactions in section 2.4 in comparison to current bounds on these operators, as well as the limits possible with noble gas detectors.

2.2 Models

In order to study the potential reach of detectors such as CDMSLite, we must consider the possible operators that generate DM-electron interactions, as well as the current constraints on the corresponding parameters. We restrict ourselves to the simplest extensions to the SM, but the DM sector could contain other scenarios (such as [59–63]), possible signals of which should be studied in future work. We also focus specifically on the case of light DM ($m_e \lesssim m_\chi \lesssim 10$ GeV), in order to find unexplored parameter space for these simple models.
2.2.1 Dipole Moments

The simplest extension to the SM is for DM and electrons to interact electromagnetically. Due to constraints on the possible electric charge of DM \[64\], the lowest dimensionality available for electromagnetic interactions corresponds to the dimension-five dipole moment operators

$$L_{\text{dipole}} = -\frac{i}{2} \bar{\chi} \sigma^{\mu\nu} (\mu_\chi + d_\chi \gamma^5) \chi F_{\mu\nu}, \quad (2.1)$$

where $\mu_\chi$ and $d_\chi$ correspond to the DM magnetic and electric dipole moments, respectively. These dipole moments correspond to a cutoff scale ($d_\chi \sim \Lambda^{-1}$) and arise from loop interactions involving heavy charged particles, such as those shown in Figure 2.1. These dipole moments are then easily generated in models where DM carries a conserved charge, such as asymmetric dark matter \[65\]. As an illustrative example, we consider two new heavy intermediaries (a fermion and a scalar), in the limit where both have approximately the same mass $M$ and coupling to DM $g$. We then
obtain (based on calculations similar to [70]) the dipole moment
\[ d_\chi \approx \frac{eg^2}{8\pi^2 M}, \quad (2.2) \]
with the same approximate form for \( \mu_\chi \). There might be some worry that loops involving these heavy charged intermediaries, shown in Figure 2.2, would push the natural DM mass beyond the MeV or GeV scale. The contribution from this diagram is
\[ \delta m_\chi \approx \frac{g^2 M}{16\pi^2}. \quad (2.3) \]

The important feature of this expression is that decreasing the coupling \( g \) between DM and the heavy intermediaries decreases the effective scale contributing to the DM mass. However, this decrease in \( g \) actually increases the effective scale contributing to the DM dipole moment. This means that for a generic set of heavy charged intermediaries, a large effective dipole scale does not imply a large mass contribution, provided the coupling with DM is small. For example, a charged fermion-scalar pair with \( M \approx 500 \text{ GeV} \) and \( g \approx 0.2 \) would contribute \( \delta m_\chi \approx 100 \text{ MeV} \) and \( d_\chi \approx 3 \times 10^{-4} \text{ TeV}^{-1} \). As we will show in section 2.4, the enhanced cross-sections of dipole interactions at low momentum transfer make them the strongest candidate for
detection with CDMSLite, with experimental sensitivity to effective mass scales \( \lesssim 10^3 \) TeV.

### 2.2.2 Effective Pointlike Vertex

The next simplest extension is the dimension-six effective four-fermion vertex, which corresponds to the exchange of a very massive mediator (such as a scalar or vector) which is then integrated out of the theory. An example is the vector-channel operator

\[
\mathcal{L}_{\text{point}} = \frac{1}{\Lambda^2} \bar{\chi} \gamma^\mu \chi \tilde{\psi}_e \gamma_\mu \psi_e, \tag{2.4}
\]

where \( \Lambda \) corresponds to the cutoff of this effective theory (roughly the mass of the intermediary particle). The strongest constraints on these pointlike interactions come from collider experiments such as LEP \cite{71,73}. For example, the vector interaction above is currently restricted to a cutoff scale \( \Lambda \gtrsim 480 \) GeV. Calculations based on the method presented in section \ref{sec:2.3} indicate that CDMSLite would only be able to search for pointlike interactions up to the scale \( \Lambda \approx 200 \) GeV. The weakness of this projected sensitivity is due to the lack of enhancement for DM pointlike scattering at low recoil energies. Semiconductor detectors will then be no more sensitive to dimension-six DM interactions than collider experiments. However, higher scale physics which generates dimension-six operators will very often generate dipole moment operators, as well, whose recoils suffer from less suppression. As will be shown in section \ref{sec:2.4}, dipole operators will therefore allow CDMSLite to probe scales far beyond the reach
possible with pointlike interactions, either in direct detection or collider experiments. In light of these facts, we do not consider pointlike interactions for the remainder of this chapter.

2.2.3 Broken $U(1)$

The final possibility for simply extending the SM is to introduce new light particles to the theory. In this case, DM can interact via a broken $U(1)$ gauge interaction, with the corresponding dimension-four operator

$$\mathcal{L}_{A'\chi} = g_A A'_\mu \tilde{\chi} \gamma^\mu \chi,$$  \hspace{1cm} (2.5)

where $A'_\mu$ is the DM gauge field. A DM-electron interaction could then result from kinetic mixing of amplitude $\epsilon$ between $A'_\mu$ and the SM photon, which can be diagonalized to give a DM gauge boson-electron coupling term

$$\mathcal{L}_{A'\ell} = -\epsilon e A'_\mu \bar{\psi}_\ell \gamma^\mu \psi_\ell.$$  \hspace{1cm} (2.6)

This interaction also has a large parameter space available for semiconductor detectors, which was discussed in [57]. We consider this interaction later in some detail for the sake of thoroughness and comparison with that work, but our main focus is on the potential exploration of new physics through DM dipole moments.
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2.3 Detection Rate

As a simpler conceptual example, we first consider DM ionizing a single isolated atom. We then turn to DM interacting with a semiconducting lattice to excite an electron from the valence band to the conduction band. Our approaches to these two cases are rather similar, as are the resulting cross-sections. This topic is similar to [57,74,75], but differs in the treatment of the lattice bandstructure and in the inclusion of momentum transfer to nuclei.

2.3.1 Basic Kinematics

First, we briefly review the kinematics of electron recoil interactions. Our convention for momenta is to use $\vec{p}$ to indicate incoming momenta and $\vec{k}$ for outgoing momenta. Also, since the DM velocity $v_\chi \sim 10^{-3}$, we use simpler nonrelativistic kinematics.

Interactions are classified by the recoil energy $E_R$, defined as the kinetic energy of the outgoing electron

$$E_R = \frac{k_e^2}{2m_e}. \quad (2.7)$$

In the lab frame, the electron (initially in a bound state) can have a nonzero incoming momentum $\vec{p_e}$, with the probability for this momentum determined by the electron’s momentum space wavefunction $\tilde{\psi}$. However, the initial energy for the electron is simply $-E_B$, the binding energy associated with its initial state $\psi$. We also choose
the lab frame to be such that the nucleus has no initial momentum ($\vec{p}_N = 0$).

The momenta values we consider are much lower ($\sim$ keV) than the nuclear masses of silicon and germanium, such that the final kinetic energy of the nucleus can be neglected. The resulting energy conservation equation can be rewritten as

\[ k_\chi^2 = p_\chi^2 - 2m_\chi(E_R + E_B). \]  

We can then calculate the minimum possible momentum transfer $q$ necessary to ionize an electron with binding energy $E_B$. For the case of semiconductors, with $E_B \sim 1$ eV, the DM must at least transfer momentum $q \sim 1$ keV for ionization to be possible. For noble gases, with a larger $E_B \sim 10$ eV, the minimum transfer necessary is $q \sim 10$ keV. This increase in momentum transfer away from the inverse Bohr radius and into the form factor regime suppresses the noble gas detection rates, reducing their experimental sensitivity in comparison with semiconductors. This suppression in the case of dipole moment interactions is given in detail in equations (2.16) and (2.18).

2.3.2 Single Atom Ionization

Our example initial state consists of free DM and a bound hydrogenic atom. If the atom is ionized by the DM-electron interaction, then the final state consists of the recoiling DM, escaping electron, and remaining nucleus. For calculational simplicity, we model the nuclear final state as a plane wave. However, the final state of the nucleus is in fact not relevant to the cross-section. Rather, what matters is that the nucleus, due to its large mass, can absorb momentum at negligible energy cost.
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From the perspective of the DM-electron system, the recoiling nucleus then breaks momentum conservation while preserving energy conservation.

The electron’s resulting wavefunction is also deformed by the presence of the charged nucleus, causing it to deviate from a simple plane wave. This deformation causes a substantial enhancement to the cross-section, which can be approximated by combining a plane wave final state with a momentum-dependent enhancement factor. This factor is similar to the standard treatments of beta decay \[76\] and Sommerfeld enhancement \[77\] (for a clear review, see \[78\]), as well as the work on noble gas detectors in \[57\]. This enhancement factor can be found by exactly solving the Dirac equation for a free electron in the presence of a Coulomb potential, and comparing the solution to that of a plane wave, yielding

\[
F(k_e) = \frac{2\pi\nu}{1 - e^{-2\pi\nu}},
\]

where \(\nu\) is the \(k_e\)-dependent factor

\[
\nu = \frac{Z_{\text{eff}}m_e\alpha}{k_e}.
\]

In order to calculate the cross-section for DM to ionize the atom, we then simply need to look at the usual scattering formula \[79\]

\[
d\sigma = \frac{1}{|v_{\text{rel}}|} \left( \prod_f \frac{d^3k_f}{(2\pi)^3} \right) 2\pi \delta(E_f - E_i) |\langle f | H | i \rangle|^2.
\]

Using this formula, we can then write out the full cross-section for our ionization process, assuming (as stated earlier) that the lab frame corresponds to the nucleus
rest frame and using the approximation \( m_e << m_N \),

\[
\frac{d\sigma}{dE_R} \approx \frac{16a^2d^2k_eF(k_e)}{\pi v^2_chi} \left[ \ln \left( \frac{1 + a^2q^2_{\text{min}}}{a^2q^2_{\text{min}}} \right) - \frac{6a^4q^4_{\text{min}} + 15a^2q^2_{\text{min}} + 11}{6(1 + a^2q^2_{\text{min}})^3} \right].
\]

(2.14)

The term \( v_chi \) refers to the incoming velocity of the DM particle (in the lab frame), and the various momenta have the following definitions

\[
k_e = \sqrt{2m_eE_R},
\]

\[
q_{\text{min}} = m_chi - \sqrt{m^2chi^2 - 2m_chi(E_R + E_B)}.
\]

(2.15)

In order to understand the form factor suppression for momentum transfer above \( a^{-1} \), we can take the further limit of \( aq_{\text{min}} >> 1 \), obtaining

\[
\frac{d\sigma}{dE_R} \rightarrow \frac{4d^2k_eF(k_e)}{\pi v^2chi^6q^3_{\text{min}}}.
\]

(2.16)
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The second cross-section corresponds to the similar magnetic dipole moment (MDM) interaction. Using the same approximations and variables as the EDM case, we find the cross-section

$$\frac{d\sigma}{dE_R} \approx \frac{64\alpha^2 a^2 \mu^2 k_e F(k_e)}{3\pi v^2} \left[ \ln \left( \frac{1 + a^2 q^2_{\text{min}}}{a^2 q^2_{\text{min}}} \right) - \frac{12a^4q^4_{\text{min}} + 30a^2q^2_{\text{min}} + 19}{12(1 + a^2 q^2_{\text{min}})^3} \right]. \quad (2.17)$$

Not surprisingly, this cross-section is quite similar to that of an EDM, but is suppressed by an additional approximate factor of $\alpha^2$. This factor corresponds to the average velocity of the bound electron ($\langle v_e \rangle \sim \alpha$). We can also obtain the similar form factor

$$\frac{d\sigma}{dE_R} \rightarrow \frac{16\alpha^2 \mu^2 k_e F(k_e)}{3\pi v^2 a^4 q^6_{\text{min}}}. \quad (2.18)$$

The final cross-section corresponds to the broken $U(1)$ DM gauge interaction, for which we consider two limiting regimes. The first corresponds to a heavy mediator, with mass $m_A$ much greater than the momentum transfer, such that the interaction is effectively pointlike. In this limit, the cross-section is

$$\frac{d\sigma}{dE_R} \approx \frac{128\lambda^2 k_e F(k_e)}{3v^2 m_A^4} \left( \frac{1}{1 + a^2 q^2_{\text{min}}} \right)^3, \quad (2.19)$$

where $\lambda$ is the effective DM-electron coupling

$$\lambda = \epsilon \sqrt{\frac{g^2}{4\pi}}. \quad (2.20)$$

The second limit corresponds to a light mediator, with mass much less than the momentum transfer, such that $A_{\mu}'$ is effectively massless. This yields the cross-section

$$\frac{d\sigma}{dE_R} \approx \frac{128\lambda^2 a^4 k_e F(k_e)}{3v^2} \left[ \frac{3}{a^2 q^2_{\text{min}}} + \frac{9a^4 q^4_{\text{min}} + 21a^2 q^2_{\text{min}} + 13}{(1 + a^2 q^2_{\text{min}})^3} - 12 \ln \left( \frac{1 + a^2 q^2_{\text{min}}}{a^2 q^2_{\text{min}}} \right) \right]. \quad (2.21)$$
2.3.3 Semiconductor Valence Band

We now consider an atom in a semiconducting lattice. For an electron in any periodic potential, the delocalized eigenfunctions of the Hamiltonian can be expressed in terms of localized wavefunctions, in the form \[80,81\]

\[
\psi_b(\vec{x}) = \frac{1}{\sqrt{N}} \sum_n e^{i\vec{b}\cdot\vec{x}_n} \phi(\vec{x} - \vec{x}_n),
\]

where \(N\) is the total number of lattice sites and \(\vec{b}\) is a wavevector with components related to the dimensions \(L_i\) of the lattice by the relationship

\[
b_i = \frac{2\pi n_i}{L_i},
\]

where \(i\) runs over the values \(x, y,\) and \(z,\) and \(n_i\) is any integer from 1 to \(N_i\), the number of lattice sites in the \(i\)-direction \((N = N_xN_yN_z)\). The wavefunction \(\phi(\vec{x} - \vec{x}_n)\) is a localized Wannier wavefunction centered around each individual lattice site (located at \(\vec{x}_n\)).

In the tight-binding, linear combination of atomic orbitals (LCAO) approximation \([82]\), the Wannier wavefunctions \(\phi\) are written in the basis of free atomic orbitals. These coefficients are very small for all atomic orbitals except those near the bound-state energy of \(\psi_b\) \([80]\), which for valence band states are the highest occupied \(s\)- and \(p\)-states. These outermost states can be reasonably approximated with hydrogenic wavefunctions, due to the screening of the nuclear charge by the inner core electrons.

For the final state, the valence band electron is excited into the conduction band, where it can be treated as an approximately free electron, with two corrections. Due
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to the weak periodic potential of the lattice, a conduction band electron propagates
with the effective mass \( m_e^* = f_e m_e \). The correction factor \( f_e \) is an element-dependent
factor determined by the direction of the electron momentum in the lattice and the
energy curvature along the conduction band. We use the density of states average
values for \( f_e \) corresponding to the very edge of the conduction band. This gives an
approximate estimate for the interaction cross-section, which will only be slightly
modified by a more exact calculation. We use \( f_e = 1.1 \) for silicon and \( f_e = 0.6 \) for
germanium [83,84].

The second correction comes from the presence of the positively charged hole
remaining in the valence band. The Coulomb interaction between these charges causes
the same enhancement as the atomic case, where the effective charge \( Z_{eff} \) felt by the
outgoing electron is simply that of the remaining hole.

With these particular initial and final states, we can use a similar approach to the
free atom considered earlier, writing the \( \vec{b} \)-dependent interaction cross-section

\[
d\sigma_{\vec{b}} = \frac{F(k_e)}{|\vec{v}_X|} \left( \prod_f \frac{d^3k_f}{(2\pi)^3} \right) (2\pi)^4 \delta^4(k_f - p_i) \left| \tilde{\psi}_X(\vec{p}_X - \vec{k}_X - \vec{k}_e) \tilde{H}(\vec{k}_X - \vec{p}_X) \right|^2 . \tag{2.24}
\]

For interactions localized to a single lattice site (momentum transfer of \( O(a^{-1}) \)),
the cross terms for wavefunctions of different sites are negligible. This cross-section
can then be expressed in terms of interactions with a single local Wannier wave-
function. We therefore approximate cross-sections involving highly delocalized states
spread throughout the entire lattice by calculating cross-sections involving a single
localized state, repeated periodically at the $N$ sites of the lattice,

$$d\sigma_b \approx F(k_e) \frac{\left| {v}_X \right|}{2\pi^3} \left( \prod_f \frac{d^3k_f}{(2\pi)^3} \right) (2\pi)^4 \delta^4(k_f - p_i) \left| \tilde{\phi}(\vec{p}_X - \vec{k}_X - \vec{k}_e) \tilde{H}(\vec{k}_X - \vec{p}_X) \right|^2. \quad (2.25)$$

Our full cross-section for a single lattice site is then simply the average of the individual $d\sigma_b$,

$$d\sigma = \frac{1}{N} \sum_b d\sigma_b. \quad (2.26)$$

In this approximation, the $\vec{b}$-dependence for each $d\sigma_b$ is contained entirely in the initial state binding energy $E_B$. Our total cross-section then changes from an average over all possible $\vec{b}$ to an integral over all possible $E_B$,

$$d\sigma \approx \int dE_B \rho(E_B) d\sigma(E_B), \quad (2.27)$$

where $\rho(E_B)$ is an experimentally determined density of states (based on \[82, \[85\] and shown in Figure 2.3) accounting for the fact that some $E_B$ values correspond to more $\psi_b$ states than others. This density of states then serves as an efficiency factor for scattering at various binding energies. For example, there is zero detection efficiency in germanium at the minimum $E_B$ of 0.7 eV, but the efficiency rapidly increases for slightly larger $E_B$. Note that our final result does not contain any directional dependence resulting from the lattice structure, but rather gives the directionally-averaged behavior of the total cross-section.

The cross-sections for electrons in a semiconducting lattice are then accurately approximated by those of a free hydrogenic bound state, in a weighted average over initial-state binding energies, with an altered final-state electron mass.
2.3.4 Detection Rate

Once the interaction cross-section is known, the total rate of detection (typically written in units of events/day/kg/eV) can be calculated using the following expression:

\[
\frac{dR}{dE_R} = \frac{\rho_\chi \eta_e}{m_\chi} \int d^3v_\chi f(\vec{v}_\chi) v_\chi \frac{d\sigma}{dE_R},
\]

(2.28)

where \( \rho_\chi \) is the DM mass density and \( \eta_e \) is the valence electron number density per unit mass of the detector. For the DM density we use the value \( \rho_\chi = 0.3 \text{ GeV/cm}^3 \). The function \( f(\vec{v}_\chi) \) is the DM velocity distribution in the lab frame, meaning that we need to account for the Earth’s average velocity \( \vec{v}_E \) through the galaxy. We use the conventional Maxwellian distribution, truncated at a maximum escape velocity of \( v_{\text{esc}} \) (in the average rest frame of the galaxy),

\[
f(\vec{v}_\chi) = \frac{1}{k} \exp \left( -\frac{(\vec{v}_\chi + \vec{v}_E)^2}{v_0^2} \right),
\]

(2.29)
where \( k \) is a normalization factor chosen such that \( \int d^3v \, f(\vec{v}_\chi) = 1 \),
\[
k = (\pi v_0^2)^{3/2} \left[ \text{erf} \left( \frac{v_{\text{esc}}}{v_0} \right) - \frac{2}{\sqrt{\pi}} \frac{v_{\text{esc}}}{v_0} e^{-v_{\text{esc}}^2/v_0^2} \right]. \tag{2.30}
\]

We use the following values for the velocity parameters: average Earth velocity \( v_E = 240 \text{ km/s} \), average DM velocity \( v_0 = 230 \text{ km/s} \), and DM escape velocity \( v_{\text{esc}} = 600 \text{ km/s} \) \([86]\). These values give us \( k = 2.504 \times 10^{-9} \) (in units with \( c = 1 \)).

Performing the angular integrals (of which our cross-sections are independent), we obtain the following expression
\[
\frac{dR}{dE_R} = \frac{\rho_\chi \eta_e \pi v_0^2}{m_\chi k v_E} \left[ \int_{v_{\text{min}}}^{v_+} dv_\chi v_\chi^2 e^{-\left(\frac{v_\chi - v_E}{v_0} \right)^2} \frac{d\sigma}{dE_R} - \int_{v_{\text{min}}}^{v_-} dv_\chi v_\chi^2 e^{-\left(\frac{v_\chi + v_E}{v_0} \right)^2} \frac{d\sigma}{dE_R} \right], \tag{2.31}
\]
where \( v_{\pm} = v_{\text{esc}} \pm v_E \) and \( v_{\text{min}} \) is the minimum DM velocity necessary to induce an interaction of recoil energy \( E_R \),
\[
v_{\text{min}} = \sqrt{\frac{2(E_R + E_B)}{m_\chi}}. \tag{2.32}
\]

## 2.4 Sensitivities

### 2.4.1 Approach

Using these detection rates, we can estimate the possible sensitivity for a detector such as CDMSLite. We assume a flat background rate of 1 event/day/kg/keV, which was provided as an experimental estimation of the expected background \([58]\). This background rate is due to residual radioactivity, which produces high energy gamma rays that can Compton scatter in the detector. To determine if the experimental
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reach is limited by the background, we also include the sensitivity possible using germanium with no background. Due to the small number of events, we find that reducing the background beyond our estimate does not have a substantial effect on the exclusion sensitivity.

The proposed detection method involves the measurement of single electron-hole pairs, assuming that the energy deposited in the initial recoiling electron will prompt the formation of secondary electron-hole pairs. This method of detection will limit the energy resolution to the average energy per electron-hole pair, which is approximately 3 eV \[58,87\]. We do not consider this process in detail, and instead focus on the initial deposition of energy into a single electron. The possible interactions we consider are all peaked very strongly at low \(E_R \sim \text{eV}\), falling off quickly with growing recoil energy. Because of this, we focus solely on recoils with the lowest energy \((E_R < 9 \text{ eV})\) where the signal is most competitive with the background. For CDMSLite, we assume the detection setup of CDMS II \[88\], with 4.4 kg of germanium and 1.1 kg of silicon. We also consider the sensitivity of each material separately for the sake of comparison, rather than combine data for a total exclusion limit. Using the approach of \[89\], we can find the 95% confidence level sensitivity possible after an experimental run of one year.

For comparison, we also include the possible sensitivity of measuring electron recoils in a xenon-based detector. This approximate sensitivity is calculated using the ionization cross-sections derived earlier, combined with atomic data from \[90\].
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For simplification, we only consider interactions with the valence electrons in the 5s- and 5p-states, which dominate the overall cross-section, and use $Z_{eff} = 1$ for the Sommerfeld enhancement. We assume a detector mass of 1 kg, a runtime of 1 year, and negligible background, in order to find the maximum possible reach of noble gas detectors.

2.4.2 Results

Results are shown in Figure 2.4 for dipole moments and in Figure 2.5 for broken $U(1)$ models, with sensitivities for silicon, germanium, and xenon detectors. As mentioned before, we consider the mass range $600 \text{ keV} < m_\chi < 10 \text{ GeV}$. The upper limit is due to current nuclear recoil experiments, most of which provide substantial exclusions down to $m_\chi \sim 10 \text{ GeV}$ [28]. The lower limit is simply due to the limited energy available for recoils at such light masses, which suppresses the possible signal.

For each interaction, we show the strongest current experimental and astrophysical constraints on the relevant parameter space. There are also detailed constraints placed by supernova cooling [91,92] and BBN [93] on DM with mass $m_\chi \lesssim 10 \text{ MeV}$, but the full calculation of those constraints for these particular models is beyond the scope of this work. For this reason, we simply indicate in our plots the mass value below which these additional bounds potentially apply. As a conceptual reference for the plots, the minimum $d_\chi$ and $\mu_\chi$ values excluded by germanium without background for $m_\chi = 10 \text{ MeV}$ correspond to $\langle \sigma v \rangle \approx 10^{-45} \text{ cm}^2$. 

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Figure 2.4: Exclusion sensitivity at 95% confidence level possible after 1 year, for (a) electric and (b) magnetic dipole moments. The solid lines assume a background of 1 event/day/kg/keV, while the dashed lines assume no background. Areas above the curves for germanium (red), silicon (blue), and xenon (brown) would be excluded. Regions in gray are already excluded for all models of DM by other experiments or astrophysical data. Masses to the left of the dashed black line are potentially constrained by supernova cooling and BBN. While a detailed calculation of these constraints on lighter masses is beyond the scope of this work, it is unlikely the entire region is fully excluded.

As discussed in the beginning of section 2.3, the larger binding energy present in xenon necessitates a larger momentum transfer. This results in an increased form
Figure 2.5: Exclusion sensitivity at 95% confidence level possible after 1 year, for effective $U(1)$ coupling $\lambda = \epsilon \sqrt{g_2^2 \chi^4 / 4\pi}$ with (a) $m_A = 10$ MeV and (b) $m_A = 1$ meV. The solid lines assume a background of 1 event/day/kg/keV, while the dashed lines assume no background. Areas above the curves for germanium (red), silicon (blue), and xenon (brown) would be excluded. Regions in gray are already excluded for all models of DM by other experiments or astrophysical data. Masses to the left of the dashed black line are potentially constrained by supernova cooling and BBN. While a detailed calculation of these constraints on lighter masses is beyond the scope of this work, it is unlikely the entire region is fully excluded.

factor suppression, reducing the experimental reach of noble gases in comparison with that of semiconductors, as seen in both Figures 2.4 and 2.5.
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Figure 2.6: Detection rates (in events/eV) after 1 year, for $m_\chi = 100$ MeV. These rates assume detector masses of 4.4 kg for germanium (red) and 1.1 kg for silicon (blue). (a) Electric dipole with $d_\chi = 10^{-5}$ TeV$^{-1}$. (b) Magnetic dipole with $\mu_\chi = 10^{-3}$ TeV$^{-1}$. (c) Heavy broken $U(1)$ with $m_A = 10$ MeV and $\lambda = 10^{-7}$. (d) Light broken $U(1)$ with $m_A = 1$ meV and $\lambda = 10^{-14}$.

For each of these exclusion limits, germanium provides somewhat weaker limits. This is caused by our assumption of the same background per unit mass for both materials, which places a stronger restriction on germanium, due to its heavier nuclear mass. Germanium’s reach is also slightly reduced by its smaller effective electron mass in the conduction band, but does become more competitive with silicon at lower DM masses, due to its smaller band gap.

For EDM and MDM, the strongest general constraints on lighter masses come from colliders such as LEP [94]. The parameter space for larger masses (> 1 GeV) is also
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probed by the direct detection experiments XQC \cite{95,96}, CRESST \cite{97}, CDMS \cite{25}, and XENON10 \cite{98}. Direct detection rates for MDM are suppressed by the extra factor of $v^2$ relative to EDM limits. This effect applies to both current nuclear recoil experiments and CDMSLite, while collider limits do not suffer from this suppression \cite{99}.

Current dark matter annihilation searches such as Fermi LAT \cite{100}, and bounds on dark matter annihilation rates in the early universe from WMAP and ACT \cite{101} also provide limits to DM dipole moments. However, these results can only constrain symmetric DM, and do not apply to asymmetric models of DM. The resulting constraints are strong for MDM interactions, but the case of EDM annihilations is much more suppressed, making those irrelevant for our purposes.

As discussed in section \ref{sec:2.2}, models which generate dipole moments will also generate pointlike four-fermion interactions. Current bounds on these pointlike operators placed by collider experiments or current direct detection results could then potentially constrain dipole moments, as well. However, the charged particles which generate dipole moments will generically not couple directly to electrons or quarks. The generation of an effective vertex between DM and electrons would therefore require the exchange of a Z boson, in addition to the loop of charge intermediaries. The resulting effective operator would be substantially suppressed by both a loop factor and the Z boson propagator. Models which generate the dipole moments considered in this chapter ($\Lambda \gtrsim 1 \text{ TeV}$) would therefore be unconstrained by current
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bounds on pointlike four-fermion operators. However, the coupling of light DM to Z bosons which contributes to these operators is constrained by measurements of the Z width \[102\]. If the resulting Z dipole moments are approximately the same order as the EDM or MDM, we estimate the dipole moment scale would be constrained to be \( \gtrsim 2 \text{ TeV} \). These EDM and MDM bounds are very model-dependent, and therefore are not included in Figure 2.4.

As mentioned earlier, there are two limiting cases for a broken \( U(1) \). The first corresponds to a heavy mediator, for which we consider \( m_A = 10 \text{ MeV} \), and the second to a light mediator, for which we use 1 meV. These sample masses were selected for comparison with \[57\].

The strongest constraints on \( \lambda \), defined previously in equation (2.20), come from a combination of astrophysical data, due to the fact that \( \lambda \) depends on both the DM self-coupling and the \( A'_\mu \)-photon mixing. Details on the \( U(1) \) mixing constraints can be found in \[7\], while the strongest constraints on DM self-coupling are explained in \[57,103,104\]. The larger mass regions are also limited by the same direct detection experiments as dipole moments.

In Figure 2.6, we also provide example detection rates for each model considered above, assuming a DM mass of 100 MeV, as well as the same runtime (1 year) and detector masses (4.4 kg for Ge, 1.1 kg for Si). In the case of larger background, annual modulation in these rates due to the Earth’s motion around the Sun would provide an important confirmation of a potential DM signal. We estimate that the annual mod-
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ulation would be approximately 6% of the baseline average for the case of both EDM and MDM, which is larger than the traditional modulation associated with a velocity-independent cross-section. This difference arises due to the initial electron wavefunction, which provides substantial $q$-dependence (and therefore velocity-dependence) in the final cross-section.

2.5 Conclusions

We have examined the potential direct detection reach of semiconductor-based experiments such as CDMSLite. Experimental sensitivity to the production of single electron-hole pairs can dramatically improve the detectable energy range over that of traditional nuclear recoil methods. This enhanced energy range kinematically allows electron recoils to probe the parameter space of light dark matter, a region which remains largely inaccessible to nuclear recoils. The small energy gap present in semiconducting bandstructure also provides materials such as silicon and germanium with a substantial advantage relative to noble gases. Motivated by these prospects, we have considered the possible interactions between electrons and dark matter, many of which fit naturally into weak-scale extensions to the SM.

We have found that semiconductor detectors are sensitive to a large range of uninvestigated parameter space, specifically for interactions such as dipole moments, which are enhanced at low recoil energies. Such dipole moments are naturally generated in many extensions of the SM and are generically expected in models where
the dark matter carries a conserved charge, such as asymmetric dark matter \cite{65, 69}. Electromagnetic moments provide a unique glimpse into higher-scale physics, and we have found that CDMSLite can extend our current reach by orders of magnitude, up to scales as large as $10^3$ TeV. Light dark matter with dipolar or new gauge interactions remains a well-motivated alternative to the traditional heavy WIMP scenario, and CDMSLite would present a substantial opportunity to explore this possibility.
Chapter 3

New Light Species and the CMB

3.1 Introduction

The Cosmic Microwave Background (CMB) is one of the only probes we have of physics in the early universe. Through a detailed mapping of anisotropies in the temperature of those photons which decoupled from visible matter in the era of recombination, we are able to determine the relativistic energy density in that era. From this, we gain information about the number of light species in our universe. In the massless limit, we can accomplish this by a fit to only one number, the relativistic degrees of freedom $g_*$. This parameter is often expressed in terms of an effective number of neutrinos, $N_{\text{eff}}$, defined such that in the Standard Model (SM) of particle physics $N_{\text{eff}}$ is roughly the number of neutrino generations. Beyond the Standard Model (BSM) physics models which contain new light species with masses
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\(\mathcal{O}(\text{eV})\) or less can contribute to this measurement. Consequently, we have new terrain in which to test the SM through its prediction of \(g_* = 3.38\), corresponding to an \(N_{\text{eff}}\) of \(3.046^{106}_{116}\).

There has been a statistically insignificant but consistent excess in the measured value of \(g_*^{17, 19, 117, 118}\). Prior to the results from the Planck satellite, the most precise reported measurement was \(g_* = 3.69 \pm 0.16\), corresponding to \(N_{\text{eff}} = 3.71 \pm 0.35^{117}\), coming from a combination of data from the South Pole Telescope (SPT) and the Wilkinson Microwave Anisotropy Probe (WMAP). A similar excess is present in measurements from the Atacama Cosmology Telescope (ACT) \(^{19}\). Very recently, however, the Planck collaboration released the first results from its measurement of CMB anisotropies, obtaining a result of \(g_* = 3.50 \pm 0.12\), corresponding to \(N_{\text{eff}} = 3.30 \pm 0.27^{16}\). Future Planck results will continue to improve the precision of this measurement, with a projected final \(g_*\) sensitivity of \(\pm 0.09^{101, 119}\). In addition, future measurements of the polarization of the CMB are projected to constrain \(g_*\) to within \(\pm 0.02\), corresponding to constraints on \(N_{\text{eff}}\) of \(\pm 0.044^{101}\). We are entering an era of being able to contrast the SM prediction for \(g_*\) with the predictions of BSM physics models containing new light species to an unprecedented precision.

The power of this probe of new physics is that in any BSM theory containing new species with masses \(\ll 0.1\,\text{eV}\) which were once in thermal equilibrium with the SM, the effect of these species is contained in a single number, the correction \(\Delta g_*\) to the SM prediction for \(g_*\). Therefore, a map from the parameters of a BSM model to the
number $\Delta g_*$ can be constructed in order to determine the consistency of regions of the parameter space with the measured value of $g_*$. Although useful approximations of such a map exist [120,121], we are entering the exciting era of precision cosmology experiments, and consequently it has become imperative to form precise theoretical predictions. The subject of this chapter is the precise numerical computation of this map of model parameters to $\Delta g_*$ for a wide variety of natural, minimal BSM theories containing new light or massless species. We approach this problem in a largely model-independent effective field theory framework to fully characterize the effects of all such models.

Although there are other existing constraints on new light species present in the early universe coming from the study of Big Bang Nucleosynthesis (BBN) [122,124], this probe does not have the same resolving power as the Planck satellite. Unlike BBN, Planck and future polarization experiments have the power to probe the actual values of the couplings of new light species to the SM, as we shall demonstrate in this work. Even in the absence of a signal for new physics from future experiments, the results of this work provide new constraints on the couplings of SM species to new light particles which are competitive with, and sometimes even surpass, existing constraints from other areas of physics. This establishes a new arena for testing the predictions of BSM physics models with new light species.

The recent results for $g_*$ from Planck are in tension with independent measurements of the Hubble expansion rate today [16]. Specifically, combining those mea-
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measurements with the results from Planck leads to a preference for higher values of \( g_* \) than quoted above. Therefore, these results are not capable of confirming or rejecting the hypothesis of new light degrees of freedom being present in the early universe. Regardless, in order to demonstrate the constraining power of measurements of the CMB, we proceed as if this tension were not present. Motivated by the Planck results given above, we proceed by supposing that values of \( g_* \geq 3.74 \) \( (N_{eff} \geq 3.84) \) are excluded at the 95% confidence level. We interpret our results in this framework in order to illustrate how further data could be utilized.

In section 3.2 of this chapter, we review the relevant details of the determination of \( g_* \) using the CMB, as well as details of thermodynamics in an expanding universe, providing a framework for the rest of this work. In section 3.3, we discuss all BSM physics models compatible with our criteria of naturalness and minimality. Specifically, we discuss the parameters which provide the interaction strength between various fields in the SM and the new light species present in the model. We present the current experimental constraints on each of these scenarios, as well as our findings for the contribution of each new light species to \( g_* \) as a function of the parameters in the underlying theory. We also interpret the viable parameter space of each model in terms of the aforementioned interpretation of the recent results from the Planck satellite, placing additional constraints on theories using this new CMB data.
3.2 Methodology

We study the effects of adding new light or massless particles to the SM on the evolution of the universe and the CMB. Specifically, we investigate new particles which at some time in the early universe were in equilibrium with the SM and decouple prior to recombination. Translating between additional fields in the Lagrangian and the measurement of the effective number of light degrees of freedom, $g_*$, requires a detailed analysis of the quasi-thermal evolution of the universe. The effects of new light degrees of freedom depend on both when and how they decouple from the thermal bath. As we shall see, a direct measurement of anisotropies in the CMB then leads to a resultant measure of $g_*$ at recombination.

In this section, we first review how light species predominantly affect the CMB, namely via Silk damping and the early integrated Sachs-Wolfe (ISW) effect. We also review the thermodynamics of the early universe, as well as the effects of decoupling and other non-equilibrium events. We then discuss the range of decoupling temperatures which can significantly impact the CMB. Finally, we briefly review the most important existing constraint on new light degrees of freedom, namely their effect on Big Bang Nucleosynthesis. As this section is predominantly a review, readers familiar with early universe thermodynamics can potentially skip to the summary provided in subsection 3.2.6.
3.2.1 Relativistic Species and the CMB

The early universe was not perfectly homogeneous, but instead had small perturbations in the distribution of energy density, which are currently believed to be seeded by inflation. These regions of under- or overdensity correspond to small perturbations in the metric away from the pure Friedmann-Robertson-Walker form. CMB anisotropies provide a direct measurement of these early universe perturbations, whose distribution and structure are sensitive to the thermodynamic conditions leading up to recombination. The CMB therefore gives us insight on the properties and structure of the universe in its infancy. The measurement of $g_*$ using the CMB is performed through a precise determination of the expansion rate, $H$, in the era of recombination. The relationship between $H$ and $g_*$ arises because the expansion rate is determined solely by the total energy density, $\rho$, and the curvature. Increasing the value of $g_*$ at a fixed temperature leads to a larger overall $\rho$, which then leads to more rapid expansion. Silk damping is sensitive to the value of $H$ leading up to and during recombination, while the early ISW effect is affected by the evolution of $H$ once photons are effectively free-streaming, which lasts from recombination onwards. For more detailed and thorough explanations of these effects than those presented here, consult [10–12] and references therein.
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Silk Damping

Prior to recombination, protons, electrons, and photons interacted very strongly to form a tightly-coupled plasma. Despite the high frequency of interactions, the mean free path for photons was nonzero, and photons were able to diffuse outward. The rate of photon diffusion grew as the protons and electrons combined into hydrogen, up until the point of last scattering. The overall diffusion scale at the end of recombination is therefore predominantly determined by the mean free path during recombination and the duration of recombination. The diffusion of photons results in a partial thermalization of the baryon-photon plasma, damping any inhomogeneities on scales smaller than the photon diffusion length. This reduction of inhomogeneities below some length scale in turn leads to a damping of temperature anisotropies, commonly called Silk damping [125], above some multipole moment $l_d$. A larger value for $H$ then leads to a decrease in the amount of time available for this diffusion, restricting the damping to smaller angular scales and reducing the magnitude of the damping. An increase in $g_*$ would therefore lead to reduced Silk damping, or equivalently a larger damping moment.

Any map between the predicted diffusion length and the precise value for $l_d$ is sensitive to experimental uncertainty in the angular distance to the last scattering surface. In practice, it is simpler to remove this uncertainty by considering the ratio of $l_d$ to the smaller sound horizon moment $l_s$. This sound horizon arises independently of photon diffusion, due to the spread of inhomogeneities in the baryon-photon plasma.
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These oscillations propagate at the corresponding speed of sound, setting an acoustic oscillation length scale at recombination. The addition of new light species reduces the time for these inhomogeneities to spread, which increases the value of $l_s$, in addition to the increase in $l_d$. These two processes, photon diffusion and sound wave propagation, have different time dependencies. This difference results in an increase of the ratio $l_s/l_d$ as $H$ grows, leading to damping of more of the acoustic peaks, despite the fact that the overall damping has been reduced.

**Early Integrated Sachs-Wolfe Effect**

Following recombination, photons propagate freely without scattering but pass through points of matter over- or underdensity. If the gravitational potential of these inhomogeneities is constant in time, there is no net effect on the CMB photons. However, if the gravitational potential has any time-dependence, the photons will experience some net loss or gain in energy as they pass through a single gravitational perturbation and be red- or blueshifted as a result. The alteration of CMB anisotropies due to time-dependent gravitational potentials is the ISW effect [126].

The evolution of gravitational potentials is determined by the expansion rate, which depends on the overall particle content. In a universe consisting solely of non-relativistic, pressureless matter, the competing effects of gravitational clustering and universe expansion cancel, such that potentials are time-independent. However, any nonnegligible pressure alters the expansion rate such that the potentials do evolve.
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with time. There are therefore two points in time at which the ISW effect could contribute to the CMB. The first occurs when the universe contains a nonnegligible radiation density, which is the case immediately following recombination. This alteration to the CMB shortly after its formation is commonly referred to as the early ISW effect. The second era corresponds to the point at which the vacuum energy becomes a significant fraction of the total energy density. This second case, which begins near modern times, is the late ISW effect.

Unsurprisingly, new light species increase the radiation energy density following recombination, altering $H$ and enhancing the early ISW effect. Specifically, the presence of additional species causes gravitational potentials to evolve more rapidly, resulting in more substantial red- and blueshifts to CMB photons passing through these evolving potentials. On very small scales, photons will pass through multiple such potentials, and the net effect cancels. However, the potentials rapidly become time-independent, such that photons are unable to pass through multiple large-scale perturbations before this effect ends. An increase to $g_*$ therefore enhances the variance in temperature anisotropies on angular scales corresponding to the largest structures immediately following recombination. The size of the largest structures at this point coincides with the acoustic horizon, such that the early ISW effect leads to an increase in the first acoustic peaks of the CMB. In practice, this effect is measured by comparing the height of the first acoustic peak to that of latter peaks.

The effects of Silk damping and early ISW, which can be seen in figure 3.1
Figure 3.1: Projected CMB anisotropy power spectrum for three different values of $g_*$ (or equivalently $N_{\text{eff}}$). The addition of new light degrees of freedom increases the height of the first peak through the early ISW effect and decreases the height of later peaks through Silk damping. The power spectrum, and therefore these effects, are measured by multiple observational experiments, such as the Planck satellite. These spectra were calculated using CAMB [127,128]. The magenta, blue, and orange curves (dark gray, black, and light gray curves, when viewed in black and white) correspond to an $N_{\text{eff}}$ of 3, 4, and 5, respectively.

complementary means of measuring $H$, and therefore $g_*$, near recombination. However, they are still sensitive to two different points in time. Silk damping probes $H$ prior to and during recombination, while the early ISW effect is sensitive to $H$ immediately after recombination. This has two important consequences for constraints on light species. The first is that experiments which focus on precision measurement at smaller values of $l$, such as WMAP, are sensitive mainly to the early ISW effect.
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The resulting constraints are therefore more limited experimentally by the effects of cosmic variance. Experiments which instead focus on anisotropies at larger $l$, such as ACT, SPT, and Planck, are predominantly sensitive to Silk damping and are less affected by cosmic variance.

The second consequence to note is that particles with masses near the temperature scale of recombination ($\sim 0.1 - 1$ eV) will potentially contribute very different signals to these two sets of experiments. The detection of such species would involve a detailed analysis of each individual effect, rather than a simple fit to all the experimental data. While we consider massless particles for the majority of this work, we will return to this possibility later in subsection 3.3.6.

3.2.2 Early Universe Thermodynamics

As mentioned earlier, CMB measurements of light species are predominantly sensitive to the relativistic energy density, which is characterized by an effective number of relativistic degrees of freedom $g_*$. For more details on the material discussed in this subsection, see [129–131]. We can then define $g_*$ in terms of $\rho_{\text{rel}}$, the energy density of all relativistic species, and a reference temperature $T$, which we take to be the photon temperature ($T \equiv T_\gamma$),

$$ \rho_{\text{rel}} \equiv g_* \frac{\pi^2}{30} T^4. \quad (3.1) $$

The light species content of the SM, which consists of photons and neutrinos, can then be used to make a prediction for the measured value of $g_*$ at recombination.
This prediction can be written in the form
\[ g_* = g_\gamma + \frac{7}{8} g_\nu N_{\text{eff}} \left( \frac{T_\nu}{T} \right)^4 = 2 + \frac{7}{8} \cdot 2 \cdot N_{\text{eff}} \left( \frac{4}{11} \right)^{4/3}, \quad (3.2) \]
where the factor of \( \frac{7}{8} \) is due to the effect of Fermi-Dirac statistics on energy density, and \( T_\nu \) is the calculated neutrino temperature assuming neutrinos instantaneously decouple from the rest of the SM at \( T \sim \text{MeV} \). The parameter \( N_{\text{eff}} \) is the effective number of neutrino species. This historically defined parameter, which is 3.046\(^4\) for the SM, is often used to parametrize the effect of any light species other than photons on \( g_* \). The contribution of neutrinos and any new light species to \( g_* \) is given solely by \( N_{\text{eff}} \). Any measured deviation from the SM prediction of \( g_* = 3.38 \) would then indicate the need for new physics.

This chapter calculates the full contribution \( \Delta g_* \) of new light species present in a large number of beyond the SM theories. This contribution to the relativistic degrees of freedom is found by calculating the energy density of new species near the point of recombination. The contribution can also be expressed as a change to \( N_{\text{eff}} \) as
\[ \Delta N_{\text{eff}} = \frac{8}{7} \frac{\Delta g_*}{g_\nu} \left( \frac{T}{T_\nu} \right)^4 \approx 2.2 \Delta g_* . \quad (3.3) \]
To find the energy density of a light species at recombination, we must track the evolution of its phase space density \( f(t,p) \). This form for the distribution function

\(^1\)This effective number of neutrinos is defined such that if neutrinos truly did decouple instantaneously, \( N_{\text{eff}} \) would be 3. However, detailed calculations have shown this to not be the case, and the actual energy density of neutrinos is slightly larger than in the instantaneous decoupling approximation due to their interactions with annihilating electrons. This then results in the slightly larger predicted value for \( N_{\text{eff}} \). For details on these calculations, see [106–116].

\(^2\)By light, we mean \( m \ll \text{eV} \). The contribution of species with masses \( \sim \text{eV} \) is more complicated, as we shall discuss later.
relies on the assumption that the universe is homogeneous and isotropic. We must first determine \( f \) at high temperatures, when the new species is in equilibrium with the SM, then calculate the changes to \( f \) as the universe expands and cools, with various species annihilating or decoupling.

As the universe expands, the evolution of each individual phase space density is controlled by both the rate of expansion \( H \equiv \frac{\dot{a}}{a} \), where \( a(t) \) is the scale factor for the expanding universe, and the rate of interaction with the other particle species. This dependence is expressed using the Boltzmann equation

\[
E \frac{\partial f}{\partial t} - H p^2 \frac{\partial f}{\partial E} = C[f],
\]

where \( p = |\vec{p}| \) and the collision functional \( C[f] \) accounts for changes to \( f \) due to interactions. If we assume that the dominant interactions will consist of 2-to-2 scattering, then \( C[f] \) for some new species \( X \) is defined as the sum over all such possible interactions involving \( X \). If each interaction process is time-reversal invariant,

\[
C[f_X] = \frac{1}{2} \sum_{X,i\rightarrow j,k} \left( \prod_{s=i,j,k} \frac{g_s d^3p_s}{(2\pi)^3 2E_s} \right) (2\pi)^4 \delta^4(p) S |\mathcal{M}|^2 \Omega(f_X, f_i, f_j, f_k),
\]

with the squared amplitude \( |\mathcal{M}|^2 \) averaged over the spins of both incoming and outgoing particles. The term \( S \) corresponds to a symmetry factor whose value is \( \frac{1}{2} \) when \( j \) and \( k \) are identical particles, to avoid overcounting of states in the phase space.

---

3As discussed earlier, the universe is in fact not perfectly homogeneous or isotropic, and the distribution functions therefore have some spatial and directional dependence. However, these deviations are quite small in magnitude, and any resulting correction to the CMB is below the experimental resolution. Consequently, any inhomogeneities and anisotropies in the distribution functions are negligible for our purposes.
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integral, and is 1 otherwise. The $\Omega(\{f\})$ function is the phase space weighting term

$$\Omega(f_X, f_i, f_j, f_k) = f_j f_k (1 \pm f_X)(1 \pm f_i) - f_X f_i (1 \pm f_j)(1 \pm f_k),$$

(3.6)

where the $\pm$ term is $+$ for bosons (Bose enhancement) and $-$ for fermions (Pauli exclusion). The collision terms therefore couple together the Boltzmann equations for various particle species.

A detailed treatment of the full evolution of species in the early universe can be found in [132]. For our purposes, the most important fact is that during non-equilibrium events, specifically the decoupling or annihilation of a species, the momentum dependence of the collision functional $C[f]$ can alter the phase space density of a decoupling species away from the standard thermal distributions. For these cases, a general phase space density must be numerically evolved in time to find the precise contribution to $g_*$ at lower temperatures. The focus of this work includes both decoupling and annihilation, necessitating our numerical treatment.

So far we have treated the expansion of the universe as an independent process, but it is in fact coupled to the evolution of its particle content through the Einstein field equations. Assuming a flat, isotropic, and homogeneous universe, we obtain the Friedmann equations,

$$H^2 = \frac{8\pi G}{3} \rho,$$

$$\frac{\partial \rho}{\partial t} = -3H(\rho + P),$$

(3.7)

where $\rho$ and $P$ refer to the total energy density and pressure of the full particle content. The Boltzmann equations and Friedmann equations then combine to give a
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coupled set of integro-differential equations governing the full evolution of the early universe.

3.2.3 Decoupling, Recoupling, and the Redistribution of Entropy

While a full solution to the Boltzmann and Friedmann equations is necessary to understand the detailed evolution of any species $X$ and its exact contribution to $g_*$, we can first gain a qualitative understanding by considering the approximation of instantaneous decoupling. Once we have developed this conceptual framework, we will then turn to more precise statements about the complete evolution of distribution functions.

In the instantaneous decoupling approximation, the point of decoupling can be found by comparing the rate of expansion $H$ to the rate of interaction $\Gamma_X$, defined as

$$\Gamma_X = \sum_{j,k \rightarrow X,i} \frac{n_j n_k}{n_X} \langle \sigma v \rangle_{j,k \rightarrow X,i},$$

(3.8)

where $\langle \sigma v \rangle$ is the thermally-averaged cross-section for any interaction $j, k \rightarrow X, i$. This average cross-section can be formally defined as

$$\langle \sigma v \rangle_{j,k \rightarrow X,i} = \int \left( \prod_{s=X,i,j,k} g_s \frac{d^3 p_s}{(2\pi)^3 2E_s} \right) (2\pi)^4 \delta^4(p) S |\mathcal{M}|^2 \frac{f_j f_k}{n_j n_k} (1 \pm f_X)(1 \pm f_i).$$

(3.9)

Note that the symmetry factor $S$ now includes an additional factor of $\frac{1}{2}$ if the initial state consists of identical particles, as well as the original $\frac{1}{2}$ for an identical-particle
final state. The full set of thermally-averaged cross-sections can be related to the collisional term $C[f]$ via

$$
\int g_X \frac{d^3 p_X}{(2\pi)^3 E_X} C[f_X] = \sum_{X,i,j,k} (n_j n_k \langle \sigma v \rangle_{j,k \rightarrow X,i} - n_X n_i \langle \sigma v \rangle_{X,i \rightarrow j,k}).
$$

(3.10)

Conceptually, $\Gamma_X$ corresponds to the rate of production per particle for species $X$. As the universe expands, both $H$ and $\Gamma_X$ will decrease, though generically at different rates. If $\Gamma_X$ decreases more quickly than $H$, then it is possible for a species originally in equilibrium to ‘freeze out’ and decouple from the remainder of the SM.

Conversely, if $H$ decreases more quickly than $\Gamma_X$, a species originally out of equilibrium may actually recouple to the SM. In this case, however, $X$ will generically not have the same temperature as the SM, if it even has a well-defined temperature, prior to recoupling. Instead, the initial distribution will depend on any other particle content that could potentially couple to $X$, making this scenario very model-dependent.

In this framework, the point of instantaneous decoupling/recoupling is defined simply as the temperature at which $\Gamma_X = H$. It is common to assume that species are in full equilibrium prior to decoupling, then evolve freely immediately after freezing out. This approximate description is correct only when all relevant species are relativistic and originally in full equilibrium. However, if $X$ decouples during other nonequilibrium processes, such as nonrelativistic annihilation, the full set of Boltzmann equations must be used.

Once $T$ drops below the mass of any particle, that species begins to annihilate away, with the number density quickly falling to a negligible amount. The entropy
of the annihilating species is redistributed amongst the remaining interacting species, such that the temperature of all remaining species decreases less quickly than would be the case in free expansion. If \( X \) has decoupled from the SM prior to this annihilation, it will not participate in the resulting entropy redistribution, and therefore reaches a temperature lower than that of the SM following the annihilation.

To determine the impact of these entropy redistributions, we need to track the relativistic entropy density \( s \) as a function of temperature. If the entropy density of all SM species in equilibrium (excluding \( X \)) was initially \( s_0 \) when \( X \) instantaneously decoupled from the SM at temperature \( T_0 \), conservation of total entropy gives us the resulting temperature ratio following an entropy redistribution. This ratio can be expressed as a function of the entropy density \( s \) of all species in equilibrium at any future temperature \( T \),

\[
\frac{T_X}{T} = \left( \frac{s/T^3}{s_0/T_0^3} \right)^{1/3}.
\]

(3.11)

In practice, because the entropy of annihilating species is only being distributed amongst relativistic species in full thermal equilibrium, it is much simpler and equivalent to instead use the relativistic degrees of freedom, rather than \( s/T^3 \), to calculate the ratio

\[
\frac{T_X}{T} = \left( \frac{g_*^{\text{after}}}{g_*^{\text{before}}} \right)^{1/3},
\]

(3.12)

where \( g_*^{\text{before}} \) and \( g_*^{\text{after}} \) are the relativistic degrees of freedom of all SM species in equilibrium immediately before and after the entropy redistribution. This decrease

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in relative temperature also decreases the $\Delta g_*$ due to $X$,

$$\Delta g_* = \Delta g_{*0} \left( \frac{g_{*\text{after}}}{g_{*\text{before}}} \right)^{4/3}, \tag{3.13}$$

where $\Delta g_{*0}$ is simply the initial contribution of $X$ to $g_*$ at $T_0$. For multiple entropy redistributions, the overall ratio $\frac{T_X}{T}$ can be found simply by multiplying together the ratios from each individual redistribution, giving the full contribution of $X$ to $g_*$.  

Again, this discussion has made the simplifying assumption of instantaneous decoupling. In general, we cannot simply use comparisons of $\Gamma_X$ to $H$ to determine the exact evolution of the phase space density $f_X(t, E)$ if the species $X$ decouples during nonequilibrium processes. Our treatment must instead be made more precise by numerically solving the Boltzmann equation for $X$, as well as the Friedmann equations, which govern the evolution of the SM temperature $T(t)$ and the expansion scale factor $a(t)$. More details on our numerical treatment can be found in appendix B.

The evolution of a given model of new light species is determined by calculating the collision functional $C[f_X]$ in terms of the model parameters, such as the suppression scale $\Lambda$ of nonrenormalizable operators in an effective theory. This interaction term then governs the process of decoupling $X$ from the SM. Any SM annihilation and entropy redistribution that occurs after this decoupling reduces the change in effective degrees of freedom $\Delta g_*$ at the point of recombination. The contribution to $g_*$ for a specific model can be found by using the resulting $f_X$ near the point of recombination to calculate the energy density $\rho_X$. Solving this contribution in terms of generic couplings establishes a direct relationship between model parameters and $\Delta g_*$.  

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It is important to note that in this work we consider the effective field theory of each model at very low energies (as low as $\sim$ MeV). In order to match to any full UV theory which generates the operators in this effective theory, one should in principle treat operator couplings as Wilson coefficients and run these couplings from the high energy theory down to the scale of interest using the renormalization group. We assume that this running has already been done when we write down our effective operators, such that we are working with the matched coefficient.

3.2.4 Relevant Decoupling Temperatures

For new light species to currently be detectable with the CMB, they must decouple at low enough temperatures such that their contribution $\Delta g_*$ is within the experimental sensitivity of Planck [16]. The full dependence of $\Delta g_*$ on the decoupling temperature for various particle types is shown in figure 3.2. This functional dependence is calculated in the instantaneous decoupling approximation by using eq. (3.13) in combination with $g_*$ of the SM as a function of temperature, which is shown in figure 3.3.

As we see in figure 3.2, for a species to be within the sensitivity of Planck, it must decouple at temperatures $T \lesssim 200$ MeV, which corresponds to the approximate scale of the QCD phase transition (see [133] and references therein for details). Prior to this point, quarks and gluons are the relevant degrees of freedom for the QCD sector, such that the total number of SM degrees of freedom is $g_* = 61.75$. As the universe cools
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Figure 3.2: Additional light degrees of freedom $\Delta g^*$ at recombination for a new light species as a function of the decoupling temperature (in the instantaneous decoupling approximation), calculated using eq. (3.13). The contribution of various particle species is shown, specifically a real scalar boson (magenta), a Weyl fermion (blue), a real gauge boson (orange), and a Dirac fermion pair (green). The dashed line indicates the current sensitivity of the Planck observational experiment [16]. The gray region corresponds to the QCD phase transition, where the precise evolution of $g^*(T)$ for the SM is not well-understood. The provided values of $\Delta g^*$ should therefore only be interpreted qualitatively in that region.

to lower temperatures, the SM transitions to a regime where mesons and baryons are the appropriate degrees of freedom. Specifically, the relevant hadrons present below the QCD phase transition are pions and charged kaons, such that $g^* = 19.25$. This significant reduction in the degrees of freedom results from the rapid annihilation or decay of any more massive hadrons which may have formed during the transition. The QCD phase transition therefore corresponds to a large redistribution of entropy
Figure 3.3: Effective degrees of freedom $g_*$ in the SM as a function of temperature. The gray region corresponds to the QCD phase transition, where the precise evolution of $g_*(T)$ is not well-understood. The provided values of $g_*$ should therefore only be interpreted qualitatively in that region.

into the remaining degrees of freedom, such that any species which decouples from the SM prior to the transition will not contribute significantly to the CMB.

In principle, it is possible to discover species which decouple during the QCD phase transition, as those species could contribute values of $\Delta g_*$ above the experimental sensitivity. However, the precise details of this phase transition are not well-understood because of, e.g., strong coupling effects, and this transition is an area of active study (see [134, 136] and references therein). Consequently, we do not know how to make precise predictions for $\Delta g_*$ for species decoupling in this era. These computations are beyond the scope of our work, so we choose to restrict our focus to species which
decouple after the QCD phase transition.

For new species which do decouple immediately after this point, the calculation of $\Delta g_*$ is sensitive to whether the species couples to leptons or to quarks. Species which couple solely to leptons have a straightforward decoupling process, as all relevant interactions are sufficiently weakly renormalized. Species which couple to quarks will then couple to pions and kaons, whose couplings can be strongly renormalized. We must restrict ourselves to quark and meson couplings which involve conserved currents, as these are then protected against strong renormalization effects. For this set of couplings, we can still make precise predictions for the contribution of new light species which couple to quarks, even at temperatures immediately below the QCD phase transition.

3.2.5 Big Bang Nucleosynthesis

Most models which include additional light degrees of freedom will have other model-dependent constraints, such as those from collider signals or various astrophysical observations. Arguably the most important model-independent bound other than that of the CMB is that placed by Big Bang Nucleosynthesis (BBN). The measurement of the primordial relic abundance of light elements formed by BBN provides an independent probe of new light species, although at times earlier than recombination. While here we only give a brief summary of the relevant aspects of BBN, an excellent introduction to the topic can be found in [122–124].
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The resulting abundances of the light elements, particularly helium-4 ($^4\text{He}$), are sensitive to the number density of neutrons at the start of BBN. When neutrons and protons were in full equilibrium, the number of neutrons relative to that of protons continued to fall due to their mass splitting. The neutron abundance is then determined by the point at which the weak interactions, which interconvert protons and neutrons, freeze out. A larger expansion rate results in earlier freezeout, which in turn leads to a larger number of neutrons and therefore more $^4\text{He}$.

The precise value of $H$ at the time of BBN, which would be increased by the presence of additional light species, therefore determines the relic abundance of $^4\text{He}$. This abundance is often expressed in terms of the so-called ‘helium mass fraction’

$$Y_P \equiv \frac{4n_{\text{He}}}{n_H + 4n_{\text{He}}}. \quad (3.14)$$

Observational determinations of $Y_P$ therefore provide another means of constraining the relativistic energy density of the early universe, though it is important to remember that these constraints apply at a different period of time than those placed by direct CMB measurements of $g_*$. The SM prediction for the primordial helium abundance is $Y_P = 0.2487 \pm 0.0006$ [122], and this prediction can be tested by both extracting the primordial abundance from direct observations of the modern helium abundance and observing the effects of $Y_P$ on CMB anisotropies.

Multiple primordial helium extractions have yielded results near $Y_P = 0.240 \pm 0.006$ [122], which are consistent with SM predictions, but two recent observational studies have indicated a higher abundance of $Y_P = 0.2565 \pm 0.0010$ (stat) $\pm 0.0050$.
(syst) \[137\] and \(Y_P = 0.2561 \pm 0.0108 \,[138]\), which are consistent with a larger rate of expansion. This in turn allows for the presence of new light species. In addition, combined CMB constraints from SPT and WMAP are consistent with \(Y_P = 0.296 \pm 0.030 \,[18]\), and combined results from Planck and WMAP are consistent with \(Y_P = 0.266 \pm 0.021 \,[16]\). These results are therefore currently incapable of either completely confirming or excluding the existence of new light species, but instead increase the importance of the precision CMB measurements of \(g_\ast\) possible with future experiments.

Lastly, it is important to note that there is tension between the SM prediction and observational determinations of the abundance of lithium-7 (\(^7\)Li), with a lower observationally inferred primordial \(^7\)Li abundance than that predicted by BBN. Unfortunately, this discrepancy is not immediately remedied simply by the presence of new light species, and the detailed model-building necessary to address this tension is beyond the scope of this work. However, the \(^7\)Li problem does present another exciting opportunity for the possible discovery of new physics \([139,143]\).

### 3.2.6 Summary

We have now introduced the framework necessary for the remainder of this chapter. The focus of this work is the effects of light species in BSM theories on the CMB, which we determine by computing the energy density of the new light species at recombination. We specifically concern ourselves with species which were in thermal
equilibrium with the SM and then decouple after the QCD phase transition, potentially during the annihilation of a SM species. Any species which decouples from the SM before the QCD phase transition cannot be probed by the Planck satellite, as its energy density is much smaller than that of the SM. The energy density of light species is calculated by numerically solving the coupled Boltzmann and Friedmann equations, found in eqs. (3.4) and (3.7), in order to compute the potentially nonthermal distribution function of the new species. The distribution function immediately following decoupling can then be used to calculate the energy density at recombination, which determines $g_*$ using eq. (3.1).

3.3 Models

In this section, we consider the set of models which can contribute to the CMB measurement of $g_*$, mainly restricting ourselves to models where the additional degrees of freedom were in thermal equilibrium immediately following the QCD phase transition.\footnote{We only consider models with light degrees of freedom. It is possible to construct models where heavier species mimic the effects of light degrees of freedom through a nonzero pressure resulting from non-equilibrium distribution functions.} Such models must either contain new species with mass $\lesssim$ eV or alter the neutrino energy density. While there are a very large number of possible models one could write down, we choose to restrict ourselves to those which are both minimal and natural.

\footnote{There are models where out-of-equilibrium effects such as decays generate a contribution to $g_*$, but no generic model-independent statements can be made about such scenarios, so we do not consider them in this work.}
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We consider a model to be minimal if it contains the smallest possible hidden sector in the low-energy theory. In particular, this restricts our discussion to models of elementary particles, ignoring the possibility of light composite states. We then direct our attention to the low-energy effective field theory (EFT) and ignore any additional particle content which may arise at higher energies, as these are irrelevant for our calculations.

For this work, we define naturalness as technical naturalness. We therefore require that the size of quantum corrections not exceed the size of the physical observables in the theory, i.e. $|\delta \lambda / \lambda| < 1$ for all parameters $\lambda$, as large corrections require an artificial fine-tuning of parameters.

A large number of potential models of light species are unnatural, due to large corrections to the mass of that new species. There are two predominant methods of suppressing quantum corrections to a particle’s mass. The first method is the introduction of an additional symmetry which prohibits the existence of a mass term for that species. The second option is the use of strong dynamics in a hidden sector to generate large anomalous dimensions for mass terms, such that those terms become irrelevant operators, giving rise to a vanishing mass in the low-energy EFT. However, most models of the latter type tend to contain a relatively rich spectrum, violating our minimality principle. Although this is an interesting direction for future research, it is outside the class of models we consider. We therefore focus solely on theories of light species which contain a protective symmetry.
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The classes of possible new light species can be divided up by spin, as this restricts
the protective symmetries available. We progress through each possible case, from
spin-0 to spin-2, considering all minimal, natural models. For each model, we then
scan over all allowed couplings, numerically solving the Boltzmann and Friedmann
equations to calculate the full process of decoupling for any species which decouples
during SM entropy redistributions. The details of our numerical approach can be
found in appendix B. Using the resulting distribution function after the decoupling of
our new light species, we then calculate the energy density at recombination, which
is reported as the contribution to \( g_* \) as a function of the coupling parameters of
the theory. This calculation of \( \Delta g_* \) is specifically done in the massless limit, and is
accurate for new species with \( m \ll \text{eV} \). In subsection 3.3.6, we briefly discuss the
potential effects of non-negligible masses.

As discussed in subsection 3.2.4, each new light species must also decouple after
the QCD phase transition in order to be constrained by Planck, which limits the
dimensionality of the operators we choose to consider. If our new species couples
to the SM with an operator of scaling dimension \( d \), the operator is suppressed by
\( \Lambda^{4-d} \), where \( \Lambda \) is the approximate cutoff scale of the EFT. Dimensional analysis
then indicates that, given independent experimental constraints, only operators of
dimension \( d \lesssim 6 \) will be able to maintain equilibrium between a new species and the
SM until after the QCD phase transition.

Finally, we discuss possible extensions to the SM which do not contain new light
species, but instead alter the neutrino distribution, through such means as decay or neutrino asymmetry. These models then enhance the neutrino energy density relative to SM predictions, leading to an increase in $g_*$.  

### 3.3.1 Spin-0: Goldstone Boson

The first possibility for new light species is a spinless scalar boson. However, the mass of any new scalar particle is generically sensitive to quantum contributions resulting from interactions. While supersymmetry could potentially preserve the naturalness of scalar masses, the observed particle spectrum indicates that any couplings between the SM and new light scalars would mediate supersymmetry-breaking mass terms significant enough to require fine-tuning. The only viable symmetry which can protect the mass term of such light scalar bosons is then a shift symmetry, $\phi \rightarrow \phi + \epsilon$. This is precisely the symmetry present in the Goldstone modes of a spontaneously broken global symmetry. In the limit of an exact global symmetry, the mass of the corresponding Goldstone boson is restricted to be zero, with any quantum corrections forbidden by the symmetry. Even if the symmetry is inexact, the mass of the pseudo-Goldstone is proportional to the symmetry-breaking terms in the original Lagrangian, rather than the cutoff of the effective theory. We therefore restrict ourselves to the study of Goldstone bosons, as other theories of light scalars are generically tuned and unnatural. These particles arise in many theories, such as the QCD axion and the so-called ‘String Axiverse’ of string compactifications \footnote{155}.  

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Since we only discuss thermodynamics after the QCD phase transition, the only allowed interactions are those with leptons, mesons, baryons, and the photon. In this low energy effective theory, any combination of such couplings may conceivably be allowed. We explore all of these possibilities, finding that current collider and astrophysical bounds are such that almost all scenarios are excluded, and minimal models of Goldstone bosons must have a negligible impact on $g_\ast$. There are small corners in (flavor-dependent) parameter space which are still viable and in which they could in principle have a small but non-negligible impact on the effective number of relativistic degrees of freedom. We conclude that unless we are very lucky, the addition of a natural massless or near massless scalar will have, at best, a tiny impact on the CMB and thus would require significant advances in our ability to measure $g_\ast$.

First, we consider couplings to leptons. Due to the shift symmetry, any coupling between an exact Goldstone boson and SM fermions must only contain derivatives of the field $\phi$. We parameterize our effective field theory as

$$\mathcal{L} \supset 1/2(\partial_\mu \phi)^2 + \frac{\partial_\mu \phi}{2\Lambda} \bar{\psi}_L \gamma^\mu \gamma^5 \psi_L + \frac{\partial_\mu \phi}{2\Lambda} \bar{\psi}_R^{c\dagger} \gamma^\mu \gamma^5 \psi_R^{c\dagger} + h.c.$$  (3.15)

Using identities found in [156], we can also write our Lagrangian in Dirac notation, resulting in

$$\mathcal{L} \supset 1/2(\partial_\mu \phi)^2 - \frac{\partial_\mu \phi}{\Lambda} \bar{\Psi} \gamma^\mu \gamma^5 \Psi + h.c.$$  (3.16)

In this form, it is simple to see that the interaction is specifically a derivative coupling between $\phi$ and the axial current of $\Psi$. One might suspect that some theories could
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potentially generate a similar coupling between $\phi$ and the vector current for $\Psi$. However, any interaction of that form must vanish due to vector current conservation. The conservation of the axial current is broken by the mass term for $\Psi$, meaning that the axial coupling does not similarly vanish. However, this does imply that any interaction rate involving the axial coupling is necessarily proportional to the fermion mass $m$, and thus vanishes in the $m \to 0$ limit.

In simple UV completions of this effective field theory, the couplings of $\phi$ to the SM are flavor-blind. More sophisticated UV model-building could potentially result in flavor-specific couplings. However, a flavor-specific basis generically leads to interactions which mix generations. There are greatly restrictive constraints coming from flavor physics, as we shall discuss briefly below.

Due to the $\Lambda$ suppression of the derivative couplings, the interaction rate between $\phi$ and leptons will be dominated by processes which only involve one Goldstone interaction term, shown in figure 3.4. Note that, as this dominant process involves the emission/absorption of a photon, the interaction rate $\Gamma_\phi$ has no dependence on the coupling between $\phi$ and neutrinos. Because of this, the only relevant lepton interactions for $\phi$ are those with electrons and muons.

In the relativistic limit, dimensional analysis would expect the interaction rate between $\phi$ and a SM lepton $\Psi$ to scale as $\Gamma_\phi \sim T^3 \Lambda^2$. However, the broken axial symmetry for $\Psi$ restricts the interaction rate to take the form $\Gamma_\phi \sim \frac{m^2 T}{\Lambda^2}$. The expansion rate will therefore drop more quickly than the interaction rate as the universe expands. This
implies that $\phi$ could have been out of thermal equilibrium after the time of global
symmetry breaking, and come back into thermal equilibrium with the SM leptons at
some point before the leptons annihilated.

As the universe cools to temperatures comparable to the relevant lepton mass,
this simplified form for the interaction rate will be substantially modified and needs
to be computed numerically. The interaction rate $\Gamma_{\phi}$ will begin to drop rapidly as the
leptons annihilate away, redistributing their entropy amongst the remaining coupled
species. If $\Lambda$ is very large and substantially suppresses $\Gamma_{\phi}$, $\phi$ will not have recoupled by
the time the leptons annihilate, meaning that $\phi$ will forever remain out of equilibrium.
Additional couplings beyond the lepton-only couplings we consider in this paragraph
would be needed in order to have SM-$\phi$ interactions. For each lepton, there is then
some maximum $\Lambda$ for recoupling. Any Goldstone boson with a larger $\Lambda$ will not be
reheated by the entropy redistribution and therefore cannot substantially contribute
to $g_*$ at recombination.

The distribution function for $\phi$ prior to recoupling is dependent on the original
process of decoupling at high energies, which is then sensitive to details of the UV theory, including the relative timings of global symmetry breaking and inflation. It is then impossible to make fully model-independent predictions for the contribution of $\phi$ to $g_*$ in the case where $\phi$ only couples to leptons. However, in a large class of models, $\phi$ will also couple to quarks and photons, which results in qualitatively different evolution.

For the case where $\phi$ has similar couplings to quarks, we must examine the resulting interactions between $\phi$ and mesons, specifically pions and charged kaons, since we consider temperatures below the QCD phase transition. At these temperatures, however, the number density of kaons will be much lower than that of pions, such that any $\phi$-kaon interactions will be subdominant. We can then simply focus on those couplings which involve pions. Following [157], the original coupling of $\phi$ to quarks can be rewritten in terms of the axial quark current. After the phase transition to mesons, interactions with the quark current are replaced by those with the axial pion current, which is safe from QCD renormalization effects. The full Lagrangian can be expanded to leading order in $\Lambda$ and $f_\pi$ and subsequently studied, and depends on the details of the flavor structure in the UV. We assume flavor-blind couplings, as flavor-specific couplings in the UV do not alter our predictions for the thermodynamic properties of $\phi$, but such couplings must obey additional constraints coming from flavor physics. With this assumption, we find that those terms which dominate
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the interaction rate between $\phi$ and pions are

$$\mathcal{L} \supset -\frac{2r_m}{3f_\pi\Lambda} \pi^+ \pi^- \partial_\mu \phi \partial^\mu \pi^0 + \frac{r_m}{3f_\pi\Lambda} \pi^0 \partial_\mu \phi \partial^\mu \pi^- + \frac{r_m}{3f_\pi\Lambda} \partial_\mu \phi \partial^\mu \pi^+. \quad (3.17)$$

We have defined the ratio $r_m = \frac{m_d - m_u}{m_d + m_u}$, where $m_u$ and $m_d$ are the up and down current-quark masses. We use the approximate value $r_m = 1/3$, based on lattice QCD calculations [102], as well as the convention $f_\pi = 93$ MeV. Interactions of this type will potentially keep the Goldstone boson $\phi$ in thermal equilibrium until the pions fully annihilate and redistribute their equilibrium, depending on the suppression scale $\Lambda$.

The corresponding interaction rates decrease more rapidly than the expansion rate, leading to the freezing out of the $\phi$-$\pi$ interactions.

Finally, there can be couplings of $\phi$ to photons via operators of the form

$$\mathcal{L} \supset -\frac{e^2}{32\pi^2\Lambda_\gamma} \phi F^{\mu\nu} \tilde{F}_{\mu\nu}. \quad (3.18)$$

This operator arises because the axial symmetry in question can be anomalous. This $\Lambda_\gamma$ is not necessarily precisely the same as the $\Lambda$ which couples $\phi$ to SM fermions, though their orders of magnitude are similar in a large number of UV completions. This is because the operator can be induced by loops of SM fermions. The additional loop factor in the parameterization of $\Lambda_\gamma$ is present because in these cases, the operator appears in the Lagrangian suppressed by a loop factor relative to the fermion couplings. As mentioned earlier, depending on the UV structure of the model, this operator may or may not be present in the low-energy theory. Similar to pion couplings, this operator gives rise to a rate such that $\phi$-$\gamma$ interactions freeze out as we
go to lower temperatures.

We now outline the constraints on these scenarios, working in a general framework with no assumptions regarding the operator or flavor structure of couplings in the UV. The bounds are best stated in terms of the effective operators

$$\mathcal{L} \supset -\frac{\partial \mu \phi}{\Lambda_f} \bar{\Psi}_f \gamma^\mu \gamma^5 \Psi_f - \frac{e^2}{32\pi^2 \Lambda_\gamma} \phi F^{\mu\nu} \tilde{F}_{\mu\nu},$$

(3.19)

where $\Psi_f$ can either be a charged lepton or the proton. The strongest bounds for these models come primarily from observations of stellar and supernova cooling, which will also greatly constrain other models within this work. The production of new light species which interact weakly enough to escape the interior of a star provides an efficient energy loss mechanism, affecting both stellar cooling and evolution. Comparison of SM predictions to astrophysical observations then provides a strong constraint on the interactions of such new species. The resulting constraints for Goldstone interactions are

$$\Lambda_e \gtrsim 2.9 \times 10^9 \text{ GeV},$$

$$\Lambda_p \gtrsim 3.5 \times 10^9 \text{ GeV},$$

$$\Lambda_\gamma \gtrsim 1.2 \times 10^7 \text{ GeV}.$$  

(3.20)

More details about these bounds can be found in [158–161]. The relationship between the effective proton scale $\Lambda_p$ and the UV quark coupling scale $\Lambda \equiv \Lambda_q$ present in eq. (3.17) depends on phenomenological parameters in the baryon chiral Lagrangian, as well as details of the UV theory. However, $\Lambda_p$ and $\Lambda_q$ are related by an $\mathcal{O}(1)$ number. Consequently, we use the conservative bound $\Lambda_q \gtrsim 5 \times 10^8 \text{ GeV}$. 

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In addition, there are constraints on the set of off-diagonal operators schematically of the form \( \frac{1}{\Lambda_{\mu e}} \partial_\mu \phi \mu e \partial_\mu \phi \gamma^\mu \gamma^5 \bar{e} \) coming from \( \mu \rightarrow e + E \) \[102\]. These bounds restrict \( \Lambda_{\mu e} \gtrapprox 1.6 \times 10^9 \) GeV. \( (3.21) \)

These off-diagonal operators’ contributions to early universe thermodynamics are not significantly different from that of muon couplings during the era following the QCD phase transition. Consequently, we do not consider this case to be qualitatively distinct from the case with muon couplings, but considerably more constrained, and so we do not consider these operators further.

Finally, there are direct constraints on \( \Lambda_\mu \) also coming from observations of supernovae. We take the average temperature within the core of a supernova to be \( T \approx 30 \) MeV \[159\], which allows for the presence of a non-negligible muon abundance. We can therefore apply the same cooling bounds to muon couplings, with a small suppression due to the lower muon number density. Based on \[159\], we calculate the approximate bound

\[ \Lambda_\mu \gtrapprox 2.0 \times 10^6 \text{ GeV}. \] \( (3.22) \)

In order to consider the general list of all possible models, we present our results for each interaction separately. For a large number of models, multiple such interactions will be present, such that these results will be even more restrictive.

**Electrons/Photons:** Electron interaction rates are suppressed by \( \frac{m_e}{T} \), and photon interaction rates are suppressed by the loop factor \( \frac{e^2}{32\pi^2} \), such that these two
Figure 3.5: $\Delta g_*$ due to a single Goldstone boson which interacts with only pions. The contribution to $g_*$ at recombination is given as a function of the effective scale $\Lambda$, which suppresses this interaction. The gray region for $\Lambda \gtrsim 5 \times 10^6$ GeV corresponds to models which decouple during the QCD phase transition. The provided values of $\Delta g_*$ should therefore only be interpreted qualitatively in that region. Supernova and star cooling constraints on this scenario limit $\Lambda \gtrsim 10^9$ GeV, and so this plot demonstrates that the Goldstone must have decoupled during or before the QCD phase transition.

Heavily-constrained interactions do not play a role in the thermal evolution of Goldstone bosons.

**Pions:** For Goldstone bosons to be in thermal equilibrium with pions and receive any of the pion entropy redistribution, the coupling suppression scale must be $\Lambda \lesssim 5 \times 10^6$ GeV. This is illustrated in figure 3.5. The maximum possible $\Lambda$ necessary is far below the bound on $\Lambda$ quoted above, and therefore the decoupling of the Goldstone must have happened during or before the QCD phase transition, making...
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Figure 3.6: \( \Delta g^* \) due to a single Goldstone boson which interacts with at least pions and muons. The contribution to \( g^* \) at recombination is given as a function of the effective scale \( \Lambda \), which suppresses this interaction. The blue-gray region for \( \Lambda \gtrsim 1.5 \times 10^7 \) GeV corresponds to models which decouple during the QCD phase transition. The provided values of \( \Delta g^* \) should therefore only be interpreted qualitatively in that region. Supernova and star cooling constraints on this scenario limit \( \Lambda \gtrsim 10^9 \) GeV, and so this plot demonstrates that the Goldstone must have decoupled during or before the QCD phase transition.

The Goldstone not a viable candidate for a contribution to \( g^* \) in theories containing only pion interactions.

**Muons:** In the case of muon-only couplings, it is not possible to give well-defined initial conditions for the Goldstone boson distribution function just prior to the recoupling of the Goldstone boson to muons. For all reasonable initial configurations of the Goldstone distribution function, the maximum contribution possible would result from thermalization of the Goldstone bosons with muons, leading to a contribution of
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$\Delta g_* = 0.26$, or $\Delta N_{\text{eff}} = 0.57$. We plan to pursue more precise predictions in future work.

However, if these couplings are present in conjunction with couplings to pions, then it is possible to study the decoupling of Goldstones from the SM, as the Goldstones had been in thermal equilibrium in the era leading up to muon annihilation. In order for a Goldstone to have received any entropy at all from SM annihilations following the QCD phase transition, it must have coupled with $\Lambda < 1.5 \times 10^7$ GeV. This is illustrated in figure 3.6. While such couplings are allowed for muon interactions, this range is below the pion bounds quoted above, and therefore the Goldstone is not a viable candidate for a contribution to $g_*$ in this scenario.

To summarize, there are no parts of the minimal, natural parameter space where the Goldstones had been in thermal equilibrium with the SM through the QCD phase transition which do not directly conflict with bounds coming from star and supernova cooling. As such, the effects of Goldstone bosons on the CMB in the predictive part of the parameter space are well below the sensitivity of the Planck satellite. One can, however, have couplings to only the muon with $\Lambda$ in the narrow window between $2.0 \times 10^6$ GeV and $1.5 \times 10^7$ GeV, and still obtain a nontrivial contribution to $\Delta g_*$, though it is not possible to give well-defined, model-independent initial conditions for the Goldstone distribution function in this scenario. Therefore, the only viable set of theories must contain a highly specific hierarchy of couplings, such that interactions with muons are much stronger than those with other SM fields present after the QCD
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phase transition, without the generation of significant off-diagonal couplings.

3.3.2 Spin-$\frac{1}{2}$: Light Fermion

Natural models of light spin-$\frac{1}{2}$ fermions are made more easily than those containing light scalar bosons. This naturalness can arise due to chiral symmetry, which corresponds to a rotation of the field by an arbitrary phase, $\chi \rightarrow e^{i\alpha} \chi$. This symmetry permits any fermion gauge and kinetic terms, but forbids Majorana mass terms. Even if chiral symmetry is explicitly broken by the presence of a small fermion mass, corrections to this mass parameter are in general proportional to the original value, eliminating the need for any fine-tuning. Similarly, Dirac mass terms can be protected by an axial symmetry.

Because of this protective symmetry, there are many allowed interactions for light fermions. The possible models include interactions with SM gauge bosons, either through direct gauge couplings or dipole moments, as well as interactions with SM fermions through effectively pointlike operators, which result from the exchange of heavy intermediary particles.

Gauge Interactions

One possibility is that a new light fermion $\chi$ is charged under the SM gauge groups. The coupling strength of a fermion in any representation of $SU(3)_C$ or $SU(2)_L$ is completely fixed by the representation theory of these groups. While $\chi$ could
naïvely have any value of hypercharge, the prospect of gauge unification indicates that hypercharge values are also discrete. Any new fermion in non-trivial representations of the SM gauge groups will therefore couple with the same strength as the SM fermions. Light species which possess electromagnetic or color charge are completely excluded. The only remaining option is a neutral fermion, which must couple to the \( Z \), but these light fermions are excluded by measurements of the \( Z \)-width \[102\]. As such, light fermions in any non-trivial representation of the SM gauge groups are excluded as potential candidates for contributions to \( g_* \).

However, if \( \chi \) instead coupled to some new gauge boson, kinetic mixing between this new field and the SM gauge bosons would lead to mixing-suppressed SM gauge couplings for \( \chi \). Any such ‘millicharged’ light fermion therefore requires the existence of a new gauge boson, which would also contribute to \( g_* \). We consider the details of new gauge bosons and the resulting millicharged interactions in subsection 3.3.3.

**Dipole and Anapole Moments**

While a new fermion cannot carry SM charges, \( \chi \) could still interact via dimension-5 dipole moment operators. The only nontrivial dipole interactions between \( \chi \) and SM gauge bosons are those with the hypercharge gauge boson, which are of the form

\[
\mathcal{L} \supset -\frac{1}{\Lambda} B_{\mu\nu} \chi_L \sigma^{\mu\nu} \chi_R^c + h.c., \tag{3.23}
\]

where the structure of these operators is such that we must introduce two new Weyl fermions, \( \chi_L \) and \( \chi_R^c \). These interactions can arise from loops involving heavy charged
intermediaries, whose mass and couplings set the dipole moment scale $\Lambda$.

However, the charged intermediary loops that generate this operator necessarily preserve only the vector $U(1)$ global symmetry of $\chi$, which is precisely the symmetry structure allowed by a Dirac mass term $m_\chi L \chi_R^c$. Therefore, any UV completion reducing to the theory containing the Lagrangian terms of eq. (3.23) must also allow for a Dirac mass term. It is not apparent how to create a UV completion of this model which induces only a dipole term corresponding to large mass scales, while generating the Dirac mass $\lesssim$ eV in a natural fashion. Experimental constraints from star cooling observations [162] currently limit $\chi$ dipole moments to

$$\Lambda \gtrsim 10^9 \text{ GeV.} \quad (3.24)$$

Due to the resulting large separation of scales in this highly constrained EFT, we do not consider a theory with new light species possessing a SM dipole moment to be a viable, natural candidate for a contribution to $\Delta g_*$. 

A similar interaction term corresponds to the anapole moment and charge radius operators, which are of the form

$$\mathcal{L} \supset -\frac{1}{\Lambda^2} \chi^\dagger \tilde{\sigma}^\mu \chi \partial^\nu B_{\mu\nu}. \quad (3.25)$$

Such interactions are dimension-6 and only require the existence of a single new Weyl fermion $\chi$. New species with such interactions were discussed in the context of dark matter in [163]. Unlike dipole moments, such anapole moment interactions do not break chiral symmetry and are therefore compatible with new light or massless
species, not just nonrelativistic dark matter.

Assuming vanishing boundary terms, the anapole interaction can be rewritten as

\[ \mathcal{L} \supset \frac{1}{\Lambda^2} \partial_\mu (\chi^\dagger \tilde{\sigma}^\mu \chi) \partial_\nu B^\nu + \frac{1}{\Lambda^2} \chi^\dagger \tilde{\sigma}^\mu \chi \partial^2 B_\mu, \]  

which then results in couplings between \( \chi \) and both the photon and \( Z \). Similar to the case of Goldstone bosons, processes involving the first interaction term will be proportional to \( m_\chi \), as this interaction involves the divergence of a current which is conserved in the limit \( m_\chi \to 0 \). As \( m_\chi \ll T \) for all cases we consider, such processes are greatly suppressed and this particular interaction is irrelevant to our discussion.

The second interaction term is not similarly suppressed but instead has the form of a gauge coupling with additional momentum dependence. The dominant process involving this interaction is the exchange of a photon between \( \chi \) and SM fermions. In such processes, the extra powers of momentum in this operator will cancel with those of the photon propagator, resulting in an amplitude of the same form as four-fermion interactions between \( \chi \) and the SM. While the full models generating four-fermion interactions are very different from those which generate anapole moments, the phenomenology and the resulting bounds on the suppression scale \( \Lambda \) will be very similar for both models. The results for four-fermion interactions, which are discussed in the following subsection, can therefore easily be applied to models involving anapole moment interactions.
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Four-Fermion Interactions

Another possibility for EFT interactions of light fermions is the dimension-6 couplings of a single Weyl fermion $\chi$ or a Dirac pair of fermions $\mathbf{X}$ to SM fermions. Such couplings can arise due to the exchange of a massive scalar or vector boson. Spontaneously broken gauge symmetries, which generate such massive interactions, are present in a large class of theories. Two well-motivated examples are the addition of light sterile neutrinos which couple to the SM via a new massive gauge boson $Z'$, corresponding to a spontaneously broken $U(1)$ \[164\ 167\], and theories where the axino, the supersymmetric partner of the axion, remains light and interacts with the SM via other heavy superpartners.

As we will see below, light Weyl fermions with dimension-6 couplings are strong candidates for significant contributions to $g_\ast$. Interactions suppressed by scales $\Lambda \sim 2$ TeV will keep new species in equilibrium until after the QCD phase transition, leaving such species with a detectable energy density at recombination. The strongest independent bounds on such models are placed by collider experiments, which will continue to probe the relevant parameter space. These theories will then potentially be discovered or fully excluded with the LHC.

In Dirac notation, the possible four-fermion operators present after electroweak
symmetry breaking (EWSB) take four forms,

$$\frac{1}{\Lambda^2} \bar{XX}\bar{\Psi}\Psi$$ (Scalar),
$$\frac{1}{\Lambda^2} \bar{XX}\bar{\Psi}\gamma^5\Psi$$ (Pseudoscalar),
$$\frac{1}{\Lambda^2} \bar{XX}\bar{\Psi}\gamma^\mu\gamma^\nu\Psi$$ (Vector),
$$\frac{1}{\Lambda^2} \bar{XX}\bar{\Psi}\gamma^\mu\gamma^5\Psi$$ (Axial),

where $\Psi$ corresponds to any SM fermion and the suppression scale $\Lambda$ arises from the mass and couplings of the exchanged intermediary. We instead discuss the couplings of a Weyl fermion $\chi$ below, as our results can simply be scaled by a factor of 2 to account for the two fermions in the Dirac case. As all four operators are dimension-6, the interaction rate will drop more quickly than $H$, leading to the decoupling of $\chi$ from the SM as the universe cools.

Couplings of this new fermion to quarks can induce couplings to pions in the low-energy theory. However, any interaction arising from the scalar or pseudoscalar operators will not be protected against strong renormalization effects, such that we cannot make precise theoretical predictions for $\Delta g_*$. If such a species is independently discovered, potentially in collider experiments, and these interactions are precisely determined, a detailed calculation could then be performed. In addition, the vector or axial interactions are such that mesons will have no charge under such couplings, with no induced pion couplings in the EFT.

\[\text{\textsuperscript{6}Vector or axial interactions between pions and $\chi$ would result from models with couplings which are not flavor-blind. Such couplings can only arise from the spontaneous breaking of nonabelian gauge groups which do not commute with flavor symmetry. Such models require a significantly larger particle content, thus violating our minimality requirement.}\]
Figure 3.7: $\Delta g_*$ due to a single Weyl fermion which interacts with the SM via the exchange of a massive vector boson. The contribution to $g_*$ at recombination is given as a function of the effective scale $\Lambda$, which suppresses this interaction. Constraints on this operator are given for interactions with electrons (purple) and quarks (blue), which come from the LEP and LHC collision experiments. The green band indicates couplings which are 95% excluded by a Planck result of $g_* = 3.50 \pm 0.12$. The gray region for $\Lambda \gtrsim 5$ TeV corresponds to models which decouple during the QCD phase transition. The provided values of $\Delta g_*$ should therefore only be interpreted qualitatively in that region. The results for scalar, pseudoscalar and axial couplings are effectively the same. The results for a Dirac fermion are double those given in this figure, indicating that they must have decoupled during or before the QCD phase transition to be compatible with the Planck data.

For each of these interactions, therefore, we can scan over possible effective suppression scales. The resulting contribution to $g_*$ as a function of $\Lambda$ is given in figure 3.7 for only the vector coupling, as the results for all four models are equivalent to within 5%. Therefore, any distinction between these models is below the experimental resolution of Planck. We also assume identical couplings to electrons and muons. For the possible case of flavor-specific couplings, the resulting $\Delta g_*$ will be the same as
the flavor-blind case, where the equivalent flavor-blind $\Lambda$ is the smallest flavor-specific $\Lambda$.

The strongest experimental constraints on such couplings are indicated in figure 3.7 for both electron and quark interactions. These bounds come primarily from $E_T + $ monojet/monophoton searches at LEP and the LHC, again assuming universal coupling to quarks. The LHC bounds specifically came from 10 fb$^{-1}$ of data, so we expect these experimental results to improve in the near future. Details of these exclusion limits can be found in [71,168,169]. Couplings to muons are largely unconstrained in a flavor-specific model, but in the universal coupling case, constraints on any species would therefore limit the muon interactions.

As we see, theories with effective suppression scales $\Lambda \gtrsim 5$ TeV decouple prior to the muon entropy redistribution and are therefore predominantly affected by the QCD phase transition. As such, our results beyond those scales can only place an approximate upper bound on the possible contribution $\Delta g_\ast$. However, there is a range of potential suppression scales below 5 TeV but above the current experimental bound which is compatible with the constraints coming from a Planck measurement of $g_\ast = 3.50 \pm 0.12$. A model with a light Weyl fermion with dimension-6 interactions with SM fermions is therefore a viable model for substantial contributions to $g_\ast$. Our results indicate that a Dirac fermion contributes double what a Weyl fermion does at the same $\Lambda$, and that scalar, pseudoscalar, vector and axial vector operators give the same results to within 5%. Consequently, Dirac fermions must have decoupled
before or during the QCD phase transition in order to be compatible with the data from Planck. Future results from the LHC will continue to probe these interaction scales, providing an independent means of discovery or exclusion of such models.

### 3.3.3 Spin-1: Gauge Boson

A massless spin-1 particle has fewer degrees of freedom than a massive one, and thus perturbative quantum effects cannot generate a mass, rendering a massless spin-1 particle technically natural. These gauge bosons are then automatic candidates for new light species. While gauge bosons can potentially acquire a mass through the Higgs mechanism, masses at scales $\lesssim eV$ are generically unnatural, unless there is a more complicated particle content\footnote{It is, of course, possible to Higgs the group at the TeV-scale, but have such a small gauge coupling that its mass is sub-eV ($g \lesssim 10^{-12}$). However, such a small gauge coupling implies that it will only recouple at very low temperatures, and even then, only to neutrinos. As neutrinos would have already decoupled from the SM, such interactions can only redistribute the neutrino energy density and cannot increase the total energy density. Thus there are no contributions of such a model to $g_*$.}. However, such non-minimal solutions are beyond the scope of this work, so we assume that any additional vector bosons are precisely massless. Similar to the case of a Goldstone boson, the corresponding gauge structure automatically restricts the available interactions for light spin-1 particles. The only possible operators are direct gauge couplings or dipole moment interactions with SM fermions, as well as kinetic mixing with SM gauge bosons.
As we will show, new massless gauge bosons with renormalizable couplings to SM fermions are viable candidates for contributions to $g_*$. Long-range force constraints greatly restrict the possible direct couplings of SM fermions charged under new gauge groups, such that these interactions must be too weak to contribute to $g_*$. However, such couplings can still arise due to kinetic mixing between the new and SM gauge fields. For such mixing to give rise to non-negligible $\Delta g_*$, there must also be new fermions charged under the new gauge group. The additional fermions obtain millicharged couplings to SM gauge fields, with astrophysical constraints such that these fermions must have masses $\gtrsim$ MeV. Such models are sensitive to the details of the full UV theory, as the hidden sector must come into equilibrium with the SM after originally being completely decoupled. The class of viable models is then constrained to a particular region of model-dependent parameter space.

For minimality, we consider the addition of a single new $U(1)$ gauge boson $A'$, with associated field strength $A'^{\mu\nu}$. The new field $A'$ can kinetically mix with the hypercharge gauge boson $B$ with the following operator

$$\mathcal{L} \supset -\frac{\epsilon}{2} A'^{\mu\nu} B_{\mu\nu},$$

(3.28)

where $\epsilon$ is simply a dimensionless mixing parameter. Such hidden sector $U(1)$ gauge bosons which mix with hypercharge arise naturally in many models [68,171,178].

This term indicates that our originally defined fields $A'$ and $B$ are not propagation
eigenstates, and must be redefined to diagonalize the propagation basis. If both gauge bosons are precisely massless, then there is always a linear combination of gauge fields which does not couple to the SM. We can always define this linear combination as $A'$ and the orthogonal combination as $B$, such that $A'$ does not couple to the SM.

However, if $A'$ originally interacts with some new fermion $\chi$, any field redefinition will generically result in couplings between $\chi$ and the SM gauge bosons. The new fermion can then act as an intermediary between $A'$ and the SM, keeping all species in equilibrium. The most minimal theory involving new direct gauge couplings must contain both a new gauge boson $A'$ and a new Dirac fermion $\chi$, with the resulting interaction terms

$$\mathcal{L} \supset -\epsilon g_A \cos \theta_W \bar{\chi} A \chi - \epsilon g_A \sin \theta_W \bar{\chi} Z \chi - g_A \bar{\chi} A' \chi. \quad (3.29)$$

We see that after the field redefinition $\chi$ interacts with both the photon and $Z$, with interaction strength that depends on the coupling $g_A$ of $\chi$ to $A'$, the original mixing $\epsilon$ between $A'$ and $B$, and the weak mixing angle $\theta_W$.

This particular choice of basis is technically arbitrary. It is also possible to redefine gauge fields such that the SM fermions possess millicharged couplings to $A'$ and $\chi$ possesses no couplings to the photon. The physics must be and is independent of the choice of basis. These rotations do not affect any physical observable, provided the observable is phrased in a basis-independent manner. Thermodynamic observables such as the overall energy density of massless gauge bosons, and therefore their contribution to $g_*$, are also basis-independent. We specifically choose to work in the
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basis of eq. (3.29), where $A'$ does not interact with the SM, because the resulting early universe thermodynamics are more transparent. However, it is important to stress that the same results are true, but less obvious, in other bases.

The dominant interactions between $\chi$ and the SM are dimension-4 gauge couplings with the photon, as any interactions with the $Z$ are suppressed at temperatures below the weak scale. Dimensional analysis then implies that at temperatures large compared to $m_\chi$, the interaction rate is linear in temperature, $\Gamma_\chi \sim T$. Similar to the Goldstone couplings to leptons, this means that at high temperatures $\chi$ will be fully decoupled from the SM and then potentially recouples as the universe cools and the expansion rate drops when $\epsilon \ll 1$. Unlike the Goldstone case, $\chi$ is always interacting with $A'$, provided the $A'$ coupling is sufficiently large, such that $\chi$ and $A'$ can maintain equilibrium distributions. Therefore, the hidden sector has a well-defined temperature. The precise ratio of temperature of the hidden sector to the temperature of the SM prior to the recoupling of the two sectors is model-dependent, as more complicated hidden sectors will generally result in a wide range of possible temperatures. Consequently, we choose to explore a wide range of such initial ratios.

The thermodynamics are sensitive to whether $A'$ and $\chi$ are in equilibrium, rather than the precise coupling $g_A$, so we can simply fix the value of $g_A$ to be sufficiently large, without loss of generality. We select the value $g_A^2 = 0.1$, but our final results can be simply related to other values of $g_A$. Once $g_A$ is fixed, there are only three remaining parameters that can change: the kinetic mixing $\epsilon$, the new fermion mass
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\(m_\chi\), and the ratio of initial temperatures \(T_{hid}/T\).

Multiple star and supernova cooling observations, as well as various collider results, place significant constraints on millicharged fermions (for details see [7, 174]). Specifically, models with \(m_\chi \lesssim 100\) keV are restricted to \(\epsilon \lesssim 10^{-13}\), such that these species will never thermally couple to the SM for all reasonable initial values of \(T_{hid}\).

Light millicharged fermions can therefore not directly contribute to \(g_*\), but more massive fermions can instead indirectly alter the CMB by maintaining equilibrium between the SM and \(A'\), which then contributes a nonnegligible \(\Delta g_*\). However, for this to occur, we need \(m_\chi \lesssim 150\) MeV, such that \(\chi\) is still present below the QCD phase transition. This therefore limits us to a very narrow range of allowed masses \(m_\chi\) for models of millicharged species which affect the CMB. For models of this type, \(\chi\) must couple to the SM prior to or during its annihilation, otherwise the hidden sector will again never couple to the SM. This limits the possible values for \(\epsilon\) and \(T_{hid}\) for any given mass \(m_\chi\), in addition to constraints placed by independent observational and experimental bounds.

To illustrate the general behavior of these models, we consider four possible fermion masses within the allowed mass range. The corresponding results are shown in figure 3.8. In each case, the millicharged fermion has mass \(m_\chi \gtrsim 10\) MeV, which are unconstrained by star and supernova cooling observations and therefore have the largest available ranges for \(\epsilon\) and \(T_{hid}\). The lowest mass shown in figure 3.8 is actually \(m_\chi = 50\) MeV, as the results are equivalent for masses between 10-50 MeV.
For each of these cases, we scan over possible values for the mixing parameter $\epsilon$, as well as possible values for the original hidden sector temperature $T_{hid}$ when the SM temperature $T = 200$ MeV. We specifically consider $T_{hid}$ below the SM temperature $T$, assuming a minimal hidden sector model containing less particle content than the SM. The hidden sector will thus be colder due to fewer entropy redistributions. This procedure involved a modified version of the original code, the details of which can be found in appendix B.

As we see in figure 3.8, there is a basic pattern to the dependence of $\Delta g_*$ on both $\epsilon$ and $T_{hid}$. For very small values of $\epsilon$, the hidden sector is never coupled to the SM, and $A'$ receives all of the $\chi$ entropy redistribution. The contribution to $g_*$ is then dependent solely on the energy available in the hidden sector. The energy density increases as the initial temperature increases relative to the SM temperature. Initial temperatures of $T_{hid} \gtrsim \frac{4}{5}T$ are excluded by a Planck result of $g_* = 3.50 \pm 0.12$.

For increasing values of $\epsilon$, the hidden sector begins to couple with the SM, until at large values the two sectors quickly become fully coupled, regardless of the initial temperature $T_{hid}$. In this regime, the contribution of $A'$ to $g_*$ is precisely that of a new gauge boson which is originally coupled to the SM then decouples before the electron annihilation, $\Delta g_* \approx 0.5$.

In the transitional region from completely decoupled to completely coupled, the contribution rapidly climbs to $\Delta g_* \approx 1$, then rapidly decreases to the fully coupled limit for large $\epsilon$. This enhanced contribution to $g_*$ corresponds to a fortuitous com-
Figure 3.8: $\Delta g_*$ due to a single gauge boson which couples to a new fermion with mixing-induced SM gauge couplings. The contribution to $g_*$ at recombination is given as a function of both the mixing parameter $\epsilon$ and the hidden sector temperature $T_{hid}$ when the SM temperature $T = 200$ MeV. Results are presented for (a) $m_\chi = 50$ MeV, (b) $m_\chi = 75$ MeV, (c) $m_\chi = 100$ MeV, and (d) $m_\chi = 125$ MeV. Blue and purple regions to the left of the black line are allowed by a Planck measurement of $g_* = 3.50 \pm 0.12$, although these regions were never in thermal equilibrium with the SM. Regions to the right of the black line are excluded by this result from Planck.

Combination of coupling and mass values, in which $A'$ participates in the muon entropy redistribution but is able to receive all of the $\chi$ entropy. This occurs because $\chi$ briefly
couples to the SM, sharing the muon entropy, then quickly decouples as the muon and \( \chi \) number densities begin to plummet, such that the SM receives none of the \( \chi \) entropy. The result is a superheated population of \( A' \) bosons, which contain a large fraction of the total energy density.

While the majority of this behavior has been largely \( m_\chi \)-independent, we do observe a slight decrease in the transitional region values of \( \Delta g_* \) as the \( \chi \) mass increases. Larger fermion masses result in the hidden sector decoupling earlier from the SM, and therefore receiving less of the muon entropy. Finally, there are no major distinctions between \( m_\chi \sim 20 \text{ MeV} \) and \( m_\chi \sim 50 \text{ MeV} \), as these masses are proximate to neither the muon nor the electron mass.

For initial hidden sector temperatures below \( \frac{T}{20} \), the behavior will be largely unchanged from the low-temperature results presented here. Theories with small mixing parameters will remain fully decoupled and contribute negligibly to \( g_* \), while theories with larger \( \epsilon \) values will rapidly reach equilibrium with the SM, such that their contribution \( \Delta g_* \) is insensitive to the initial temperature.

We find that for any value of the initial temperature, \( \epsilon \) is restricted to be \( \lesssim 10^{-8} \) when there is a Dirac fermion \( \chi \) with masses between \( 10 - 150 \text{ MeV} \), forcing the SM and the hidden sector to never have been in thermal equilibrium. In addition, a scenario with \( m_\chi \lesssim 10 \text{ MeV} \) is inconsistent with constraints from star and supernovae cooling, and one where \( m_\chi \gtrsim 150 \text{ MeV} \) causes the \( A' \)'s to decouple before or during the QCD phase transition. The result in the absence of \( \chi \) is the same result as obtained
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by raising the $\chi$ mass and integrating it out of the theory; there exists a basis in which there are no couplings between the hidden sector and the SM, thus preventing the thermalization of $A'$. The scenario of new gauge bosons which mix with SM hypercharge is therefore further constrained by results from Planck.

Dipole Moments

While it is always possible to eliminate any mixing-induced renormalizable couplings between SM fermions and a new unbroken gauge boson $A'_\mu$, there could generically still be higher-order nonrenormalizable couplings after integrating out $A'$-SM interaction mediators in the full theory. If the low-energy effective theory contains no light species charged under $U(1)_{A'}$, then the dominant interactions between $A'$ and the SM are of the form

$$L \supset -\frac{1}{M^2} A'_\mu \psi_R^c \sigma^{\mu\nu} h^\dagger \psi_L + h.c.,$$  \hspace{1cm} (3.30)

where $A'^{\mu\nu}$ is again the associated field-strength tensor and $M$ is the mass scale associated with the heavy species integrated out of the theory. After EWSB, the expansion of the Higgs field about its expectation value $v$ will lead to dipole moment interactions of the form

$$L \supset -\frac{v}{M^2} A'_\mu \psi_R^c \sigma^{\mu\nu} \psi_L + h.c. \rightarrow -\frac{1}{\Lambda} A'_\mu \psi_R^c \sigma^{\mu\nu} \psi_L + h.c.,$$  \hspace{1cm} (3.31)

where we have now defined an effective dipole scale $\Lambda \equiv \frac{M^2}{v}$. Without knowledge of the full UV theory, it’s possible for the resulting dipole interactions to have generic
flavor structure, rather than be flavor-blind. We consider all such structure in this section.

The induced dipole couplings for pions must involve a composite pion operator which is antisymmetric in its two Lorentz indices. If the quark dipole moments are flavor-blind, such that the up and down quarks have the same couplings to $A'$, then all such antisymmetric operators vanish. If, instead, the dipole couplings are not flavor-blind, interactions between $A'$ and pions will potentially appear. However, there is no symmetry protecting against renormalization of such operators. We then expect these pion interactions to be strongly renormalized, thereby preventing us from making robust predictions about such contributions to $g_\ast$. We therefore focus solely on the dipole couplings of $A'$ to leptons.

With the interaction Lagrangian of the form

$$\mathcal{L} \supset -\frac{1}{\Lambda_f} \bar{\Psi}_f \sigma^{\mu\nu} \Psi_f A'_\mu \nu,$$

(3.32)

where $\Psi_f$ can either be an elementary lepton or a composite nucleon, we obtain the bounds

$$\Lambda_e \gtrsim 2.0 \times 10^{10} \text{ GeV},$$

(3.33)

$$\Lambda_{p,n} \gtrsim 9.8 \times 10^9 \text{ GeV}.$$  

These bounds again come from star and supernova cooling, and details can be found in \[158, 179, 180\]. There are also constraints on off-diagonal couplings $\Lambda_{\mu e}$ coming from $\mu \to e + \bar{\nu}$ \[179\], which limit such couplings to

$$\Lambda_{\mu e} \gtrsim 2.3 \times 10^9 \text{ GeV}.$$  

(3.34)
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For direct muon constraints, we again calculate the approximate supernova cooling bounds

\[ \Lambda_\mu \gtrsim 2.7 \times 10^6 \text{ GeV}. \]  

(3.35)

We find that the electron-only and electron-muon off-diagonal coupling scenarios are constrained to decouple before or during the QCD phase transition, preventing \( A' \) from contributing to \( g_* \) in this part of parameter space. However, when the coupling of \( A' \) to those species in the SM present after the QCD phase transition is dominated by its coupling to the muon, there is still a potentially allowed range for \( \Lambda_\mu \). Our results for muon-dominated couplings are shown in figure [3.9]. We find that \( \Lambda_\mu < 10^7 \text{ GeV} \) in order for the \( A' \) to remain coupled after the QCD phase transition, and consequently contribute to \( g_* \). This requires a significant hierarchy between the electron-\( A' \) coupling and the muon-\( A' \) coupling, but such a hierarchy is compatible with an MFV-like framework, as the hierarchy does not need to be much larger than \( y_e/y_\mu \). Values of \( \Lambda_\mu \lesssim 10^6 \text{ GeV} \) are inconsistent with a Planck result of \( g_* = 3.50 \pm 0.12 \), providing constraints which are approximately equivalent to those placed by supernova observations.

### 3.3.4 Spin-\( \frac{3}{2} \): Gravitino

Any model of supergravity contains the gravitino, which is the unique elementary spin-\( \frac{3}{2} \) particle. If supersymmetry were unbroken, the gravitino would be precisely massless. In a method similar to that of gauge symmetries, the spontaneous breaking
Figure 3.9: $\Delta g_*$ due to an $A'$ which interacts primarily with muons. The contribution to $g_*$ at recombination is given as a function of the effective scale $\Lambda$, which suppresses this interaction. Constraints on this muon interaction resulting from observations of supernova cooling are given in purple, restricting $\Lambda \gtrsim 2.7 \times 10^6$. The gray region for $\Lambda \gtrsim 10^7$ GeV corresponds to models which decouple during the QCD phase transition. The provided values of $\Delta g_*$ should therefore only be interpreted qualitatively in that region. The green region corresponds to values of $\Lambda$ excluded by a Planck result of $g_* = 3.50 \pm 0.12$, which are comparable to, but slightly weaker than, the constraints placed by supernova cooling.

of supersymmetry gives rise to a massless fermion, the Goldstino, which then becomes the longitudinal mode of the gravitino. As a result, the gravitino acquires a mass $m_{3/2} \sim \frac{F}{M_{pl}}$, where $F$ is generally the largest supersymmetry breaking scale squared in the theory. The gravitino can potentially remain a light degree of freedom for sufficiently low supersymmetry-breaking scales.

Naively, the gravitino would interact solely with gravitational strength and would
therefore decouple at very high temperatures. However, at energy scales far above the gravitino mass, the Goldstino equivalence theorem ensures that the longitudinal components of the gravitino interact with Goldstino strength, potentially maintaining equilibrium with the SM down to lower temperatures. As is well known [181], the Goldstino couplings to the SM are of the form

\[ \mathcal{L} \supset -\frac{1}{F^2} \chi^\dagger \sigma_\mu \partial_\nu \chi T^{\mu\nu}, \]  

where \( T^{\mu\nu} \) is the stress-energy tensor comprised of SM fields. While this coupling is no longer gravitationally suppressed, it is still a dimension-8 operator, such that the gravitino will still decouple above the QCD phase transition for all viable supersymmetry-breaking parameters \( F \) and not contribute significantly to \( g_* \).

3.3.5 Spin-2: Graviton

The unique elementary spin-2 particle is the graviton. The graviton interacts solely with gravitational strength, such that it either decouples from the SM at very high temperatures or is never even in thermal equilibrium. Similar to the discussion of subsection 3.2.4, the contribution to \( g_* \) of gravitons which decouple at such large temperatures is well below the sensitivity of Planck.

3.3.6 Models with Light Masses

Up to this point, we have considered any new species to be precisely massless, which allows their contribution to \( g_* \) to be directly computed from the distribu-
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tion function near recombination. This approximation is valid for any particles with masses $m \ll eV$, for these particles will still be fully relativistic during and shortly after the formation of the CMB. This range of validity can be explicitly seen in figure 3.10, which shows the ratios of both the energy density and pressure of a massive particle which decouples at high temperatures to those of a massless particle which decouples at the same high temperature, all as a function of the particle’s mass over the relevant temperature. For any given temperature, such as that of recombination, we can then use these simple ratios to determine the range of masses which can be treated as negligible, such that our massless approximation is valid.

It is still possible for there to be natural models with $m \sim eV$. One example is the addition of 1-3 light sterile neutrinos, which are motivated by multiple short baseline oscillation results suggesting the existence of neutrino mass splittings distinct from those required to fit solar and atmospheric neutrino data (for details, see [8] and references therein). Various analyses of these sterile neutrino models can be found in [182–190].

The presence of nonzero masses alters the relation between the energy density and the pressure of a species, such that the full effects cannot be captured by a single number $\Delta g_*$. In order to interpret the constraints of CMB measurements on species which become nonrelativistic during recombination, we must consider the resulting differences between Silk damping and early ISW. Both of these processes are sensitive to the precise evolution of $H$, whose time-dependence is sensitive to the mass of new
Figure 3.10: Ratios of the (a) energy density $\rho$ and (b) pressure $P$ of a massive particle which decouples at some high temperature to those of a massless particle which decouples at the same temperature, expressed as a function of $m/T$, where $m$ is the particle mass and $T$ is the temperature of interest. These calculations assume that the particle decoupled such that it maintained an equilibrium distribution, specifically the Bose-Einstein (blue) or Fermi-Dirac (red) distribution. At temperatures below the mass of the particle, the pressure of the massive particle rapidly drops, while the energy density rapidly becomes much larger than that of a massless particle. The resulting deviations of physical observables, such as the expansion rate $H$, can be extracted from these ratios to see the sensitivity of such observables to the particle’s mass.

light species. In addition, there are new mass-dependent effects which can arise, such as alterations to the matter power spectrum and to gravitational lensing of the CMB,
which are similar to the effects caused by nonzero neutrino masses. Such discussion is beyond the scope of our current work, but more details can be found in [16][19].

Silk damping is primarily sensitive to the overall expansion rate, and therefore the overall energy density, near the point of recombination. Any additional light species will add more energy density than is predicted solely by the SM. However, if such species have non-negligible masses, these new particles behave as relatively hot dark matter, as they will have become nonrelativistic by the modern era. Consequently, they contribute to measurements of $\Omega_{DM} h^2$ today, whereas they did not impact the CMB in the same fashion as standard cold dark matter. The exact contribution of a massive species to Silk damping is therefore sensitive to the amount of dark matter in our universe, which is dominated by uncertainty in the overall dark matter content, and a more careful analysis of the effects of new light species is needed.

Similarly, the early ISW effect is sensitive to the radiation/matter ratio following the formation of the CMB. New massive species will be transitioning to a nonrelativistic distribution during this period, behaving as neither pure radiation nor pure matter. Again, the exact prediction of early ISW effects is also dependent on the precise energy density of cold dark matter.

The main complication to the calculation of $\Delta g_*$ for such models arises from the use of the specific $\Lambda$CDM framework in calculating cosmological parameters from CMB data, in which the mass of dark matter is significantly higher than the temperature of recombination. This leads to model-dependence in the reported bounds,
which do not necessarily exclude models which fall outside of this framework.

It is important to stress that the difficulty arises due to uncertainty in the precise expansion rate and dark matter content, not due to any calculational uncertainty in the new light species sector. For example, one can precisely calculate the decoupling of light sterile neutrinos which potentially accommodate the recent short baseline results. We used both the normal and inverted hierarchy best-fit models of [192], which includes two light sterile neutrinos, and calculated the evolution of the two mostly-sterile mass eigenstates. We find that, for both hierarchies, these species decouple from the SM near the end of the muon entropy redistribution, such that they would contribute $\Delta N_{\text{eff}} \approx 1.9$ if they were massless. However, due to their non-negligible mass, the effects of these particles on the CMB is not fully characterized simply by a contribution to $N_{\text{eff}}$ or $g_*$. In order to fully probe the effect of models such as this on the CMB, a more general analysis of the CMB anisotropy data must be taken, which includes the possibility of nonzero masses for various additional light species. Such an analysis is beyond the scope of this chapter, but will be pursued in future work.

3.3.7 Models without New Light Species

Up to this point, we have considered the addition of new light species to the SM in order to increase the total relativistic energy density, $\rho_{\text{rel}}$, at recombination. The other possibility is a modification of the distribution functions of light species already present in the SM. The distribution function of photons is well-established as
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Bose-Einstein by measurements of the CMB, such that the energy density of photons at recombination is known to high precision. The only remaining option is therefore a modification of the distribution function of neutrinos. If this change were to occur while neutrinos were still in equilibrium with the SM, then interactions with SM species would thermalize the distribution functions, washing out any original alteration. Therefore, new physics must only affect the neutrino energy density at temperatures below the MeV scale. Here we briefly discuss the possible mechanisms which can alter the distribution function of neutrinos to increase \( g_\ast \) at recombination: a neutrino asymmetry, interactions with new massive species, and new interactions between neutrinos and the remaining SM species.

A simple modification to the distribution function of a species is the introduction of a chemical potential \( \mu \), such that

\[
    f(t, E) = \frac{1}{e^{(E-\mu)/T} + 1},
\]

for fermions, with \( \mu \rightarrow -\mu \) for antifermions. Such a chemical potential results in an asymmetry between the number of particles and antiparticles. Once a species has fully decoupled and freely evolves, the Boltzmann equation constrains \( f \) to remain solely a function of \( a(t)p \), such that \( \xi \equiv \frac{\mu}{T} \) is then time-independent. We can then express any resulting effects in terms of this constant \( \xi \).

Although the neutrino distribution function is no longer Fermi-Dirac after decoupling from the SM, we shall assume it is for illustrative purposes. The total energy
density stored in neutrinos and antineutrinos with nonzero $\xi$ is given by

$$\rho_\nu = -\frac{3N_\nu T_\nu^4}{\pi^2} \left( \text{Li}_4(-e^\xi) + \text{Li}_4(-e^{-\xi}) \right) = \frac{7N_\nu \pi^2 T_\nu^4}{120} \left( 1 + \frac{30\xi^2}{7\pi^2} + \frac{15\xi^4}{7\pi^4} \right). \quad (3.38)$$

The presence of a nontrivial chemical potential for neutrinos would therefore increase the energy density, thereby increasing $g_*$. The electron neutrino chemical potential affects the neutron-to-proton ratio prior to the start of BBN through reactions of the form $p + \bar{\nu} \rightarrow n + \bar{e}$. This ratio directly affects the helium-4 abundance after BBN, and so bounds can be placed on the electron neutrino chemical potential. Furthermore, since all neutrino mass eigenstates contain some wavefunction overlap with the electron neutrino, all of the neutrino mass eigenstate chemical potentials are constrained. The result is $\xi \lesssim 0.1$ for each of the three neutrino species (193 194 and references therein).

A second possibility is the interaction of some new massive species with neutrinos. This heavy species can alter the neutrino distribution through annihilation or decay 195. For the case of annihilation, the new species must interact predominantly with neutrinos and possess a mass $\lesssim 10$ MeV, such that the resulting entropy redistribution occurs after neutrinos decouple from the SM. For the case of decay, the heavy species must have fully decoupled at some higher temperature, leaving a significant relic energy density, with a decay rate such that it decays predominantly to neutrinos after neutrino decoupling but prior to recombination. These decays would then significantly alter the neutrino distribution, creating a large number of neutrinos with energies comparable to the particle mass. For both cases, the mass, number density, and
coupling to neutrinos for this new species determine the precise contribution to $g_\ast$, making these scenarios highly model-dependent.

Finally, the existence of higher-dimensional operators coupling the neutrinos to other SM species could potentially maintain thermal equilibrium between neutrinos and the SM until lower temperatures. A later point of neutrino decoupling would result in a larger share of the electron entropy being distributed to neutrinos, raising their energy density. The possible interactions with the lowest dimensionality are electromagnetic dipole moments or four-fermion interactions between neutrinos and electrons, which are significantly constrained by star cooling \cite{162} and the LEP collider \cite{71}.

\section{Conclusions}

The Standard Model of particle physics represents our current knowledge of the quantum field theory that best describes all short-distance interactions down to $10^{-17}$ cm. Knowing that this model is incomplete leads us to search for fundamental particles outside the Standard Model. While the search for heavier particles continues at colliders, we focus on another class of new physics – light, stable particles – which can be probed via their effects on cosmology, most strikingly on the Cosmic Microwave Background. In this article, we have surveyed what we call the most ‘natural’ (or least contrived) models and their parameter spaces. By doing so we lay out the reach of current and future experiments detailing the power spectra in the Cosmic Microwave
Background and other probes of the initial density perturbations and cosmological parameters.

We have been able to analyze the effects on the radiation density of the universe of new light degrees of freedom which decouple after the QCD phase transition. This includes species that decouple at ‘complicated’ cosmological times, such as the time around which the muon becomes non-relativistic. We are able to compute the energy density, and consequently $\Delta g_*$, to an accuracy of 1%. This allows us to place constraints on the couplings in those well-motivated BSM effective models which contain new light degrees of freedom, which are competitive with constraints coming from other areas of physics. We do this using a program which solves the Boltzmann and Friedmann equations for the case of one new light species, calculating the resulting evolution of that species’ distribution function, while approximating the SM species using fully thermalized equilibrium distributions and only considering the effects of leading order interaction terms. Using these calculations, we have demonstrated the ability of Planck and future experiments to place exclusion limits on all natural, minimal models with new light species. The compatibility of each model with the recent Planck results is given in table 3.1.

Higher levels of calculational accuracy could be achieved if we used a different numerical algorithm which was better adapted for the integro-differential equations considered in this chapter, or used a larger and finer momentum grid. In addition, loop corrections to the amplitudes, three-body final states, and finite-temperature
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QFT effects all contribute at the 0.1% level. If much higher precision is ever achieved observationally, potentially through next-generation polarization measurements, then these improvements would be warranted. Such a high-precision measurement of $g_*$ would better reveal degrees of freedom which decouple before or during the QCD phase transition. In such a scenario, this measurement, combined with independent measurements of the nature and couplings of a new light degree of freedom could potentially even allow us, in this way, to probe the structure of the QCD phase transition.

The future work we intend to pursue is the inclusion of the mass effects on different observables in the Cosmic Microwave Background. While this is only relevant in a narrow mass range (close to recombination temperatures), it turns out to be quite important for a number of specific models, such as those of sterile neutrinos. The more accurately we can describe their impact on the ISW effect and on Silk damping, the greater the possibility of finding a ‘smoking gun’ for such models.

If we coarsely divide the types of possible undiscovered particles into four types, categorized by stable or unstable and light or heavy, this work is an attempt to help push forward our probe of one of these types – new stable light particles. As the challenge to build new, more powerful high-energy colliders intensifies, it is exciting to see this new frontier mature as an additional source of information about the world beyond the Standard Model.
### Table 3.1: Compatibility of those natural, minimal models considered here with the recent results of the Planck satellite, $g_\ast = 3.50 \pm 0.12$ and $N_{\text{eff}} = 3.30 \pm 0.27$ [16]. While the current Planck results are in tension with other observational measurements, future experiments will greatly improve the precision and reach of these exclusion limits.

<table>
<thead>
<tr>
<th>Model</th>
<th>Operator</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goldstone bosons</td>
<td>$\frac{1}{\lambda} \partial_\mu \phi \bar{\Psi} \gamma^\mu \gamma^5 \Psi$</td>
<td>Flavor-blind: Decouple during/before QCD PT Muon-only: $\Lambda &gt; 2 \times 10^3$ TeV</td>
</tr>
<tr>
<td>Four-fermion V</td>
<td>$\frac{1}{\lambda} \chi^\dagger \bar{\chi} \gamma^\mu \Psi \gamma^\mu \Psi$</td>
<td>Weyl: $\Lambda &gt; 1$ TeV</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{\lambda} \bar{X} \gamma^\mu X \Psi \gamma^\mu \Psi$</td>
<td>Dirac: $\Lambda &gt; 5$ TeV</td>
</tr>
<tr>
<td>$U(1)'$</td>
<td>$e e \bar{\chi} A\chi$</td>
<td>$\epsilon &lt; 10^{-8}$ for $10 \text{ MeV} \leq m_\chi \leq 150 \text{ MeV}$ $m_\chi &gt; 150 \text{ MeV}$: Decouple during/before QCD PT</td>
</tr>
<tr>
<td>$A'$-dipole</td>
<td>$\frac{1}{\lambda} A'_{\mu\nu} \bar{\Psi} \sigma^{\mu\nu} \Psi$</td>
<td>Flavor-blind: Decouple during/before QCD PT Muon-only: $\Lambda &gt; 3 \times 10^3$ TeV</td>
</tr>
<tr>
<td>Massive Particles</td>
<td>Any</td>
<td>Inconclusive; mass-dependent</td>
</tr>
<tr>
<td>(e.g. Sterile Neutrinos)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Chapter 4

Universality of Long-Distance AdS Physics from the CFT Bootstrap

4.1 Introduction

Spacetime is a set of coordinate labels associated with the states and operators of a quantum mechanical system. It becomes a useful concept when the Hamiltonian of the system is approximately local in these coordinate labels. One need not resort to holography to find examples; for instance, this line of thinking underlies the reconstruction of extra dimensions from their Kaluza-Klein spectra. One can produce even more elementary examples by studying the ‘emergence’ of the coordinate label $x$ from an abstract interacting harmonic oscillator defined in terms of creation and annihilation operators.
In this spirit, the conformal bootstrap \cite{196–198} and related techniques \cite{199–201} have recently led to a rigorous, non-perturbative proof \cite{51} of the cluster decomposition principle in AdS$_{d+1}$ for all unitary $d \geq 3$ CFTs. Both AdS cluster decomposition and the leading corrections to it, including long-distance gravitational and gauge forces, are the AdS spacetime interpretation of a CFT theorem. The theorem pertains to the operator content of the operator product expansion (OPE) in the large angular momentum limit.

In this chapter we will explain the AdS interpretation in more detail, review the theorem and its proof, and then study its generalization to CFT$_2$/AdS$_3$. We will show that in a certain semi-classical limit of 2d CFTs it is possible to generalize the theorem. In particular, we will derive the existence of deficit angles in AdS$_3$ from the properties of Virasoro conformal blocks. We will also study the CFT dual of a light object interacting with a BTZ black hole \cite{55}.

The goal of the analysis is to use the conformal bootstrap to constrain the dynamics of an emergent AdS theory in a limit where a pair of objects are well-separated in AdS. The geodesic distance between the AdS objects will be extremely large and in particular, it may be much larger than the radius of curvature of the AdS theory. One should therefore think of the results as demonstrating super-AdS scale locality.\footnote{This is in contrast to analyses that demonstrate sub-AdS scale locality after making various additional assumptions about the CFT \cite{202–207}.}

Below, as in \cite{51}, we will formulate a more precise criterion along these lines that we

\begin{footnotesize}
\footnote{We emphasize that we are not assuming anything about the existence of an actual description in terms of fields, strings, etc. propagating in AdS. All our claims about AdS will follow as consequences of the CFT spectrum and OPE.}
\end{footnotesize}
Figure 4.1: This figure indicates the correspondence between a descendant operator/state in the CFT and a center-of-mass wavefunction in AdS. The relationship is entirely kinematical; it follows because the conformal group is the isometry group of AdS. A primary state would have its center of mass at rest near $\rho = 0$, the origin of AdS in the metric of equation (4.9).

To motivate our criterion for cluster decomposition, we rely on some basic facts about the kinematics of ‘objects’ in AdS, which we discuss in more detail in section 4.2. The AdS kinematic facts that we will invoke follow almost entirely from the role of the conformal symmetry group as the isometry group of AdS. We define an ‘object’ in AdS as a state created by any primary operator in the CFT with definite dimension and angular momentum. The wavefunction for the center-of-mass of an object can be
uniquely determined, and it is mainly supported near the origin of AdS. All possible
center-of-mass motions in AdS arise as linear combinations of conformal descendant
states, as pictured in Figure 4.1. In other words, center-of-mass wavefunctions in AdS
fill out a single irreducible representation of the conformal group.

Next we would like to understand how to construct a CFT state corresponding
to a pair of well-separated objects in AdS. Naively one might try acting on the
vacuum with two primaries, $\mathcal{O}_A$ and $\mathcal{O}_B$, but how can we create a large separation
between objects $A$ and $B$? There is no CFT state where the objects are far apart
and permanently at rest in AdS, because the AdS potential would cause them to
fall towards each other. However, if we give the pair of objects a large relative
orbital angular momentum, then the centrifugal force will keep them far apart. A
rough definition of cluster decomposition can now be provided: given the existence
of primaries $\mathcal{O}_A$ and $\mathcal{O}_B$ in a CFT, there also exist primary operators with large
angular momentum $\ell$ that create states with the appearance of objects $A$ and $B$,
spinning around each other at large $\ell$ in AdS, with vanishingly small interactions.
Such a state is pictured in figure 4.2.

We must clarify what we mean when we say the objects are non-interacting in
the limit of wide separation. If their interactions are negligible, then the interaction
or ‘binding’ energy of the two-object state must be negligible as well. The Dilatation
operator of the CFT must split up into two pieces that act separately on objects
$A$ and $B$. This translates into the statement that the anomalous dimension of the
two-object state should vanish. In precise terms, given two CFT primary operators, $O_A$ and $O_B$, their OPE should contain primary operators $[O_A O_B]_{n,\ell}$ with dimensions

$$\Delta_{AB}(n, \ell) = \Delta_A + \Delta_B + 2n + \ell + \gamma_{AB}(n, \ell),$$

such that $\gamma_{AB}(n, \ell) \to 0$ as $\ell \to \infty$. Here $n$ is an additional quantum number that parameterizes the eccentricity of the orbits in the semi-classical limit, so it allows for relative boosts between the objects.

This is exactly the spectrum of ‘double-trace’ states in a generalized free theory (GFT). These are theories whose correlators are entirely determined by two-point Wick contractions, as we discuss in section 4.3.2. For our present purposes it is more useful to define GFTs as the dual of free quantum field theories in AdS, since this definition emphasizes that GFTs describe non-interacting objects in AdS. In the limit $\ell \to \infty$, not only the anomalous dimensions, but also the OPE coefficients of $[O_A O_B]_{n,\ell}$ with $O_A$ and $O_B$ should approach those of a generalized free theory. In other words, at large angular momentum the CFT should have a spectrum and OPE coefficients that match GFT. When these criteria are all satisfied, we say that the AdS dual satisfies the cluster decomposition principle.

Crucially, this implies that at large angular momentum, the Hilbert space of the CFT has the structure of a Fock space. In other words, associating creation and annihilation operators $a_{A,i}^\dagger, a_{B,i}^\dagger$ and $a_{A,i}, a_{B,i}$ with the $i$-th descendants of $O_A$ and $O_B$, it is meaningful to write the state $[O_A O_B]_{n,\ell}$ as $c_{n,\ell,i,j} a_{A,i}^\dagger a_{B,j}^\dagger |0\rangle$, where $c_{n,\ell,i,j}$ is the appropriate ‘Clebsch-Gordan coefficient’ for irreducible representations of the
Figure 4.2: This figure shows two objects created by CFT operators $O_A$ and $O_B$ orbiting each other at large angular momentum, and therefore at large separation, in AdS. A major goal will be to show that such states exist and to describe their properties.

conformal group. The Dilatation operator $D$, which is the Hamiltonian for radial evolution, acts at large $\ell$ as

$$D = \sum_i (\Delta_{A,i} a_{A,i}^\dagger a_{A,i} + \Delta_{B,i} a_{B,i}^\dagger a_{B,i}).$$  \hspace{1cm} (4.2)

When we study AdS in global coordinates, this is the time translation operator, or in other words, the Hamiltonian.

As shown in [51] and reviewed in section 4.3, all CFTs in $d \geq 3$ satisfy this cluster decomposition principle. This result generalizes earlier results found in perturbation theory in large classes of CFTs [199,208,209]. It is consistent with, though clearly stronger than, our experience with weakly coupled field theories in AdS$_{\geq 4}$. 

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Specifically, potentials between particles due to the exchange of massless fields fall off exponentially in proper distance at large separation. In fact, when the lowest-twist \((\tau = \Delta - \ell)\) operator appearing in both the \(O_A^* O_A\) and \(O_B^* O_B\) OPE is a conserved current, such as \(T_{\mu\nu}\), the leading anomalous dimension at large angular momentum is \([51,200,201]\)

\[
\gamma_{AB}(\ell) \propto \frac{1}{\ell^{d-2}}. \tag{4.3}
\]

The constant of proportionality is determined by the central charge of the current and the charges of \(O_A, O_B\). In the case where this conserved current is the energy-momentum tensor, we verify that the numerical value of the coefficient exactly matches the prediction from semi-classical gravity in AdS. Thus “Newtonian” gravity in AdS is a generic long-distance feature for any CFT in \(d \geq 3\).

More generally, if operators with twist \(\tau_m < d - 2\) are present, the correction behaves like \(\gamma_{AB}(\ell) \propto \ell^{-\tau_m}\). By unitarity, the twist cannot be less than \(\frac{d-2}{2}\) for scalars, and cannot be less than \(d - 2\) for operators with spin \(\ell \geq 1\). Violations of the unitarity bound could produce forces that grow at long-distance, so unitarity is intimately connected with AdS locality.

The key observation that allows us to obtain these constraints is that individual conformal blocks\(^3\) in the decomposition of the four-point CFT correlator

\[
\langle O_A^*(x_1)O_A(x_2)O_B(x_3)O_B^*(x_4) \rangle \tag{4.4}
\]

\(^3\) For readers unfamiliar with the conformal bootstrap, we give a brief overview in section \[4.3.1\]. For a more thorough review, see e.g. [198].
Figure 4.3: One can only obtain an s-channel singularity in a scattering amplitude via an infinite sum of t-channel partial waves as $\ell \to \infty$. The same physical point, adapted to AdS/CFT, underlies the proof of cluster decomposition and the derivation of long-range forces from the CFT bootstrap.

predict singularities in the $\mathcal{O}_A^* \mathcal{O}_A \to \mathcal{O}_B^* \mathcal{O}_B^*$, or ‘s-channel’ that cannot be reproduced by any sum over a finite number of spins in the decomposition in the $\mathcal{O}_A \mathcal{O}_B \to \mathcal{O}_A \mathcal{O}_B$, or ‘t-channel’. An analogous phenomenon in scattering theory is indicated in Figure 4.3. These singularities occur in the limit $x_{12}^2 \to 0$, which is often referred to as a "light-cone" limit since the position $x_2$ is being brought onto the light-cone of the position $x_1$. In the s-channel, these singularities are controlled by the exchange of operators with minimum twist, which generically includes the identity operator 1 and conserved currents.

The situation becomes both more difficult and richer in $d = 2$, as we discuss in section 4.4. On the one hand, this difficulty can already be seen from the exchange of weakly coupled massless fields in AdS$_3$, where the potential at long distances no longer falls off at wide separation; we discuss AdS$_3$ dynamics in detail in sections 4.2.2 and 4.2.3. This is related to the fact that the minimum twist of operators allowed by unitarity in $d = 2$ is zero, so the leading correction from equation (4.3) to the
anomalous dimension does not decay at large angular momentum $\ell$. More precisely, in $d = 2$, the Virasoro algebra implies that there are infinite towers of zero-twist operators, which are the (anti-)holomorphic descendants of any (anti-)holomorphic primary operator, and these contribute singularities at the same order as the identity operator. At a minimum, the spectrum always contains the holomorphic and anti-holomorphic descendants of the identity operator itself.

Therefore to make progress in $d = 2$ we must take these contributions into account, which means we must determine the Virasoro conformal block for the identity operator. Fortunately we can use technology that has been specifically developed to exploit the full Virasoro symmetry. In particular, by focusing on the case of large central charge $c$, we can use powerful techniques to calculate various contributions to correlators, and in particular the contribution from the OPE exchange of any number of products of the energy-momentum tensor. The conformal blocks holomorphically factorize, so in such a calculation we can focus on the holomorphic piece. In all cases, we are looking at the conformal block for an operator with weight $h_p$ contributing to the the four-point function $\langle \mathcal{O}_A(0)\mathcal{O}_A(z)\mathcal{O}_B(1)\mathcal{O}_B(\infty) \rangle$ of operators $\mathcal{O}_A, \mathcal{O}_B$ with weight $h_A, h_B$. In the semi-classical limit $c \to \infty$ and formally $\frac{h_A}{c}, \frac{h_B}{c}$ fixed, the conformal blocks $\mathcal{F}(z)$ take the form

$$\mathcal{F}(z) = \exp \left( -\frac{c}{6} f(z) \right)$$

(4.5)

for a function $f(z)$ that depends on $c$ only through the various ratios $h/c$. In the
limit \( h_A \ll c, h_p \ll c \) but keeping \( h_B/c \) arbitrary, we find

\[
\frac{c}{6} f(z) = (2h_A - h_p) \log \left( \frac{1 - (1 - z)^{\alpha_B}}{\alpha_B} \right) + h_A (1 - \alpha_B) \log(1-z) + 2h_p \log \left( \frac{1 + (1 - z)^{\alpha_B}}{2} \right),
\]

where \( \alpha_B \equiv \sqrt{1 - 24h_B/c} \), and we neglect terms of order \( O(h_A^2/c^2, h_p^2/c^2) \). Further results using these methods for the conformal blocks are presented in appendix E.

The identity conformal block is the special case of (4.6) with \( h_p = 0 \). In AdS_3, this captures the exchange of arbitrary numbers of gravitons in the semi-classical (large \( m_{\text{pl}} \)) limit. By taking appropriate limits of the positions \( x_i \), one can reinterpret the four-point function equivalently as the two-point function of \( O_A \), not in the vacuum state, but in the state created by a heavy operator. A remarkable fact is that in this semi-classical limit, we find that the identity conformal block exactly reproduces the two-point function for the light operator \( O_A \) in a CFT at finite temperature

\[
\langle O_B | O_A(it) O_A(0) | O_B \rangle = \frac{(\pi T_B)^{2h_A}}{\sinh^{2h_A}(\pi T_B t)},
\]

set by the conformal weight of the heavy operator \( O_B \)

\[
T_B = \frac{\sqrt{24h_B/c - 1}}{2\pi},
\]

where we have conformally mapped (4.7) to radial time coordinates \( t = -\log(z) \).

An identical formula with \( h_A, T_B, z \rightarrow \bar{h}_A, \bar{T}_B, \bar{z} \) holds for the anti-holomorphic piece \( \bar{F}(\bar{z}) \) of the identity conformal block, so for spinning operators \( O_B \) one finds distinct left- and right-moving temperatures. The effective temperatures \( T_B, \bar{T}_B \) obtained here from the bootstrap match the semi-classical temperature of a black hole in AdS_3.
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with mass and spin given by the conformal weights of $O_B$. Consequently, the effect of multi-$T_{\mu\nu}$ exchange (i.e., multi-graviton exchange in AdS$_3$) between a light “test mass” and a heavy operator has exactly the same effect that the BTZ black hole geometry has on light fields in AdS$_3$. This provides a derivation of a version of the Eigenstate Thermalization Hypothesis [213,214] for CFT$_2$ at large central charge.

Because we take the large $c$ limit, the results we obtain in 2d have a more limited range of applicability than in $d \geq 3$, where we made no assumptions whatsoever about the CFT other than unitarity and the OPE. However, in the large $c$ limit we have a transparent physical interpretation in AdS$_3$, and we can prove striking results about the dual dynamics, including the presence of deficit angles from particles in AdS$_3$, as well as the modes in a BTZ black hole background. A summary of the results from our bootstrap analyses follows.

**Summary: CFT$_d$ with $d \geq 3$**

It is convenient to state the results [51,201] in terms of the anomalous dimension

$$\gamma_{AB}(n, \ell) \equiv \Delta_{AB} - (\Delta_A + \Delta_B + 2n + \ell)$$

and the OPE coefficients $c_{AB}(n, \ell)$ for the operator $[O_A O_B]_{n,\ell}$. These operators are implicitly defined by the proof that in the limit of large $\ell$, there exists a sequence of operators with the stated properties for every integer $n$. We begin with the result for the general case, which assumes only unitarity and the OPE:

$$\gamma_{AB}(n, \ell) \sim \frac{\gamma_n}{\ell m} \quad \text{General:} \quad P_{AB}(n, \ell) \sim P_{GFT}(n, \ell) \left(1 + \mathcal{O}(\gamma_{AB}(\ell, n))\right)$$
In the above expression, the symbol $\sim$ denotes the behavior in the limit of large $\ell$. The function $P_{\text{GFT}}(n, \ell)$ is the OPE coefficient-squared in generalized free theories; the explicit expression can be found in [215]. $\tau_m$ is defined as the smallest twist of any operator that appears in both the $\mathcal{O}_A^* \mathcal{O}_A$ and $\mathcal{O}_B^* \mathcal{O}_B$ OPE, and by unitarity this cannot be less than $\frac{d-2}{2}$.

Using the results of [216], it is convenient to separate out the case of CFTs whose correlators are exactly those of free fields, and all other CFTs. The reason is that only the former case can have conserved currents with spin $\ell \geq 3$, so eliminating this one essentially trivial case allows us to restrict the minimal twist $\tau = d - 2$ operators to spin-1 currents and the energy-tensor. The result in this large class of CFTs is:

$$\gamma_{AB}(n, \ell) \sim \frac{\gamma_{\text{grav}} + \gamma_{\text{gauge}}}{\ell^{d-2}}$$

$$\gamma_{\text{grav}} \approx -\frac{2^d \pi G_N (\Delta_A \Delta_B)^{\frac{d}{2}}}{\text{vol}(S^{d-1})(d-1)}$$

$$\gamma_{\text{gauge}} \propto q_A q_B$$

The coefficients $\gamma_{\text{grav}}$ and $\gamma_{\text{gauge}}$ can be calculated in the CFT by using the Ward identities to constrain the coefficients of conserved currents in the $\mathcal{O}_A^* \mathcal{O}_A$ OPE in terms of the charge of $\mathcal{O}_A$, which for a spin-1 current is defined above as $q_A$, and for $T_{\mu\nu}$ is the dimension $\Delta_A$. For simplicity we have approximated $\gamma_{\text{grav}}$ in the limit of large $\Delta_A$ and $\Delta_B$. The conserved current contributions can be interpreted in terms of AdS parameters by using their relation to the CFT central charges at weak coupling; in section 4.2.1 we perform this matching in $d = 4$ for the gravitational term and find complete agreement.
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Summary: CFT

In the limit where $h_A h_B/c$ is fixed while $h_A / c$ and $h_B / c \to 0$ as $c \to \infty$, the Virasoro conformal block for the identity is particularly simple. Assuming the identity is the only zero-twist primary being exchanged, the bootstrap leads to:

$$h_A, h_B \ll c: \gamma_{AB}(n, \ell) = -24 \frac{h_A h_B}{c} = -4G_N E_A E_B$$

The above anomalous dimension gets corrections at order $O\left(\frac{h^3}{c^2}, \frac{nh}{c}\right)$. As indicated in the final equality above, this agrees exactly with the binding energy for two test masses in linearized gravity in AdS$_3$.

We can also go beyond this “test mass” limit, and analyze the bootstrap constraints in the limit that $h_B / c$ is fixed but $h_A / c$ is small. It is well known that AdS$_3$ has a gap in energy of $\frac{1}{8G_N}$ between the vacuum and the lightest BTZ black hole. Below this threshold, masses in AdS$_3$ just create local conical “deficit angle” singularities. Using the relation $c = \frac{3}{2G_N}$, this energy gap translates to a threshold in the weight of a scalar operator at $h = \bar{h} = \frac{c}{24}$. It is convenient to separate our results into $h_B > \frac{c}{24}$ and $h_B < \frac{c}{24}$, i.e. into weights that correspond to AdS geometries above and below threshold for a BTZ black hole. As we review in section 4.2.2, the deficit angle created by a particle with mass $2h_B$ in AdS$_3$ is just $\Delta \phi = 2\pi(1 - \sqrt{1 - 24h_B/c})$. In this more general limit, we find:

$$h_B < \frac{c}{24} : \tau_{AB}(\ell, n) \sim 2 \left(h_B + \sqrt{1 - 24h_B/c(h_A + n)}\right) = E_B + \left(1 - \frac{\Delta \phi}{2\pi}\right) E_A$$

$$h_B > \frac{c}{24} : \tau_{AB}(\ell, n) = \text{dense} \sim 2h_B + 4\pi i T_{BTZ}(h_A + n)$$
where we have listed the case of scalar $\mathcal{O}_A$ and $\mathcal{O}_B$, for simplicity.

The energy spectrum below the BTZ black hole threshold exactly matches the semi-classical result from AdS$_3$ with a deficit angle $\Delta\phi$, as we discuss in more detail in section 4.2.2. The spacing between modes becomes vanishingly small as one approaches the BTZ threshold at $h_B = c/24$. Above the BTZ threshold we derive a dense discretum of twists in the large $\ell$ spectrum of the $\mathcal{O}_A\mathcal{O}_B$ OPE. One can also identify the spectrum of BTZ quasi-normal modes. For this, one should use a basis not of primary operators (which must have real and positive dimensions by unitarity), but rather of in and out states, obtained in practice by adopting an appropriate $i\epsilon$ prescription. As shown in equation (4.7), the semi-classical identity conformal block matches the two-point function evaluated in a thermal background, so the full spectrum$^4$ of BTZ quasinormal modes can be reproduced \[217\].

4.2 Defining Long-Distance AdS Physics in CFT

Terms

In this section we will formulate a version of the AdS cluster decomposition principle and translate it into a statement about the spectrum and OPE of a CFT. Brief in situ reviews of some necessary aspects of AdS/CFT \[48,218,219\] will be given where required.

\[^4\text{Our methods are generally only reliable for the large angular momentum modes.}\]
We will be considering CFTs in radial quantization, taking the Dilatation operator $D$ as the Hamiltonian. Since the angular momentum generators commute with $D$, we label CFT states according to their scaling dimension $\Delta$, which is their $D$ eigenvalue, and their angular momentum quantum numbers, which we denote by $\ell$. In this basis the momentum generators $P_\mu = -i\partial_\mu$ act as raising operators of the dimension $\Delta$, while the special conformal generators $K_\mu$ act as lowering operators. Irreducible representations of the conformal group are labeled by the quantum numbers of a primary state, which is a state annihilated by all the $K_\mu$. Descendant states are created by acting with $P_\mu$ on a primary. In radial quantization, local operators can be identified with the states they create on a tiny circumscribing ball.
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We will study AdS$_{d+1}$ in global coordinates, with metric

$$ds^2 = \frac{R_{\text{AdS}}^2}{\cos^2 \rho} \left( dt^2 - d\rho^2 - \sin^2 \rho d\Omega^2 \right).$$  \hspace{1cm} (4.9)

This coordinate system has a natural correspondence with a CFT in radial quantization, as pictured in Figure 4.4. We identify the unit $d$-vector $\hat{\Omega}$ with coordinates on a sphere about the origin in the CFT, and $e^t$ with the radius of the sphere. The Dilatation operator generates $t$-translations, so that bulk energies correspond to CFT dimensions via

$$\Delta_{\text{CFT}} = E_{\text{AdS}} R_{\text{AdS}}.$$  \hspace{1cm} (4.10)

The other global conformal generators also correspond to AdS isometries. For the most part we will work in units with $R_{\text{AdS}} = 1$, although we will occasionally reintroduce the AdS length for clarity and emphasis.

Conformal invariance uniquely determines an AdS$_{d+1}$ wavefunction for the center of mass coordinate of any primary or descendant state, as pictured in Figure 4.1. This is a general result; it follows because the conformal symmetries form the isometry group of AdS, so there is a one-to-one map between conformal representations and AdS coordinates. A primary wavefunction must be annihilated by all the special conformal generators $K_\mu$, and this provides $d$ distinct first order differential equations that must be satisfied by a primary wavefunction in AdS$_{d+1}$. In the scalar case primary wavefunctions necessarily take the form

$$\psi_{\text{prim}}(t, \rho, \Omega) = e^{i\Delta t} \cos^\Delta \rho.$$  \hspace{1cm} (4.11)
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Since the Dilatation operator \( D = -i \partial_t \) we see that the undetermined parameter \( \Delta \) is the scaling dimension of the state.

Equation (4.11) describes a wavefunction centered at \( \rho = 0 \), falling off quickly at large distances, with a characteristic rate set by \( \Delta \). In the large \( \Delta \) limit this can be approximated by a Gaussian wavepacket at the center of AdS, with a width \( \sim 1/\sqrt{\Delta} \). It is held in place by the effect of the AdS curvature. Descendant state wavefunctions filling out a full irreducible representation of the conformal group can be computed by acting on the primary wavefunction with the raising operator \( P^\mu \), the CFT momentum generator. A typical descendant state is portrayed in Figure 4.1.

Let us be a bit more precise about the kinematics of the descendant states. The AdS wavefunction for the center of mass of a state descending from a scalar primary is (see e.g. [204, 220])

\[
\psi_{n,\ell,j}(t, \rho, \Omega) = \frac{e^{-iE_{n,\ell}t}}{N_{\Delta n\ell}} Y_{\ell j}(\Omega) \sin\rho \cos^\Delta \rho \frac{\sin^2 \rho}{\Gamma(\Delta + n - \frac{d-2}{2})} \frac{\Gamma(\Delta + n + \ell + \frac{d}{2})}{\Gamma(\Delta + n + \ell)} \cdot (4.12)
\]

with normalizations

\[
N_{\Delta n\ell} = (-1)^n \sqrt{\frac{n! \Gamma^2(\ell + \frac{d}{2}) \Gamma(\Delta + n - \frac{d-2}{2}) \Gamma(\Delta + n + \ell)}{\Gamma(n + \ell + \frac{d}{2}) \Gamma(n + \ell) \Gamma(\Delta + n + \ell)}} \cdot (4.13)
\]

where \( E_{n,\ell} = \Delta + 2n + \ell \). The two quantum numbers \( n \) and \( \ell \) index changes in the twist and angular momentum, respectively, where the twist \( \tau \equiv \Delta - \ell \). If we consider the simple case of \( n = 0 \) and \( \ell \gg \Delta \gg 1 \), corresponding to minimal twist and large angular momentum, then we find that the norm of the wavefunction has a maximum.
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at a geodesic distance

\[ \langle \kappa \rangle \approx \frac{R_{\text{AdS}}}{2} \log \left( \frac{2\ell}{\Delta} \right) \]  

from the center of AdS, with a width of order \( R_{\text{AdS}}/\sqrt{\Delta} \) in \( \langle \kappa \rangle \). In this limit the wavefunction represents an object in a circular orbit about the center of AdS.

The preceding discussion of CFT states and AdS center-of-mass wavefunctions was completely general. Now let us specialize for a moment and consider CFTs with AdS duals whose spectra include weakly coupled particles. The 2-particle primary states in such an AdS theory are dual to operators that we will represent as \([O_1O_2]_{n,\ell}\) in the CFT, where \(O_1\) and \(O_2\) are primaries that create single-particle states.

The primary operators \([O_1O_2]_{n,\ell}\) create 2-particle states whose center of mass is supported near our chosen origin at \(\rho = 0\) in AdS, but the pair of particles themselves can have a large relative motion. In particular, we can study the state where the particles both orbit the center of AdS precisely out of phase, so that they are opposite each other across the center of AdS. This configuration is pictured in Figure 4.5. The particles are very well-separated at large \(\ell\), because they are balanced across the center of AdS. In the case of free particles the primary operators \([O_1O_2]_{n,\ell}\) have dimension

\[ \Delta_1 + \Delta_2 + 2n + \ell. \]

This CFT scaling dimension corresponds to the rest mass of the two AdS particles plus a contribution from the kinetic energy of their relative motion.

\(^5\)The geodesic distance \(\kappa\) from the center of AdS is related to the \(\rho\) coordinate by \(\sinh \kappa = \tan \rho\).
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In the case of a pair of non-interacting AdS objects, including the case of free particles, we can work out the kinematics exactly. In the appendices of [215, 221] it was shown how to decompose a primary operator \([O_1O_2]_{n,\ell}\) in a generalized free theory into the descendants of \(O_1\) and \(O_2\). This is identical to decomposing 2-particle primary wavefunctions into sums of products of one-particle descendant wavefunctions in AdS. In the case of \(n = 0\) one finds

\[ [O_1O_2]_{\ell} = \sum_{\ell_1 + \ell_2 = \ell} s_{\ell_1, \ell_2} \left( \partial_{\mu_1} \cdots \partial_{\mu_{\ell_1}} O_1 \right) \left( \partial_{\nu_1} \cdots \partial_{\nu_{\ell_2}} O_2 \right) \]  

(4.16)

with coefficients

\[ s_{\ell_1, \ell_2} = \frac{(-1)^{\ell_1}}{\ell_1! \ell_2! \Gamma(\Delta_1 + \ell_1) \Gamma(\Delta_2 + \ell_2)}. \]  

(4.17)

This means that at large \(\ell\), the CFT primary \([O_1O_2]_{\ell}\) is dominated by contributions from descendants with

\[ \ell_1 \approx \frac{\ell}{2} \left( 1 + \frac{\Delta_2 - \Delta_1}{2\ell - \Delta_1 - \Delta_2} \right) \approx \frac{\ell}{2} + \frac{\Delta_2 - \Delta_1}{4}. \]  

(4.18)

We see that at large angular momentum, such operators are composed of pairs of descendants of \(O_1\) and \(O_2\) with nearly equal angular momenta. The relation (4.18) will be useful for the semi-classical gravity calculations that follow in section 4.2.1.

The operators \([O_1O_2]_{n,\ell}\) always appear in the OPE of \(O_1\) and \(O_2\) if the conformal theory is a generalized free theory. If the theory is perturbative in either an AdS coupling (e.g. \(1/N\)) or some weak coupling in the CFT, then these operators are also

---

\(\footnote{A generalized free theory is the conformal theory dual to a free field theory in AdS. It can also be described as a CFT whose correlators can all be obtained by Wick contractions into 2-point correlators.} \)
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guaranteed to exist [199] and to make an appearance in the $\mathcal{O}_1(x)\mathcal{O}_2(0)$ OPE. But away from free theory they will acquire an anomalous dimension $\gamma(n, \ell)$.

From the AdS viewpoint, this anomalous dimension arises due to the interaction energy between the two objects. This means that at large $\ell$ we can use the relationship between $\langle \kappa \rangle$ and $\ell$ from equation (4.14) to write the total dimension of $[\mathcal{O}_1\mathcal{O}_2]_{n,\ell}$ as

$$\Delta_1 + \Delta_2 + 2n + \ell + \gamma(n, \ell(\kappa)), \quad (4.19)$$

where $\kappa$ is the geodesic distance between the objects in AdS. Since $\ell$ grows exponentially with $\kappa$, the strength of the AdS interaction at large distances is determined by the magnitude of the anomalous dimensions $\gamma(n, \ell)$ at very large $\ell$. In perturbative examples the anomalous dimension $\gamma(n, \ell)$ falls off as a power-law in $\ell$ as $\ell \to \infty$ in the case $d \geq 3$.

Do operators like $[\mathcal{O}_1\mathcal{O}_2]_{n,\ell}$ always exist in the OPE of $\mathcal{O}_1$ and $\mathcal{O}_2$ in any CFT? If so, then every CFT has a Hilbert space that can be interpreted in terms of states moving in AdS. The anomalous dimensions $\gamma(n, \ell)$ would give information about the properties of AdS interactions, with the large $\ell$ behavior corresponding to the effects of long-range forces in AdS.

We are finally ready to formulate our version of the AdS cluster decomposition principle as a statement about the OPE and the CFT spectrum: In the OPE of any two primary operators $\mathcal{O}_1$ and $\mathcal{O}_2$, for each non-negative integer $n$, there exists an infinite tower of operators $[\mathcal{O}_1\mathcal{O}_2]_{n,\ell}$ in the limit that $\ell \to \infty$, with dimension $\Delta_1 + \Delta_2 + 2n + \ell + \gamma(n, \ell)$ where $\gamma(n, \ell) \to 0$ as $\ell \to \infty$. Furthermore, one can show
that

\[ \gamma(n, \ell) = \frac{\gamma_n}{\ell^{\tau_m}}, \quad (4.20) \]

where \( \tau_m \) is the twist of the minimal twist operator appearing in the OPE of both \( O_1 \) with \( O_1^\dagger \) and \( O_2 \) with \( O_2^\dagger \). Generically \( \tau_m \leq d - 2 \), since the energy momentum tensor \( T_{\mu\nu} \) always appears in both of these OPEs, and in fact it is straightforward to go beyond equation (4.20) to derive the anomalous dimension at subleading order in \( 1/\ell \). In section 4.2.1 we will give an explicit computation of the long-distance gravitational effects for \( d \geq 3 \), which match the universal contribution from \( T_{\mu\nu} \) that we will obtain from the CFT bootstrap in section 4.3.4.

This theorem has been proven \([51, 201]\) for all CFT\( \geq 3 \), without any assumptions beyond unitarity. However, our formulation of the cluster decomposition principle is false in the case of AdS\( _3 \)/CFT\( _2 \). In fact, the 2d Ising model provides an explicit counter-example \([51]\).

We will see what goes wrong in section 4.2.2 but the intuition from AdS\( _3 \) is simple. Gravitational effects in \( 2 + 1 \) dimensions lead to deficit angles surrounding massive sub-Planckian objects, and these deficit angles can be detected from arbitrarily large distances. This means that they make finite corrections to the spectrum of operator dimensions, so that \( \gamma(n, \ell) \) approaches a finite constant \( \gamma(n) \) as \( \ell \to \infty \). The CFT\( _2 \) interpretation is that the presence of zero twist operators, such as the Virasoro descendants of the identity, imply that in equation (4.20) we have \( \tau_m = 0 \). However, with proper caveats we will show that a modified theorem holds, and that we can
compute the finite anomalous dimensions $\gamma(n)$ directly from the CFT bootstrap in two dimensions. We study the AdS$_3$ expectations for deficit angles in section 4.2.2. Then in section 4.2.3 we will obtain even more interesting expectations when we consider BTZ black holes. We will review the fact that there are no stable orbits about these objects, so we do not expect that cluster decomposition can hold above the BTZ threshold. However, what we can expect is a thermal spectrum of quasi-normal modes. In the remainder of this work we will then provide a universal CFT proof of these results without making further reference to AdS expectations.
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4.2.1 AdS$_{\geq 4}$: the Newtonian Gravitational Potential

In this section we will compute the shift in energy due to the gravitational interactions between very distant, uncharged, scalar masses in AdS$_{\geq 4}$. This corresponds to the CFT computation of the anomalous dimension of the primary operator $[\mathcal{O}_1\mathcal{O}_2]_{n,\ell}$ in the OPE of primaries $\mathcal{O}_1$ and $\mathcal{O}_2$, in the large $\ell$ limit. We will derive this anomalous dimension directly from the CFT bootstrap in section 4.3.4 and find that the results match.

The idea of the calculation is to do perturbation theory in the inverse distance between the objects, resulting in a ‘Newtonian’ approximation in AdS. This approximation is good only when $d \geq 3$, because gravitational interactions do not fall off with distance in $2+1$ bulk dimensions; in $2+1$ dimensions, one must further assume that $G_N$ is sufficiently small. We will obtain the first order energy shift by computing the expectation value of the gravitational interaction Hamiltonian using the unperturbed wavefunction for the orbiting object. First we will compute the interaction Hamiltonian (gravitational potential) at large distances due to the presence of a point mass, and then we will evaluate the expectation value.

In AdS$_{\geq 4}$, the AdS-Schwarzschild metric $^{222}$ is the solution to Einstein’s equations in the presence of a spherically symmetric, uncharged mass. In $d+1$ dimensions it is

$$ds^2 = U(r)dt^2 - \frac{1}{U(r)}dr^2 - r^2d\Omega^2,$$

(4.21)
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where

\[ U(r) = 1 - \frac{\mu}{r^{d-2}} + \frac{r^2}{R_{\text{AdS}}^2} \]  

(4.22)

and the mass of the black hole is

\[ M = \frac{(d-1)\Omega_{d-1}\mu}{16\pi G_N} \]  

(4.23)

where \( \Omega_{d-1} = \text{vol}(S^{d-1}) \). This coordinate system is useful because \( \sqrt{-g} \) is independent of \( M \), so only \( g^{00} \) and \( g^{rr} \) are affected by the mass \( M \). We need compute only to first order in \( M \), since this is equivalent to expanding in the inverse distance.

The energy shift to first order in \( M \) is then

\[ \delta E_{\text{orb}} = \langle n, \ell_{\text{orb}} | \delta H | n, \ell_{\text{orb}} \rangle \]

\[ = -\frac{\mu}{4} \int d^d r \left( \frac{r^{2-d}}{(1 + r^2)^2} (\partial_t \phi)^2 + r^{2-d} (\partial_r \phi)^2 \right) \Omega \]  

(4.24)

The two pre-factors of \( \frac{1}{2} \) in the above equation come from the normalization of the action for a scalar field in AdS and the inclusion of both the scalar and gravitational energy shifts (see e.g. [223]). We have attached an ‘orb’ label to emphasize that we are currently studying one mass, described by the scalar field \( \phi \), orbiting a second mass \( M \) at the origin of AdS. This is not a primary state in the CFT, since its center of mass is not at rest, and so we will need to translate this result to obtain the anomalous dimension of a primary operator \( [\mathcal{O}_1 \mathcal{O}_2]_{n,\ell} \).

Using the wavefunctions from equation (4.12) transformed to \( r = \tan \rho \) coordi-
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nates, we find

$$\delta E_{\text{orb}}(n, \ell_{\text{orb}}) = -\frac{\mu}{4} \int \frac{dr}{N_{n\ell_{\text{orb}}}^2} \left( \frac{E_{n\ell_{\text{orb}}}^2}{(1 + r^2)^2} |\psi_{n\ell_{\text{orb}}}(r)|^2 + (\partial_r \psi_{n\ell_{\text{orb}}}(r))^2 \right),$$

(4.25)

where $\psi_{n\ell_{\text{orb}}}(r)$ just includes the $r$ dependence of the wavefunctions. Taking the $n = 0$ case as an example and expanding the result as $\ell \to \infty$, we find the two terms

$$\delta E_{\text{orb}}(0, \ell_{\text{orb}}) = -\frac{4\pi G N M \Delta}{\Omega_{d-1}} \left( \frac{\Gamma(\Delta)}{2\Gamma(-\frac{d}{2} + \Delta + 1)} \left( \frac{1}{\ell_{\text{orb}}} \right)^{\frac{d-2}{2} + 1} \left( \frac{1}{\ell_{\text{orb}}} \right)^{\frac{d-2}{2}} \right)$$

(4.26)

and clearly the first term is dominant at large $\ell_{\text{orb}}$. This follows from the familiar fact that the Newtonian approximation requires us to keep track only of shifts in the metric component $g_{tt}$. In fact, we could have obtained this energy shift to leading order at large $\Delta$ via a computation in classical gravitational perturbation theory.

Equation (4.26) is not yet the formula of interest, since it is the energy shift associated with a configuration where one mass is at rest at the center of AdS, while the other orbits. To get the energy shift or anomalous dimension of the primary operator $[\mathcal{O}_1 \mathcal{O}_2]_{n, \ell}$, we need to use equation (4.18) to relate the double-trace primary to this ‘orbit’ state. In the semi-classical limit the orbit state has the same energy shift as a primary with equal geodesic separation between the two objects, so that

$$\kappa_{\text{orb}} = \kappa_1 + \kappa_2$$

(4.27)

with

$$\kappa_1 = \frac{1}{2} \log \left( \frac{\ell_{\text{prim}}}{\Delta_1} \right) \quad \text{and} \quad \kappa_2 = \frac{1}{2} \log \left( \frac{\ell_{\text{prim}}}{\Delta_2} \right).$$

Using equation (4.14) for the geodesic radius of an orbit, the angular momentum of orbit can be related to that of the primary by

$$\ell_{\text{orb}} = \frac{\ell_{\text{prim}}^2}{2\Delta_1}.$$

(4.28)
Taking $\ell_{\text{prim}} \to \ell$, $M \approx \Delta_1$, and $\Delta = \Delta_2$, we find a semi-classical energy shift
\[
\delta E(0, \ell) \approx -\frac{2^d \pi G_N (\Delta_1 \Delta_2)^{\frac{d}{2}}}{\Omega_{d-1} (d-1)} \left( \frac{1}{\ell} \right)^{d-2}
\]
in the approximation that $\ell \gg \Delta_1, \Delta_2 \gg 1$. In the case of $d = 4$, using the relation $c = \frac{\pi}{8 G_N}$, this gives
\[
\gamma(0, \ell) \approx -\frac{1}{6} \frac{(\Delta_1 \Delta_2)^2}{c} \left( \frac{1}{\ell} \right)^2,
\]
which matches the result we will derive from the CFT bootstrap in section 4.3.4.

As a final consideration, one might ask if these AdS$_{\geq 4}$ configurations are unstable due to the emission of gravitational and other radiation.\footnote{We thank Gary Horowitz for discussions of this point.} For a variety of reasons we expect that radiation will be an extremely small effect at large $\ell$. First, it is worth emphasizing that unlike binary star systems in our own universe, the pair of objects we consider here are held in their orbit by the AdS curvature. The gravitational binding energy between the objects vanishes at large $\ell$ even though the orbital period remains constant. Each object in the orbiting pair closely resembles a conformal descendant, as indicated in equation (4.16), and such states are exactly stable. This means that an emitted graviton would have to ‘know’ about both objects, despite their very large separation, and so we would expect emission to be exponentially suppressed. Furthermore, since the gravitational binding energies vanish at large $\ell$, while gravitons in AdS have an energy or dilatation gap $d/R_{\text{AdS}}$, considerations of energy and angular momentum conservation also suggest that graviton emission
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should be an exponentially suppressed process. Thus we expect that our orbiting pairs will have a highly suppressed radiation rate at very large $\ell$.

4.2.2 Deficit Angles in AdS$_3$ from Sub-Planckian Objects

Although there are no propagating gravitons in 2 + 1 dimensional gravity, Einstein’s equations have well-known, non-trivial solutions [224,225] in the presence of sources. In particular, a point particle of sub-Planckian mass placed in AdS$_3$ will produce a deficit angle at its location, while the spacetime remains locally AdS$_3$ everywhere else. This explicit solution for a particle at the origin can be written as

$$ds^2 = \frac{(1 - 8G_NM)}{\cos^2(\rho)} \left( dt^2 - \frac{d\rho^2}{1 - 8G_NM} - \sin^2(\rho) d\theta^2 \right), \quad (4.31)$$

where $M$ is the mass of the particle and $\theta \in [0, 2\pi)$. This looks exactly like the usual AdS$_3$ metric except for the presence of an angular deficit of $2\pi(1 - \sqrt{1 - 8G_NM})$, which is $\approx 8\pi G_NM$ in the limit $G_NM \ll 1$. We have made our choice for the normalization of $t$ and $\theta$ so that these coordinates have the usual relationship with CFT coordinates in radial quantization. In particular, the Dilatation operator $D = i\partial_t$.

Now let us compute the energy shift of a particle in AdS$_3$ due to the presence of the deficit angle. In fact, there is no computation to do. The usual bulk wavefunctions $\psi_{n\ell}(t, \rho, \theta)$ in AdS$_3$ from equation $(4.12)$ are also the wavefunctions in our AdS-deficit
spacetime if we send

$$\Delta \to \Delta \sqrt{1 - 8G_N M}, \quad n \to n\sqrt{1 - 8G_N M}, \quad \ell \to \ell.$$  \hspace{1cm} (4.32)

In particular, this means that the eigenspectrum for a scalar field in this spacetime is

$$E_{n,\ell} = (\Delta + 2n)\sqrt{1 - 8G_N M} + \ell.$$  \hspace{1cm} (4.33)

An interesting feature of this equation is that as $8G_N M \to 1$ the spectrum of twists, labeled by $n$, becomes more and more closely spaced, until we obtain a dense spectrum at $8G_N M = 1$, the BTZ black hole threshold. In section 4.4.1 we will derive this result in CFT$_2$ with large central charge in the large $\ell$ limit, without making reference to AdS$_3$.

It is also useful to consider an expansion in the limit that $G_N M \ll 1$. The anomalous dimension obtained from this limit should be doubled in order to incorporate the effect of the second particle on the first, but it is multiplied by a factor of $1/2$ once we account for the shift in the gravitational action of the deficit angle solution, as discussed in section 4.2.1. Thus we have a prediction that in the limit

$$1, n \ll \Delta_1, \Delta_2 \ll c,$$

we should find an anomalous dimension

$$\gamma(n, \ell) \approx -\frac{6}{c} \Delta_1 \Delta_2$$  \hspace{1cm} (4.34)

for the shift in dimension of large $\ell$ operators that dominate the OPE of $\mathcal{O}_1$ and $\mathcal{O}_2$, where we have identified $c = \frac{3}{2G_N}$ for the case [226] of AdS$_3$/CFT$_2$.  

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4.2.3 Quasi-Normal Mode Spectrum from Super-Planckian Objects

In AdS\(_3\) there exist the well-known BTZ black hole solutions. As our last example we will be interested in the quantum mechanical spectrum associated with a sub-Planckian object moving with large angular momentum around a 2 + 1 dimensional black hole. We can approach this question by studying the scalar wave equation in the BTZ background.

The BTZ metric for a spinless, uncharged black hole is

\[
\begin{align*}
\text{ds}^2 &= (r^2 - r_+^2) dt^2 - \frac{dr^2}{r^2 - r_+^2} - r^2 d\phi^2.
\end{align*}
\] (4.35)

The black hole has a horizon at the coordinate \(r = r_+\). Unlike in the case of higher dimensional AdS black holes, there are no timelike geodesics in this spacetime that avoid entering the black hole horizon. This is easy to see from the metric of equation (4.21), which naturally accords with the BTZ metric when \(d = 2\). Timelike geodesics in this metric can be characterized by a radial equation

\[
\begin{align*}
\dot{r}^2 &= E^2 - V(r) \quad \text{where} \quad V(r) = \left(1 - \frac{\mu}{r^{d-2}} + r^2\right) \left(1 + \frac{\ell^2}{r^2}\right)
\end{align*}
\] (4.36)

We see that when \(d = 2\), the effective potential \(V(r)\) is always monotonic in the presence of a BTZ black hole (when \(\mu > 1\)), whereas \(V(r)\) can have a stable minimum in \(d > 2\). So there are no classical orbits about the BTZ black hole, sharply differentiating the behavior in AdS\(_3\) from AdS\(_{\geq 4}\).
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The wave equation for a scalar with squared mass \( m^2 = \Delta(\Delta - 2) \) in the spinless BTZ background has solutions of the form

\[
\phi(t, r, \phi) = e^{-i\omega t + i\ell \phi} U_{\omega \ell}(r),
\]

(4.37)

where the normalizable radial wavefunction is

\[
U_{\omega \ell}(r) = \left( r^2 - r_+^2 \right) \frac{i\omega}{2r_+} \binom{\frac{i\ell + i\omega}{2r_+} + \frac{1}{2} \Delta, \frac{i\ell - i\omega}{2r_+} + \frac{1}{2} \Delta, \frac{r_+^2}{r^2}}{2} \binom{\frac{i\ell}{2r_+} + \frac{1}{2} \Delta, \Delta, \frac{r_+^2}{r^2}} \right). \quad (4.38)
\]

One can check that these solutions analytically continue to the pure AdS\(_3\) solutions of equation (4.12) if one takes \( r_+ \to i \).

The BTZ-background solutions differ in an important qualitative way from those for a scalar in empty AdS\(_3\). The BTZ solutions are oscillatory in \( \log r \), whereas the AdS\(_3\) solutions are exponentially suppressed as \( \log r \to -\infty \). As a consequence, even at very large \( \ell \), the BTZ solutions are not suppressed in the vicinity of the black hole horizon. This is the quantum mechanical reflection of the absence of stable orbits. This behavior sharply distinguishes the BTZ solutions from those in \( d \geq 3 \), as in the latter we can make the orbital lifetime as large as desired by taking the limit of large angular momentum.

This feature of the solutions has an immediate consequence for the quasi-normal mode frequencies \( \omega \). To determine these frequencies we need to impose some sort of boundary condition on the radial wavefunction, and quasi-normal modes are taken to be purely ingoing solutions at \( r_+ \), the black hole horizon. Imposing this boundary
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condition leads to

\[ \omega_{n,\ell} = \ell + i r_+ (\Delta + 2n), \] (4.39)

where \( \ell \) is the angular momentum, and \( n \) is another quantum number analogous to that which labels the twist in the pure AdS\(_3\) case. In fact this is just the analytic continuation of equation (4.33). We see that for all values of the angular momentum \( \ell \) the AdS\(_3\) energies have a constant, finite imaginary part. We therefore expect that after diagonalizing the CFT\(_2\) Dilatation operator we will obtain a dense spectrum of twists \( \tau \equiv \Delta - |\ell| \). This matches expectations from equation (4.33), which showed that as the mass of a deficit angle approaches the minimal BTZ mass, the spectrum of twists becomes more and more closely spaced. We also expect to reproduce the quasi-normal mode spectrum (4.39) after analytic continuation in radial time of the CFT correlators. We will prove both of these predictions using the CFT\(_2\) bootstrap in section 4.4.2.

The fact that there are no stable orbits around a BTZ black hole has the surprising consequence that one cannot make stable configurations of multiple deficit angle singularities orbiting each other, if their total mass is above the BTZ mass threshold. This is all the more surprising since above \( d = 2 \), we know how to make such states by spreading high-energy particles diffusely throughout space and giving them large angular momentum. To understand this phenomenon better, let us see qualitatively why such a state forms a black hole in AdS\(_3\). Consider the case of \( k \) identical particles each with large angular momentum \( \ell \), so that they are well-separated. The
Schwarzschild radius for a state with dimension $E$ is

$$r_+ = \sqrt{8G_N E - 1},$$  \hspace{1cm} (4.40)

On the other hand, each of the $k$ particles is localized at a radial coordinate $r \approx \sqrt{\frac{\ell}{\Delta}}$. The $k$-particle state has total dimension $E \approx k(\Delta + \ell)$, and so the condition for them to be outside their Schwarzschild radius is

$$r > r_+ \Rightarrow \frac{\Delta + \ell}{\Delta} > 8G_N k(\Delta + \ell),$$  \hspace{1cm} (4.41)

or equivalently, since $\Delta$ and $\ell$ are positive,

$$k\Delta < \frac{1}{8G_N}.$$  \hspace{1cm} (4.42)

The important point is that increasing $\ell$ does not help in satisfying this condition. Once the total rest mass $k\Delta$ of the particles increases beyond the BTZ mass threshold $1/8G_N$, black hole formation cannot be avoided by increasing the angular momentum, as was possible in higher dimensions.

### 4.3 Review of the Bootstrap Derivation for $d \geq 3$

Despite the various technical details we will discuss along the way, the argument presented here is conceptually quite simple. In any CFT, the correlation function of four scalar operators can be expressed in terms of a series of basis functions, known as conformal partial waves or conformal blocks, in a calculation directly analogous to the standard partial wave expansion of scattering amplitudes. This expansion can
be performed in any of three channels, yielding different expressions which must be identical. The equality of these expressions is referred to as the conformal bootstrap equation, which is a powerful tool used to constrain the structure of any CFT. The bootstrap was originally developed in the case of CFT$_2$ [196, 197], and has recently seen extensive analytic [51, 52, 202, 207, 228, 230] and numerical [51, 198, 231, 248] application in CFT$_{\geq 3}$.

We consider the bootstrap equation in a particular kinematic limit, the lightcone OPE limit, such that the left-hand side of the equation has a manifest singularity, as pictured in figure 4.6. This singularity simply arises from the disconnected correlator, and would correspond to free propagation in a scattering amplitude. The lightcone OPE singularity must be reproduced by the other side of the bootstrap equation, but this can arise only from an infinite sum of conformal blocks, with very specific scaling behavior. Since conformal blocks correspond to the exchange of definite states in the theory, this analysis has far-reaching implications for the structure of the Hilbert space and the spectrum of the Dilatation operator.

We will provide a brief review of the arguments in [51], which specifically studied correlators involving a single primary operator $\phi$. We will give the argument for the case of two distinct scalar primaries $\phi_1$ and $\phi_2$, but the core of the analysis will remain the same, such that interested readers may consult [51] for details and for a more rigorous proof.
Figure 4.6: This figure indicates the form that the bootstrap equation takes in the lightcone OPE limit where the conformal cross-ratio \( u \to 0 \). The first and dominant term on the left-hand side comes from the exchange of the 1 operator, and corresponds to ‘free propagation’ or 2-point Wick contraction. The other terms indicate the exchange of low-twist operators, such as the energy-momentum tensor.

4.3.1 Bootstrap Recap

In a CFT, the product of any two local operators can be rewritten using the operator product expansion (OPE), which is a sum over all primary operators in the theory,

\[
\phi_1(x)\phi_2(0) = \sum_\mathcal{O} \lambda_{\mathcal{O}} C_{\mathcal{O}}(x, \partial) \mathcal{O}(0). \tag{4.43}
\]

The function \( C_{\mathcal{O}}(x, \partial) \) corresponds to the contribution of all operators in the conformal multiplet associated with the primary operator \( \mathcal{O} \), and its structure is completely fixed by conformal invariance. The OPE coefficients \( \lambda_{\mathcal{O}} \) are theory-dependent and undetermined by conformal invariance. The OPE can be used within a four-point correlation function, rewriting the correlator as a sum over the exchange of irreducible representations of the conformal group. The contribution of each representation, associated with the primary operator \( \mathcal{O} \) of dimension \( \Delta \) and spin \( \ell \), is referred as the
conformal block $g_{\tau,\ell}(u, v)$, where $\tau = \Delta - \ell$ is the twist of $\mathcal{O}$ and the conformally-invariant cross-ratios $u$ and $v$ are defined as

$$ u = \left( \frac{x_{12}x_{34}}{x_{24}x_{13}} \right)^2, \quad v = \left( \frac{x_{14}x_{23}}{x_{24}x_{13}} \right)^2, \quad (4.44) $$

with $x_{ij} = x_i - x_j$. In terms of these conformal blocks, the four-point correlator takes the form

$$ \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \frac{1}{x_{12}^{\Delta_1 + \Delta_2} x_{34}^{\Delta_3 + \Delta_4}} \left( \frac{x_{24}}{x_{14}} \right)^{\Delta_{12}} \left( \frac{x_{14}}{x_{13}} \right)^{\Delta_{34}} \sum_{\tau, \ell} P_{\tau, \ell} g_{\tau, \ell}(u, v), \quad (4.45) $$

where $x_i$ is the position of $\phi_i$, $\Delta_{ij} = \Delta_i - \Delta_j$, and the conformal block coefficient $P_{\tau, \ell}$ is proportional to the product of OPE coefficients $\lambda_{12} \lambda_{34} \lambda_{\mathcal{O}}$.

In this expansion, we specifically took the OPE of the products $\phi_1 \phi_2$ and $\phi_3 \phi_4$. However, we could have instead taken the OPE of $\phi_1 \phi_4$ and $\phi_2 \phi_3$. The conformal bootstrap equation is simply the statement that these two different expansions, or channels, give the same correlator

$$ \frac{1}{x_{12}^{2\Delta_1} x_{34}^{2\Delta_2}} \sum_{\tau, \ell} P_{\tau, \ell} g_{\tau, \ell}(u, v) = \frac{1}{(x_{14}x_{23})^{\Delta_1 + \Delta_2}} \left( \frac{x_{13}x_{24}}{x_{12}^2} \right)^{\Delta_{12}} \sum_{\tau, \ell} P_{\tau, \ell} g_{\tau, \ell}(v, u), \quad (4.46) $$

where we have taken the first two operators to be $\phi_1$ with dimension $\Delta_1$, and the latter two to be $\phi_2$ with dimension $\Delta_2$, as this will be the case we examine below.

The bootstrap equation for $\langle \phi_1 \phi_1 \phi_2 \phi_2 \rangle$ provides a strong constraint on the spectrum and OPE coefficients of the CFT.
4.3.2 The Bootstrap in Generalized Free Theories

As a simple but far-reaching example of our bootstrap argument, we will consider four-point correlation functions in a generalized free theory (GFT), where all correlators are determined by 2-point Wick contractions. GFTs can also be defined as the dual correlators derived from free quantum field theories in AdS. In the case where we consider two different operators \( \phi_1 \) and \( \phi_2 \) we simply have

\[
\langle \phi_1(x_1)\phi_1(x_2)\phi_2(x_3)\phi_2(x_4) \rangle = \frac{1}{x_{12}^{2\Delta_1}x_{34}^{2\Delta_2}}.
\] (4.47)

We can also express this correlator as an expansion in conformal blocks. This calculation is trivial in the ‘s-channel’, as the only contribution in the series is from the identity,

\[
\frac{1}{x_{12}^{2\Delta_1}x_{34}^{2\Delta_2}} \sum_{\tau,\ell} P_{\tau,\ell}(11,22) g_{\tau,\ell}(11,22)(u,v) = \frac{1}{x_{12}^{2\Delta_1}x_{34}^{2\Delta_2}},
\] (4.48)

where the superscripts for \( P_{\tau,\ell} \) and \( g_{\tau,\ell} \) simply indicate that this channel corresponds to the OPE of \( \phi_1\phi_1 \) and \( \phi_2\phi_2 \). However, the expansion in the ‘t-channel’ takes a very different form, setting up the non-trivial equality of equation (4.46), which we can write as

\[
u^{-\frac{1}{2}(\Delta_1+\Delta_2)} = \nu^{-\frac{1}{2}(\Delta_1+\Delta_2)} u^{-\frac{1}{2}\Delta_{12}} \sum_{\tau,\ell} P_{\tau,\ell}(12,12) g_{\tau,\ell}(12,12)(v,u).
\] (4.49)

If we consider this expression in the limit \( u \ll v \ll 1 \), we see that the left side contains a very specific power-law singularity \( u^{-\frac{1}{2}(\Delta_1+\Delta_2)} \). This singularity must be reproduced by the right side, and our focus will be on precisely how it is reproduced.
Since we are considering a GFT, the only primary operators appearing in this conformal block expansion are the operators $[\phi_1\phi_2]_{n,\ell}$, which schematically take the form

$$[\phi_1\phi_2]_{n,\ell} \sim \phi_1 \partial^{2n} \partial_{\mu_1} \cdots \partial_{\mu_\ell} \phi_2,$$  \hspace{1cm} (4.50)

with fixed twist $\tau_n = \Delta_1 + \Delta_2 + 2n$. As discussed in [51], the corresponding conformal blocks $g_{\tau_n,\ell}(v, u)$ are known exactly and possess at most a logarithmic divergence in the limit $u \to 0$. For the bootstrap equation to be satisfied, the full sum over the t-channel conformal blocks must not converge uniformly in $u$ and $v$.

In order to understand this series, we need to study the conformal blocks at large $\ell$, in the limit $u \ll v \ll 1$. In fact, we need the specific limit $\ell \to \infty$ with $\ell \sqrt{u}$ fixed. As shown in appendix C, in this limit the conformal blocks at fixed $\tau$ take the approximate form

$$g_{\tau,\ell}(v, u) \approx 2^{\tau+2\ell} v^{\tau} u^{\Delta_1/2} \sqrt{\frac{\ell}{\pi}} K_{\Delta_1/2}(2\ell \sqrt{u}),$$  \hspace{1cm} (4.51)

where $K_{\nu}(y)$ is a modified Bessel function. We see that at small $v$ the lowest twist terms $(n = 0)$ will dominate. In addition, the universal prefactor of $u^{-\frac{1}{2}\Delta_1}$ in eq. (4.49) will cancel with a corresponding term from each conformal block, such that the only remaining $u$-dependence arises from the Bessel function.

Let us now consider the conformal block coefficients $P_{\tau_n,\ell}$, specifically for the minimal twist operators. As shown in [213], these coefficients can be calculated
precisely in a generalized free theory, and for \( n = 0 \) take the form

\[
P_{\tau_0,\ell} = \frac{(\Delta_1)\ell(\Delta_2)\ell}{\ell!(\Delta_1 + \Delta_2 + \ell - 1)\ell},
\]

where \((q)_x = \frac{\Gamma(q+x)}{\Gamma(q)}\) is the rising Pochhammer symbol. In the large \( \ell \) limit, these coefficients take the approximate form

\[
P_{\tau_0,\ell} \approx \frac{4\sqrt{\pi}}{\Gamma(\Delta_1)\Gamma(\Delta_2)}\frac{\ell^{\Delta_1+\Delta_2-\frac{3}{2}}}{2^{\tau_0+2\ell}}.
\]

Combining these results, the sum of large \( \ell \) conformal blocks can be approximated as

\[
v^{-\frac{1}{2}(\Delta_1+\Delta_2)}u^{-\frac{1}{2}\Delta_1+\Delta_2} \sum_{\tau_n,\ell} P_{\tau_n,\ell} g_{\tau_n,\ell}(v, u) \approx \frac{4}{\Gamma(\Delta_1)\Gamma(\Delta_2)} \sum_{\ell} \ell^{\Delta_1+\Delta_2-1} K_{\Delta_1\Delta_2}(2\ell\sqrt{u}).
\]

This sum over large \( \ell \) can be further approximated as an integral, which we can write as

\[
\sum_{\text{large } \ell} \ell^{\Delta_1+\Delta_2-1} K_{\Delta_1\Delta_2}(2\ell\sqrt{u}) \approx u^{-\frac{1}{2}(\Delta_1+\Delta_2)} \int d\ell \ell^{\Delta_1+\Delta_2-1} K_{\Delta_1\Delta_2}(2\ell),
\]

where we are specifically considering the limit of large \( \ell \) at fixed \( \ell\sqrt{u} \). As we can see, the large \( \ell \) conformal blocks perfectly replicate the \( u \to 0 \) behavior present on the left side of eq. (4.49).

The takeaway lesson from this discussion is that the full sum of large \( \ell \) conformal blocks contains a singularity in \( u \) that is not present in any individual term. This singularity was required by the bootstrap equation, and it is simply the result of a 2-point Wick contraction, also known as the exchange of the identity operator, or ‘free propagation’.
4.3.3 Lightcone OPE Limit and Cluster Decomposition

Let us now study the existence and properties of large $\ell$ operators in any CFT $\geq 3$. Separating the contribution of the identity operator, the bootstrap equation can be written as

$$u^{-\frac{1}{2}(\Delta_1 + \Delta_2)} \left( 1 + \sum_{\tau,\ell} P_{\tau,\ell} u^\tau f_{\tau,\ell}(u, v) \right) = v^{-\frac{1}{2}(\Delta_1 + \Delta_2)} \sum_{\tau,\ell} P_{\tau,\ell} v^\tau f_{\tau,\ell}(v, u), \quad (4.56)$$

where we have again suppressed the channel superscripts and rewritten the conformal blocks as $g_{\tau,\ell}(u, v) = u^\tau v^{\frac{1}{2}\Delta_1} f_{\tau,\ell}(u, v)$ to highlight their behavior at small $u, v$.

For $d \geq 3$, unitarity separates the twist of the identity from that of all other operators, placing the bounds

$$\tau \geq \begin{cases} \frac{d-2}{2} & (\ell = 0), \\ d - 2 & (\ell \geq 1). \end{cases} \quad (4.57)$$

With these bounds in mind, we can see that the identity provides the dominant contribution to the left side of eq. (4.56) in the limit $u \rightarrow 0$. In fact, in this limit the left side of the bootstrap equation for any CFT is approximately the same as in GFT. Our arguments will again hinge on the simple statement that the right side must reproduce this contribution from the identity in the limit $u \ll v \ll 1$. This statement can be written as the approximate constraint

$$1 \approx \left( \frac{u}{v} \right)^{\frac{1}{2}(\Delta_1 + \Delta_2)} \sum_{\tau,\ell} P_{\tau,\ell} v^\tau f_{\tau,\ell}(v, u) \quad (u \rightarrow 0), \quad (4.58)$$

where we have suppressed the superscripts on $P_{\tau,\ell}$ and $f_{\tau,\ell}(v, u)$, as we will only consider conformal blocks in the t-channel for the remainder of this discussion.
We can clearly see that the $u,v$-dependence of the right side of eq. (4.58) must vanish in the appropriate limit. Just as in the case of GFT, the $u$-dependence cannot be reproduced by any individual conformal block, so it must come from the full infinite sum. We again need to consider the large $\ell$ portion of this expression, as demonstrated in [51].

As discussed in appendix C, the large $\ell$ conformal blocks in the limit $u \ll 1$ on the right-hand side of equation (4.58) can be approximated as

$$g_{\tau,\ell}(v, u) \approx v^\tau k'_{2\ell}(1 - z) F^{(d)}(\tau, v),$$

(4.59)

where $k'_{2\ell}(x) = x^\beta F_1(\beta - \frac{1}{2}\Delta_{12}, \beta - \frac{1}{2}\Delta_{12}; 2\beta; x)$, $z$ is defined by $u = z\bar{z}, v = (1 - z)(1 - \bar{z})$, and the $d$-dependent function $F^{(d)}(\tau, v)$ is positive and analytic near $v = 0$, though its exact form will be unimportant for our discussion. Note that the limit $z \to 0$ at fixed $\bar{z}$ is equivalent to $u \to 0$ at fixed $v$. For the remainder of this section, we will be using $z$ rather than $u$, as this greatly simplifies the discussion.

Note that in this limit the $z,\ell$-dependence of the conformal blocks factorizes from the $v,\tau$-dependence, such that we may consider the cancellation of each piece separately. Since the conformal blocks are completely theory-independent, the function $k'_{2\ell}(1 - z)$ takes the same approximate form as in GFT. The total sum over $\ell$ must then produce the divergence of $z^{-\frac{1}{2}(\Delta_1 + \Delta_2)}$ necessary to cancel the prefactor in eq. (4.58).

What about the $v$-dependence? As mentioned above, the function $F^{(d)}(\tau, v)$ approaches a finite positive value as $v \to 0$. In this limit, the $v$-dependence of each
large $\ell$ conformal block is approximately $v^\tau$. Since this dependence must cancel the prefactor of $v^{-\frac{1}{2}(\Delta_1+\Delta_2)}$, we can obtain a bound on the possible twists that dominate in the large $\ell$ sum. While this is already a powerful restriction, we can make a much stronger statement. In order to reproduce the left side of eq. (4.58), there must be a contribution from an infinite number of operators of increasing spin with $\tau \to \Delta_1 + \Delta_2$ as $\ell \to \infty$. The constraint on the twists comes from the need to cancel the $v$-dependence, while the requirement for an infinite tower of these operators comes from the need to cancel the $z$-dependence.

We can actually take this argument one step further. Consider the conformal block associated with any primary operator in this infinite tower of operators with $\tau \approx \Delta_1 + \Delta_2$. We can then expand this conformal block as a series in $v$,

$$g_{\tau,\ell}(v, u) \approx v^{\frac{\tau}{2}k_{2\ell}}(1-z)F^{(d)}(\tau, 0) + O(v^{\frac{\tau}{2}+1}). \quad (4.60)$$

However, to obtain eq. (4.58) we only had to take the $z \to 0$ limit, which means that this equality must hold to all orders in $v$. There must then be another conformal block which cancels the $O(v^{\frac{\tau}{2}+1})$ term. More specifically, this additional conformal block must correspond to an operator with twist $\tau' = \tau + 2 \approx \Delta_1 + \Delta_2 + 2$. We can continue this process at every level in this power series, each time requiring the existence of a new operator with twist $\tau_n \approx \Delta_1 + \Delta_2 + 2n$ to cancel the other $O(v^{\frac{\tau}{2}+n})$ terms.

This argument applies to every operator in our infinite tower at $\tau \approx \Delta_1 + \Delta_2$. This tells us that, for each non-negative integer $n$, the large $\ell$ spectrum of any CFT must
include an infinite tower of operators with twists approaching \( \tau_n = \Delta_1 + \Delta_2 + 2n \). We refer the reader to [51] for a mathematically rigorous version of these arguments.

4.3.4 Anomalous Dimensions and Long-Range Forces in AdS

In this section we will explain how subleading corrections to the \( u \to 0 \) lightcone OPE limit of the bootstrap equation make it possible to constrain the anomalous dimensions and OPE coefficients of the operators \( [\phi_1 \phi_2]_{n,\ell} \). This means that we can use the bootstrap to derive the effects of long-range forces in AdS\( \geq 4 \). The universal exchange of \( T_{\mu\nu} \) in the bootstrap leads to the universal long-range gravitational potential in AdS.

So far we have considered only the dominant s-channel behavior due to the identity. However, we can extend our argument by considering the subleading contributions of conformal blocks associated with the CFT’s minimal nonzero twist \( \tau_m \). In the limit of small \( u \), these minimal twist operators provide a correction to the left side of the bootstrap equation,

\[
1 + \sum_{m=0}^{2} P_m u^{\frac{r_m}{2}} f_{m,\ell_m}(u, v) \approx \left( \frac{u}{v} \right)^{\frac{1}{2}(\Delta_1 + \Delta_2 - \frac{\ell}{2})} \sum_{\tau,\ell} P_{\tau,\ell} v^{\tau} f_{\tau,\ell}(v, u) \quad (u \to 0). \tag{4.61}
\]

We have limited this sum to \( \ell_m \leq 2 \), because higher spin operators either possess twist greater than that of the energy-momentum tensor or couple \([216, 249]\) as in a free field theory.

However, it is worth emphasizing that the \( u \to 0 \) contribution of all operators on the left-hand side with \( \tau < \Delta_1 + \Delta_2 \) must be matched by the large \( \ell \) sum on the right-
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hand side. This follows because these operators on the left-hand side of the bootstrap equation create a power-law singularity in $u$, while any finite sum of conformal blocks on the right-hand side can only produce a logarithmic singularity in $u$. This means that one could use the bootstrap to compute the OPE coefficients and dimensions of the $[\phi_1\phi_2]_{n,\ell}$ operators contributing on the right-hand side to $\mathcal{O}\left(\frac{1}{r^{\Delta_1+\Delta_2}}\right)$ in the large $\ell$ limit.

In the limit $u \ll 1$, the minimal twist conformal blocks can be written as

$$g_{\tau_m,\ell}(u, v) \approx u^{\tau_m} (1 - v)^{\ell_m} {}_2F_1\left(\frac{\tau_m}{2} + \ell_m, \frac{\tau_m}{2} + \ell_m; \tau_m + 2\ell_m; 1 - v\right). \quad (4.62)$$

As we can see, these blocks factorize into a $u$-dependent piece, with simple scaling behavior, and a $v$-dependent piece, which can be expanded in a power series at small $v$ by using the relation

$$_2F_1(\beta, \beta; 2\beta; 1 - v) = \frac{\Gamma(2\beta)}{\Gamma^2(\beta)} \sum_{n=0}^{\infty} \left(\frac{\beta}{n!}\right)^2 v^n \left(2(\psi(n + 1) - \psi(\beta)) - \ln v\right), \quad (4.63)$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. The precise form of this expansion is largely irrelevant to our discussion. All that matters to us is the presence of logarithmic terms of the form $v^n \ln v$. Since eq. (4.61) is true to all orders in $v$, these terms must again be replicated by the t-channel conformal blocks.

To see how these logarithmic terms are reproduced by the right side of equation (4.61), we shall consider the situation where one of the special twist values $\tau_n = \Delta_1 + \Delta_2 + 2n$ is approached by a single tower of operators $\mathcal{O}_{\tau_n,\ell}$ which at large $\ell$ are separated by a twist gap from all other operators in the spectrum. For simplicity,
we will specifically consider the case where there is one operator accumulating near $	au_n$ for each $\ell$, and the corresponding conformal block coefficients approach those of GFT. However, this approach can be generalized to more complicated scenarios [51].

Generically, the twists $\tau(n, \ell)$ for this tower of operators will not be precisely $\tau_n$. Instead, they will be shifted by some anomalous dimension $\gamma(n, \ell) = \tau(n, \ell) - (\Delta_1 + \Delta_2 + 2n)$. For sufficiently large $\ell$, we can expand the associated conformal blocks in terms of the anomalous dimension to obtain the approximate form

$$g_{\tau_n + \gamma, \ell}(v, u) \approx v^{\frac{z}{2}} \left(1 + \frac{\gamma(n, \ell)}{2} \ln v\right) k'_2(1-z) F^{(d)}(\tau_n, v).$$  

(4.64)

We see that the logarithmic terms due to minimal twist operators in the s-channel are replicated by the anomalous dimensions of large $\ell$ operators in the t-channel. By matching both sides of the bootstrap equation, we can then constrain the form of $\gamma(n, \ell)$.

While it is clear that we can match the $v$-dependence of both sides, we still need to consider the $z$-dependence. As we can see in eq. (4.61), the right side must not only cancel the original factor of $z^{\frac{1}{2}(\Delta_1+\Delta_2)}$, it must produce an additional factor of $z^{\frac{z_m}{2}}$. Just like in GFT, we simply need to consider the contribution of conformal blocks at large $\ell$. Focusing on only the relevant terms, we need the approximate relationship

$$z^{\frac{z_m}{2} - \frac{1}{2}(\Delta_1+\Delta_2)} \sim \sum_{\text{large } \ell} \gamma(n, \ell) \ell^{\Delta_1+\Delta_2-1}K_{\Delta_{12}}(2\ell \sqrt{z}).$$  

(4.65)

Since we are considering the $\ell \to \infty$ limit, we can approximate the anomalous dimension with its leading $\ell$ dependence $\gamma(n, \ell) \approx \gamma_n \ell^a$, such that we obtain
\[
\sum_{\text{large } \ell} \gamma_n \ell^{a+\Delta_1+\Delta_2-1} K_{\Delta_12}(2\ell \sqrt{z}) \approx \gamma_n z^{-\frac{1}{2}(a+\Delta_1+\Delta_2)} \int d\ell \ell^{a+\Delta_1+\Delta_2-1} K_{\Delta_12}(2\ell). \quad (4.66)
\]

Matching this to the left side of the bootstrap, we see that \( a = -\tau_m \), such that the anomalous dimension takes the asymptotic form

\[
\gamma(n, \ell) \approx \frac{\gamma_n}{\ell^{\tau_m}} \quad (\ell \to \infty),
\]

where the \( \ell \)-independent coefficient \( \gamma_n \) can be determined by carefully matching the \( v^n \ln v \) terms on both sides.

As a simple example, consider the case of stress-energy tensor exchange in \( d = 4 \), which has \( \tau_m = \ell_m = 2 \) and \( P_m = \frac{\Delta_1 \Delta_2}{360c} \). Matching all terms proportional to \( \ln v \), we then obtain the approximate relation

\[
-\frac{u\Delta_1 \Delta_2}{6c} \left( 1 + 4v + v^2 \right) \approx \frac{u^{\Delta_2}}{v^{\frac{1}{2}(\Delta_1+\Delta_2)}} \sum_{n,\ell} P_{\tau_n,\ell} v^{\frac{\tau_n}{2} k_2 \ell} (1-z) F^{(4)}(\tau_n, v). \quad (4.68)
\]

Note that the conformal block coefficients \( P_{\tau_n,\ell} \) are approximately those of GFT, which in the limit \( \Delta_1, n \ll \Delta_2 \ll \ell \) take the form

\[
P_{\tau_n,\ell} \approx \frac{(\Delta_1)_n}{n! 2^{\Delta_2}} P_{\tau_n}. \quad (4.69)
\]

As every term is proportional to \( P_{\tau_n,\ell} \), we can evaluate the sum over \( \ell \) to cancel the \( z \)-dependence of both sides, yielding the relation

\[
-\frac{\Delta_1(\Delta_1-1)\Delta_2^2}{6c} \left( 1 + 4v + v^2 \right) \approx (1-v)^{\Delta_2} \sum_n \frac{(\Delta_1)_n}{n! 2^{\tau_n}} \gamma_n v^n F^{(4)}(\tau_n, v). \quad (4.70)
\]
In this particular limit, we can also apply the results of appendix \[251\] to the \(d = 4\) conformal blocks derived in \[251\] to obtain the approximation

\[
F^{(4)}(\tau_n, v) \approx 2\tau_n (1 - v)^{\Delta_{12}^{-1}}. \tag{4.71}
\]

Using this result, we then have the simplified expression

\[
- \frac{\Delta_1(\Delta_1 - 1)\Delta_2^2}{6c} \left( \frac{1 + 4v + v^2}{(1 - v)^{\Delta_1+1}} \right) \approx \sum_n \frac{(\Delta_1)_n}{n!} \gamma_n v^n. \tag{4.72}
\]

If we expand the left side as a series in \(v\), we can then match corresponding terms from the two series to determine the anomalous dimension coefficients \(\gamma_n\). For the terms with \(n \ll \Delta_1\), this takes the simple form \(\gamma_n \approx -\frac{(\Delta_1\Delta_2)^2}{6c}\), which matches precisely with the AdS gravity computation that produced equation \(4.30\) in the \(\Delta_1, \Delta_2 \gg 1\) limit.

This has a nice physical interpretation in terms of the picture of section \(4.2.1\): when \(n \ll \Delta_1, \Delta_2 \ll \ell\) the variation of \(n\) does not significantly alter the distance between objects 1 and 2 in AdS, and so \(\gamma_n\), which corresponds to the gravitational binding energy, is independent of \(n\).

This same approach can be applied to theories with an arbitrary number of minimal twist primary operators. For example, the general \(n = 0\) anomalous dimension coefficient is

\[
\gamma_0 \approx -\frac{2\Gamma(\Delta_1)\Gamma(\Delta_2)}{\Gamma(\Delta_1 - \frac{\tau_m}{2})\Gamma(\Delta_2 - \frac{\tau_m}{2})} \sum_{\ell_m} P_m \frac{\Gamma(\tau_m + 2\ell_m)}{\Gamma^2(\frac{\tau_m}{2} + \ell_m)}. \tag{4.73}
\]

Furthermore, as noted above, we could in principle use the existence of the singularity \(u^{\frac{1}{2}(\tau - \Delta_1 - \Delta_2)}\) on the left-hand side of equation \(4.61\) to match the large \(\ell\)
anomalous dimensions and OPE coefficients to $O\left(\frac{1}{2\Delta_1+\Delta_2}\right)$ on the right-hand side. For large values of $\Delta_1$ or $\Delta_2$ this could be extremely powerful.

### 4.4 Virasoro Blocks and the Lightcone OPE Limit

We would like to generalize the bootstrap arguments from the previous section to the case of CFTs in $d=2$, which possess an infinite-dimensional Virasoro symmetry. For $d \geq 3$, our argument relied on the fact that once a CFT correlator in the OPE limit is decomposed into conformal blocks, it can then be expanded in increasing powers of $u$, beginning with the identity contribution,

$$
\langle \phi_1 \phi_1 \phi_2 \phi_2 \rangle = u^{-\frac{1}{2}(\Delta_1+\Delta_2)} + \sum_{\tau, \ell} P_{\tau, \ell} u^{\frac{1}{2}(\tau-\Delta_1-\Delta_2)} f_{\tau, \ell}(u,v). \quad (4.74)
$$

Two features were crucial for the analysis – firstly that $\tau \geq \frac{d}{2} - 1 > 0$, so that the identity was clearly separated from the contributions of other operators, and secondly, that there were only a finite number of conformal block contributions at the minimum twist $\tau_m > 0$. Neither of these properties holds in the case of 2d CFTs. So it is not surprising that many 2d CFTs, including the 2d Ising model \[51\], violate the conclusions of the theorem we proved for $d \geq 3$. In fact we saw in sections 4.2.2 and 4.2.3 that explicit AdS$_3$ calculations provide different expectations for the large spin spectrum in CFT$_2$.

We will overcome the aforementioned hurdles by computing the Virasoro conformal blocks in various semi-classical limits and then using them in a more general
Figure 4.7: This figure suggests how the exchange of all descendants of the identity operator in the Virasoro algebra corresponds to the exchange of all multi-graviton states in $\text{AdS}_3$. This is sufficient to build the full, non-perturbative $\text{AdS}_3$ gravitational field entirely from the CFT$_2$.

lightcone OPE bootstrap analysis. Due to the technical nature of the computation of the blocks themselves, we have confined these calculations to the appendices, with the general method described in appendix E and the specific computations in appendix F and G. We also provide a more straightforward brute force computation in a more restricted limit in appendix D. With the blocks in hand, the bootstrap analysis proceeds along the same line of reasoning that we saw in section 4.3, although with qualitatively different conclusions.

Let us now briefly discuss the bootstrap equation in CFT$_2$. In $d = 2$ we can make use of holomorphic factorization to discuss operators of general spin; nevertheless we will mostly discuss scalar external operators for simplicity and uniformity with section 4.3. Correlators of local operators in CFT$_2$ can be expanded in Virasoro conformal blocks corresponding to the exchange of irreducible representations of the Virasoro group. Each of these Virasoro blocks (or Virasoro partial waves) is associated with
a primary operator $O_{h,\bar{h}}$ of scaling dimension $\Delta = h + \bar{h}$ and spin $\ell = |h - \bar{h}|$. The Virasoro block decomposition of a four-point correlation function takes a very similar form to the conformal block expansion in $d \geq 3$, namely

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \frac{1}{x_{12}^{\Delta_1 + \Delta_2} x_{34}^{\Delta_3 + \Delta_4}} \left( \frac{x_{24}}{x_{14}} \right)^{\Delta_{12}} \left( \frac{x_{14}}{x_{13}} \right)^{\Delta_{34}} \sum_{h,\bar{h}} P_{h,\bar{h}} V_{h,\bar{h}}(u, v), \quad (4.75)$$

where $P_{h,\bar{h}}$ is the set of theory-dependent Virasoro block coefficients and $V_{h,\bar{h}}$ are the Virasoro blocks. The bootstrap equation can then be written in terms of Virasoro blocks as

$$\frac{1}{x_{12}^{2\Delta_1} x_{34}^{2\Delta_4}} \sum_{h,\bar{h}} P_{h,\bar{h}} V_{h,\bar{h}}(u, v) = \frac{1}{(x_{14}x_{23})^{\Delta_1 + \Delta_2}} \left( \frac{x_{13}x_{24}}{x_{12}^2} \right)^{\Delta_{12}} \sum_{h,\bar{h}} P_{h,\bar{h}} V_{h,\bar{h}}(v, u), \quad (4.76)$$

where we are specifically considering the Virasoro block decomposition of a correlator with only two independent scalar operators $\phi_1, \phi_2$. We have written the bootstrap equation in terms of $x_i$ and the cross-ratios $u$ and $v$ to make contact with section 4.3, but it is often more natural to use variables $z$ and $\bar{z}$, with $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$, since the full two-dimensional conformal group breaks up into holomorphic and anti-holomorphic Virasoro algebras.

### 4.4.1 AdS$_3$ Deficit Angles from Semi-Classical Virasoro Blocks

Factoring out the contribution due to the identity operator, we can rewrite the CFT$_2$ bootstrap equation as

$$V_{0,0}(u, v) + \sum_{h,\bar{h}} P_{h,\bar{h}} V_{h,\bar{h}}(u, v) = \left( \frac{u}{v} \right)^{\frac{1}{2}(\Delta_1 + \Delta_2)} u^{-\frac{1}{2}\Delta_{12}} \sum_{h,\bar{h}} P_{h,\bar{h}} V_{h,\bar{h}}(v, u). \quad (4.77)$$
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We can clearly see the first difference between 2d CFTs and those in higher dimensions. In our previous discussion, the contribution of the identity operator was simple, with no additional $u, v$-dependence. More concretely, there was no extended ‘conformal block’ associated with the identity, but only a single, trivial operator. This is not the case in 2d, as we now have contributions from all of the descendants of the vacuum. In terms of the global conformal symmetry, these descendants are simply the states that we obtain by acting with the stress-energy tensor on the CFT\textsubscript{2} vacuum.

As in section 4.3, we are specifically interested in studying eq. (4.77) in the light-cone OPE limit $u \to 0$. In order to make their small $u$ behavior manifest, the Virasoro blocks can be rewritten as

$$V_{h,\bar{h}}(u, v) = u^{\frac{\tau}{2}} F_{h,\bar{h}}(u, v). \tag{4.78}$$

where $F$ is analytic at small $u$. The left side of the bootstrap equation will be dominated by operators with zero twist. However, unitarity no longer forbids additional operators with $\tau = 0$. The stress-energy tensor is an example, but it has already been included in the Virasoro identity block. Any other local primary operator with $h = 0$ or $\bar{h} = 0$, and therefore of zero twist, will be a conserved current. We will limit our discussion to theories with no additional continuous global symmetries, such that the only zero twist operators are contained within the identity Virasoro block. In the small $u$ limit, eq. (4.77) can be written as

$$V_{0,0}(u, v) \approx \left(\frac{v}{u}\right)^{\frac{1}{2}(\Delta_1 + \Delta_2)} u^{-\frac{1}{2}\Delta_{12}} \sum_{\tau, \ell} P_{\tau, \ell} g_{\tau, \ell}(v, u) \quad (u \to 0). \tag{4.79}$$
We have chosen to explicitly write the t-channel or right-hand side in terms of global conformal blocks. We discuss the limitations of this approximation below, when it becomes relevant, with most calculations confined to appendix G.

We want to study the behavior of \( \mathcal{V}_{0,0} \) in the \( u \to 0 \) limit. Unlike global conformal blocks, there is no general closed-form expression for Virasoro blocks. However, as discussed in appendix F, the approximate structure of these blocks can be determined in the semi-classical limit where the CFT central charge \( c \to \infty \) with

\[
1 \ll \Delta_1 \ll c \quad \text{and} \quad \frac{\Delta_2}{c} \quad \text{fixed.} \tag{4.80}
\]

In the semi-classical limit, the \( u \to 0 \) form of the identity block is approximately

\[
\mathcal{V}_{0,0}(u,v) \approx \alpha^{\Delta_1} v^{-\frac{1}{2} \Delta_1 (1-\alpha)} \left( \frac{1-v}{1-v^{\alpha}} \right)^{\Delta_1}, \tag{4.81}
\]

where we have defined \( \alpha \equiv \sqrt{1-\frac{12 \Delta_2}{c}} \). We have assumed that \( \phi_i \) are scalar operators with \( \Delta_i = h_i + \bar{h}_i = 2h_i \), although it is easy to generalize to the case with \( h_i \neq \bar{h}_i \) using the results of appendix F and holomorphic factorization. The identity Virasoro block contains new \( v \)-dependence that arises due to the Virasoro descendants of the vacuum. However, we can also see that in this limit the left side of eq. (4.79) is completely independent of \( u \), which tells us that the right side must also have no \( u \)-dependence.

As in higher dimensions, it is impossible for any one conformal block to cancel the \( u \)-dependent prefactor. One might wonder if this remains true in CFT\(_2\), where the Virasoro blocks replace the simpler global blocks. We argue in appendix G that
it does. Specifically, at both $\ell \gg \Delta_2, c$ and for $\ell \ll \Delta_2, c$ we find that the individual t-channel Virasoro blocks $\mathcal{V}(v, u)$ do not contain a sufficiently strong singularity as $u \to 0$ to reproduce the identity block in the t-channel. Furthermore, there is a natural interpolation between the large and small $\ell$ behavior. So although our approximations do not allow for a rigorous proof, we expect that there must be an infinite sum of large $\ell$ Virasoro blocks on the right-hand side of equation (4.77) to reproduce the singularity as $u \to 0$.

Now let us study the bootstrap equation in the limit $u \ll v \ll 1$, using the global blocks on the right-hand side so that we can write

$$1 \approx \alpha^{-\Delta_1} u^{\frac{1}{2}(\Delta_1+\Delta_2)} v^{-\frac{1}{2}(\alpha \Delta_1+\Delta_2)} u^{-\frac{1}{2} \Delta_{12}} \sum_{\tau, \ell} P_{\tau, \ell} g_{\tau, \ell}(v, u).$$  \hspace{1cm} (4.82)

We are specifically interested in the large $\ell$ conformal blocks, which in this limit take the same approximate form as in higher dimensions,

$$g_{\tau, \ell}(v, u) \approx 2^{\tau+2\ell} u^{\frac{1}{2} \Delta_{12}} \frac{\sqrt{\ell}}{\pi K_{\Delta_{12}}(2\ell \sqrt{u})}. \hspace{1cm} (4.83)$$

We can easily see that this discussion will be very similar to our arguments from section 4.3. The overall prefactor of $u^{-\frac{1}{2} \Delta_{12}}$ will be cancelled by each individual block, but the necessary power of $u^{-\frac{1}{2}(\Delta_1+\Delta_2)}$ can only be produced by an infinite tower of large $\ell$ conformal blocks. In every 2d CFT with large $c$, there must then exist an infinite spectrum of large $\ell$ global conformal blocks, just as in higher dimensions.

However, things become much more interesting if we look at the $v$-dependence. In the small $v$ limit, the conformal blocks approximately scale as $v^{rac{\tau}{2}}$. Since this $v$-
dependence must cancel with the overall prefactor, we again obtain bounds on the possible twists which can dominate in the large $\ell$ sum. More importantly, there must be an infinite tower of large $\ell$ operators with twist $\tau \to \alpha \Delta_1 + \Delta_2$ as $\ell \to \infty$.

This behavior is very different from that of CFTs in higher dimensions. In the large $\ell$ limit, we would naively expect the spectrum to approach that of GFT, with $\tau \approx \Delta_1 + \Delta_2$. Phrased in terms of AdS, we would expect the binding energy of two particles to vanish in the long-distance limit. Instead, we see the presence of a universal ‘anomalous dimension’, or binding energy, which does not vanish in the large $\ell$ limit. As discussed in section 4.2.2, this is precisely the behavior we would associate with a deficit angle in AdS, with the corresponding energy shift

$$\Delta_1 \to \Delta_1 \sqrt{1 - \frac{12 \Delta_2}{c}} = \Delta_1 \sqrt{1 - 8 G_N M}, \quad (4.84)$$

where we have identified $\Delta_2 = M$ and $c = \frac{3}{2 G_N}$.

Let us now extend our argument by considering the bootstrap at arbitrary $v$. We will find it convenient to use $v$ and $z$ as variables, instead of $v$ and $u$ as above. As discussed in appendix C, conformal blocks with twist $\tau \sim \Delta_2 \gg \Delta_1$ can be approximated as

$$g_{\tau,\ell}(v, u) \approx 2^{\tau + 2\ell} v^{\frac{\tau}{2}} (1 - v)^{\frac{\Delta_{12}}{2}} u^{\frac{\Delta_{12}}{2}} \sqrt{\frac{\ell}{\pi}} K_{\Delta_{12}}(2\ell \sqrt{z}), \quad (4.85)$$

where we have made no assumptions about the size of $v$. Inserting this into the bootstrap equation and expanding the identity Virasoro block as a series in $v^\alpha$, we
can obtain the relation
\[
v_1^{\frac{1}{2}(\Delta_1+\Delta_2)} \sum_{n=0}^{\infty} \frac{(\Delta_1)_n}{n!} v^n \approx \alpha^{-\Delta_1} z^{\frac{1}{2}(\Delta_1+\Delta_2)} \sum_{\tau,\ell} P_{\tau,\ell} 2^{\tau+2\ell} v^\tau \sqrt{\frac{\ell}{\pi}} K_{\Delta_1}(2\ell v). \tag{4.86}
\]
In order for the \(v\)-dependence of both sides to match, there must be at least one primary operator with approximate twist
\[
\tau_n \approx \alpha (\Delta_1 + 2n) + \Delta_2, \tag{4.87}
\]
for every non-negative integer \(n\). In order for the \(z\)-dependence of both sides to also match, there must actually be an infinite tower of primary operators with increasing spin for every twist \(\tau_n\). Note that since these twists \(\tau_n\) have non-integer spacings, they must correspond to distinct Virasoro primaries.

We find that the large \(\ell\) spectrum of any CFT with large central charge matches that of the operators \([\phi_1 \phi_2]_{n,\ell}\) in a generalized free theory, but with the rescalings
\[
\Delta \rightarrow \Delta \sqrt{1 - 8G_N M}, \quad n \rightarrow n \sqrt{1 - 8G_N M}, \quad \ell \rightarrow \ell. \tag{4.88}
\]
As discussed in section 4.2.2, this is precisely the spectrum associated with a probe orbiting a deficit angle in AdS\(_3\). Using only the bootstrap equation for a 2d CFT, we have rediscovered the universal long-distance effect of gravity in AdS\(_3\).

### 4.4.2 BTZ Quasi-Normal Modes from Semi-Classical Virasoro Blocks

Now we will consider the spectrum of twists \(\tau_n\) in the case where one of the external operators is above the BTZ mass threshold, i.e. \(\Delta_2 > \frac{c}{12}\). In this case \(\alpha\) is imaginary,
so we will define $\beta \equiv \sqrt{\frac{12\Delta_2}{c} - 1} = -i\alpha$. For small $z$ with fixed $v$, the bootstrap equation now takes the form

$$\left(v^{-i\beta/2} - v^{i\beta/2}\right)^{-\Delta_1} \approx (i\beta)^{-\Delta_1} z^{\frac{1}{2}(\Delta_1+\Delta_2)} \sum_{\tau,\ell} P_{\tau,\ell} 2^{\tau+2\ell} v^\frac{1}{2}(\tau-\Delta_2) \sqrt{\frac{\ell}{\pi}} K_{\Delta_1}(2\ell\sqrt{z}).$$

(4.89)

At large spin, an infinite sum over spins is still necessary in order to cancel the prefactor of $z^{\frac{1}{2}(\Delta_1+\Delta_2)}$, and this completely constrains the large $\ell$ behavior of $P_{\tau,\ell}$. Thus, taking the $z \to 0$ limit, we can simplify to

$$\left(v^{-i\beta/2} - v^{i\beta/2}\right)^{-\Delta_1} \approx \int d\tau \bar{P}_\tau v^\frac{1}{2}(\tau-\Delta_2),$$

(4.90)

where $\bar{P}_\tau$ is the remaining $\ell$-independent piece of $P_{\tau,\ell}$, and we have replaced the sum on twists with an integral, without loss of generality. To constrain the spectrum of twists, we can take $v = e^{-s}$ and perform an inverse Laplace transform of each side, obtaining

$$\bar{P}_{\Delta_2+2\delta} = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{ds}{2\pi i} e^{s\delta} \frac{1}{\sin^{\Delta_1}(s\beta/2)}.$$  

(4.91)

where we have rewritten the twists as $\tau = \Delta_2 + 2\delta$. Taking $\gamma = \frac{\pi}{\beta}$ to avoid the poles in the denominator, this integral can be evaluated and one finds

$$\bar{P}_{\Delta_2+2\delta} = \frac{\Gamma(\frac{\Delta_1}{2} + i\frac{\delta}{\beta})\Gamma(\frac{\Delta_1}{2} - i\frac{\delta}{\beta})}{2\beta\Gamma(\Delta_1)}.$$  

(4.92)

This is regular for all real $\delta$, and thus indicates that there is a dense spectrum of twists. This result is consistent with the fact that the separation between twists in the deficit angle spectrum from the previous section approaches zero as $\Delta_2 \to \frac{c}{12}$, the BTZ threshold.
To connect to the spectrum of quasi-normal modes for the BTZ black hole, we want to look not for eigenstates of the Dilatation operator, but rather for asymptotic ‘in’ and ‘out’ states. We thus need to Wick rotate $v = e^{-s} \rightarrow e^{-is(1+i\epsilon)}$. One can then read off the spectrum from the poles of (4.92). In this case $v^{-i\beta} \ll v^{i\beta}$ at large $s$, so it is already manifest from a small $v^{-i\beta}$ expansion of (4.90) that the spectrum of twists is

$$\tau_n \approx i\beta(\Delta_1 + 2n) + \Delta_2 = 2\pi iT_{\text{BTZ}}(\Delta_1 + 2n) + \Delta_2,$$

(4.93)

which is just the natural analytic continuation of (4.87) to imaginary $\alpha = i\beta$. This reproduces the spectrum of BTZ quasi-normal modes, as in [217]. We emphasize that these are universal results for the large $\ell$ spectrum of any CFT$_2$ with large $c$ and no twist zero Virasoro primaries aside from the identity.

### 4.5 Discussion

What does it mean for a bulk gravitational spacetime to emerge holographically from a CFT?

One approach views the AdS geometry as an ever-present feature of a CFT state. The Ryu-Takayanagi formula [252, 253] exemplifies this viewpoint beautifully, as it associates bulk geometry with entanglement entropy in the CFT, even in the unperturbed vacuum. The disadvantage of this philosophy is its static nature, for it does not readily yield information about bulk dynamics, especially the locality of
interactions in AdS. The concept of a geometric distance between physical objects
is important only because local interactions fall off with distance; without locality
geometry loses much of its meaning.

In this work we have taken a complementary approach, interpreting bulk geometry
as a derivative idea, defining it purely in terms of the dynamics of localized objects.
From this point of view spacetime coordinates are simply a set of approximate, a
posteriori labels that can be consistently applied to operators or states as they evolve
with time. The S-Matrix program in flat spacetime and the reconstruction of AdS
effective field theory from CFT correlators exemplify this philosophy, and in both
cases we have a host of information about the necessary and sufficient conditions on
amplitudes for a local bulk theory. In this approach, one attempts to “hear the shape”
of the geometry by looking at the spectrum of its excitations – for example, in this
chapter we worked with energy and angular momentum eigenstates. These can be
translated and combined to form local wavepackets in AdS, which in turn can then
be used to probe the geometry in a more direct way. In many cases these states and
their local interpretation are already familiar, and our approach has the advantage of
connecting geometry to AdS locality in an essential way.

Applying the CFT bootstrap to 2d large central charge theories has allowed us
to derive general, non-perturbative results that are ripe for interpretation in terms of
AdS$_3$ dynamics. We saw that the exchange of the identity and its Virasoro descen-
dants, which can be interpreted in AdS$_3$ as multi-graviton states, creates an effect
identical to the presence of either a deficit angle or BTZ black hole background.

Virasoro primaries with dimension $h_\phi > \frac{c}{24}$ create a universal background in which “light” primaries with dimension $h_\phi \ll c$ have thermal correlators, as shown in equation (4.7). This can be viewed as a derivation of the Eigenstate Thermalization Hypothesis \cite{213, 214} for CFT\textsubscript{2} at large central charge, although it is important to keep in mind that it will receive corrections from $1/c$ effects and, away from the light-cone limit, from other conformal blocks. The corrections from other operators could cancel, since $\phi_1 \phi_1 \rightarrow \phi_2 \phi_2$ conformal block coefficients can have either sign, or alternatively the OPE coefficients of these operators might simply be small. The suppression of these corrections may be related to both eigenstate thermalization and ‘no hair’ theorems for black holes.

It would be interesting to investigate this approximate thermality for more general correlators of light primaries in future work, by studying Virasoro conformal blocks for $n$-point correlators \cite{254}. There should also be a nice confluence of the methods employed here with entanglement entropy methods: by taking the light operators to be “twist” operators at the edge of an interval, one might compute the entanglement entropy in the background of a “heavy” $h_\phi > \frac{c}{24}$ operator and reconstruct the corresponding bulk geometry by using the Ryu-Takayanagi formula. It is also interesting to note the connection to results on the general instability of excitations of AdS\textsubscript{3} with energy above the BTZ threshold \cite{255}. We have seen that the universal background created by any “heavy” operator with $h_\phi > \frac{c}{24}$ produces a spectrum of modes with
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an instability even at large distances. Our results are limited to the case where the excitations are in the “test mass” limit, meaning they do not back react on the geometry, but further results on the identity operator conformal block would allow one to generalize beyond this case.

Formally, the lightcone OPE limit gives reliable information only about the primaries of large angular momentum $\ell$, but in the case where there is a large gap in twists between the identity operator and the remaining primary operators in the theory, we have obtained results that accord with the BTZ geometry for all values of $\ell$. This indicates that in AdS$_3$, all states above the BTZ threshold look like black holes, up to corrections embodied by the exchange of higher twist operators. It suggests that up to the horizon, all black hole states look nearly identical, arguing against any proposal for quantum gravity that would lead to large non-local modifications of the dynamics outside the horizon.

It would be interesting to sharpen these claims for 2d CFTs with a small number of low twist $\tau \sim \mathcal{O}(1)$ primaries, and more generally to explore the corrections from primaries with twist $\tau > 0$ in the bootstrap equation. These contributions will have a sub-dominant impact on the large spin operators that we have identified, but they would make it possible to estimate the range of interaction length scales in AdS$_3$. In particular, one might try to control the behavior in the vicinity of a black hole. It is also possible to study CFT$_2$ with $\tau = 0$ operators besides the identity, namely conserved currents. In that case one would need to include the contributions of the
entire zero twist sector at once, including operators of higher spin, perhaps via a
generalization of the monodromy method. One could also study the identity block in
theories with a more general $\mathcal{W}_N$ algebra structure $^{256,257}$. It would be particularly
interesting to see if the $\mathcal{W}_N$ blocks can be interpreted as thermal correlators, since
it might shed light on whether the AdS duals of these theories have black hole-like
states $^{258,259}$.

The semi-classical approximation to the Virasoro identity block contains all the
information we need to reconstruct a dynamical AdS$_3$ geometry. However, it would be
fascinating to explore the corrections to thermality embedded in the (unknown) exact formula for the Virasoro blocks, by going beyond the semi-classical approximation
of the monodromy method. Formulas for the blocks based on other approximation
methods should be able to shed light on this question. In particular, the recursion
relations $^{260,262}$ for the OPE limit might be used directly, perhaps even numerically, or else they might be transformed $^{263}$ to the lightcone OPE limit. It seems
reasonable to expect that the AdS$_3$ interpretation we have uncovered will persist in
irrational CFTs with finite $c > 1$, so it would be interesting to examine finite central
charge Virasoro blocks in general, or simply in the lightcone OPE limit. As we have
argued, this limit by itself provides a great deal of information about the spectrum
of the CFT.
Appendix A

Ionization of Atoms by Dark Matter

Here we present a more detailed description of ionization by dark matter. Our interaction system consists of the DM ($\vec{x}_X$), electron ($\vec{x}_e$), and nucleus ($\vec{x}_N$). Since the electron and nucleus originally form a bound state, we can change coordinate systems to more clearly show that this is a 2- to 3-body scattering process (bound atom + DM $\rightarrow$ electron + nucleus + DM). These coordinates are similar to the relative and center-of-mass coordinates used in strictly two-body systems. First we define new coordinates for the atomic electron-nucleus system,

\begin{align*}
\vec{x}_a &= \vec{x}_e - \vec{x}_N, \\
\vec{x}_A &= \frac{m_e}{m_e + m_N} \vec{x}_e + \frac{m_N}{m_e + m_N} \vec{x}_N.
\end{align*}

(A.1)
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We can then use these to define new coordinates for the full DM-atom system,

\[ \vec{x}_r = \vec{x}_\chi - \vec{x}_A, \]
\[ \vec{x}_R = \frac{m_\chi}{m_\chi + m_A} \vec{x}_\chi + \frac{m_A}{m_\chi + m_A} \vec{x}_A. \]

These new coordinates make it simple to write the system’s initial state, as well as the relative velocity necessary for our scattering equation,

\[ \Psi_i(\vec{x}_a, \vec{x}_r, \vec{x}_R) = \psi(\vec{x}_a) e^{i\vec{p}_r e^{i\vec{p}_R}}, \]
\[ \vec{v}_{rel} = \frac{\vec{p}_r}{\mu_r} = \frac{\vec{p}_\chi}{m_\chi} - \frac{\vec{p}_A}{m_A}, \]
\[ (A.3) \]

while the final state can easily be written in either coordinate system

\[ \Psi_f(\vec{x}_a, \vec{x}_r, \vec{x}_R) = e^{i\vec{k}_a e^{i\vec{k}_r e^{i\vec{k}_R}}. \]
\[ (A.4) \]

Now we should be more careful and worry about normalizing these free state wavefunctions, but whatever normalization factors we use now will cancel out and not affect our final interaction cross-section, so we will ignore them. We can now use these coordinates to simplify the matrix element for the electron recoil, assuming the interaction Hamiltonian \( H \) is only a function of the relative displacement between the DM and electron

\[ |\langle f | H | i \rangle|^2 = (2\pi)^3 \delta^3(\vec{k}_f - \vec{p}_a) \left| \frac{\mu_a}{m_e} \right| \left| \frac{\vec{k}_a + \frac{\mu_a}{m_e} (\vec{k}_r - \vec{p}_r)}{\vec{H}(\vec{k}_r - \vec{p}_r)} \right|^2, \]
\[ (A.5) \]

where \( \vec{H}(\vec{p}) \) can be found by simply taking the nonrelativistic limit of the interaction amplitude \( \mathcal{M}(\vec{p}) \), with all spinors normalized to 1. The last assumptions we need to make to simplify our final cross-section formula is that the incoming momentum of
the nucleus is zero and \( m_e \ll m_N \). We can then integrate over the outgoing nuclear momentum and convert back to the more intuitive physical coordinates, resulting in

\[
d\sigma = \frac{1}{|\vec{v}_{\text{rel}}|} \left( \prod_f \frac{d^3k_f}{(2\pi)^3} \right) 2\pi \delta(E_f - E_i) |\langle f|H|i\rangle|^2
\]

\[
\approx \frac{1}{|\vec{v}_\chi|} \frac{d^3k_\chi}{(2\pi)^3} \frac{d^3k_e}{(2\pi)^3} 2\pi \delta(E_f - E_i) \left| \tilde{\psi}(\vec{p}_\chi - \vec{k}_\chi - \vec{k}_e) \right|^2 \left| \tilde{H}(\vec{p}_\chi - \vec{k}_\chi) \right|^2.
\]

In the limit \( m_N \to \infty \), the fixed nucleus breaks translation invariance, and therefore momentum conservation. Perhaps a more straightforward interpretation is that the nucleus can absorb any finite momentum at negligible energy cost, due to the \( m_N \) suppression in its kinetic energy. Either way, energy conservation is now the only constraint on the electron-DM system.

We can further simplify our final expression by taking the limit \( k_e \ll q \), where \( q \) is the magnitude of the momentum transfer \( (\vec{q} = \vec{p}_\chi - \vec{k}_\chi) \). We can check that this limit is valid by comparing the expressions for \( k_e \) and the minimum possible \( q \) for a given incoming \( v_\chi \),

\[
k_e = \sqrt{2m_eE_R},
\]

\[
q_{\text{min}} = m_\chi v_\chi - \sqrt{m_\chi^2v_\chi^2 - 2m_\chi(E_R + E_B)}.
\]

We then see that \( q_{\text{min}} \) can be further reduced either by increasing \( m_\chi \) or \( v_\chi \) or by decreasing \( E_R \) or \( E_B \). If we then consider the most extreme case, with \( m_\chi = 10 \text{ GeV} \), \( v_\chi = v_{\text{esc}} \), \( E_B = 1 \text{ eV} \), and \( E_R = 1 \text{ eV} \), we find that \( k_e \approx q_{\text{min}} \approx 1 \text{ keV} \). However, the density of states for such a small binding energy is negligible, especially in the case of silicon. Increasing the binding energy to values with higher detection efficiency will only increase the ratio of \( q_{\text{min}} \) to \( k_e \), as will decreasing \( m_\chi \) or \( v_\chi \). For the bulk
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of the relevant parameter space, then, \( k_e \) will be substantially smaller than even the minimum possible momentum transfer, making this approximation valid.

As a pedagogical example, we now choose a simple point-vertex interaction and use the ground-state hydrogen wavefunction for \( \psi \). After enforcing energy conservation and integrating over the trivial angles, we find the expression

\[
\frac{d\sigma}{dE_R} \approx \frac{16a^2k_em_e}{\Lambda^4\pi^2v_\chi} \int d\cos\theta_\chi (1 + a^2(p_\chi - k_\chi)^2)^{-4}.
\]

(A.8)

The naïve expectation would be for this integral to be \( \mathcal{O}(1) \) at low energy recoil (and therefore low momentum transfer), but it turns out that there is a large suppression for most of the range of integration. This can be understood as a phase space suppression for any angle that deviates away from forward scattering (or alternatively, for any momentum exchange \( q \) that deviates away from the minimum value \( q_{\text{min}} = p_\chi - k_\chi \)).

This means that our final expressions will be dominated by terms corresponding to \( q_{\text{min}} \). In this example, we obtain

\[
\frac{d\sigma}{dE_R} \approx \frac{8ak_em_e}{3\Lambda^4\pi^2v_\chi^2} \left[ (1 + a^2(p_\chi - k_\chi)^2)^{-3} - (1 + a^2(p_\chi + k_\chi)^2)^{-3} \right]
\]

(A.9)

The second apparent form of suppression comes in the factor of \( m_e \) in the numerator, rather than some form of atom-DM reduced mass. This can be understood by considering the limiting behavior as \( m_e \to \infty \). In this limit, the cross-section rapidly grows for recoil energies \( E_R \approx 0 \), and is heavily suppressed for all others, converging on a delta function centered at zero recoil energy (an ionized electron with no kinetic
energy). This limiting behavior should have been expected to appear, so it is not surprising to see the factor $k_e m_e$ in our final answer.

Applying this same approach to the models considered in chapter 2, we obtain the following full cross-sections

\[
\frac{d\sigma_{EDM}}{dE_R} = \frac{16a^2d^2k_e}{\pi v^2} \left[ \frac{6a^4(p_e + k_e)^4 + 15a^2(p_e + k_e)^2 + 11}{6(1 + a^2(p_e + k_e)^2)^3} - 2 \ln \left( \frac{p_e - k_e}{p_e + k_e} \right) \right]
\]

\[
+ \ln \left( \frac{1 + a^2(p_e - k_e)^2}{1 + a^2(p_e + k_e)^2} \right) - \frac{6a^4(p_e - k_e)^4 + 15a^2(p_e - k_e)^2 + 11}{6(1 + a^2(p_e - k_e)^2)^3} \right]
\]

\[
\frac{d\sigma_{MDM}}{dE_R} = \frac{64a^2a^2\mu^2 k_e}{3\pi v^2} \left[ \frac{6a^4(p_e + k_e)^4 + 15a^2(p_e + k_e)^2 + 11}{6(1 + a^2(p_e + k_e)^2)^3} - 2 \ln \left( \frac{p_e - k_e}{p_e + k_e} \right) \right]
\]

\[
+ \ln \left( \frac{1 + a^2(p_e - k_e)^2}{1 + a^2(p_e + k_e)^2} \right) - \frac{6a^4(p_e - k_e)^4 + 15a^2(p_e - k_e)^2 + 11}{6(1 + a^2(p_e - k_e)^2)^3} \right]
\]

\[
+ \frac{1}{4}(1 + a^2(p_e - k_e)^2)^{-3} - \frac{1}{4}(1 + a^2(p_e + k_e)^2)^{-3} \right]
\]

\[
\frac{d\sigma_{U(1)}}{dE_R} = \frac{512\lambda^2a^4k_e}{v^2(a^2m^2_A - 1)^4} \left[ \frac{1}{(a^2m^2_A - 1)^4} \ln \left( \frac{(1 + a^2(p_e - k_e)^2)(m^2_A + (p_e + k_e)^2)}{(1 + a^2(p_e + k_e)^2)(m^2_A + (p_e - k_e)^2)} \right) \right]
\]

\[
+ \frac{(a^2m^2_A - 1)^2 - 3(a^2m^2_A - 1)(1 + a^2(p_e - k_e)^2) + 9(1 + a^2(p_e - k_e)^2)^2}{12(1 + a^2(p_e - k_e)^2)^3}
\]

\[
- \frac{(a^2m^2_A - 1)^2 - 3(a^2m^2_A - 1)(1 + a^2(p_e + k_e)^2) + 9(1 + a^2(p_e + k_e)^2)^2}{12(1 + a^2(p_e + k_e)^2)^3}
\]

\[
+ \frac{1}{4a^2(m^2_A + (p_e - k_e)^2)} - \frac{1}{4a^2(m^2_A + (p_e + k_e)^2)} \right]
\]

If we then take the appropriate limits of these cross-sections, we obtain the approximate results given earlier.
Appendix B

Details of Boltzmann Equation

Code

In order to solve the Boltzmann equation to obtain the distribution functions and consequently \( \Delta g_i \), a numerical code was written in C++. The code evolves a universe forward in time subject to some initial boundary conditions. The inputs to the code are the masses \( m_i \), statistics \( \sigma_i \) and degrees of freedom of species \( g_i \) in the universe, as well as the initial temperature \( T_i \). Interactions between the various species in the SM sector are strong enough that it is safe to assume that the distribution functions are Fermi-Dirac or Bose-Einstein until down to temperatures well below their mass, as discussed in subsection 3.2.2. At that point, their number density has become low enough that their interactions to our new species have frozen out, and we work in an effectively radiation-dominated universe, so the error in the distribution
function does not affect our evolution. In order to work with $\mathcal{O}(1)$ numbers for the distribution functions which are decoupling, we track $v(p,t) \equiv v$ instead, defined implicitly through the equation $f(p,t) = \frac{1}{e^{v-\sigma}}$, where $\sigma$ is 1 for bosons and $-1$ for fermions. The code solves for the following quantities:

- $v_i(p,t)$ for all BSM species $i$
- $T_{SM} = T_\gamma \equiv T$ (as we work above the neutrino decoupling temperature)
- $\rho_i, P_i, n_i$ for all species $i$
- $H$ and $a$

The following equations are used to solve for the aforementioned quantities:

\begin{equation}
E \frac{\partial v}{\partial t} - H p E \frac{\partial v}{\partial p} = \frac{\partial v}{\partial f} C[f], \tag{B.1}
\end{equation}

where we discuss computation of $C[f]$ below,

\begin{equation}
\frac{\partial \rho_{tot}}{\partial t} = y_{SM} \frac{\partial T_{SM}}{\partial t} + \frac{\partial \rho_\chi}{\partial t} = -3H(\rho_{tot} + p_{tot}), \tag{B.2}
\end{equation}

where $y_{SM} = \sum_{i \subset SM} y_i$ and

\begin{equation}
y_i = \frac{g_i}{2\pi^2} \frac{1}{T^2} \int_0^\infty dp \ p^2 E^2 e^{v_i} f_i^2, \tag{B.3}
\end{equation}

\begin{equation}
H = \sqrt{\frac{8\pi G}{3}} \rho_{tot}, \tag{B.4}
\end{equation}

\begin{equation}
\frac{\partial a}{\partial t} = aH, \tag{B.5}
\end{equation}

\begin{equation}
\rho_i = \frac{g_i}{2\pi^2} \int_0^\infty dp \ p^2 Ef_i, \tag{B.6}
\end{equation}

\[171\]
APPENDIX B. DETAILS OF BOLTZMANN EQUATION CODE

\[ P_i = \frac{g_i}{2\pi^2} \int_0^\infty dp \frac{p^4}{3E} f_i, \quad (B.7) \]

\[ n_i = \frac{g_i}{2\pi^2} \int_0^\infty dp \ p^2 f_i, \quad (B.8) \]

\[ g_{*,i} = \frac{30\rho_i}{\pi^2 T^4}. \quad (B.9) \]

The various quantities are tracked over 2000 timesteps, spaced logarithmically. We begin at \( T = 200 \) MeV and typically end at around 1 MeV. As we only have earlier time information, time derivatives that cannot be computed analytically are typically computed by forming an interpolating polynomial to the previous four pieces of data and taking an exact derivative of the resulting polynomial. Furthermore, this technique is also used to obtain an estimate of the next value of the variable in question, in order to improve the accuracy of the code. We found that this technique is considerably more accurate at numerically evaluating derivatives than more elementary finite difference methods.

Our distribution function is evaluated on a grid of 100 momentum-steps, logarithmically spaced between 10 keV and 10 GeV. All derivatives with respect to momentum that cannot be computed analytically are computed with the interpolating polynomials method described previously.

The algorithm used is as follows:

- Set boundary conditions: Bose-Einstein or Fermi-Dirac distributions for every species with temperature \( T_i \), and \( a_i = 1 \)
- Compute all initial \( \rho_i, p_i, n_i, H, g_{*,i} \)
Appendix B. Details of Boltzmann Equation Code

- Compute the next SM temperature with eq. (B.2)

Iterate the following at timestep \( j \) until 2000 timesteps:

- Approximate \( H_j \) at the next timestep by extrapolating the previous values of \( H(t) \)

- Solve the Boltzmann equations of all BSM species \( i \) for \( v_{i,j} \), described more thoroughly below

- Compute all remaining undetermined parameters, as well as \( H(t_j) \)

- Compute the next SM temperature with eq. (B.2)

The Boltzmann eq. (B.1) was solved using a generalization of a predictor-corrector method. The objective was to simultaneously vary \( v_i \) at all points on the momentum grid, attempting to minimize the quantity

\[
\sum_{i \subset \text{BSM}} \sum_{k=1}^{100} \frac{1}{p_k} \left[ E_k \frac{\partial v_i(p_k)}{\partial t} - H p_k E_k \frac{\partial v_i(p_k)}{\partial p_k} - \frac{\partial v_i}{\partial f_i} \bigg|_{p_k} C(f_i(p_k)) \right]. \tag{B.10}
\]

Because the collisional integral is the most computationally intensive part of the algorithm, we attempted to minimize the number of calls of it. This was accomplished by primarily studying the effects of the variation of \( v_i \) on the left-hand side of the Boltzmann equation, as the right-hand side varied more slowly, only recomputing \( C \) when we had settled on a \( v \) that minimized the local error

\[
\left| E_k \frac{\partial v_i(p_k)}{\partial t} - H p_k E_k \frac{\partial v_i(p_k)}{\partial p_k} - \frac{\partial v_i}{\partial f_i} \bigg|_{p_k} C(f_i(p_k)) \right|, \tag{B.11}
\]
APPENDIX B. DETAILS OF BOLTZMANN EQUATION CODE

with a relative accuracy of \( \approx 10^{-6} \).

The collisional integral in eq. (3.5) has been computed by following the method devised by [108]. We briefly summarize the algorithm; see [108] for more details. Define the following quantities:

- The angle between \( \vec{p}_1 \) and \( \vec{p}_2 \) is \( \alpha \)
- The angle between \( \vec{p}_1 \) and \( \vec{p}_3 \) is \( \theta \)
- The azimuthal angle between \( \vec{p}_2 \) and \( \vec{p}_3 \) is \( \beta \)
- \( x = \cos \alpha \)
- \( z = \cos \theta \)
- \( Q = m_1^2 + m_2^2 + m_3^2 - m_4^2 \)

In the amplitude \( |M|^2 \), we plug in

\[
p_1 \cdot p_2 = E_1 E_2 - |\vec{p}_1||\vec{p}_2|x, \quad (B.12)
\]

\[
p_1 \cdot p_3 = E_1 E_3 - |\vec{p}_1||\vec{p}_3|z, \quad (B.13)
\]

\[
p_1 \cdot p_4 = m_1^2 + E_1 E_2 - |\vec{p}_1||\vec{p}_2|x - E_1 E_3 + |\vec{p}_1||\vec{p}_3|z, \quad (B.14)
\]

\[
p_2 \cdot p_3 = E_1 E_2 - |\vec{p}_1||\vec{p}_2|x - E_1 E_3 + |\vec{p}_1||\vec{p}_3|z + Q/2, \quad (B.15)
\]

\[
p_2 \cdot p_4 = E_1 E_3 - |\vec{p}_1||\vec{p}_3|z + m_2^2 - Q/2, \quad (B.16)
\]

\[
p_3 \cdot p_4 = E_1 E_2 - |\vec{p}_1||\vec{p}_2|x - m_3^2 + Q/2. \quad (B.17)
\]
APPENDIX B. DETAILS OF BOLTZMANN EQUATION CODE

We can change variables to $|\vec{p}_1|, |\vec{p}_2|, |\vec{p}_3|, \vec{p}_4$, $x, z, \beta$ and $\mu$, where $\mu$ is an integration variable parameterizing the $SO(2)$ rotational symmetry about $\vec{p}_1$. The collisional integral has no dependence on $\mu$, and it can therefore be done trivially. After using the momentum-conserving delta function to integrate $\vec{p}_4$, we can use the energy-conserving delta function to integrate $\beta$. Now that there are no more four-vectors in our expression, we switch notation $|\vec{p}_i| \to p_i$. After some algebra, it has been shown that $C$ can be written in the form

$$C(f(p_1)) = \int_0^\infty dp_2 \int_0^\infty dp_3 \frac{p_2^2 p_3^2 \Omega(f) F}{(2\pi)^5 16 E_2 E_3},$$

where $\Omega$ was defined before as $f_3 f_4 (1 + \sigma_1 f_1)(1 + \sigma_2 f_2) - f_1 f_2 (1 + \sigma_3 f_3)(1 + \sigma_4 f_4)$ and $F = F(p_1, p_2, p_3)$ is

$$F = \int_{-1}^1 dz \int_{x_-}^{x_+} dx \frac{|M|^2(x, z)}{\sqrt{a(z)x^2 + b(z)x + c(z)}} \Theta(A),$$

where

$$a(z) = (-4p_2^2(p_1^2 + p_3^2)) + (8p_2^2p_1p_3) z,$$

$$b(z) = (p_1p_2(8\gamma + 4Q)) + p_2p_3 (8p_1^2 - 8\gamma - 4Q) z + (-8p_1p_2p_3) z^2,$$

$$c(z) = (4p_2^2p_3^2 - 4\gamma^2 - 4\gamma Q - Q^2) + (-p_1p_3(8\gamma + 4Q)) z + (-4p_3^2(p_1^2 + p_2^2)) z^2,$$

$$\gamma = E_1 E_2 - E_3(E_1 + E_2),$$

$$x_\pm = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}.$$  \hspace{1cm} \text{(B.24)}

Note that since $a \leq 0$, we know that $x_+ \geq x_-$. If we define

$$z_\pm = \frac{1}{2p_1p_3} \left(-2\gamma - 2p_2^2 - Q \pm 2p_2\sqrt{2\gamma + p_1^2 + p_2^2 + p_3^2 + Q}\right),$$

\hspace{1cm} \text{(B.25)}
then $\Theta(A)$ is 1 when $2\gamma + p_1^2 + p_2^2 + p_3^2 + Q > 0$, $z_+ > -1$, and $z_- < 1$, and is 0 otherwise.

The amplitudes $|\mathcal{M}|^2$ were computed with the assistance of Tracer \cite{264}. After the substitutions above, $|\mathcal{M}|^2$ can be written as a rational function in $x$ and $z$. We first expand it in $x$, and integrate it analytically with Mathematica. Afterwards, if it is possible to analytically integrate with respect to $z$, we do so and store the results at all $10^6$ combinations $\{p_1, p_2, p_3\}$. Otherwise, we numerically integrate with respect to $z$ at all $10^6$ points in the resulting phase space. These are stored and then loaded into our C++ code.

The code has been verified to give the same answer for $T_{\text{eff},\nu}(p)$ as that given in \cite{106}, giving the same values for $\Delta g_*$ to the percent level. In addition, the results were computed and compared to all cases where it is possible to use the instantaneous decoupling approximation or otherwise solve the problem analytically, and agreement was again found to the percent level or better in all cases. Percent-level accuracy is more precise than the resolution of the Planck satellite, and so we do not quote theoretical errors throughout chapter \ref{chap:planck}.

In subsection \ref{sec:modified}, we reference a modified version of the code suitable for tracking two separate thermalized sectors which undergo partially thermalizing interactions. The structure of the code is very similar to the code outlined above, but with a few changes:

- The initial conditions are $T_{SM}$, $T_{hid}$ and $\epsilon$. 
APPENDIX B. DETAILS OF BOLTZMANN EQUATION CODE

- The distribution functions are always Bose-Einstein or Fermi-Dirac, and so only the two temperatures are tracked, eliminating the need for storing any distribution functions.

- We use a modified set of equations to track the distribution functions:

\[
\frac{\partial T_{SM}}{\partial t} = -\frac{3H(\rho_{SM} + P_{SM})}{y_{SM}} - \Gamma, \quad (B.26)
\]

\[
\frac{\partial T_{hid}}{\partial t} = -\frac{3H(\rho_{hid} + P_{hid}) + \Gamma}{y_{hid}}, \quad (B.27)
\]

\[
\Gamma = \frac{g}{2\pi^2} \int_0^\infty dp_1 \, p_1^2 \, C[f_1(p_1)]. \quad (B.28)
\]

- A semi-implicit Euler method was used to compute the temperatures at the next timesteps, in order to minimize the amount of time spent computing \( \Gamma \).
Appendix C

Properties of Global Conformal Blocks

Our arguments rely on some key properties of t-channel global conformal blocks, specifically in the small $u$, large $\ell$ limit. The majority of these properties were discussed quite thoroughly in [51], but were restricted to the case where all four scalar operators in the correlation function are identical. In this appendix, we generalize this discussion to consider two distinct scalar primary operators $\phi_1, \phi_2$. We will specifically focus on the most relevant case of $d = 2$, though this discussion can easily be extended to general spacetime dimensions by following [51].
APPENDIX C. PROPERTIES OF GLOBAL CONFORMAL BLOCKS

C.1 Factorization at Large $\ell$ and Small $u$

In general, the t-channel conformal block expansion of a four-point correlator can be written as

$$\langle \phi_1(x_1)\phi_1(x_2)\phi_2(x_3)\phi_2(x_4) \rangle = \frac{1}{(x_{14}x_{23})^{\Delta_1+\Delta_2}} \left( \frac{x_{13}x_{24}}{x_{12}^2} \right)^{\Delta_{12}} \sum_{\tau,\ell} P_{\tau,\ell} g_{\tau,\ell}(v,u). \quad (C.1)$$

For the specific case of $d = 2$, the global conformal blocks in this expansion take the form

$$g_{\tau,\ell}(v,u) = k'_{\tau+2\ell}(1-z)k'_{\tau}(1-\bar{z}) + k'_{\tau+2\ell}(1-\bar{z})k'_{\tau}(1-z). \quad (C.2)$$

where $k'_{2\beta}(x) \equiv x^{\beta}F_1(\beta - \frac{1}{2}\Delta_{12}, \beta - \frac{1}{2}\Delta_{12}; 2\beta; x)$. Because we are specifically considering the regime with $(1-\bar{z}) < 1$, the second term will be exponentially suppressed at large $\ell$, such that we may ignore it.

We can use the integral representation of hypergeometric functions to rewrite the general function $k'_{2\beta}(1-z)$ as

$$k'_{2\beta}(1-z) = \frac{1}{B(\beta \pm \frac{1}{2}\Delta_{12})} \int_0^1 \frac{dt}{t(1-t)} \left( \frac{(1-z)t(1-t)}{1-t(1-z)} \right)^\beta \left( \frac{(1-t)(1-t(1-z))}{t} \right)^{\frac{1}{2}\Delta_{12}} \quad (C.3)$$

where the prefactor is the beta function $B(x \pm y) = \frac{\Gamma(x+y)\Gamma(x-y)}{\Gamma(2x)}$. For the case where $\beta = \frac{\tau}{2} + \ell$, we can see that the integrand of this expression factorizes into a $\tau$-dependent piece and an $\ell$-dependent piece. When $\ell$ is large, this integrand will be sharply peaked at $t_* = \frac{1-\sqrt{z}}{1-z}$. As the $\tau$-dependent piece of the integrand will vary slowly over this region, we can safely approximate that part with its value at $t = t_*,$

$$\left( \frac{(1-z)t_*(1-t_*)}{1-t_*(1-z)} \right)^\frac{\tau}{2} \sim \left( \frac{1-z}{1+\sqrt{z}} \right) + O(1/\sqrt{\ell}). \quad (C.4)$$
APPENDIX C. PROPERTIES OF GLOBAL CONFORMAL BLOCKS

If we use Stirling’s approximation for the beta function prefactor and take the small $z$ limit, which is equivalent to small $u$, we find

$$k'_{r+2\ell}(1 - z) = 2^r k'_{2\ell}(1 - z) \times \left(1 + O(\sqrt{z}, 1/\sqrt{\ell})\right).$$  \hfill (C.5)

In this limit, we can therefore see that the $\tau$-dependence of 2d conformal blocks factorizes from the $\ell$-dependence,

$$g_{\tau,\ell}(v, u) = k'_{2\ell}(1 - z) 2^r k'_{\tau}(v) \times \left(1 + O(\sqrt{z}, 1/\sqrt{\ell})\right),$$  \hfill (C.6)

where we have used the fact that $1 - \bar{z} = v + O(z)$ in the small $z$ limit.

As discussed in [51], this factorization behavior can be generalized to higher space-time dimensions, such that we obtain

$$g_{\tau,\ell}^{(d)}(v, u) = k'_{2\ell}(1 - z) v^z F^{(d)}(\tau, v) \times \left(1 + O(\sqrt{z}, 1/\sqrt{\ell})\right),$$  \hfill (C.7)

where $F^{(d)}(\tau, v)$ is a $d$-dependent analytic function which is regular and positive at $v = 0$.

C.2 Further Approximations at Small $u$

The function $k'_{2\ell}(1 - z)$ can be approximated further if we consider the limit $\ell \to \infty$ with the product $y \equiv z\ell^2$ fixed such that $y \lesssim O(1)$,

$$k'_{2\ell}(1 - z) = \frac{\Gamma(2\ell)}{\Gamma^2(\ell)} \int_0^1 \frac{dt}{t(1 - t)} t^{\ell - \frac{1}{2}\Delta_{12}} (1 - t)^{\Delta_{12}} e^{-\frac{itv}{\pi^{1/2}}} \times (1 + O(1/\ell)), \hfill (C.8)$$
APPENDIX C. PROPERTIES OF GLOBAL CONFORMAL BLOCKS

where we have again used Stirling’s approximation to simplify the $\Gamma$-functions. The evaluation of this integral can be greatly simplified by defining the new variable $s \equiv \frac{ty}{\ell(1-t)}$,

$$\frac{\Gamma^2(\ell)}{\Gamma(2\ell)} k^{\prime}_{2\ell}(1-z) = \left(\frac{y}{\ell}\right)^{\Delta_{12}} \int_0^\infty \frac{ds}{s^{\Delta_{12}+1}} e^{-s\frac{z}{s}} \times \left(1 + O(1/\ell)\right)$$

$$= 2z^{\frac{1}{2}\Delta_{12}} K_{\Delta_{12}}(2\ell \sqrt{z}) \times \left(1 + O(1/\ell)\right),$$

where $K_x(y)$ is a modified Bessel function of the second kind. We stress that this approximation breaks down when $y \gg 1$, but provides a valid description in the regime with $y \lesssim O(1)$.

C.3 Global Conformal Blocks in the Heavy/Light Probe Limit

So far, we have made no assumptions about the twists or external scaling dimensions associated with these global conformal blocks. However, in this work we are especially interested in pairs of scalar primaries $\phi_1, \phi_2$ in the limit $\Delta_2 \gg \Delta_1$, such that the relevant conformal block twists are $\tau \gtrsim \Delta_2$. To make this manifest, we can rewrite the twists as $\tau = \Delta_2 + \delta$. With this change of variables, the function $k^{\prime}_\tau(v)$ takes the form

$$k^{\prime}_\tau(v) = v^{\frac{1}{2}(\Delta_2 + \delta)} F_1 \left(\Delta_2 + \frac{1}{2}(\delta - \Delta_1), \Delta_2 + \frac{1}{2}(\delta - \Delta_1); \Delta_2 + \delta; v\right). \quad \text{(C.10)}$$
APPENDIX C. PROPERTIES OF GLOBAL CONFORMAL BLOCKS

Using a Pfaff transformation, this can be rewritten as

\[ k'_{\tau}(v) = v^\frac{\tau}{2} (1 - v)^{\frac{\Delta_{12} - \tau}{2}} \binom{\Delta_{12} + \frac{1}{2}(\delta - \Delta_{1}); \frac{1}{2}(\delta + \Delta_{1}); \Delta_{2} + \delta; \frac{v}{v - 1}}{1} \] (C.11)

In the limit \( \Delta_{1}, \delta \ll \Delta_{2} \), the hypergeometric function greatly simplifies, such that this function is approximately

\[ k'_{\tau}(v) = v^\frac{\tau}{2} (1 - v)^{\Delta_{12}} \left( 1 + O(\delta/\Delta_{2}, \Delta_{1}/\Delta_{2}) \right). \] (C.12)

This extremely simple result is explained, in the general Virasoro context, in appendix G. It arises because the exchange of the primary dominates over all descendant exchanges.
Appendix D

Direct Approach to Virasoro

Conformal Blocks

In this appendix, we present one method for determining the structure of the identity Virasoro block, specifically in the semi-classical limit $c \to \infty$. This ‘direct’ approach relies solely on the Virasoro algebra to construct the identity block as a sum over all possible intermediate graviton states in AdS$_3$. While the reach of this approach is rather limited in comparison to the monodromy method discussed in appendix E, it serves as a useful and very elementary test of those more general results. Also, we use these methods in appendix G to show that the Virasoro conformal blocks greatly simplify in a certain semi-classical limit relevant for the right-hand side of the bootstrap equation (4.77).
APPENDIX D. DIRECT APPROACH TO VIRASORO BLOCKS

D.1 Virasoro Blocks and Projection Operators

For any correlation function, we can always insert the identity operator as a sum over all possible intermediate states $|\alpha\rangle$ of the theory,

$$
\langle \phi_1(x_1)\phi_1(x_2)\phi_2(x_3)\phi_2(x_4) \rangle = \sum_{\alpha} \langle \phi_1(x_1)\phi_1(x_2)|\alpha\rangle \langle \alpha|\phi_2(x_3)\phi_2(x_4) \rangle. \tag{D.1}
$$

This statement is of course true in any theory, and does not rely on the presence of any conformal symmetry. However, for the case of a 2d CFT, the states $|\alpha\rangle$ can be organized into irreducible representations of the Virasoro group, each of which is associated with a Virasoro primary operator $O_{h,\bar{h}}$,

$$
\langle \phi_1(x_1)\phi_1(x_2)\phi_2(x_3)\phi_2(x_4) \rangle = \sum_{h,\bar{h}} \sum_{\alpha_{h,\bar{h}}} \langle \phi_1(x_1)\phi_1(x_2)|\alpha_{h,\bar{h}}\rangle \langle \alpha_{h,\bar{h}}|\phi_2(x_3)\phi_2(x_4) \rangle, \tag{D.2}
$$

where the states $|\alpha_{h,\bar{h}}\rangle$ are those states created by $O_{h,\bar{h}}$ and its Virasoro descendants.

This separation of states into representations of the Virasoro group is precisely the Virasoro conformal block decomposition of a correlation function,

$$
\sum_{h,\bar{h}} \sum_{\alpha_{h,\bar{h}}} \langle \phi_1(x_1)\phi_1(x_2)|\alpha_{h,\bar{h}}\rangle \langle \alpha_{h,\bar{h}}|\phi_2(x_3)\phi_2(x_4) \rangle = \frac{1}{x_{12}^{\Delta_1} x_{34}^{2\Delta_2}} \sum_{h,\bar{h}} P_{h,\bar{h}} V_{h,\bar{h}}(u, v), \tag{D.3}
$$

such that we can associate each Virasoro block with a particular projection operator

$$
P_{h,\bar{h}} = \sum_{\alpha_{h,\bar{h}}} |\alpha_{h,\bar{h}}\rangle \langle \alpha_{h,\bar{h}}|. \tag{D.4}
$$

The descendant states $|\alpha_{h,\bar{h}}\rangle$ are created by acting with various linear combinations of the Virasoro generators $L_m, \bar{L}_n$ on the state $|h, \bar{h}\rangle = O_{h,\bar{h}}|0\rangle$, where these generators
APPENDIX D. DIRECT APPROACH TO VIRASORO BLOCKS

obey the algebra

\[
[L_m, \bar{L}_n] = 0,
\]

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m,-n},
\]

\[
[\bar{L}_m, \bar{L}_n] = (m - n)\bar{L}_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m,-n}.
\]

(D.5)

Note that \(L_{-1}, L_0, \) and \(L_1\) form the holomorphic global conformal subalgebra, and \(c\) drops out of their commutation relations. Because the holomorphic generators \(L_m\) commute with all of the antiholomorphic \(\bar{L}_n,\) we can simultaneously diagonalize one generator from each set, which we choose to be the operators \(L_0, \bar{L}_0.\) Our basis states \(|\alpha_{h,\bar{h}}\rangle\) can then be expressed as a tensor product of eigenstates of \(L_0\) with eigenstates of \(\bar{L}_0,\)

\[
|\alpha_{h,\bar{h}}\rangle = |\alpha_h\rangle \otimes |\bar{\alpha}_{\bar{h}}\rangle.
\]

(D.6)

Similarly, the projection operator \(P_{h,\bar{h}}\) can be written as the tensor product

\[
P_{h,\bar{h}} = \sum_{\alpha_h} |\alpha_h\rangle \langle\alpha_h| \otimes \sum_{\bar{\alpha}_{\bar{h}}} |\bar{\alpha}_{\bar{h}}\rangle \langle\bar{\alpha}_{\bar{h}}| = P_h \otimes \bar{P}_{\bar{h}},
\]

(D.7)

which tells us that the Virasoro block can be written as the product

\[
\mathcal{V}_{h,\bar{h}}(u, v) = \mathcal{V}_h(z)\bar{\mathcal{V}}_{\bar{h}}(\bar{z}),
\]

(D.8)

where \(u = z\bar{z}\) and \(v = (1 - z)(1 - \bar{z}).\)

As these functions are invariant under any global conformal transformation, we can simplify their calculation by choosing coordinates such that we obtain the relation

\[
\langle \phi_1(\infty)\phi_1(1)P_h\phi_2(z, \bar{z})\phi_2(0)\rangle = \langle \phi_1(\infty)\phi_1(1)\rangle \langle \phi_2(z, \bar{z})\phi_2(0)\rangle P_h\mathcal{V}_h(z),
\]

(D.9)

with a similar relation for \(\bar{\mathcal{V}}_{\bar{h}}(\bar{z}).\)
D.2 Semi-Classical Graviton Basis

Everything we discussed in the previous section is exact, with no assumptions about the 2d CFT or the primary operator associated with the Virasoro block. Theoretically, any Virasoro block could be constructed in this fashion, by finding the associated projection operator and acting within a particular correlation function. In practice, though, this process is prohibitively difficult for general operators in a general theory. We will therefore restrict our focus to the identity Virasoro block in theories with large central charge.

The identity operator has $h = \bar{h} = 0$ and its associated state is the vacuum $|0\rangle$. The descendant states which make up the projection operators $\mathcal{P}_0, \bar{\mathcal{P}}_0$ are therefore linear combinations of $L_m, \bar{L}_n$ acting on the vacuum. Because the identity is a Virasoro primary, the vacuum is annihilated by all the ‘lowering’ operators $L_m, \bar{L}_m$ with $m > 0$. In addition, the vacuum transforms trivially under the global conformal group, so it is also annihilated by all the global operators, such that we have

$$L_m|0\rangle = \bar{L}_m|0\rangle = 0 \quad (m = -1, 0, 1). \quad (D.10)$$

Our projection operators will therefore consist of states created by generators of the form $L_{-m}, \bar{L}_{-m}$ with $m \geq 2$. We will restrict our discussion to the holomorphic projector $\mathcal{P}_0$, but all of our results will also apply to the antiholomorphic $\bar{\mathcal{P}}_0$.

One obvious basis to use is the ‘graviton’ basis, consisting of the states

$$|\alpha_0\rangle = \frac{L_{-m_1} \cdots L_{-m_n}|0\rangle}{\sqrt{N_{\{m_i,k_i\}}}}, \quad (D.11)$$
where $\mathcal{N}_{\{m_i,k_i\}}$ is simply a normalization factor. To avoid redundancy, we will use the ordering convention $m_1 > \cdots > m_n$. In terms of AdS, these basis states can be loosely interpreted as $k$-graviton states, where $k = \sum_i k_i$, though in AdS$_3$ gravitons are not propagating degrees of freedom in the bulk.

In order to work in this basis, we need to determine an expression for the normalization factors $\mathcal{N}_{\{m_i,k_i\}}$. For example, let us consider the normalization of a general $k$-graviton state,

$$\mathcal{N}_{m_1 \cdots m_k} = \langle L_{m_k} \cdots L_{m_1} L_{-m_1} \cdots L_{-m_k} \rangle,$$

where again we have the ordering convention $m_1 \geq \cdots \geq m_k$. To determine the precise form of this factor, we simply need to use the structure of the Virasoro algebra to commute each $L_{m_i}$ term through to the far right, where it then annihilates the vacuum. Starting with $L_{m_1}$, we obtain

$$\mathcal{N}_{m_1 \cdots m_k} = \langle L_{m_k} \cdots L_{m_2} (L_{-m_1} L_{m_1} + [L_{m_1}, L_{-m_1}]) L_{-m_2} \cdots L_{-m_k} \rangle$$

$$= \langle L_{m_k} \cdots L_{m_2} (L_{-m_1} L_{m_1} + 2m_1 L_0) L_{-m_2} \cdots L_{-m_k} \rangle$$

$$+ \frac{c}{12} m_1(m_1^2 - 1) \mathcal{N}_{m_2 \cdots m_k}.$$  \hspace{1cm} (D.13)

The $L_0$ originating from $[L_{m_1}, L_{-m_1}]$ can easily be commuted through the remaining operators, resulting in

$$\langle L_{m_k} \cdots L_{m_2} (2m_1 L_0) L_{-m_2} \cdots L_{-m_k} \rangle = 2m_1 \left( \sum_{i=2}^{k} m_i \right) \mathcal{N}_{m_2 \cdots m_k}. \hspace{1cm} (D.14)$$

Since we are considering the limit $c \to \infty$ at fixed $m_i$, this term will be subdominant, such that we can safely ignore it.
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As we continue to commute $L_{m_1}$ through the remaining operators, we can immediately see that the only non-negligible terms are those which arise if $m_i = m_1$. We then obtain the semi-classical recursion relation

$$N_{m_1 \cdots m_k} \approx \frac{c}{12} m_1(m_1^2 - 1) \left( 1 + \sum_{i=2}^{k} \delta_{m_1 m_i} \right) N_{m_2 \cdots m_k}. \quad (D.15)$$

Using this recursion relation, we can then obtain an approximate expression for every normalization factor in the semi-classical limit,

$$N(m_i, k_i) = \langle L^k \rangle \prod_{i=1}^{n} \left( \frac{k_i! m_i^2 (m_i^2 - 1) (1 + \delta_{m_i})}{k_i} \right), \quad (D.16)$$

where again $k = \sum_i k_i$.

In general, we cannot actually use these $k$-graviton states to construct our projection operators, because this basis is not orthogonal. For example, consider the inner product

$$\langle L_{m+n} \rangle \sqrt{N_{m+n}} \approx \frac{n(n^2 - 1)(2m + n)}{\sqrt{p^2 - 1) n^2 - 1)(\frac{c}{12} m(m^2 - 1)(1 + \delta_{mn}) + 2mn)} \delta_{p,m+n}. \quad (D.17)$$

Though these are two distinct states, their inner product is clearly nonzero for $p = m + n$. However, this expression vanishes to leading order in the semi-classical limit $c \to \infty$,

$$\langle L_{m+n} \rangle \sqrt{N_{m+n}} \approx \frac{1}{\sqrt{c}}, \quad (D.18)$$

such that these two states become approximately orthogonal. This behavior is in fact quite general, and applies to all inner products of distinct graviton states. At
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some level, this is rather unsurprising, as the limit $c \to \infty$ in a CFT is equivalent to the limit $G_N \to 0$ in AdS, such that interactions between gravitons are greatly suppressed. Our basis is therefore approximately orthogonal in the large-$c$ limit, and we can construct the approximate projection operator

$$\mathcal{P}_0 \approx \sum_{\{m_i, k_i\}} \frac{L_{-m_1}^{k_1} \cdots L_{-m_n}^{k_n} |0\rangle \langle 0| L_{m_n}^{k_n} \cdots L_{m_1}^{k_1}}{N_{\{m_i, k_i\}}}.$$  \hspace{1cm} (D.19)

D.3 $T_{\mu\nu}$ Correlators and the Identity Virasoro Block

We can now use our approximate projector to determine the holomorphic identity block through the relation

$$\mathcal{V}_0(z) = \frac{\langle \phi_1(\infty) \phi_1(1) \mathcal{P}_0 \phi_2(z) \phi_2(0) \rangle}{\langle \phi_1(\infty) \phi_1(1) \rangle \langle \phi_2(z) \phi_2(0) \rangle}.$$ \hspace{1cm} (D.20)

Since we are working in the graviton basis, we need to calculate correlation functions of the form

$$\langle \phi_1(\infty) \phi_1(1) L_{-m_1}^{k_1} \cdots L_{-m_n}^{k_n} \rangle, \langle L_{m_n}^{k_n} \cdots L_{m_1}^{k_1} \phi_2(z) \phi_2(0) \rangle.$$ \hspace{1cm} (D.21)

Our approach will be quite similar to the normalization factor calculations in the previous section. We can simply commute the Virasoro generators through the various scalar operators $\phi_i$, using the commutation relation

$$[L_{-m}, \phi_i(w)] = h_i (1 - m) w^{-m} \phi_i + w^{1-m} \partial \phi_i,$$ \hspace{1cm} (D.22)

where $h_i$ is the holomorphic scaling dimension of $\phi_i$ and $w = x^0 + i x^1$. For a review of this and various related techniques for computing these correlators see e.g. [53].
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As a simple example of this process, let us consider a general one-graviton correlation function. Using this commutation relation, we can obtain the expression

\[ \langle \phi_i(w_1) \phi_i(w_2) L_{-m} \rangle = -\langle [L_{-m}, \phi_i(w_1)] \phi_i(w_2) \rangle - \langle \phi_i(w_1) [L_{-m}, \phi_i(w_2)] \rangle \]

\[ = (h_i(m - 1)(w_1^{-m} + w_2^{-m}) - w_1^{1-m} \partial_1 - w_2^{1-m} \partial_2) \langle \phi_i(w_1) \phi_i(w_2) \rangle. \]  

(D.23)

If we use the known two-point correlation function

\[ \langle \phi_i(w_1) \phi_i(w_2) \rangle = \frac{1}{|w_{12}|^{4h_i}}, \]  

(D.24)

we can calculate the exact one-graviton correlator,

\[ \langle \phi_i(w_1) \phi_i(w_2) L_{-m} \rangle = h_i \left( (m - 1)(w_1^{-m} + w_2^{-m}) + \frac{2(w_1^{1-m} - w_2^{1-m})}{w_{12}} \right) \langle \phi_i \phi_i \rangle. \]  

(D.25)

Similarly, we can obtain the other correlation function

\[ \langle L_m \phi_i(w_1) \phi_i(w_2) \rangle = h_i \left( (m + 1)(w_1^m + w_2^m) - \frac{2(w_1^{1+m} - w_2^{1+m})}{w_{12}} \right) \langle \phi_i \phi_i \rangle. \]  

(D.26)

Combining all of these results, we find the full one-graviton contribution to the identity block

\[ \mathcal{V}_0^{(k=1)}(z) = \sum_{m=2}^{\infty} \frac{\langle \phi_1(\infty) \phi_1(1) L_{-m} \rangle \langle L_m \phi_2(z) \phi_2(0) \rangle}{\mathcal{N}_m(\phi_1(\infty) \phi_1(1)) \langle \phi_2(z) \phi_2(0) \rangle} \]

\[ = 12 \frac{h_1 h_2}{c} \sum_{m=2}^{\infty} \frac{(m - 1)^2}{m(m^2 - 1)} z^m = 2 \frac{h_1 h_2}{c} z^2 F_1(2, 2; 4; z), \]  

(D.27)

which is the precise form of the global conformal block of the one-graviton global conformal primary \( L_{-2} \). This result is unsurprising, because the other one-graviton operators \( L_{-m} \) are all global conformal descendants of \( L_{-2} \).
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Let us now consider the more general $k$-graviton correlator,

$$\langle \phi_i(w_1)\phi_i(w_2)L^{k_1}_{-m_1} \cdots L^{k_n}_{-m_n} \rangle. \quad (D.28)$$

Just as before, we can commute the various $L_{-m_i}$ operator through the two scalar operators to obtain the general expression

$$\prod_{j=1}^{n} \left( h_i (m_j - 1)(w_1^{-m_j} + w_2^{-m_j}) - w_1^{1-m_j}\partial_1 - w_2^{1-m_j}\partial_2 \right)^{k_j} \langle \phi_i(w_1)\phi_i(w_2) \rangle. \quad (D.29)$$

These differential operators clearly do not commute, and computing the resulting expression will generally become intractable. However, if we consider the limit $c \to \infty$ at fixed $\frac{h_i}{\sqrt{c}}$, we only need to consider terms with leading powers of $h_i$. The result then simplifies to the approximate form

$$h^k_i \prod_{j=1}^{n} \left( (m - 1)(w_1^{-m} + w_2^{-m}) + \frac{2}{w_1 w_2} (w_1^{1-m} - w_2^{1-m}) \right)^{k_j} \langle \phi_i(w_1)\phi_i(w_2) \rangle. \quad (D.30)$$

We emphasize that the rest of this section will be studying the limit $h_1, h_2, c \to \infty$ with $h_1/c \to 0$ and $h_2/c \to 0$ but $h_1 h_2/c$ fixed and finite.

We can now determine the general $k$-graviton contribution to the identity Virasoro block, which is associated with the approximate projection operator

$$P^{(k)}_0 \approx \sum_{\{m_i, k_i\}} \frac{L^{k_1}_{-m_1} \cdots L^{k_n}_{-m_n} |0 \rangle \langle 0 | L^{k_n}_{m_n} \cdots L^{k_1}_{m_1}}{\mathcal{N}_{\{m_i, k_i\}}}. \quad (D.31)$$

Inserting this projection operator into the four-point correlator, we obtain

$$\mathcal{V}^{(k)}_0(z) \approx \left( \frac{12h_1 h_2}{c} \right)^k \sum_{\{m_i, k_i\}} \prod_{i=1}^{n} \frac{(m_i - 1)^{2k_i}}{k_i! m_i^2 (m_i^2 - 1)^{k_i}} z^{k_i m_i}. \quad (D.32)$$
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Now the crucial step is to note that the contribution of each of the $k$-gravitons commutes with the others, so we can write the entire $k$-graviton piece of the conformal block in the limit of interest as

$$V_0^{(k)}(z) \approx \frac{1}{k!} \left( \frac{12 h_1 h_2}{c} \sum_{m=2}^{\infty} \frac{(m-1)^2}{m(m^2-1)} z^m \right)^k . \quad (D.33)$$

The expression in parentheses is precisely the one-graviton contribution we found earlier. Now when we sum over $k$, we find that the result exponentiates! Thus we have determined the full expression for the identity holomorphic block in our restricted semi-classical limit

$$V_0(z) \approx \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{2 h_1 h_2}{c} z^2 \left. \binom{2}{2} F_1(2; 2; 4; z) \right)^k \right) = \exp \left[ \frac{2 h_1 h_2}{c} z^2 \left. \binom{2}{2} F_1(2; 2; 4; z) \right] \right) , \quad (D.34)$$

with a similar result for the antiholomorphic block $\bar{V}_0(\bar{z})$. In the limit we are considering, with $c \to \infty$ with $h_1 h_2 / c$ fixed, this result for the identity Virasoro conformal block should hold for all values of $z$. 

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Appendix E

Review of Monodromy Method for the Virasoro Blocks

In this appendix we provide a self-contained review of what we refer to as the ‘monodromy method’ for computing Virasoro conformal partial waves in the semi-classical limit. Although the method may be well known to experts, we have included this appendix for the sake of completeness. Our discussion closely follows [254, 265]. We will now give a brief sketch of the main ideas behind the monodromy method, and then we will discuss each step in detail in the subsections that follow.

The semi-classical limit is defined as the large central charge limit $c \to \infty$ with the ratios $h/c$ of conformal dimensions to the central charge kept finite. It is believed that in this limit, the Virasoro conformal partial waves take the form

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|\alpha\rangle\langle\alpha|\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \mathcal{F}_\alpha(x_i) \approx e^{-\frac{c}{6}f(x_i)}, \quad (E.1)$$
where \( f(x_i) \) approaches some fixed function of \( x_i \) and the various ratios \( h/c \) in the semi-classical limit. The \( \approx \) sign indicates that we are dropping subleading corrections in our \( c \to \infty \) limit. As far as we know this statement has not been rigorously proven, but we will see very good evidence for it below by making use of Liouville theory. In the much more restrictive limit of appendix D we essentially gave a proof by computing an explicit sum over states. In what follows we will simply assume this semi-classical scaling behavior.

The next step is to insert into the correlator a ‘light’ operator \( \hat{\psi}(z) \) whose dimension is fixed as \( c \to \infty \). We will argue that the leading semi-classical behavior is unchanged, but the conformal block is multiplied by a wavefunction \( \psi(z) \):

\[
\Psi(x_i, z) \equiv \langle O_1 O_2 | \alpha \rangle \langle \alpha | \hat{\psi}(z) O_3 O_4 \rangle = \psi(z, x_i) F_\alpha(x_i). \tag{E.2}
\]

Note that \( \psi(z, x_i) \) is just a function, whereas \( \hat{\psi} \) is an operator. This formula defines \( \psi(z, x_i) \); the content of the equation is that \( \psi \) and its derivatives are \( \mathcal{O}(e^{\alpha}) \). This is extremely powerful, because we can take \( \hat{\psi} \) to be any light operator we like, including one of the degenerate operators in the theory. In particular, we can choose an operator that obeys the shortening condition

\[
\left( L_{-2} - \frac{3}{2(2h_{\psi} + 1)} L_{-1}^2 \right) |\psi\rangle = 0. \tag{E.3}
\]

Acting with \( \left( L_{-2} - \frac{3}{2(2h_{\psi} + 1)} L_{-1}^2 \right) \) on \( \hat{\psi} \) inside \( \Psi(z_i, z) \) then implies the differential equation in the \( z \) variable

\[
\psi''(z) + T(z)\psi(z) = 0, \tag{E.4}
\]

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where $T(z)$ is given by

$$
\frac{c}{6} T(z) = \frac{h_1}{z^2} + \frac{h_2}{(z-x)^2} + \frac{h_3}{(1-z)^2} + \frac{h_1 + h_2 + h_3 - h_4}{z(1-z)} - \frac{c}{6} \frac{c_2(x) x(1-x)}{z(z-x)(1-z)} \tag{E.5}
$$

after setting $x_1 = 0, x_2 = x, x_3 = 1, x_4 = \infty$, with $c_2 = \frac{\partial}{\partial x_2} f(x_i)$.

As a final step, it turns out that $\hat{\psi}(z)$ must have a specific monodromy, again because the degeneracy of $\hat{\psi}$ is very constraining. In particular, if we study the OPE

$$
\mathcal{O}_3(0) \mathcal{O}_4(x) = \sum_{\beta} c_{34\beta}(x) \mathcal{O}_\beta(0) \tag{E.6}
$$

inside $\langle \alpha | \hat{\psi}(z) \mathcal{O}_3 \mathcal{O}_4 \rangle$ in [E.2], the shortening condition [E.3] implies that only operators $\mathcal{O}_\beta$ with one of two different possible weights $h_\beta$ can contribute. Thus, moving $\psi(z)$ around a cycle that encloses $x_1$ and $x_2$ must have monodromy consistent with these two weights. This is sufficient to determine $c_2(x)$, and therefore $f(x)$.

Now we will go through each of these points in more detail.

### E.1 Scaling of the Semi-Classical Action

The first key point is that conformal blocks at large central charge are believed to behave like $\sim e^{-\frac{1}{6} f}$, i.e.

$$
\lim_{c \to \infty} \frac{1}{c} \log F = -\frac{1}{6} f(x_i) < \infty. \tag{E.7}
$$

One piece of evidence for this result, and the origin of the term ‘semi-classical’ limit, comes from Liouville theory. This is a theory with action

$$
S = \frac{1}{4b^2} \int d^2 x \sqrt{g} \left( g^{\alpha\beta} \partial_\alpha \phi_c \partial_\beta \phi_c + 2(1 + b^2) R \phi_c + 16\lambda e^{\phi_c} \right), \tag{E.8}
$$
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where $R$ is the Ricci scalar and $b$ is a parameter related to the central charge $c$ by

$$c = 1 + 6 \left( b + \frac{1}{b} \right)^2 \overset{b \ll 1}{\approx} 6b^{-2}. \quad (E.9)$$

The Liouville theory has a continuous spectrum, with correlators that receive contributions from conformal blocks of arbitrary dimension and spin, so it is a useful laboratory for studying conformal blocks. Roughly speaking, we can obtain semi-classical conformal blocks by projecting them out of Liouville correlators.

At small $b$ and fixed $\lambda$, the equation of motion for $\phi_c$ is

$$\partial \bar{\partial} \phi_c = 2\lambda e^{\phi_c}, \quad (E.10)$$

with boundary condition $\phi_c \sim -2 \log(z\bar{z}) + O(1)$ at $z \to \infty$, so $\langle \phi_c \rangle \sim O(e^b)$. Thus, at small $b$, the action should have a semi-classical limit

$$S_{cl} \overset{b \ll 1}{=} \frac{3c}{2} \int d^2x \sqrt{g} \left( g^{\alpha\beta} \partial_\alpha \phi_c \partial_\beta \phi_c + 2R \phi_c + 16\lambda e^{\phi_c} \right), \quad (E.11)$$

which implies the scaling in (E.7).

Primary operators in Liouville theory can be constructed by taking exponentials, i.e.

$$V_\alpha \equiv e^{\phi_c}. \quad (E.12)$$

The weight of such an operator is $h_V = \alpha(b + \frac{1}{b} - \alpha) \overset{b \ll 1}{\approx} \alpha \sqrt{\frac{2}{b}} - \alpha^2$. Thus, in order to take $c \to \infty$ with $h_V/c$ fixed, we take $\alpha \sim O(\sqrt{c})$. Taking $\alpha = \frac{b}{2}$, we can solve for $\alpha$ in terms of $h_V$, finding $\alpha = \frac{1}{2}(1 \pm \sqrt{1-24h_V/c})$. So these “heavy” operators can
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be written as

\[ V = e^{\alpha \phi_c} = \exp \left( c \frac{(1 \pm \sqrt{1 - 24hV/c})}{12} \phi_c \right). \]  

(E.13)

When we insert one of these operators in the path integral, it has the effect of shifting

\( S_d \) of the Liouville theory by \( \mathcal{O}(c) \), and of shifting the equations of motion for \( \phi_c \) by \( \mathcal{O}(c^0) \). This argument falls short of a proof of the scaling of \( f(x_i) \) because we have only estimated the scaling of the correlators. We need to project the correlators onto conformal blocks to determine the scaling of \( \log \mathcal{F} \), and so we have not proven that the individual blocks themselves scale as desired.

If we want to construct a light operator, with dimension that scales like \( c^0 \), then we should take \( \alpha \sim \frac{1}{\sqrt{c}} \sim b \). Such operators are of the form \( V = e^{\mathcal{O}(c^0)\phi_c} \), and their insertions only shift the semi-classical Liouville action by \( \mathcal{O}(c^0) \).

E.2 Insertion of the Degenerate Operator

The claim that correlators behave like \( e^{-\hat{\psi}f} \) in the semi-classical \( c \to \infty \) has far-reaching consequences once we ask what happens when we insert additional light operators \( \hat{\psi} \), \textit{i.e.} operators with dimensions \( \sim \mathcal{O}(1) \), in correlators. The effect of adding such an operator is to multiply the correlator by a wavefunction \( \psi(z, x_i) \) for the position of the insertion of \( \hat{\psi} \):

\[ \sum_k \langle O_1 O_2 | \alpha; k \rangle \langle \alpha; k | \hat{\psi}(z) O_3 O_4 \rangle = \psi(z, x_i) \sum_k \langle O_1 O_2 | \alpha; k \rangle \langle \alpha; k | O_3 O_4 \rangle, \]  

(E.14)
where we have made the sum over descendant states explicit via the $k$ label. In the above equation, as in all sums over states of the form $\sum_i |i\rangle\langle i|$, there is implicit position dependence in the sum, because the states must be inserted on a ball that separates the fields on the left from the fields on the right; equivalently, one can write the OPE in terms of sums over operators. One can take the above equation as a definition of $\psi(z, x_i)$; as stated above, the content of the equation is that $\psi(z, x_i) \sim O(e^{c})$. We can investigate this assumption by using the definition of the conformal blocks as a sum over states. Define

$$\psi_k(z, x_i) \equiv \frac{\langle \alpha; k | \hat{\psi}(z) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle}{\langle \alpha; k | \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle}, \quad (E.15)$$

so that

$$\langle \mathcal{O}_1 \mathcal{O}_2 | \alpha; k \rangle \langle \alpha; k | \hat{\psi}(z) \mathcal{O}_3 \mathcal{O}_4 \rangle = \psi_k(z, x_i) \langle \mathcal{O}_1 \mathcal{O}_2 | \alpha; k \rangle \langle \alpha; k | \mathcal{O}_3 \mathcal{O}_4 \rangle \quad (E.16)$$

Let $k_0$ be the lowest level so that $\psi_{k_0}$ in (E.15) does not vanish. Then, equation (E.14) follows if $\frac{\psi_k(z)}{\psi_{k_0}(z)}$ is $O(e^c)$ at $c \to \infty$ for a light operator $\hat{\psi}$. To understand why this should be true, we will first assume that $\psi_{k_0}$ is of order $O(e^c)$, due to $\hat{\psi}$ being a light operator. Then, we can look at how $\psi_k$ for general $k$ is related to $\psi_{k_0}$ by examining
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the action of the Virasoro operator $L_m$ inside the correlator:

$$\langle \alpha; k_0|L_m \hat{\psi}(z)O_3O_4 \rangle = \sum_{i=3,4,\tilde{z}} \left( \frac{(m-1)h_i}{x_i^m} - \frac{1}{x_i^{m-1}} \partial_i \right) \langle \alpha; k_0|\hat{\psi}(z)O_3O_4 \rangle$$

$$= \sum_{i=3,4,\tilde{z}} \left( \frac{(m-1)h_i}{x_i^m} - \frac{1}{x_i^{m-1}} \partial_i \right) \psi_{k_0}(z, x_3, x_4) \langle \alpha; k_0|O_3O_4 \rangle$$

$$\cong \psi_{k_0}(z, x_i) \sum_{i=3,4} \left( \frac{(m-1)h_i}{x_i^m} - \frac{1}{x_i^{m-1}} \partial_i \right) \langle \alpha; k_0|O_3O_4 \rangle$$

$$= \psi_{k_0}(z, x_i) \langle \alpha; k_0|L_m O_3O_4 \rangle. \quad (E.17)$$

The key step in in the third line, where “$a \cong b$” means $a/b = \mathcal{O}(e^0)$. This step is justified because we can take $h_z$ and $\partial_z$ as $\mathcal{O}(e^0)$ since $\hat{\psi}$ is a light operator and $\psi_{k_0}$ is $\mathcal{O}(e^0)$, whereas $h_3, h_4$ and $\partial_3, \partial_4 \sim \mathcal{O}(c)$. Dividing both sides of this equation by $\langle \alpha; k_0|L_m O_3O_4 \rangle = \langle \alpha; m+k_0|O_3O_4 \rangle$ and being a bit schematic with the indices labeling the level of the descendants, we obtain

$$\psi_{k_0+m}(z, x_i) = \psi_{k_0}(z, x_i), \quad (E.18)$$

whose consequence is (E.14).

E.3 Differential Equation from the Degeneracy Condition

Next, we want to explore the consequences of the shortening condition (E.3) for correlators of $\hat{\psi}$ with four heavy operators. The idea is that (E.3) becomes a differ-
entential equation for the correlator (see e.g. [53])

\[
0 = \left( \frac{3}{2(2h_\psi + 1)} \partial_z^2 + \sum_{i=1}^{4} \left( \frac{h_i}{(z - x_i)^2} + \frac{1}{z - x_i} \partial_i \right) \right) \langle O_1 O_2 \hat{\psi} O_3 O_4 \rangle
\]

where in the second line we have used the weight of the degenerate operator

\[
h_\psi = -\frac{1}{2} - \frac{3b^2}{4},
\]

and \( c \approx \frac{6}{b^2} \) at \( b \ll 1 \). We would like to argue that this equation is satisfied not only for the correlator, but for each of its constituent conformal blocks. The justification for this is that each conformal block has a different monodromy in \( z \), determined by the weight of the block itself. So we have

\[
0 = \left( \frac{c}{6} \partial_z^2 + \sum_{i=1}^{4} \left( \frac{h_i}{(z - x_i)^2} + \frac{1}{z - x_i} \partial_i \right) \right) \psi(z, x_i) e^{-\frac{c}{6} f(x_i)}
\]

where

\[
T(z, x_i) = \sum_{i=1}^{4} \frac{\epsilon_i}{(z - x_i)^2} - \frac{c_i}{z - x_i}, \quad c_i \equiv \frac{\partial}{\partial x_i} f, \quad \epsilon_i \equiv \frac{6h_i}{c},
\]

and we have again used the fact that \( \psi \sim \mathcal{O}(e^{\phi}) \), so we can neglect \( \partial_i \) derivatives acting on it. Finally, \( T(z, x_i) \) itself is further constrained by a conformal Ward identity, as it is exactly the wavefunction that arises when we compute the \( \langle \hat{T}(z) O_1 O_2 O_3 O_4 \rangle \) five-point function, where the energy-momentum tensor \( \hat{T}(z) \) should not be confused
with its wavefunction $T(z,x_i)$:

$$
\langle \hat{T}(z)\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4 \rangle = \sum_{i=1}^{4} \left( \frac{h_i}{(z-x_i)^2} + \frac{1}{z-x_i} \partial_i \right) \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4 \rangle \\
= -\frac{c}{6} T(z,x_i) \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4 \rangle.
$$

(E.23)

Therefore $T(z,x_i)$ must decay like $z^{-4}$ as $z \to \infty$, which implies three constraints:

$$
\sum_i c_i = 0, \quad \sum_i \left( c_i x_i - \frac{6h_i}{c} \right) = 0, \quad \sum_i \left( c_i x_i^2 - \frac{12h_i}{c} x_i \right) = 0.
$$

(E.24)

Taking $x_1 = 0, x_2 = x, x_3 = 1, x_4 = \infty$ then leads us to equation (E.5).

### E.4 Constraint on $h_\beta$ and Monodromy

Finally, we need to constrain the monodromy of $\psi(z)$ to determine the function $f(x_i)$ which defines the semi-classical conformal block. First, let us consider the constraint of the shortening condition for $\hat{\psi}$ on three-point functions

$$
V_{\alpha\beta\psi} = \langle \mathcal{O}_\alpha(x_1)\mathcal{O}_\beta(x_2)\hat{\psi}(x_3) \rangle = \frac{C_{\alpha\beta\psi}}{x_{12}^{(h_\alpha + h_\beta - h_\psi)} x_{13}^{(h_\alpha + h_\psi - h_\beta)} x_{23}^{(h_\psi + h_\beta - h_\alpha)}}.
$$

(E.25)

It is straightforward to act on this with the appropriate shortening operator for $\hat{\psi}$ to see

$$
0 = \left( -\frac{3}{2(2h_\psi + 1)} \partial_3^2 + \sum_{i=1,2} \left( \frac{h_i}{(x_3-x_i)^2} + \frac{1}{x_3-x_i} \partial_i \right) \right) V_{\alpha\beta\psi} \\
= \left( \frac{(2h_\psi + 1) (h_\alpha + h_\beta) - 3(h_\alpha - h_\beta)^2 + h_\psi^2 - h_\psi}{4h_\psi + 2} \right) V_{\alpha\beta\psi} x_{12}^2 x_{13}^2 x_{23}^2.
$$

(E.26)

\footnote{This is easy to see by taking the $\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4 \supset c_{1234T}(x_i)T(0)$ OPE.}
APPENDIX E. REVIEW OF MONODROMY METHOD

One can solve this algebraic equation for \( h_\beta \) as a function of \( h_\alpha \) and set \( h_\psi = -\frac{1}{2} - \frac{3b^2}{4} \).

In the limit \( b \ll 1 \) with \( h_\alpha b^2 \) fixed, one finds

\[
h_\beta - h_\alpha - h_\psi = \frac{1}{2} \left( 1 \pm \sqrt{1 - 4b^2 h_\alpha} \right).
\]

(E.27)

We want to know the monodromy of \( \psi(z) \) as \( \hat{\psi} \) encircles \( x_1 \) and \( x_2 \) in the four-point function \( \langle O_1 O_2 | \alpha \rangle \langle \alpha | \hat{\psi} O_3 O_4 \rangle \). To relate this to the argument above, we take the OPE of \( O_3 O_4 = \sum_\beta c_{3\beta}^4 O_\beta \). Our analysis of the 3-pt function shows that \( \sum_\beta c_{34\beta} \langle \alpha | \hat{\psi} O_\beta \rangle \) gets contributions only from \( O_\beta \) with \( h_\beta \) such that \( \langle O_\alpha(y) \hat{\psi}(z) O_\beta(\frac{x_3 + x_4}{2}) \rangle \sim (z - y)^{1+\sqrt{1-24h_\alpha}/c} \) as \( z \) encircles \( y \). Since the sum over states \( |\alpha\rangle \) arises from the \( O_1 O_2 \) OPE, this cycle must enclose both \( x_1 \) and \( x_2 \) when we apply it to \( \psi(z) \). Thus, under a cycle encircling \( x_1 \) and \( x_2 \) but not \( x_3 \) and \( x_4 \), the solutions to the differential equation (E.21) must have monodromy

\[
M = \begin{pmatrix}
e^{i\pi(1+\Lambda_\alpha)} & 0 \\
0 & e^{i\pi(1-\Lambda_\alpha)}
\end{pmatrix} = -\begin{pmatrix}
e^{i\pi\Lambda_\alpha} & 0 \\
0 & e^{-i\pi\Lambda_\alpha}
\end{pmatrix},
\]

(E.28)

in a basis that diagonalizes \( M \), where \( \Lambda_\alpha = \sqrt{1 - 24h_\alpha/c} \). This fact combined with the results of the previous subsection allows us to determine the semi-classical conformal block using the monodromy method. Note that for the identity or vacuum conformal block this means that \( M \) must be the 2 \( \times \) 2 identity matrix, which is identical in all bases. This leads to further simplifications for the monodromy method when applied to the identity conformal block.
Appendix F

Computing Virasoro Blocks via the Monodromy Method

We will now use the monodromy method reviewed in appendix E to compute the Virasoro conformal blocks in a semi-classical limit more general than that which was considered in appendix D. Specifically, we will be able to determine the conformal block for a primary of weight $h_p$ in a correlator of the form

$$\langle \phi_1(0)\phi_1(x)\phi_2(1)\phi_2(\infty) \rangle$$

in the limit that

$$c \to \infty, \text{ and } \frac{h_i}{c} \text{ fixed,}$$

followed by a perturbative expansion to linear order in $h_1/c$ and $h_p/c$, but working non-perturbatively in $h_2/c$. Note that working to linear order in $h_1/c$ in the com-
putation of \( f \) for the Virasoro block \( F = e^{-\frac{c}{6} f} \) means that we are neglecting terms of order \( h^2_1/c \) in the exponent of \( F \). To use the monodromy method we are already neglecting order one terms in the exponent of \( F \), so strictly speaking, we need to take \( h^2_1/c \lesssim 1 \) for a self-consistent approximation. This makes it possible to use the CFT bootstrap to study AdS\(_3\) setups where a probe object orbits a finite mass deficit angle or a BTZ black hole. For the reader just looking to find the results, the formulas we compute for the conformal blocks are equations (F.22) and (F.24).

**F.1 S-Channel Virasoro Blocks**

As discussed in appendix E, we would like to solve the differential equation

\[
\psi''(z) + T(z)\psi(z) = 0 \tag{F.3}
\]

where \( T(z) \) is given by equation (E.5). Then we must impose that the pair of solutions for \( \psi \) (there are two, since the differential equation is second order) have monodromy according to (E.28) when we take \( z \) around 0 and \( x \); this determines the function \( c_2(x) \). Once \( c_2 \) is fixed we can use the relation \( c_2 = \frac{\partial}{\partial x} f(x) \) to determine the semi-classical conformal block

\[
F(x_i) \approx e^{-\frac{c}{6} f(x_i)} \tag{F.4}
\]

For our particular semi-classical limit let us define \( \epsilon_i \equiv \frac{6h_i}{c} \). We write the solutions for \( \psi \) as

\[
\psi = \psi^{(0)} + \epsilon_1 \psi^{(1)} + \epsilon_2 \psi^{(2)} + \ldots \tag{F.5}
\]
APPENDIX F. COMPUTING VIRASORO BLOCKS VIA MONODROMY

Then we can write
\[
T(z) = \frac{\epsilon_2}{(1-z)^2} + \epsilon_1 \left( \frac{1}{z^2} + \frac{1}{(z-x)^2} + \frac{2}{z(1-z)} - \frac{\epsilon_2}{\epsilon_1} \frac{x(1-x)}{z(z-x)(1-z)} \right). \tag{F.6}
\]
We can immediately solve the differential equation for \( \psi^{(0)} \) to find the two solutions
\[
\psi_{1,2}^{(0)}(z) = (1-z)^{\frac{1\pm\sqrt{1-4\epsilon_2}}{2}} \tag{F.7}
\]
Notice that the exponent transitions from real to complex exactly when the large mass \( h_2 \) develops a horizon in AdS_3. To see this, recall that \( c = \frac{3}{2G} \) so we have
\[
m_2 = 2h_2 = \frac{\epsilon_2 c}{3} = \frac{\epsilon_2}{2G} \tag{F.8}
\]
Thus, exactly at \( \epsilon_2 = \frac{1}{4} \), the mass reaches the critical mass \( \frac{1}{8G} \) to make a BTZ black hole.

To solve for \( \psi \) at higher orders in \( \epsilon_1 \), it is useful to use our zeroth order solutions in order to reduce the second order differential equation to a first order differential equation using the method of variation of parameters. In this method, given an inhomogeneous ODE of the form
\[
y''(z) + a(z)y(z) = b(z) \tag{F.9}
\]
and two solutions \( y_i(z) \) to the homogeneous ODE \( y''(z) + a(z)y(z) = 0 \), we can find a solution of the form
\[
y_p(z) = f_1(z)y_1(z) + f_2(z)y_2(z) \tag{F.10}
\]
through
\[
f_1'(z) = -\frac{y_2(z)b(z)}{W(z)}; \quad f_2'(z) = \frac{y_1(z)b(z)}{W(z)} \tag{F.11}
\]
APPENDIX F. COMPUTING VIRASORO BLOCKS VIA MONODROMY

where

\[ W(z) \equiv y_1(z)y_2'(z) - y_1'(z)y_2(z). \] (F.12)

is the Wronskian determinant.

To bring our problem into this form, we divide up \( T \) into a zero-th order piece \( T^{(0)} \) and a correction \( T^{(1)} \):

\[
T = T^{(0)} + \epsilon_1 T^{(1)} + \epsilon_1^2 T^{(2)} + \ldots
\]

\[
T^{(0)} = \epsilon_2 \frac{1}{(1-z)^2}
\]

\[
T^{(1)} = \left( \frac{1}{z^2} + \frac{1}{(z-x)^2} + \frac{2}{z(1-z)} - \frac{c_2^{(1)}}{\epsilon_1} \frac{x(1-x)}{z(z-x)(1-z)} \right). \] (F.13)

At linear order in \( \epsilon_1 \), our differential equation takes the form

\[
(\psi^{(1)}_i)' = -T^{(1)}\psi^{(0)}_i. \] (F.14)

Now we can determine \( \psi^{(1)}_i \). We simply need to integrate

\[
\psi^{(1)}_i = \psi^{(0)}_i \int dz \frac{-\psi^{(0)}_2(-T^{(1)}\psi^{(0)}_i)}{W} + \psi^{(0)}_2 \int dz \frac{\psi^{(0)}_1(-T^{(1)}\psi^{(0)}_i)}{W}. \] (F.15)

These integrals can be performed in closed form in terms of logarithms and hypergeometric functions, which allows one to read off their monodromy properties.

We want to demand that the solutions \( \psi^{(1)}_i \) transform with eigenvalues given by (E.28) as \( z \) encircles 0 and \( x \) in order to determine the function \( c_2(x) \). The method of variation of parameters automatically gives \( \psi^{(1)} \) in a form that is decomposed into a basis of the zero-th order solutions multiplied by coefficients that are functions of

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z. Let us analyze the coefficient of $\psi_1^{(0)}$ first, since it is simpler:

$$
\int dz \frac{-\psi_2^{(0)}(-T_1^{(1)}\psi_1^{(0)})}{W} = \frac{\left(\frac{\epsilon_2}{\epsilon_1}(1 - x) + 1\right) \log(\frac{z}{z-x}) + \frac{(x-2)z+x}{z(z-x)}}{\sqrt{1 - 4\epsilon_2}}.
$$

(F.16)

It is easy to see that this returns to itself after a rotation of $z = re^{i\phi}$ with $\phi$ from 0 to $2\pi$ if $r > x$, since we never cross the branch cut of the logarithm. This can also be seen by noting that the two poles of the integrand at $z = 0$ and $z = x$ have opposite residues. This means that this term does not contribute to the monodromy of $\psi^{(1)}$.

Now consider the second term:

$$
\int dz \frac{\psi_2^{(0)}(-T_1^{(1)}\psi_1^{(0)})}{W} = \int dz \frac{(1 - z)^{\alpha_2} \left(\frac{c_2(x-1)xz(x-z)}{\epsilon_1} - (x^2 - 2xz)(z + 1) - 2z^2\right)}{z^2\alpha_2(x - z)^2}
$$

(F.17)

After either a direct evaluation, or an examination of the residues of the poles at $z = 0$ and $z = x$, we find that under a $2\pi$ phase rotation, the integral shifts by a monodromy ($\delta M_{0x}$)$_{12}$ given by

$$
(\delta M_{0x})_{12} = \frac{2\pi i}{\alpha_2} \left(\alpha_2 - 1\right) - \left(\frac{c_2(x)}{\epsilon_1}(x - 1) - \alpha_2 - 1\right) (1 - x)^{\alpha_2} + \frac{c_2(x)}{\epsilon_1}(x - 1),
$$

(F.18)

where $\alpha_2 \equiv \sqrt{1 - 4\epsilon_2}$. The calculation for $\psi_2^{(1)}$ follows from the same calculation but with $\alpha_2 \rightarrow -\alpha_2$. At this order, we have therefore found the monodromy matrix is

$$
\delta M_{0x} = \begin{pmatrix}
0 & (\delta M_{0x})_{12} \\
(\delta M_{0x})_{21} & 0
\end{pmatrix},
$$

(F.19)

where $(\delta M_{0x})_{21}[\alpha_2] = -(\delta M_{0x})_{12}[-\alpha_2]$. The eigenvalues of $M_{0x}$ at this order are therefore $1 \pm \left[(\delta M_{0x})_{12}(\delta M_{0x})_{21}\right]^{1/2}$. By inspection of (E.28) expanded to linear order
in $h_p$, we can therefore identify $\sqrt{(\delta M_{0x})_{12}(\delta M_{0x})_{21}}$ as $2i\pi \epsilon_p$, or equivalently

$$(\delta M_{0x})_{12}(\delta M_{0x})_{21} = -4\pi^2 \epsilon_p^2. \quad (F.20)$$

This equation can easily be solved for $c_2$:

$$c_2 = \frac{\epsilon_1 (-1 + \alpha_2 + (1 - x)^{\alpha_2}(1 + \alpha_2)) \pm \alpha_2 (1 - x)^{\frac{\alpha_2}{2}} \epsilon_p}{(1 - x)(1 - (1 - x)^{\alpha_2})}. \quad (F.21)$$

Finally, this can be integrated to get the conformal block at $O(\epsilon_1, \epsilon_p)$ and any $\epsilon_2$. We choose the integration constant and the sign of $\pm$ in the above equation so that $f(z) \sim 2(\epsilon_1 - \epsilon_p) \log(z)$ at $z \sim 0$, to obtain

$$f(z) = (2\epsilon_1 - \epsilon_p) \log\left(\frac{1 - (1 - z)^{\alpha_2}}{\alpha_2}\right) + \epsilon_1 (1 - \alpha_2) \log(1-z) + 2\epsilon_p \log\left(\frac{1 + (1 - z)^{\frac{\alpha_2}{2}}}{2}\right). \quad (F.22)$$

This gives us the conformal block in the limit we desired, where one operator $h_1$ is a ‘test mass’ and the other operator of dimension $h_2 \propto c$ would create a finite deficit angle or a BTZ black hole in AdS.

Let us pause to note the approximations we have made. Aside from the limit $c \to \infty$ with $h_i/c$ fixed, we have also expanded the function $f$ in the conformal block $\mathcal{F} \approx e^{-\frac{\pi}{2}f}$ in $h_1/c$. Since we have only computed $f$ to first order in $h_1/c$, we are dropping terms of order $h_1^2/c^2$, which means that we have ignored effects of order $h_1^2/c$ in the exponent. By pushing the monodromy method further and working to higher order in $h_1/c$, we could control these neglected terms. However, the monodromy method always neglects terms of order $1 \ll c$ in the exponent of $\mathcal{F}$. 

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As a check, we can look at the identity block $\epsilon_p = 0$ and compare to our results from the direct approach. Replacing $\alpha_2 = \sqrt{1 - 4\epsilon_2}$ and expanding to $O(\epsilon_2)$ to compare with the result of appendix D, one finds

$$\frac{1}{\epsilon_1} f(z) = 2 \log(z) - \frac{\epsilon_2}{3} z^2 F_1(2, 2, 4, z) + O(\epsilon_2^2). \quad (F.23)$$

We see that the second term matches, as expected. We have also checked that equation (F.22) agrees with the recursion relation method [254, 261, 262] when we expand in small $z$.

### F.2 S-Channel Virasoro Blocks at Quadratic Order

We can also obtain the conformal blocks at order $O(\epsilon_1^2, \epsilon_p^2)$ if we set $\epsilon_2 = \epsilon_1 = \epsilon$. To do this, we take our first order solutions in the limit of small $\epsilon_2$ and substitute them back into (F.15). The resulting expression for $\psi_{1,2}^{(2)}$ simply contains logarithms and dilogarithms, and thus the monodromy can straightforwardly be matched to (E.28) at second order in $h_p/c$. We find the result:

$$f^{(1)}(z) = (2\epsilon - \epsilon_p) \log(z) + 2\epsilon_p \log\left(\frac{1 + \sqrt{1 - z}}{2}\right),$$

$$f^{(2)}(z) = 2 \left(2\epsilon^2 - \epsilon_p^2\right) \log(1 - z) + 4\epsilon_p^2 \log\left(\frac{1}{2} \left(\sqrt{1 - z} + 1\right)\right) + 2 \left(z (\epsilon_p - 2\epsilon)^2 + \log(1 - z) \left(\epsilon_p - 2\sqrt{1 - z}\epsilon\right)^2\right). \quad (F.24)$$
A feature of this result is that $\epsilon_p$ terms contain no divergences at $z \to 1$ at this order:

$$f(1 - y) \overset{y \ll 1}{\sim} 4\epsilon^2 \log(y) + \mathcal{O}(y^0, \epsilon^3, \epsilon_p^3).$$

(F.25)
Appendix G

T-Channel Virasoro Blocks

In this appendix we will study the Virasoro blocks in the t-channel, based on the primary exchange

\[ \langle \phi_1 \phi_2 | \mathcal{O}_p \rangle \langle \mathcal{O}_p | \phi_1 \phi_2 \rangle \]  \hspace{1cm} (G.1)

We will analyze the particular semiclassical heavy/light or probe limit \[266\], where \( h_1^2 \ll h_2, h_p, c \). We will study two further limits which, when combined, are sufficient for discussions of the t-channel blocks on the right-hand side of the bootstrap equation \((4.77)\) in section \[4.4\]. For the first limit, we define \( \delta h \equiv h_p - h_2 \) and then we assume \( \delta h^2 \ll h_2, c \). This is the limit that is relevant for the anti-holomorphic part of the Virasoro blocks. In the second limit we take \( h_p \gg h_2, c \) in order to obtain 2d Virasoro blocks with large spin and fixed twist. This is discussed at the end of this appendix.

The Virasoro blocks greatly simplify in the first limit, so that they are dominated solely by the exchange of the primary \( \mathcal{O}_p \). To see this, note that the three-point...
APPENDIX G. T-CHANNEL VIRASORO BLOCKS

The function is

\[ \langle \phi_1(y_1) \phi_2(y_2) \mathcal{O}_p(y_3) \rangle = \frac{1}{y_{12}^{h_1-\delta h} y_{13}^{2h_2+\delta h-h_1} y_{23}^{h_1+\delta h}}. \]  

(G.2)

Now, when we act on \( \mathcal{O}_p \) with \( L_{-n} \) and take \( y_3 \to 0 \), we find

\[ \frac{\langle \phi_2(\infty) \phi_1(1) L_{-n} | \mathcal{O}_p \rangle}{\langle \phi_2(\infty) \phi_1(1) | \mathcal{O}_p \rangle} = nh_1 + \delta h. \]  

(G.3)

Similarly, the conjugate gives

\[ \frac{\langle \mathcal{O}_p | L_n \phi_1(z) \phi_2(0) \rangle}{\langle \mathcal{O}_p | \phi_1(z) \phi_2(0) \rangle} = z^n (nh_1 + \delta h). \]  

(G.4)

The point is that both of these ratios of 3-pt functions are proportional to \( h_1 \) and \( \delta h \), but they never involve \( h_2 \) or \( c \). This persists if we study more general descendant states.

These computations are relevant for the t-channel blocks if we study a modified version of the ‘graviton basis’ of equation (D.11), where we also include the \( L_{k_{-1}} \) operators. This is necessary because the state \( \mathcal{O}_p|0\rangle = |\mathcal{O}_p\rangle \) will not be annihilated by these global conformal generators. So we have a modified version of the projector in equation (D.19)

\[ P_{\mathcal{O}_p} \approx \sum_{\{m_i, k_i\}} \frac{L_{-m_1}^{k_1} \cdots L_{-m_n}^{k_n} L_{-1}^{k_0} | \mathcal{O}_p \rangle \langle \mathcal{O}_p | L_1^{k_0} L_{m_n}^{k_n} \cdots L_{m_1}^{k_1}}{N_{\mathcal{O}_p}^{\{m_i, k_i\}}}, \]  

(G.5)

which we might use to compute the Virasoro block. The modified normalization

\[ N_{\mathcal{O}_p}^{\{m_i, k_i\}} = \langle \mathcal{O}_p | L_1^{k_0} L_{m_n}^{k_n} \cdots L_{m_1}^{k_1} L_{-m_n}^{k_n} \cdots L_{-m_1}^{k_1} L_{-1}^{k_0} | \mathcal{O}_p \rangle \]  

(G.6)

has a single important feature – namely that in this particular semiclassical limit, we obtain an extra factor of either \( c \) or \( h_p \approx h_2 \) from each additional \( L_m \). Thus the
APPENDIX G. T-CHANNEL VIRASORO BLOCKS

The contribution of descendants to this Virasoro block is always suppressed as a power of one of the ratios

$$
\frac{h_1^2}{h_2}, \frac{\delta h_2^2}{h_2}, \frac{h_1^2}{c}, \frac{\delta h_2^2}{c} \ll 1,
$$

which are small in the probe limit. So in the t-channel, in this heavy / light probe semiclassical limit, not only is it sufficient to use the global blocks for the 2d bootstrap; in fact, it is sufficient to simply use the OPE limit, or the result of primary exchange!

To use the Virasoro blocks at high spin, as is necessary in section 4.4, we also need to study a very different limit where

$$
h_p \gg h_1, h_2.
$$

Combining an anti-holomorphic Virasoro block with $\bar{h}_p \approx \bar{h}_2$ and a holomorphic block with $h_p \gg h_1, h_2$ allows us to construct a block with twist $\tau \approx \Delta_2$ but with large $\ell = h_p - \bar{h}_p$. Fortunately this large $h_p$ limit has already been studied [262], see appendix D of [265] for a thorough discussion using the monodromy method. The result is that

$$
\mathcal{F}(z) \sim (16q)^{h_p - \frac{c}{2}} \theta_3(q) \frac{e^{-8h_1 - 8h_2} z^{h_1 - h_2} (1 - z)^{\frac{c}{2} - h_1 - h_2}}{\sqrt{\pi} K(z)},
$$

where

$$
q = e^{-\pi K(1-z)/K(z)}, \quad \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \sqrt{\frac{2}{\pi} K(z)}, \quad \frac{1}{2} \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-zt)}}.
$$
APPENDIX G. T-CHANNEL VIRASORO BLOCKS

This is the result for the operators inserted at $x_1 = 0, x_2 = z, x_3 = 1, x_4 = \infty$. To apply this to the t-channel of the bootstrap equation, we need to map to $x_1 = 0, x_2 = \infty, x_3 = z, x_4 = 1$, which corresponds to

$$F(z) \to \frac{1}{z^{2h_1}} F\left(1 - \frac{1}{z}\right). \quad (G.12)$$

Expanding near $z \sim 0$, we find

$$F(z) \sim z^{2h_1 - 2h_1}, \quad (G.13)$$

which should be compared with the singularity $z^{-2h_1}$ of the identity block in the s-channel. Clearly there is a mismatch in the power of the singularity, and since $c \gg 1$, the singularity of the Virasoro blocks is much weaker in the limit $h_p \gg h_1, h_2, c$ at small $z$.
Appendix H

Calculation of Deficit Angle Spectrum

In this appendix, we present a more detailed calculation of the results discussed in section 4.4.1. Specifically, we will use the 2d bootstrap equation to place bounds on the coefficients of $t$-channel global conformal blocks. These bounds provide rigorous evidence that the large $\ell$ spectrum of 2d CFTs with large central charge matches that of deficit angles in AdS$_3$.

H.1 Bootstrap Equation in the Lightcone OPE Limit

In the limit $u \ll v \ll 1$, the bootstrap equation takes the approximate form

$$1 \approx \alpha^{-\Delta_1} z^{\frac{1}{2} \left(\Delta_1 + \Delta_2 - \Delta_{12}\right)} v^{-\frac{1}{2} \left(\alpha \Delta_1 + \Delta_2\right)} \sum_{\tau, \ell} P_{\tau, \ell} k'_{2\ell} (1 - z) 2^\tau v^\tau k'_\tau (v).$$

(H.1)
where \( k'_{2\beta}(x) = x^\beta_2 F_1(\beta - \frac{1}{2}\Delta_{12}, \beta - \frac{1}{2}\Delta_{12}; 2\beta; x) \). The left side of this expression is clearly constant and finite, so the \( u, v \)-dependence of the right side must also vanish. The small \( v \) behavior of each term in this series is approximately \( v^{\frac{1}{2}(\tau - \alpha\Delta_1 - \Delta_2)} \), which greatly constrains the possible twists \( \tau \) that can dominate at large \( \ell \).

In particular, there must exist operators with \( \tau \approx \alpha\Delta_1 + \Delta_2 \) in order to produce a constant result in the limit \( v \to 0 \). For the right side to also be independent of \( u \approx z \), there must actually be an infinite tower of conformal blocks with twist accumulating at \( \alpha\Delta_1 + \Delta_2 \) as \( \ell \to \infty \), such that the full sum introduces a power-law singularity in \( z \) not possessed by any individual term. In the small \( v \) limit, where these conformal blocks provide the dominant contribution, we can approximate the bootstrap equation as

\[
1 \approx 2^{\tau_0} \alpha^{-\Delta_1} z^{\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_{12})} \sum_\ell P_{\tau_0, \ell} k'_{2\ell}(1 - z),
\] (H.2)

where \( P_{\tau_0, \ell} \) can be formally thought of as the sum of all conformal block coefficients with twist within some small range centered about \( \tau_0 \equiv \alpha\Delta_1 + \Delta_2 \).

Following the work of [51], the sum over \( \ell \) can be written as an integral over a conformal block coefficient density \( f_0(\ell) \),

\[
\sum_\ell P_{\tau_0, \ell} k'_{2\ell}(1 - z) = \int_0^\infty d\ell \ f_0(\ell) \ k'_{2\ell}(1 - z),
\] (H.3)

where \( f_0(\ell) \) is defined as

\[
f_0(\ell) \equiv \sum_{\ell'} P_{\tau_0, \ell'} \delta(\ell - \ell').
\] (H.4)
APPENDIX H. CALCULATION OF DEFICIT ANGLE SPECTRUM

In the following section we will derive bounds on the structure of $f_0(\ell)$ which indicate that it is of the form

$$f_0(\ell) = A_0 \frac{\Gamma^2(\ell)}{\Gamma(2\ell)} \ell^{\Delta_1 + \Delta_2 - 1}. \quad (H.5)$$

Assuming this form, we can rewrite the bootstrap equation as

$$1 \approx 2^{\tau_0 + 1} \alpha^{-\Delta_1} z^{\frac{1}{2}(\Delta_1 + \Delta_2)} A_0 \int_0^{\infty} d\ell \ell^{\Delta_1 + \Delta_2 - 1} K_{\Delta_1}(2\ell \sqrt{z}). \quad (H.6)$$

This expression can be used to fix the value of $A_0$, which in turn provides the result

$$P_{\tau_0, \ell} \approx \frac{4 \sqrt{\pi} \alpha^{\Delta_1}}{2^{\tau_0 + 2\ell} \Gamma(\Delta_1) \Gamma(\Delta_2)} \ell^{\Delta_1 + \Delta_2 - \frac{3}{2}} \approx 2^{\Delta_1(1-\alpha)} \alpha^{\Delta_1} P_{GFT}^{\Delta_1 + \Delta_2, \ell} \quad (\ell \gg 1). \quad (H.7)$$

At large $\ell$, the approximate conformal block coefficients have been related to those of GFT, with an $\alpha$-dependent coefficient. This is not strictly obligatory, since we are only constraining the accumulation at large $\ell$, and not the contribution of each individual block, but it provides a plausible expectation. It should be noted that we are using the Virasoro blocks in the semi-classical limit, so this result will be corrected by $1/c$ effects.

We can extend this argument to higher twists by considering the bootstrap equation to all orders in $v$,

$$\alpha^{\Delta_1} v^{-\frac{1}{2} \Delta_1(1-\alpha)} \left( \frac{1 - v}{1 - v^\alpha} \right)^{\Delta_1} \approx \left( \frac{u}{v} \right)^{\frac{1}{2}(\Delta_1 + \Delta_2)} u^{-\frac{1}{2} \Delta_2} \sum_{\tau, \ell} P_{\tau, \ell} g_{\tau, \ell}(v, u), \quad (H.8)$$

which can be rewritten in the more useful form

$$(1 - v^\alpha)^{-\Delta_1} \approx \alpha^{-\Delta_1} z^{\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_1)} v^{-\frac{\tau_0}{2}} (1 - v)^{-\Delta_1} \sum_{\tau, \ell} P_{\tau, \ell} g_{\tau, \ell}(v, u), \quad (H.9)$$
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where we have used the relation \( u \approx z(1 - v) \). We can now subtract the \( \tau_0 \) contributions from both sides of this expression. Since we are specifically working in the limit \( \Delta_1 \ll \Delta_2 \), such that \( \tau_0 \approx \Delta_2 \), we can use the approximate global conformal blocks derived in appendix C to calculate the approximate \( \tau_0 \) contribution,

\[
\sum_{\ell} P_{\tau_0,\ell} g_{\tau_0,\ell}(v, u) \approx \alpha^{\Delta_1} z^{-\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_{12})} v^{\frac{\Delta_1}{2}} (1 - v)^{\Delta_{12}}. \tag{H.10}
\]

Notice that this expression is of precisely the right form to cancel the overall prefactor, such that the \( \tau_0 \) contribution is simply 1, with no subleading corrections in \( v \). Our modified bootstrap equation then becomes

\[
(1 - v^\alpha)^{-\Delta_1} - 1 \approx \alpha^{-\Delta_1} z^{\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_{12})} v^{-\frac{\Delta_1}{2}} (1 - v)^{-\Delta_{12}} \sum_{\tau > \tau_0, \ell} P_{\tau,\ell} g_{\tau,\ell}(v, u). \tag{H.11}
\]

We can now repeat our earlier procedure with this modified bootstrap equation. Expanding the left side as a power series in \( v^\alpha \) and taking the small \( v \) limit, we obtain the relation

\[
\Delta_1 v^\alpha \approx \alpha^{-\Delta_1} z^{\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_{12})} v^{-\frac{\Delta_1}{2}} \sum_{\tau > \tau_0, \ell} P_{\tau,\ell} k'_{2\ell}(1 - z)^{2^\tau} v^{\frac{1}{2}k'_{2\ell}}(v). \tag{H.12}
\]

For this expression to be satisfied, there must be an infinite tower of conformal blocks with twist \( \tau \approx \alpha(\Delta_1 + 2) + \Delta_2 \). To find the corresponding conformal block coefficients, we can again consider the limit \( v \to 0 \), where these operators are the dominant contribution,

\[
\Delta_1 \approx 2^{\tau_1} \alpha^{-\Delta_1} z^{\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_{12})} \sum_{\ell} P_{\tau_1,\ell} k'_{2\ell}(1 - z), \tag{H.13}
\]
where we have introduced the generalized notation \( \tau_n \equiv \alpha (\Delta_1 + 2n) + \Delta_2 \). We can also define a generalized conformal block coefficient density \( f_n(\ell) \), such that

\[
\sum_\ell P_{\tau_n,\ell} k_{2\ell}'(1-z) = \int_0^\infty d\ell \, f_n(\ell) k_{2\ell}'(1-z). \tag{H.14}
\]

The bounds we will derive in the following section indicate that this more general density is also of the form

\[
f_n(\ell) = A_n \frac{\Gamma^2(\ell)}{\Gamma(2\ell)} \ell^{\Delta_1 + \Delta_2 - 1}. \tag{H.15}
\]

Assuming this form for our case of \( n = 1 \), the modified bootstrap equation becomes

\[
\Delta_1 \approx 2^{\Delta_1 + 1} \alpha^{-\Delta_1} z^{\frac{1}{2}(\Delta_1 + \Delta_2)} A_1 \int_0^\infty d\ell \, \ell^{\Delta_1 + \Delta_2 - 1} K_{\Delta_1 \Delta_2}(2\ell \sqrt{z}). \tag{H.16}
\]

Solving this expression for \( A_1 \), we then find the conformal block coefficients

\[
P_{\tau_1,\ell} \approx \frac{4\sqrt{\pi} \Delta_1 \alpha^{\Delta_1}}{2^{\Delta_1 + 2} \Gamma(\Delta_1) \Gamma(\Delta_2)} \ell^{\Delta_1 + \Delta_2 - 3} \quad (\ell \gg 1). \tag{H.17}
\]

We therefore find coefficients of a very similar form to those for \( n = 0 \). Inspired by those previous results, let’s compare this expression to the coefficients of GFT \[215\],

\[
P_{A_1+\Delta_2+2n,\ell} = \frac{\Gamma(\Delta_1)}{n! 2^{2n} \Gamma(\Delta_1 + \Delta_2)} P^{GFT}_{A_1+\Delta_2+2,\ell}, \quad (\ell \gg 1) \tag{H.18}
\]

where we have specifically taken the limit \( \Delta_1, n \ll \Delta_2 < \ell \). We therefore have the relation

\[
P_{\tau_1,\ell} \approx 2^{(\Delta_1 + 2)(1-\alpha)} \alpha^{\Delta_1} P^{GFT}_{\Delta_1+\Delta_2+2,\ell} \quad (\ell \gg 1). \tag{H.19}
\]

with the same caveat as above, namely that we can really only constrain the large \( \ell \) accumulation, and not the contribution of each individual term.
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We can continue to repeat this procedure to find the coefficients for increasing values of $n$. To see this most clearly, we expand the left side of the bootstrap equation as a series in $v^\alpha$,

$$\sum_{n=0}^{\infty} \frac{\Delta_1^n}{n!} v^{n\alpha} \approx \alpha^{-\Delta_1} z^{1/2(\Delta_1+\Delta_2-\Delta_{12})} v^{-\frac{\tau_0}{2}} (1-v)^{-\Delta_{12}} \sum_{\tau,\ell} P_{\tau,\ell} g_{\tau,\ell}(v,u). \quad (H.20)$$

For $n \ll \Delta_2$, each individual $v^{n\alpha}$ term in the series on the left corresponds to the full contribution of the $\tau_n$ tower of conformal blocks on the right side. Our procedure can be iterated to find the corresponding coefficients $P_{\tau_n,\ell}$, but we can already see the full answer from this expression. The factor of $\frac{(\Delta_1^n)}{n!}$ in the power series is precisely the factor needed to reproduce the appropriate GFT coefficients, such that we obtain the general relation

$$P_{\tau_n,\ell} \approx 2^{(\Delta_1+2n)(1-\alpha)} \alpha^{\Delta_1} P_{\Delta_1+\Delta_2+2n,\ell}^{GFT} \quad (\ell \gg 1). \quad (H.21)$$

We therefore see that in the limit $\alpha \to 1$, with vanishing deficit angle, the large $\ell$ spectrum of operators and conformal block coefficients for any CFT with large central charge perfectly reproduces that of a generalized free theory. This is precisely what we would expect, as it corresponds to the $c \to \infty$ limit with fixed $\Delta_1$ and $\Delta_2$.

H.2 Bounds on Coefficient Density

We will now place bounds on the asymptotic behavior of the conformal block coefficient density $f_n(\ell)$. More specifically, we will prove that given a function $\mathcal{L}_n(z)$,
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defined as

\[ \mathcal{L}_n(z) \equiv \int_0^\infty d\ell \, f_n(\ell) k^\prime_{2\ell}(1 - z), \quad (H.22) \]

which behaves like \( z^{\frac{1}{2}(\Delta_{12} - a)} \) at small \( z \), then there exist numbers \( A_L, A_U \) such that

the integrated density

\[ F_n(L) \equiv \int_0^L d\ell \, \frac{\Gamma(2\ell)}{\Gamma^2(\ell)} f_n(\ell), \quad (H.23) \]

is bounded at large \( L \) by

\[ A_U L^a \geq F_n(L) \geq A_L \frac{L^a}{\ln L}. \quad (H.24) \]

This discussion will be almost identical to a similar proof in [51], which interested readers may consult for more details.

First, we consider the upper bound. For simplicity, we define the function

\[ h(\ell, z) \equiv \frac{\Gamma^2(\ell)}{\Gamma(2\ell)} k^\prime_{2\ell}(1 - z), \quad (H.25) \]

which is a positive, decreasing function of \( \ell \) at any fixed \( z \). Since the integrand of \( \mathcal{L}_n(z) \) is non-negative, we can place the bound

\[ \mathcal{L}_n(z) \geq h(L, z) F_n(L), \quad (H.26) \]

for any value of \( L \). As shown in appendix C at large \( L \) and fixed \( \lambda \equiv L \sqrt{z} \) the function \( h(L, z) \) takes the approximate form

\[ \lim_{L \to \infty} h(L, z) = 2 z^{\frac{1}{2} \Delta_{12}} K_{\Delta_{12}}(2\lambda). \quad (H.27) \]
Combining these two results, we obtain the upper bound
\[ F_n(L) \leq \frac{L^a}{2\lambda^a K_{\Delta_{12}}(2\lambda)} \quad (L \gg 1). \] (H.28)

The parameter \( \lambda \) is arbitrary and \( L \)-independent, such that we can identify this upper bound as \( A_U L^a \).

We now turn to the lower bound. Since \( h(\ell, z) \) is a decreasing function of \( \ell \), the related function
\[ \tilde{h}(\ell, z) \equiv -\frac{\partial}{\partial \ell} h(\ell, z), \] (H.29)
is positive for all \( \ell \). Using this new function, we can write \( \mathcal{L}_n(z) \) as
\[ \mathcal{L}_n(z) = \int_0^\infty dL \, F_n(L) \tilde{h}(L, z). \] (H.30)

In the small \( z \) limit, this integrand is dominated by the contribution at large \( L \), such that the integral is unaffected by a shift in the lower limit of integration. We can then define a new function,
\[ \tilde{\mathcal{L}}_n(z) = z^{-\frac{1}{2} \Delta_{12}} \int_{L_0}^\infty dL \, F_n(L) \tilde{h}(L, z), \] (H.31)
which approaches \( z^{-\frac{a}{2}} \) at small \( z \). For sufficiently large \( L \), the function \( h(L, z) \) has the approximate form
\[ h(L, z) \approx z^{\frac{1}{2} \Delta_{12}} \sqrt{\frac{\pi}{L}} e^{-2L\sqrt{z}} \sqrt{\frac{1}{L - \frac{4}{\sqrt{z}}}}, \] (H.32)
such that the function \( z^{-\frac{1}{2} \Delta_{12}} \tilde{h}(L, z) \) is independent of \( \Delta_{12} \) at large \( L \). Since \( \tilde{\mathcal{L}}_n(z) \) is also independent of \( \Delta_{12} \), the asymptotic behavior of \( F_n(L) \) must be \( \Delta_{12} \)-independent.
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We shall therefore specifically consider the simplifying case $\Delta_{12} = 0$, as the resulting bound will generalize to arbitrary $\Delta_{12}$ in the large $L$ limit.

Since $F_n(L)$ is an integral over non-negative terms, $F_n(L_1) > F_n(L_0)$ for $L_1 > L_0$, such that we can obtain

$$\tilde{\mathcal{C}}_n(z) = \int_{L_0}^{L_1} dL F_n(L) \tilde{h}(L, z) + \int_{L_1}^{\infty} dL F_n(L) \tilde{h}(L, z)$$

$$\leq F_n(L_1) \left( h(L_0, z) - h(L_1, z) \right) + A_U \int_{L_1}^{\infty} dL L^a \tilde{h}(L, z) \quad (H.33)$$

$$\leq F_n(L_1) h(L_0, z) + A_U \int_{L_1}^{\infty} dL L^a \tilde{h}(L, z).$$

In the limit of large $\lambda \equiv L_1 \sqrt{z}$, the second integral takes the approximate form $A_U' e^{-2\lambda}$, for some new coefficient $A_U'$. As for the first term, in the small $z$ limit the function $h(L_0, z)$ is proportional to $\ln(1/z)$, such that we have

$$\tilde{\mathcal{C}}_n(\lambda/L_1) \leq A'_L F_n(L_1) \ln(L_1/\lambda) + A'_U e^{-2\lambda}, \quad (H.34)$$

where $A'_L$ is some constant coefficient. In the limit of large $\lambda$, the second term can be made arbitrarily small, such that we obtain the lower bound

$$F_n(L) \geq \frac{L^a}{A'_L \lambda^a \ln(L)} \quad (L \gg 1). \quad (H.35)$$

As the parameter $\lambda$ is arbitrary, we can then identify this lower bound as $A_L L^a / \ln(L)$.

Note that if the lower bound could be improved, such that we could prove that $F_n(L) \propto L^a$, we would expect the coefficient density $f_n(\ell)$ to have the asymptotic form

$$f_n(\ell) \approx A_n \frac{\Gamma(\ell)}{\Gamma(2\ell)} \ell^{a-1} \quad (\ell \gg 1). \quad (H.36)$$
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Vita

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