TOPOLOGY OPTIMIZATION UNDER NONLINEAR MECHANICS

by

Reza Lotfi

A dissertation submitted to The Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy.

Baltimore, Maryland

Nov, 2013

© Reza Lotfi 2013

All rights reserved
Abstract

Topology optimization is a demonstrated tool for the design of components with enhanced mechanical properties. Despite tremendous advances in topology optimization methods over the past 25 years, however, the vast majority of topology optimization work assumes linear elastic governing mechanics. This dissertation proposes algorithms for designing components and cellular materials with optimized properties governed by nonlinear mechanics.

Optimizing for properties in the nonlinear regime poses several challenges, both fundamental and practical. The general ill-posedness of maximum stiffness formulations leads to numerical instabilities of solution mesh dependence, well-known for problems governed by linear mechanics and persisting for problems governed by nonlinear mechanics. Other instabilities are associated with modeling low density elements and include stress singularities and excessive element distortion. Computational expense must also be addressed, as the large dimensionality of topology optimization problems coupled with iterative nonlinear finite element analysis becomes
computationally prohibitive. These issues are circumvented herein using projection-based topology optimization methods that enable separation of the analysis and design spaces and subsequent manipulation of the spaces to achieve stability and efficiency as needed. While similar approaches proposed in literature have shown significant sensitivity to user-defined optimization parameters, the nature of the nonlinear projection is shown to provide more robust and stable performance in the context of nonlinear mechanics.

The proposed algorithms are then used to design periodic cellular materials with optimized nonlinear response properties, including maximized energy absorption considering geometric and material nonlinearities. Effective elastic properties and symmetry of the bulk material are estimated using elastic homogenization under the assumption of infinite periodicity. Nonlinear properties are estimated using finite periodicity. This leads to a unit cell topology optimization problem with analysis conducted over two different domains. The target material system is a Bulk Metallic Glass cellular material whose constituent properties are approximated with an elastoplastic material model. Several new topologies are presented that are shown to offer improved nonlinear performance when compared to topologies optimized for elastic properties and traditional honeycomb patterns.

Advisor: Dr. James K. Guest

Committee members: Dr. Takeru Igusa, Dr. Somnath Ghosh
Acknowledgements

I would like gratefully thank my advisor Dr. James K Guest for all the supports and encouragements during my research. He is not only my advisor but also a great friend in my life. His patience and understanding was the basic factor to achieve this accomplishment. He gave me the opportunity to think independently and apply my ideas in different parts of this research.

I would also like to thank Dr. Takeru Igusa who showed me how we can solve very complicated physical problems using statistical analysis. I always appreciate his patience to move me step by step to the higher levels.

I would like to thank the department of civil engineering at Johns Hopkins University and my committee members, Specially Dr. Somnath Ghosh and Dr. Lori Graham-Brady, for their valuable comments and suggestions.

And my special thanks to my parents for encouraging me in all stages of my life.

To my mother, Fatemeh, and my father, Nurolah
# Contents

Abstract ii

Acknowledgements iv

List of Tables ix

List of Figures x

1 Circumventing Numerical Instabilities Associated With Low Density Elements 7

1.1 Introduction ......................................................... 7

1.2 Solid-Only Modeling in Topology Optimization ...................... 12

1.2.1 Heaviside Projection Equations .............................. 12

1.2.2 Problem Formulation ........................................... 14

1.2.3 Sensitivities .................................................... 16

1.2.4 Inherent Element Reintroduction .............................. 18

1.2.5 Sensitivity Plots ............................................... 18
1.2.6 Solids-Only Finite Element Analysis .................................. 24
1.3 Linear Elastic Modeling ...................................................... 24
  1.3.1 Cantilever Beam Example ............................................. 25
  1.3.2 Compliant Inverter Example ......................................... 28
  1.3.3 3D Pile Cap ............................................................. 30
1.4 Summary ........................................................................... 32

2 Topology Optimization For Nonlinear Mechanics .................. 33
  2.1 Introduction .................................................................... 33
  2.2 Objective Functions ........................................................ 35
  2.3 Geometrically Nonlinear Topology Optimization ............... 36
  2.4 Sensitivities .................................................................... 38
  2.5 Long Cantilever Example .................................................. 39
  2.6 Material Nonlinear Optimization ....................................... 41
    2.6.1 Sensitivity Analysis .................................................... 44
    2.6.2 Beam Examples ........................................................ 46
  2.7 Minimize Peak Stress ....................................................... 50
  2.8 Comparison of Optimized Structures ................................. 55
  2.9 Summary ........................................................................ 56

3 Topology Optimization For Cellular Structures .................... 59
  3.1 Introduction .................................................................... 59
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.4.1 Climatological Relative Humidity</td>
<td>116</td>
</tr>
<tr>
<td>A.5 Vertical Variation In Relative Humidity</td>
<td>123</td>
</tr>
<tr>
<td>A.5.1 Comparison With MLS Data</td>
<td>126</td>
</tr>
<tr>
<td>A.5.2 Relationship with Other Fields</td>
<td>127</td>
</tr>
<tr>
<td>A.6 Relative Contribution From Changes In Specific Humidity And Temper</td>
<td>129</td>
</tr>
<tr>
<td>A.7 Probability Distribution Functions</td>
<td>134</td>
</tr>
<tr>
<td>A.8 Summary</td>
<td>142</td>
</tr>
<tr>
<td>A.9 Acknowledgement</td>
<td>143</td>
</tr>
</tbody>
</table>
List of Tables

1.1 Objective function and number of equations for cantilever example . . 27
1.2 Objective function and number of equations for compliant inverter ex-
ample . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 28
1.3 Objective function and number of equations for 3D cap example . . 31

2.1 Statistical characteristics of optimized structure for p-norm stress
based objective function and compliance . . . . . . . . . . . . . . . . 54

3.1 Number of symmetry planes for different material types . . . . . . . 61
3.2 Mechanical properties of optimized structure . . . . . . . . . . . . . 69
3.3 Compare mechanical behavior of structures for 12.5% volume fraction 80
3.4 Compare mechanical behavior of structures for 25% volume fraction . 83
3.5 Sensitivity of mechanical properties of optimized structures to cell
numbers . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 88
3.6 Sensitivity of mechanical properties of optimized structures to cell
numbers . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 89

4.1 Geometrical properties for corrugated plates . . . . . . . . . . . . . 98
4.2 Maximum vertical load . . . . . . . . . . . . . . . . . . . . . . . . . 100
4.3 Continue Maximum vertical load . . . . . . . . . . . . . . . . . . . 101
4.4 Maximum vertical load . . . . . . . . . . . . . . . . . . . . . . . . . 102
4.5 Continue Maximum vertical load . . . . . . . . . . . . . . . . . . . 103
4.6 Maximum vertical load . . . . . . . . . . . . . . . . . . . . . . . . . 103
4.7 Von Mises stresses and stress-strain curve . . . . . . . . . . . . . . 104
4.8 Von Mises stresses and stress-strain curve . . . . . . . . . . . . . . 105
4.9 Material nonlinearity effect in load-deflection results . . . . . . . . . . 106
4.10 Material nonlinearity effect in load-deflection results . . . . . . . . . . 106
List of Figures

1.1 Distortion of elements in finite element analysis. Yoon and Kim (2005b) ................................................. 8
1.2 Element connectivity parametrization(ECP) in Yoon and Kim (2005a) method .................................................. 10
1.3 (a)Projection domain for design variable i and (b) element e neighborhood set ............................................. 13
1.4 Cantilever beam .......................................................... 22
1.5 Cantilever beam .......................................................... 23
1.6 a) Initial material distribution b)final material distribution .......................................................... 23
1.7 Optimized structure for a) void modeling b) $\rho_t = 10^{-3}$ c) $\rho_t = 10^{-2}$ .......................... 26
1.8 Number of equations to be solved in each optimization iterations ............................................. 27
1.9 Solid-only modeling method for a)Inverter example. b) Considering void modeling and c) $\rho_t = 10^{-3}$ d) $\rho_t = 10^{-2}$ ................................................. 29
1.10 3D pile cap design domain ................................................. 30
1.11 Pile cap solutions, using a) void modeling and b) solid-only modeling with $\rho_t = 10^{-3}$ ................................................. 31

2.1 Distortion of elements in finite element analysis. Yoon and Kim (2005b) ................................................. 34
2.2 Long cantilever beam .......................................................... 39
2.3 Optimal structure for cantilever example ................................................. 40
2.4 Clamped sides beam example (Maute et al. (1998)) ................................................. 48
2.5 Final structures .......................................................... 48
2.6 Load-displacement diagram ................................................. 48
2.7 Simple supported beam(Maute et al. (1998)) ................................................. 49
2.8 Final structures .......................................................... 49
2.9 Load-displacement diagram ................................................. 49
2.10 Optimized structure, resulting Von-Mises stresses contour, and probability density function of stresses for different objective functions ................................................. 53
2.11 Optimized structures using different methods and objectives ................................................. 57
2.12 Force displacement curve for optimized structure

3.1 Material properties which are used as the objective function

3.2 Physical representation of bulk modulus in 3D

3.3 Optimized structure and Von-Mises stresses in the structure and stress-strain curve

3.4 Deflected states and Von-Mises stress contours for maximized elastic modulus structure corresponding to points in Figure 3.3(c)

3.5 Deflected states and Von-Mises stress contours for maximized elastic modulus structure corresponding to points in Figure 3.3(d)

3.6 Optimized structure for shear modulus and Poisson’s ratio

3.7 Optimized structures for elastic energy and total energy dissipation

3.8 Deformed structures, stress distribution and stress-strain curve

3.9 Deformed structures and stress distribution for the honeycomb structure under horizontal loading

3.10 Deformed structures and stress distribution for the honeycomb structure under vertical loading

3.11 Deformed structures and stress distribution for the structure which optimized for total energy

3.12 Deformed structures and stress distribution for the structure which optimized for elastic energy

3.13 Deformed structures and stress distribution for the structure which optimized for bulk modulus

3.14 Deformed structures and stress distribution for the structure which optimized for elastic modulus

3.15 Stress-strain diagram for optimized structures for total energy dissipation

3.16 Stress-strain diagram for optimized structures for elastic energy dissipation

3.17 Honeycomb experimental test and stress strain curve (Sarac et al. (2012))

3.18 Deformed structure and Von-Mises stresses for Honeycomb and optimized structure for total energy dissipation

3.19 Stress-strain diagram for honeycomb structure and optimized structure for total energy dissipation

3.20 Experimental structures for optimized structure and honeycomb (Schroers Lab at Yale University)

3.21 Stress-strain diagram for optimized structure and honeycomb (Schroers Lab at Yale University)

3.22 Experimental structures for optimized structure and honeycomb (Schroers Lab at Yale University)

3.23 Stress-strain diagram for optimized structure and honeycomb (Schroers Lab at Yale University)
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1 Composite with corrugated layers, geometric properties and volume</td>
<td>98</td>
</tr>
<tr>
<td>4.2 Manufacturing corrugated structures in the lab (Schroers (2010))</td>
<td>99</td>
</tr>
<tr>
<td>4.3 Von Mises stresses for an example of each type</td>
<td>99</td>
</tr>
<tr>
<td>4.4 3D corrugated plates in compression</td>
<td>104</td>
</tr>
<tr>
<td>4.5 3D corrugated plates in Bending</td>
<td>104</td>
</tr>
<tr>
<td>A.1 Climatology mean for (a)DJF (b)All seasons</td>
<td>118</td>
</tr>
<tr>
<td>A.2 Climatology mean for (a)JJA (b) All seasons</td>
<td>119</td>
</tr>
<tr>
<td>A.3 Pacific circulations (Walker and Hadley) for (a)El-Nino (b)La-Nina</td>
<td>119</td>
</tr>
<tr>
<td>A.4 Data analysis of relative humidity at 300hPa (a)EOF1 (b)ENSO</td>
<td>121</td>
</tr>
<tr>
<td>A.5 Time series of EOF1 and ENSO</td>
<td>122</td>
</tr>
<tr>
<td>A.6 Vertical cross section along a)equator (b)25\textdegree N (c)25\textdegree S</td>
<td>124</td>
</tr>
<tr>
<td>A.7 Relative humidity sensitivity section for (a)120\textdegree E (b)180\textdegree (c)140\textdegree W (d)80\textdegree W</td>
<td>125</td>
</tr>
<tr>
<td>A.8 Horizontal maps of ENSO sensitivity at levels of (a)150hPa (b)300hPa (c)500hPa (d)700 hPa</td>
<td>125</td>
</tr>
<tr>
<td>A.9 Maps of MLS and AIRS differences (a)relative humidity AIRS @ 300hPa (b)relative humidity MLS @261 hPa (c)relative humidity AIRS @ 200hPa (d)relative humidity MLS @178 hPa</td>
<td>126</td>
</tr>
<tr>
<td>A.10 Maps of sensitivity and upward winds for (a)relative humidity (b)specific humidity (c)temperature</td>
<td>127</td>
</tr>
<tr>
<td>A.11 Maps of sensitivity in (a) SST (b) OLR on relative humidity colormap</td>
<td>128</td>
</tr>
<tr>
<td>A.12 Map of contribution of T, H2O , second order and difference with exact value</td>
<td>130</td>
</tr>
<tr>
<td>A.13 Vertical cross Section of contribution of specific humidity , temperature and relative humidity for 180\textdegree</td>
<td>131</td>
</tr>
<tr>
<td>A.14 WaterVapor &amp; Temperature , Latitudinal section at 180\textdegree</td>
<td>131</td>
</tr>
<tr>
<td>A.15 Vertical cross section of contribution of specific humidity , temperature and relative humidity for 140\textdegree W</td>
<td>132</td>
</tr>
<tr>
<td>A.16 Water vapor and temperature , longitudinal section at 140\textdegree W</td>
<td>133</td>
</tr>
<tr>
<td>A.17 Vertical cross section of contribution of specific humidity , temperature and relative humidity for 120\textdegree W</td>
<td>133</td>
</tr>
<tr>
<td>A.18 Vertical cross section of contribution of specific humidity , temperature and relative humidity for 120\textdegree E</td>
<td>134</td>
</tr>
<tr>
<td>A.19 Regions with high changes in the mean values of relative humidity difference</td>
<td>135</td>
</tr>
<tr>
<td>A.20 Probability density function of relative humidity at the tropical region</td>
<td>136</td>
</tr>
<tr>
<td>A.21 Number of points in the dry and wet regions</td>
<td>136</td>
</tr>
<tr>
<td>A.22 Changes in relative humidity , specific humidity and temperature</td>
<td>137</td>
</tr>
</tbody>
</table>
A.23 Pdfs of (a) relative humidity (b) specific humidity (c) temperature in region 1, 4 and 6 ......................................................... 138
A.24 Pdfs of (a) relative humidity (b) specific humidity (c) temperature in region 7, 8 and 9 ......................................................... 139
A.25 Relative humidity Pdfs for tropical region ........................................ 140
A.26 Relative humidity Pdfs for central pacific ..................................... 141
A.27 Relative humidity Pdfs for central pacific ..................................... 141
Introduction

Since the seminal work of Bendsoe and Kikuchi (1988), the potential for topology optimization as a powerful, free-form design tool has been evident. The freedom to add or remove material resources from any point in the design domain means material quantities and system connectivity are simultaneously optimized. This design freedom has enabled the discovery of high performance solutions, ranging from structural components (Proos et al. (2001)) to devices (Sigmund and Jensen (2003)) and multifunctional materials (Torquato et al. (2005), Guest and Prvost (2006) governed by multiple physics.

The first step in topology optimization is discretization of the design domain. This meshing must ultimately be driven by the governing mechanics, but generally uses either solid (continuum) finite elements or discrete members such as truss or frame elements. In continuum topology optimization, the goal is to determine the optimal phase composition in each elemental domain. In traditional solid-void design, each element carries an indicator variable $\rho_e$, known as the volume fraction, that identifies the element as either solid material ($\rho_e = 1$) or a void ($\rho_e = 0$). The resulting optimization can be stated in general form as:
minimize $\varphi$ $f(\rho_1, \rho_2, \ldots, \rho_n)$

subject to $h_i(\rho_1, \rho_2, \ldots, \rho_n) = 0$

$g_i(\rho_1, \rho_2, \ldots, \rho_n) \leq 0$

$0 \leq \rho_e \leq 1$

where $f$ is the objective function, which for mechanical system design is typically weight or mechanical properties like strain energy; $h$ are the equality constraints that at a minimum include the equations representing the governing mechanics; $g$ are the inequality constraints that may include design specifications, such as mass constraint or minimum bound on a mechanical property; and the final constraint indicates the design variable $\rho_e$ must be binary.

Although straightforward in concept, the topology optimization problem (1) is fundamentally and numerically challenging to solve. It is a large scale binary programming problem that for maximum stiffness problems is ill-posed (Haber et al. (1996)): stiffness can generally be improved by decreasing the size and increasing the number of holes, leading to non-convergent behavior. This has led the vast majority of research to focus on linear elastic mechanics. The most commonly solved topology optimization formulation is minimum compliance, where the goal is to minimize internal strain energy (maximize stiffness) in a structure of fixed mass for given load case and boundary conditions. The resulting problem is solved by relaxing the binary condition, using a nonlinear interpolation model for the material properties, known
as the Solid Isotropic Material with Penalization (SIMP) method (Bendsoe (1989), Zhou and Rozvany (1991)), that penalizes magnitudes between zero and one, and uses gradient-based optimizers to make design changes. Ill-posedness is circumvented by restricting the design space through (for example) a perimeter constraint (Haber et al. (1996)) or restricting minimum length scale of designed features (Poulsen (2002), Guest et al. (2004)).

While advanced methods have now been developed for solving such linear elastic problems, resulting solutions are optimal only for the problem as mathematically posed - only for the linear elastic response. This may lead to solutions that are (for example) stiff but brittle, or stiff without redundant load paths, ultimately leading to designs that are impractical from engineering points of view.

Only relatively recently has methodologies for performing topology optimization under nonlinear mechanics been considered. Swan and Kosaka (1997) and Maute et al. (1998) simultaneously developed sensitivity analysis schemes for considering elastoplastic materials. Buhl et al. (2000) considered geometric nonlinearity in topology optimization and showed there are large differences between solutions optimized under linear and geometrically nonlinear optimization, particularly for structures subjected to snap-through behavior. Sigmund (2001a) and Sigmund (2001b) emphasized the importance of considering large deformations when designing electro mechanical devices such as actuators. C.J. Chang (2007) considered a simplified nonlinear system by defining tensile and compressive members in the system. Gaynor et al. (2013)
used piecewise linear stress-dependent constitutive models in the context of concrete
design, mainly to focus tension and compression into different structural elements,
and Bogomolny and Amir (2012) considered elastoplastic behavior to improve ulti-
mate strength of concrete. With the exception of the concrete work where there is
no void phase, each of these approaches rely on manipulating the numerical analy-
sis to circumvent numerical instabilities that may result due to nonlinear modeling
of low volume fraction elements, including stress singularities and excessive element
distortions. This may be one explanation for the relative small body of work consid-
ering nonlinear mechanics. Another reason is simply the large computational expense
associated with nonlinear topology optimization.

We seek to address these issues using projection-based topology optimization
methods that enable separation of the analysis and design spaces. These spaces can
then be modified independently to achieve stability and computationally efficiency
as needed. In particular, we show later in this chapter that low density regions need
not be modeled to maintain the ability to have material reintroduced. This is not
a new idea: Bruns and Tortorelli (2003) demonstrated that eliminating void mod-
eling stabilized the nonlinear analysis. However, previous similar approaches have
shown significant sensitivity to user-defined optimization parameters. The nature of
the nonlinear projection is shown in chapter two to provide more robust and stable
performance in the context of nonlinear mechanics. The algorithm is evaluated in the
context of design of macro scale components, such as beams.
The method is then extended in chapter three to design cellular materials considering geometric and material nonlinearities. Topology optimization has been used by several researchers to design materials with optimized effective (homogenized) properties, including elastic properties such as negative Poissons ratio (Sigmund (1994b), Sigmund (1995a)) thermoelastic (Sigmund and Torquato (1996), Sigmund and Torquato (1997a), Sigmund and Torquato (1997b)), piezoelectric (Nelli Silva et al. (1997), Silva et al. (1997), Nelli Silva et al. (1998); Sigmund and Torquato (1999)), fluid permeability (Guest and Prvost (2006), Guest and Prvost (2007)), and stiffness-thermal conductivity (Challis et al. (2008)). Topology optimization of composites and functionally graded materials can also be found in the literature (Gibiansky and Sigmund (2000); Rubio et al. (2011)). These works, however, all consider linear properties, enabling analysis (and design) of a single unit cell to estimate effective bulk properties through homogenization. In the absence of nonlinear homogenization approaches, we turn to finite periodicity to estimate the nonlinear properties of the cellular material system. Effective elastic properties and symmetries are estimated using elastic homogenization as dictated by the problem formulation, leading to a unit cell topology optimization problem with analysis conducted over two different domains: the unit cell for elastic properties and structure with finite periodicity for the nonlinear properties. Optimized solutions are compared to well-known honeycomb structures and simulation results compared to experimental studies on honeycombs composed of Bulk Metallic Glass (BMG) (Sarac et al. (2012)), our target material.
Finally, we note that appendix contains a body of work done by the author concerning the statistical analysis and data modeling of a climate shift trend known as El Nino Southern Oscillation (ENSO). ENSO is a large weather circulation in the tropical region which effects regions from Australia to South America. In particular, we seek to identify local data trends related to such parameters as water vapor, temperature, and relative humidity, that serve as large scale indicators or drivers of weather circulations in a large scale. Although perhaps unrelated topology optimization component of this thesis, the work represents a major effort by the author and falls under the category of mining big data for trends and governing parameters, which is becoming an important trend in materials research and design. It is therefore included in the appendix.
Chapter 1

Circumventing Numerical Instabilities Associated With Low Density Elements

1.1 Introduction

An important drawback of material distribution approaches is the requirement of modeling entire domain, including void regions. Elements in the void regions are structurally insignificant in the analysis and they are removed in the manufacturing of the final design. They are, however, still needed in the optimization process for reintroduction of material as design evolves. It can be therefore be said that they are necessary for the design portion of optimization process, not the analysis. In
fact, elements of negligible volume fraction are quite detrimental to analysis as they
(1) maximize the system to be solved and thus computational expense and (2) are
susceptible to numerical instabilities under nonlinear mechanics, such as excessive
distortions under finite deformations. (2.1) A number of works have therefore focused
on combatting these two issues.

Decreasing computational cost of analysis is a long-standing goal of developers in
topology optimization. Finite element analysis of structure is the driving cost in
topology optimization, often taking 90% of the computational cost in solving an op-
timization problem. Researches have tried reduce this cost by (i) reducing the cost
of linear solver or (ii) reducing the size of the system to be solved. Examples of
reduction in finite element cost include advanced solvers (Wang et al. (2003)) and
efficient reanalysis techniques (Amir and Sigmund (2011), Amir et al. (2009), Amir
et al. (2010)). Changing the size of problem often requires re-meshing such as uni-
form re-meshing (Kim and de Weck (2005)) or adaptive design dependent re-meshing
(Stainko (2006), de Sturler E (2008), Bruggi and Verani (2011)) where both independent design and state variables share the same mesh. Alternatively, design variables and state variable use separate meshes, allowing different discretization to be applied to design variable and analysis spaces. (Maute and Ramm (1995), Guest (2007), Guest and Smith Genut (2010) and Nguyen et al. (2010)).

There are also works focused on circumventing numerical instabilities due to low stiffness elements. Bruns and Tortorelli (2001) suggested hyper-elastic constitutive model for structures undergoing large deformations, stiffening elements with large deformation to prevent distortion. Also there were methods suggested to remove low stiffness elements from analysis domain. Buhl et al. (2000) proposed removing degrees of freedom associated with low stiffness elements from Newton-Raphson convergence criteria, thereby preventing oscillations associated with nodes of void elements. Another method proposes adding zero-length elements to the elastic continuum elements. The link stiffness is defined in a way to find optimized structure. In this method which was suggested by Yoon and Kim (2005a) void regions are represented by flexible links and thus distortion was localized in the links. Therefore solid elements do not go to plastic region and they are not affected in the optimization process. This method, however, doubles number of elements in the domain, needs more iterations to reach to the final result, and requires post processing to determine topology.

In this chapter we emphasize separation of design and analysis domain. And we
Figure 1.2: Element connectivity parametrization (ECP) in Yoon and Kim (2005a) method

propose simply removing elements with low stiffness from analysis domain and using solid-only modeling. This reduces the size of finite element systems to solve and avoids numerical instabilities localized in these elements. The idea of solids-only modeling is certainly not new to topology optimization. This has long been a selling point of Evolutionary Structural Optimization approaches (Xie YM (1997), Querin (1998)) and discrete implementations of level set methods (Allaire et al. (2004), Wang et al. (2003), Challis (2010)), with savings associated with the latter be specifically quantified in Challis and Guest (2009) for fluids. Maute and Ramm (1995) implemented formal re-meshing scheme for material distribution. Relative density isolines were computed on the design space and threshold line selected to represent the structural boundary. As this boundary evolved, the structural domain was re-meshed. This adaptive scheme was then extended to problems governed by elastoplastic material models (Maute et al. (1998)).

In this work we remove low stiffness elements, defined as elements with volume
fraction $\rho^e$ less than threshold density, denoted here as $\rho_t$ from the analysis mesh. Bruns and Tortorelli (2003) implemented a similar idea by performing a Gaussian elimination on the full global stiffness matrix to highlight degrees of freedom leading to instabilities, which are then stabilized by prescribing deformations. Herein, artificial boundary conditions are used to reduce the number of free degrees of freedom to be solved.

The main part of this method in distribution of material is that the design and analysis spaces are separate and discretized on different meshes. The independent design variables are then mapped, or projected, onto the finite element space to determine the physical design for analysis. This means regions of the analysis mesh may be made inactive but maintain the ability to reappear provided the corresponding region of the design variable space remains active. In Bruns and Tortorelli (2003), the mapping of the spaces was done with a linear density filter of radius $r_{\text{min}}$. Here, we use nonlinear (Heaviside) projection methods. It is shown, through a simple sensitivity analysis, that this nonlinearity more rapidly enables the reintroduction of material into void regions. This seems to prevent convergence issues and false optimum reported in Bruns and Tortorelli (2003). A key point of emphasis is that this material reintroduction is driven purely by formal sensitivity analysis. This makes the method different than soft-kill methods such as BESO where element reintroduction is driven by heuristic rules, as discussed in detail by Zhou and Rozvany (1991).
The proposed algorithm is demonstrated on several benchmark problems governed by linear and nonlinear (geometric or material) mechanics. No instances of numerical instabilities were detected, including issues of islanding and disconnected load paths reported in Bruns and Tortorelli (2003).

1.2 Solid-Only Modeling in Topology Optimization

Solid-only element method is developed based on the Heaviside projection method.

1.2.1 Heaviside Projection Equations

The physical design variables $\rho_e$ represent the volume fraction of the elements in the finite element mesh, with $\rho_e = 0$ representing a void element and $\rho_e = 1$ representing a solid element. These variables are defined on the finite element mesh and therefore serve as the blueprint for the optimized design. The Heaviside projection methodology expresses these element volume fractions as a function of an independent design variable field $\phi$. If a design variable $\phi_i$ indicates material placement ($\phi_i > 0$), material is projected radially over a distance $r_{min}$ onto element space $\rho_e$, creating a circular solid feature of radius $r_{min}$, defined as the minimum length scale of structural members (see Figure 1.3, Guest et al. (2004)). Thus it can be said, that a design variable
Figure 1.3: (a) Projection domain for design variable $i$ and (b) element $e$ neighborhood set

$i$ has the ability to introduce solid material into any element within a distance $r_{\text{min}}$.

It is useful for our discussion here to formally define this set as the neighborhood set $N^i_\phi$ as follows

\[ \text{element } e \in N^i_\phi \quad \text{if} \quad ||x_i - \bar{x}^e|| \leq r_{\text{min}} \]  \hspace{1cm} (1.1)

where $x_i$ is the location of design variable $i$, $\bar{x}^e$ is the location of the centroid of $e$, and the subscript $\phi$ emphasizes the set is defined from the perspective of the design variable.

From the perspective of an element $e$, every design variable within a distance $r_{\text{min}}$ has the ability to turn $e$ into a solid element. This subset of design variables is shown in Figure 1.3b and is defined as the neighborhood set as follows:

\[ \text{element } e \in N^i_\phi \quad \text{if} \quad ||x_i - \bar{x}^e|| \leq r_{\text{min}} \]  \hspace{1cm} (1.2)

The projection of $\phi$ onto an elemental $\rho^e$ is performed here using the standard regularized Heaviside Projection function (Guest et al. (2004)):

\[ \rho^e = 1 - e^{-\beta \mu^e(\phi)} + \mu^e(\phi)e^{-\beta} \]  \hspace{1cm} (1.3)
where $\beta$ is curvature parameter dictating the aggressiveness of the Heaviside function and $\mu^e$ indicates the intensity of design variable projection onto the element, computed as

$$
\mu^e = \frac{\sum_{i \in N^e_p} \phi_i w(x_i - \bar{x}^e)}{\sum_{i \in N^e_p} w(x_i - \bar{x}^e)}
$$

(1.4)

where $w$ is the weighting function, typically expressed as a function of distance between the independent design variable and element centroid (Bruns and Tortorelli (2001)). For a $\beta$ chosen sufficiently large, a design variable $\phi_i > 0$ will lead to $\mu^e > 0$ and therefore $\rho^e = 1$ for all $e$ in $N^i\phi$. Herein we discuss both standard distance-based ($w_d$) and uniform weight functions ($w_u$), defined as:

$$
w_d(x_i - \bar{x}^e) = \begin{cases} 
\frac{r_{\text{min}} - ||x_i - \bar{x}^e||}{r_{\text{min}}} & \text{if } i \in N^e_p \\
0 & \text{otherwise}
\end{cases} \quad w_u(x_i - \bar{x}^e) = \begin{cases} 
1 & \text{if } i \in N^e_p \\
0 & \text{otherwise}
\end{cases}
$$

1.2.2 Problem Formulation

The volume constrained linear elastic topology optimization problem of a design domain $\Omega$ may be stated in general as

$$
\begin{align*}
\min_{\phi} & \quad f = L^T d \\
\text{subject to} & \quad R(\phi, d) = K(\phi)d - F = 0 \\
& \quad \sum_{e=1}^{n_{\text{elem}}} \rho_i v_i \leq V \\
& \quad 0 \leq \phi_i \leq \phi_{\text{max}} \quad \forall i \in \Omega
\end{align*}
$$

(1.5)
where $\mathbf{d}$ are the nodal displacements, $\mathbf{R}$ is the equilibrium residual (shown above for linear elastic mechanics), $\mathbf{F}$ are the applied nodal loads, $V$ the allowable volume of material, $v^e$ is elemental volume, $\phi_{\text{max}}$ is upper design variable bound, and $\mathbf{K}$ is the global stiffness matrix assembled from element stiffness matrices $\mathbf{K}^e$ in the usual manner. The vector $\mathbf{L}$ dictates the displacements to be optimized. For example, $\mathbf{L} = \mathbf{F}$ produces the minimum compliance problem and a unit vector $\mathbf{L}$ containing a one at a single degree of freedom minimizes a single displacement, such as the case for the well-known compliant inverter problem.

The Solid Isotropic Material with Penalization (SIMP) method (Bendsoe (1989), Zhou and Rozvany (2001)) is used in this work, making element stiffness matrices take the following form:

$$
\mathbf{K}^e(\phi) = (\rho^e(\phi)^n + \rho^e_{\text{min}})\mathbf{k}_0^e
$$

(1.6)

where $n \geq 1$ is the SIMP exponential penalty term and $\mathbf{k}_0^e$ is the stiffness matrix of a solid element. Traditionally, void elements are included in the finite element analysis and thus $\rho^e_{\text{min}}$ is chosen as a small positive number to prevent singularity of the global stiffness matrix. This is not required in solids-only modeling and thus $\rho^e_{\text{min}} = 0$ herein.
1.2.3 Sensitivities

The sensitivity of the objective function with respect to independent design variables simply follows the chain rule:

\[
\frac{\partial f}{\partial \phi_i} = \sum_{e \in N_i} \frac{\partial f}{\partial \rho_e} \frac{\partial \rho_e}{\partial \phi_i} \tag{1.7}
\]

where the derivative of element volume fractions with respect to the design variable is:

\[
\frac{\partial \rho_e}{\partial \phi_i} = (\beta e^{-\beta \mu_e} \phi + e^{-\beta}) \frac{\partial \mu_e}{\partial \phi_i} \tag{1.8}
\]

with

\[
\frac{\partial \mu_e}{\partial \phi_i} = \frac{w(x_i - x_e)}{\sum_{i \in N_p} w(x_i - x_e)} \tag{1.9}
\]

The derivative of the objective function with respect to the physical design variables is found using the adjoint method and shows:

\[
f_A = f - \lambda(Kd - F) \tag{1.10}
\]

Where \( \lambda \) is adjoint variable. The sensitivity then

\[
\frac{\partial f_A}{\partial \rho^e} = \frac{\partial f}{\partial \rho^e} - \lambda(Kd - F) \tag{1.11}
\]

\[
\frac{\partial f_A}{\partial \rho^e} = L^T \frac{\partial d}{\partial \rho^e} - \lambda(\frac{\partial K}{\partial \rho^e} \frac{\partial d}{\partial \rho^e} + K \frac{\partial d}{\partial \rho^e} - \frac{\partial F}{\partial \rho^e}) \tag{1.12}
\]

The term \( \frac{\partial d}{\partial \rho^e} \) is difficult to compute and this is eliminated by choosing \( \lambda \) that solves:
\[(L^T - \lambda K) \frac{\partial d}{\partial \rho^e} = 0 \]  \hspace{1cm} (1.13)

\[L^T - \lambda^T K = 0 \]  \hspace{1cm} (1.14)

Equation 1.12 then simplifies to:

\[\frac{\partial f_A}{\partial \rho^e} = -\lambda \left( \frac{\partial K}{\partial \rho^e} d - \frac{\partial F}{\partial \rho^e} \right) \]  \hspace{1cm} (1.15)

Assuming applied load \( F \) is not a function of density, this simplifies filter to:

\[\frac{\partial f}{\partial \rho^e} = -\eta(\rho^e(\phi))^{n-1} \lambda^T K\phi d^e \]  \hspace{1cm} (1.16)

where \( \lambda^e \) is the elemental component of vector \( \lambda \) that solves \( K(\phi)\lambda = L \).

Equation (1.7) is the key to element reintroduction in the solids-only modeling scheme. It indicates that any element within a distance \( r_{min} \) of the design variable \( \phi_i \) is capable of creating a nonzero sensitivity for that design variable. This means design variables in un-modeled void space can be driven to project material if they are within a distance \( r_{min} \) of the structural boundary. As will be shown, such design variables then project material over a distance \( r_{min} \), potentially allowing material to develop over a distance of \( 2 \times r_{min} \) into the void space.
1.2.4 Inherent Element Reintroduction

With the equations in place, let us explore the properties of analysis element removal and reintroduction under the objective of minimizing compliance. We consider the cantilever domain shown in Figure 1.4 meshed with 1000 finite elements (element size $h = 1$ unit) and create a straight beam topology as the initial guess. We will examine the sensitivity profiles for linear sensitivity filtering (Sigmund (1994a)), linear density filtering ($\beta = 0$) (Bruns and Tortorelli (2001), Bourdin (2001)), and Heaviside projection. In all cases $r_{\min} = 2.5$ and $\beta = 30$ in the case of Heaviside Projection, a typical magnitude. To simplify comparisons, design variables are placed at element centroids and their magnitudes indicated as constant over the element domain, although it is noted the design variables are actually points in space and thus have no physical volume representation. The color scaling for each individual plot is normalized against the maximum value in plot.

1.2.5 Sensitivity Plots

The design variable distributions and resulting topologies for the three cases are shown in the first two rows of figure 1.5. For sensitivity filtering, the design and topology variables are identical ($\rho = \phi$). Although the linear sensitivity filter is not capable in general of achieving a 0-1 topology, we use a 0-1 distribution as the initial guess. A direct comparison of properties is difficult as the solid-void interface
is sharp in Heaviside projection and blurry in linear density filtering. To maintain a fair comparison, we use topologies with the same volume of material and define the structural boundary threshold in linear density filtering as $\rho^e = 0.5$, which is the reason the design variable distributions are different.

The derivatives $\frac{df}{d\rho^e}$ are shown in the third row of figure 1.5 and from equation (1.16) are simply elemental strain energies scaled by the coefficient $\eta(\rho^e)^{\eta-1}$. This coefficient is (or approaches) zero magnitude for void (or low-density) elements when the SIMP exponent value exceeds one. As shown in figure 1.5 for $\eta = 3$, this drives the elemental derivatives to zero regardless of the strain energy in these low-density elements. This supports the logic that low-density elements need not be modeled as their contribution to the design variable sensitivities is negligible.

The design variable sensitivities are shown in the fourth row of figure 1.5, and in the case of the sensitivity filter represent the filtered sensitivities. The key observation here is that the design variable sensitivities are nonzero in regions extending a distance $r_{\min}$ beyond the topological boundary. This means that design variables located in void space may achieve nonzero sensitivities, and therefore maintain the capability to change magnitude, even if the nearest elements have zero magnitudes of $\frac{df}{d\rho^e}$. This supports the idea that material may reappear in elements even if they are removed from the finite element analysis.

The final row illustrates reach of these sensitivities, illustrating how they would intend to change topology. These are visualized by simply treating the normalized
design variable sensitivities as design variables, and thus in the case of linear density filtering and Heaviside projection passing them through the linear filter and regularized Heaviside functions, respectively. This is not a rigorous estimation as we neglect effect of constraints such as upper bounds, but rather illustrates the geographic extent to which topological changes are possible. In the case of the sensitivity filter, topological growth is limited to a distance of \( r_{\text{min}} \) from the original topological boundary. The reach of the linear density filter and projection methods, however, is \( 2 \times r_{\text{min}} \). Design variables within a distance \( r_{\text{min}} \) may change, subsequently leading to topological growth a distance \( r_{\text{min}} \) from these variables. In the case of the linear density filter, however, the intensity of this growth falls off rapidly. The largest sensitivities \( \frac{df}{d\phi} \) are located on the interior of the topology. Elements at or beyond a distance \( r_{\text{min}} \) from the structural boundary see little intensity due to the relatively small weights in the sensitivity calculation, which are then damped further by the small weights in density filter. In the case of Heaviside projection, however, the sensitivity intensity is strongest at the topological boundary and the Heaviside function amplifies the potential impact of small sensitivities, leading to a stronger reach at further distances than the linear density filter. This is clearly seen in the figure where the red spike is centered on the topological boundary and extends to a distance of \( 2 \times r_{\text{min}} \).

To summarize, for a SIMP exponent greater than one, topological changes essentially grow from structural boundaries in all cases, enabling void material that is not modeled to be reintroduced. This growth is limited to a distance of \( r_{\text{min}} \) in the case
of the sensitivity filter and a distance of $2 \times r_{\text{min}}$ in the case of linear density filtering and Heaviside projection. A key difference in the latter two are that the potential rate of material reintroduction. The linear filter has a nonzero but little effect beyond a distance of $r_{\text{min}}$ from the structural boundary, while the Heaviside projection does not reach such levels until a distance of $2 \times r_{\text{min}}$. This enables reintroduction of material more quickly, as well as the fundamental benefit of achieving (near) 0-1 topologies.

While the proceeding figures motivate and justify the use of solids-only modeling, the rate of material reintroduction is significantly less in case where intermediate volume fractions are unpenalized ($\rho = 1$). In such cases, the derivatives $\frac{df}{d\rho}$ are nonzero at all locations and thus neglecting to model the voids causes potentially valuable sensitivity information to be lost. Table 1.1 displays this information for the cantilever beams shown in figure 1.4 when modeling the whole domain and the solids-only. Of course, the solids-only modeling approach is capable of obtaining the same solution through boundary growth, it will certainly require more iterations to arrive at this point. The use of solids-only modeling for unpenalized problems ($\eta = 1$) is thus less appealing.

Finally, we note that the plots in figure 1.5 assumed a linear distance weighting function ($w_d$) as it is the most commonly used. It is widely known that using uniform weighting ($w_u$) reduces grey regions at topological boundaries for Heaviside Projection (Guest et al. (2004)). For completeness, plots of figure 1.5 are repeated in figure 1.6.
for this case. We also note that the work of Bruns and Tortorelli (2003) used a linear density filter with Gaussian weights, which would essentially have an effect between the linear and uniform weighting schemes.

Figure 1.4: Cantilever beam
Figure 1.5: Cantilever beam

(a) Design Variable $\phi$

(b) Topology $\rho^*$

(c) $-df/d\rho^*$

(d) Sensitivity $-df/d\phi$

(e) Potential topological influence of sensitivity

Figure 1.6: a) Initial material distribution b) final material distribution
1.2.6 Solids-Only Finite Element Analysis

As discussed in the introduction, researchers have proposed different methods for neglecting void element analysis. Maute and Ramm (1995) used re-meshing, Buhl et al. (2000) modified nonlinear convergence criteria, and Bruns and Tortorelli (2003) stabilized the stiffness matrix following Gaussian elimination. These methods require a threshold ($\rho_t$) to be set below which element stiffness is considered negligible. In this work, we simply introduce artificial boundary conditions to degrees of freedom that are surrounded completely by void elements. This is achieved by looping over the elements and marking the nodes of elements whose stiffness is to be modeled ($\rho_e > \rho_t$). Nodes that are unmarked receive a temporary boundary condition. Equation numbering and finite element assembly proceed in the standard manner (e.g., Hughes 2000), although it is noted the assembly routine need not check the equation numbers of void elements (including along the structural interface). This process is performed at each design iteration where the solids-topology changes.

1.3 Linear Elastic Modeling

We begin by considering linear elastic benchmark problems. The primary advantage of solids-only modeling for such problems is the reduction in the number of free degrees of freedom and thus linear equations to be solved. In all cases, designs
begin with a uniform distribution of material and thus these benefits will not be realized in the initial few iterations. After extensive numerical testing, the threshold of $\rho_t = 0.001$ was chosen as a robust while still efficient magnitude. Two- and three-dimensional problems use four-node quadrilateral plane stress and eight-node brick elements, respectively. Material is assumed isotropic with Young's modulus of one and Poisson's ratio of 0.3. All problems are solved using MMA optimizer (Svanberg (1987), Svanberg (1995)) and constant SIMP and Heaviside parameters of $\eta = 3, \beta = 30$ (Guest et al. (2011)).

### 1.3.1 Cantilever Beam Example

The cantilever beam of figure 1.4 is solved using domain dimensions of $L = 40$ and $H = 25$, meshed using $240 \times 150$ finite elements. Allowable material volume $V$ is 50% of the domain and length scales of $r_{\text{min}} = 2$ and $r_{\text{min}} = 1$ are considered. Figure 1.7 displays solutions found when modeling the entire domain (void and solid) and when using solids-only modeling with $\rho_t$ of 0.01 and 0.001. The solutions are consistent. The benefits of this approach are shown in Figure 1.8, which displays the number of linear equilibrium equations solved at each design iteration. It is clear that the larger the removal threshold the fewer equations need to be solved as the algorithm progresses and topology develops. The upward spikes in these plots indicate material be added into void (unmodeled) elements, confirming the ability for material to reappear. In this, and most other minimum compliance problems we considered, the
overall trend is strongly downward and these spikes are few in number and small in magnitude.

In figure 1.8 optimization steps to the final results are shown. This figure shows how threshold density can effect number of equations to solve and number of iteration to reach final result. Increasing $\rho_t$ leads to more aggressive truncation and fewer equations at the risk of converging to lower quality local minimum. Choosing suitable value of threshold density is therefore critical.

The final topologies, void modeling and $\rho_t = 10^{-3}$ are quite similar but $\rho_t = 10^{-2}$ shows a different final structure. Though topologically different, there objective function magnitudes are very close. We calculate number of equations needs to be solved, in order to compare computational cost for each example. In table 1.1 average number of equations ($Neq$) per iteration are shown. Increasing $\rho_t$ from 0 to $10^{-2}$ decreases the number of equations about 40%. Table 1.1 shows that final objective function does not affected by threshold value in large scale.

Figure 1.7: Optimized structure for a) void modeling b) $\rho_t = 10^{-3}$ c) $\rho_t = 10^{-2}$
Figure 1.8: Number of equations to be solved in each optimization iterations

Table 1.1: Objective function and number of equations for cantilever example

<table>
<thead>
<tr>
<th></th>
<th>void modeling</th>
<th>$\rho_t = 10^{-3}$</th>
<th>$\rho_t = 10^{-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective function</td>
<td>60.26</td>
<td>60.36</td>
<td>60.59</td>
</tr>
<tr>
<td>Average $\frac{N_{eq}}{Iteration}$</td>
<td>72480</td>
<td>52591</td>
<td>50033</td>
</tr>
</tbody>
</table>
1.3.2 Compliant Inverter Example

The well-known compliant inverter problem is shown in figure 1.9(a) and meshed using $240 \times 170$ finite elements. Allowable material volume $V$ is 25% of the domain and length scales of $r_{\text{min}} = 2$ and $r_{\text{min}} = 1$ are considered. Figure 1.9 displays solutions using the same $\rho_t$ thresholds and table 1.2 the number of equations to be solved in each design iteration. As in the previous example, the topologies are consistent and the number of equations decreases using solids-only modeling. As the element sensitivities are positive and negative in the inverter problem, we do see more spikes in the number of equation iteration history, suggesting the need for void elements recover material is more critical than the minimum compliance examples.

The difference between results, however, is noticeable. Displacement value for $\rho_t = 10^{-3}$ is better than for void modeling. However, increasing $\rho_t = 10^{-2}$ leads to convergence to trivial local minimum. Therefore, aggressive magnitudes of $\rho_t$ must be avoided for compliant inverter.

Table 1.2: Objective function and number of equations for compliant inverter example

<table>
<thead>
<tr>
<th></th>
<th>Void modeling</th>
<th>$\rho_t = 10^{-3}$</th>
<th>$\rho_t = 10^{-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final deformation</td>
<td>$-2.10$</td>
<td>$-2.14$</td>
<td>$0.00$</td>
</tr>
<tr>
<td>Average $\frac{Neq}{\text{Iteration}}$</td>
<td>$58079$</td>
<td>$50771$</td>
<td>$47136$</td>
</tr>
</tbody>
</table>
Figure 1.9: Solid-only modeling method for a) Inverter example. b) Considering void modeling and c) $\rho_t = 10^{-3}$ d) $\rho_t = 10^{-2}$
1.3.3 3D Pile Cap

A 3d pile cap example is shown in figure 1.10. Symmetry is leveraged to model one-quarter of the domain, which is meshed using $80 \times 80 \times 80$ finite elements resulting in approximately 3,000,000 free degrees of freedom for the full domain.

![3D pile cap design domain](image)

Final objective values for the optimized structure and average number of equations to be solved per iteration are shown in table 1.3. Final compliance for both cases are close together and removing elements does not effect solution optimality. There are small differences in the structures, especially near boundaries. In the table average number of equations are shown to compare with default method of modeling the entire domain. This average is the ratio of total number of equations to the number of iterations.
Figure 1.11: Pile cap solutions, using a) void modeling and b) solid-only modeling with $\rho_t = 10^{-3}$

Table 1.3: Objective function and number of equations for 3D cap example

<table>
<thead>
<tr>
<th></th>
<th>Void modeling</th>
<th>$\rho_t = 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final Compliance</td>
<td>4.380</td>
<td>4.346</td>
</tr>
<tr>
<td>Average $\frac{N_{eq}}{\text{Iteration}}$</td>
<td>3,227,988</td>
<td>1,872,316</td>
</tr>
</tbody>
</table>
1.4 Summary

The idea of eliminating void modeling, originally proposed by Bruns and Tortorelli (2003), is revised and shown to enable computational savings in linear elastic design problem. Convergence issues, such as disconnected load paths originally reported in Bruns and Tortorelli (2003) were not observed due to robustness of nonlinear projection, which was also demonstrated. To readily reintroduce material into the void space. Solution quality, however, was shown to suffer when void elements were aggressively removed. For the challenging inverter problem, this threshold was smaller than minimum compliance problems, where even $\rho_t = 10^{-10}$ found quality solutions.
Chapter 2

Topology Optimization For Nonlinear Mechanics

2.1 Introduction

Moving from linear elastic properties to nonlinear mechanics adds new challenges to the topology optimization problem. A small number of papers considered geometric or material nonlinearity in topology optimization of structures. Computational expense necessarily increases due to the requirement of iterative analysis. Also, numerical instabilities occur due to low density elements. For example, low density elements subjected to large deformation may undergo distortion (Figure 2.1), leading to negative Jacobian (Yoon and Kim (2005b)).

In the last chapter we introduced a method to separate design variables and analysis
Figure 2.1: Distortion of elements in finite element analysis. Yoon and Kim (2005b)

mesh and eliminate modeling of elements with low stiffness elements of the type that
would undergo these excessive distortions. Ghost elements consist of elements without mechanical contribution.

Von-Mises criteria is used as the failure criteria in our examples. Von-Mises yield surface is a smooth surface which is not related to the element density. Therefore sensitivity of yield stress is removed from sensitivity analysis. We should note, however, that this criteria is not applicable for large group of materials such as ceramic or polymers. Finally, we note that nonlinear response is dependent on load combinations and order in which these loads are applied. Herein, this ordering is assumed known.

In the second part of this chapter we consider stress based topology optimization using p-norm stress optimization. We also show how statistical properties of stress distribution change due to altering p-norm value in the objective function. There are
relatively little researches on the stress based topology optimization. Rozvany and Kirsch (1995) introduced a new optimality method DCOC to do stress base topology optimization. Baumgartner et al. (1992) proposed a stress based optimization method based on biological growth concept. There are also numerical issues in stress base optimization methods which studied by researchers in this area (Yang and Chen (1996), Cheng and Guo (1997) and Le et al. (2010)) At the end we compare results of p-norm stress optimization with nonlinear optimization to determine if expensive nonlinear optimization can be approximated with linear stress based optimization.

2.2 Objective Functions

In this chapter we propose different nonlinear optimization methods. Different objective function is considered for every method. The main objective functions are complimentary compliance and strain energy absorption which are based on the gradually application of force or displacement on the structure. In the geometrically nonlinear topology optimization we use final compliance as the objective function (2.1). Nonlinear geometry analysis needs to be done in incremental steps. The residual value is the residual for these increments. But the objective function is calculated with final displacement and force on the structure.

\[ f = F_{\text{final}}^T d \]  

(2.1)

Cumulative strain energy is used as the objective function in the material nonlinear
topology optimization method. This objective function is defined according to change in stress and strain of the structure.

\[ f = -\int_{\Omega} \sigma^T d\varepsilon d\Omega \]  

(2.2)

In the stress based optimization method we use p-norm of stresses with different \( p \) values as the objective function.

\[ f = \int_{\Omega} \sigma^p_{\text{vm}} d\Omega \]  

(2.3)

### 2.3 Geometrically Nonlinear Topology Optimization

When we consider geometric nonlinearity in the analysis process the objective function and constraints can be different from elastic analysis optimization. One such objective function is final compliance the optimization problem is shown in equation 2.4.

\[
\begin{align*}
\text{minimize} & \quad f = F^T d \\
\text{subject to} & \quad R(\rho, d) = 0 \\
& \quad \sum_{e=1}^{nelem} \rho^e v^e \leq V \\
& \quad \rho_{\text{min}} \leq \rho^e \leq 1
\end{align*}
\]

(2.4)

This equation is similar to equation (2.5) in chapter two. The only difference is in the residual force \( R(\rho, d) \), which is nonlinear. This value is zero for linear optimization due
to one step solving problem. But geometrically nonlinear optimization is an iterative method to reach zero value for the residual force. Residual force is defined as the error in the equilibrium equation which is shown in equation 2.5.

\[
R = F - \int B^T \sigma dV
\]  

(2.5)

Where \( B \) is the strain-displacement matrix, and \( \sigma \) is second Piola stress. The \( B \) matrix changes due to strain values including second order terms.

\[
d\epsilon = Bd
\]  

(2.6)

We use advanced \( B \) matrix to get higher order strains for Green-Lagrangian strains. The higher order strains equation is shown in equation 2.7.

\[
\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})
\]  

(2.7)
2.4 Sensitivities

Again, we use adjoint method to find sensitivities of objective function and constraints. The residual force at the equilibrium should be zero. Therefore we can use adjoint method for structures with large deformations.

\[ f_A = F^T d + \lambda^T R \]

(2.8)

\[ \frac{\partial f_A}{\partial \rho_e} = F^T \frac{\partial d}{\partial \rho_e} + \lambda^T \left( \frac{\partial R}{\partial d} \frac{dd}{\partial \rho_e} + \frac{\partial R}{\partial \rho_e} \right) \]

(2.9)

As before, we select the adjoint operator that achieves

\[ (F^T - \lambda K_T) \frac{dd}{d\rho_e} = 0 \]

(2.10)

meaning the following equation is satisfied

\[ K_T \lambda = F \]

(2.11)

leaving

\[ \frac{\partial f_A}{\partial \rho_e} = \lambda^T \left( \frac{\partial R}{\partial \rho_e} \right) \]

(2.12)


2.5 Long Cantilever Example

We assume long cantilever beam example to evaluate geometrically nonlinear optimization. In this example we consider final compliance as the objective function. Optimized results for this method are compared with linear optimization method in Figure 2.3.

![Figure 2.2: Long cantilever beam](image)

In nonlinear optimization methods the final structure depends on the applied load. Increasing load value results in large deformations and new topology. Moreover, Geometrically nonlinear analysis is an iterative analysis to find the equilibrium equation with zero value of residual vector.
Figure 2.3: Optimal structure for cantilever example
2.6 Material Nonlinear Optimization

When material nonlinearity is considered in the optimization process, the objective function and parameters get more difficult. Buhl et al. (2000), Maute et al. (1998) suggested some mechanical characteristics as the objective function for nonlinear topology optimization. Some of these mechanical characteristics are cumulative compliance, complimentary elastic work and compliance. Compliance is usually used as the objective function for elastic topology optimization. The drawback of final compliance as the objective function is that the structure may collapse under the load before it reaches the end deformation. In this section we use complimentary compliance (strain energy) as the objective function in the optimization problem. The optimization problem is shown in equation 2.13. Complimentary compliance is the total energy that structure can dissipate during mechanical deformation. We minimize negative strain energy to optimize for complimentary energy.

\[
\begin{aligned}
\text{minimize} & \quad f = -\int\int_{\Omega} \sigma^T d\epsilon d\Omega = -\int\int_{\Omega} K^T dU d\Omega \\
\text{subject to} & \quad \sum_{e=1}^{nelem} \rho_i v_i \leq V \\
& \quad 0 \leq \phi_i \leq \phi_{\text{max}} \quad \forall i \in \Omega
\end{aligned}
\] (2.13)

Equation 2.13 indicates that nonlinear optimization is a path dependent problem. We can set up virtual work equation for the objective function considering boundary
\[ \int_\Omega \delta \varepsilon^T \sigma d\Omega - \int_\Omega \delta u^T \rho \sigma d\Omega + \int_\Omega \delta u^T \rho \dot{u} d\Omega - \int_\Omega \delta u^T \bar{t} d\Omega = 0 \quad (2.14) \]

Parameters in this equation are stress \( \sigma \), strain \( \varepsilon \), displacement \( u \), acceleration \( \ddot{u} \) and traction \( t \) on the boundary. We neglect body force and acceleration in the next step. Then we add constraints in the Kuhn-Tucker condition and find equation 2.15.

\[ \frac{\partial}{\partial \rho_i} \int_\Omega \sigma^T d\epsilon d\Omega + \lambda v_i = 0 \quad (2.15) \]

\[ \frac{\partial}{\partial \rho_i} \int_\Omega K^T dU d\Omega + \lambda v_i = 0 \quad (2.16) \]

Where \( \lambda \) is the Lagrangian multiplier, \( v_i \) is the element volume. We apply non-linear finite element analysis and a displacement control iterative method. When the nonlinear analysis finishes, a new structure is proposed according to the sensitivity of elements. If \( \rho^e \) is the density of element, we assume the SIMP exponential relationship between material properties and density of elements according to equation 2.17.

\[ D = \rho^\beta_i D_0 \quad H = \rho^\beta_i H_0 \quad \sigma_y = \rho^\beta_i \sigma_{y_0} \quad (2.17) \]

Maute et al. (1998) used higher exponent values for yield stresses. We, however, did not see this need and therefore use similar values of \( \beta \) for all material parameters.
D is the elastic constitutive matrix, H is the plastic hardening matrix and, \( \sigma_y \) is the yield stress. We use Von-Mises yield criteria for our nonlinear analysis. This criteria is shown in equation 2.18.

\[
F = \sqrt{3J_2 - k} \quad (2.18)
\]

\[
k = \sigma_y + H \quad (2.19)
\]

where \( J_2 = \frac{1}{2}S_{ij}S_{ji} \), \( S_{ij} \) is deviatoric stress. Total strains are composed of elastic and plastic strains.

\[
d\epsilon = d\epsilon^e + d\epsilon^p \quad (2.20)
\]

Where \( d\epsilon^e \) are elastic strains and \( d\epsilon^p \) plastic strains that are needed to determine yield point.

\[
d\epsilon = BdU \quad (2.21)
\]

Then we can find stress changes due to equation 2.22.

\[
d\sigma = D^{ep}d\epsilon \quad (2.22)
\]

Where \( D^{ep} \) is the elastoplastic constitutive matrix. Equation 2.23 shows general equation for elasto plastic constitutive matrix.

43
\[ D^{ep} = D^e - \frac{D^{e}n_g L_n^{T} D^e}{H_L + n_g^{T} D^{e} n_L} \]  \hspace{1cm} (2.23)

\[ K^T = \int_{\Omega} B D^{ep} B d\Omega \]  \hspace{1cm} (2.24)

The equations above are used to perform the nonlinear finite element analysis.

### 2.6.1 Sensitivity Analysis

After defining governing mechanical problems, we need to do sensitivity analysis to find the derivative of objective functions and constraints with respect to design variables.

The derivative of material parameters related to the density is shown in equation 2.25.

\[ \frac{\partial D}{\partial \rho_i} = \beta \rho_i^{\beta - 1} D_0 \quad \frac{\partial H}{\partial \rho_i} = \beta \rho_i^{\beta - 1} H_0 \quad \frac{\partial \sigma_{y}}{\partial \rho_i} = \beta \rho_i^{\beta - 1} \sigma_{y_0} \]  \hspace{1cm} (2.25)

and equation 2.26 shows derivative of the objective function with respect to \( \rho \).

\[ \frac{\partial f}{\partial \rho_i} = -\frac{\partial}{\partial \rho_i} \int_{\Omega} \int_{\epsilon} \sigma^T d\epsilon d\Omega = -\frac{\partial}{\partial \rho_i} \int_{\Omega} \int_{\sigma} d\sigma^T d\epsilon d\Omega \]  \hspace{1cm} (2.26)

Where derivative of stress can be evaluated with

\[ \frac{\partial d\sigma}{\partial \rho_i} = \frac{\partial (D^{ep} d\epsilon)}{\partial \rho_i} = \frac{D^{ep}}{\partial \rho_i} d\epsilon + D^{ep} \frac{\partial d\epsilon}{\partial \rho_i} \]  \hspace{1cm} (2.27)

We then replace in the virtual work equation with Kuhn-Tucker coefficient and solve for the equation to find the derivatives.
\[
\int_\Omega \delta \epsilon^T \sigma d\Omega - \lambda \int_\Gamma \delta u^T \hat{t} d\Gamma = 0 \quad \text{(2.28)}
\]

\[
\frac{\partial f}{\partial \rho_i} = -\int_\Omega \int_{\epsilon} d\epsilon^T \frac{\partial D^{rp}}{\partial \rho_i} d\epsilon d\Omega - 2 \int_\Gamma \frac{\partial \lambda}{\partial \rho_i} \hat{t} d\Gamma
\]

Where \( d\lambda \) is equal to zero for the load controlled algorithm and the sensitivity equation just depends on \( \frac{\partial D^{rp}}{\partial \rho_i} \). But it is hard to apply load controlled analysis for non-linear topology optimization because stiffness during initial iterations and typically small. Applying loads instead of displacements elements may lead to some elements undergoing to very large strains and leading to numerical instabilities. Therefore we apply displacement controlled analysis instead of load controlled. Displacement values at the boundaries replace constant forces. In this case sensitivity value for \( \frac{\partial d\lambda}{\partial \rho_i} \) is nonzero and \( \frac{\partial \hat{u}}{\partial \rho} = 0 \).

Then we can simplify \( \frac{\partial d\lambda}{\partial \rho_i} \) to equation 2.30

\[
\frac{\partial d\lambda}{\partial \rho_i} = -\int_\Omega d\epsilon^T \frac{\partial D^{rp}}{\partial \rho_i} d\epsilon d\Omega = \int_\Omega dU^T \frac{\partial K^T}{\partial \rho_i} dU d\Omega \] (2.30)

And we need to calculate sensitivity of \( D^{rp} \) respect to the densities. Following equations display sensitivity of each parameter respect to density. The derivative of each parameter respect to design variables can easily be calculated with chain rule and using Heaviside projection equations. The derivative of the normal to the yield surface is zero \( \left( \frac{\partial n}{\partial \rho_i} = 0 \right) \).
\[
\frac{\partial d\lambda}{\partial \rho_i} = -\int d\epsilon^T \frac{\partial D_{ep}^{T}}{\partial \rho_i} d\epsilon \frac{d\Omega}{t} du = \int dU^T \frac{\partial K^T}{\partial \rho_i} dU \frac{d\Omega}{t} du \quad (2.31)
\]

\[
\frac{\partial D_{ep}^{T}(D)}{\partial \rho_i} = \frac{\partial D}{\partial \rho_i} - \frac{(\frac{\partial D_{ep}^{T}}{\partial \rho_i} n n^T D + D n n^T \frac{\partial D}{\partial \rho_i}) (H + n^T D n) - (H + n^T \frac{\partial D}{\partial \rho_i} n) (D n n^T D)}{(H + n^T D n)^2} \quad (2.32)
\]

The yield stress part of the elasto plastic tangent modulus does not depend on density. While hardening is related to densities, shown as follows:

\[
\frac{\partial D^c_{ep}(\sigma_y)}{\partial \rho_i} = 0 \quad (2.33)
\]

\[
\frac{\partial D^c_{ep}(H)}{\partial \rho_i} = \frac{(\frac{\partial H}{\partial \rho_i}) (D n n^T D)}{(H + n^T D n)^2} \quad (2.34)
\]

We now have all the equations needed for sensitivity analysis and can set up nonlinear topology optimization.

### 2.6.2 Beam Examples

Two structures are optimized with elastic and plastic methods in this section. The first structure is clamped on both sides and subjected to load in the middle. Beam length and depth is 20 and 5 units, respectively. Material properties for this structure
are listed in equation 2.35.

\[ \text{Young's Modulus} \quad E = 180000 \]

\[ \text{Yield stress} \quad \sigma_y = 360 \]

\[ \text{Poisson's ratio} \quad \nu = 0.30 \]  

(2.35)

Symmetry is used to model only one half of the structure. Final optimized structures for linear elastic material and elasto-plastic material are shown in figure 2.9. The main difference between these structures is element thickness and number of elements in both structures. In the elastic optimized structure two thick elements carry loads to the supports and build up a large elastic stiffness for whole structure. However, the optimized structure for total energy shows less stiffness in the elastic region but higher energy dissipation due to secondary members which activate in the last steps. Figure 2.9 shows the load-displacement diagram for the two structures under nonlinear mechanics. The area under this curve indicates amount of mechanical energy dissipation due to deformation in the structure. This value is 10% higher for structure "optimized under nonlinear mechanic" compare to structure "optimized under linear mechanics".

The second example is a simply supported beam loaded in the middle. This example also shows similar behavior for the two results. Optimized structure under nonlinear mechanic has more redundancy which activates in large deformation and keeps strength of structure higher compare to optimized structure under linear
Finally, we note that solids-only modeling using the Heaviside projection prevents numerical instabilities during the nonlinear analysis of the structure and significantly reduces computational expense.
Figure 2.7: Simple supported beam (Maute et al. (1998))

(a) Elastic material
(b) Elasto plastic material

Figure 2.8: Final structures

Figure 2.9: Load-displacement diagram
2.7 Minimize Peak Stress

Despite computational savings associated with solid-only modeling, topology optimization problems under nonlinear mechanics remain relatively expensive. Among different elastic methods we found that stress based method can make a good approximation for nonlinear topology optimization. We examine stress based method in this section and solve previously considered examples for comparison to solutions found using full nonlinear analysis.

There are different approaches to minimizing peak stress. One common way is to consider stress of each element as a constraint in the optimization problem. This method gets very expensive as number of finite elements increases. It is also not practical because it is mesh dependent and infeasible for some load and boundary conditions. For example, elements are susceptible to large stresses at the boundaries and load points. When we force constraints for allowable stress in all elements, some of boundary elements fail to satisfy the constraints leading to an early feasible set. Researchers suggested different relaxation methods to solve this problem. Cheng and Guo (1997) introduced \( \epsilon \)-relaxation and Rozvany and Sobieszczanski-Sobieski (1992) used smooth envelope functions. We use one of relaxation methods called p-norm method in this section. In this method we consider norm of stresses in the objective function and we can tighten or loosen constraints with increasing and decreasing p value respectively. This method is also practical in mechanical design because in a large structure if
some elements yield in a loading points, mechanical characteristics of the structure may not change in large scale. Equation 2.39 shows optimization problem for p-norm of stresses.

\[
\begin{align*}
\text{minimize} && f &= \int_{\Omega} \sigma_{vm}^p d\Omega \\
\text{subject to} && \sum_{e=1}^{nelem} \rho_i v_i &\leq V \\
&& 0 &\leq \phi_i \leq \phi_{max} \forall i \in \Omega
\end{align*}
\]

where

\[
\sigma_{vm} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx} \sigma_{yy} + 3\sigma_{xy}^2} = (\sigma' T \sigma)^{0.5}
\]

\[
T = \begin{pmatrix}
1 & -0.5 & 0 \\
-0.5 & 1 & 0 \\
0 & 0 & 3
\end{pmatrix}
\]

Sensitivity of stress based optimization problem is more complicated than the linear optimization. We use adjoint method to simplify sensitivity equations as it is shown in equation 2.39.

\[
\begin{align*}
f_A &= f - \lambda (Kd - F) \\
\frac{df_A}{d\rho} &= \lambda^T \left( \frac{df}{d\rho} - \frac{dK}{d\rho} u \right) + \frac{\partial f_A}{d\rho} \\
K^T \lambda &= \sum_{e=1}^{nelem} \frac{\partial F_{\sigma}}{d\sigma_{vm}} \left( \frac{\partial \sigma_{vm}}{d\sigma_{e}} \frac{\partial \sigma_{e}}{d\sigma} \right)^T
\end{align*}
\]
Where:
\[
\frac{\partial d\sigma_{vm}}{d\sigma_e} = \nabla g^T DBu
\]
\[
\nabla g^T = [2\sigma_{xx} - \sigma_{yy}, 2\sigma_{yy} - \sigma_{xx}, 6\sigma_{xy}]
\]
\[
\frac{\partial d\sigma_e}{d\sigma_e} = pp^{p-1}D_0Bu
\]  

(2.40)

In this section we apply p-norm stress based optimization problem to the benchmark L-bracket problem. This example is 240 × 120 L-bracket with loading on the sides and boundaries at the top part. We use symmetric properties of the domain and apply optimization on half of the domain.

Table 2.10(a) shows final material distributions for optimized structure and p-norm stress based optimization. It is also shown how the stress distribution changes in the structure through changing p values. Probability density functions are also shown with dash lines indicate $\mu - \sigma$, $\mu$ and $\mu + \sigma$ values. When we increase p value in the problem the effect of high stresses dominates in the objective function. Table 2.1 shows some of statistical properties of the final optimized structures. This include mean and standard deviation of stress as well as stress values for last 1% and 0.1% of the elements. P-norm stresses have main effect on the 0.1% of stresses at heigh values. This value has a direct relationship with chosen p values.

In the next section we examine relationship between stress based optimization results and those of nonlinear topology optimization. These are evaluated using nonlinear finite element analysis with negligible hardening values.
Figure 2.10: Optimized structure, resulting Von-Mises stresses contour, and probability density function of stresses for different objective functions

We can capture this pattern of load transferring for low nonlinearity cases with the stress based topology optimization. In the next section, we apply stress based topol-
Table 2.1: Statistical characteristics of optimized structure for p-norm stress based objective function and compliance

<table>
<thead>
<tr>
<th>Objective function</th>
<th>Statistical properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimize compliance</td>
<td>$\mu = 128.9$</td>
</tr>
<tr>
<td></td>
<td>$\text{Var} = 1357$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{\text{max}} = 697.8$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{1%} = 202.9$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{0.1%} = 315.0$</td>
</tr>
<tr>
<td></td>
<td>$M = 0.068$</td>
</tr>
<tr>
<td></td>
<td>$V_f = 30.0%$</td>
</tr>
<tr>
<td>Minimize Mean stress</td>
<td>$\mu = 129.3$</td>
</tr>
<tr>
<td></td>
<td>$\text{Var} = 1564$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{\text{max}} = 760.9$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{1%} = 214.1$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{0.1%} = 358.0$</td>
</tr>
<tr>
<td></td>
<td>$M = 0.058$</td>
</tr>
<tr>
<td></td>
<td>$V_f = 29.8%$</td>
</tr>
<tr>
<td>Minimize P5 stresses</td>
<td>$\mu = 132.0$</td>
</tr>
<tr>
<td></td>
<td>$\text{Var} = 1659$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{\text{max}} = 474.6$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{1%} = 231.4$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{0.1%} = 270.4$</td>
</tr>
<tr>
<td></td>
<td>$M = 0.058$</td>
</tr>
<tr>
<td></td>
<td>$V_f = 29.9%$</td>
</tr>
</tbody>
</table>

ogy optimization on previous example and try to capture nonlinear behavior of the structure with this method.
2.8 Comparison of Optimized Structures

In this section we compare results of topology optimization under elastic and inelastic mechanic with results of stress based topology optimization. We apply stress based topology optimization on the beam example which explained in section 3.3.2. In this example we found pattern of optimized structure changes due to elasto plastic behavior of material.

Figure 2.11 shows the optimized structures designed with different methods of topology optimization. Then elastic minimum compliance optimization result is similar to p-2 norm result and the nonlinear optimization result is similar to the P-5 norm result.

It is not unexpected as nonlinear optimization is mostly governed by regions with high stresses which go to plastic region first.

In the optimized structure for elastic behavior two small handles which work in tension without buckling help to increase elastic modulus of structure. They also increase buckling load of main members. These members are removed in p-2 norm optimization because of low effect on stress distributions. But both structures have similar pattern for other elements.

Optimized structure for nonlinear mechanics and p-5 norm both have members in tension and compression. These members make the structure indeterminate with more elements which activate after first yielding. When we increase loading on the
structure combination of buckling and yielding happens in the members. These members add redundancy which prevent large drop after first yielding in the members. In the optimized structure for maximum energy dissipation there are more redundant members which keep high strength of structure for larger deformations. But in p-norm 5 stress based optimized structure, there are few members in tension and compression which activate in post buckling deformations. We should note that in this structure compressive members yield first before lower level members go plastic region.

2.9 Summary

In this chapter, it was shown that solid-only modeling eliminate numerical instabilities associated with elemental distortion in nonlinear finite element analysis, while also reducing computational cost.

Various objective functions for geometric and material nonlinearity methods were discussed, as well as associated sensitivity analyses. Then we introduced stress based optimization method as a proxy for material nonlinear topology optimization while this compared reasonably well, it is emphasized that this is only expected for the final compliance objective functions.
Figure 2.11: Optimized structures using different methods and objectives

Figure 2.12: Force displacement curve for optimized structure
Simple example showed how much results can change when we move from linear optimization to nonlinear optimization problem. This method opens new area of mechanical modeling of viscous material, fracture in the optimization problem.

I also applied stress based optimization method to show this method can replace low ductility examples. Final results for stress based and energy based methods compared and suggested to use stress based optimization method with higher p values to get results of ductile structures with nonlinear optimization.
Chapter 3

Topology Optimization For Cellular Structures

3.1 Introduction

In chapter three we applied topology optimization under nonlinear mechanics to design structures such as beams. In this chapter we focus on design of cellular materials. The idea is to design the unit cell (small scale) such that the cellular exhibits optimized effective properties at large scale. Researchers have suggested different methods which works for both microscopic and macroscopic scales.

We use inverse homogenization method to design new patterns with targeted elastic properties and symmetries. This method is developed for different objective functions such as elastic modulus, bulk modulus, and Poisson’s ratio. (Sigmund (1995b), Sigm-
mund (2000)), as well as thermoelastic (Sigmund and Torquato (1996)), piezoelectric (Nelli Silva et al. (1997)), fluid permeability (Guest and Prvost (2006)) and stiffness thermal conductivity (Challis et al. (2008)).

Although inverse homogenization is very popular in design of cellular structures, it is limited to linear properties. We therefore begin by considering elastic material properties and move to more complicated nonlinearities in the structure. Some of the difficulties of nonlinear optimization method were mentioned in chapter three.

In this chapter there are numerical instabilities of nonlinear constraints in addition to the previous difficulties in nonlinear analysis. Therefore nonlinear constraints are applied to get constituitive symmetries in cellular scale including isotropic and cubic symmetry. These symmetric properties are well defined for elastic materials in homogenization problems. Here, we apply these symmetric properties to the optimization problems under nonlinear mechanics.

At the end of this chapter we compare some analytical results of structure with results of experimental tests performed by collaborators at Yale university (Schroers (2010)).

In this chapter we start with homogenization method in elastic region and then move to plastic region with symmetric constraints.
3.2 Elastic Symmetrics

We begin defining by geometric symmetry in microscopic scale level on the cellular structure. For this purpose we use elastic constitutive matrix to force symmetry in this level. The elastic constitutive matrix is a forth order tensor with 81 constants. This number reduces to 21 because of symmetry in strain tensors and stress tensors. Symmetry planes decrease number of independent variables in the constitutive matrix. There are infinite planes of symmetry in isotropic material property, and it can be shown that the constitutive matrix has two independent parameters in this case. Other symmetry constraints have different number of cutting planes.

<table>
<thead>
<tr>
<th>Symmetry property</th>
<th>Number of symmetric planes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic</td>
<td>infinite</td>
</tr>
<tr>
<td>Monoclinic</td>
<td>1</td>
</tr>
<tr>
<td>Cubic</td>
<td>4</td>
</tr>
</tbody>
</table>

In this chapter we use isotropic or cubic constraints for all the examples. An isotropic and cubic(square) symmetric material can be shown to have the following stiffness.
Equation 3.4 shows isotropic constraint on the constitutive matrix.

\[ g_{\text{cubic}} = (C_{11} - C_{22})^2 + C_{13}^2 + C_{23}^2 = 0 \]  

Equation 3.4 shows isotropic constraint on the constitutive matrix.

\[ g_{\text{iso}} = (C_{11} - C_{22})^2 + (C_{11} - C_{12} - 2C_{33})^2 + (C_{22} - C_{12} - 2C_{33})^2 + C_{13}^2 + C_{23}^2 = 0 \]  

In the optimization problems we replace this constraint with an inequality constraint. It helps to satisfy constraint and remain in the feasible region during the optimization process. This inequality constraint is shown in equation 3.5.

\[ g_{\text{iso}} = (C_{11} - C_{22})^2 + (C_{11} - C_{12} - 2C_{33})^2 + (C_{22} - C_{12} - 2C_{33})^2 + C_{13}^2 + C_{23}^2 < \epsilon B^2 \]  

\( \epsilon \) is the tolerance limit and B is bulk modulus. \( \epsilon \) defines the acceptable tolerance for this constraint. Tolerance limit of \( \epsilon \) usually changes between 0.0001 to
0.001 for the following examples and Bulk modulus coefficient scales the constraint values.

Some mechanical properties which we use as objective functions in this chapter can be expected as functions of stiffness tensor as follows:

\[ \text{Elastic Modulus} = C_{11}^H - \frac{2(C_{11}^H)^2}{C_{22}^H} + C_{22}^H - \frac{2(C_{11}^H)^2}{C_{11}^H} \]  \hspace{1cm} (3.6)

\[ \text{Bulk Modulus} = \frac{C_{11}^H + C_{22}^H}{2} + \frac{C_{12}^H}{2} \]  \hspace{1cm} (3.7)

\[ \text{Poisson's Ratio} = 2 \times \frac{C_{12}^H}{C_{11}^H + C_{22}^H} \]  \hspace{1cm} (3.8)

\[ \text{Shear Modulus} = C_{33}^H \]  \hspace{1cm} (3.9)

Figure 3.1 illustrates properties which we use as objective function in our optimization problems. First we start with two main properties in the linear region, elastic modulus and bulk modulus. In following sections we move to nonlinear properties such as total and plastic energy absorption. Elastic modulus is a property of structure which defines stiffness of structure for applied loads. Stiffer structures show less deformations in applied loads.

Bulk modulus is an elastic property which defines stiffness of the structure for hydrostatic loads. It is not possible to show this property in the stress strain curve from vertical loading case. We show the physical behavior of bulk modulus in figure 63.
3.2. Equation 3.2 displays the mathematical calculation of bulk modulus.

$$K = -V \frac{dP}{dV}$$  \hspace{1cm} (3.10)

Bulk modulus is the stiffness of structure for the hydrostatic load cases. Therefore optimized structure for bulk modulus has the highest stiffness for hydrostatic load application.
3.3 Elastic Homogenization

In this section we explain some basics of homogenization method. In this method we find large scale properties with respect to microscopic scale characteristics. The goal is to determine the effective stiffness tensor $C^H$ of the material with the following elastic behavior:

$$\sigma = C^H \epsilon$$

(3.11)

Where $\sigma$ and $\epsilon$ are stress and strain in large scales.

Using homogenization method for optimization problems first introduced by Benoussan (1978) and Palencia (1980). They tried to find a mechanical properties of macroscopic scale designing material pattern in microscopic scale. Equation 3.12 shows the effective elasticity tensor of a periodic material in energy form.
\[ E^x_{ijkl}(x) = E_{ijkl}(x, y) \quad y = \frac{x}{\epsilon} \] (3.12)

\[ C^H_{pqrs} \epsilon^{0(kl)}_{pq} \epsilon^{0(rs)}_{ij} = \frac{1}{|Y|} \int_Y \left( C_{pqrs} \epsilon^{0(kl)}_{pq} - \epsilon^*(x^{kl})_{pq} \right) \left( \epsilon^{0(rs)}_{pq} - \epsilon^*(x^{ij})_{pq} \right) dy \] (3.13)

Where \( C_{pqrs} \) is the elasticity tensor of the solid material, \( \epsilon^{0(pq)}_{lk} \) are the test strain fields, \( \epsilon^*(pq)_{kl} \) are the fluctuation strains. Fluctuation strains define by strain-displacement relations. \( x^{kl} \) is found through solution to the following base cell problem:

\[ \int_Y C^H_{ijpq} \frac{\partial x_{kl}^{pq}}{y_p} \frac{\partial v_i}{y_j} dY = \int_Y C^H_{ijpq} \epsilon^{0(kl)}_{pq} \frac{\partial v_i}{y_j} dY \quad \forall v \in \hat{V} \quad \hat{V} = \{ v : v \ is \ Y-periodic \} \] (3.14)

We can simplify the constitutive matrix in the finite element format as below:

\[ C^H_{ijkl} = \frac{1}{|Y|} \sum (d^{(ij)}_0 - d^{(ij)}_e) K^e(\rho^e)(d^{(kl)}_0 - d^{(kl)}_e) \] (3.15)

Where \( K^e(\rho^e) \) is the stiffness matrix of element \( e \) and \( d^{(ij)}_e \) is the nodal displacement for element \( e \).

Several optimization methods developed based on homogenization method (Bendsoe (1989), Wang et al. (2003), Eschenauer and Olhoff (2001), Suzuki and Kikuchi (1991)) In these optimization problems they tried to find a configuration for microscopic scale which gives an optimized characteristics in macroscopic scale. For this purpose various macroscopic scale properties can be
considered as the objective function in the optimization problem. In this chapter we use inverse homogenization method to optimize structures for elastic modulus and bulk modulus to make a base structure for future comparisons to compare results of nonlinear optimization.

We should note that homogenization method is based on linear properties of material and we can not extend it for nonlinear region. In this chapter we use energy optimization method from last chapter considering nonlinear mechanics but we also add geometric constraints.

### 3.4 Maximum Elastic Modulus

In this section we optimize for elastic properties of structure under isotropy constraint. In this linear region we evaluate constitutive matrix.

The homogenized constitutive matrix is calculated according to equation 3.15. Equation 3.16 shows general optimization problem for elastic modulus due to isotropic constraints on the structure. As explained in the introduction this constraint is applied with an inequality equation. All the elements in the optimization equation are
homogenized values.

$$\max_{\varphi} E^H(C^H(\rho))$$

subject to: $g_{symm} \leq 0$

$$\sum_{e=1}^{nelem} \rho_i v_i \leq V$$

$$0 \leq \phi_i \leq \phi_{max} \quad \forall i \in \Omega$$

(3.16)

Where $\epsilon$ is the tolerance limit for isotropic constraint.

Another objective function we are considering in linear optimization problem is bulk modulus. Equation 3.17 shows the optimization problem with the constitutive matrix elements. Again, we should notice that all the elements are homogenized values.

$$\min_{\varphi} B^H(C^H(\rho))$$

subject to: $g_{symm} \leq \epsilon$

$$\sum_{e=1}^{nelem} \rho_i v_i \leq V$$

$$0 \leq \phi_i \leq \phi_{max} \quad \forall i \in \Omega$$

(3.17)

The materials will be evaluated under vertical loading, as done by experiment at Yale. We apply nonlinear finite element analysis for all the optimized structures. We use material properties of bulk metallic glass for all examples. These materials are special crystalized metals which form in high temperature under pressures. We explain more characteristics of the materials in the next section with experimental results. Some of the properties for Bulk metallic glass materials (BMG) are as below:
\[ E = 86900 \text{MPa}, \quad \nu = 0.375, \quad \sigma_y = 1475 \text{MPa} \]

Optimization for elastic and bulk modulus is well-understood and solutions found for BMG material properties are shown in Figure 3.12(c) and 3.12(b), respectively. Table (3.2) displays estimated homogenized properties for optimized structure and figure 3.12(b) shows stress-strain curve for these structures considering nonlinear analysis. In this figure pattern of structure with four cells is used for finite element analysis. Results of finite element analysis show that mechanical properties are clearly differentiable between structures optimized for bulk modulus and elastic modulus.

<table>
<thead>
<tr>
<th>Objective function</th>
<th>Elastic modulus</th>
<th>Bulk modulus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic modulus</td>
<td>6.514</td>
<td>3.356</td>
</tr>
<tr>
<td>Bulk modulus</td>
<td>0.100</td>
<td>3.360</td>
</tr>
</tbody>
</table>

As we expected the optimized structure for elastic modulus contains large horizontal and vertical elements. During the optimization problem we find pattern of each cell as the optimize structure. These structures are optimized for elastic properties but we apply nonlinear finite element analysis to find mechanical behavior of structures beyond linear region. However these structures are designed for linear region they have good energy absorption characteristic in nonlinear region.

We can distinguish special properties from stress-strain curve. The optimized structure for elastic modulus has very higher initial stiffness with a sudden drop after 1%
strain. Both high stiffness and high failure point are special characteristics of this structure. Therefore this structure is applicable for high initial load but not large energy dissipation. Optimized structure for bulk modulus has higher energy absorption property with lower initial stiffness. This structure can support 12MPa stresses to very large strains of 40% which makes it very practical for barrier and impact proof structures. Three points of stress-strain curve selected to show what is stress situation of each element at every point. First point shows the highest elastic behavior in both structures. "The maximum elastic modulus topology” structure reaches this point in small deformation while "the maximum bulk modulus topology” structure behaves linearly until most of vertical elements reach yield point. The optimized structure for bulk modulus has lower maximum stress but in a large deformation path. In this structure there are initial deformations due to elastic behavior of structures and larger deformations due to plastic strains. Final failure point for both structures happen at failure of elements under axial and it happens due to yielding of some members in a row. This happens after large deformations in the ”the maximum bulk modulus topology” structure but low deformations in ”the maximum elastic modulus topology” structure.
Figure 3.3: Optimized structure and Von-Mises stresses in the structure and stress-strain curve
Figure 3.4: Deflected states and Von-Mises stress contours for maximized elastic modulus structure corresponding to points in Figure 3.3(c)

Figure 3.5: Deflected states and Von-Mises stress contours for maximized elastic modulus structure corresponding to points in Figure 3.3(d)
Other elastic properties which we use as an objective function are Poisson ratio and shear modulus. (Figure 3.6) These properties can be related to the homogenized constitutive matrix elements with nonlinear relationships. Equations 3.8 and 3.9 show formula for these properties. This figure shows one cell of the structure. Each structure consists of repetitive pattern on this cell in two dimension.

(a) Objective: Minimize $\nu^H$  \hspace{1cm} (b) Objective: Maximize $G^H$

Figure 3.6: Optimized structure for shear modulus and Poisson’s ratio
3.5 Optimization Of Nonlinear Properties

In this section we optimize the nonlinear response properties of cellular materials while imposing elastic symmetries.

3.5.1 Total Energy Dissipation

As mentioned in the introduction, there are no widely accepted methods for nonlinear homogenization. In this section we therefore use finite periodicity and design materials using developed approaching in previous chapter. Considered objective functions include total energy dissipation which is a nonlinear, path dependent objective function.

Total energy dissipation is extensively explained in chapter three. In this chapter we added constraint of isotropic symmetry to the problem for the cellular structure. To have a better understanding of energy absorption in the mechanical behavior of structure we refer you to figure 3.1. From this figure we can find that strain energy absorption is the area under the curve. Therefore structure which has larger strain energy absorption shows combination of higher strains and stresses in the structure. Equation 3.18 shows general optimization problem for total energy absorption with isotropic constraints.
minimize \( f = - \int_\Omega \sigma^T d\epsilon d\Omega \)

subject to: \( R(\rho, \epsilon) = 0 \)

\[ g_{symm} \leq 0 \]

\[ \sum_{e=1}^{nelem} \rho_i v_i \leq V \]

\[ 0 \leq \phi_i \leq \phi_{max} \quad \forall i \in \Omega \]

Where it is emphasized that unit cell homogenization is used to compare elastic \( g_{sym} \) and finite periodicity the objective function.

Figure 3.7(b) displays topology optimized for total energy absorption for square symmetric and 25% volume fractions. This result will be analyzed and discussed in section 3.6.

### 3.5.2 Maximum Elastic Energy

After finding optimized structure for total energy dissipation we note that most of internal members yield at small loads and stress drops significantly after maximum load point. This is not usually desirable for engineers to have a very high energy absorption for small strains and suddenly drop after maximum load.

In this section elastic energy is considered as the objective function. The idea is to find more elastic buckling in the members of structure before yielding failure happens in the elements. Generally more local and global buckling makes elastic deformation
before plastic deformations. Therefore we consider the elastic part of total energy in the optimization problem and we expect elastic buckling to be the primary energy dissipation mechanism, more so then yielding. Equation 3.19 shows the optimization equation for this problem.

\[
\begin{align*}
\text{minimize } f &= - \int_\Omega \sigma^T_{\text{elas}} d\epsilon_{\text{elas}} d\Omega \\
\text{subject to: } R(\rho, \epsilon) &= 0 \\
g_{\text{symm}} &\leq 0 \\
\sum_{e=1}^{n_{\text{elem}}} \rho_i v_i &\leq V \\
0 &\leq \phi_i \leq \phi_{\text{max}} \quad \forall i \in \Omega
\end{align*}
\]

(3.19)

Another option is to minimize plastic energy dissipation, thereby (hopefully) encouraging elastic energy dissipation. However, this method exhibited difficulties of moving from initial guess to the final step. In this method initial distribution makes large plastic strains in low density elements. It makes very large values for objective function in the initial steps compared to final steps.

Optimized structures for total energy and elastic energy are shown in figure 3.7. Volume fraction for these structures are 25% and structures are optimized maximum for 5% strain in the structure. Figure 3.8 shows the strain-stress diagram for both structures up to 10% strain in the structures. As well as the stress distribution at the final step.
Optimized structure for elastic energy and total energy introduce two types of structures which have different governing failure modes. Different layers of members in the maximum elastic energy structures transfer load in a larger path and fails mostly due to buckling in the elements.

Figure 3.7: Optimized structures for elastic energy and total energy dissipation

Figure 3.8: Deformed structures, stress distribution and stress-strain curve
3.6 Comparison

In addition to comparing optimized topologies with each others, we also analyze and discuss honeycomb topologies, which are well-known to dissipate energy efficiently. Lots of studies have been done on honeycomb periodic structures. Also large number of engineering products such as automotive, airplanes, buildings use of honeycomb patterns (Hutmacher et al. (2001), Gibson (1989), Neubert (1980), Chamis et al. (1986)). Wadley et al. (2003) introduced a method to fabricate periodic sandwich panel and found equations of failure for some kinds of periodic structures. They also compared results of experimental test for the optimized structure and foam based structures and conclude that optimized structures showed several times better results than foam structures. We compare results of foam materials with designed pattern materials like honeycomb.

Failure study of honeycomb structures have been done by Zarei and Kroger (2008), Failure of honeycomb structures generally happens due to crash in members. Crash force as well as energy absorption of these structures are higher than foam structures with the same volume fraction of material. Also these structures fail in more stable deformed shape compare to others. Liu et al. (2006) introduced designs for tubular structures for sandwich panel and attempt to optimize angle and length of elements to get better performance compare to honeycomb structures. Boucher et al. (2013) studied damping properties of honeycomb structures. They added new pattern of
materials to the honeycomb structure and increased damping characteristics of optimized structure for dynamic loads.

Table 3.3 summarizes the nonlinear response of honeycombs, topologies optimized for elastic properties and topologies optimized for nonlinear properties. The stress-strain response is shown, along with deformed shape at peak strains. The optimized structure for elastic modulus shows very high linear stiffness \((E^H = 5720 \frac{N}{mm^2})\) but very low energy dissipation in plastic region. On the other hand the structure optimized for total energy dissipation has more plastic deformation which increases energy dissipation in the structure. However, the elastic modulus \((E^H = 4010Nmm^2)\) is lower than the case of elastic modulus structure.

In comparison to optimized structures honeycomb topologies offer lower elastic modulus and energy dissipations. In addition to high energy dissipation in honeycomb structures large ductility is also important for these structures. Table 3.3 show that energy absorption in optimized structures are twice as honeycomb structure. Therefore cellular structure design based on the topology optimization method has better characteristics for energy absorption. Stress-strain curve for optimized structure terminates at 20% strain. This is because our finite element analysis lacks consideration of contact, which is clearly seen in the final deformed shape. Our collaborator at Yale is producing these structures with real bulk metallic glass material to capture very large strains in the structures. It is also important to notice that experimental results
Table 3.3: Compare mechanical behavior of structures for 12.5% volume fraction

<table>
<thead>
<tr>
<th>Objective</th>
<th>Optimized structures</th>
<th>Stress-strain</th>
<th>Deformed structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Honeycomb</td>
<td><img src="image" alt="Honeycomb structure" /></td>
<td><img src="image" alt="Stress-strain graph" /></td>
<td><img src="image" alt="Deformed structure" /></td>
</tr>
<tr>
<td>Honeycomb</td>
<td><img src="image" alt="Honeycomb structure" /></td>
<td><img src="image" alt="Stress-strain graph" /></td>
<td><img src="image" alt="Deformed structure" /></td>
</tr>
<tr>
<td>Maximize $B^H$</td>
<td><img src="image" alt="Maximize $B^H$ structure" /></td>
<td><img src="image" alt="Stress-strain graph" /></td>
<td><img src="image" alt="Deformed structure" /></td>
</tr>
<tr>
<td>Total energy</td>
<td><img src="image" alt="Total energy structure" /></td>
<td><img src="image" alt="Stress-strain graph" /></td>
<td><img src="image" alt="Deformed structure" /></td>
</tr>
<tr>
<td>Maximize $E^H$</td>
<td><img src="image" alt="Maximize $E^H$ structure" /></td>
<td><img src="image" alt="Stress-strain graph" /></td>
<td><img src="image" alt="Deformed structure" /></td>
</tr>
</tbody>
</table>
showed honeycomb structures have large recovery in the structure after unloading. If we find similar behavior for optimized structure in the experimental test we can introduce this design as a structure with super mechanical properties for engineering structures.

The optimized structure for elastic modulus has thick members in x and y direction which increase the elastic stiffness of structure but leads to dramatic loss of stiffness at onset of yielding. Local plasticity combines with buckling causing large reduction in stiffness of the whole structure. Therefore in this structure we can capture very high elastic stiffness with low ductility and post elastic energy absorption.

Diagonal members in the optimized structure for energy dissipation adds stiffness for shear and torsional deformations. They also make shorter free member length and increase the buckling load for the member.

Optimized structure for bulk modulus and honeycomb structure shows similar behavior in nonlinear region. Both structures have similar energy dissipation and similar failure modes. In both structures, most of nonlinearity is focused at the nodes with reduction in momentum capacity in the member. If the experimental results show similar recovery in the structure it can be a good replacement for honeycomb structure because of simpler structural pattern. Generally all the mechanical characteristics which are used in optimization problem can be easily captured in the stress-strain diagrams in table 3.3.
Table 3.4 summarizes topologies optimized using the same objectives but with a larger allowable volume fractions of 25%. The same trends are observed here. The honeycomb structures still show good stiffness and ductility in the nonlinear region. The optimized structure for maximum energy dissipation have larger stress capacity and larger mechanical energy absorption. Optimized structure for maximum elastic energy have smaller total energy dissipation, but the elastic deformation is the governing deformation for this structure. The optimized structure for elastic modulus and bulk modulus have large elastic stiffness and low post elastic stiffness.
Table 3.4: Compare mechanical behavior of structures for 25% volume fraction

<table>
<thead>
<tr>
<th>Objective</th>
<th>Structures</th>
<th>Stress-strain</th>
<th>Deformed structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Honeycomb</td>
<td><img src="image1" alt="Honeycomb" /></td>
<td><img src="image2" alt="Honeycomb" /></td>
<td><img src="image3" alt="Honeycomb" /></td>
</tr>
<tr>
<td>Maximize $B^H$</td>
<td><img src="image4" alt="Maximize B^H" /></td>
<td><img src="image5" alt="Maximize B^H" /></td>
<td><img src="image6" alt="Maximize B^H" /></td>
</tr>
<tr>
<td>Total energy</td>
<td><img src="image7" alt="Total energy" /></td>
<td><img src="image8" alt="Total energy" /></td>
<td><img src="image9" alt="Total energy" /></td>
</tr>
<tr>
<td>Maximize $E^H$</td>
<td><img src="image10" alt="Maximize E^H" /></td>
<td><img src="image11" alt="Maximize E^H" /></td>
<td><img src="image12" alt="Maximize E^H" /></td>
</tr>
<tr>
<td>Elastic energy</td>
<td><img src="image13" alt="Elastic energy" /></td>
<td><img src="image14" alt="Elastic energy" /></td>
<td><img src="image15" alt="Elastic energy" /></td>
</tr>
</tbody>
</table>
Figure 3.9: Deformed structures and stress distribution for the honeycomb structure under horizontal loading

Figure 3.10: Deformed structures and stress distribution for the honeycomb structure under vertical loading
Figure 3.11: Deformed structures and stress distribution for the structure which optimized for total energy.

Figure 3.12: Deformed structures and stress distribution for the structure which optimized for elastic energy.
Figure 3.13: Deformed structures and stress distribution for the structure which optimized for bulk modulus

Figure 3.14: Deformed structures and stress distribution for the structure which optimized for elastic modulus

86
3.7 Convergence Under Finite Periodicity

A key assumption is that finite periodicity is capturing the "effective" nonlinear properties of the cellular material system. Table 3.5 and 3.15 display the stress-strain response for varying number of unit cells of topologies optimized for energy absorption under square and isotropic symmetry constraints. The tables are summarized in Figure 3.15 and 3.16, show that the shape of stress-strain response curve is relatively consistent for finite periodicity represented by $2 \times 2$, $4 \times 4$, and $6 \times 6$ unit cell patterns. However, it is noted that the magnitude of the peak stress in the case of table 3.5 does vary significantly. It is because of effect of horizontal load applied at the top in addition to the vertical displacement. This was likewise observed for honeycomb structures. Computational capacity limits us from optimizing at $6 \times 6$ distribution, but we do not expect solutions to change as the effects appear to be related through linear scaling.
Table 3.5: Sensitivity of mechanical properties of optimized structures to cell numbers

<table>
<thead>
<tr>
<th></th>
<th>2 Cells</th>
<th></th>
<th>4 Cells</th>
<th></th>
<th>6 Cells</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 3.15: Stress-strain diagram for optimized structures for total energy dissipation
Table 3.6: Sensitivity of mechanical properties of optimized structures to cell numbers

<table>
<thead>
<tr>
<th></th>
<th>2 Cells</th>
<th>4 Cells</th>
<th>6 Cells</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stress/MPa</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Strain/mm</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 3.16: Stress-strain diagram for optimized structures for elastic energy dissipation
3.8 Experimental Results

In this section we compare experimental results with analytical results and use further analytical simulation to find effect of material and geometric parameters on the final mechanical behavior of these structures.

Mechanical properties of honeycomb structures change with different material properties. In this section we model honeycomb structures and compare results from numerical analysis with the experimental testing. Volume fraction for honeycomb structures are 12.5%. The material which is used in honeycomb structures are bulk metallic glass (BMG). This is a special metallic alloy highly ordered arrangement of atoms. Because of special cooling procedure an amorphous structure build up.

Honeycomb structures have large flexibility and recoverable characteristic after large deformation. These properties are evaluated in an experimental test by Sarac et al. (2012). The stress-strain relationships for honeycomb structures are shown in Figure 3.17. Volume fraction clearly has an effect on strength of structure, but does not seem to significantly change ductility. Our analysis results for honeycomb of volume fraction of 12.5% is shown in Figure 3.22(a). We see the predicted curve strongly resemble the experimental results in Figure 3.17.

In this section we compare results of nonlinear behavior for honeycomb structure and optimized structure for total energy dissipation. Same material has been used
for analysis of both structures. The optimized structure shows higher energy dissipation and higher fracture forces compare to honeycomb structure. Peak stress is three times larger the energy dissipation is two times larger. The deflected shapes illustrate that the optimized structure uses vertical and horizontal elements to transfer loads to the supports and diagonal members act as bracing to avoid buckling and torsional movement of main members. This helps high energy dissipation occur because of vertical members and more stable structure raised with diagonal elements. Diagonal elements are mostly in tension avoiding rotation in main members. Also these members increase stiffness of structure for shear forces. Mechanical analysis in figure 3.18 shows how failure path changes from shear failure in honeycomb structure to axial failure in vertical elements in optimized structure.
Experimental tests have been done on the optimized structures for total energy absorption. Figure 3.20 shows the structure built with BMG material in 36 cells. Stress-strain diagram in Figure 3.21 indicate that optimized structure absorb energy
more than 10% compared to honeycomb structure. This energy absorption increases to higher values for higher strains. In addition to the higher energy absorption, these structures show three times higher strength and stiffness. These properties make the optimized structure more valuable in building structures.

Figure 3.20: Experimental structures for optimized structure and honeycomb (Schroers Lab at Yale University)

Figure 3.21: Stress-strain diagram for optimized structure and honeycomb (Schroers Lab at Yale University)

We also compared results for 25% volume fraction. Again for this structure, energy absorption increased more than 10% and higher values for yield stress and initial
stiffness. Figure 3.22 shows the optimized structure and honeycomb structure for volume fraction of 25%. The stress-strain diagram in Figure 3.23 indicates these increments in mechanical properties.

Figure 3.22: Experimental structures for optimized structure and honeycomb (Schroers Lab at Yale University)

![Experimental structures](image1)

![Experimental structures](image2)

Figure 3.23: Stress-strain diagram for optimized structure and honeycomb (Schroers Lab at Yale University)

![Stress-strain diagram](image3)

Both volume fraction results show higher mechanical properties not only in energy absorption but also in strength and stiffness.
3.9 Summary

In this chapter, nonlinear optimization is applied for cellular structures with constitutive symmetry. Different objective functions were considered and compared to honeycomb structures to show how much improvement produced by using topology optimization.

Also, in this chapter some experimental results are compared with computational predictions. It was also shown that relatively small instances of finite element periodicity were sufficient to capture the general shape of the nonlinear stress-strain response, though did underestimate capacity in one case. This trend can not necessarily be considered for all materials and objectives, but appeared sufficient here. At this time of this thesis, we were still awaiting fabrication and experimental testing of our optimized topologies.
Chapter 4

Corrugated Structures

4.1 Introduction

The first step towards 3D BMG materials is corrugated sheets. We therefore examine the effect of corrugation and bound strength between layers on corrugated structure mechanical properties.

Another type of structures which are widely used as light weight and high stiffness structures are corrugated structures. In this chapter we analyze some parameters which has effect on stiffness and strength of these structures such as length, thickness and material properties. Corrugated structures are developed during the years to reduce weight of structures by special patterns. This is a base of moving to the optimized structure in three dimension. We start with one cell analysis of corrugated structures and move to multi layers and multi
cell structures.

4.2 Corrugated Structures

Composite structures are light weight structures used for covering mechanical parts or making walls in buildings. It is important to find structures with maximum mechanical properties and minimum material usage. Therefore companies working on composite structures try to find configurations which are easily constructable and minimum material usage. In this section we evaluate one of most common composite structures with corrugated plates to find mechanical properties and give suggestions for experimental studies to our collaborator at Yale University.

In figure 4.1 three common types of composite with corrugated plates are shown. Volume fraction of each type is also calculated for weight constraints.

Manufacturing of these structures depend on the layers and connections. Usually they start with a flat plate and try to find the new configuration by bending, welding and cuttings. One of simple methods to build corrugated plates are shown in figure 4.2. This method is done by Waniuk et al. (2001) to define plates in a specific configuration and connect two layers with heat and pressure on both layers.

Mechanical properties of these three types of corrugated structures are evaluated in this section. Table 4.1 shows geometrical variables are initiated with specific constructible values. These values are suggested by the experimental challenges in
making corrugated plates.

Figure 4.1: Composite with corrugated layers, geometric properties and volume fraction (Thill et al. (2010))

<table>
<thead>
<tr>
<th>L(mm)</th>
<th>b(mm)</th>
<th>t(mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>100</td>
<td>0.1</td>
</tr>
<tr>
<td>10</td>
<td>150</td>
<td>0.2</td>
</tr>
<tr>
<td>200</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Corrugated structures are made in the lab with deforming plates on dented molds. Figure 4.2 shows how two layers of corrugated plates are made in the lab.

Twenty-six corrugated plates are analyzed with Abaqus to find variation in me-
Figure 4.2: Manufacturing corrugated structures in the lab (Schroers (2010))

Mechanical characteristics due to changes in geometrical properties.

Figure 4.3: Von Mises stresses for an example of each type

Navtruss panels show about two times higher values in peak stresses. Also peak stress happens in higher strains. This was expected because diamond corrugated plates consist of two layers with a small connection in the middle. Triangular and Navtruss both have similar values of high stresses and large deformation in the plastic region. Therefore final results for different corrugated structures do not change from
Table 4.2: Maximum vertical load

Effect of out of plane length is very low for all the cases. It means changing in the length of panels does not change the stress strain curve in the mechanical test. Thickness and length of members both have effect on the controlling load. Failure in
the elements can happen because of buckling or yielding. The $\frac{L}{t}$ ratio defines which criterial governs in the failure.

After evaluating one layer corrugated structured structures we move to the real size of experimental tests. These layers of corrugated plates are going to be built and tested in the lab. In this section we try to find effect of main members and connections on the mechanical behavior of panels. Plastic behavior of each part can have effect on the final results.

The following examples are three layer corrugated plates in compression and bending. Connection elements attached two layers of plates together. Material properties of connection elements change due to construction process. Different ratio of stiffness...
Table 4.4: Maximum vertical load

<table>
<thead>
<tr>
<th></th>
<th>$L = 5\text{mm}$</th>
<th></th>
<th>$L = 10\text{mm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b100t.1</td>
<td><img src="image" alt="Graph" /></td>
<td>b100t.2</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>b150t.1</td>
<td><img src="image" alt="Graph" /></td>
<td>b150t.2</td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>b120t.1</td>
<td><img src="image" alt="Graph" /></td>
<td>b120t.2</td>
<td><img src="image" alt="Graph" /></td>
</tr>
</tbody>
</table>

and strength are considered for the connection part. Ratios which are considered in this section are 20%, 40% and 60%.

Results of nonlinear analysis compared with results of linear elastic analysis. Ma-
### Table 4.5: Continue Maximum vertical load

<table>
<thead>
<tr>
<th>Configuration</th>
<th>$L = 5\text{mm}$</th>
<th>$L = 10\text{mm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b200t.1</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
</tr>
<tr>
<td>b200t.2</td>
<td><img src="image3" alt="Graph" /></td>
<td><img src="image4" alt="Graph" /></td>
</tr>
</tbody>
</table>

### Table 4.6: Maximum vertical load

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Load 1</th>
<th>Load 2</th>
<th>Load 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>L5b100t.1</td>
<td>20000</td>
<td>3000</td>
<td>98000</td>
</tr>
<tr>
<td>L5b100t.2</td>
<td>115000</td>
<td>14000</td>
<td>34000</td>
</tr>
<tr>
<td>L5b150t.1</td>
<td>32000</td>
<td>3000</td>
<td>9800</td>
</tr>
<tr>
<td>L5b150t.2</td>
<td>125000</td>
<td>13000</td>
<td>35000</td>
</tr>
<tr>
<td>L5b200t.1</td>
<td>36000</td>
<td>3000</td>
<td>11000</td>
</tr>
<tr>
<td>L5b200t.2</td>
<td>120000</td>
<td>27000</td>
<td>37000</td>
</tr>
</tbody>
</table>
Figure 4.4: 3D corrugated plates in compression

Table 4.7: Von Mises stresses and stress-strain curve

Von-Mises stresses | Stress-Strain Curve

Figure 4.5: 3D corrugated plates in Bending
Table 4.8: Von Mises stresses and stress-strain curve

<table>
<thead>
<tr>
<th>Von-Mises stresses</th>
<th>Stress-Strain Curve</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Von-Mises stresses" /></td>
<td><img src="image" alt="Stress-Strain Curve" /></td>
</tr>
</tbody>
</table>

terial nonlinearity for main members are compared with material nonlinearity in connection elements. Figure 4.10 shows that how material plasticity in different part of this composite structure has effect on stress-strain curve of whole structure. First part of all diagrams governed by elastic behavior of material in both part. Then plastic behavior of connections causes reduction in the stiffness of the structure. The final steps in plato region is governed by buckling and yielding in the main members. When we do not consider plasticity in these members we can capture high deformations but analysis stops with plasticity failure in the main diagonal members. Very large deformation in the thin members cause large strain changes and stops analysis in the middle steps. Therefore most of analysis results with nonlinearity in both materials stops about 5% strains.
Table 4.9: Material nonlinearity effect in load-deflection results

<table>
<thead>
<tr>
<th>Von-Mises stresses</th>
<th>Stress-Strain Curve</th>
</tr>
</thead>
</table>

Table 4.10: Material nonlinearity effect in load-deflection results

<table>
<thead>
<tr>
<th>Von-Mises stresses</th>
<th>Stress-Strain Curve</th>
</tr>
</thead>
</table>
4.3 Summary

In this chapter we evaluated corrugated structures as step towards optimized structures in three dimension. For this purpose the effect of different material and geometry parameters are indicated. Also the effect of imperfection evaluated in the material properties of main members and connections are considered. It is shown reduction in the material properties are mostly effects on the stiffness of corrugated panels not strength. The ratio of decrease in stiffness is related to the ratio of stiffness reduction in connection material.

In corrugated structures governing failure is buckling and yielding in the main members. Because of the length thickness ratio connection members do not fail before diagonal members.

Governing failure in the corrugated structures are yielding in the connection or buckling in the diagonal members. Therefore different configurations with same size of diagonal members and same material properties show similar behavior in failure.
Chapter 5

Future Works

Topology optimization is a developing method in the field of mechanical engineering. One group of isotropic material with Von-Mises yield criteria is discussed in this thesis but there are large area of materials needed to be implemented in the optimization method. More material properties and non-isotropic materials should be considered for the future researches. Also different scales of materials are governed by different mechanical characteristics which need more researches to apply in the optimization method.

Three dimensional modeling and optimization is a new method which needs more researches to overcome arises difficulties for compressible and non compressible materials. We started with corrugated structures in this dissertation as the base point for future comparisons.
Appendix A

Climate Changes Due To ENSO

A.1 Introduction

This chapter contains a body of work done by the author concerning the statistical analysis and data modeling of a climate shift trend known as El Nino Southern Oscillation (ENSO). ENSO is a large weather circulation in the tropical region which effects regions from Australia to South America. In particular, we seek to identify local data trends related to such parameters as water vapor, temperature, and relative humidity, that serve as large scale indicators or drivers of weather circulations in a large scale. Although perhaps unrelated topology optimization component of this thesis, the work represents a major effort by the author and falls under the category of mining big data for trends and governing parameters, which is becoming an important trend in materials research and design.
We examined the interannual variability for 9 years from 2002 to 2011 in the time series analysis. Relative humidity, water content and temperature are three parameters which we chose for this study. We focus on the geographic area of Pacific Ocean which ENSO has the most effect.

There are couple of research studies in this field the last two decades. Spencer and Braswell (1997) studied variation in the clear sky outgoing long wave radiation (OLR) in upper tropospheric due to water vapor changes and found that OLR fluctuations are mostly connected to relative humidity in the troposphere region. Bates et al. (1996) evaluated the variability of upper troposphere relative humidity and found the correlation with ENSO. They found relationship between tropical circulation during boreal winter and spring. Yun et al. (2010) studied climate changes due to ENSO in east Asia. They worked on the relation between northward-propagating intraseasonal oscillation (NPISO) and ENSO and focused on OLR changes. Gettelman et al. (2000) focused on the tropopause region. They studied OLR, Temperature and water vapor and found that ENSO is the first mode of variability for convection in the tropical pacific. They compared these changes with the community climate model (CCM3) and found that CCM3 model can not capture observed tropopause temperature anomaly. Liang et al. (2011) used AIRS and MLS data to study atmospheric variability of the temperature and water vapor. They found topical western pacific as the location with persistent deep convection and the tropical central pacific as the region of subsidence. Also showed that wind circulation has the largest effect on these parameters. The
anomalies of the temperature due to convection in the troposphere reaches 200hPa and water vapor reaches 120hPa. They showed that Tropical Central Pacific (TCP) have a significant role in the water vapor distribution.

McCarthy and Toumi (2004) studied interannual variability of the tropical troposphere relative humidity and studied effects of different components in relative humidity. They found low effect of ENSO on the tropics wide troposphere relative humidity (UTRH). Oman et al. (2011) worked on the effect of ENSO on troposphere ozone and found relationship between tropical sea surface temperature (SST) anomalies and the response of tropical tropospheric ozone. Also they introduced a linear relationship between ozone and ENSO.

This chapter introduces daily base changes in relative humidity (RH), temperature and water vapor in different region. Then two relationship between pattern effect and water vapor distribution in troposphere layer.

A.2 Data And Method

We analyzed products of Atmospheric infrared sounder (AIRS) level 3 gridded data and Microwave Limb Sounder (MLS) data collected by satellites. It is a one degree gridded data which coordinates in the range of (−180°)-(180°) in longitude and (−90°)-(90°) in latitude. There are daily and monthly data for each grid point. Level 3 daily product contains information for descending and ascending orbits. Daily data is used to find probability density function of water vapor, Temperature and relative
humidity. For average seasonal data we used monthly average data. In some points in this chapter we confirm results with MLS data. MLS data is non gridded data based on the microwave instruments. It is just used as a check point for AIRS data in this chapter.

The phenomena that we consider in my Analysis is called El-Nino southern oscillation (ENSO). ENSO time period is defined based on the Nino-3.4 index of central pacific Sea Surface Temperature(SST) anomalies in the region bounded in 5°S-5°N, 170°W and 120°W. For this purpose we do sensitivity analysis of relative humidity, water vapor, temperature, OLR and Wind with respect to ENSO to find how much this circulation has influence on weather parameters. To be sure that ENSO has the principle effect on the variability of each parameter we do Empirical Orthogonal Function(EOF) analysis. First EOF shows the main effect on the changes of each parameter. We also added probability density functions for each parameter for the first time to confirm results from sensitivity analysis and find changing parameters in days. We added structural variability to the new sta

A.3 El-Nino Southern Oscillation(ENSO)

ENSO is a large circulation and climate change that happens in the pacific ocean. ENSO has a quasi-periodic cycle in every five years and refers to the variation in the sea surface temperature at eastern pacific ocean. It happens in two phases. The warm phase, El-Nino, causes high air surface pressure in western pacific and cold
phase, La-nina, causes low air pressure in western pacific. The recharge-discharge oscillator argues that this oscillation in the equatorial heat content causes oscillation in the system.

Earliest studies on ENSO have been done by Bjerknes(1966-1969). He found evidences of long term persistence of climate anomalies associated with Walker southern oscillation. Walker circulation causes positive Sea Surface Temperature(SST) anomalies in eastern and central pacific. SST anomaly causes a never ending warm state in equatorial pacific. But Bjerknes couldn’t find what causes a turnabout from warm phase to a cold phase. In this section we explain some of famous turnover activities.

1) Delayed oscillator (Suarez and Schopf (1988), Graham and White (1988)) A Mechanism of oscillatory nature of ENSO proposed by McCarthy based on the subtropical oceanic upwelling Rossby waves at western boundaries. Surez and Schops provided the delayed oscillator as a mechanism of ENSO. The conceptual delayed model is represented by an ordinary differential equation with both positive and negative feedbacks.

\[
\frac{dT}{dt} = AT - BT(t - \eta) - \epsilon T^3
\]  

(A.1)

T is SST anomaly in the equatorial eastern pacific, A, B, \(\eta\) and \(\epsilon\) are constant values. First term is the positive feedback, Second term is the delayed negative feedback by free Rossby waves generated in eastern pacific coupling region that reflect from western boundaries returning as Kelvine waves to reverse the anomalies in eastern
pacific region. Last term is damping term which subsides effects of other terms.

2) The western pacific oscillator: During ENSO, warm SST and low Sea Level Pressure (SLP) anomalies in the eastern pacific and low Outgoing Long wave Radiation (OLR) are accompanied by the cold SST in western pacific. This model has the role of western pacific in ENSO. SST and SLP anomalies in western pacific induce equatorial western pacific wind anomalies. It can be represented as:

\[
\frac{dT}{dt} = aT + b_2T_2(t - \delta) - \epsilon_1T^3
\]

(A.2a)

\[
\frac{dh}{dt} = -cT_1(t - \lambda) - \epsilon_2h
\]

(A.2b)

\[
\frac{dT_1}{dt} = dT - \epsilon_3T_1
\]

(A.2c)

\[
\frac{dT_2}{dt} = dh - \epsilon_4T_2
\]

(A.2d)

T is SST anomaly, h is thermocline depth anomaly, T_1 and T_2 are equatorial wind stress anomalies. Gill 1980 found that condensation heating due to convection in the equatorial central pacific induces a pair of off-equatorial cyclones which increase SST in equatorial eastern pacific. It raises thermocline via Ekman pumping. Suggested values for constants are as below. \( a = 1.5 \times 10^3 m^3 N^{-1} y^{-1}, b = 9 \times 10^3 m^3 N^{-1} y^{-1}, c = 1.5 \times 10^3 m^3 N^{-1} y^{-1}, d = 3.0 \times 10^3 m^3 N^{-1} y^{-1}, e = 3.0 \times 10^{-3} m^3 N^{-1} y^{-1}, \epsilon_1 = \epsilon_2 = 1.0 \times 10^{-2} m^{-2} y^{-1}, \epsilon_3 = \epsilon_4 = 7.0 \times 10^2 m^4 N^{-2} y^{-1} \) and \( \delta = \lambda = 30 \text{days} \).
Thermocline is a thin layer in the ocean, in which temperature changes more rapidly with depth than it does in layers above or below. Thermocline in equatorial pacific ocean is normally found between 15 – 20°C isotherms. It is a main indicator of the current ENSO mode and can be used to forecast transition of the oscillation. Thermocline is naturally deeper in western pacific and shallower in eastern pacific due to downwelling of warm waters and upwelling of cold waters, respectively. A shallower thermocline provides a greater supply of cold waters from upwelling, while a deeper thermocline provides a lesser supply of cold waters for upwelling.

During El-Nino event, Kelvin waves propagate westward and weaken the easterly trades thus causes warm waters to move from west to east. Therefore thermocline becomes shallower in the western pacific and deeper in central and eastern pacific.

During El-Nino to La-Nina event, thermocline levels off as west becomes shallower and east becomes deeper. Eventually thermocline is leveled and shallowed throughout entire equatorial pacific. Any weak fluctuation in surface winds causes sea surface to cool down and promote upwelling. This may set the stage for La Nina to develop and transition may only take one month or two.

3) The recharge-discharge oscillator based on relaxation and built up of sea level Cane and Zebiak (1985) proposed this oscillator model for ENSO.

\[
\frac{dT}{dt} = CT + Dh - \epsilon T^3 \quad \text{(A.3a)}
\]
\[
\frac{dh}{dt} = -ET - R_h h
\]  

(A.3b)

During warm phase of ENSO, divergence of Sverdrup transport accompanied by equatorial central pacific wind anomalies. Discharge of equatorial central pacific wind anomalies. Discharge of equatorial heat content leads to transition phase to the cold phase. Excitation of recharge oscillator can be self-excitation or stochastic excitation.

\section*{A.4 Interannual Variability}

We use monthly mean values of relative humidity in this section. Several studies mentioned in the introduction have looked to the relative humidity changes, usually with different data set. So we compare my results with relevant previous studies in the following sections.

Also we added new analysis for vertical changes of these parameters. Most of previous works didn’t study vertical variability of each parameter. We will show that some parameters behave very differently in vertical pressure levels. We also apply daily analysis for probability distribution function (Pdf) we use daily data to compare pdf of different regions and categorize them in small groups with similar behavior.

\subsection*{A.4.1 Climatological Relative Humidity}

We examine interannual variability in tropical and subtropical relative humidity using AIRS monthly-mean data in the time range of 2002-2012.
Figures A.1 and A.2 display climatological mean of relative humidity for winter, summer and all seasons. High average values for eastern pacific in JJA months is distinguishable.

High values of relative humidity in winter times in tropical region happen around $120^0E$ and $300^0W$ and low relative humidity values happen at the subtropics. This high and low values are results of pacific Hadley circulations. In summer time Hadly circulation subsides and we can find low relative humidity at the south subtropic and higher relative humidity at North subtropic. Section from equator shows that mean values for summer and winter times are similar in this section. But the big difference is between $0 - 30^0W$. In this section we evaluate different statistical methods to find variety in relative humidity. In the mean value map of all seasons according to Walker circulation there are three convection regions. $30^0E$, $150^0E$ and $70^0W$ are three regions that convection happens. $0^0$, $40^0E$ and $120^0W$ are three regions that subsidence happens. In the subtropical region around $150^0W$ convection occurs. All these subsidences and convections are related to the Walker and Hadley pacific circulation. These circulation patterns are shown in figure A.3. Hadley circulation happens because of the Coriolis forces which is defined as

$$F = 2m\Omega|U|$$  \hspace{1cm} (A.4)

In winter time large subsidence in the tropical region around $180^0$ accomplished by large equatorial convention around $120^0E$. This change happens due to large con-
vention of Walker circulation. Hadley and Walker pacific circulation limited to the levels of upper troposphere. Therefore higher than this level complex pattern of distribution happens. In the upper troposphere from $60^\circ E$ to $180^\circ$ westward circulation happens. This circulation is limited to $800\,hPa$ in the lower part and $100\,hPa$ at top level. Except in the region about $30^\circ E$ that subsidence goes to the higher pressure levels about 1000hPa.

In figure A.4 sensitivity of relative humidity, First EOF of relative humidity and difference between two extreme years are compared. Those three show similar pattern for level of 300 hpa.
Figure A.2: Climatology mean for (a) JJA (b) All seasons

Figure A.3: Pacific circulations (Walker and Hadley) for (a) El-Nino (b) La-Nina
Figure A.1 displays mean and standard deviation of relative humidity mostly effected by ENSO in winter months not summer times. Mean values for summer months has very different pattern and less correlation with ENSO.

In this part we want to find the variability of relative humidity with respect to ENSO. For this purpose we use sensitivity analysis. In sensitivity analysis the ratio of relative humidity value at each point is calculated with respect to ENSO value. Figure A.4 shows pattern of sensitivity at 300hPa. There are clearly regions with high positive and negative sensitivities. positive sensitivity means relative humidity value goes up when ENSO increases and negative value is vice versa.

The method of empirical orthogonal function (EOF) is used to find the dominant patterns in the time. EOF analysis is based on decomposition of basis functions. In this method we can find the principle component and time series. It is an orthogonal linear transformation that transfer data to the new coordinate which greatest variance by any projection is on the first coordinate. Assume $X^T$ as the input data. We use decomposition to find eigenvectors and eigenvalues of $X$.

$$X = U S V^T$$  \hspace{1cm} (A.5)

$U$ is matrix of eigenvectors for covariance matrix $XX^T$. $S$ is the diagonal matrix of eigenvalues of $XX^T$ and $V$ is the matrix of eigenvectors of $X^T X$. EOF analysis of data $X$ shows the basis component of $X$ and separates the main components of data.
First EOF of relative humidity shows similar pattern to the sensitivity pattern. It means that ENSO is the first important phenomenon makes changes in relative humidity pattern after seasonal oscillation. Time series diagram for ENSO and first EOF (A.5) shows high correlation.

Figure A.4 shows that sensitivity of relative humidity with respect to ENSO, First EOF of relative humidity and difference of relative humidity values for two extreme winters have similar pattern.

Figure A.5 shows correlation between first EOF of relative humidity and ENSO.
index 3.4. Correlation for the first EOF is 80% and next EOFs have correlation less than 15%.

We assumed that relative humidity sensitivity pattern is mostly affected by the extreme years that EL-Nino and La-Nina happens. In figure A.4 regions with high and low sensitivity found by differences of two years and show changes in the circulation from one year to the next year. We have also found sensitivity in this period without considering two extreme years. The interesting result shows that ENSO still has dominant effect on these years. We calculate wind circulation in all period of ENSO to find how sensitivity pattern changes due to wind circulations.
A.5 Vertical Variation In Relative Humidity

We found similar patterns of sensitivity, Difference and first EOF for relative humidity in the last section. In the previous section we found similar patterns for sensitivity, Difference of relative humidity for two years and first EOF for relative humidity. In this section we choose sensitivity for future statistical analysis and show some cross sections for sensitivity of relative humidity in the regions with large differences. The variability described above is approximately the same for upper and lower levels.

Figure A.6 shows the sensitivity pattern for three longitudinal sections at equator, 25N and 25S. Sensitivity pattern does not follow previous pattern for pressure levels less than 200hPa. It is because of reverse circulation pattern at this level.

Figure A.7 shows high sensitivity at 120°E for the tropic and subtropics. At 180° reverse pattern in the tropic and subtropics are clearly defined. 140°W shows completely different pattern. Higher sensitivity values happen at upper stratosphere and higher sensitivity around equator and lower sensitivity at south tropic upper stratosphere. 80°W shows similar pattern to 120°E. Two circulations at subtropics are distinguishable. In cross sections at 140°W and 80°W lower troposphere shows different pattern than upper troposphere. It is mostly because Hadley circulation does not go to the high levels in stratosphere. Figure A.8 displays sensitivity patterns for lower and upper troposphere. Sensitivity pattern at upper troposphere is not influenced by the ENSO phenomena.
Figure A.6: Vertical cross section along (a) equator (b) $25^0 N$ (c) $25^0 S$
Figure A.7: Relative humidity sensitivity section for (a)120°E (b)180° (c)140°W (d)80°W

Figure A.8: Horizontal maps of ENSO sensitivity at levels of (a)150hPa (b)300hPa (c)500hPa (d)700 hPa
A.5.1 Comparison With MLS Data

Figures A.9(a) and A.9(b) show sensitivity maps of relative humidity for MLS and AIRS data. These figures show that MLS has the same interannual variability with different methods of measurement. AIRS data can be confirmed by MLS data at different pressure levels. Comparison of two years difference for MLS and AIRS relative humidity shows similar features at 300 hPa and same off equator maximum over western pacific. Different values happen because of different methods of measurement and data resolution of each method. AIRS data has more points of measurement and less data missing. Therefore we use AIRS data in the next sections.
In this section we want to find what causes interannual variability in relative humidity in the ENSO time period.

Figure A.11 shows that first EOF of OLR and relative humidity have similar patterns with different high values at most sensitive region. First EOF of OLR mostly follows pattern of specific humidity. Therefore these two patterns have been affected in similar ways by weather circulations.
We use wind directions to understand how these parameters are affected by weather circulations. In the regions with high sensitivities the upward wind shows higher values and lower values in low sensitive regions. Upward circulation has the most important role in sensitivity changes. Plan of sensitivity values and winds are shown in figure A.10. Wind directions and values follow sensitivity contours. It is clear that most of changes in the values of relative humidity, specific humidity and OLR happens because of in plane and out of plane wind circulation.

Figure A.10 is the map with upward wind contours and inplane winds. Figure A.11 shows SST and OLR on the relative humidity contours. This figure indicates that changes in SST a
A.6 Relative Contribution From Changes In Specific Humidity And Temperature

In this section we decompose different parameters in relative humidity to find effect of each item. Equations A.6 to A.9 display important part of each component in relative humidity.

\[ \Delta R = \Delta RH_T + \Delta RH_W + \frac{\Delta RH_T \cdot \Delta RH_W}{RH} + \epsilon \]  \hspace{1cm} (A.6)

\[ \Delta RH_T = \frac{(e_s \cdot \bar{RH})}{e_s + e_s} - \bar{RH} \]  \hspace{1cm} (A.7)

\[ \Delta RH_W = \frac{(\Delta e_s + e_s) \cdot (\bar{RH} + \Delta RH)}{e_s} - \bar{RH} \]  \hspace{1cm} (A.8)

\[ \Delta RH_S = \frac{\Delta RH_T \cdot \Delta RH_W}{RH} \]  \hspace{1cm} (A.9)

This contribution is shown in figure A.12. Contribution of second order term is almost zero. If we compare two years differences of relative humidity and contribution of each parameter we can distinguish regions which mostly affected by water vapor changes and regions which highly affected by temperature changes. These parameters do not follow same patterns in different regions.

To get a better understanding of changes in these parameters, longitudinal and
latitudinal cross sections are shown for couple of important regions. At the upper troposphere, contribution of each parameter does not follow the pattern from below. As it is explained before, circulation effect does not go to low pressures in vertical levels.

Figures A.13 and A.14 show high effect of specific humidity on the relative humidity in the tropic region and effect of temperature on relative humidity in north subtropic.

Figure A.12 shows temperature, water vapor and second order parts of the relative humidity. Figure A.12(c) defines the difference between sum of all these parts and the exact value of relative humidity.
Figure A.13: Vertical cross Section of contribution of specific humidity, temperature and relative humidity for 180°

Figure A.14: WaterVapor & Temperature, Latitudinal section at 180°
Figures A.15 and A.16 display temperature effects in relative humidity at high levels in stratosphere. In the lower levels contribution of water vapor causes reverse effects on relative humidity.

Figure A.17 and A.18 show effects of water vapor in the tropic region and temperature in the subtropic regions. In this section at the higher pressure levels the effect of temperature gets more important.
Figure A.16: Water vapor and temperature, longitudinal section at 140°W

Figure A.17: Vertical cross section of contribution of specific humidity, temperature and relative humidity for 120°E
A.7 Probability Distribution Functions

As we named in the last section there are couple of studies on ENSO and its effect on different weather parameters. Previous studies have focused on monthly or seasonal means, but it is important to find difference in daily variability imposed by ENSO. Using daily data for two extreme winters characterizes the change in daily variability. Monthly AIRS data used to find seasonal changes in relative humidity, temperature and OLR. Now we use daily data to get probability density function for each parameter. First we focus on the regions with the large differences in the mean values. In figure A.19 we marked regions with the most changes in the mean value of relative humidity. PDFs over whole tropics is very similar for 2009 and 2010 winters. what is happening here is redistribution of moisture but not a change of tropic-wide relative

Figure A.18: Vertical cross section of contribution of specific humidity , temperature and relative humidity for 120°E
Figure A.19: Regions with high changes in the mean values of relative humidity difference.

Figure A.20 shows that total probability density function distribution in the tropical region does not change between two extreme regions. It means monthly redistribution causes large changes between these years.

Figure A.22 shows the changes of relative humidity, H2O and temperature during the time in the tropic region. It shows the constant increasing in the relative humidity with time in nine years.

Figures A.21 indicate that number of dry and wet grids in the tropic region is not
Figure A.20: Probability density function of relative humidity at the tropical region

Figure A.21: Number of points in the dry and wet regions
Figure A.22: Changes in relative humidity, specific humidity and temperature
Figure A.23: Pdfs of (a) relative humidity (b) specific humidity (c) temperature in region 1,4 and 6

relate to ENSO values.

region 6 shows the region with the small changes in the relative humidity during the EL-Nino and La-Nina winters. This is the region where temperature effect and water vapor effect are on the opposite directions.

For smaller (55) regions there can be very different PDFs for different phases of ENSO.

We go to the detail of subtropic regions with are defined in the second assumption. We can see that in this region temperature effect on the relative humidity is important and very higher than water vapor effect.
Figure A.24: Pdfs of (a) relative humidity (b) specific humidity (c) temperature in region 7,8 and 9
Figure A.25: Relative humidity Pdfs for tropical region
Figure A.26: Relative humidity Pdfs for central pacific

Figure A.27: Relative humidity Pdfs for central pacific
A.8 Summary

We did EOF analysis on the OLR, Relative Humidity and Temperature at the upper troposphere. First EOF results shows similar pattern as the sensitivity respect to ENSO.

We found regions of changes in relative humidity with affected by changes in water vapor and regions which are more influenced by temperature changes.

PDFs show regions with large differences in the diagram from one EL-Nino year to the next La-nina one.

ENSO has dominant effects even on weather parameters in pacific region. Rossbey waves around 150°W causes large differences in temperature in this region. Other references didn’t mention about this effect. Wave pattern is different from recharge-discharge method. We did EOF analysis on OLR, relative humidity and temperature at upper troposphere. Sensitivity and EOF analysis of relative humidity, temperature and water vapor shows regions with high changes in the value. Pacific region around 180° shows convergence and eastern pacific around 120°E shows divergences. In the longitude section of 180° from one year to the next year Pdf diagram changes. Not only mean value but also standard deviation changes i two years. High changes in relative humidity in western pacific is mostly controlled by changes in temperature. In the probability density functions we found that effects of water vapor and temperature
are in the opposite direction in this region because of Rossbey circulation. This difference is not similar for north and south subtropics. North subtropic shows high changes in mean and standard deviation of temperature while south subtropic shows high changes in water vapor than temperature.

\section*{A.9 Acknowledgement}

I acknowledge NASA’s support for all the researches in this chapter on the climate change due to ENSO. Also I should thank Professor Darryn Waugh for giving me access to the data center in the department of earth and planetary sciences.


V. J. Challis, A. P. Roberts, and A. H. Wilkins. Design of three dimensional isotropic


A. Gettelman, J. R. Holton, and A. R. Douglass. Simulations of water vapor in the


J. K. Guest and J. H. Prvost. Design of maximum permeability material struc-


C. Le, J. Norato, T. Bruns, C. Ha, and D. Tortorelli. Stress-based topology optimiza-


E. C. Nelli Silva, J. S. Ono Fonseca, and N. Kikuchi. Optimal design of periodic


K. Suzuki and N. Kikuchi. A homogenization method for shape and topology opti-


H. N. G. Wadley, N. A. Fleck, and A. G. Evans. Fabrication and structural per-


S. G. Xie YM. *Evolutionary structural optimization (ESO) using a bidirectional algorithm*, volume 42. MCB UP Ltd, 1997.


K. Y. Yun, K. Seo, and K. Ha. Interdecadal change in the relationship between enso


Biography

Reza Lotfi, born in Kerman and raised in Esfahan, Iran, graduated in 2001 from Helli high school. I entered Tehran university for bachelor degree in civil engineering. After four years of course works I continued in master degree in structural and earthquake engineering. My research was optimization of braced frame structures under seismic loads. Working for two years helped me get the knowledge of engineering applications. In 2009 I continued my research in optimization of structures for nonlinear mechanics at Johns Hopkins University. I graduated in 2013 and followed my career as the structural software developer.