Micro-mechanical Modeling of Brittle Materials under Dynamic Compressive Loading

by

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Abstract

Micro-mechanical modeling of brittle dynamic failure provides a physical insight into the relationship between microstructure, which dictates key failure mechanisms, and material performance. This dissertation addresses key issues in micromechanical models associated with brittle dynamic failure, in particular damage-induced material anisotropy, efficiency of the micromechanics model to enable larger-scale implementation, and non-local modeling to represent strain softening materials. The dominant failure mechanism for brittle materials under compressive loads is the growth and coalescence of wing-cracks; therefore, the micromechanics model based on wing-cracks is the focus of this work.

Based on a wing-crack RVE model, analytical closed-form solutions for the instantaneous effective anisotropic compliance (or stiffness) of a damaged material under compression are derived through both kinematic and energetic approaches. These solutions are functions of the geometric measurements of the wing-cracks and the friction coefficient. Application of the model to tensile loads is a straightforward simplification of the proposed model. Finite element models of periodic wing-cracks
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verify the analytical results, confirming the effectiveness of the analytical solution. Combining this anisotropic compliance with an established micro-mechanics model that addresses crack growth, the stress-strain relationship of a brittle material under compressive loading is established.

In upscaling to a macro-scale computational model, this micro-mechanically based constitutive model is evaluated at each individual integration point of the mesh. For models with many integration points, this micromechanical analysis for every integration point at every time step is computationally prohibitive. An upscaling technique is proposed here to tackle this issue. Instead of repeatedly performing the complete constitutive model for each integration point, a Taylor series expansion is applied to approximate the micromechanics damage process. Such a methodology can be deployed to enhance the efficiency of the macro-scale models as well as the statistical estimations of the local stress and strength of the material with heterogeneous nature.

To the end of properly modeling the strain softening behavior at the highly damaged stage, a conventional nonlocal finite element method is investigated by simulating a series of benchmark problems.

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Dedication

To my family, particularly my parents and my wife Ling.
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Chapter 1

Introduction

Brittle materials such as ceramics, concrete, rock and ice, exhibit highly non-linear and complex overall response to external loads before total failure. Understanding the mechanisms and the material response during the failure process is important for engineering practice. For instance, ceramics are used as armour materials for their high hardness, stiffness and strength under impacts from explosion and intrusion; studying the failure processes in ceramics can help improve the performance of protection gear. Similarly, studying the failure of concrete can help civil engineers design and construct reliable structures. On the other hand, measures that promote the failure of rocks help with extraction of natural resources such as ore, oil and gas; such measures also rely on a fundamental understanding of the failure mechanisms in brittle materials.

In order to enhance the performance of brittle materials under compressive loads,
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especially under dynamic loads, a multidisciplinary team is required. A typical re-
search cycle to enhance material performance can be demonstrated by the Materials
By Design concept, as shown in Fig. 1.1. A research group with expertise in material
processing provides the material samples to experimentalists, who test and measure
the micro-structural characteristics and the mechanical properties of the material.
Based on the experimental observations, theoretical models of the relevant failure
mechanisms are proposed to explain the observations. More physics based numerical
models are then developed and applied to explore potential implementations in mate-
rial performance. After being properly verified and validated, a successful model will
provide feedback to the material processing group regarding material characteristics
that will enhance performance.

![Figure 1.1: Illustration of the Materials By Design concept](image)

The contribution of the present work generally lies in the development of a more
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physically based model of failure mechanisms. However, knowledge regarding the material microstructure and the mechanical properties discovered from experimental observations are necessary to maintain a realistic view of the relevant mechanisms. Therefore, a brief introduction on the relevant experimental studies and the observed failure mechanisms are presented here.

1.1 Failure of Brittle Materials under Compressive Load

Due to the limited available plastic deformation, the typical failure mode of brittle materials is associated with crack growth and coalescence. On the micro-scale the cracks can be nucleated from various sources, such as soft/hard inclusions, weak grain boundaries, voids, unprocessed ingredients (such as graphite), undesired phases, and so on. These nucleation sites of micro-cracks are generally called the flaws. For ceramics and concretes the flaws are generally introduced by the processing procedures and are often unavoidable, although their sizes, shapes and numbers might be controllable. The crack growth processes leading to damage are profoundly related to such pre-existing flaws at the micro-scale.

The growth paths of cracks are generally governed by the highest rate of energy dissipation, but they are also sensitive to the applied load, the localization of material inhomogeneity, the interaction of stress fields with other flaws/cracks and the material
boundary. The development of cracks is highly directional. Although brittle materials may exhibit isotropic macro-scale properties in the virgin state due to the random orientations of crystal grains and the pre-existing flaws, preferential crack growth direction under applied loads causes the constitutive behavior of the solid to become highly anisotropic.

For ceramic materials such as B4C and AlN under uniaxial compressive load, the cracks develop along the primary loading direction, and result in the so-called axial splitting phenomenon [2]. Numerous experimental studies with different experimental techniques (for example, the Kolsky bar test illustrated in Fig. 1.2) have reported that under compressive loads without confinement, specimens fail by axial cracking (or splitting) accompanied by a significant inelastic volume increment and dilatancy transverse to the loading direction (see Fig. 1.3 and representative works for example, [3], [4], [5], [6], [7], [8], [11], [9]). Brace and Bombolakis [3] proposed a frictional sliding
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model which connects the observed phenomena of axial cracking and dilatation with the pre-existing flaws. Under uni-axial compressive loads, pre-existing flaws are closed and the material tends to slide on the flaw surfaces that are inclined relative to the primary loads. This sliding increases the stress intensity at the tips/edges of the pre-existing flaws and opens new crack surfaces when the critical stress intensity factor is reached. This mode of crack growth is generally referred to as wing-cracking, and has been observed in experiments (see Fig. 1.4). The path of the wing-cracks follows the maximum energy release rate, and aligns with the maximum compressive loading direction once it is fully developed. Ultimately, failure in the axial splitting mode is considered to be the consequence of the growth and coalescence of such wing-cracks associated with pre-existing flaws.

Figure 1.3: Stress-time history with high speed camera images of the failure process under uniaxial dynamic compression (Hu et.al) [9]. Arrows in the high speed camera images indicate the development of the cracks along loading direction.

Different aspects of the frictional sliding model have been heavily investigated by
Figure 1.4: Wing-cracks in Columbia Resin CR 39 (material for eyeglass lenses) under uniaxial compressive load, Nemat-Nasser and Horii [10].
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authors including [11], [12], [13], [10], [2], [14], [15], [16], [17], [18], [19] and many others. In particular, [10] derived the analytical expression of the stress intensity factors at the tips of the wing-cracks and solved the orientation of incremental crack growth; [17], [18] investigated the impact of the strain rate on the evolution of damage and the interaction from other wing-cracks.

However, if confinement is applied to one of the directions transverse to the primary load, development of cracks show different patterns, for instance the cracks may grow along the confinement direction. These studies also observed that lateral confinement strongly affects the inelastic behavior and enhances the material strength.

The strength of the brittle materials is sensitive to the applied loading rate (either stress or strain) for both tensile and compressive loads. Studies of the mechanical response of brittle material under dynamic load relies on the experiments that generate high stress/strain rates in the specimen, such as the Kolsky bar (or similarly split-Hopkinson bar) test. Experimental studies using this approach include some for ceramic materials [20], [9], [1], [21], [22], some for concrete: [23], [24], [25], and some for geological materials [26] [27] [28]. These experimental works show that the strength of a brittle material increases with the applied stress/strain rate, if a certain rate threshold is exceeded. The rate dependence is associated with the interrelation between material inertia and the crack growth rate. Brittle materials withstand the load until total collapse and failure due to crack saturation, while the crack growth rate is bounded by an upper limit, which may be related to the Rayleigh wave speed.
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Under a high loading rate, propagation of large cracks is rate limited, so the magnitude of stress in the solid continues to rise to a high level and cracks also nucleate from smaller flaws. As a result, leading is absorbed and dissipated through the multiple opening crack surfaces, which leads to the higher strength. An increase in the number and overall length of cracks and a decrease in the resulted fragment sizes as a result of an increasing applied loading rate are widely reported from the above studies, which supports on this hypothesis.

A positive volume change under compressive load (also known as bulking, dilation or dilatancy) and the residual strain after removal of the applied load are other interrelated features of brittle damage (see e.g., [4], [29], [22], [30], [28], [9], [31]). As the micro-cracks open under the applied load, the overall volume increases. When the applied load is removed, the deformation cannot be fully reversed due to the hindrance of friction or interference, resulting in a residual strain.

Based on the above discussion on the experimental studies of the failure of brittle material under dynamic loads, the following features can be identified:

• Inelastic response is dominated by crack development and coalescence.

• Pre-existing flaws have a profound impact on strength.

• Local variation in the flaw population lead to randomness and localization of failure.

• Axial splitting is a primary mode of failure under compression.
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- Strain rate effect on strength.

- Bulking (volume dilatancy) is observed during the damage process.

As illustrated in Fig. 1.1 numerical models can be developed based on experimental observations to study the failure of brittle material. For the numerical models of brittle materials under compressive loads, challenges include the following:

(a) Identifying a physics-based constitutive relationship that accurately describes the stress-strain relations in the brittle material during a damage process.

(b) Maximizing efficiency of the model in order to increase the feasibility of implementing the above constitutive relationships for large scale modeling.

(c) Developing models that address the highly nonlinear behavior of brittle damaged materials.

This thesis tackles these challenges. In the following section these challenges are discussed individually.
1.2 Constitutive Relationship of Damaged Brittle Materials under Compressive Loads

For the failure of brittle materials, the constitutive relationship can be decomposed into two parts: (a) material properties (compliance or stiffness tensor) with respect to damage, which is used to identify the local stress state on cracks; (b) development of micro-cracks under applied loads, including the crack growth direction and rate, the interactions among the cracks. Different models of constitutive relationships for the damaged brittle material have been developed for various applications, and these models can be categorized into two types: phenomenological damage model, and micromechanical models.

Phenomenological models describe the brittle material with elasto-plastic properties under prescribed stress-strain relationships as the damage process proceeds. The widely used phenomenological constitutive laws for modeling ceramic materials under high loading rate and confined condition are the Johnson-Holmquist (JH) model and its subsequent modified version (JH2) ([32, 33, 34, 35]). These models are relatively simple, easy to implement and highly efficient to represent the constitutive description in the model. However, these phenomenological models do not reflect either the microstructure or the normative material properties and therefore cannot provide the
physical insight of the damage process. Furthermore, the coefficients and exponents in the phenomenological models need to be recalibrated based on the experiment data if the material and the applied load / boundary conditions are changed. In other words, the empirical models are not truly predictive since the model needs to be recalibrated for different physical loading scenarios.

**Figure 1.5:** Multiscale modeling incorporating a micromechanical model. (a) Macroscale modeling, at each integration point incorporating the constitutive relationship generated by the micromechanical model; (b)-(d) Micromechanical (wing-crack) model for brittle materials under dynamic compressive load proposed by Paliwal and Ramesh [36]. (b) Development of wing-cracks under a compressive load; (c) Isolation of individual wing-cracks in elliptical regions, assigning damaged compliance outside the ellipse; (d) Using a self-consistent scheme, resolve stress field around individual wing-cracks and evaluate the crack growth / damage rate.

Micromechanical models incorporate the physical aspects of the material that govern the damage process, in particular degrading the effective moduli of the material based on a quantified damage measurement. The constitutive description of this type embodies the characteristics of the microstructure (such as the density, length and orientations of the microcracks) and the material properties (e.g. the yielding stress, toughness, density, wave speed and so on). Fracture mechanics is typically incorpo-
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rated to capture the crack growth process. Generally the micromechanical models are more complicated and less efficient than the phenomenological ones. But supported by the rapidly enhancing computational capability and more optimized calculation schemes, micromechanical models incorporating more sophisticated mechanisms and on the finer length scales have been the trend in the field of computational mechanics for the past decades. When implemented in macroscale modeling, this kind of approach is generally referred to as multiscale modeling. A typical multiscale modeling scheme is illustrated in Fig. 1.5 where (a) represents a macroscale simulation, such as finite element modeling, and (b)-(d) illustrate a micromechanical model which assesses the constitutive behavior at individual integration points.

Deshpande and Evans (2008) [37] and their subsequent work [38] characterized three principal inelastic mechanisms that control the constitutive properties of brittle materials under the damage process: (a) plasticity due to lattice dislocation or twinning, (b) nucleation and development of microcracks, and (c) granular plasticity when the cracks are heavily coalesced with each other (saturated). Depending on the load path and boundary conditions, different mechanisms may be triggered in sequence. Recent studies on the performance of ceramics under extremely high loading rate also consider amorphization [39] and thermal effects as the weakening mechanisms of brittle material. The present work focuses on the microcracking mechanism, since it is the primary contributor to failure of brittle materials, and the major parts of the inelastic response.
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On the constitutive response of cracked brittle solids under tension, Budiansky and O’Connell (1976) \cite{Budiansky1976} derived a 2D homogeneous isotropic relationship between effective elastic modulus and damage under hydrostatic tensile loads, with the assumption that the solid contains a statistically large number of cracks with random orientations and various shapes. Kachanov (1980) \cite{Kachanov1980} obtained the increment of potential energy due to the presence of penny-shape cracks, which leads to anisotropic compliance through differentiation of the applied loads. Following Hudson’s crack interaction theory \cite{Hudson1983, Hudson1984, Hudson1985}, Grechka and Kachanov (2006) \cite{Grechka2006} derived an anisotropic constitutive relationship by adding interaction terms to the potential in \cite{Kachanov1980} and obtained a tensor form of the compliance with respect to crack damage.

Regarding compressive failure, Walsh (1965) \cite{Walsh1965} provided early studies of the effect of pre-existing flaws on the elastic modulus and Poisson’s ratio under compressive loads. Many authors \cite{Walsh1965, Cattaneo1967, Cattaneo1968, Cattaneo1970, Cattaneo1971} analyzed the behavior of solids undergoing the wing-crack damage mechanism, and developed expressions for the constitutive relations during the wing-crack development process. As a common practice of these authors, stress intensity factors at the crack tip are derived to bridge the damage with constitutive relations. The results obtained from these analyses generally describe the dynamic relations between crack growth and the material response, but they suffer from the lack of a closed form expression for the effective compliance with respect to the instantaneous damage state. Therefore, these solutions were difficult to implement in micro-mechanical modeling. Furthermore, comparing with experi-
mental results, the dilatation predicted by these models for the damage process were generally underestimated.

In recent years, micro-mechanical models with decomposed analyses of effective material properties and the compressive damage process have been developed. Paliwal and Ramesh (2008) [36] developed a 2D model (see Fig. 1.5b-d) tackling the constitutive property of brittle solids under dynamic compressive load, in which the level of damage (typically denoted as $\Omega$) due to the development of wing-crack was updated through a self-consistent scheme, so that interactions among different flaws are taken into account. This model is capable of capturing the influence of the statistics of pre-existing flaws and the applied strain rate on the stress-strain response. The damage due to wing-crack development was measured by multiplication of the flaw density with the square of the wing-crack lengths. The evaluated damage parameter $\Omega$ is substituted into a damage-compliance relationship, derived from Budiansky and O’Connell’s solution [40], and then the complete constitutive relationship (stress-strain curve) is obtained. More details about the PR model will be discussed in Chapter 4. Based on this work, Hu et al. (2014)[47] developed a 3D numerical model adapting the wing-crack mechanism to simulate material response under multiaxial loading conditions. Katcoff and Graham-Brady (2014) [48] implemented the self-consistent micro-mechanical model to study the compressive dynamic failure of brittle materials with circular flaws. Tonge (2014) [49] applied a similar methodology as part of a three dimensional macro-scale model of brittle dynamic failure.
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The effective compliance with respect to the damage parameter under a compressive loading state are required for all of these above mentioned models. The solutions of effective material properties, such as those from [40] or [45], have been applied in these models. However, these readily available damage-compliance relationships were developed for cracked solids under tensile loads, while the rigorous closed form solutions for compressively damaged solid are still missing. The tensile solutions of effective properties generally overestimate the compliance in the loading direction, but underestimate the dilatation at a given damage state, and thus are not suitable for describing the mechanical response of a damaged material under dynamic compressive loads.

1.3 Implementation of the Constitutive Relationship in Multi-Scale Modeling

Computational models that explicitly track the development of individual cracks, such as the cohesive element methods [50, 51] and the extended finite element method [52], require very fine element discretization (sub-RVE level) and significant computational effort. Therefore, they are infeasible for modeling brittle dynamic failure on structural scale, due to the excessive number of cracks and the complicated crack paths.

On the other extreme, models based on fully homogenized materials, in which the
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constitutive behavior is assumed to be spatially uniform throughout the whole material, are computationally efficient and have been successful when modeling materials with linear elastic, elastic-plastic and mild non-linear constitutive properties. However, due to the influence of the pre-existing flaws on the constitutive relations and the local random fluctuations of flaw characteristics, failure can occur in localized, instantaneous and catastrophic manner. The globally homogenized models fail to capture the heterogeneities of the material that lead to these localizations of failure. 

As an intermediate approach that falls between these two extremes, the idea of local homogenization of microstructure is applied in this work, which will allow for spatial variations in constitutive properties yet will not require explicit representation of individual cracks. The idea is that each integration point of the finite element mesh has a microstructural region associated with it, that can be characterized in terms of larger scale characteristics such as crack/flaw density, size distribution, etc., and which can be analyzed using micromechanical tools. Representative works can be referred to, e.g., [54, 55, 56, 57, 58].

Brannon et al. [55] demonstrated that by introducing the random fluctuations of constitutive properties to the finite element mesh, numerical modeling of dynamic impact on ceramic materials provides a much more realistic cracking pattern. In the present work, a similar approach of randomly fluctuating the constitutive properties is developed, and the distribution of strength predicted by such an approach is
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investigated.

On the other hand, if the micro-mechanical model is performed to generate the constitutive response at every time step for every element / integration points, the computational effort for the entire hierarchical model can be infeasibly large. A computationally efficient methodology is needed for circumventing the repeated computation.

Pope [59] developed a so called in situ adaptive tabulation (ISAT) method for modelling the combustion chemistry reactions that evolve with time. Using this method, the initial states, the constitutive result (mapping) and the gradients with respect to the initial states are stored in a data base as the calculation is performed. When results are required for another integration point with different values of the initial state, depending on the error of accuracy, the system either returns tabulated results from the data base, or directly performs the complex constitutive model. Efficiency enhancement for this ISAT method can reach a factor of 1000. In the field of computational solid mechanics, Arsenlis et al. [60] implemented the ISAT, while Barton et al. [61], Knap et al. [62] developed a Adaptive Sampling method to perform multi-scale modeling with plastic material property.

To shed light on the efficiency of macroscale modeling of brittle dynamic failure incorporating local fluctuations, a store-retrieve methodology similar to the ISAT has been developed. Instead of repeatedly performing the complete micro-mechanical models with varied flaw parameters, constitutive properties are obtained by adjusting
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the reference results according to the difference of input flaw parameters, through a Taylor series expansion. In this way, the efficiency of the whole macro-scale model is greatly enhanced, while the heterogeneous nature of the material can be retained.

The flaw characteristics and the statistics, which are required in the micro-mechanical constitutive models, can be identified in the global sense using techniques such as ultrasound, digital image characterization and x-ray tomography (e.g., [63, 64, 65, 31]). However, the local fluctuations of these properties are difficult to identify through experimental methods. Numerical methods that incorporating probability and statistics are usually applied to generate data sets with local fluctuations based on the global data for micro-mechanical modeling [49].

1.4 Modeling Issues Associated with the Strain-Softening Behavior

When the crack growth process becomes unstable, the effective moduli are rapidly degraded and the brittle solid fails in catastrophic manner. Experimental studies on failure of transparent brittle material under compressive load revealed that significant internal cracking takes place well before the material reaches its strength, then the stress-strain curve presents negative slope as the material collapses [66, 1, 9] (see Fig. 1.3). Such post-peak response is generally referred to as strain softening.

Under quasistatic load the failure process is abrupt, and thus it is usually not
necessary to capture the degrading constitutive properties in numerical modeling. However under dynamic load, the post-peak softening is a more gradual process and thus should be taken into account.

When incorporating the softening behavior for the homogenized micro-structure at each integration point, conventional finite element methods face some issues. The negative tangent moduli leads to a non-positive definite stiffness matrix, making the solution unstable. Regardless of the analyst’s choice of mesh size, the model converges with decreasing mesh size to a physically incorrect solution, which is known as mesh sensitivity. Additional measurements are required to overcome this mesh sensitivity.

Element deletion is a simple and effective measurement, but with the significant drawback of not conserving mass. Cohesive zone FEM and Extended FEM (XFEM) are useful for tracking a small number of individual cracks, but are very costly when numerous cracks exist. Other effective tools include the Material Point Method (MPM) and the non-local finite element method.

The Material Point Method (MPM) stores information, such as the nodal velocity and the effective mass, at the material points; at each time step the information is mapped to background grids which are fixed in space, at which the gradient and equations of motion are solved. The computational results on the grids, such as the stress gradient and grid velocity, are mapped back to the material points to update the position. MPM is capable of handling complicated geometries, dynamic conditions and complex constitutive response, while avoiding mesh entanglement issues.
Tonge (2014) [49] performed multiscale modeling using the Material Point Method for brittle materials under dynamic compressive load, in which two different damage mechanisms at the micro-scale were incorporated: a wing-crack model for the early damage stage when interaction between individual cracks is mild, and then a granular flow mechanism when the cracks have saturated.

Another approach is to introduce an additional length scale into the constitutive relationship. In the nonlocal theories [68, 69, 70] it is proposed that the value of the stress and strain at a point depend not only on the local evaluations, but also on the gradients within a nearby region. The length scale is associated with the size of this effective region. In practice such a length scale (or localization limiter) may be introduced in integral form [68, 71, 72] as a shear band or through additional layers of imbricate elements, or in differential form [70, 73] as explicit terms of gradients in the constitutive relationship.

Although successful in stabilizing the solution and removing the mesh sensitivity issue, some limitations exist in the nonlocal finite element methods. Firstly, the treatment of the boundaries is not well posed. For integral methods such as using imbricate finite elements, the imbricate elements near the model boundaries have to be truncated, while for differential methods the parameter gradients should be assigned, which are usually not well known. Secondly the introduction of a localization limiter is not effective for incorporating micro-structural heterogeneity. Lastly, although the solution is stabilized, the introduction of the additional length-scale limits the
accuracy of the result. In some circumstances improved accuracy cannot be achieved by refining the size of local element, if the prescribed length-scale is fixed.

This thesis we explores the feasibility of incorporating the integral nonlocal finite element method for modeling heterogeneous brittle materials with strain softening behavior. Some interesting features of the nonlocal method are revealed, but it is concluded that the original integral nonlocal method is not suitable for the current modeling needs.

1.5 Organization of this Thesis

This thesis tackles various challenges in numerical modeling of the failure of brittle materials. Chapter 2 describes a derivation of the effective compliance of brittle materials under a compressive damaged process, in which the wing-crack model is incorporated. Closed-form expressions of the instantaneous compliance tensor with respect to the damage related variables (flaw densities, flaw orientations, lengths of developed cracks, etc.) are obtained. Comparisons among previous solutions of cracked solids and our newly derived solution are also presented.

Chapter 3 presents a series of finite element models of the wing-crack problem. From the results of the finite element modeling we confirm the verification of our analytical solutions, and obtain analytical expressions for the crack compliance tensor $R$. 
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In Chapter 4 the closed-form solution of the damage-compliance relationship is combined with the micromechanical model introduced by Paliwal an Ramesh [36], and a complete constitutive model of brittle dynamic failure is presented.

A store-retrieve methodology (ISAT) to enhance the computational efficiency for micro-mechanical modeling of brittle dynamic failure is presented in Chapter 5. The ISAT method is applied to enhance the efficiency of evaluating the constitutive model under with perturbed flaw statistics, and it is also implemented in statistical studies that analyse the connection between random variation in the flaw population and the local mechanical response.

As an attempt to incorporate the highly non-linear constitutive relationship of brittle dynamic failure into hierarchical multiscale modeling, an investigation on non-local finite element modeling is described in Chapter 6.

Finally, Chapter 7 summarizes this thesis, and presents discussions and suggestions for future work.
Chapter 2

Damage-Compliance Relationship of Brittle Materials Under Compressive Loads

2.1 A Brief Review on Mechanism-Based Inelastic Response of Cracked Colid

As discussed in the first chapter, the models describing the effective material properties for cracking damaged brittle material can be distinguished as two types: the phenomenologically based models, for example the Johnson Holmquist model \[35\], and the more physical mechanism-based models. Since the mechanism-based models
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more directly reflect the relationship of the damage processes to the microstructure,
we focus the current work on this type of model. In this section a brief review of the
mechanism-based models for micro-cracking damaged solids is presented.

2.1.1 Damage parameter

For brittle damaged solids, the micro-cracks are considered the primary source of inelastic behavior of the material. It is useful to extract the characteristics of the full crack population (such as length, number density, orientations and so forth) into a single mathematical expression for damage, in order to ease implementation of damage into numerical models. The most commonly used crack damage parameter, also called the crack damage measure or the crack density parameter, was introduced by Bristow (1960) [74] and Walsh (1965) [46]. For three-dimensional solids containing a population of $N_s$ penny-shaped cracks, the damage parameter is given by:

$$\Omega = \frac{1}{V} \sum_{j=1}^{N_s} s_j^3,$$

(2.1)

where $s_j$ is the radius of crack $j$, and $V$ is the volume of the Representative Volume Element (RVE). For a two-dimensional solid containing rectilinear cracks, the similar damage measure is:

$$\Omega = \frac{1}{A} \sum_{j=1}^{N_s} s_j^2,$$

(2.2)

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where $s_j$ is the half-length of crack $j$, and $A$ is the representative area. If all the cracks in the solid are of the same size $s$, the damage parameter for the 3D solid is simplified as:

$$\Omega = \frac{N_s}{V} s^3 = \eta s^3,$$  \hspace{1cm} (2.3)

where $N_s$ denotes the number of cracks in the RVE, and $\eta$ is the number density of cracks per unit volume. A similar expression form is found for a 2D solid with the square of half the crack length $s$.

If multiple crack sizes are considered, generally the crack sizes are discretized into a number ($N_{bins}$) of families, and each crack that is categorized into the family $k$ is approximated by the the representative crack size $s_k$. The damage parameter is then reformulated as:

$$\Omega = \eta \sum_k^{N_{bins}} g(s_k) s_k^3,$$  \hspace{1cm} (2.4)

where $g(s_k)$ is the probability mass function (PMF) that indicates the fraction of cracks in bin $k$, and $s_k$ is the representative crack length for bin $k$. Naturally, the total probability law should be fulfilled: $\sum_k g(s_k) \leq 1$. It may not be necessary to cover the whole range of flaw sizes in the material when doing modeling, thus the “≤” sign is applied here.

A similar way to express damage with a distribution of crack sizes is to put together
the total crack density $\eta$ and the size distribution function $g(s_k)$:

$$\Omega = \sum_{k=1}^{N_{\text{size}}} \eta_k s_k^3, \quad \eta_k = \eta g(s_k), \quad (2.5)$$

thus, $\eta_k$ denotes the number density of crack family $k$ in the RVE.

While the scalar damage parameter in Eqs. (2.1) or (2.5) is appealing because of the simplicity in the expression, damage often exhibits a directional characteristics due to preferential crack orientation. This motivates the use of a tensor based expression for damage. A second rank damage parameter tensor $\Omega$ is defined by Kachanov (1980) [41] as follows:

$$\Omega = \frac{1}{V} \sum_{j=1}^{N_s} s_j^3 n_j n_j, \quad (2.6)$$

where $n$ is a unit vector normal to the crack, and $s_j$ is the half length of crack $j$. Comparing with the scalar damage parameter shown in Equ. (2.1), the damage parameter tensor contains information about the distribution of both crack orientations and sizes in the RVE, while the scalar counterpart only contains the information about the crack sizes. The scalar and tensor forms can be interrelated by:

$$\Omega = \text{tr}(\Omega). \quad (2.7)$$

Therefore, the damage tensor proposed by Kachanov [41] can be considered a natural
2.1.2 General solutions for cracked solid under tensile load

Assume that an undamaged solid consists of a homogeneous, isotropic and purely elastic matrix material, with elastic modulus $E_0$ and Poisson’s ratio $\nu_0$. The general effective mechanical response of a damaged representative volume element (RVE) can be expressed by the Hooke’s law:

\[ \epsilon = S\sigma = (S_0 + \Delta S)\sigma, \quad (2.8) \]

or alternatively:

\[ \sigma = C\epsilon = (C_0 - \Delta C)\epsilon, \quad (2.9) \]

where \( \epsilon \) and \( \sigma \) are the second-rank tensors denoting the averaged strain and stress of the RVE, respectively. \( S \) denotes the effective compliance tensor and \( C \) denotes the effective stiffness tensor, which are both fourth-rank tensors. \( S_0 \) and \( C_0 \) denote the parameters associated with the intact (without any cracks) solids, and \( \Delta S \) and \( \Delta C \) denote the changes of compliance or stiffness due to the presence of cracks. The
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strain tensor can be explicitly split into an elastic part $\epsilon^e$ and an inelastic part $\epsilon^i$:

$$\epsilon = \epsilon^e + \epsilon^i = S_0 \sigma + \epsilon^i, \quad \epsilon^i = \Delta S \sigma.$$  \hspace{1cm} (2.10)

Due to the small contribution of plasticity (such as the lattice dislocation or twinning) to the inelastic deformation in brittle material, here we generally ignore the effect by plasticity during the cracking process. Therefore, the inelastic strain $\epsilon^i$ is primarily contributed by the kinetics on the crack faces. Denoting the discontinuous displacement on the crack surfaces as $u$, the inelastic strain $\epsilon^i$ can be written as:

$$\epsilon^i = \frac{1}{2V} \sum_k \int_k (n \otimes u + u \otimes n) dA,$$  \hspace{1cm} (2.11)

The discontinuous displacement $u$ is also called the crack opening displacement (COD). The integration of displacement can be replaced by the averaged value of the COD $\bar{u}$ over the whole crack surface, so that the above equation can be simplified as:

$$\epsilon^i = \frac{1}{2V} \sum_k [(n \bar{u} + \bar{u} n) A]_k$$  \hspace{1cm} (2.12)

The averaged COD can be interrelated to the far-field stress $\sigma$ by a second-rank tensor $Z$:
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\[ \tilde{u} = \mathbf{n} \cdot \sigma \cdot Z = P \cdot Z. \]  

(2.13)

where \( P \) is the resolved traction on the crack surface due to the far-field stress \( \sigma \).

\( Z \) in the above formula is known as the crack compliance tensor. The physical sense and mathematical derivations of \( Z \) was first tackled by Budiansky and O’Connell (1976) [40] for circular and elliptic cracks, and later was introduced by Schoenberg (1980) [77] as an explicit variable.

Equation (2.13) relies on an important assumption that interactions in the stress field of different cracks can be ignored, so that each individual crack can sense the applied far field stress \( \sigma \). Therefore, the traction on the individual crack surfaces can be resolved based on the far-field stress and the crack normal.

In order to find the effective material property, it is helpful to formulate the elastic energy stored in the cracked solid under applied load \( \sigma \):

\[
\begin{align*}
f & = \frac{1}{2} \sigma : \epsilon = \frac{1}{2} \sigma : (S_0 + \Delta S) : \sigma \quad (2.14a) \\
& = \frac{1}{2} \sigma : S_0 : \sigma + \frac{1}{2V} \sigma : \Delta S : \sigma \\
& = f_0 + \Delta f 
\end{align*}
\]

(2.14b)

(2.14c)

in which \( S_0 \) is the compliance of uncracked solid; \( \mathbf{n} \) and \( A \) denote the normal and area of crack surface, respectively. \( f \) is also called the elastic potential, and \( f_0 \) is the
elastic potential of the intact (uncracked) solid. $\Delta S$ and $\Delta f$ denotes the changes of effective compliance and the elastic potential resulting from the presence of cracks, respectively.

The change of compliance due to the presence of cracks is then obtained by differentiating the change of potential function $\Delta f$ in Eq. (2.14b) with respect to the stress tensor [41], [78], [79]. For a 3D problem it is found out to be:

$$\Delta S = \frac{1}{V} \sum_k (\mathbf{n} \cdot \mathbf{Z} \cdot \mathbf{n}A)_k,$$

(2.15)

and for a 2D problem:

$$\Delta S = \frac{2}{A} \sum_k (\mathbf{n} \cdot \mathbf{Z} \cdot \mathbf{ns})_k.$$

(2.16)

In what follows we’ll discuss the solutions for 3D and 2D problem separately, and for each dimension the cases of random (isotropic solution) and parallel (anisotropic solution) crack orientations are reviewed.

### 2.1.2.1 Three-dimensional solution

Consider a local coordinate system on the circular crack, as shown in Fig. 2.1. The bases are denoted as $\mathbf{n}$, $\mathbf{r}$ and $\mathbf{t}$, with $\mathbf{n}$ the crack normal and the other two inside the crack plane. The crack compliance tensor $\mathbf{Z}$ can be reformulated through
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spectral decomposition [40] [80]

\[ Z = Z_n n n + Z_r r r + Z_t t t \]  \hspace{1cm} (2.17)

Figure 2.1: Local coordinate system of a circular crack

Kachanov (1992[80],1993[81]) extracted the explicit expressions of \( Z \) based on the work of Budiansky and O’Connell (1976) [40]. For a dry circular crack with a radius \( s \), the three components of \( Z \) are given by:

\[ Z_n = \frac{16s(1 - \nu_0^2)}{3\pi E_0}, \quad Z_r = Z_t = \frac{16s(1 - \nu_0^2)}{3\pi E_0(1 - \nu_0/2)}. \]  \hspace{1cm} (2.18)

Since \( Z_n = (1 - \nu_0/2)Z_r \), the crack compliance tensor \( Z \) can be reformulated as:

\[ Z = Z_r (I - \frac{\nu_0}{2} n n). \]  \hspace{1cm} (2.19)
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in which \( I \) is the identity.

Although not exactly equivalent, the values of the three components of \( Z \) are relatively close, especially when the Poisson ratio of the matrix material is small. Therefore, in many cases they can be approximated by the same expression of \( Z_r \), and the crack tensor becomes \( Z_r \) times the identity.

Substituting the expression of \( Z \) in Eq. (2.19) back to the solution of \( \Delta S \) (Eq. (2.15)), and reorganizing the formula, the following expression for the change of compliance in the dyadic form is obtained:

\[
\Delta S_{ijlm} = \frac{8(1 - \nu_0^2)}{3E_0(2 - \nu_0)}(\Omega_{il}\delta_{jm} + \Omega_{im}\delta_{jl} + \Omega_{jl}\delta_{im} + \Omega_{jm}\delta_{il} + 4\beta_{ijlm}),
\]

\((i, j, l, m = 1, 2, 3)\) \( (2.20) \)

where \( \delta_{jm} \) is the Kronecker delta, and \( \Omega \) is the tensor form of the damage parameter, which is calculated by Eq. (2.6).

\( \beta_{ijlm} \) in Eq. (2.20) is a fourth-rank tensor, which can be approximated by \( 15 \):

\[
\beta_{ijlm} = -\nu_0(\Omega_{il}\delta_{jm} + \Omega_{im}\delta_{jl} + \Omega_{jl}\delta_{im} + \Omega_{jm}\delta_{il})/8.
\]

\((2.21)\)

In many cases \( \beta \) can be simply ignored due to its relatively small amount.

If the normal of cracks in the solids are of random orientations, the effective material properties are isotropic (given that the matrix material itself is isotropic). Closed form expressions of the compliance tensor can be obtained by evaluating the
averaged damage tensor $\Omega$ for all the crack orientations, and then substituting $\Omega$ into Eq. (2.20). The normal components of the compliance tensor is found to be [40, 80]:

$$S_{iii} = \frac{1}{E_0} \left[ 1 + \frac{16(1-u_0^2)(10-3u_0)}{45(2-u_0)} \Omega \right], \quad (i = 1, 2, 3), \quad (2.22)$$

where $\Omega$ is the scalar damage parameter. If the cracks are of the identical flaw size $s$, $\Omega$ calculated by Eq. (2.3); otherwise, Eq. (2.4) or (2.5) may be applied to the multiple flaw size system. The off-diagonal components (related to the Poisson’s ratio) remain unchanged [40, 80]:

$$S_{ij ij} = -\frac{\nu_0}{E_0}, \quad (i \neq j, i, j = 1, 2, 3), \quad (2.23)$$

and the shear components are found to be:

$$S_{ij ij} = \frac{1}{G_0} \left[ 1 + \frac{32(1-u_0^2)(5-u_0)}{45(2-u_0)} \Omega \right], \quad (i \neq j, \quad i, j = 1, 2, 3), \quad (2.24)$$

These expressions are also found in the work by Budiansky and O’Connell 1976 [40] for dry circular cracks.

If the cracks are parallel with each other, the effective material properties become anisotropic. Assuming the crack normal is aligned with the $e_2$ direction, then the closed form expressions of the compliance are obtained by substituting $\Omega = \eta s^3 e_2 e_2$.
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into Eq. (2.6):

\[
S_{1111} = S_{3333} = \frac{1}{E_0}, \quad (2.25a)
\]

\[
S_{1122} = S_{3322} = -\frac{\nu_0}{E_0}, \quad (2.25b)
\]

\[
S_{2222} = \frac{1}{E_0} \left[ 1 + \frac{8(1 - \nu^2)(4 - \nu_0)}{3(2 - \nu_0)} \Omega \right] \Omega, \quad (2.25c)
\]

\[
S_{1212} = S_{2323} = \frac{1}{G_0} \left[ 1 + \frac{4(1 - \nu)(10 - 2\nu_0)}{3(2 - \nu_0)} \Omega \right] \Omega. \quad (2.25d)
\]

2.1.2.2 Two-dimensional solution

As a simplified case, effective properties of cracked solids in two-dimensional space have also been well studied. For a rectilinear crack with half length \( s \), the crack compliance tensor \( Z \) can be written as:

\[
Z = Z_{n \text{nn}} + Z_{s \text{ss}}, \quad (2.26)
\]

with \( n \) and \( s \) the normal to and trajectory direction of the crack. The values of the two crack compliance components are equivalent:

\[
Z_n = Z_s = \frac{\pi s}{E_0^*} \quad (2.27)
\]

where \( E_0^* \) is the equivalent elastic modulus in 2D; \( E_0^* = E_0 \) for plane stress and \( E_0^* = E_0/(1 - \nu_0^2) \) for plane strain, with \( E_0 \) and \( \nu_0 \) denoting the elastic moduli and
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the Poisson’s ratio of the intact isotropic material, respectively.

Substituting the above crack compliance \( Z \) into the general solution Eq. (2.16), an explicit expression of the change of compliance is obtained:

\[
\Delta S = \frac{\pi}{2E_0^*}(\Omega_{il}\delta_{jm} + \Omega_{jm}\delta_{il} + \Omega_{jl}\delta_{im} + \Omega_{im}\delta_{jl}), \quad (i, j, l, m = 1, 2), \tag{2.28}
\]

where \( \Omega \) is the damage tensor for 2D problem:

\[
\Omega = \frac{1}{A}\sum_k (s^2nn)_k. \tag{2.29}
\]

For a solid containing randomly orient cracks, averaged compliance can be obtained through integrating the contribution of compliance increment of every possible orientation. The result is isotropic, with the expressions given as follows:

\[
S_{1111} = S_{2222} = \frac{1}{E_0^*}(1 + \pi\Omega), \tag{2.30a}
\]

\[
S_{1122} = -\frac{\nu_0}{E_0^*}, \tag{2.30b}
\]

\[
S_{1212} = \frac{1}{G_0^*}(1 + \pi\Omega). \tag{2.30c}
\]

For parallel cracks, assume the crack normal parallel with the \( e_2 \) direction, we
have the explicit expressions:

\[ S_{1111} = \frac{1}{E_0}, \]  
\[ S_{2222} = \frac{1}{E_0} (1 + 2\pi \Omega), \]  
\[ S_{1122} = -\frac{\nu_0}{E_0}, \]  
\[ S_{1212} = \frac{1}{G_0} (1 + 2\pi \Omega). \]

\[ \text{(2.31a)} \]
\[ \text{(2.31b)} \]
\[ \text{(2.31c)} \]
\[ \text{(2.31d)} \]

\[ 2.1.3 \quad \text{Inelastic response of cracked solid under compressive damaged} \]

Under compressive load, contact and frictional sliding occur at the discontinuous surfaces in brittle materials and make the problem more complicated. Many authors investigated the inelastic behavior of brittle material under compressive load, for instance [13], [15], [11], [7], [16], [82], [47], [49]; however, there have not been many solutions in the literature regarding the instantaneous effective material properties with respect to the internal damage which are directly feasible for micromechanical modeling. In this subsection, some of these solutions are briefly reviewed.
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2.1.3.1 Nemat-Nasser and colleagues’ solutions

Nemat-Nasser and colleagues thoroughly investigated brittle solids under compressive loads from different aspects ([10] [83] [2] [14] [16] [17] [18]). Many of these studies are based on the wing-crack damaged model. In particular, Nemat-Nasser and Obata [16] related the inelastic strain tensor with the sliding displacement and derived a closed form solution of the effective material properties based on the stress intensity factors and kinematic arguments. Our analysis confirms part of the studies, but also shows its inaccuracy due to over-simplification of the wing-crack model. Since a detailed analysis and comparison of this solution will be presented in Section 2.8.1 we do not unfold the discussion at this point.

2.1.3.2 Tonge’s solution

Tonge [2014] [49] performed 3D multiscale modeling on brittle materials under dynamic compressive load using Material Point Method (MPM), in which wing-crack damage is incorporated as one of the major damage mechanisms. Due to the lack of a functioning damage-compliance relationship for the wing-crack problem, Tonge developed his own version of effective compliance based on the Kachanov’s anisotropic solutions for 3D cracked solids under tensile load, as reviewed in Section 2.1.2.1. To account for the higher stiffness and the strong lateral dilantancy associated with compressive load, an interactive term $Z_c$ is introduced to the elastic potential in Eq.
and thus the elastic potential becomes:

\[
f = \frac{1}{2} \sigma : S_0 : \sigma + \frac{1}{2V} \sigma : \sum_k \left[ n \cdot Z \cdot n A + Z c g(n) \right]_k : \sigma,
\]  

\[ (2.32) \]

in which \( g(n) \) is a function of the crack normal, given by:

\[
g(n) = nnI + Inn - 2nnnn,
\]  

\[ (2.33) \]

where \( I \) is the second-rank identity.

The interactive term \( Z c g(n) \) controls the strength of the coupling between the crack compliance in the normal and trajectory directions. Conditions are applied to the values of \( Z c \) for a valid energy dissipation during the damage process; in the original work [49] it was chosen to be:

\[
Z c = -\frac{Z_n}{8},
\]  

\[ (2.34) \]

where \( Z_n \) was defined in Eq. (2.18).

Similar to the tensile problem, a uniformly random crack orientation leads to an isotropic solution for the effective compliance. After averaging over all possible flaw orientations, the following expressions for the compliance are achieved:
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\[
S_{iii} = \frac{1}{E_0} \left[ 1 + \frac{(1 - \nu_0^2)(72 - 20\nu_0)}{45(1 - \nu_0/2)} \right], \quad (i = 1, 2, 3); \quad (2.35)
\]

\[
S_{iij} = \frac{1}{E_0} \left[ -\nu_0 - \frac{32(1 - \nu_0^2)}{45(1 - \nu_0/2)} \right], \quad (i \neq j, i, j = 1, 2, 3); \quad (2.36)
\]

\[
S_{ijj} = \frac{1}{G_0} \left[ 1 + \frac{(1 - \nu_0^2)(88 - 20\nu_0)}{45(1 - \nu_0/2)} \right], \quad (i \neq j, i, j = 1, 2, 3). \quad (2.37)
\]

In these expressions, \( \Omega \) is given by:

\[
\Omega = \sum_k \eta_k (s + l)_k^3, \quad (2.38)
\]

in which \( s \) is the half-size of the pre-existing flaw and \( l \) denotes the length of the crack that newly developed during the damage process.

Comparing with Kachanov’ solutions of Eqs. (2.22) through (2.24), the most significant difference in Tonge’s solution lies in the expression of \( S_{iijj} \), which increases with the damage parameter, while in the counterpart of Kachanov’s solution \( S_{iijj} \) remains a constant of \(-\nu_0/E_0\).

For an anisotropic solution, assume \( e_1 \) is the direction of applied compressive load, and all the crack normals are parallel to the \( e_2 \) direction (perpendicular to loading). Therefore, in Eq. (2.32) and (2.33) \( n = e_2 \); differentiating Eq. (2.32) with respect to the corresponding stress components gives the expressions of individual components of the compliance:
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\[ S_{1111} = \frac{1}{E_0}, \quad (2.39) \]
\[ S_{2222} = \frac{1}{E_0} \left[ 1 + \frac{16(1 - \nu^2_0)}{3} \Omega \right], \quad (2.40) \]
\[ S_{1122} = \frac{1}{E_0} \left[ -\nu_0 - \frac{2(1 - \nu^2_0)}{3} \Omega \right], \quad (2.41) \]
\[ S_{1212} = \frac{1}{G_0} \left[ 1 + \frac{16(1 - \nu^2_0)}{3(1 - \nu_0/2)} \right] \Omega. \quad (2.42) \]

In the work of Tonge 2014, both the isotropic and anisotropic solutions were applied in the multi-scale modeling of compressive damage. At the microscale, in order to determine the stress intensity factor on the crack tips of each flaw families and thus updating the crack length at each time step, a self-consistent scheme was implemented, in which the anisotropic solution was applied to described the property of the smeared damaged matrix material. With the updated crack lengths, the scalar damage parameter was evaluated and then substitute to the isotropic solution to describe the properties of the material at the discretization (macro-) scale. The implementation of isotropic solution at discretization scale is to circumvent the inconsistency of stress state that may occur if different load paths were applied.

### 2.1.3.3 Hu’s solution

In the work of Hu et. al. (2014) [47], the classical solution of cracked solids under tensile load (Eq. (2.20)) is modified to describe the inelastic response of damaged
solids under compression:

\[
\Delta S_{ijlm} = \frac{8(1 - \nu_0^2)}{3E_0(2 - \nu_0)} (\Omega_{il} \delta_{jm} + \Omega_{im} \delta_{jl} + \Omega_{jl} \delta_{im} + \Omega_{jm} \delta_{il} + \chi tr(\Omega) \frac{3}{\delta_{ij} \delta_{kl}}),
\]

\[(i, j, l, m = 1, 2, 3) \tag{2.43}\]

In this inelastic relationship, a term associated with the trace of the tensor damage parameter \( \Omega \) is added to tackle the interaction effects among different directions. With this interaction term, the components of compliance in the direction transverse to the loading direction is increased with respect to the scalar measure of the damage parameter. The effect of the interaction term is controlled by a tunable parameter \( \chi \). When \( \chi = 0 \), the Kachanov’s solution is recovered; if \( \chi > 0 \), all the compliance components except the the shear terms are increased proportional to the trace of the damage tensor.

In this work, the damage tensor \( \Omega \) is increased by an accumulative manner when modeling the compressive failure of brittle material. At the initial state the damage tensor is calculated following Eq. [2.6], in which the sizes and normal directions of the pre-existing flaws are input for the evaluation. At a given time step during the loading process, the increment of the damage tensor is calculated by:

\[
\Delta \Omega = \frac{3\Delta l}{V} \sum_k [(s + l)^2 n_i n_i]_k \tag{2.44}
\]

where \( \Delta l \) denotes the increment of the length of the wing-crack at the time step, and
Similar to the Grechka and Kachanov’s solution \[45\], the tensor form damage parameter $\Omega$ simplifies tracking the damage directions and evaluating the change of compliance due to the anisotropic damage.

Based on the simple wing-crack model, under uniaxial compressive load along the $e_1$ direction, the wing-cracks developed along $e_1$, then the normal of the wing-crack face $n_l$ lies in the plane of $e_2 \otimes e_3$. Therefore, the components associated with $e_2$ and $e_3$ increases while the rest remains the initial value. If the classical solution Eq. (2.20) is applied, compliance components associated with the 11 direction ($S_{1111}$, $S_{1122}$, $S_{1133}$ and their symmetric counterparts) remain constants during damage process. By adding the interactive term with $\chi$, the magnitudes of the compliance in all directions including $S_{1111}$ (governing the softening along the loading direction), $S_{1122}$ and $S_{1133}$ (related to bulking, transverse dilatation) increase, which capture the material response for compressive failure.

Nevertheless, this form of inelastic mechanical response suffers from some inconsistencies. For example, by adding the interactive term with the trace of damage tensor, the solution for compression predicts a higher value of compliance than the classical solution for tensile loads, which counters the logic that tensile compliance should be higher than compressive compliance, due to the lack of contact at the discontinued surfaces. Furthermore, the fitting parameter $\chi$ is somewhat arbitrary and not easily connected to the physical behavior of the material. Also the effect of the
friction coefficient on the flaw surface is not reflected in this solution, which should affect the sliding along the flaw surface that causes the development of wing-cracks.

In the following sections, we present our own theoretical study on the instantaneous two-dimensional compliance tensor for the wing-crack model, and provide a comparison with the mechanism-based solutions discussed here.

2.2 Analytical Solutions for Wing-Crack Damaged Solid

Under uniaxial compressive loads, wing-cracks are developed at the tips of pre-existing flaws due to the frictional sliding on the flaw faces. The growth direction of a wing-crack is controlled by the maximum energy release rate (see for example, [10]). As shown in Figure 2.2a, a wing-crack starts at about 70° to the flaw surface, and then gradually realigns with the principal compressive loading direction as it grows. We call this "realignment stage". For ease of analysis, the realignment stage is usually ignored and the wing-cracks are simplified into straight line cracks aligned at an angle towards the primary loading direction, see Fig. 2.2b. Some authors, for instance [16] and [18], further simplified the model into a straight crack, replaced the pre-existing flaw by forces applied in the center, as shown in Fig. 2.2c. In this case, the magnitude of the force is the flaw size multiplied by the resolved shear stress at the flaw surface.

Our analysis will be based on the first simplified model with a straight wing-crack
shown in Fig. 2.2b. Latter sections will address these assumptions, and will show that the straight crack model in Fig. 2.2c does not provide a good approximation for the material response.

Our solution is limited to two dimensions, although extension to a three-dimensional solution may be possible using appropriate dimensionality laws. Consider a solid containing numerous straight through-thickness rectilinear flaws (no wing-cracks at this stage). The internal volume (area) of the flaws at the unloaded state is considered to be zero, and the undamaged material is assumed to be isotropic linear elastic. Since
plasticity is ignored here, the inelastic part of the strain tensor $\epsilon^i$ is solely associated with the displacement discontinuity vector $u$ across the flaws, which is also known as the crack opening displacement (COD). Inelastic strains can be related to the COD following a summation of the kinematic discontinuity contributed by all $n$ cracks and flaws contained within a representative area $A$ ([75] [76]):

$$\Delta \bar{\epsilon}^i = \frac{1}{A} \sum_{k=1}^{n} (n \otimes u + u \otimes n) s_k,$$

(2.45)

in which $s_k$ denotes the half-length of flaw $k$, $n_k$ denotes the unit vector normal to the cracked surfaces associated with flaw $k$, and $u_k$ represents the averaged COD along the discontinued surfaces associated with flaw $k$, even without the bar on top (same as what follows). In the special case of a material in which all flaws are of the same length ($2s$) and orientation ($\phi$), this inelastic strain simplifies to:

$$\Delta \bar{\epsilon}^i = \eta s (n \otimes u + u \otimes n),$$

(2.46)

in which $\eta = n/A$ is the number density of flaws, $n$ denotes the unit vector normal to the flaws, and $u$ represents the averaged COD along the discontinued surfaces associated with the flaws.

To simplify the following derivations, a local $n - s$ coordinate system with the
bases in the normal $n$ and tangential $s$ directions of the flaw surface is adopted (Fig. 2.3). If a uniform traction $P = \begin{bmatrix} P_n \\ P_s \end{bmatrix}$ is applied to the internal surfaces of the flaw, an associated crack opening displacement $u = \begin{bmatrix} u_n \\ u_s \end{bmatrix}$ develops. The relationship between the COD $u$, associated with strain as shown in Eq. (2.45), and the applied traction $P$, associated with stress, is necessary to calculate the increment of compliance $\Delta S$ in Eq. (2.10), as shown in Fig. 2.3. Similar to Eq. (2.13), $u$ and $P$ can be interrelated through a second rank tensor $R$ by the expression:

$$u = \begin{bmatrix} u_n \\ u_s \end{bmatrix} = 2sRP, \quad (2.47)$$

where $s$ is the half-length of the flaw and $R$ is the normalized crack compliance tensor:

$$R = \begin{bmatrix} R_{nn} & R_{ns} \\ R_{ns} & R_{ss} \end{bmatrix}. \quad (2.48)$$

For non-interactive cracks under tensile loads, this relationship has been well studied (see, e.g. [40], [77], [80]) and has been discussed in Section 2.1.2. Different from Eq. (2.13), we extract the flaw size $2s$ from the expressions of $Z$, in other words $R$ is the crack compliance tensor $Z$ (Eq. (2.13)) normalized by the size of the crack/flaw; therefore $R$ can be called the normalized crack compliance tensor. $R$ can be
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Figure 2.3: Applied traction $P$ and resulting displacement $u$ on a flaw surface in the local $n - s$ coordinate system.

expressed by the decomposition

$$R = R_{nn} n \otimes n + R_{ns} (n \otimes s + s \otimes n) + R_{ss} s \otimes s,$$  \hspace{1cm} (2.49)

where $n$ and $s$ are the basis of the local coordinate system. Comparing with the $Z$ tensor in Eq. (2.27), the off-diagonal term $R_{ns}$ is added to tackle the interaction between the normal and shear components due to the presence of wing-cracks.

The components of $R$ depend on the crack size and shape. For example, for a through-thickness slit crack in two-dimensional space within an isotropic material,

$$R_{nn} = R_{ss} = \pi / 2E^*_0, \hspace{0.5cm} R_{ns} = 0,$$  \hspace{1cm} (2.50)

where $E^*_0 = E_0$ for plane stress and $E^*_0 = E_0 / (1 - \nu_0^2)$ for plane strain, with $E_0$ and
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\( \nu_0 \) denoting the elastic modulus and the Poisson’s ratio of the intact isotropic material, respectively [80]. For more complex geometries like the wing-crack, analytical expressions are less readily available.

2.2.1 Kinematic approach for wing-crack problem

The two-dimensional wing-crack problem is a special case of the general cracked solid problem. We consider an infinite solid with a periodic array of slit flaws, all with identical length \( 2s \), wing-crack length \( l \) and orientation \( \phi \), as shown in Fig. 2.4a. The spacing between flaws is \( L \). The flaw density is defined as number of flaws per area, \( \eta = n/A \), with \( A \) the size of representative volume element (RVE) and \( n \) the number of flaws within this RVE. For the periodic arrangement of the flaws, each flaw with its surrounding region can be considered as an RVE, as shown in Fig. 2.4b, and therefore \( \eta = 1/L^2 \). We consider \( \bar{\sigma}_{11} \) the primary compressive load, \( |\bar{\sigma}_{11}| >> |\bar{\sigma}_{22}| \) and \( \bar{\sigma}_{12} = 0 \). Straight wing-cracks emanate from the tips of the pre-existing flaws and are oriented along the primary loading direction \( \bar{\sigma}_{11} \).

The instantaneous compliance associated with such a solid is a function of the flaw density \( \eta \), flaw size \( s \), orientation \( \phi \), wing-crack length \( l \) and the friction coefficient \( \mu \) between the flaw surfaces. Because we assume the material is linear elastic, the compliance is independent from the magnitude of the applied load. Since the plastic zone at the crack (flaw) tips is ignored, the displacement discontinuity on the crack and flaw surfaces are the primary contributors to the inelastic strain, and Eq.
Figure 2.4: Periodic wing-crack model: flaw length $2s$, wing-crack length $l$, and orientation $\phi$ of every wing-crack are identical in a periodic array of wing-cracks spaced a distance $L = 1/\sqrt{\eta}$ apart.

Equation (2.45) is applicable. The summation in Eq. (2.45) should be conducted over all the discontinued surfaces. Specifically, each wing-crack RVE contains one flaw and two branches of wing-cracks. Denote the unit normal on the flaw as $\mathbf{n}_s$, while the unit normal on the wing-cracks $\mathbf{n}_l$, we have:

$$\mathbf{n}_s = \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix}, \quad \mathbf{n}_l = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (2.51)$$

Based on the superposition principle, the total crack opening displacement can be decomposed into two parts, as shown in Fig. 2.5b: the crack opening displacement $\mathbf{u}_s$ associated only with the average sliding displacement on the flaw, and the wing-crack
opening displacement $u_l$ when the flaw surface is fixed. The first part of the motion $u_s$ leads to an opening displacement on the wing-crack, the average of which can be approximated by $\xi u_s$. Therefore, Eq. (2.45) can be written explicitly as:

$$
\Delta \bar{\epsilon}^i = \frac{\eta}{2} [2s(n_s \otimes u_s + u_s \otimes n_s) + 2\xi l(n_l \otimes u_s + u_s \otimes n_l) + 2l(n_l \otimes u_l + u_l \otimes n_l)].
$$

(2.52)

Figure 2.5: Illustration of the crack opening displacement on the flaw and wing-cracks. (a) total crack opening displacement due to sliding displacement on the flaw interface and to the far-field stresses. (b) decomposition of crack opening displacement into two parts: (1) crack opening displacement due only to the sliding displacement $u_s$ on the flaw, and (2) wing-crack opening displacement that occurs in the absence of any sliding displacement ($u_s = 0$), equivalent to a fixed flaw surface.

In the work of [16], $\xi$ was approximated by $1/2$. Since the sliding motion along the flaw surface is the only permissible degree of freedom of the wing-crack model,
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\( \mathbf{u}_s \) is expressed in the global coordinate system as:

\[
\mathbf{u}_s = u_s \begin{bmatrix} \sin \phi \\ -\cos \phi \end{bmatrix}
\] (2.53)

where \( u_s \) the magnitude of the averaged sliding displacement in the flaw direction, to be solved later. The relation between \( \mathbf{u}_l \) and the applied far-field stress can be obtained from the classical elastic solution of line-cracks with crack density \( 2\eta \) and a half length of each wing-crack branch \( l/2 \):

\[
\mathbf{u}_l = \frac{\pi l}{2E_0} \overline{\mathbf{\sigma}} \mathbf{n}_l = \frac{\pi l}{2E_0} \begin{bmatrix} \overline{\sigma}_{22} \\ -\overline{\sigma}_{12} \end{bmatrix}
\] (2.54)

Substituting the unit normals \( \mathbf{n}_s \) and \( \mathbf{n}_l \), and Eqs. (2.53) and (2.54) into Eq. (2.52), we have the explicit expression of \( \overline{\mathbf{\epsilon}}^i \) in global coordinate system:

\[
\Delta \overline{\mathbf{\epsilon}}^i = \eta su_s \begin{bmatrix} \sin 2\phi & -\cos 2\phi \\ -\cos 2\phi & -\sin 2\phi \end{bmatrix} + \xi \eta u_s l \begin{bmatrix} 0 & \sin \phi \\ \sin \phi & -2\cos \phi \end{bmatrix} + \eta \frac{\pi l^2}{2E_0} \begin{bmatrix} 0 & \overline{\sigma}_{12} \\ \overline{\sigma}_{12} & 2\overline{\sigma}_{22} \end{bmatrix}
\] (2.55)

This expression turns out to be identical with the solution in Eq. (2.8) of the work Nemati-Nasser and Obata [16] after translating to a consistent notation, except the coefficient of the last term is reduced by half, since the wing-crack branches are treated here as two individual cracks of length \( l \), while in [16] they are treated as single crack.
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of length $2l$ (see Fig. 2.5).

The additional compliance $\Delta S$ due to wing-cracking can be derived by differentiating $\bar{\epsilon}^i$ by $\bar{\sigma}$:

$$\Delta S_{ijkl} = \frac{\partial \bar{\epsilon}_{ij}}{\partial \bar{\sigma}_{kl}}. \quad (2.56)$$

Applying the above differentiation to the corresponding components in (2.55), the individual terms of $\Delta S_{ijkl}$ are derived and listed as follows:

$$\Delta S_{1111} = \frac{\partial \bar{\epsilon}_{11}}{\partial \bar{\sigma}_{11}} = 2s\eta \sin \phi \cos \phi \frac{\partial u_s}{\partial \bar{\sigma}_{11}} \quad (2.57)$$

$$\Delta S_{2222} = \frac{\partial \bar{\epsilon}_{22}}{\partial \bar{\sigma}_{22}} = -2\eta \cos \phi (s \sin \phi + \xi l) \frac{\partial u_s}{\partial \bar{\sigma}_{22}} + \frac{\eta \pi l^2}{E_0} \quad (2.58)$$

$$\Delta S_{1122} = \frac{\partial \bar{\epsilon}_{11}}{\partial \bar{\sigma}_{22}} = 2s\eta \sin \phi \cos \phi \frac{\partial u_s}{\partial \bar{\sigma}_{22}} \quad (2.59)$$

$$\Delta S_{2211} = \frac{\partial \bar{\epsilon}_{22}}{\partial \bar{\sigma}_{11}} = -2\eta \cos \phi (s \sin \phi + \xi l) \frac{\partial u_s}{\partial \bar{\sigma}_{11}} \quad (2.60)$$

$$\Delta S_{1212} = \frac{\partial \bar{\epsilon}_{12}}{\partial \bar{\sigma}_{12}} = -\eta (s \cos 2\phi - \xi l \sin \phi) \frac{\partial u_s}{\partial \bar{\sigma}_{12}} + \frac{\eta \pi l^2}{2E_0} \quad (2.61)$$

2.2.2 Energy balance approach

As an alternative solution besides the kinematic approach, the damage-compliance relationship may also be analyzed from energetic points of view, although eventually the same results are obtained. Such an energetic approach is presented in this sub-
Consider the intact material with the compliance $S_0$. The density of external work applied to the intact material under applied load $\bar{\sigma}$ can be expressed by:

$$U_0 = \frac{1}{2}(S_0\bar{\sigma}) : \bar{\sigma}$$

(2.62)

in which the subscript 0 represents the energy state and the material properties for the material without any flaw and cracks. The nomenclature is consistent throughout this document. The elastic strain energy density under the applied load is equivalent to $U_0$.

After introducing the flaws and the associated wing-cracks, the softening of material leads to additional external work done:

$$\Delta U = \frac{1}{2}(S - S_0)\bar{\sigma} : \bar{\sigma} = \frac{1}{2}\Delta S\bar{\sigma} : \bar{\sigma}$$

(2.63)

where $\Delta S$ is the change of compliance tensor. The close form expressions of $\Delta S$ with respect to flaw and wing-crack parameters is what we after.

The averaged elastic strain energy density of the material within an RVE is given by:

$$E = \frac{1}{2}\eta \int \sigma : \epsilon dV$$

(2.64)

where $\eta$ is the flaw density, $\sigma$ and $\epsilon$ are the local stress and strain tensors. The change
of elastic strain energy after the introduction of flaws and wing-cracks is denoted as $\Delta E$.

As the material moves on the flaw and crack surfaces, the virtual work done by external load can be approximated by:

$$W = \eta \int (\bar{\sigma} n_c) \cdot u_c \, dl_c.$$  \hspace{1cm} (2.65)

in this equation, $n_c$ denotes the normal of discontinued surfaces (flaw and crack). Therefore, $\bar{\sigma} n_c$ gives the traction on the discontinued surfaces. $u_c$ denotes the material displacement on the discontinued surfaces. The integration is performed along the whole discontinued surfaces.

Consider the presence of friction on the flaw surface, and assume the work done by friction is $W_f$. Based on energy conservation, the additional work input to the system due to the introduction of flaw and wing-cracks is equivalent to the change of elastic strain energy density plus the work done by friction:

$$\Delta U = \Delta E + W_f$$ \hspace{1cm} (2.66)

Imagine we apply the reversed resolved traction on the discontinued surfaces, so that the crack opening displacements are closed and the system restored to the intact state as if no flaws and cracks were introduced. During this procedure, the energy
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conservation can be described by:

\[ 2\Delta U = \Delta E + W_f + W \]  

(2.67)

Therefore, we have:

\[ \Delta U = W \]  

(2.68)

Specifically for the wing-crack problem as shown in Figure 2.5, the normal of the flaw \( n_s \) and the normal of the wing-crack \( n_l \) have been defined in Eq. (2.51). The crack opening displacement (COD) \( u \) can be separated into two parts by the application of superposition principle, as shown in Figure 2.5: displacement on the two branches of wing-cracks \( u_l \) while fixing the flaw surface, and the displacement on the flaw surface \( u_s \) and the induced part on the wing-cracks. The induced displacement on the crack can be approximated as \( \xi u_s \). Since each branch of the wing-crack is considered an individual crack, \( u_l \) corresponds to the crack characteristics of halved length but twice density. Therefore, Equ. 2.65 can be re-written as:

\[ W = \frac{\eta}{2} [2s(\sigma n_s) \cdot u_s + 2l(\sigma n_l) \cdot (\xi u_s + u_l)]. \]  

(2.69)

Combining Equ. (2.68) and (2.69), we have the closed form solution: for \( \Delta S \):
\[ \frac{1}{2}(\Delta S \bar{\sigma} ) : \bar{\sigma} = \frac{\eta}{2} [2s(\bar{\sigma}n_u) \cdot u_s + 2l(\bar{\sigma}n_l) \cdot (\xi u_s + u_l)] \] (2.70)

Since the sliding motion on the flaw surface is the only permissible degree of freedom of the wing-crack model, \( u_s \) can be expressed as \( u_s \begin{bmatrix} \sin \phi \\ -\cos \phi \end{bmatrix} \), with \( u_s \) the magnitude of averaged sliding displacement.

According to classical elastic solutions of line cracks, the averaged COD along the wing-cracks (with half-size \( l/2 \)) is related to the allied remote stress by:

\[ u_i = \frac{\pi l}{2E_0} \bar{\sigma}n_i = \frac{\pi l}{2E_0} \begin{bmatrix} \bar{\sigma}_{12} \\ \bar{\sigma}_{22} \end{bmatrix} \] (2.71)

The expressions of individual component of \( \Delta S \) can be obtained by applying the corresponding term of far field stress individually. Consider applying an uniaxial load \( \bar{\sigma}_{11} \), while keeping stress terms in other directions zero. The product \( \bar{\sigma}n_i \) in Eqn. (2.70) gives a zero vector, and the whole equation is simplified to be:

\[ \Delta S_{1111} \bar{\sigma}_{11}^2 = 2s \eta \bar{\sigma}_{11} \cos \phi \sin \phi u_s \] (2.72)

It is worth noting that \( \bar{\sigma}_{11} \cos \phi \sin \phi \) equals to the resolved shear stress on the flaw
surface, and multiplying by $2s u_s$ gives us the virtual work done on the surface of a flaw.

Equ. (2.70) can be re-written into a differential form with respect to the perturbation of stress tensor $\tilde{\sigma}$:

$$
 \frac{1}{2} (\Delta S \partial\sigma) : \sigma = \frac{\eta}{2} [2s(\tilde{\sigma} n_s) \cdot \partial u_s + l(\sigma n_l) \cdot \partial(u_s + 2u_l)],
$$

(2.73)

where the perturbation of $u_l$ becomes a function of perturbation of far field applied load:

$$
 \partial u_l = \frac{\pi l}{2E_0} \partial \tilde{\sigma} n_l = \frac{\pi l}{2E_0} \begin{bmatrix}
 \partial \tilde{\sigma}_{12} \\
 \partial \tilde{\sigma}_{22}
 \end{bmatrix}
$$

(2.74)

Following the same derivation in the previous sub-section, the differential form of Equ. (2.72) can be written as Eq. (2.57). Similarly, perturbing the corresponding stress components $\partial \tilde{\sigma}_{22}$ and $\partial \tilde{\sigma}_{12}$ individually in Equ. (2.73), we obtain the expressions of individual compliance components from Eq. (2.58) through (2.61).

The expressions $\partial u_s / \partial \tilde{\sigma}_{ij}$ are required to obtain the closed form solutions for the compliance tensor components. Since plasticity is ignored, the magnitude of $u_s$ can be considered a linear superposition of terms containing $\tilde{\sigma}_{11}$, $\tilde{\sigma}_{22}$ and $\tilde{\sigma}_{12}$. Previous authors (e.g. [11, 16, 19]) obtained the expression of $u_s$ by equating the stress intensity factors $K_I$ at the crack tips based on different approaches, such as through COD or
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related to applied loads. In this analysis we will present a novel derivation for \( u_s \).

### 2.3 Kinematic Relations for Flaws and Wing-Cracks

The general traction-displacement relationship expressed by Eq. (2.47) is applicable to the wing-crack problem. Under a compressive loading state, since no plasticity is considered, the kinematics on the flaw can be described as a superposition of the effects of the macro-scale applied stress and of the tractions on the flaw surface. For instance, rigid contact on the flaw is ensured by requiring that the macro-scale compressive stress and the traction on the flaw cause an equal but opposite displacement in the normal direction of the flaw surface, leading to the combined normal displacement \( u_n = 0 \). We seek the magnitude of the displacement in the shearing direction, \( u_s \), under this constraint on \( u_n \).

Using the superposition described above, the solution of \( u_s \) is found by first applying the external load without contact, allowing the material to overlap freely (State 1), and then applying the reactive traction necessary to ensure that the combined normal displacement \( u_n \) is zero, thus recovering the contact on the flaw (State 2). This decomposition is illustrated in Fig. 2.6 and the deflection associated with each State 1 and State 2 are detailed in what follows.

**State 1:** This is a fictitious state, in which the contact on the flaw surfaces is
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Figure 2.6: Total average relative displacements between the opposite sides of the flaw, \( u_s \), decomposed into deflection due to the applied stresses without contact (State 1) and deflection due to reactive tractions on the flaw surfaces that ensure that the normal deflection \( u_n = 0 \) (State 2).

Artificially removed and the opposite sides of the flaw are allowed to penetrate each other. In this state, even if a compressive far-field load is applied, the crack faces deflect with the same magnitude but opposite direction as seen under tensile loads.

The average relative displacement vector between the upper and lower flaw surfaces is now denoted as \( u_1 \), with decompositions \( u_{1n} \) and \( u_{1s} \) in the local coordinate system.

The products of the stress tensor and the flaw/crack normal yield the resolved tractions on the corresponding discontinued surface. In other words, the resolved tractions along the flaw are \( \bar{\sigma}n_s \) and the resolved tractions along the wing crack are \( \bar{\sigma}n_l \). Since the motion on the flaw surface is not hindered by contact or friction, the kinematic relationship between the displacement on the flaw surfaces \( u_1 \) resulting from the applied far-field stress \( \bar{\sigma} \) is expressed using Eq. (2.47): 

\[
\begin{bmatrix}
u_{1n} \\
u_{1s}
\end{bmatrix} = R\bar{\sigma}(2sn_s + 2\xi l_n), \quad (2.75)
\]
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where $\xi$ is the same parameter as in Eq. (2.52). If Eq. (2.75) is differentiated by the corresponding stress components and then substituted into Eq. (2.57) through (2.61), the material response under tensile load is obtained. When $l = 0$, Eq. (2.75) for crack opening displacement under tensile load is identical with the solutions given by other authors, such as [80] and [45].

State 2: This state represents the recovery of the discontinued surfaces to the true physical state, in which the opposite sides of the flaw are in contact and can only move relative to each other in the direction of the flaw surfaces. Assuming that the traction on the flaw surface is denoted $P_2$, with $P_{2n}$ and $P_{2s}$ the components in the normal and shear directions of the flaw, respectively, the deflection $u_2$ is expressed:

$$u_2 = \begin{bmatrix} u_{2n} \\ u_{2s} \end{bmatrix} = 2sR P_2$$  \hspace{1cm} (2.76)

Because there is not contact on the wing-cracks, we only consider contact tractions at the flaw interface in this expression. Since $P_2$ results from the rigid contact between opposite sides of the flaw, we use the following relationship:

$$-\mu |P_{2n}| < P_{2s} < \mu |P_{2n}|$$  \hspace{1cm} (2.77)

in which $\mu$ is the coefficient of friction. Depending on the sliding direction on the flaw
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at State 2, the shear traction $P_{2s}$ varies between $\pm \mu |P_{2n}|$. Under a monotonically increasing load, we can assume $P_{2s} = \mu P_{2n}$.

Superimposing the kinematic relations of State 1 and State 2 leads to the following relations in the local coordinate system:

$$u_1 + u_2 = \begin{bmatrix} 0 \\ u_s \end{bmatrix}.$$ (2.78)

Combining Eqs. (2.75), (2.76) and (2.78), the sliding displacement $u_s$ can be solved in terms of the applied far-field stress and the normalized crack compliance tensor $R$. The relationship between $u_s$ and individual stress components are derived in the following sub-section.

2.4 Perturbed $u_s$ in Terms of Perturbations in $\bar{\sigma}$

Because we assume contact at the flaw interface and wing-cracks aligned in the $x_1$-direction, $\bar{\sigma}_{11}$ is assumed to be the primary compressive stress, $|\bar{\sigma}_{11}| >> |\bar{\sigma}_{22}|$ and $|\bar{\sigma}_{11}| >> |\bar{\sigma}_{12}|$. Perturbing these stresses around their original values and evaluating the perturbations in the displacement $u_s$, the derivatives in Eqs. (2.57) through (2.61) can be evaluated. The magnitude of $\bar{\sigma}_{11}$ at the initial state should be sufficiently large
so that the flaw surfaces do not lose contact under the stress perturbation. In this section, perturbations of \( u_s \) due to perturbations of each of the stress perturbation components are developed.

When \( \sigma_{11} \) is perturbed, the resolved perturbing traction on the wing-cracks surface \( \partial \bar{\sigma} n_l \) is zero, since the wing-crack branches are aligned in the \( x_2 \)-direction. The resolved perturbing tractions on the flaw and wing-crack surfaces are therefore given by:

\[
\partial \bar{\sigma}_n = \partial \bar{\sigma}_{11} \begin{bmatrix} \cos^2 \phi \\ \cos \phi \sin \phi \end{bmatrix}, \quad \partial \bar{\sigma}_n = 0.
\] (2.79)

Substituting Eq. (2.79) into (2.75), the perturbed crack opening displacement \( u_1 \) is:

\[
u_1(\partial \bar{\sigma}_{11}) = 2s \partial \bar{\sigma}_{11} \cos \phi \begin{bmatrix} \cos \phi R_{nn} + \sin \phi R_{ns} \\ \cos \phi R_{ns} + \sin \phi R_{ss} \end{bmatrix}.
\] (2.80)

Similarly, if \( \bar{\sigma}_{22} \) is perturbed, the resolved perturbing tractions and the resulting \( u_1 \) in the local coordinate system are given by:

\[
\partial \bar{\sigma}_n = \partial \bar{\sigma}_{22} \begin{bmatrix} \sin^2 \phi \\ -\cos \phi \sin \phi \end{bmatrix}, \quad \partial \bar{\sigma}_n = \partial \bar{\sigma}_{22} \begin{bmatrix} \sin \phi \\ -\cos \phi \end{bmatrix},
\] (2.81)
Finally, if $\bar{\sigma}_{12}$ is perturbed, we find the following similar expressions:

$$\partial \bar{\sigma}_{ns} = \partial \bar{\sigma}_{12} \begin{bmatrix} 2 \sin \phi \cos \phi \\ -(\cos^2 \phi - \sin^2 \phi) \end{bmatrix}, \quad \partial \bar{\sigma}_{nl} = \partial \bar{\sigma}_{12} \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix}, \quad (2.83)$$

$$u_1(\partial \bar{\sigma}_{12}) = \partial \bar{\sigma}_{12} \begin{bmatrix} 2s \\ \sin 2\phi R_{nn} - \cos 2\phi R_{ns} \\ \sin 2\phi R_{ns} - \cos 2\phi R_{ss} \end{bmatrix} + 2\xi l \begin{bmatrix} \cos \phi R_{nn} + \sin \phi R_{ns} \\ \cos \phi R_{ns} + \sin \phi R_{ss} \end{bmatrix}. \quad (2.84)$$

The recovery displacement $u_2$ is obtained by substituting the normal traction due to contact $P_{2n}$ and the resulting shear traction due to friction $\mu P_{2n}$ into Eq. (2.76):

$$u_2 = 2s \begin{bmatrix} R_{nn} & R_{ns} \\ R_{ns} & R_{ss} \end{bmatrix} \begin{bmatrix} P_{2n} \\ \mu P_{2n} \end{bmatrix}. \quad (2.85)$$

Combining the above $u_2$ with $u_1$ in Eq. (2.80), (2.82) or (2.84), and substituting into Eq. (2.78), the perturbation $\partial u_s$ is found in terms of the perturbation of each
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individual stress component:

\[
\partial u_s(\partial \bar{\sigma}_{11}) = \partial \bar{\sigma}_{11} 2 s \cos \phi (\sin \phi - \mu \cos \phi) \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}}, \tag{2.86}
\]

\[
\partial u_s(\partial \bar{\sigma}_{22}) = -\partial \bar{\sigma}_{22} (2 \xi l + 2 s \sin \phi) \cos \phi + \mu \sin \phi \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}}, \tag{2.87}
\]

\[
\partial u_s(\partial \bar{\sigma}_{12}) = -\partial \bar{\sigma}_{12} [2 s (\cos 2 \phi + \mu \sin \phi) + 2 \xi l (\sin \phi - \mu \cos \phi)] \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}}. \tag{2.88}
\]

To be consistent with [16], we adopt the value \(\xi = 1/2\) for \(u_s(\partial \bar{\sigma}_{22})\). However, for \(u_s(\partial \bar{\sigma}_{12})\) we choose \(\xi = 1/3\), which provides a good fit to the finite element modeling results that are described in the next section.

Substituting Eqs. (2.86) through (2.88) into Eqs. (2.57) through (2.61) provides expressions for \(\Delta S\):

\[
\Delta S_{1111} = \eta (2 s)^2 \sin \phi \cos^2 \phi (\sin \phi - \mu \cos \phi) \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}}, \tag{2.89}
\]

\[
\Delta S_{2222} = \eta (l + 2 s \sin \phi)^2 \cos \phi (\cos \phi + \mu \sin \phi) \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}} + \frac{\eta \pi l^2}{E_0}, \tag{2.90}
\]

\[
\Delta S_{1122} = -\eta (l + 2 s \sin \phi) 2 s \sin \phi \cos \phi (\cos \phi + \mu \sin \phi) \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}}, \tag{2.91}
\]

\[
\Delta S_{2211} = -\eta (l + 2 s \sin \phi) 2 s \cos^2 \phi (\sin \phi - \mu \cos \phi) \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}}, \tag{2.92}
\]

\[
\Delta S_{1212} = \eta (s \cos 2 \phi - l \sin \phi/2) F(s, l, \phi) \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}} + \frac{\eta \pi l^2}{2 E_0}, \tag{2.93}
\]

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where the function $F(s, l, \phi)$ is defined as:

$$F(s, l, \phi) = 2s(\cos 2\phi + \mu \sin \phi) + \frac{2}{3}l(\sin \phi - \mu \cos \phi).$$  \hfill (2.94)

It is worth noting that if there is no friction on the flaw surface ($\mu = 0$), $\Delta S_{2211}$ and $\Delta S_{1122}$ are identical:

$$\Delta S_{2211} = \Delta S_{1122} = -\eta(l + 2s \sin \phi)2s \sin \phi \cos^2 \phi \frac{R_{nn}R_{ss} - R_{ns}^2}{R_{nn}}. \hfill (2.95)$$

However, if the friction coefficient is not zero, the values of $\Delta S_{2211}$ and $\Delta S_{1122}$ are unequal and therefore the compliance tensor $S$ is unsymmetric.

At this point, $\Delta S$ has been explicitly derived with respect to the flaw / crack geometric parameters, the friction coefficient and the components of the normalized crack compliance tensor $R$, which describes the relationship between the applied traction and resulting crack opening displacement on the flaw and crack surfaces. Although $R$ has been addressed by many authors for regular-shaped cracks in both two and three dimensions (e.g. [40], [77], [11, 12, 45]), $R$ is not well understood for wing-cracks. It is reasonable to assume that the $R$ tensor is a function of the geometric parameters, including the flaw size $s$, the flaw orientation $\phi$, the flaw number density $\eta$ and the wing-crack length $l$. Solving the $R$ tensor analytically is not straightforward, and therefore it is calculated based on finite element models of the wing-crack, which will be discussed in the next chapter.

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2.5 Simplified Solution

Equations of the compliance components Eqs. (2.89) through (2.93) contain a common expression $R_{comb}$, which is a combination of the components of the $R$ tensor and the friction coefficient $\mu$:

\[ R_{comb} = \frac{R_{nn}R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}} \]  

(2.96)

If $R_{comb}$ can be fit by a closed form expressions with respect to the wing-crack geometric parameters ($s$, $l$, $\eta$) in the similar way as the individual $R$ components, the evaluation of $\Delta S$ can be simplified, and the influence of $\Delta S$ by the wing-crack geometric parameters becomes more straightforward.

Since $\mu \frac{R_{ns}}{R_{nn}} << 1$, $R_{comb}$ can be reorganized and approximated by:

\[ R_{comb} = \left( R_{ss} - \frac{R_{ns}^2}{R_{nn}} \right) \left( 1 - \mu \frac{R_{ns}}{R_{nn}} \right) = R_{c1} - \mu R_{c2}, \]  

(2.97)

(2.98)

with the individual combined expressions:

\[ R_{c1} = R_{ss} - \frac{R_{ns}^2}{R_{nn}}, \]  

(2.99)

\[ R_{c2} = \left( R_{ss} - \frac{R_{ns}^2}{R_{nn}} \right) \frac{R_{ns}}{R_{nn}}. \]  

(2.100)
The closed form expression for $R_{c1}$ can be fitted with the aid of the commercial software program Eureqa ([84]), and the following expressions with respect to the normalized flaw size $s^*$, crack length $l^*$ and flaw orientation $\phi$ are proposed:

$$R_{c1}^* E_0^* = \frac{\pi}{2} + 3l^* + \frac{0.1 \cos 2\phi + 0.45}{0.07 + \sin \phi s^*} - \frac{0.1 \cos 2\phi + 0.45}{0.07 + \sin \phi s^* + l^*}$$  \hspace{1cm} (2.101)

In many circumstances $R_{c2}$ can be generally ignored due to its relatively small amount, particularly when the friction coefficient $\mu$ is also small. Otherwise, since $R_{ss}/R_{nn} \approx 1$, $R_{c2}$ can be approximated by $R_{ns}$ with Equation (3.6).

### 2.6 Wing-Crack with Realignment

The derivations in this paper so far have considered the wing-crack to be aligned with the primary loading direction as soon as it initiates from the tip of the pre-existing flaw, as in Fig. 2.2b. In reality, the crack growth direction is governed by the maximum energy release rate and exhibits a curved shape at the beginning, as demonstrated in Fig. 2.2a. [10] showed that a wing-crack initiates with an angle of about 70° from the surface of pre-existing flaw and gradually redirects to the primary loading direction as it continues to grow.

The distance between the two branches of wing-cracks $2s_p$ is illustrated in Fig. 2.7. For a straight wing-crack model the length of this region equals to $2s \cos \phi$. When the wing-cracks align with the central line, $2s_p$ reduces to zero. By considering
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wing-crack realignment, $2s_p$ reduces to a fraction of the projected flaw size. We refer to this fraction as $p_s = s_p/s \cos(\phi)$, which is estimated by following [10] and plotted in Fig. 2.7 along with a linear least square fit:

$$p_s(\phi) = -8.97 \times 10^{-3} \phi + 1.13,$$

(2.102)

where $\phi$ is in the unit of degree.

![Figure 2.7: Shape of the wing-crack: (a) distance between aligned wing-crack branches $2s_p$; (b) the projection coefficient $p_s = s_p/(s \cos \phi)$ based on [10], along with a linear least square fit by Eq. (2.102).](image)

The exact shape of the wing-crack depends on the material properties, such as fracture toughness and friction on the flaw surface. The wing-crack can be considered fully aligned with the primary loading direction when $l/s > 2$, and the distance between the wing-crack and the central line can be taken as $s_p$. When the wing-crack is fully aligned with the loading direction, the sliding displacement on the flaw
surface resulting from a perturbation in the primary compressive stress, \( u_s(\partial\sigma_{11}) \), is proportional to \( p_s \).

The impacts of the wing-crack realignment will be discussed in the next chapter, with the illustration of FEM results. Here we propose the amendment to the quantities of \( u_s(\partial\sigma_{11}) \), \( S_{1111}, S_{2211} \) and \( S_{1122} \) by multiplying these expressions with \( p_s \) to reflect crack realignment:

\[
\Delta S_{1111} = \eta p_s (2s)^2 \sin \phi \cos^2 \phi (\sin \phi - \mu \cos \phi) \frac{R_{nn}R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}},
\]

(2.104)

\[
\Delta S_{2211} = -\eta p_s (l + 2s \sin \phi) 2s \cos^2 \phi (\sin \phi - \mu \cos \phi) \frac{R_{nn}R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}},
\]

(2.105)

\[
\Delta S_{1122} = -\eta p_s (l + 2s \sin \phi) 2s \sin \phi \cos \phi (\cos \phi + \mu \sin \phi) \frac{R_{nn}R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}},
\]

(2.106)

while \( S_{2222} \) is generally not affected by crack re-alignment.
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2.7 Inelastic Response for Wing-Cracked Material under Tensile Loads

Now let’s consider the mechanical response of wing-cracked damaged solid under tensile load. This solution is applicable to the following scenario: a brittle solid is first compressed and damaged as the wing-cracks developed, then suddenly the external load is flipped over to be tensile. Such a scenario may occur when a compressive wave hits a free boundary and bounced back as a tensile wave. Same as the case of compressive solution, we look for the instantaneous mechanical response with respect to geometrical parameters of the wing-cracks such as the flaw density \( \eta \), flaw size \( s \), orientation of flaw normal \( \phi \) and the length of wing-crack \( l \).

Under tensile loads, the flaw surfaces are detached from each other, and thus the solid can be treated as the ordinary cracked solid. Indeed this is the same problem with the State 1 back in Section 2.3 in which the contact on the flaw surface is fictitiously removed to provide transition for solving the flaw sliding displacement under compressive load.

Along the wing-crack’s normal direction, the compliance component \( S_{2222} \) can be approximated by substituting the projected length of the discontinued surfaces \( s \sin \phi + l \) into the classical solution:

\[
\Delta S_{2222} = \frac{2\eta \pi}{E_0} (s \sin \phi + l)^2 = \frac{\pi}{2E_0} (s^* \sin \phi + l^*)^2; \quad (2.107)
\]
while for $S_{1212}$, the total length of the path of discontinuity $s + l$ can be applied, which leads to the following equation:

$$\Delta S_{1212} = \frac{2\eta \pi}{E_0^*} (s + l)^2 = \frac{\pi}{2E_0^*}(s^* + l^*)^2.$$  \hfill (2.108)

Other compliance components can be solved by interrelating the opening displacement on the discontinued surfaces with the applied load $\sigma_{11}$. Since the wing-cracks are developed along the loading direction $e_1$ and thus no relative displacement is caused, only the flaw surface need to be accounted for the opening displacement. The compliance components are then given by:

$$\Delta S_{1111} = 2s\eta \cos \phi \frac{\partial u_1}{\partial \sigma_{11}}$$  \hfill (2.109)

$$\Delta S_{1122} = \Delta S_{2211} = -2s\eta \sin \phi \frac{\partial u_2}{\partial \sigma_{11}}$$  \hfill (2.110)

where $u_1$ and $u_2$ are the displacements along the directions of $e_1$ and $e_2$ in the global coordinate system, respectively. The crack opening displacement can be interrelated to the applied load $\sigma_{11}$ through the crack compliance tensor $R$. Substituting the resolved normal and shear stress on the flaw surface $\sigma_{11} \cos^2 \phi$ and $\sigma_{11} \sin \phi \cos \phi$ into
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Eq. (2.47), the opening displacement in the local coordinate system is given by:

\[
\mathbf{u} = \begin{bmatrix}
    u_n \\
    u_s 
\end{bmatrix} = 2s\sigma_{11} \begin{bmatrix}
    R_{nn} & R_{ns} \\
    R_{ns} & R_{ss}
\end{bmatrix} \begin{bmatrix}
    \cos^2 \phi \\
    \sin \phi \cos \phi
\end{bmatrix}
\] (2.111)

in which the subscript \( n \) and \( s \) denote the normal and tangential direction of the flaw, respectively. Transferring to the global coordinate system there are:

\[
u_1 = u_n \cos \phi + u_s \sin \phi, \quad (2.112a)
\]
\[
u_2 = u_n \sin \phi - u_s \cos \phi. \quad (2.112b)
\]

Substituting \( \mathbf{u} \) into Eq. (2.109) and (2.110), the changes of the compliance components are found to be:

\[
\Delta S_{1111} = \eta (2s)^2 \cos^2 \phi (R_{nn} \cos^2 \phi + 2R_{ns} \sin \phi \cos \phi + R_{ss} \sin^2 \phi) \quad (2.113)
\]
\[
\Delta S_{2211} = -\eta (2s)^2 \sin \phi \cos \phi [(R_{nn} - R_{ss}) \sin \phi \cos \phi + R_{ns} (\cos^2 \phi - \sin^2 \phi)] \quad (2.114)
\]
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2.8 Comparison with Other Solutions

2.8.1 Comparison with Nemat-Nasser and Obata’s solution of two-dimensional wing-crack problem

As presented in the Section 2.2.1, the expressions for inelastic strain Eq. 2.55 is very similar to the solution developed by Nemat-Nasser and Obata [16], which can be expressed by:

\[
\Delta \bar{\epsilon} = \eta s u_s \begin{bmatrix} 
\sin 2\phi & -\cos 2\phi \\
-\cos 2\phi & -\sin 2\phi 
\end{bmatrix} + \frac{\eta u_s l}{2} \begin{bmatrix} 
0 & \sin \phi \\
\sin \phi & 2 \cos \phi 
\end{bmatrix} + \frac{\eta \pi l^2}{E_0} \begin{bmatrix} 
0 & \bar{\sigma}_{12} \\
\bar{\sigma}_{12} & 2\bar{\sigma}_{22} 
\end{bmatrix},
\]

(2.115)

As discussed earlier, in our solution the coefficient of the second term 1/2 is replaced by a more general parameter \( \chi \), and the coefficient of the third term is reduced by half.

The compliance \( \Delta S \) from Eqs. (2.57) to (2.61) are obtained by differentiating the above inelastic strain \( \bar{\epsilon} \) by the corresponding stress terms, where explicit expressions for \( \partial u_s / \partial \bar{\sigma}_{11} \), \( \partial u_s / \partial \bar{\sigma}_{22} \) and \( \partial u_s / \partial \bar{\sigma}_{12} \) are required. Different from the kinematic approach incorporating the crack compliance \( R \) that presented in current work, [16] solved these gradient terms from fracture mechanics perspective. In what follows, a
brief comparison is presented.

In [16] as well as many other previous works, solutions for $\frac{\partial u_s}{\partial \bar{\sigma}_{22}}$ were derived essentially by solving the stress intensity at the tips of the wing-crack. The stress intensity factor due to the crack opening displacement was derived and expressed in Equation 2.10 in [16]. Translated into notational context in this work, it reads:

$$K_I(u) = \frac{E^*}{2\sqrt{\pi(l + 0.54s\pi^2/32)}}(u_s \cos \phi + u_n \sin \phi) \quad (2.116)$$

This equation is applicable to any straight crack (both Figure 2.2b and c) with displacements $u_s$ and $u_n$ at length $l$.

Recall the straight crack model with concentrated force in the center, as shown in Figure 2.2c, which is the ultimate simplification of the wing-crack model by many authors. For this model the stress intensity factor due to far field stress was expressed in (2.9) in [16], and it reads (assuming the crack aligned with $\bar{\sigma}_{22}$):

$$K_I(\bar{\sigma}) = \frac{4s}{\sqrt{2\pi(l + 0.54s)}}\sigma_{22} \cos^2 \phi \sin \phi - \mu \cos \phi \quad (2.117)$$

This equation is only applicable to the straight crack model (Figure 2.2c), not to the straight wing-crack model. However, it is applicable to the central aligned wing-crack model. In figure 2.8 we plot the $K_I$ exported from FE modelling results, $K_I(\sigma_{22})$ with the same value of $\sigma_{22}$ value in FE modelling, and $K_I(u_s)$ with $u_s$ from the straight wing-crack FE modelling result.
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Figure 2.8: $K_I$ results predicted by [16] and corresponding FEM results

By equating the stress intensity factors, i.e. $K_I(u_s) = K_I(\sigma_{22})$, an explicit expression for $u_s$ is obtained:

$$u_s = \frac{1}{E_s^*} 2s \sigma_{22} \cos\phi (\sin\phi - \mu \cos\phi) \sqrt[3]{\frac{8(l + 0.54s^2/32)}{l + 0.54s}}. \quad (2.118)$$

However, due to the conditions of the $K_I$ equations discussed above, the $u_s$ obtained by the above expression is only applicable to the centrally aligned wing-crack model. A comparison between the above $u_s$ solution and the finite element results of the two types of models are plotted in Figure 2.9, which clearly shows the agreement of Equation 2.118 with the results by the central aligned model, but discrepant to the results from the straight wing-crack model.

As shown earlier in the chapter, compliance terms $S_{2222}$ and $S_{1122}$ depend on $u_s(\sigma_{22})$. The $u_s(\sigma_{22})$ expressions derived from [16] are generally constants, which leads
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Figure 2.9: $u_s$ results predicted by [16] and corresponding FEM results to underestimated changes of $S_{2222}$ and $S_{1122}$ with respect to the wing-crack damage. The solution of $u_s(\sigma_{22})$ provided here can better reflect the damage-compliance relationship. In the next chapter, by using the finite element modeling as a tool for verification, comparison between these two solutions will be continued.

2.8.2 Comparison with other three-dimensional solutions through dimension reduction

In this section we compare our inelastic material response under compressive load with other solutions for cracked solids which were introduced in Section 2.1 specifically the followings:

- Classical solution in Kachanov’s form for random crack orientations (isotropic);
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- Classical solution in Kachanov’s form for parallel cracks (anisotropic);
- Tonge’s isotropic and anisotropic solutions;
- Hu’s solution
- Liu’s solution for wing-cracked solid under compression
- Liu’s solution for wing-cracked solid under tension

In order to make a valid comparison, all the compared solutions should be adapted to the same scenario, which we define as follows. At the initial state, rectilinear through-thickness flaws with the identical size $2s$ were homogeneously located in a solid in 2D space; the orientations of the flaw normal for anisotropic solutions are parallel, and for isotropic solutions are uniformly random. Under applied loads, cracks nucleated at the tips of the flaws. For anisotropic solutions the cracks grows along the $e_1$ direction, and thus the surface normal is in the $e_2$ direction; for isotropic solutions the cracks extend within the plane of the flaw from which they nucleated.

The classical isotropic and anisotropic solutions for a 2D cracked solid are well suited for the above scenario, thus they can be directly implemented without additional processing. However, for the rest of the above solutions, additional processing is needed before they can be reasonably compared with each other.

Tonge and Hu’s solutions were established based on 3D problem, in which the flaws are assumed to be circular disks. Comparing Kachanov’s general solutions for 3D circular cracks Eq. (2.20) (omitting the unsymmetric term with coefficient $\beta$)
and 2D rectilinear cracks Eq. (2.28), besides the different expressions of the damage tensor, these two solutions can be interrelated with a scalar multiplier. The multiplier can be identified as:

\[
C_{32a} = \frac{3\pi(2 - \nu_0)}{16(1 - \nu_0)}.
\] (2.119)

Here the subscript 32 indicates the transformation from 3D to 2D solution, and the subscript \(a\) denotes the anisotropic solution. The same value of the multiplier can be confirmed by comparing the anisotropic compliance change with Eq. (2.25c) and Eq. (2.31b).

For isotropic solutions, in addition to the multiplier found between the general 3D and 2D solutions, averaging over all possible flaw orientations gives another scaling difference of \(\pi/2\). So the coefficient to transfer an isotropic solutions from 3D to a 2D becomes:

\[
C_{32i} = \frac{3\pi^2(2 - \nu_0)}{32(1 - \nu_0)}.
\] (2.120)

Therefore, to convert the solutions by Tonge and Hu from 3D problem to 2D, these solutions are multiplied by the corresponding coefficients \(C_{32a}\) for Tonge’s anisotropic and Hu’s solutions, \(C_{32i}\) for Tonge’s isotropic solution), and the 3D damage parameter is replaced by the 2D counterpart.
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Hu and Liu’s solutions contain explicit terms of the flaw orientation $\phi$, the angle between the applied uniaxial load and the flaw normal. For Hu’s model, the normal of the flaw is preserved within the plane of $e_1 \otimes e_2$; as the load applied in the $e_1$ direction, the normal of the wing-crack remains in the $e_2$ direction. Furthermore, to give an evaluation accounting for the contribution from all the possible flaw orientations within the plane $e_1 \otimes e_2$, numerical averaging is performed for Hu and Liu’s solutions. The simplest numerical averaging method with homogeneous distribution of $\phi$ is applied. We discretize the range from 0 to $\pi/2$ into $n$ values with identical interval, and evaluate the compliance solutions with the angle $\phi$ assigned each of these values; the averaged results is a summation of these evaluations multiplied by the weighting coefficient $1/n$.

After the above described processing, the different versions of inelastic solutions are evaluated, and the normalized results of individual compliance components $S_{1111}$, $S_{2222}$ and $S_{2211}$ are shown in Fig. 2.10 and Fig. 2.11. Two different sets of flaw statistics are applied, one with the normalized flaw size $s^* = 2s\sqrt{\eta} = 0.05$ (or equivalently, the initial damage parameter $\Omega = \eta s^2 = 6.25 \times 10^{-4}$), the other with $s^* = 0.5$ (or initially $\Omega = 6.25 \times 10^{-2}$). For Hu’s solution we let the interactive parameter $\kappa = 2$.

In addition to the compliance results, the results effective Poisson’s ratio $\nu_{12}$ is also calculated and shown in Fig. 2.12. $\nu_{12}$ interrelates the strain in the transverse direction to the load $e_2$ to the strain in $e_1$ under the uniaxial load along the $e_1$
Figure 2.10: Comparing different versions of inelastic relationship under cracked damage, initial normalized flaw size $s^* = 0.05$. 
Figure 2.11: Comparing different versions of inelastic relationship under cracked damage, initial normalized flaw size $s^* = 0.5$. 

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direction, which is given by the expression:

\[ \nu_{12} = -\frac{S_{2211}}{S_{1111}} = -\frac{\epsilon_{22}}{\epsilon_{11}} \]  

(2.121)

It’s worth noting that in anisotropic material response \( \nu_{12} \neq \nu_{21} \), for \( \nu_{21} \) is given by

\[ \nu_{21} = \frac{S_{1122}}{S_{2222}}. \]  

(2.122)

Specifically for the wing-crack problem, \( S_{1122} \) can be considered equal to \( S_{2211} \) if the friction on the flaw faces is ignored, but \( S_{2222} \) is much larger than \( S_{1111} \) due to the directional growth of wing-cracks, and thus making \( \nu_{12} \) totally different values from \( \nu_{21} \) except at the initial state.

In the results we can see that, for isotropic solutions (both classical (Kachanov’s form) and Tonge’s solutions) give the same results of \( S_{1111} \) and \( S_{2222} \), and both are changed proportional to the second order of crack length \( l \). The absolute values of \( S_{2211} \) by Tonge’s solutions also continuously increase with \( l^2 \), while the classical solutions remains constants, indicating that Tonge’s isotropic solution capably captures the dilatation of the solid due to the damage.

In the anisotropic solutions by Kachanov and Tonge, the changes of compliance in the normal direction of the crack surface (\( S_{2222} \)) are twice as much as the change of the isotropic solutions, while the compliance components along the crack direction \( S_{1111} \) generally remain constants. By Kachanov’s solution, the interaction component \( S_{2211} \)
Figure 2.12: Effective Poisson’s ratio, \( \nu_{12} = -S_{2211}/S_{1111} \).
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is also unchanged during damage, but by Tonge’s solution its magnitude increases with about the same amount as Tonge’s isotropic solution, proportional to $l^2$.

Both of our (Liu’s) solutions for tensile and compressive loads are anisotropic. The tensile solution for small normalized flaw size gives similar results with the classical anisotropic solutions, but with slightly less magnitude of the component $S_{2222}$, and increasing absolute values of $S_{1111}$ and $S_{2211}$ instead of constant. For larger normalized flaw size, the discrepancy between the counterparts of our tensile solution and the classical anisotropic one are enlarged. A closer examination on $S_{1111}$ (Fig. 2.10a and Fig. 2.11a) reveals that it starts with about the values with isotropic solutions by Kachanov and Tonge (this is reasonable since our solution is processed through numerical averaging for all flaw orientations), then diverges with an increment proportional to first order of $l$, rather than $l^2$ of the other two solutions.

The normal compliance components $S_{1111}$ and $S_{2222}$ predicted by our compressive solution changes with the magnitudes about half of the counterparts of our tensile solution, indicating a higher stiffness of the solid under compressive loads than under tensile loads. However, component $S_{2211}$ of the compressive solution changes faster than the tensile solution, indicating a stronger interactions between the two strain components and thus the bulking effect under compressive damage. This is due to the sliding displacement on the flaw surface. The effective Poisson’s ratio plotted in Fig. 2.12 also shows the rising interactions predicted by our compressive solution.
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Hu’s solution with $\kappa = 0$ can be considered the same with the classical anisotropic solution (Kachanov’s form). With a positive value of $\kappa$, the interactive term causes the magnitudes of all the compliance components increase on the base of the classical anisotropic solutions. Therefore, the change of $S_{1111}$ and $S_{2211}$ by Hu’s solution are proportional to the trace of damage and thus second order of crack length $l$. The corresponding effective Poisson’s ratio $\nu_{12}$ shown in Fig. 2.12 also rises with the crack length, indicating the transverse dilatancy increases under the uniaxial loading and damage process.

As we discussed earlier in the introduction part, although capable of capturing the features of anisotropy response and lateral dilatation, Hu’s solution predicts higher magnitudes of compliance components $S_{2222}$ than the counterparts of the classical solution for tensile load, and thus physically not reasonable of being a solution for compressive problem. By comparing with our compressive solution which is supported by the corresponding finite element modeling, Hu’s solution can be amended to better adapt to the mechanics. Some qualitative suggestions are provided here. Change of the compliance components along to the crack normals $S_{2222}$ due to the damage tensor can be reduced by half, and the interactive term controlled by $\kappa$ should be subtracted from $S_{2222}$ instead of adding to it; for the compliance components along the (wing-) crack direction ($S_{1111}$), the interactive term should be added with magnitude proportional to the square root of the trace of damage tensor $\Omega$ (thus proportional to $l$); and then the value of $\kappa$ can be tuned to fit $S_{1111}$ and $S_{2211}$. 

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2.9 Summary

The closed form expressions of the damage-compliance relationship for two-dimensional
wing-crack problem are summarized as follows:

\[
\Delta S_{1111} = \eta p_s (2s)^2 \sin \phi \cos^2 \phi (\sin \phi - \mu \cos \phi) \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}},
\]
(2.123)

\[
\Delta S_{2222} = \eta (l + 2s \sin \phi)^2 \cos \phi (\cos \phi + \mu \sin \phi) \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}} + \frac{\eta \pi l^2}{E^*},
\]
(2.124)

\[
\Delta S_{1122} = -\eta p_s (l + 2s \sin \phi) 2s \sin \phi \cos \phi (\cos \phi + \mu \sin \phi) \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}},
\]
(2.125)

\[
\Delta S_{2211} = -\eta p_s (l + 2s \sin \phi) 2s \cos^2 \phi (\sin \phi - \mu \cos \phi) \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}},
\]
(2.126)

\[
\Delta S_{1212} = \eta F(s, l, \phi) \frac{R_{nn} R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}} + \frac{\eta \pi l^2}{2E^*},
\]
(2.127)

where the function \( F(s, l, \phi) \) is defined as:

\[
F(s, l, \phi) = (s \cos 2\phi - l \sin \phi/2) \left[ 2s(\cos 2\phi + \mu \sin \phi) + \frac{2}{3}l(\sin \phi - \mu \cos \phi) \right],
\]
(2.128)

and \( p_s \) is the realignment amendment coefficient, given by:

\[
p_s(\phi) = -8.97 \times 10^{-3} \phi + 1.13.
\]
(2.129)
If the flaw size $s$ and wing-crack length are replaced by the normalized parameters:

$$s^* = 2s\sqrt{\eta}, \quad l^* = 2l\sqrt{\eta},$$

then the above expressions for the change of compliance can be re-formulated as:

$$\Delta S_{1111} = p_s s^2 \sin \phi \cos^2 \phi (\sin \phi - \mu \cos \phi) \frac{R_{nn}R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}}$$

$$\Delta S_{2222} = \left(\frac{l^*}{2} + s^* \sin \phi\right)^2 \cos \phi (\cos \phi + \mu \sin \phi) \frac{R_{nn}R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}} + \frac{\pi l^*^2}{4E_0^*}$$

$$\Delta S_{1122} = -p_s \left(\frac{l^*}{2} + s^* \sin \phi\right) s^* \sin \phi \cos \phi (\cos \phi + \mu \sin \phi) \frac{R_{nn}R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}}$$

$$\Delta S_{2211} = -p_s \left(\frac{l^*}{2} + s^* \sin \phi\right) s^* \cos^2 \phi (\sin \phi - \mu \cos \phi) \frac{R_{nn}R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}}$$

$$\Delta S_{1212} = F(s^*, l^*, \phi) \frac{R_{nn}R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}} + \frac{\pi l^*^2}{8E_0^*}$$

where the function $F(s^*, l^*, \phi)$ is defined as:

$$F(s^*, l^*, \phi) = \left(\frac{s^*}{2} \cos 2\phi - \frac{l^*}{4} \sin \phi\right) \left[s^* (\cos 2\phi + \mu \sin \phi) + \frac{1}{3} l^* (\sin \phi - \mu \cos \phi)\right].$$

$$\Delta S_{1212} = F(s^*, l^*, \phi) \frac{R_{nn}R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}} + \frac{\pi l^*^2}{8E_0^*}$$

In the above equations, $R_{nn}$, $R_{ss}$ and $R_{ns}$ are the components of a normalized crack compliance tensor $\mathbf{R}$, which is introduced to relate the applied stress on the flaw surface and the resulted displacement in the local coordinate system. Their analytical expressions will be discussed in the next chapter, with the aid of finite
A common expression $R_{comb}$ which consists of the components of $R$ and friction coefficient $\mu$ can be extracted from the above $\Delta S$ expressions:

$$R_{comb} = \frac{R_{nn}R_{ss} - R_{ns}^2}{R_{nn} + \mu R_{ns}} \approx R_{c1} - \mu R_{c2}. \quad (2.137)$$

The approximated expression of $R_{c1}$ is found to be:

$$R_{c1} \cdot E_0^* = \frac{\pi}{2} + 3l^* + \frac{0.1\cos 2\phi + 0.45}{0.07 + \sin \phi s^*} - \frac{0.1\cos 2\phi + 0.45}{0.07 + \sin \phi s^* + l^*}. \quad (2.138)$$

$R_{c2}$ can be either ignored when $\mu$ is small, or approximated by $R_{ns}$.

Tensile solution for the wing-crack model is also obtained as the intermediate step to the compressive result, in with the contact on the flaw surface is artificially removed:

$$\Delta S_{1111} = \eta (2s)^2 \cos^2 \phi (R_{nn} \cos^2 \phi + 2R_{ns} \sin \phi \cos \phi + R_{ss} \sin^2 \phi); \quad (2.139)$$

$$\Delta S_{1122} = -\eta (2s)^2 \sin \phi \cos \phi [(R_{nn} - R_{ss}) \sin \phi \cos \phi + R_{ns} (\cos^2 \phi - \sin^2 \phi)]; \quad (2.140)$$

$$\Delta S_{2211} = \Delta S_{1122}; \quad (2.141)$$

$$\Delta S_{2222} = \frac{2\eta \pi}{E_0^*} (s \sin \phi + l)^2; \quad (2.142)$$

$$\Delta S_{1212} = \frac{2\eta \pi}{E_0^*} (s + l)^2. \quad (2.143)$$
and with the normalized flaw size and crack length:

\[ \Delta S_{1111} = s^2 \cos^2 \phi (R_{nn} \cos^2 \phi + 2R_{ns} \sin \phi \cos \phi + R_{ss} \sin^2 \phi); \]  
\[ \Delta S_{1122} = -s^2 \sin \phi \cos \phi [(R_{nn} - R_{ss}) \sin \phi \cos \phi + R_{ns} (\cos^2 \phi - \sin^2 \phi)]; \]  
\[ \Delta S_{2211} = \Delta S_{1122}; \]  
\[ \Delta S_{2222} = \frac{\pi}{2E_0^*}(s^* \sin \phi + l^* \cos \phi)^2; \]  
\[ \Delta S_{1212} = \frac{\pi}{2E_0^*}(s^* + l^*)^2. \]

Comparisons with the solutions provided by other authors are also presented.

In the next chapter, finite element modeling on the wing-crack RVE will be presented to further investigate the effective compliance of brittle materials under compression. In particular, the solutions derived in this chapter will be verified.
Chapter 3

Finite Element Modeling and Solution Verification

Based on the periodic wing-crack array described in section 2.2.1 and shown in Fig. 2.4a, an \( L \times L \) region containing a single flaw is taken as an RVE for the wing-crack damaged material if periodic boundary conditions are applied on the region boundaries. The periodic displacement boundary conditions are defined by the following equations:

\[
\begin{align*}
\mathbf{u}|_{\Gamma_{43}} - \mathbf{u}|_{\Gamma_{12}} &= \mathbf{u}_4 - \mathbf{u}_1, \quad (3.1a) \\
\mathbf{u}|_{\Gamma_{23}} - \mathbf{u}|_{\Gamma_{14}} &= \mathbf{u}_2 - \mathbf{u}_1, \quad (3.1b)
\end{align*}
\]
where $u_i$ is the displacement vector at node point $i$ shown in Fig. 2.4b, and $\Gamma_{ij}$ denotes the boundaries, also shown in Fig. 2.4b. By investigating the strain response $\bar{\epsilon}$ of the unit cell containing a single flaw under an applied load $\bar{\sigma}$, the compliance tensor is evaluated.

Finite element models based on the RVE shown in Fig. 2.4b are created in Abaqus. The geometric parameters in the model scale to the flaw spacing $L$. Therefore, the normalized geometries $s^* = 2s/L$ and $l^* = 2l/L$ are treated as independent variables. For notational simplicity in the description of the parameters in the FE model, these variables are grouped into an input parameter set $V = [s^*, l^*, \phi, \mu]$.

The finite element model enables the following tasks:

**Task 1.** Based on the strain response predicted by the finite element model, calculate the components of $S$ with respect to input parameters $V$.

**Task 2.** Use the finite element models calculate $\partial u_s(\partial \bar{\sigma})$ with respect to $V$, and input the results into Eqs. (2.57) through (2.60). Compare the resulting $S$ with those obtained in Task 1 in order to verify Eqs. (2.57) through (2.60).

**Task 3.** Use the finite element model to obtain $R_{nn}$, $R_{ss}$ and $R_{ns}$ with varied input $V$. Substitute these results into Eqs. (2.89) through (2.93), and compare the resulting $S$ results with those from Task 1 in order to verify Eqs. (2.89) through (2.93).

**Task 4.** Find analytical expressions that approximately fit $R$ obtained from the finite element model in Task 3. Substitute the analytical expressions for $R$ into Eqs. (2.89) through (2.93), and compare the resulting $S$ results with those from Task 1 in
order to verify these analytical expressions.

3.1 Finite Element Model Set-up

The assembled finite element model is shown in Fig. 3.1. The model consists of separate parts that represent different sides of the flaw and wing-cracks. These parts are tied together by contact pairs at the flaw surfaces, by tie constraints above and below the wing-cracks, and untied along the length of the wing-cracks themselves. Although the periodic boundary conditions defined in Eq. (3.1) are applied on the FE model, the artificial stress localization at the boundaries can be significant when the tips of the wing-cracks are close to the model boundary, making it inconsistent with a true RVE model. In order to alleviate this inconsistency, a series of \( mL \times nL \) models were generated with \( m > 1 \) and \( n > 1 \). The results of this study showed that the \( 2L \times L \) model shown in Fig. 3.1, with two wing-crack blocks aligned in the \( x_1 \) direction, sufficiently represents an accurate RVE.

The wing-cracks are simulated by removing the Tie constraints on the designated edges, as shown in red in Fig. 3.1b. During each individual run, the length of the wing-crack \( l \) is fixed. Multiple simulations are carried out with varied \( V \) input parameter sets.

The boundary conditions are also shown in Fig. 3.1b. The periodic boundary conditions are assigned to designated nodes on the boundaries, which are shown as
Figure 3.1: Finite element model of two RVE: (a) Notations and boundary conditions. Along the flaw surfaces (denoted as $D_f$), the model assumes contact, either with or without friction. Along the wing-crack (denoted as $D_w$), the model assumes the degrees of freedom on opposite sides of the crack are not tied. Along the wing-crack lines above and below the wing-crack (denoted as $D_t$), the model assumes that the degrees of freedom are fully tied. Periodic boundary conditions for the displacement, as defined in Eq. (3.1), are assigned along the boundaries ($\Gamma$). (b) Geometries of the model, showing flaw size $2s$, flaw orientation $\phi$, wing-crack length $l$. (c) Mesh and load. Refinement near the flaw and wing-crack. (d) Typical contour plot of lateral stress $\sigma_{22}$. 

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CHAPTER 3. FINITE ELEMENT MODELING AND SOLUTION VERIFICATION

yellow circles. $u_x = 0$ is defined on the vicinity of points 1 and 4, and $u_y = 0$ is defined on the vicinity of points 1 and 2.

The model is meshed with coarse elements on the boundaries and fine elements on the flaw and crack surfaces, as shown in Fig. 3.1c. Coarse meshing on the boundaries not only reduces the calculation efforts, but also reduced the number of nodes on which the periodic boundary definitions are applied. The mesh uses triangular elements with second order shape functions. Static compressive tractions are applied on the edges $\Gamma_{12}$ and $\Gamma_{43}$ as the principal loads, while tractions with minor and perturbing magnitudes are applied onto the edges $\Gamma_{14}$ and $\Gamma_{23}$ to fulfill the tasks, which will be discussed individually in the following subsections.

In these finite element models, the material properties of Aluminium Nitride are adopted. The material properties and the range of input parameter sets applied to the FE simulations are listed in the following table.

<table>
<thead>
<tr>
<th>$E$</th>
<th>$\nu$</th>
<th>$\mu$</th>
<th>$L$</th>
<th>$\eta = 1/L^2$</th>
<th>$2s/L$</th>
<th>$2l/L$</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>300 GPa</td>
<td>0.24</td>
<td>[0, 0.7]</td>
<td>200 $\mu$m</td>
<td>$2.5 \times 10^{-7}/m^2$</td>
<td>[0.05, 0.4]</td>
<td>[0, 1]</td>
<td>[30°, 70°]</td>
</tr>
</tbody>
</table>

**Table 3.1:** Modeling parameters used in finite element model
3.2 Task 1: FE Calculation of Compliance

Since the model is two dimensional, Eq. (2.8) is reorganized into the following form:

\[
\begin{bmatrix}
\bar{\epsilon}_{11} \\
\bar{\epsilon}_{22} \\
\bar{\epsilon}_{12}
\end{bmatrix} =
\begin{bmatrix}
S_{1111} & S_{1122} & 0 \\
S_{2211} & S_{2222} & 0 \\
0 & 0 & S_{1212}
\end{bmatrix}
\begin{bmatrix}
\bar{\sigma}_{11} \\
\bar{\sigma}_{22} \\
\bar{\sigma}_{12}
\end{bmatrix}
\]

(3.2)

Natural boundary conditions are applied to the FE model, so the averaged stresses \( \bar{\sigma}_{11}, \bar{\sigma}_{22} \) and \( \bar{\sigma}_{12} \) are known. The averaged strain components \( \bar{\epsilon}_{11}, \bar{\epsilon}_{22} \) and \( \bar{\epsilon}_{12} \) are obtained from the calculated nodal displacements on the boundaries. If \( S_{2211} \) and \( S_{1122} \) are treated as two independent variables, there are four unknown parameters left to be determined by the first two individual equations in Eq. (3.2), hence at least two sets of loading definitions \( \begin{bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \end{bmatrix} \) are required to obtain one set of solutions for the normal components of the compliance matrix. A pure shear boundary condition is also used to calculate the shear compliance \( S_{1212} \). Representative normalized finite element predictions of \( S \) without friction (\( \mu = 0 \)) are shown as solid lines in Fig. 3.2, for various input parameter sets \( V \).

The modeling results confirm that the constitutive relations of wing-crack damaged material is highly anisotropic, specifically:

1. The compliance \( S_{2222} \) transverse to the primary compression increases much more
Figure 3.2: Predictions of compliance $S$. Solid lines: $S$ calculated directly from the FE model; dashed lines: $S$ predicted from Eqs. (2.57)-(2.61), using $\partial u_s/\partial \bar{\sigma}_{ii}$ obtained from FE modelling: (a) $S_{1111}, \mu = 0$; (b) $S_{1111}, \mu = 0.2$ (c) $S_{2222}, \mu = 0$; (d) $S_{2222}, \mu = 0.2$; (e) $S_{2211}, \mu = 0$; (f) $S_{1212}, \mu = 0$. 

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than $S_{1111}$ parallel to the primary compression as wing-crack length $l$ increases.

2. The compliance $S_{2211}$ also increases rapidly with crack length $l$, indicating the significant change of equivalent Poisson’s ratio, leading to volume dilatation of the material during the wing-crack damage process.

3. Shear compliance $S_{1212}$ also increases significantly with crack length $l$.

4. Friction with coefficient $\mu$ at the flaw interface leads to a moderate decrease in $S_{1111}$ relative to the case without friction.

### 3.3 Task 2: FE Calculation of Sliding Displacement and Associated Compliance

The sliding displacement on the flaw surface $u_s$ is extracted from the finite element model under multiple loading conditions and various input parameters $V$. Based on these results, $\Delta u_s/\Delta \bar{\sigma}_{ij}$ is calculated and substituted into the terms $\partial u_s/\partial \bar{\sigma}_{ij}$ which appear in Eqs. (2.57) through (2.60), providing an approximation to $S$. The corresponding solutions are shown as dashed lines in Fig. 3.2. The results show good agreement with the $S$ obtained directly from the finite element in Task 1, indicating that Eqs. (2.57) through (2.60) are reasonable predictors of the finite element results in Task 1.
3.4 Task 3: Obtaining Effective Properties Based on Finite Element Predictions of Crack Compliance

In order to obtain expressions for the crack compliance tensor $R$, the traction $P$ and the corresponding displacement discontinuity vector $u$ on the flaw surface are extracted from the finite element simulation, for a variety of input parameters $V$. The resulting predictions of $R_{nn}$, normalized by the effective moduli $E^*_0$, for orientation $\phi = 50^\circ$, are shown in Fig. 3.3. Other components of $R$ are obtained similarly. Substituting $R$ into Eqs. (2.89) through (2.93), Fig. 3.4 compares the resulting compliance $S$ (in dash-dot lines) to the compliance obtained in Task 1 (in solid lines). The agreement between these sets of results is generally very good, except for some small flaw sizes ($s^* = 0.05$) there are some overestimation in $S_{2222}$. Also discrepancies between the modeling and the analytical results at large crack length $l^*$ are observable, which are likely due to the interaction between adjacent wing-cracks, which is not accounted for in the analytical derivation of Eqs. (2.89) through (2.92).
Figure 3.3: Crack compliance component $R_{nn}$ predicted by the FE model, corresponding to a flaw orientation $\phi = 50^\circ$. 
Figure 3.4: Predictions of compliance $S$. Solid lines: $S$ calculated directly from the FE model; dash-dot lines: $S$ predicted from Eqs. (2.89)-(2.93), using $R$ obtained from FE model: (a) $S_{1111}, \mu = 0$; (b) $S_{1111}, \mu = 0$; (c) $S_{2222}, \mu = 0$; (d) $S_{2222}, \mu = 0$; (e) $S_{2211}, \mu = 0$; (f) $S_{1212}, \mu = 0$. 

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3.5 Task 4: Generating Approximate Analytical Expressions for Crack Compliance

Using expressions for the $R$ tensor obtained in Task 3, analytical approximations to $R$ are obtained in this task. Analysis of the data reveals that the values for $R_{nn}$ and $R_{ss}$ conform closely to a universal curve for a given flaw orientation $\phi$ by applying a shifted crack length defined by:

$$l_p = l^* + \kappa s^*,$$

(3.3)

where $\kappa$ is a fitting coefficient. In Fig. 3.5a and b, $R_{nn}$ and $R_{ss}$ for $\phi = 30^\circ$, $\phi = 50^\circ$ and $\phi = 70^\circ$ are analyzed and plotted against the shifted crack length $l_p$, respectively, with $\kappa = 1.5 \sin \phi$ for $R_{nn}$ and $\kappa = 0.78 \cos \phi$ for $R_{ss}$. These results show that for a model with a given flaw orientation, $R_{nn}$ and $R_{ss}$ agree well with a universal curve for a variety of flaw sizes. When the tips of the wing-cracks approach the model boundaries ($l_p \approx 1$), $R_{nn}$ and $R_{ss}$ deviate from the universal curve and become much larger. We consider that such deviation is caused by the interaction with neighbouring wing-cracks, which is not considered in the analytical expression for $R_{nn}$ and $R_{ss}$.

To confirm the results of $R_{nn}$ and $R_{ss}$, Finite element models of the wing-crack
Figure 3.5: $R_{nn}$ and $R_{ss}$ calculated from the finite element model, as a function of a shifted crack length $l_p$ defined in Eq. (3.3), for flaw orientations $\phi = 30^\circ$, 50$^\circ$, and $70^\circ$.

Figure 3.6: $R$ calculated from finite element results of rotated wing-crack model. (a) Wing-crack RVE with rotated coordinate system, $\phi = 30^\circ$, $s^* = 0.05$; periodic boundary conditions for the displacement, as defined in Eq. (3.1), are assigned to the nodes marked as circles along the boundaries; tractions (either normal or shear) applied to the flaw surface, and $R$ is calculated based on the resulted displacement. (b) Comparing $R_{nn}$ results by rotated and original wing-crack model.
RVE of a different direction is created, as shown in Fig. 3.6a. Periodic boundary conditions for the displacement, as defined in Eq. (3.1), are assigned to the RVE boundaries. Such a RVE model represents a solid with the same flaw density with the model presented in Fig. 2.4 but with a different arrangement. In such arrangement for small flaw size (such as \( s^* = 0.05 \)), when the tips of the wing-crack approach the RVE boundary there are further distances with other crack tips, therefore the interaction between the tips poses less influence to the effective properties. The modeling results of \( R_{nn} \) processed by the shifted crack length shows good agreement with the previous results (Fig. 3.6) when the crack length is small (\( l^* < 0.8 \)); and when the crack tip near the RVE boundary (\( l^* \approx 1 \)) there is no obvious increased slope. Therefore, we may conclude that the deviation of \( R_{nn} \) from the universal curve is due to the interaction of crack tips.

With the aid of the commercial software program Eureqa [84], the following expressions with respect to the normalized flaw size \( s^* \), crack length \( l^* \) and flaw orientation \( \phi \) are proposed to approximate the crack compliance components \( R_{nn} \) and \( R_{ss} \):

\[
R_{nn} E^* = \frac{\pi}{2} + s^* + 4.58 \sin \phi l^* + \frac{0.23}{0.0625 + 1.5s^* \sin \phi} - \frac{0.23}{0.0625 + l^* \tan \phi + 1.5s^* \sin \phi}
\]  

(3.4)
\[ R_{ns}E^* = \frac{\pi}{2} + s^* + 3l^* + \frac{0.085 \cos 2\phi + 0.367}{0.0625 + 0.78s^* \sin \phi} - \frac{0.085 \cos 2\phi + 0.367}{0.0625 + l^* + 0.78s^* \sin \phi} \] (3.5)

The \( R_{ns} \) results do not conform to universal curves by shifting the \( l \)-axis, and no general solution that could fit the data with high confidence could be reached. Indeed, \( R_{ns} \) highly depends on the boundary conditions and wing-crack model arrangement. An expression which could provide a good match with finite element results when the flaw sizes are relatively large (\( s^* > 0.25 \)) is proposed below:

\[ R_{ns}E^* = -\frac{0.8 \cos \phi l^*}{l^* + s^* - 0.04} - 7 \cos \phi s^* l^* \] (3.6)

We note the other extreme that if \( s^* = 0 \), \( R_{ns} = 0 \). The magnitude of \( R_{ns} \) increases with \( s^* \) and \( l^* \) but decreases with \( \phi \), and typically it is much less than \( R_{nn} \) and \( R_{ss} \). For example, given \( \phi = 50^\circ \), \( s^* = 0.27 \) and \( l^* = 0.75 \), the ratio \( |R_{ns}/R_{nn}| \) is about 0.13, and it is smaller at the earlier stage of the wing-crack development.

Substituting the \( R \) tensor into Eq. (2.89) through (2.93) provides explicit expressions for \( S \). Results of \( S_{1111} \), \( S_{2222} \) and \( S_{2211} \) by finite element modeling and analytical solutions are compared in Fig. 3.7, and of \( S_{1212} \) in Fig. 3.8, for both the cases with and without friction on the flaw surface. The analytical solutions are in good agreement with finite element modeling results for crack lengths \( l^* < 0.5 \), for some orientations and flaw sizes may even upto 0.8, when interactions begins to play an important role. Again the discrepancies at longer wing-crack lengths \( l^* \) is likely
Chapter 3. Finite Element Modeling and Solution Verification

Figure 3.7: Predictions of compliance $S$. Solid lines: $S$ calculated directly from the FE model; dotted lines: $S$ predicted from Eqs. (2.89)-(2.93), using $R$ obtained from analytical expressions Eqs. (3.4)-(3.6): (a) $S_{1111}, \mu = 0$; (b) $S_{1111}, \mu = 0.2$ (c) $S_{2222}, \mu = 0$; (d) $S_{2222}, \mu = 0.2$; (e) $S_{2211}, \mu = 0$; (f) $S_{2211}, \mu = 0.2$. 

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due to the interaction between adjacent wing-cracks, which cannot be captured by
the analytical solutions in Eqs. (2.89) through (2.93).

The presence of friction not only changes the magnitudes of individual compliance
terms, but also leads to the unsymmetry of the compliance \( S \). Examples of the off-
diagonal terms \( S_{1122} \) and \( S_{2211} \) are shown in Fig. 3.9 for different values of friction
coefficients. When \( \mu = 0 \) the magnitudes of \( S_{1122} \) and \( S_{2211} \) are identical with each
other at a given crack length; but when \( \mu > 0 \), \( S_{1122} \) and \( S_{2211} \) are highly diverse. For
both cases, our analytical solutions provides predictions with good agreement with
the FEM results.

Under uniaxial load along the \( x_1 \) direction, the effective Poisson’s ratio \( \nu_{12} \) which
represent the ratio of strain in the transverse direction to the one along loading
direction \( (\epsilon_{22}/\epsilon_{11}) \), can be evaluated by \( \nu_{12} = S_{2211}/S_{1111} \). It is an indication of the
Figure 3.9: Comparing $S_{1122}$ and $S_{1122}$ for varied friction coefficient. $\phi = 50$, $s^* = 0.27$. Analytical solution obtained from Eq. (2.91) and (2.92) using $R$ from Eqs. (3.4)-(3.6): (a) $\mu = 0$, (b) $\mu = 0.2$, (c) $\mu = 0.7$. 

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CHAPTER 3. FINITE ELEMENT MODELING AND SOLUTION VERIFICATION

Figure 3.10: Results of effective Poisson’s ratio $\nu_{12} = S_{2211}/S_{1111}$, normalized flaw size $s^* = 0.27$. $\nu_0 = 0.24$ for intact material. Solid lines: $\nu_{12}$ calculated from the results of FE model; dotted lines: $\nu_{12}$ obtained from analytical solutions. (a) $\mu = 0$, (b) $\mu = 0.2$.

3.6 Extending the Wing-Cracks Beyond the RVE

When we designed the finite element model for the wing-crack RVE, extension of the wing-crack beyond the RVE boundary to the neighbouring RVE area has been taken into account. To this end, the tie constraint along the path of the wing-cracks are released. As long as the periodic boundary displacement conditions applied to
CHAPTER 3. FINITE ELEMENT MODELING AND SOLUTION VERIFICATION

the model boundaries, the model can be considered the RVE for the whole material.

Typical contour of the stress $\sigma_{22}$ is shown in Fig. 3.11a, in which the stress intensity can be observed at the tips of the wing-cracks. Comparisons of the compliance $S$ from the FE modeling and analytical results predicted by Eqs. (2.89)-(2.93), using $R$ obtained from analytical expressions Eqs. (3.4)-(3.6) are shown in Fig. 3.11b-d. The tips of the wing-cracks cross the boundaries when the total crack length $l_n = 1$. Beyond this point, the compliance $S$ predicted by FE modeling deviate from our analytical results. Again such deviations can be considered the results of interaction between nearby wing-cracks. The deviations are not significant, and eventually stabilized as the results from the two methods show parallel curves when $l_n \approx 1.6$. Therefore, the analytical solution still provides reasonably good prediction to the effective compliance.

3.7 Applying Simplified Solution of Crack Compliance

In Section 2.5 we introduced the simplified solutions with the combined term $R_{comb}$, and proposed an approximated expression. Here compliance components predicted by the approximated $R_{comb}$ are shown in Fig. 3.12 and again compared with the results directly calculated from FE modeling. For most cases the agreement is still reasonably good.
CHAPTER 3. FINITE ELEMENT MODELING AND SOLUTION VERIFICATION

Figure 3.11: Predictions of effective compliance of the wing-cracks extended beyond the RVE boundaries. (a) Contour of $\sigma_{22}$ calculated by finite element model of extended wing-crack, $\phi = 40^\circ$, $s^* = 0.3$, $\mu = 0$; (b) $S_{1111}$; (c) $S_{2222}$; (d) $S_{2211}$. 
Figure 3.12: Predictions of compliance $S$. Solid lines: $S$ calculated directly from the FE model; dashed lines: $S$ predicted from the simplified analytical expressions using $R_{comb}$, with $R_{c1}$ predicted by Eqs. 2.138 and $R_{c2} = 0$: (a) $S_{1111}, \mu = 0$; (b) $S_{1111}, \mu = 0.2$ (c) $S_{2222}, \mu = 0$; (d) $S_{2222}, \mu = 0.2$; (e) $S_{2211}, \mu = 0$; (f) $S_{1212}, \mu = 0$. 111
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3.8 Effect of the Realignment Stage

The impacts of the wing-crack realignment on $u_s(\bar{\sigma}_{11})$ and $S_{1111}$ are illustrated in Fig. 3.13. Finite element models for moderately realigned ($p_s = 0.7$) and fully central-aligned ($p_s = 0$) wing-cracks are shown in Fig. 3.13a and b, while the results of $u_s(\bar{\sigma}_{11})$ and $S_{1111}$ are plotted in Fig. 3.13c and d. For the case of central alignment, after a small initial increase, $u_s(\bar{\sigma}_{11})$ and $S_{1111}$ reach a constant value much lower than the straight wing-crack model predicts.

These analytical solutions for $S_{1111}$ predicted by Eq. 2.104 are compared to finite element predictions for flaw orientations $\phi = 30^\circ$ and $50^\circ$, with moderate realignment in Fig. 3.14. Using the projection coefficient $p_s$, the amended expression for $S_{1111}$ improves over results that assume $p_s = 1$, providing a good fit to the true results. Similar results are found for $\Delta S_{2211}$.

The $u_s$ and $S_{1111}$ results predicted by the moderately re-aligned model lies in between. The results of $u_s$ and $S_{1111}$ predicted by Nemat-Nasser and Obata [1988] are also plotted in Fig. 3.13, which are in good agreement with the results of central aligned wing-crack model.

3.9 Summary

In this chapter, we carried out finite element modeling on the wing-crack RVE to explore the features of wing-crack damaged solids, and verified our analytical so-
CHAPTER 3. FINITE ELEMENT MODELING AND SOLUTION VERIFICATION

Figure 3.13: FE models and results for a wing-crack with realignment, $\phi = 50$, $s^* = 0.05$. (a) FE model of moderately realigned wing-crack ($p_s = 0.7$); (b) FE model of centrally aligned wing-crack ($p_s = 0$); (c) Comparing $u_s$ predicted by FE modeling for different wing-crack re-alignments and \([16]\) (d) Comparing $S_{1111}$ predicted by FE modeling for different wing-crack re-alignments analytical results.
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Figure 3.14: Comparison between normalized $S_{1111}$ found from FE model with wing-crack realignment to normalized $S_{1111}$ found from the analytical model with wing-crack realignment and with straight wing-cracks ($p_s = 1$): (a) Flaw orientation $\phi = 30^\circ$, $p_s = 0.85$; (b) Flaw orientation $\phi = 50^\circ$, $p_s = 0.7$.

...solutions derived in the previous chapter. Periodic displacement boundary conditions are applied on the region of the model boundaries. Instead of explicitly modeling the crack growth process, the development of the wing-cracks are simulated by releasing the tie constraints on the pre-defined paths of the wing-cracks. Properties such as the effective compliance, sliding displacement on the flaw, crack compliance $R$, effective Poisson’s ratio and etc., are calculated based on the FE results. Comparisons between the modeling and our analytical results show good agreement for these above mentioned properties, and thus confirms the validity of our analysis in the previous chapter. Even when the wing-crack propagate beyond the boundary of the RVE, our analytical solution can still provide reasonable results of the effective compliance.

Features of the wing-crack damaged solid are explored from. Both the results predicted by FEM and analytical solution confirm the followings: unsymmetric in
the compliance (also stiffness) tensor occurs when friction is presence in the flaw sliding; increasing values of effective Poisson’s ratio, which indicates the dilatancy or bulking during the wing-crack damage process.

Wing-crack models with the realignment stage are also created and analyzed. The central aligned wing-cracks shows constant compliance in the loading direction \( (S) \), and the compliance of the wing-crack models with moderate realignment (using the data of \( p_s \) from [10]) lies in between the results of straight wing-crack and the central aligned one. We show that the solution by [16] agrees with the central aligned wing-crack model, while our solution incorporating the realignment coefficient \( p_s \) shows agree with the more realistic moderate realigned results.
Chapter 4

Implementation of
Damage-Compliance Solutions in
Micro-mechanical Model

4.1 State of the Art: Micro-Mechanical Model for Brittle Dynamic Failure

We are interested in modeling the failure process of brittle materials (e.g., ceramic) under dynamic loading environment. Based on the wing-crack damage mechanism, Paliwal and Ramesh [36] developed a self-consistent model to tackle the interacting micro-crack damage under uniaxial compressive loads (PR model). This model suc-
CHAPTER 4. IMPLEMENTATION OF DAMAGE-COMPLIANCE SOLUTIONS IN MICRO-MECHANICAL MODEL

cessfully captured the influence of strain rate, density and size distribution of the pre-existing flaws to the mechanical response. Hu et al. (2014) \[47\] corrected an error in the solution of self-consistent scheme in the original PR model, and developed a 3D modelling framework incorporating the wing-crack damage mechanism by introducing a second order damage tensor. Katcoff and Graham-Brady (2014) \[48\] apply this 2D wing-crack model to study circular pore flaws with multiple flaw sizes. Tonge (2014) \[49\] incorporated the mechanisms of wing-crack and granular flows into the Material Point Method (MPM) for the early and latter phase of the damage process, and performed simulations of brittle material with stochastic flaw distributions under dynamic loads.

In this work we implement the damage-compliance relationship derived in Chapter 2 into the two-dimensional PR model with the corrected self-consistent scheme \([47]\). We assume all the flaws are rectilinear slits with the same orientation relative to the loading direction. Multiple flaw sizes are incorporated, while the densities and size distribution are locally perturbed with a statistical technique which will be address in the next section.

In what follows is a brief review of the PR model. The micromechanical evaluate the constitutive relationship at each integration point of a macro-scale finite element model. The self-consistent scheme incorporated in the PR model is illustrated in Fig.1.5b-d. Readers can also refer to the flowchart in Graham-Brady (2010) \[53\] for more details. The model assumes the presence of an elliptical region surrounding
CHAPTER 4. IMPLEMENTATION OF DAMAGE-COMPLIANCE SOLUTIONS IN MICRO-MECHANICAL MODEL

each individual flaw, inside of which the material is undamaged (Fig. 1.5b). Material outside of the ellipse is assigned the compliance properties accounting for the damage associated with the entire flaw population (Fig. 1.5c). The stress field $\sigma_e$ inside the ellipse is calculated using a classical elasticity solution for an elastic elliptical inclusion in an elastic matrix. The resolved stress inside the ellipse $\sigma_e$ is imposed on the individual wing-crack model to compute the resolved shear stress at the initial flaw surface $2s$ and the associated stress intensity factor induced at the crack tips (Fig. 1.5d). The crack growth rate at time step $t_{n+1}$ is governed by the Rayleigh wave speed and the critical stress intensity factor $K_{IC}$ with the following equation [85]:

$$i(t_{n+1}) = \frac{C_R}{\alpha} \left( \frac{K_I(t_{n+1}) - K_{IC}}{K_I(t_{n+1}) - K_{IC}/2} \right)^\gamma,$$  \hspace{1cm} (4.1)

where $C_R$ is the Rayleigh wave speed of the material, $\alpha$ and $\gamma$ are fitting parameters for the crack growth rate and $K_{IC}$ is the mode I critical stress intensity factor (fracture toughness) of the material. The mode I stress intensity $K_I$ at the crack tip is solved by ([14]):

$$K_I(t_{n+1}) = \frac{-2s\tau_{e,\text{eff}}(t_{n+1})\cos\phi}{\sqrt{\pi(l(t_n) + 0.27s)}} + \sigma_{22}(t_{n+1})\sqrt{\pi l(t_n)},$$  \hspace{1cm} (4.2)

in which $s$ is the half length of the pre-existing flaw (see Fig. 2.2a), $\tau_{e,\text{eff}}$ is the local shear stress resolved on the flaw surface, and $\phi$ is the angle between the normal of
The crack length is updated at every time step \( t_{n+1} \) by the following equations:

\[
l_k(t_{n+1}) = l_k(t_n) + \dot{l}_k(t_{n+1}) \Delta t.
\] (4.3)

Here the subscript \( k \) represents the crack length correspond to flaw family \( k \) associated with initial half-length \( s_k \). The global damage measure at time step \( t_{n+1} \) is denoted as \( \Omega(t_{n+1}) \). In \[74, 46, 40\], a damage measure widely used for cracked solids was adopted:

\[
\Omega(t_{n+1}) = \sum_k \eta_k l_k(t_{n+1})^2,
\] (4.4)

in which \( \eta_k \) is the area density of flaws in family \( k \). Relationships between flaw densities of an individual family \( \eta_k \) and the total flaw density of the material \( \eta_{tot} \) are expressed by:

\[
\eta_{tot} = \sum_k \eta_k,
\] (4.5)

\[
\eta_k = g(k) \eta_{tot}
\] (4.6)

where \( g(k) \) is the probability mass function (PMF) associated with flaw family \( k \).

The modulus of the damaged material follows the work of Budiansky and O’Connell \[40\], in which a linear relationship is assumed between the elastic modulus and dam-
\[ \sigma(t_{n+1}) = (E_0 - E_1 \Omega(t_{n+1})) \epsilon(t_{n+1}), \]

(4.7)

where \( E_0 \) is the modulus of undamaged material, and following [40]:

\[ E_1 = \frac{\pi^2}{30} (1 + \nu)(5 - 4\nu)E_0. \]

(4.8)

The calculation of crack lengths, global damage and global stress proceeds in the next time step \( t_{n+2} \) based on these values in the current time step \( t_{n+1} \), and continues until a pre-defined global damage limit is exceeded. Beyond this limit the calculation becomes unstable, and other physics-based mechanisms, such as granular flow, should be incorporated into the analysis (as in models by Deshpande and Evans [37] and Tonge [49]).

To briefly summarize, at every time step in the Paliwal and Ramesh model, the following evaluation tasks are carried out:

1. Evaluate the stress state inside the ellipse \( \sigma_e \) through complex variable method;

2. Evaluate the stress intensity factor and update the wing-crack length (Eqs. (4.1)-(4.3));

3. Evaluate the damage measure and the effective compliance (Eqs. (4.4)-(4.8));
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(4) Update the stress state of the matrix for the next time step (Eq. (4.7)).

While the structure of the micromechanics model is sound, there were some errors found in the implementation of the self-consistent model that required attention before proceeding with updating the model. This correction is discussed in the next section.

4.2 Erratum on Previous PR Model

This section describes changes to the micromechanics model that resulted from errors found in its implementation. While these alterations lead to quantitative changes in the resulting stress-damage-time results, it is important to emphasize that the basic approach and resulting trends remain the same. In particular, the following is presented here:

- The errors in the previous work are associated with the solution to an elliptical elastic inclusion of isotropic material in an anisotropic elastic medium. This solution is described in Appendix A of the original paper [36]; therefore, a modified version of Appendix A is given here with the changes highlighted.

- Applying the corrected micromechanical model assuming an isotropic damage model (as in Eqs. (4.7) and (4.8)) leads to unrealistic results. Therefore, an anisotropic damage model that is more readily associated with the axial splitting
failure mode, such as the one developed in the previous two chapters or some
simplified approximations, should be applied.

In the following subsections, the first bullet is addressed by showing the correction
in Section 4.2.1 followed by a finite element verification in Section 4.2.2. Section 4.3
will address the second bullet above.

4.2.1 Analytical derivation for two-phase problem

Consider a two-phase problem defined as follows. An elliptical inclusion with
isotropic elastic properties is embedded in an infinite orthotropic elastic matrix, which
is subjected to far field principal stresses $N_1$ and $N_2$, as shown in Fig. 4.1. The semi-
major and semi-minor axes of the ellipse are denoted as $a$ and $b$, respectively, which
are normalized as $ab = 1$. The matrix ($\Omega_m$) and the inclusion ($\Omega_i$) are perfectly
bonded at the interface ($\Gamma$). The solution of the stress field inside the inclusion is
obtained through the complex variable method.

Following Green and Zerna (1968) [86] in Chapter 6, we can write the coordinates
in complex space with respect to the Cartesian coordinates:

$$
\begin{align*}
  z &= x + iy, & \bar{z} &= x - iy \\
  z_i &= z + \gamma_i \bar{z}, & \bar{z}_i &= \bar{z} + \bar{\gamma}_i z
\end{align*}
$$

(4.9)
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Figure 4.1: Illustration of two-phase elliptical inclusion problem.

with

$$|\gamma_1| < 1, \quad |\gamma_2| < 1. \quad (4.10)$$

For a body with anisotropic material properties, $\gamma_1$ and $\gamma_2$ are either real or complex conjugates. They can be solved through the following characteristic equations:

$$s_{22}^2 \alpha^4 - 2(s_{22}^{11} + 2s_{12}^{12}) \alpha^2 + s_{11}^{11} = 0, \quad (4.11)$$

with the roots

$$\alpha_k = \frac{1 + \gamma_k}{1 - \gamma_k}, \quad (k = 1, 2). \quad (4.12)$$
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In Eq. 4.11, $s_{kl}^{ij}$ are the components of the planer compliance tensor $S$ of the material:

$$S = \begin{bmatrix}
  s_{11}^{11} & s_{12}^{12} & 0 \\
  s_{21}^{12} & s_{22}^{22} & 0 \\
  0 & 0 & s_{12}^{12}
\end{bmatrix} \quad (4.13)$$

So far, the derivation follows exactly that in the original PR model [36]. However, in the code used to generate results in the original paper the term $s_{12}^{12}$ was mistakenly assigned a value twice that which was stipulated in the original derivation [86]. $s_{12}^{12}$ should be given by $s_{12}^{12} = 1/4\mu = (1 + \nu)/2E$ in isotropic media.

The stress and displacement field can be solved with Airys stress function. Following Green and Zerna (1968), the Airys stress function in complex space can be expressed as:

$$\phi = \Omega(z_1) + \bar{\Omega}(\bar{z}_1) + \omega(z_2) + \bar{\omega}(\bar{z}_2), \quad (4.14)$$

where $\Omega$ and $\omega$ are called complex potentials. For anisotropic matrix materials, the displacement field $D_m$, tractions $P_m$ can be expressed as follows:

$$D_m = u_x + iu_y = \delta_1 \Omega'(z_1) + \delta_1 \bar{\Omega}'(\bar{z}_1) + \delta_2 \omega'(z_2) + \delta_2 \bar{\omega}'(\bar{z}_2), \quad (4.15)$$

$$P_m = X + iY = 2i[\gamma_1 \Omega'(z_1) + \bar{\Omega}'(\bar{z}_1) + \gamma_2 \omega'(z_2) + \bar{\omega}'(\bar{z}_2)], \quad (4.16)$$
while for the stress field in complex space $\Theta$ and $\Phi$, there are:

\[
\begin{align*}
\Theta &= \sigma_{xx} + \sigma_{yy} = 4\gamma_1\Omega''(z_1) + 4\bar{\gamma}_1\bar{\Omega}''(\bar{z}_1) + 4\gamma_2\omega''(z_2) + 4\bar{\gamma}_2\bar{\omega}''(\bar{z}_2) \\
\Phi &= \sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = -4\gamma_1^2\Omega''(z_1) - 4\bar{\Omega}''(\bar{z}_1) - 4\gamma_2^2\omega''(z_2) - 4\bar{\omega}''(\bar{z}_2)
\end{align*}
\]

The prime denotes differentiation with respect to the argument. In these expressions,

\[
\begin{align*}
\delta_1 &= (1 + \gamma_1)\beta_2 - (1 - \gamma_1)\beta_1, \\
\delta_2 &= (1 + \gamma_2)\beta_1 - (1 - \gamma_2)\beta_2 \\
\bar{\rho}_1 &= (1 + \gamma_1)\beta_2 + (1 - \gamma_1)\beta_1, \\
\bar{\rho}_2 &= (1 + \gamma_2)\beta_1 + (1 - \gamma_2)\beta_2
\end{align*}
\]

where

\[
\begin{align*}
\beta_1 &= s_{21}^{11} - s_{22}^{12}a_1, \\
\beta_2 &= s_{21}^{22} - s_{22}^{21}a_2.
\end{align*}
\]

For problems involving elliptical boundaries, it is convenient to apply a conformal transformation to map the axes of coordinates in the $z$-plane onto a circle of $|\zeta| = 1$ in the $\zeta$-plane:

\[
z = c\zeta + \frac{d}{\zeta}, \quad \zeta = e^{\xi + i\eta}
\]

in which

\[
c = \frac{a + b}{2}, \quad d = \frac{a - b}{2}.
\]
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$a$ and $b$ are the normalized longer and shorter semi-axes of the ellipse respectively, and $a \cdot b = 1$. $c$ and $d$ are real and positive and $c > d$. On the boundary of the ellipse we have $|z| = |\zeta| = 1$. Taking the two roots of Eq. 4.11 into account, two complex variables $\zeta_1$ and $\zeta_2$ are defined by the transformations:

\[
\begin{align*}
  z_1 &= z + \gamma_1 \bar{z} = (c + \gamma_1 d)\zeta_1 + (d + \gamma_1 c)/\zeta_1 \\
  z_2 &= z + \gamma_2 \bar{z} = (c + \gamma_2 d)\zeta_2 + (d + \gamma_2 c)/\zeta_2
\end{align*}
\]

in which $\zeta_k$ ($k = 1, 2$) are defined in such a way that when $|\zeta_k| \to \infty$, $|z| \to \infty$. Also, the singularities of the transformation from the $z_k$ to the $\zeta_k$ plane are inside the $|\zeta_k| = 1$ circle.

Now the complex potential functions with respect to $\zeta_k$ can be re-written as:

\[
\Omega'(z_1) = f(\zeta_1), \quad \Omega'(z_2) = g(\zeta_2)
\]

with the following form of complex potential for the matrix:

\[
f(\zeta_1) = H_1 \zeta_1 + \frac{G_1}{\zeta_1}, \quad g(\zeta_2) = H_2 \zeta_2 + \frac{G_2}{\zeta_2},
\]

where $H_1$ and $H_2$ are given by:

\[
H_1 = (B + iC)(c + \gamma_1 d), \quad H_2 = (B' + iC')(c + \gamma_2 d).
\]
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Here $B, C, B', C'$ are given by the prescribed far field stress state and can be solved by the following equation set:

$$
\begin{align*}
N_1 + N_2 &= 4\gamma_1(B + iC) + 4\gamma_1(B - iC) + 4\gamma_2(B' + iC') + 4\gamma_2(B' - iC') \\
(N_1 - N_2)e^{2i\alpha} &= -4\gamma_1^2(B + iC) - 4(B - iC) - 4\gamma_2^2(B' + iC') - 4(B' - iC')
\end{align*}
$$

(4.26)

where $N_1$ and $N_2$ are the principal stresses at infinity and $\alpha$ is the angle between $N_1$ and the $x$-axis. $G_1$ and $G_2$ in Eq. (4.24) are unknowns that will be determined from the boundary conditions.

Substituting Eqs. (4.23) - (4.26) into Eqs. (4.14) - (4.17), the displacement and traction solutions on the elliptical interface of $|\zeta| = 1$ are obtained as:

$$
\begin{align*}
D_m &= (\delta_1 H_1 + \rho_1 \bar{G}_1 + \delta_2 H_2 + \rho_2 \bar{G}_2)\zeta + (\delta_1 G_1 + \rho_1 \bar{H}_1 + \delta_2 G_2 + \rho_2 \bar{H}_2)\bar{\zeta} \\
P_m &= 2i[(\gamma_1 H_1 + \bar{G}_1 + \gamma_2 H_2 + \bar{G}_2)\zeta + (\gamma_1 G_1 + \bar{H}_1 + \gamma_2 G_2 + \bar{H}_2)\bar{\zeta}] 
\end{align*}
$$

(4.27) (4.28)

Until this point, the equations and derivations have followed closely the original PR model as discussed in the appendix of [36]. However, the original version of the derivation treated the inclusion as orthotropic, when it is meant to be isotropic. This leads to more significant changes to the resulting equations for the inclusion, which are derived next. For the isotropic material inside the inclusion, the two roots of Eq.
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are real and identical, so Eq. (4.20) and (4.21) are sufficient for the conformal transformation. Applying subsequent equations with real and identical roots leads to erroneous results. Therefore, these equations must be re-derived in a manner that is consistent with the isotropic material. At the elliptical interface the three circles $|\zeta_1| = 1, |\zeta_2| = 1$ and $|\zeta| = 1$ coincide and are given by:

$$\zeta_1 = \zeta_2 = \zeta = e^{i\eta}, \quad \bar{\zeta} = 1/\zeta$$

(4.29)

The form of the displacement and traction solutions within the elliptical inclusion differ from the anisotropic case shown in Eq. (4.14) and (4.15). Referring to Chapter 8 of Green and Zerna (1968), these are:

$$\mu D_i = \kappa \Omega(z) - z\bar{\Omega}'(\bar{z}) - \bar{\omega}'(\bar{z}),$$

(4.30)

$$P_i = 2i[ze^{i\eta}(\bar{z}) + \Omega(z) + \bar{\omega}'(\bar{z})].$$

(4.31)

In the displacement solution $D_i$, $\mu = E/2(1+\nu)$ is the shear modulus. We have

$$\kappa = 3 - 4\nu$$

(4.32)
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for plane strain, and

\[ \kappa = \frac{3 - \nu}{1 + \nu} \quad (4.33) \]

for plane stress.

According to Eshelby’s theory the stress fields are uniform inside the elliptical inclusion, so the complex potential solutions can be written as:

\[ \Omega(z) = A_1 z, \quad \Omega'(z) = A_1, \quad \bar{\omega}'(\bar{z}) = A_2 \bar{z}. \quad (4.34) \]

Substituting the above equations into Eq. (4.30) and (4.31) and applying the conformal transformation (4.20):

\[ \mu D_i = [(\kappa - 1) A_1 c - A_2 d] \zeta + [(\kappa - 1) A_1 d - A_2 c] \bar{\zeta} \quad (4.35) \]

\[ P_i = 2i[(2A_1 c + A_2 d) \zeta + (2A_1 d + A_2 c) \bar{\zeta}] \quad (4.36) \]

In order to determine the complex potentials, the following boundary conditions are applied:

1. The far field principal stresses are prescribed as \( N_1 \) and \( N_2 \);

2. Across the elliptical interface between the matrix and inclusion, the continuity condition of displacement and tractions should be satisfied, i.e. \( D_m = D_i \) and \( P_m = P_i \) at \( |\zeta_1| = |\zeta_2| = |\zeta| = 1 \).
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Condition 1 has been addressed through Eq. (4.26), which leads to the solutions of $H_1$ and $H_2$ through (4.25). From condition 2 we can establish traction and displacement continuity on the interface to solve:

$$\gamma_1 H_1 + \bar{G}_1 + \gamma_2 H_2 + \bar{G}_2 = 2A_1 c + A_2 d, \quad (4.37a)$$
$$\gamma_1 G_1 + \bar{H}_1 + \gamma_2 G_2 + \bar{H}_2 = 2A_1 d + A_2 c, \quad (4.37b)$$
$$\delta_1 H_1 + \rho_1 \bar{G}_1 + \delta_2 H_2 + \rho_2 \bar{G}_2 = \frac{\kappa - 1}{\mu} A_1 c - \frac{A_2}{\mu} d, \quad (4.37c)$$
$$\delta_1 G_1 + \rho_1 \bar{H}_1 + \delta_2 G_2 + \rho_2 \bar{H}_2 = \frac{\kappa - 1}{\mu} A_1 d - \frac{A_2}{\mu} c, \quad (4.37d)$$

where $A_1$ and $A_2$ are either real or complex. The above equation set contains four equations and four unknown complex variables $G_1, G_2, A_1, A_2$ and can be easily solved. For convenience we take the conjugate form of the first and third equation while solving the equation set. Once $A_1$ and $A_2$ are known, according to Section 8.1 of Green and Zerna (1968) the stress field in the inclusion can be obtained by:

$$\sigma_{rr} + \sigma_{\theta\theta} = 8A_1, \quad (4.38a)$$
$$\sigma_{rr} - \sigma_{\theta\theta} + 2i\sigma_{r\theta} = -4A_2. \quad (4.38b)$$

Considering the $(x, y)$ directions as specified in Fig. 4.1 stresses in polar coordinates are transformed to Cartesian coordinates assuming that $\theta = 0$. Solving Eqs. (4.38) and applying the coordinate transformation leads to expressions for the
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uniform values of $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$ in the inclusion:

$$\sigma_{xx} = \text{real}(4A_1 - 2A_2),$$
$$\sigma_{yy} = \text{real}(4A_1 + 2A_2),$$
$$\sigma_{xy} = -\text{imag}(2A_2).$$

(4.39a)  
(4.39b)  
(4.39c)

To briefly summarize, the primary modifications to Appendix A of [36] are found in two parts:

1. Adopting the correct value of $s_{12}^{12}$ in Eq. (4.11). The correct value should be $1/4\mu$ which is in accordance with Green and Zerna (1968) [86], while in the [36] the value $1/2\mu$ was applied. The incorrect value of $s_{12}^{12}$ leads to incorrect solutions of $\gamma_1$ and $\gamma_2$ and thus affect the entire solution for the complex potential.

2. In the work of [36] the anisotropic solution of complex potential was applied to the matrix as well as to the inclusion, even though the inclusion was assumed to be an isotropic material. However, the equations corresponding to an anisotropic material cannot be solved with isotropic material property input, because the roots of Eq. (4.11) are real and identical in this case. Therefore, in the updated model the isotropic solutions are applied to the inclusion in order to enforce traction and displacement continuity at the interface.
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4.2.2 Verification through finite element analysis

In order to verify the modifications to the analytical derivations described in the previous subsection, finite element models are created in Abaqus based on the conceptual two-phase model set-up, as shown in Fig. 4.2, with $a$ and $b$ the longer and shorter semi-axis respectively. Symmetric boundary conditions are defined on the left and bottom edges of the assembly. The matrix and inclusion are connected with the "Tie" constraint on the interface $\zeta$, therefore the continuities of displacement and traction across the interface are satisfied.

Quadratic element type CPS6M is applied to the whole model. Elements in the inclusion are meshed with a much finer size than in the matrix to achieve good accuracy.

The size of the matrix $L$ is much larger than $a$ and $b$ so that tractions applied on the matrix boundaries can be considered far-field stresses. The assembly is loaded with uniaxial compressive pressure $N_1$ on the right boundary, and thus $N_1$ is aligned with the longer semi-axis $a$.

According to Eshelby’s theory [87], for such two-phase problem with damaged elastic moduli in the matrix, the stress fields inside the elliptical inclusion are homogeneous. This is confirmed by our modeling results (Fig. 4.3).

The moduli in the matrix evolve with a damage parameter $\Omega$ following the expressions from Budiansky and O'Connell 1976 [40]:

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Figure 4.2: Finite element model for verifying complex variable method derivation

Figure 4.3: Homogeneous stress field within the elliptical inclusion
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\[
E(\Omega) = E_0^* \left[ 1 - \frac{\pi^2}{30} (1 + \nu_0)(5 - 4\nu_0)\Omega \right]
\]

\[
G(\Omega) = \frac{E_0^*}{2(1 + \nu_0)} \left[ 1 - \frac{\pi^2}{60} (10 - 7\nu_0)\Omega \right]
\]

in which \(E_0^* = E_0\) for plane stress and \(E_0^* = E_0/(1 - \nu_0^2)\) for plane strain, with \(E_0\) and \(\nu_0\) the elastic modulus and the Poisson’s ratio of the intact material. \(\Omega = \eta l^2\) is the damage measure, with \(l\) the half length of wing-crack, as shown in Fig. 2.2.

These relations are applied to the moduli in different directions separately, therefore softening the matrix material in different ways. For each case, a series of finite element analyses are conducted with varied damage magnitude \(\Omega\). The corresponding stress \(\sigma_{11}^i\) and \(\sigma_{22}^i\) inside the inclusion are identified, and compared against the analytical results given by the old and new solutions of complex variable method (CVM). Based on ten different cases of moduli definitions and various aspect ratios of the elliptical inclusion, the results all agree very well with the new complex variable solutions described in the previous subsection. Results for all the cases are given in the Appendix, but two representative cases are demonstrated and discussed in what follows.

- Case 1: Isotropic elastic modulus: \(E_1 = E_2 = E(\Omega)\), shear modulus \(G = G(\Omega)\), Poisson’s ratio constant \(\mu = \mu_0\).

This is the constitutive relationship adopted by Paliwal and Ramesh [36]. Apply-
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ing the previous erroneous complex variable solutions, the stress $\sigma_{22}^i$ is predicted to be positive, indicating that a tensile stress arises in the direction transverse to the far-field uniaxial loading direction. In the context of dynamic compressive failure of brittle material, this would indicates that the crack growth process is accelerated by softening of the material at the macro-scale, which matches experimental observations. However, FE analysis and the corrected complex variable solutions predict a compressive stress transverse to the loading direction. Therefore, if the a numerical model adopts the correct complex variable solution based on an isotropic damage-moduli relationship, the crack growth process would be suppressed by a transversely compressive stress, which is inconsistent with observations of failures. Given preferential crack growth directions under compressive loading, it is more accurate to treat damage and modulus as anisotropic quantities.

\[ \sigma_{22}^i \]

\[ \Omega \]

\[ \sigma_{11} \]

\[ \sigma_{22} \]

\[ \text{FEM results, } a:b=1 \]

\[ \text{FEM results, } a:b=3 \]

\[ \text{Original CVM solutions, } a:b=1 \]

\[ \text{Original CVM solutions, } a:b=3 \]

\[ \text{Corrected CVM solutions, } a:b=1 \]

\[ \text{Corrected CVM solutions, } a:b=3 \]

\[ 0 0.1 0.2 0.3 0.4 0.5 \]

\[ -2 -1.9 -1.8 -1.7 -1.6 -1.5 -1.4 -1.3 -1.2 -1.1 -1 \]

\[ \text{Damage } \Omega \]

\[ \text{Normalized stress } \sigma_{11}^i \]

\[ \text{Normalized stress } \sigma_{22}^i \]

\[ 0 0.1 0.2 0.3 0.4 0.5 \]

\[ -0.1 -0.05 0 0.05 0.1 0.15 0.2 0.25 \]

\[ \text{Damage } \Omega \]

\[ \text{Normalized stress } \sigma_{22}^i \]

\[ \text{FEM results, } a:b=1 \]

\[ \text{FEM results, } a:b=3 \]

\[ \text{Original CVM solutions, } a:b=1 \]

\[ \text{Original CVM solutions, } a:b=3 \]

\[ \text{Corrected CVM solutions, } a:b=1 \]

\[ \text{Corrected CVM solutions, } a:b=3 \]

**Figure 4.4:** Stress in the inclusion under a far-field stress of $\sigma_{11} = 1$, assuming isotropic damage-modulus relationship (case 1). Transverse stress $\sigma_{22}^i$ in the inclusion is shown to be compressive for this case, not tensile as predicted by the previous solution. CVM is short for complex variable method.

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• Case 2: Anisotropic elastic modulus: \( E_1 = E_0, \ E_2 = E(\Omega) \), shear modulus \( G = G(\Omega) \), Poisson’s ratio \( \mu_{12} = \mu_0 * E_1/E_2 \).

In this case the elastic modulus in the loading direction is assumed to be unchanged while it is reduced in the transverse direction. Poisson’s ratio is assumed to be increasing with damage. This is somewhat similar to the anisotropic damage-compliance relations derived in Chapter 2 of this dissertation. The old CVM solution predicts compressive stress transverse to the loading direction, while FE analysis and the updated solution predict tensile stress. Therefore, the crack growth process would be accelerated in a numerical model that adopts the correct CVM solution with an anisotropic damage-stiffness (or compliance) relationship.

![Figure 4.5](image)

**Figure 4.5:** Stress in the inclusion under a far-field stress of \( \sigma_{11} = 1 \), assuming anisotropic damage-modulus relationship (case 2). Transverse stress \( \sigma_{22} \) in the inclusion is shown to be tensile for this case, while compression was predicted by the previous solution.

Based on the finite element modeling results, we may conclude that the updated complex variable solutions derived in the previous subsection accurately predict the...
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stress inside the elliptical inclusion for the two-phase problem (see Fig. 4.1). Furthermore, the anisotropic damage-compliance relationship is critical to the micro-mechanical modeling (PR model). Attempts on micro-mechanical modeling incorporating anisotropic damage-compliance will be discussed in the following sections.

4.3 Implementation of Simple Anisotropic Effective Compliance

The damage-compliance relationship applied in Paliwal and Ramesh (PR) model [36] are:

\[
S_{1111}(\Omega) = S_{2222}(\Omega) = \frac{1}{E(\Omega)},
\]
\[
S_{1122}(\Omega) = -\frac{\nu_0}{E(\Omega)},
\]
\[
S_{1212}(\Omega) = \frac{1}{G(\Omega)},
\]

with \(E(\Omega)\) and \(G(\Omega)\) taken from the Budiansky and O’Connell’s [40] isotropic solution, as shown in Eqs. (4.40) and (4.41). Applying these relations to the original (incorrect) self-consistent scheme leads to tensile stress predictions in the direction transverse to loading (Fig. 4.4), which promotes the Mode I stress intensity factor at the wing-crack tip, and therefore triggers a positive feedback between the growth of wing-cracks.
and reduction of stiffness in the overall material. This previous model successfully reproduces features that are found in the failure of brittle material under dynamic loads, including the following:

- Rapidly reduced stiffness when the internal damage is sufficiently large, leading to post-peak strain softening behaviour (negative slope of stress-strain curve after peak stress is observed);
- Strain rate dependence of the strength, specifically, higher applied strain rate leads to higher peak stress;
- Profound impact of initial flaw statistics on mechanical behaviour: higher flaw density and/or larger flaw size lead to lower peak stress.

With the capability of capturing the above physical phenomena and the merit of being a fully micromechanics-based model (in other words, the model parameters are physically meaningful rather than empirical fits), the PR model is favourable in simulating brittle dynamic failure and has been widely used in a variety of scenarios.

The previous subsection pointed out that the corrected self-consistent scheme leads to unrealistic results when implementing an isotropic compliance solution. We re-run the original and updated PR model applying the isotropic compliance-damage relationship (Eq. 4.40) assuming the material properties and flaw statistics listed in Table 4.1. The stress-strain results predicted from both the original and the updated PR model are plotted in solid black and dashed blue line, respectively, in Fig. 4.6. It
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is clear that using the corrected model, the isotropic damage-compliance relationship does not predict the expected results described earlier in this section.

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<td>$E$</td>
<td>$\nu$</td>
<td>$\mu$</td>
<td>$\rho$</td>
<td>$K_{IC}$</td>
<td>$\tau_e$</td>
<td>$\alpha, \gamma$</td>
</tr>
<tr>
<td>460 GPa</td>
<td>0.2</td>
<td>0.40</td>
<td>2300 kg/m$^3$</td>
<td>2.52 MPa$\sqrt{m}$</td>
<td>0</td>
<td>1, 1</td>
</tr>
<tr>
<td>$2s$</td>
<td>$\eta$</td>
<td>$\phi$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50 $\mu$m</td>
<td>$10^8$</td>
<td>50.7$^\circ$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1: Material properties (ceramic B4C) and flaw statistics for Paliwal-Ramesh model.

The failure of the original set-up can be reasoned as follows. As shown in the study of the matrix-inclusion problem in the previous section, under uniaxial compression, assuming an isotropic compliance in the damaged material leads to a transverse compressive stress state in the inclusion (Fig. 4.4), therefore the growth of wing-cracks is suppressed or even fully prohibited. As damage growth is suppressed, the stress continues to grow indefinitely.

Based on the study in the previous subsection, implementation of an anisotropic compliance-damage relation leads to tensile stress predicted in the transverse loading direction (Fig. 4.4), and therefore predicts crack growth promotion along the loading direction, eventually leading to axial splitting as observed in experiments. As a first step to address anisotropy, a simple damage-compliance relationship as expressed
CHAPTER 4. IMPLEMENTATION OF DAMAGE-COMPLIANCE SOLUTIONS IN MICRO-MECHANICAL MODEL

Figure 4.6: Comparison of PR model with original and corrected self-consistent scheme, implementing simple and micro-mechanical compliance-damage relations for the matrix. Material properties and flaw statistic applied are listed in Table 4.1.
CHAPTER 4. IMPLEMENTATION OF DAMAGE-COMPLIANCE SOLUTIONS IN MICRO-MECHANICAL MODEL

below is implemented:

\[
S_{1111}(\Omega) = \frac{1}{E^*_0} \left[ 1 - \alpha_f \frac{\pi^2}{30} (1 + \nu_0)(5 - 4\nu_0)\Omega \right]^{-1} \tag{4.45a}
\]

\[
S_{2222}(\Omega) = \frac{1}{E^*_0} \left[ 1 - \frac{\pi^2}{30} (1 + \nu_0)(5 - 4\nu_0)\Omega \right]^{-1} \tag{4.45b}
\]

\[
S_{1122}(\Omega) = -\nu_0 S_{2222}(\Omega) \tag{4.45c}
\]

\[
S_{1212}(\Omega) = \frac{2(1 + \nu_0)}{E^*_0} \left[ 1 - \frac{\pi^2}{60} (10 - 7\nu_0)\Omega \right]^{-1} \tag{4.45d}
\]

The compliance transverse to loading direction \(S_{2222}\) remains the same as the isotropic form (Eq. (4.40)), while for the modulus along the loading direction \(S_{1111}\) the damage term is confined by a coefficient \(\alpha_f\) with the value between 0 and 1. When \(\alpha_f = 1\), the whole constitutive solution falls back to the isotropic form. When \(\alpha_f = 0\), \(S_{1111}\) never increases with damage, and therefore the predicted stress-strain curve is a straight line until abrupt failure occurs when the wing-cracks are initiated. Modeling results applied with a range of values of \(\alpha_f\) are shown in Fig. 4.7. Reasonable results are found when \(0.1 < \alpha_f < 0.6\), although an intermediate value of 0.3 provides results more accurate with those observed in experiments. By implementing an anisotropic damage-compliance relationship, the features of the rate-dependent stress-strain curve are preserved. Therefore, anisotropic damage-compliance relationship is not only physically reasonable for capturing the preferential growth direction of the wing-cracks, but it is also required for the PR model to provide physically reasonable results.
Figure 4.7: Comparison of PR model with corrected complex variable method (CVM) solutions, implementing simple compliance-damage relations with varied $\alpha_f$ in (4.45). Material properties and flaw statistics applied are listed in Table 4.1.
CHAPTER 4. IMPLEMENTATION OF DAMAGE-COMPLIANCE SOLUTIONS IN MICRO-MECHANICAL MODEL

This simple anisotropic damage-compliance relationship is for damage parameter $\Omega < 0.6$; within this range, the components of the stiffness matrix (the inverse of the compliance matrix) monotonically decrease. However, with $\Omega > 0.6$, although the magnitudes of each compliance term continues to grow, some stiffness components derived from the inverse of the compliance tensor actually increase, making the material property spurious. Solutions to this issue could include adding further assumptions to ensure a mathematically feasible damage-compliance relationship, as well as investigating the damage-compliance solutions based on the wing-crack micro-mechanics, as previous authors did [10, 11, 12, 16]. Indeed, the need of a physics-based anisotropic damage-compliance relationship motivated our investigation in Chapter 2 and 3 of this thesis.

Implementing the micromechanical damage-compliance solutions derived in Chapter 2, assuming the parameters given in Table 4.1, predictions of stress-strain curve are generated and are plotted in the red solid curve in Fig. 4.6. Given appropriate flaw statistics and loading rate, our solution provides similar results to the original PR model, and also reserve features such as sensitivity to applied strain rate and flaw statistics. Building on this success, predictions from this fully corrected and updated micro-mechanical model are evaluated for various parameters and multiple flaw size in the next section.
4.4 Implementation of Analytical Anisotropic Effective Compliance

An analytical damage-compliance relationship has been derived in Chapter [2]. In this section the derived closed form expressions of $\Delta S$ with respect to the wing-crack geometry parameters are implemented into the PR model using the corrected self-consistent scheme. Again for simplicity, implementations in this section adopt a uniform flaw orientation with $\phi = 50.7^\circ$, which is a preferential orientation to generate wing-cracks. The analysis is easily extended to a distribution of flaw orientations, but the results do not change appreciably. The friction coefficient on the flaw surfaces is 0.4.

Recall the end of Section 4.1, where the Paliwal and Ramesh model was summarized and decomposed into four tasks at each time step. Implementation of the newly derived damage-compliance relationship, only Task (3) will be affected, while the other part of the PR model is generally the same. In our analytical solutions, changes of compliance $\Delta S$ are no longer functions of the damage measure $\Omega$ evaluated by Eq. 4.4, but functions with respect to the normalized flaw size $s^*$ and wing-crack length $l^*$ through Eqs. (2.130)-(2.130). But for the rest three tasks, the actual flaw size $s$ and crack length $l$ should still be applied to evaluate the parameters, particularly the stress intensity factor $K_I$ in Eq. (4.2).

The challenge lies on finding an appropriate way to evaluate $s^*$ and $l^*$ for the
compliance $S$. If a single flaw size is implemented, it is straightforward to evaluate the material response. The normalized flaw size $s^* = 2s\sqrt{\eta}$ can be evaluated before the scheme begins; at each time step the normalized length of the wing-crack $l^*(t_n) = 2l(t_n)\sqrt{\eta}$ is calculated. Substituting $s^*$ and $l^*(t_n)$ into the Eqs. \[(2.131)\text{ through } (2.136)\] (let $p_s = 1$), $\Delta S$ for time step $t_n$ can be evaluated.

For the case of multiple flaw sizes, evaluations of the normalized flaw and crack length $s^*$ and $l^*$ should be consistent with the total spacing of flaws. For example, consider a material that contains two families of flaws, with sizes $s_1$ and $s_2$ and with flaw densities $\eta_1$ and $\eta_2$, respectively. The following consistency condition should be satisfied: if $s_1 = s_2$ but still treated as two families (particularly, the lengths of the wing-crack are evaluated separately), the modeling result should be identical to the case of a single flaw size with density $\eta = \eta_1 + \eta_2$.

To satisfy this consistency condition, a universal normalized flaw size is evaluated as follows:

$$s^* = 2\sqrt{\sum_k \eta_k s_k^2} \quad (4.46)$$

This universal normalized flaw size is substituted into the compliance equations for all the flaw families when evaluating their own contribution of increment of the compliance $\Delta S_k$. In the context of self-consistent scheme of two-phase problem, the normalized flaw size represents the ratio of a flaw within its own control region;
therefore, such a measurement of universal normalized flaw size \( s^* \) reflects a varied size of elliptical region adaptive to the actual flaw size of each family. The semi-axes of the ellipse surrounding the flaws are proportional to the actual flaw size \( s \), so that the same normalized flaw size \( s^* \) for all the flaw families is obtained.

Similar to the wing-crack damage measure, initial damage parameters for individual flaw families and the overall system are introduced with the following expressions:

\[
\omega_k = \eta_k s_k^2, \quad \omega = \sum_k \eta_k s_k^2 = \sum_k \omega_k = s^2 / 4
\] (4.47)

The fraction of initial damage by the individual flaw families is given by:

\[
\zeta_k = \frac{\eta_k s_k^2}{\sum_k \eta_k s_k^2} = \frac{\omega_k}{\omega}
\] (4.49)

The total compliance increment is then calculated by the summation of the individual compliance increment \( \Delta S^{(k)} \) from all the flaw families multiplied by their own initial damage fractions:

\[
\Delta S = \sum_k \zeta_k \Delta S^{(k)}.
\] (4.50)
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The normalized crack length of each flaw family is now calculated by:

\[ l^* = l_k \frac{s^*}{s_k} = 2 \sqrt{\sum_k \eta_k s_k^2 l_k / s_k} \]  \hspace{1cm} (4.51)

Consider again the consistency condition described earlier, for the dual flaw family system with individual flaw sizes \( s_1 \) and \( s_2 \), densities \( \eta_1 \) and \( \eta_2 \). If \( s_1 = s_2 = s \), then the normalized flaw size \( s^* = 2s^2 \sqrt{\eta_1 + \eta_2} = 2s^2 \sqrt{\eta} \) and the normalized wing-crack length \( l^*_1 = l^*_2 = 2l \sqrt{\eta} \), both of which are consistent with the single flaw family.

During the modeling, the updated length of wing-crack is calculated through the stress intensity factor \( K_I \) using Eq. (4.2) in which the actual flaw size and wing-crack length are applied. Therefore, although different flaw families may present the same normalized flaw size \( s^* \) and normalized wing-crack length \( l^* \) at some instance, the growth rate of wing-crack still depends on the actual flaw size \( s \) and wing-crack length \( l \); specifically, the flaw family with larger flaw size gives a faster growth rate of wing-cracks. \( \zeta_k \) denotes the ratio of control area of individual flaw families in the whole system.

With \( s^* \) and \( l^*_k \) substituted into Eqs. (2.131) through (2.136) or Eqs. (2.144) through (2.148), the increment of compliance \( \Delta S_k \) for compressive or tensile loads contributed by the flaw family \( k \) can be evaluated. Then the total change of compliance is evaluated by Eq. (4.50). Consider as an example, the tensile solution of
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$\Delta S_{2222}$, which is expressed in Eq. (2.147), and let $l^*_k = 0$ for the initial stage:

$$
\Delta S_{2222} = \sum_k \zeta_k \Delta S^{(k)}_{2222} = \sum_k \eta_k \frac{s_k^2}{s^2} \Delta S^{(k)}_{2222}
$$

\[ \text{(4.52a)} \]

$$
= \sum_k \eta_k \frac{s_k^2}{s^2} \left[ \frac{\pi}{2E^*_0} \left( s^* \sin \phi_k \right)^2 \right]
$$

\[ \text{(4.52b)} \]

$$
= \frac{\pi}{2E^*_0} \sum_k \eta_k s_k^2 \sin^2 \phi_k
$$

\[ \text{(4.52c)} \]

$$
= \frac{\pi}{2E^*_0} \sum_k \omega_k \sin^2 \phi_k
$$

\[ \text{(4.52d)} \]

which is exactly the Kachanov’s 2D tensile solution of cracked solids, as shown in Eqs. (2.28) and (2.29). When damage occurs and $l^*_k$ become non-zero, there is:

$$
\Delta S_{2222} = \sum_k \zeta_k \Delta S^{(k)}_{2222} = \sum_k \eta_k \frac{s_k^2}{s^2} \left[ \frac{\pi}{2E^*_0} \left( s^* \sin \phi_k + l^*_k \right)^2 \right]
$$

\[ \text{(4.53a)} \]

$$
= \sum_k \eta_k \frac{s_k^2}{s^2} \left[ \frac{\pi}{2E^*_0} \left( s^* \sin \phi_k + s^* \frac{l_k}{s_k} \right)^2 \right]
$$

\[ \text{(4.53b)} \]

$$
= \frac{\pi}{2E^*_0} \sum_k \eta_k (s_k \sin \phi_k + l_k)^2.
$$

\[ \text{(4.53c)} \]

which recovers the Eq. (2.142). For most other compliance components, the normalized flaw size $s^*$ and crack length $l^*$ are required in order to predict the crack compliance tensor $R$ and thus the change of compliance, rather than simply providing an alternative way of evaluation of $S$. Eqs. (4.46) through (4.51) provide the evaluation of effective compliance which is in consistency with the classical Kachanov’s solutions.

Although they appear to be complicated at first glance, $s^*$ and $\zeta_k$ are evaluated only
CHAPTER 4. IMPLEMENTATION OF DAMAGE-COMPLIANCE SOLUTIONS IN MICRO-MECHANICAL MODEL

once prior to micro-mechanical modeling, and thus only \( l_k^* \) is updated at every time step, which is straightforward using Eq. (4.51).

Although no longer needed for evaluating the mechanical response of the damaged material, the damage parameter \( \Omega \) still provides a useful measure of the damage associated with the wing-cracks, which helps in comparing the damage contribution by different flaw families. The wing-crack damage parameters for individual flaw families are measured following the conventional equation:

\[
\Omega_k = \eta_k l_k^2 = \frac{\eta_k s_k^2}{s^*^2} l_k^*^2 = \zeta_k l_k^*^2 / 4
\]  
(4.54)

The total wing-crack damage parameter is calculated through the summation:

\[
\Omega = \sum_k \Omega_k = \sum_k \eta_k l_k^2 = \sum_k \zeta_k l_k^*^2 / 4.
\]  
(4.55)

Results of stress versus strain for single and dual-flaw families are shown in Fig. 4.8, 4.9 and 4.10, with the material parameters listed in the first row in Table 4.1. In case 1 - 3 (Fig. 4.8, 4.10a), single and dual flaw size systems with the same normalized flaw size \( s^* \approx 0.5 \) are tested, while in case 4 (Fig. 4.10b), the flaw size equals the statistical averaged of the case 1 and 2. Comparing with larger flaw size by lower flaw density, a system with smaller flaw size with higher density gives a higher strength of the material, which is consistent with the fracture mechanics theories, and also in agreement with the original PR model.
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Three levels of loading strain rates are applied in the models. Strain rate sensitivity are observed for all the given flaw statistics: at lower strain rates (e.g., \( \dot{\epsilon} = 10^3 \text{s}^{-1} \)), the material exhibits a lower strength, which is followed by an abrupt decrease of stress-strain slope, indicating more brittle failure. At intermediate strain rate (e.g., \( \dot{\epsilon} = 10^4 \text{s}^{-1} \)), the strength of the material is higher, and the strain softening response after the peak stress is relieved. When the strain rate is sufficiently high, the stress continues to grow with no peak stress observed.

In other words, for the micro-mechanical model using analytical material properties, there’s a upper limit of applicable strain rate for a peak stress to be observed; if the applied strain rate exceed this limit, the stress in the material would continuously increase to some unrealistic value, even if the damage parameter is allowed to grow without limit. This is different from the original PR model or the corrected model with simple anisotropic constitutive relations, in which the peak stress can always be reached even if excessive high strain rate (e.g., \( 10^6 \text{s}^{-1} \)) is applied.

Such a limit is due to the insufficient increment of compliance while competing with the increment of applied load. In our analytical solutions of mechanical response, the normal compliance in the loading direction (\( \Delta S_{1111} \) in Eq. (2.131)) is proportional to the first order of the wing-crack length \( l^1 \); but the isotropic solution by Budiansky and O’Connell [40] and its simple anisotropic variation (Eq. (4.45)) are proportional to \( l^2 \). Therefore, incorporating the analytical mechanical response that we developed,
CHAPTER 4. IMPLEMENTATION OF DAMAGE-COMPLIANCE SOLUTIONS IN MICRO-MECHANICAL MODEL

the material does not soften faster than the stress increment under high strain rate and thus is unable to provide the peak stress and negative slop in the stress-strain relationship after the peak; but it does when the other two mechanical responses are applied.

The strain limit is also related to the flaw size in the material. As shown in Fig. 4.8, for smaller flaw sizes, a high strain rate can be applied and peak stress is still observed, while for larger flaw size the same strain rate leads to continuous increment of stress.

Comparing with a single flaw size system, a multiple flaw size system is capable to capture larger variation of applied strain rate, and shows higher strain rate sensitivity of strength. This can be explained by the activation of different sizes of flaws under different applied strain rate, as shown in the plot of corresponding wing-crack damage parameter in Fig. 4.9. For lower strain rate ($\dot{\epsilon} = 10^3 \text{s}^{-1}$), the total wing-crack damage is almost completely contributed by the larger flaw family ($s = 18 \mu m$), while the portion from small flaw family is negligible. Therefore, the material behaves as if single flaw size system with only larger size of flaws. For higher strain rate ($\dot{\epsilon} = 10^4 \text{s}^{-1}$), damage from larger flaws still rises first, but as the load continues the portion from smaller flaw family also increases. When the modeling stopped at the prescribed limit of total damage, the damage portions from the individual flaw families are approximately equivalent. Since both the flaw families are activated, the material behaves as if averaged of two individual single flaw systems. Therefore, in order to
better capture the response of flaws with different sizes under varied strain rate, a
distribution of flaw sizes is necessary. A full distribution of the flaw size obtained
from microscopic characterization of the material sample should be implemented in
the micro-mechanical modeling, so that better physical reality can be achieved.

![Figure 4.8: Comparing stress-strain responses of different flaw size input, identical
normalized flaw size ($s^* = 0.45$) and initial damage ($\omega = 0.05$), varied strain rate.
Case 1: actual flaw size $s_1 = 18\mu m$, $\eta_1 = 1.54 \times 10^8/m^2$; Case 2: actual flaw size
$s_2 = 12\mu m$, $\eta_2 = 3.47 \times 10^8/m^2$. Under the same strain rate, smaller flaw size (12\mu m)
predicts a higher strength than larger flaw size (18\mu m); for the same flaw parameter,
higher strain rate resulted in higher strength; dual flaw size system predicts larger
dispersion of strength under varied strain rate than single flaw size system.]

When the friction on the flaw surface is taken into account, as shown in Fig. 3.9.
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Figure 4.9: Wing-crack damage contributions from individual flaw families of dual flaw-size system. Under low strain rate \(10^3/s\), the damage is mostly contributed by larger flaw family; Under high strain rate \(10^5/s\), the damage contributed by the individual flaw families are almost equal.

Figure 4.10: Comparing stress-strain responses of different flaw size input, varied strain rate. (a) Case 3: actual flaw size \(s_3 = 15\mu m\), \(\eta_3 = 2.22 \times 10^8/m^2\), normalized flaw size \(s_3^* = 0.45\) (same as in case 1 and 2); (b) Case 4: actual flaw size \(s_4 = 13.85\mu m\) (statistica average of case 1 and 2), \(\eta_4 = (\eta_1 + \eta_2)/2 = 2.51 \times 10^8/m^2\), normalized flaw size \(s_4^* = 0.438\). Dual flaw size system predicts larger dispersion of strength under varied strain rate than single flaw size system.
CHAPTER 4. IMPLEMENTATION OF DAMAGE-COMPLIANCE SOLUTIONS IN MICRO-MECHANICAL MODEL

discrepancy between $S_{1122}$ and $S_{2211}$ exists and increases with the fiction coefficient. This discrepancy causes issue on the self-consist scheme when determining the term $s_{22}^{11}$ in Eq. (4.11), since in the work of Green and Zerna [86] it was assumed that the compliance tensor was symmetric. As shown in Fig. 4.11, adopting different values for $s_{22}^{11}$ leads to quite different stress-strain results of the micro-mechanical model, particularly on the value of peak stress. As a compromise, we consider the average value of $S_{1122}$ and $S_{2211}$ an appropriate assignment for $s_{22}^{11}$; indeed this was how we defined $s_{22}^{11}$ in the micro-mechanical models and obtained the results in Figs. 4.6 and 4.8 - 4.10.

4.5 Micro-Mechanical Modeling without Self-Consistent Scheme

In Paliwal and Ramesh’s model [36], the interactions among different flaw families are accounted through the means of crack-matrix-effective-medium approach, in which the self-consistent scheme plays a crucial role in determining the local (effective) stress field around the individual flaws. The effective stress is substituted into Eq. (4.2) originated from Horii and Nemat-Nasser 1986 [14] in order to calculate the stress intensity factor $K_I$ (Task (2) summarized in Section 4.1), and then the crack growth rate can be evaluated.

There is a second way to calculate the stress intensity factor $K_I$ for a wing-crack
CHAPTER 4. IMPLEMENTATION OF DAMAGE-COMPLIANCE SOLUTIONS IN MICRO-MECHANICAL MODEL

Figure 4.11: Self-consistent scheme (single flaw size) adopting different values for $s^{11}_{22}$ in Eq. (4.11), and comparison with results using direct scheme; flaw size $s_0 = 12 \mu m$, flaw density $\eta = 3.5 \times 10^8/m^2$, initial damage $\omega = 0.05$, applied strain rate $\dot{\varepsilon} = 10^4 s^{-1}$. Solid black: $s^{11}_{22} = (S_{1122} + S_{2211})/2$; dotted black: $s^{11}_{22} = S_{1122}$; dashed black: $s^{11}_{22} = S_{2211}$

problem, though, which was obtained through the crack opening displacement on the flaw surface. The derivation procedure was shown in detail in Nemat-Nasser and Obata 1988 [16], with the following expression for $K_I$:

$$K_I = \frac{E^*}{\sqrt{2\pi(I + s^{**})}} (u_s \cos \phi + u_n \sin \phi) + \sigma_{22}^{*}(t_{n+1})\sqrt{\pi l(t_n)/2}, \quad (4.56)$$

where
CHAPTER 4. IMPLEMENTATION OF DAMAGE-COMPLIANCE SOLUTIONS IN MICRO-MECHANICAL MODEL

\[ s^{**} = \frac{0.27\pi^2}{32}s. \]  

\( (4.57) \)

As discussed in Chapter 2, \( u_s \) and \( u_n \) are the opening displacements of the flaw surfaces in the local coordinates of shear and normal directions. For general compressive loads we may let \( u_n = 0 \).

In Chapter 2, closed form expressions of \( u_s \) with respect to the far field stress have been derived for a single flaw size system with the wing-cracks periodically distributed in space. Interaction among the wing-cracks are taken into account through the periodic boundary conditions used to evaluate the expressions of crack compliance tensor \( R \). Although the solution more rigorously represents a material with a single flaw size, it could be considered an approximate solution for the wing-crack problem when the interaction between flaws is small, for instance when the normalized length of wing-crack \( l^* < 0.8 \). Beyond this value of wing-cracks we extrapolate this solution of compliance to provide an estimate of material response. As discussed in Section 4.4, the material reaches the strength typically before \( l^* = 1 \), therefore, this extrapolation affects the post peak behaviour of the stress-strain relations, which is beyond the scope of our investigation here.

In our analytical solutions for the crack displacement \( u_s \) and crack compliance tensor \( R \), mild interactions among the wing-cracks have been incorporated while strong
CHAPTER 4. IMPLEMENTATION OF DAMAGE-COMPLIANCE SOLUTIONS IN MICRO-MECHANICAL MODEL

interactions associated with long cracks are not considered. This is consistent with the self-consistent idea. Since these parameters are directly evaluated using the far-field stress, the self-consistent scheme can be circumvented and thus the whole micro-mechanical model becomes simplified. In contrast with the previous self-consistent scheme, we call this method direct scheme of micro-mechanical modeling.

Implementation of the direct scheme is straightforward for both single and multiple flaw size systems. Normalized flaw size $s^*$ is calculated at the initial stage using the equations (4.46), and the normalized wing-crack $l^*$ length is updated at each time step using (4.51), then substituted into the expressions of crack compliance tensor $R$. The displacement on the flaw $u_s$ is then evaluated using $R$ and the perturbed far-field stress using Eqs. (2.86) through (2.88). For the stress intensity factor $K_I$ in Eq. (4.56), the second term is associated with the effective stress normal to the wing-crack surface $\sigma_{22}$, which is now considered 0. The rest of the model is the same as that based on the self-consistent scheme, with the length of wing-crack updated, the effective compliance calculated, and progression to next time step.

Predicted stress-strain results of a single flaw size system using this direct scheme is shown in Fig. 4.11, with the comparison to the results from the self-consistent scheme, assuming the material properties in the first row of Table 4.1. The strength predicted by the direct scheme is about 10% less than then self-consistent scheme (averaged off-diagonal compliance for $s_{11}^{\text{off}}$), but roughly they are similar. Under the same applied strain rate, both schemes reach the strength at roughly the same value
of wing-crack damage parameter (for $\dot{\varepsilon} = 10^4 s^{-4}$, the strength is reached at $\Omega = 0.25$).

The direct scheme shows advantages in at least two aspects. Due to the simplified evaluations in the core process, the modeling time using the direct scheme is only about 1/8 of that using the self-consistent scheme, while there is no substantial difference in the results; therefore the direct scheme is much more efficient. In addition, the inconsistency caused by the unsymmetric compliance tensor (see Fig. 4.11) does not occur in the direct scheme, since the off-diagonal components do not affect the stress state on the flaw. When modeling the material behavior under uniaxial loads, the off-diagonal components of the compliance are not relevant.

4.6 Summary

In this chapter we explored the constitutive modeling of brittle material failure under dynamic compressive load. To this end, effective material properties (stiffness or compliance) are combined with crack growth law to provide a loading rate rate dependent stress-strain relationship.

The Paliwal and Ramesh model, which takes into account the interaction among individual wing-cracks through the self-consistent scheme, is amended through analytical re-derivation and finite element modeling verification. We showed that with the corrected solution of the self-consistent scheme, an anisotropic stiffness/compliance tensor is necessary to properly reflect the local stress state around individual
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wing-cracks.

Two sets of anisotropic material property definitions are incorporated into the amended PR model. The first definition is a simple modification from the work of Budiansky and O’Connell \[40\], in which the compliance along the loading direction is reduced to a fraction \((\alpha f)\) of its corresponding isotropic value. This effective property definition provides reasonable results when incorporated into the amended PR model if the damage measurement is within a given range. But since it is based on introducing an empirical coefficient into the isotropic model, it also lacks the physical insight for damage under compression.

Another definition of the effective compliance is the one developed in Chapter 2, which is based on rigorous analytical derivation and has been verified by FE modeling of a wing-crack RVE. The challenge to incorporating this definition lies in the implementation for a population of multiple flaw sizes, while in the original assumption the RVE contains a single pre-existing flaw. Using a consistency condition, the definitions for an effective normalized flaw size and wing-crack lengths is proposed, which takes the statistics of different flaw families into account. The modeling results properly reflect the influence of strain rate and the flaw size to the constitutive relationship. Different from the earlier effective compliance definitions, an upper limit of applied strain rate exists for the presence of peak stress when modeling with this effective compliance definition. This limit of strain rate depends on the flaw size and is generally well beyond the value that is practically applied in experiments, thus it
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doesn’t impair the validity of implementation of this effective property. In addition, the modeling results incorporating dual flaw-size show more reasonable results relative to this strain rate dependency, thus suggesting that it is preferable to incorporate a flaw size distribution in order to better capture the relevant physics.

An alternatively effective model was proposed based on the idea of finding the stress intensity factor at the wing-crack tips based on the sliding displacement on the pre-existing flaw. Applying the relationship between the sliding displacement and the applied far-field stress derived in Chapter 2, micro-mechanical modeling without the self-consistent scheme is presented, which shows similar results with the self-consistent modeling. Although the interaction among different cracks may not be reflected as well, this sliding displacement approach possess the merits of simplicity and thus high efficiency.
Chapter 5

Efficient Multiscale Modeling

Based on in situ Adaptive Tabulation

In this chapter, a methodology is proposed to enhance the computational efficiency for micromechanical constitutive modeling of brittle dynamic failure with perturbed flaw parameters. The motivation is discussed in the first section. In the second section the methodology is derived, followed with two examples of different implementations in heterogeneous macroscale modeling. In the third section this methodology is deployed to investigate the statistics of the mechanical response given the variation of flaw population, then summary and discussion is presented in the last section.
CHAPTER 5. EFFICIENT MULTISCALE MODELING BASED ON IN SITU ADAPTIVE TABULATION

5.1 Multiscale Modeling with Micromechanics

5.1.1 Heterogeneous multiscale modeling

The properties of pre-existing flaws at the micro-scale, including the size, shape, orientations and clustering, have significant impact on the mechanical properties and constitutive relationship of brittle materials. Since pre-existing flaws are heterogeneous by distributed in space, strength exhibits spatial variation, leading to localization of stress and subsequent failure. Therefore, multiscale modeling with sub-RVE (meso-scale) level elements needs to incorporate variability in the local flaw population.

A schematic of micromechanics-based multiscale modeling is provided in Fig. 5.1. Assume that each element contains a random number of flaws sampled from a given flaw size distribution. In other words, the flaw size distribution varies from one element to the next, as shown in Fig. 5.1. During the simulation at a given time step, a set of data associated with the element / integration point $I$, which contains the current loading state (stress and / or strain tensors), the material / flaw properties and the damage status, are taken as input information for the micromechanics model. Based on these inputs, the micromechanical model provides the results, including the crack length growth speed, updated crack lengths, the damage measure and the con-
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The constitutive relationships, which feed back to the global simulation. We can group these micromechanical results an output data vector $O$. The micromechanical calculations update the constitutive properties at all the elements / integration points, and the global simulation then proceed to the next time step.

To achieve a heterogeneous modeling as described above, the first step is to generate spatial fluctuation in flaw population. A stochastic modeling method is discussed in the next subsection.

5.1.2 Stochastic / probabilistic modelling of the flaw population

Spatial fluctuations of flaw density is a natural result of the randomness in flaw locations. Directly recording and applying the local flaw population from real material is unnecessary and also infeasible, due to the huge amount of required data generation. Common practice is to characterize the flaw statistics at the global scale, and accordingly generate local realizations through statistical methodologies.

Graham-Brady applied a moving-window approach to derive the variations in the local meso-scale flaw density and distribution corresponding to a given set of global statistics. In particular, flaw clustering with adjustable percentage and clustering area is realized based on a parent-child approach. In the present work we assume that the flaws are occurring randomly in space without any clustering or
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correlation. The Poisson distribution is sufficient to describe the local flaw density in this case, without resuming to the aforementioned moving window approach.

Consider the $k$th family of flaws in a ceramic material as being characterized with an averaged density $\eta_k$ at the macro-scale. Given an element of sub-RVE size $A$, the number of flaws contained in this element can be generated by a Poisson random variable with parameter $\lambda = \eta_k A$. Therefore, the mean number of flaws within area $A$ is $\eta_k A$. The random flaw density is denoted as $\eta_k'$, which equals the number of flaws divided by the area $A$. Therefore, the mean of the perturbed flaw density is given by:

$$E[\eta_k'] = \eta_k$$  \hspace{1cm} (5.1)

The variance of the number of flaws within this area is also $\eta_k A$, corresponding to a variance in the flaw density:

$$Var[\eta_k'] = \frac{\eta_k}{A}.$$  \hspace{1cm} (5.2)

This Poisson variable approach is applied to every flaw family $k$, providing a distribution of local flaw density $\eta_k'$.

Two realizations of local flaw density for multiple flaw size families are shown in Fig. 5.1. Here we assume a log-normal probability mass function described the meso-scale averaged flaw size distribution, which has been found for ceramic materials by recent characterization efforts ([65, 31]). The realizations of flaw densities for each
family fluctuates around the reference value shown as stars in Fig. 5.1. It is clear that when the sample area increases (comparing $A = 16 \text{mm}^2$ with $4 \text{mm}^2$), the random perturbation in flaw density around the reference values decrease.

Figure 5.1: Realizations of random flaw densities using Poisson’s random variable. Total flaw density $\eta_{tot} = 10^7 / \text{m}^2$, global averaged (reference) flaw densities of each flaw families follow a log-normal PMF. (a) sample size $A = 4 \text{mm}^2$, (b) sample size $A = 16 \text{mm}^2$.

As the spatial fluctuation of flaw population generated by the stochastic model described in the previous section, the next step is to incorporate the heterogeneous multiscale modeling with micromechanical constitutive model. However, a direct approach is costly. In the next section a more efficient approach is proposed to address the efficiency issue.
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5.2 Enhancing Efficiency for Heterogeneous Micromechanical Modeling

5.2.1 Introducing the \textit{in situ} adaptive tabulation (ISAT)

This section proposes a methodology to enhance the efficiency of micro-mechanical modelling using a Taylor series expansion to represent fluctuations in the underlying flaw population; we name this approach \textit{in situ} adaptive tabulation, following the nomenclature of [59]. The scheme is illustrated in Figure 5.2.

Consider a multiscale model for a ceramic material. The sample being modelled is discretized into $N$ meso-scale elements. During the modelling process, input and output data ($I$ and $O$) are transferred between the macro-scale simulation and micro-mechanical model, as illustrated in Fig. 1.5. For our micro-mechanics model, the input data set $I$ contains the flaw density and distribution, material properties, and current state of the element such as the stress and strain.

Here we denote $I'$ and $O'$ as the data of individual elements, while $I$ and $O$ denote sets of reference (or averaged) input and output data. The localized values $O'$ perturb around the reference data according to perturbations in the input values $I'$. In this work we apply the Poisson distribution to randomly generate the flaw density associated with elements, as described in Section 5.1.2. Other parameters, such as
Figure 5.2: Flowchart for \textit{in situ} adaptive tabulation (ISAT) for multiscale modelling incorporating micro-mechanical models
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the loading conditions, may also be fluctuate from the reference values.

Applying the ISAT scheme, in advance of looping through all the elements, a single micro-mechanical analysis (e.g., the PR model described in the earlier chapters) is carried out with the reference input data set $I$, and provides the results $O$. The gradients of $O$ with respect to the input variables $\frac{\partial O}{\partial I}$ are also calculated. If better accuracy is necessary, the Hessians or even higher order differentiations can be generated, at the expense of increased complexity and computational demands. These results are organized into a tabular structure using interpolation methods, so that the results can be quickly retrieved. We denote the tabulated result with a ‘∗’ subscript, and the tabulated data are stored for future use. This procedure is called the Reference Run.

Instead of repeatedly performing the micromechanical analysis for each element at each time step to determine the material response, new results $O'$ are calculated from the tabulated reference results $O^*$ through a Taylor series expansion by accounting for the difference in some key variables $x_i$:

$$O(x'_1, ..., x'_n) = O(x_1, ..., x_n) + \sum_{i=1}^{n} \frac{\partial O(x_1, ..., x_n)}{\partial x_i} \Delta x_i$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 O(x_1, ..., x_n)}{\partial x_i \partial x_j} \Delta x_i \Delta x_j + ...$$

(5.3)

where $x_i$ may belong to the input variable set $I$, or be some critical intermediate
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variables. Selection of $x_i$ and the derivation of the gradients require an in-depth understanding of the micro-mechanical model.

We call the procedure of retrieving the stored data $O^*$ and performing a Taylor series calculation as the *Transferring Run*. Compared with a complete analysis of the micro-mechanical model, the transferring run provides a close approximation in much less computational time. The accuracy of the transferring run depends on: (a) the difference of the values between the fluctuated and the reference input variables $\Delta x_i^*$ (or in general, $\Delta I$), and (b) the order of the Taylor series that approximation. To account for (a) with large $\Delta I$ perturbations, multiple reference runs should be performed with different values of $I$ that cover a full range of fluctuations. During transferring runs, the reference results that are calculated from the closest data set $I$ should be applied in the Taylor series. Regarding (b), applying higher order gradients improves the accuracy of the Taylor series approximation, yet with the trade-off of requiring more computational effort in the reference run. It is also challenging to derive higher order terms from the micro-mechanics model. Considering $n$ independent variables in the Taylor series, then there are $n$ first order gradient terms of $O$, $n(n + 1)/2$ second order gradient terms, and so forth. The balance between accuracy and computational efficiency must be considered before implementing this methodology.

In the following subsections are two examples that implement ISAT in the micro-mechanical model introduced in Chapter 4.
5.2.2 Case 1: varying initial flaw size distribution

In this first demonstrative example, we take the flaw size distribution as the only perturbed input variable, and let all other material properties and the loading condition identical to the reference values. Specifically, material properties of Aluminium Nitride, as listed in the Table 5.1 [36], are applied in the micro-mechanical model. The sample is loaded with a uni-axial compressive strain, whose magnitude increases from zero with a constant strain rate $\dot{\varepsilon} = 10^5 s^{-1}$.

<table>
<thead>
<tr>
<th>$E$</th>
<th>$\nu$</th>
<th>$\mu$</th>
<th>$\rho$</th>
<th>$K_{IC}$</th>
<th>$\tau_c$</th>
<th>$\alpha$, $\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>300 GPa</td>
<td>0.24</td>
<td>0.20</td>
<td>3673 kg/m$^3$</td>
<td>2.7 MPa√m</td>
<td>0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

Table 5.1: Material properties (ceramic AlN) and flaw statistics for micromechanics model.

The flaw orientation is assumed to be fixed at $\phi = 50.7^\circ$ to the loading direction, which is the optimum angle for wing-crack development given a friction coefficient 0.2. Pre-existing flaws with size ranging from 10$\mu$m to 40$\mu$m are accounted for in this model. The distribution of the flaw size is assumed to vary randomly from element to element. For the global material, the total density of the flaws is set to be $10^7$/mm$^2$, and the flaw size distribution is assumed to follow a lognormal function. The density of each flaw family $\eta_k$ is obtained by multiplying the total flaw density in family $k$ by the probability that a given flaw is in family $k$ (Eq. (4.6)), and the reference flaw densities are shown in the dashed line in Fig. 5.1.
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Using the Poisson’s process simulation discussed in Section 5.1.2 we can generate sample values of the local flaw density for a given area (Fig. 5.1). A micro-mechanical model is performed with both the reference (or global) and the random (or local) flaw density as input, and the resulting stress-strain curves are plotted as solid lines in Fig. 5.3a. The strength of this specific realization is 4.64\% less than the reference one.

![Stress vs strain curves](image1)

![Relative error of stress between ISAT and complete re-run](image2)

\begin{figure}[h]
\centering
\subfloat[Stress vs strain curves]{\includegraphics[width=0.4\textwidth]{image1}} \quad \subfloat[Relative error of stress between ISAT and complete re-run]{\includegraphics[width=0.4\textwidth]{image2}}
\caption{Case 1: Perturbing the flaw density input, comparison of the computational results with full calculation and ISAT. Ratio of computational time between the full calculation and ISAT is about 1000.}
\end{figure}

The next step is to apply ISAT in order to evaluate the accuracy of this approximation. At a time step $t_n$, the instantaneous crack length of each flaw family $l_k(t_n)$ is affected by the local flaw density, but more directly it is affected by the change of global damage measure $\Omega(t_n)$. It should be noted that by assuming a constant strain rate, at a given time step $t_n$, the reference and the local random samples are

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experiencing an identical external load. Therefore, $\Omega(t_n)$ is the only key variable.

Applying a Taylor series approximation to the updated crack length in terms of damage, we get:

$$l'_k(t_n) = l_k(t_n) + \frac{dl_k}{d\Omega} \Delta \Omega(t_n)$$

(5.4)

where $l'_k$ represents the crack length of family $k$ with the local (random) flaw density. To find the expression for $dl_k/d\Omega$, consider the Eq. (4.1) and define a function $f(\Omega, \epsilon, s...)$ as follows:

$$f = \frac{\partial l}{\partial \epsilon} = \frac{C_r}{\dot{\epsilon}} \left( \frac{K_I - K_{IC}}{K_I - K_{IC}/2} \right)^\gamma$$

(5.5)

In our model the value of $\gamma$ is set to be 1. At a given time step $t_n$, the change of crack length $l_i$ due to a perturbation of damage $\Delta \Omega(t_n)$:

$$\Delta l(t_n) = \int \frac{df}{d\Omega} \Delta \Omega d\epsilon = \int \left( \frac{\partial f}{\partial \Omega} + \frac{df}{dl} \frac{dl}{d\Omega} \right) \Delta \Omega d\epsilon$$

(5.6)

$$\approx \sum_n \left( \frac{\partial f}{\partial \Omega} + \frac{df}{dl} \frac{dl}{d\Omega} \right) \bigg|_n \Delta \Omega_n \Delta \epsilon_n$$

(5.7)

Through numerical evaluation:

$$\frac{dl}{d\Omega} \bigg|_{t_n} \approx \sum_n \left( \frac{\partial f}{\partial \epsilon} \bigg|_n + \frac{df}{dl} \bigg|_n \frac{dl}{d\Omega} \bigg|_{n-1} \right) \frac{\Delta \Omega_{n-1} \Delta \epsilon_n}{\Delta \Omega_n}$$

(5.8)
Note that $\Delta \Omega_{n-1}$ in the numerator varies in different time step and cannot be extracted from the summation operation. Other terms in (5.8) can be derived from (5.5):

\[
\frac{\partial f}{\partial \Omega} = C_r K_{IC} \left( K_I - \frac{K_{IC}}{2} \right)^{-2} \frac{\partial K_I}{\partial \Omega}, \quad (5.9)
\]

\[
\frac{\partial f}{\partial \Omega} = C_r K_{IC} \left( K_I - \frac{K_{IC}}{2} \right)^{-2} \frac{\partial K_I}{\partial l}. \quad (5.10)
\]

The stress intensity factor $K_I$ has been derived by [14] with the expression Eq. (4.2). Substituting into the above equations, numerical results of Eq. (5.8) can be obtained.

Referring to the evaluation of damage measure Eq. (4.4), with the new flaw densities $\eta_k$ and crack length $l_k$, the damage measure in the new system can be expressed as:

\[
\Omega'(t_n) = \Omega(t_n) + \Delta \Omega(t_n) = \sum_k (\eta_k + \Delta \eta_k)(l_k^2 + \frac{\partial l_k}{\partial \Omega} \Delta \Omega)^2, \quad (5.11)
\]

where $\Delta \eta_k$ is the random perturbation of the area density of flaws in family $k$. Reorganizing Eq. (5.11) and ignoring the higher order terms, a closed form expression for
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the damage perturbation with respect to the flaw density perturbation is obtained:

\[
\Delta \Omega(t_n) = \frac{\sum \Delta \eta_k l_k \frac{\partial \eta_k}{\partial \Omega}}{1 - 2 \sum \eta_k l_k \frac{\partial \eta_k}{\partial \Omega}} \quad (5.12)
\]

Substituting the updated \( \Omega'(t_n) \) into the constitutive relations (4.7), the stress result with the perturbed flaw density is then obtained. As shown in Fig. 5.3, the result with the ISAT matches well with the one obtained from a complete evaluation of PR model. When the strength is reached \( (\epsilon_{11} \approx 0.02) \), the relative error between the transferred result and the full calculation is about 0.2%.

We consider the computational time of the core process (excluding the parameter definition and data loading procedures, for these procedures only need to perform once for multiple sample calculations) as the measure of efficiency. Specifically in this case, the ratio of computational time between the complete calculation and the ISAT is about 1000, which may slightly vary depending on the platform in which the model is run.

5.2.3 Case 2: varying flaw statistics and loading conditions

In the previous case we assumed identical loading conditions for local elements, which is generally unrealistic in multiscale modeling. Due to the heterogeneous nature and the damage related constitutive properties, the load path for a specific local
element is never exactly the same as its neighbouring element, even if a homogeneous far-field load is applied and the sample possesses a uniform structure.

Consider an element S which contains a higher overall density of pre-existing flaws than the global averaged value, while all other elements in the sample possess the identical flaw density and distribution. Under a far-field load, the initial damage in element S is higher than in its neighbouring elements, leading to a weaker constitutive relationship. The stress and strain fields are then concentrated in element S, and relieved in the neighbouring elements. The directions of the principal stress / strain in element S are also rotated from the far-field applied load direction because of the localization. As the magnitude of the applied load continues to grow, the localization of stress and damage interact with each other, leading to a global failure starting at element S.

Therefore, to be applicable in multiscale modelling with heterogeneous material properties, the ISAT method should be able to address variations in the local load path from element to element. A solution of ISAT for the case of varying both material properties (flaw statistics) and the loading strain rate is presented as follows.

In this case, we choose the crack length of each family $l_k$ as a random parameter. At the end of the reference run, the final crack lengths $l_k(N)$ are evenly discretized into an array $l^*_k$ of size $M$. For any output parameter $f$ in $O$, which can be either a direct result or a first, second or third order gradient, the parameter $f$ can be treated as a function of the crack length, i.e. $f(l_k)$. Tabulated data $f^*$ is obtained by
interpolating the function $f(l_k)$ with respect to $l_k^*$. This interpolation is performed for each crack family. In the end, a tabulated result of $f^*$ with size $K \times M$ is obtained.

The crack growth rate of each family $\dot{l}_k$ at a given time step $t_n$ is considered as a function of the crack length $l_k$, the strain and the damage:

$$
F(l_k(t_n), \Omega(t_n), \epsilon(t_n)) = \frac{\partial l_k(t_n)}{\partial t} 
$$

(5.13)

The Taylor series that connects the reference and transferred values of $F$ can be expressed by:

$$
F(l_k, \Omega', \epsilon') \approx F^*(l_{km}^*, \Omega_{km}^*, \epsilon_{km}^*) + \frac{\partial F}{\partial \Omega} \bigg|_{km}^* \Delta \Omega + \frac{\partial F}{\partial \epsilon} \bigg|_{km}^* \Delta \epsilon + ... \\
+ \frac{\partial^2 F}{\partial \Omega^2} \bigg|_{km}^* \frac{\Delta \Omega^2}{2} + \frac{\partial^2 F}{\partial \epsilon^2} \bigg|_{km}^* \frac{\Delta \epsilon^2}{2} + \frac{\partial^2 F}{\partial \epsilon \partial \Omega} \bigg|_{km}^* \Delta \Omega \Delta \epsilon + O(\Delta \Omega^3, \Delta \epsilon^3)
$$

(5.14)

where $m$ is the index which maps the tabulated crack length $l_{km}^*$ that is closest to the current value of $l_k$, i.e.:

$$
l_{km}^* \approx l_k(t_n)
$$

(5.15)

During the transferring run at every time step, we need to identify this index $m$ for each flaw family $k$. With this index identified, the damage $\Omega_{km}^*$, the strain $\epsilon_{km}^*$, the crack length growth rate $F$ as well as all the gradient terms that correspond to the
crack length $l_k$ in the reference run can be extracted from the tabulated data base. The difference between the current and reference run for both damage and strain can then be evaluated by:

$$
\Delta \Omega = \Omega' - \Omega_{km}^* \tag{5.16}
$$

$$
\Delta \epsilon = \epsilon' - \epsilon_{km}^* \tag{5.17}
$$

With the new data of crack growth rate, the crack length at the new time step is updated by:

$$
l_k(t_{n+1}) = l_k(t_n) + F(l_k, \Omega', \epsilon') \Delta t \tag{5.18}
$$

The damage at the next time step is then evaluated using Eq. (4.4). Substituting the damage and the updated strain into the constitutive relationship of Eq. (4.7), the stress at the new time step is calculated.

We performed some example runs with varied flaw statistics and applied strain rates, which are listed in Table 5.2 and the results are shown in Fig. 5.4. The reference run is applied with the data set $I_0$, assuming the global averaged flaw size distribution shown in Fig. 5.1. The total flaw density and flaw size distribution of $I_1$ and $I_2$ perturb from the global value in $I_0$ following the Poisson’s process simulation assuming a sample area of $A = 8\text{mm}^2$. The applied strain rates in $I_1$ and $I_2$ are constants with magnitudes 10% and 250% of $I_0$, respectively. Since the constitutive
relationship predicted by the micromechanical model is highly strain-rate dependent, the result strengths are about 35% lower \( (I_1) \) and higher \( (I_2) \) than the reference run, respectively. \( I_3 \) shows the scenario described in the beginning of this subsection: the total global density is 50% higher than the reference one, and the actual flaw size distribution perturbs from this global function; the applied strain rate starts with the same value as in \( I_0 \), but gradually increases to 15% greater than \( I_0 \). All models are performed with the same time interval \( \Delta t \).

For \( I_1 - I_3 \), three separate runs are performed for each data set: a complete re-run with the micro-mechanical model and the updated parameters, ISAT with first
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<table>
<thead>
<tr>
<th>Data set $I_0$</th>
<th>Total flaw density $\eta(1/m^2)$</th>
<th>Loading strain rate $\dot{\varepsilon}_{11}(s^{-1})$</th>
<th>ISAT order of accuracy $(\Delta \varepsilon, \Delta \Omega)$</th>
<th>Efficiency gain with ISAT</th>
<th>Strength error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1$</td>
<td>$10^7$</td>
<td>$1 \times 10^5$</td>
<td>$(1, 1)$</td>
<td>24</td>
<td>6.28</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(3, 2)$</td>
<td>14</td>
<td>0.24</td>
</tr>
<tr>
<td>$I_2$</td>
<td>$10^7$</td>
<td>$2.5 \times 10^5$</td>
<td>$(1, 1)$</td>
<td>104</td>
<td>3.11</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(3, 2)$</td>
<td>22</td>
<td>0.39</td>
</tr>
<tr>
<td>$I_3$</td>
<td>$1.5 \times 10^7$</td>
<td>varying</td>
<td>$(1, 1)$</td>
<td>28</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(3, 2)$</td>
<td>18</td>
<td>0.15</td>
</tr>
</tbody>
</table>

**Table 5.2:** Input data of modeling and the results

order accuracy for $\Delta \varepsilon$ and $\Delta \Omega$ in Taylor series (results not plotted in Fig. 5.4), and ISAT with 3rd order accuracy for $\Delta \varepsilon$ and 2nd order accuracy for $\Delta \Omega$ in Taylor series (dashed lines in Fig. 5.4). The efficiency enhancement (computational time of complete re-run divided by that of ISAT) and the relative accuracy of ISAT are listed in Table 5.2. The efficiency gain varies from 10 to 100 times, and lower orders of accuracy are even more efficient, but at the cost of larger relative error.

Based on the results we can conclude that ISAT is capable of predicting the constitutive response with high accuracy and high efficiency for perturbed material properties and varying applied load.
5.3 Implementation in Statistical Study

Variation of material properties leads to variation in the mechanical response. The statistics of the mechanical response can be obtained through Monte Carlo simulation; in this context it refers to randomly generating the input data, repeatedly running the micro-mechanics model, and then analysing statistics of the computational results. Such a brute force methodology requires significant computational effort. In this section we explore an alternative much more efficient method with the aid of variable gradients derived from ISAT.

As in Section 5.2.2 we consider the PR micro-mechanics model for uni-axial loading conditions, with the assumption of constant strain rate $\dot{\epsilon}$ and flaw densities $\eta_k$ as the only random material properties.

5.3.1 Instantaneous C.O.V. of stress

Our first goal is to identify the instantaneous variance of stress, $\text{Var}[\sigma(t_n)]$. Referring to the constitutive relations (4.7), the variance of stress at a given time step $t_n$ can be estimated by:

$$\text{Var}(\sigma(t_n)) = E_1^2 \epsilon(t_n)^2 \text{Var}(\Omega(t_n))$$

The perturbation in damage $\Omega'(t_n)$ due to perturbation in the of flaw size distribution has been discussed in section 5.2.2. Expanding Eq. (5.11) and keeping the first order
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approximation, $\Omega'(t_n)$ can be estimated by:

$$\Omega'(t_n) \approx \sum_k \left( \eta_k l_k(t_n)^2 + 2\eta_k l_k(t_n) \frac{\partial l_k(t_n)}{\partial \Omega} \Delta \Omega(t_n) + l_k(t_n)^2 \Delta \eta_k \right)$$  \hfill (5.20)

$\Delta \Omega(t_n)$ has been derived in (5.12). The mean of $\Delta \Omega(t_n)$ is zero, because the mean of $\Delta \eta_k$ is zero. Therefore, the mean of the damage is found from (5.20):

$$\mathbb{E}(\Omega(t_n)) = \sum_k \eta_k l_k(t_n)^2$$  \hfill (5.21)

The variance of $\Omega(t_n)$ is calculated by taking the expected value of the square of (5.20) and subtracting the square of (5.21), which gives:

$$\text{Var}(\Omega(t_n)) = \sum_k \left[ 4\eta_k^2 l_k^2 \left( \frac{\partial l_k}{\partial \Omega} \right)^2 \mathbb{E}(\Delta \Omega^2) + l_k^4 \mathbb{E}(\Delta \eta_k^2) + 4\eta_k l_k \frac{\partial l_k}{\partial \Omega} \mathbb{E}(\Delta \eta_k \Delta \Omega) \sum_j l_j^2 \right]_{t_n}$$  \hfill (5.22)

As discussed in section 5.1.2 we assume that the flaws occur randomly in space without any clustering or correlation. Given an area $A$, the number of flaws in family $k$ is a Poisson random variable with parameter $\eta_k A$. Therefore, the expectations that
appear in (5.22) can be found as follows:

\[
\begin{align*}
E(\Delta \eta_k^2) &= \frac{\eta_k}{A} \\
E(\Delta \Omega(t_n)^2) &= \frac{1}{A} \sum_k \eta_k l_k^4 \left(1 - 2 \sum_{l} \eta_k l_k \frac{\partial \Omega}{\partial x} \right)^2 \\
E(\Delta \eta_k \Delta \Omega(t_n)) &= \frac{1}{A} \frac{\eta_k l_k^2}{1 - 2 \sum \eta_k l_k \frac{\partial \Omega}{\partial x}}
\end{align*}
\] (5.23)

Substituting (5.22) and (5.23) back into (5.19), the variance of the stress at time step \( t_n \) can be calculated.

A series of Monte Carlo simulations with multiple sample areas are conducted to verify our analyses, and the results are plotted in Fig. 5.5. The coefficient of variations (COV) of the simulated results agrees well with our analytical results. It confirms the effectiveness and efficiency of our solution: a single micro-mechanical calculation is sufficient to provide the variance of stress for any sample size \( A \).

To briefly summarize, in order to calculate the variance of the stress due to the random occurrence of flaws, we only need to calculate and store the crack lengths and the corresponding gradients at every time step with the reference flaw population, and then apply these expressions in into the derived analytical expressions.

### 5.3.2 Variance of strength

Next, consider the variance of strength (or peak stress) due to the random occurrence of flaws. The analyses in the previous subsection is not applicable for this goal,
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Figure 5.5: C.O.V. of stress for varying sample size. Dashed lines are the results obtained by Monte Carlo simulations with 1000 realizations; solid lines are calculated by our analytical solutions based on the gradients. Relative errors between the two methods are less than 5%.
since the time step (and strain) at which the peak stress occurs fluctuates as well as
the value of the peak stress. Here we identify the strength $\sigma_p$ and all the parameters
for random flaw density using the reference result and the gradients calculated with
the averaged flaw density.

During the micro-mechanical modelling, the time step $t_p$ when the strength is
reached can be identified by $\dot{\sigma}(t_p) = 0$; differentiating the constitutive relation (4.7)
with respect to time gives us the relations of the strain, damage and damage rate at
this instant:

$$
\Omega_p \dot{\epsilon} + \dot{\Omega}_p \epsilon_p = \frac{E_0}{E_1} \dot{\epsilon}
$$

(5.24)

where $\Omega_p = \Omega(t_p)$, $\dot{\Omega}_p = \dot{\Omega}(t_p)$, $\epsilon_p = \epsilon(t_p)$ and the subscript $p$ denotes the parameters
at the time step corresponding to the strength.

As the realization of the flaw densities $\eta_k$ deviate from the reference values, the
damage, damage rate and strain have different values when the peak stress occurs.
Eq. (5.24) is updated as:

$$
(\Omega_p + \Delta \Omega_p) \dot{\epsilon} + (\dot{\Omega}_p + \Delta \dot{\Omega}_p) (\epsilon_p + \Delta \epsilon_p) = \frac{E_0}{E_1} \dot{\epsilon}
$$

(5.25)

in which the $\Delta$ terms indicates the difference of the corresponding parameters to the
reference ones. Since the strain rate $\dot{\epsilon}$ is set to be constant, relations between the
perturbed variables can be estimated by subtracting Eq. (5.25) from Eq. (5.24) and
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ignoring the higher order terms:

\[ \Delta \Omega_{p} \dot{\epsilon} + \Delta \dot{\Omega}_{p} \epsilon_{p} + \dot{\Omega}_{p} \Delta \epsilon_{p} = 0 \] (5.26)

In Eq. (5.26), \( \Delta \Omega_{p} \) and \( \Delta \dot{\Omega}_{p} \) can be expressed with respect to \( \Delta \epsilon_{p} \) and the perturbed \( \eta_{k} \). When the strength is reached, assuming the same flaw density, the damage \( \Omega_{p} \) is:

\[ \Omega_{p} = \sum_{k} \eta_{k} l_{kp}^{2} \] (5.27)

where \( l_{kp} = l_{k}(t_{p}) \) is the crack length for the \( k \)th flaw family at time step \( t_{p} \) when the model reaches the peak stress. For a new flaw density \( \eta'_{k} \):

\[ \Omega'_{p} = \sum_{k} \eta'_{k} (l_{kp} + \frac{\partial l_{k}}{\partial \Omega} \bigg|_{t_{p}} \Delta \Omega(t_{p}) + \frac{\partial l_{k}}{\partial \epsilon} \bigg|_{t_{p}} \Delta \epsilon_{p})^{2} \] (5.28)

where \( \Delta \Omega(t_{p}) \) corresponds to the difference between damage predicted by the two models at the same time step \( t_{p} \) when the strength is reached using the reference flaw density (not the difference of \( \Omega_{p} \)), and should be calculated by Eq. (5.12).

Subtracting Eq. (5.28) from Eq. (5.27) and reorganizing the result by extracting the terms with \( \Delta \epsilon_{p} \):

\[ \Delta \Omega_{p} = \Delta \Omega_{r} + \alpha_{r} \Delta \epsilon_{p} \] (5.29)
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where \( \Delta \Omega_r \) and \( \alpha_r \) can be calculated by the following expressions:

\[
\Delta \Omega_r = \sum_k \Delta \eta_k \ell_{kp}^2 + 2 \sum_k \eta'_k \ell_{kp} \frac{\partial \Omega}{\partial \Omega} \bigg|_{t_p} \Delta \Omega(t_p) + \sum_k \eta'_k \left( \frac{\partial \ell_k}{\partial \Omega} \Delta \Omega \right) \bigg|_{t_p} \quad (5.30a)
\]

\[
\alpha_r = 2 \sum \eta'_k \frac{\partial \ell_k}{\partial \epsilon} \bigg|_{t_p} \left( \ell_k + \frac{\partial \ell_k}{\partial \Omega} \Delta \Omega \right) \bigg|_{t_p} \quad (5.30b)
\]

Differentiating Eq. (5.27) gives the damage rate at the peak stress with reference flaw density:

\[
\dot{\Omega}_p = 2 \sum_k \eta_k \ell_{kp} \dot{i}_{kp} \quad (5.31)
\]

Using the chain rule we have \( \dot{i}_{kp} = \frac{\partial \ell}{\partial \epsilon} \bigg|_{t_p} \dot{\epsilon} \). For a new flaw density \( \eta'_k \), \( \dot{\Omega}_p \) can be calculated by updating crack length and the gradients with the perturbing terms:

\[
\dot{\Omega}'_p = 2 \sum \eta'_k \left( \ell_{kp} + \frac{\partial \ell_k}{\partial \Omega} \Delta \Omega + \frac{\partial \ell_k}{\partial \epsilon} \Delta \epsilon_p \right) \bigg|_{t_p} \left( \frac{\partial \ell_k}{\partial \epsilon} + \frac{\partial^2 \ell_k}{\partial \Omega \partial \epsilon} \Delta \Omega + \frac{\partial^2 \ell_k}{\partial \epsilon^2} \Delta \epsilon_p \right) \bigg|_{t_p} \dot{\epsilon} \quad (5.32)
\]

Subtracting Eq. (5.32) from Eq. (5.31) and collecting the \( \Delta \epsilon_p \) terms, we find the difference of \( \dot{\Omega}_p \):

\[
\Delta \dot{\Omega}_p = \Delta \dot{\Omega}_r + \beta_r \Delta \epsilon_p \quad (5.33)
\]
where:

\[
\Delta \dot{\Omega}_r = 2 \sum_k \Delta \eta_k l kp \dot{l}_kp + 2 \sum_k \eta'_k \epsilon \left( \frac{\partial l_k}{\partial \epsilon} \frac{\partial l_k}{\partial \epsilon} \Delta \Omega + \frac{\partial^2 l_k}{\partial \epsilon^2} \Delta \Omega l_k + \frac{\partial l_k}{\partial \epsilon} \frac{\partial^2 l_k}{\partial \epsilon \partial \Omega} \Delta \Omega^2 \right) \bigg|_{t_p}
\]

(5.34a)

\[
\beta_r = 2 \sum_k \eta'_k \epsilon \left( \frac{\partial^2 l_k}{\partial \epsilon^2} \dot{l}_k + \frac{\partial l_k}{\partial \epsilon} \frac{\partial l_k}{\partial \epsilon} + \frac{\partial l_k}{\partial \epsilon} \frac{\partial^2 l_k}{\partial \epsilon \partial \Omega} \Delta \Omega + \frac{\partial l_k}{\partial \epsilon} \frac{\partial^2 l_k}{\partial \Omega \partial \epsilon} \Delta \Omega \right) \bigg|_{t_p}
\]

(5.34b)

Substituting Eqs. (5.29 and 5.33 into Eq. (5.26), the closed form solution of \( \Delta \epsilon_p \) is reached:

\[
\Delta \epsilon_p = - \frac{\Delta \Omega_r \epsilon + \Delta \dot{\Omega}_r \epsilon_p}{\alpha_r \epsilon + \beta_r \epsilon_p + \dot{\Omega}_p}.
\]

(5.35)

In the above equation, for \( \Delta \Omega_r, \alpha_r, \Delta \dot{\Omega}_r \) and \( \beta_r \), every term except the flaw densities can be calculated during the reference run. Therefore, \( \Delta \epsilon_p \) can be easily obtained by substituting the stored data and the perturbed flaw densities into the above equations, and then \( \Delta \Omega_p \) and \( \sigma'_p \) are solved in sequence through Eq. (5.29) and (4.7).

Once \( \Delta \epsilon_p \) is solved, \( \Omega_p \) and \( \sigma_p \) for the random flaw density \( \eta_k \) is obtained. In this way, new strength and the corresponding strain is calculated without running the complete micro-mechanical model and saving a great amount of computational effort.

An example of the approach is shown in Fig. 5.6. 500 realizations are generated using the Poisson’s process simulation described in section 5.1.2. Complete micro-mechanical models are performed to obtain the stress-strain relations and the strength. The data of reference run and of each sample, including the flaw statis-
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tics and the results, are stored. We then apply the flaw statistics realizations to the derived ISAT solutions, and calculate the corresponding strength. The calculated stress-strain curves and the strengths by two methods are shown in Fig. 5.6a.

The cumulative distribution function describing strength is plotted in Fig. 5.6b. From this curves we can tell that the ISAT approximation of the strength slightly deviates from the results from the full run at the lower and higher ends, but the majority of values exhibit high accuracy.

Using the Monte Carlo study, we confirm that:

(a) the perturbed strengths predicted by the updated PR model with corrected elasticity solutions ([47]) match well with the lognormal distribution (Fig. 5.6c), same as shown in [53].

b) The coefficient of variation (COV) of the strength is proportional to the inverse square root of sampling area, as shown in Fig. 5.6d.

With the perturbed flaw statistics and reference data readily generated, computing the of strength with ISAT only takes a small fraction of the time required by one single run of micromechanical model. The factors that affect the calculation time for the strength statistics study using ISAT include the number of Monte Carlo simulations, the order of the Taylor series expansion, and data generation and storage during the reference run. It is worth noting that the time step directly affects the computational time of the micromechanical model, but has no influence on the ISAT after the reference run, which further promises the computational efficiency of ISAT.
Figure 5.6: Statistical study of material strength with Monte Carlo simulations. (a)-(c): sample size $A = 4\text{mm}^2$, 500 flaw population realizations. (a) stress-strain curves and strengths calculated by two methods; (b) CDF of strength results by two methods; (c) Probability plot of strength for lognormal distribution; (d) Relationship between strength C.O.V and sample size.
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To briefly summarize, the ISAT method can be applied to Monte Carlo simulations to characterize the strength statistics without repeatedly running the complete micro-mechanical model. The only data required are the parameters and gradients in the reference run describing the mechanical state at the time step associated with the peak stress, and the perturbed flaw statistics.

Perturbation of material properties leads to variation in mechanical response. The statistics of the mechanical response can be obtained through Monte Carlo simulation; in this context it refers to randomly generating the input data and repeatedly running the micro-mechanics model, then analysing the computational results with statistics tools. Such a brute force methodology requires excessive computational efforts. In this section we explore the methods of performing high efficiency statistical analyses with the aid of variable gradients derived from ISAT.

As in section 5.2.2 we consider the PR micro-mechanics model for uni-axial loading conditions, with the assumptions of constant strain rate $\dot{\varepsilon}$ applied to the samples and flaw densities $\eta_k$ as the only perturbing material properties.

5.4 Summary and Discussion

In this chapter we present a highly efficient method (in situ adaptive sampling, ISAT) for micro-mechanical modeling of brittle dynamic failure. As flaw properties
and loading conditions are perturbed, instead of repeatedly evaluating the constitutive relationships with the micro-mechanical model, results are obtained by expanding from set of reference results based on the difference in the flaw population. Computational speed-up varies from 20 to 1000 times, depending on the complexity of the perturbed variables, and the error is small.

Statistical studies based on the micro-mechanical model also benefit from ISAT. Provided the sampling area and the perturbed flaw population, the coefficient of variant of the instantaneous stress and the strength is directly calculated without additional evaluation of the micro-mechanical model, except for the single reference run.

For brittle failure problems, the ISAT is easily applied for problems with a monotonically increasing applied load and internal damage measure. However, it may not be easily implemented in some scenarios such as cycled loads, progressive crack growth (fatigue), autonomous crack growth under fixed applied loads, etc., since the damage-load or damage-time relationships are not unique.

Throughout this chapter ISAT is applied to a relatively simple micro-mechanical model for 2D uniaxial loads. For large scale 3D simulations with complex loading and boundary conditions, derivation and numerical calculation of the gradient and Hessian of the internal parameters can be challenging tasks. Nevertheless, considering the benefit in computational efficiency, the ISAT may be worth implementing when the multi-scale model requires excessive effort in repeated micro-mechanical model
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computations.
Chapter 6

Non-local Finite Element Modeling

As noted in, for example, Fig. 1.3 after a brittle material reaches a peak stress under compressive loading, it exhibits a negative tangent stiffness. Under these conditions, special modeling methods are required to retain numerical stability. In this chapter, the nonlocal finite element modeling method is investigated as a possible approach to address this strain softening behavior.

In the first section of this chapter, the background and definition of a classical nonlocal finite element method is reviewed. Some benchmark problems implementing this nonlocal model will be presented in the second section for verification and investigation of its feature, and in the last section some conclusions will be drawn.
CHAPTER 6. NON-LOCAL FINITE ELEMENT MODELING

6.1 Introduction and Background

From the stress-strain curves of either the experiment (e.g., [1, 9, 31]) or the micromechanical modeling results (for example, Fig. 4.10), negative tangent modulus is observable when the material is heavily damaged. This kind of post-peak behaviour is generally referred to as strain softening phenomena, which is well documented in brittle materials (ceramics, concrete, rocks) when they are loaded to failure (Fig. 6.1a).

When applying the strain softening material properties in ordinary finite element models, the stiffness matrix ceases to be positive definite, leading to severe difficulties in the finite element implementations. The challenges can be summarized as:

(a) Lack of convergence: When the slope of the stress-strain curve changes from positive to negative, solutions typically do not converge.

(b) Mesh sensitivity: Once softening occurs, the results become highly mesh dependent. As shown in Fig. 6.1b, as the elements are refined, the inelastic strain distribution tends to localize in a single element with increasing magnitude, while in the remaining elements the strain is unloaded. Such mesh dependency is non-objective and is thus unreliable.

(c) Unrealistic energy dissipation: As the inelastic strain localizes in a zone of vanishing volume, the structure fails at zero energy dissipation since the energy dissipation per unit volume is finite.

To overcome these challenges associated with strain softening in a finite element
CHAPTER 6. NON-LOCAL FINITE ELEMENT MODELING

(a) Strain softening behaviour

(b) Issue of mesh sensitivity

Figure 6.1: Strain softening behaviour and issue of mesh sensitivity. (a) Strain softening behaviour of heavily damaged brittle materials. Evans and Marathe (1968) [88]; (b) Issue of mesh sensitivity when modeling the strain softening behaviour with ordinary FE method. Bazant and Chang (1987) [89]

context, the continuum damage theory was proposed in previous research, see for example Bazant, Belytschko [68, 71]. In this theory it was proposed that continuum damage is finite, and thus a localization limiter is required when modeling this type of material.

In general, two types of localization limiters have been developed. One is referred to as the differential limiter, represented by works such as [91, 92, 93, 94, 95, 96]. In this method the material constitutive relationship (stress-strain) is enhanced by including spatial gradient terms.

The other type of localization limiter is known as the integral limiter, or the spatial averaged nonlocal method, which is represented by the works of, for examples, [69, 68, 97, 71, 98]. In this type of method, the solutions (stress / strain) are determined by the local values as well as by a spatial averaging of the values over a surrounding region.
Although these two methods appear to be different approaches, they essentially share some common features from a mathematical point of view: the spatial integration terms can be expanded with a Taylor series expansion to yield a series of gradient terms. Therefore, these two types of methods present similar results and both meet some common challenges. In the present work a typical spatial integration nonlocal method is investigated to explore the feasibility of using the nonlocal method for a strain softening problem.

6.2 Spatially Averaged Nonlocal Model

6.2.1 Definition of nonlocal parameters

In order to implement the spatial integral nonlocal limiter, the concept of the imbricate finite element method was proposed by Bazant and his colleagues [69, 68]. The illustration for a one-dimensional imbricate model is shown in Fig. 6.2a, and a two-dimensional imbricate model is shown in Fig. 6.2b. The basic structure is discretized in the same way as the ordinary FE method, although in the context of the nonlocal method these elements are called the local elements. The nonlocal / imbricate elements connect nodes such that the nonlocal element contains multiple local elements. The length of the nonlocal elements (denoted as $l$) is a constant and is considered a type of material property. Therefore, the ratio between the lengths of nonlocal and local elements is a constant denoted as $\bar{n}$, except on the boundaries.
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where the nonlocal elements are typically truncated. In the 1D nonlocal models shown in Fig. 6.2a, the nonlocal elements are of twice the length of the local elements (denoted as $h$), and thus $\bar{n} = 2$; while in b the 2D nonlocal elements cover three local elements per side ($\bar{n} = 3$).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.2}
\caption{Imbricate nonlocal finite element model proposed by Bazant (1984) \cite{ Bazant1984} \cite{ Bazant1984}
(a) One dimensional model (b) Two dimensional model}
\end{figure}

The nonlocal strain is evaluated by averaging the strains of the local elements that the nonlocal element covers, giving the relationship between local and nonlocal
strains as follows:

$$\epsilon^{nl}(x) = \frac{1}{l} \int_{x-l/2}^{x+l/2} \epsilon^{l}(x + s) ds$$  \hspace{1cm} (6.1)

By applying the imbricate elements, this nonlocal strain $\epsilon^{nl}$ can be evaluated using the displacement on the nodes it connects through a finite difference calculation. Taking the one-dimensional problem as an example, the nonlocal strain at position $x$ is calculated by:

$$\epsilon^{nl}(x) = \frac{1}{l} \left[ u \left( x + \frac{l}{2} \right) - u \left( x - \frac{l}{2} \right) \right]. \hspace{1cm} (6.2)$$

Therefore, the overlaying imbricate elements ease evaluation of the nonlocal strain.

The stresses of local and nonlocal elements are evaluated separately:

$$\sigma^{l}(x) = G^{l}[\epsilon^{l}(x)], \quad \sigma^{nl}(x) = G^{nl}[\epsilon^{nl}(x)], \hspace{1cm} (6.3)$$

where $G^{l}$ and $G^{nl}$ are the pre-defined constitutive relationships for the local and nonlocal elements, respectively. These constitutive relationship will be discussed in more detail in the next subsection.

In addition, a second spatial averaging is applied to the nonlocal stress and strain.
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to evaluate broad-range parameters:

$$\sigma^b(x) = \frac{1}{l} \int_{x-l/2}^{x+l/2} \sigma^{nl}(x + s) ds,$$

(6.4)

$$\epsilon^b(x) = \frac{1}{l} \int_{x-l/2}^{x+l/2} \epsilon^{nl}(x + s) ds.$$

(6.5)

Then the total stress and strain is calculated by:

$$\sigma^t = c_f \sigma^l + (1 - c_f) \sigma^b,$$

(6.6)

$$\epsilon^t = c_f \epsilon^l + (1 - c_f) \epsilon^b,$$

(6.7)

where $c_f$ is the fraction of local parameters, $0 < c_f < 1$.

Similarly to the traditional finite element method, the nodal displacement $\mathbf{u}$ is calculated by solving the equation:

$$K \mathbf{u} = \mathbf{F},$$

(6.8)

where $K$ is the global stiffness matrix, and $\mathbf{F}$ is the global nodal force vector. In the nonlocal context, the stiffness matrix, the nodal force vector, and the traction on the
boundaries is decomposed into local and nonlocal parts as follows:

\[
K = c_f K^l + (1 - c_f) K^{nl}, \quad (6.9)
\]
\[
F = c_f F^l + (1 - c_f) F^{nl}, \quad (6.10)
\]
\[
T = c_f T^l + (1 - c_f) T^{nl}. \quad (6.11)
\]

**Figure 6.3:** Illustration of 1D imbricate nonlocal finite element model.

The local stiffness matrix and force vector are assembled following the same procedure as in traditional finite element analysis; while a special treatment is required for the nonlocal parts. Take the one-dimensional problem shown in Fig. 6.3 as example.

At the node numbered \( k \), the local nodal force \( F_k^l \) is evaluated by:

\[
F_k^l = \sum f^l = \frac{\sigma_{k+1}^l - \sigma_k^l}{h}, \quad (6.12)
\]
where $f^l$ represent the effective force vector from the local elements connected to the node $k$. The nonlocal nodal force $F_{nl}^k$ is:

$$F_{nl}^k = \sum \frac{f_{nl}^i}{\bar{n}} = \frac{\sigma_{nl}^{k+n} - \sigma_{nl}^k}{\bar{n}h}. \quad (6.13)$$

A similar approach is applied to the local and nonlocal stiffness matrices. For two dimensional problems, the nonlocal force vector and stiffness matrix can be expressed by:

$$F_{nl}^x = \sum \frac{f_{nl}^i}{\bar{n}_x}, \quad F_{nl}^y = \sum \frac{f_{nl}^i}{\bar{n}_y}, \quad (6.14a)$$
$$K_{nl}^x = \sum \frac{k_{nl}^i}{\bar{n}_x}, \quad K_{nl}^y = \sum \frac{k_{nl}^i}{\bar{n}_y}, \quad (6.14b)$$

in which $f_{nl}^x, f_{nl}^y, k_{nl}^x, k_{nl}^y$ are the element nonlocal force vectors and stiffness matrices, while $\bar{n}_x$ and $\bar{n}_y$ denote the size of the nonlocal elements in the two directions, although in general $\bar{n}_x = \bar{n}_y$.

### 6.2.2 Definition of constitutive relationship

Consider a material with a simple strain softening property, with the stress-strain curve shown in Fig. 6.4. Before yielding occurs the material exhibits linear elastic behavior with modulus $E_0$; after that, the tangent modulus becomes negative and is denoted as $E_t^l$. As the displacement increases the stress drops to zero, at which point
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a infinitesimal positive tangent modulus is assumed.

Figure 6.4: Stress strain diagram. Bazant et.al, (1984) [68].

The constitutive relationship is decomposed and assigned respectively to the local and to the nonlocal elements. Based on nonlocal theory, the local elements are always linear elastic, while the inelastic properties such as the strain softening are assigned to the nonlocal elements. Therefore, for local elements the following relationship always applies:

\[ \sigma^l = E_0 \epsilon^l \]  (6.15)

For nonlocal elements, assuming the yielding occurs at strain \( \epsilon^y \), the nonlocal strain \( \epsilon^{nl} \) can be broken into an elastic part \( \epsilon^{nl}_e \) and an inelastic part \( \epsilon^{nl}_p \). After the yield stress the inelastic strain is:

\[ \epsilon^{nl}_p = \epsilon^{nl} - \epsilon^{nl}_e = \epsilon^{nl} - \frac{\sigma^{nl}}{E_0}, \]  (6.16)
in which the nonlocal stress $\sigma^{nl}$ can be evaluated by:

$$\sigma^{nl} = E_0 \epsilon^{nl}_e = E_0 \epsilon^y + E_p (\epsilon^{nl} - \epsilon^y)$$

(6.17a)

$$= E_p \epsilon^{nl} + (E_0 - E_p) \epsilon^y.$$  

(6.17b)

It should be noted that $E_p$ denotes the inelastic (softening) tangential modulus of the nonlocal elements, which is not the same as the total inelastic tangent modulus $E^t_p$. Assuming the whole structure deforms uniformly, $E_p$ and $E^t_p$ can be interrelated by:

$$E^t_p = \frac{E_0 E_p}{c_f E_p + (1 - c_f) E_0}.$$  

(6.18)

However, since the deformation always tends to localize in a certain region when strain softening occurs, the uniform deformation assumption is not valid and thus the above relationship is not applicable. More detailed discussions regarding the definition of $E_p$ will be presented in the next section along with the one-dimensional strain softening problem.

Reformulating Eq. (6.17a) and (6.17b), we have:

$$\epsilon^{nl}_e = \frac{E_p}{E_0} \epsilon^{nl} + \left( 1 - \frac{E_p}{E_0} \right) \epsilon^y,$$

(6.19)

$$\epsilon^{nl}_p = \left( 1 - \frac{E_p}{E_0} \right) \epsilon^{nl} - \left( 1 - \frac{E_p}{E_0} \right) \epsilon^y.$$  

(6.20)

Since the broad-range strain $\epsilon^b$ is the spatial averaged term of the nonlocal strain, $\epsilon^b$
can also be decomposed into the elastic and inelastic part, and the elastic part $\epsilon^b_e$ can be evaluated by a relationship similar to Eq. (6.19):

$$\epsilon^b_e = \frac{E_p}{E_0} \epsilon^b + \left(1 - \frac{E_p}{E_0}\right) \epsilon^y. \quad (6.21)$$

The total elastic strain is then obtained by substituting the elastic parts into Eq. (6.7):

$$\epsilon^t_e = c_f \epsilon^l + \left(1 - c_f\right) \epsilon^b_e. \quad (6.22)$$

### 6.2.3 Nonlocal elements on the boundaries

In the boundary regions there are nonlocal elements that are smaller than the size of nonlocal element ($l$). Therefore, the nonlocal elements on the boundary need to be truncated. Special treatment must be applied to these nonlocal elements; and this treatment influences the entire model.

Again, we take the 1D example shown in Fig. 6.3 to demonstrate the boundary elements. In this case the size of nonlocal element is three times the local one ($\bar{n} = 3$).
The stresses of the nonlocal elements on the left boundary are given by:

\[
\sigma_{nl}^1 = \sigma_l^1, \quad (6.23a)
\]

\[
\sigma_{nl}^2 = (\sigma_l^1 + \sigma_l^2)/2, \quad (6.23b)
\]

\[
\sigma_{nl}^3 = (\sigma_l^1 + \sigma_l^2 + \sigma_l^3)/3, \quad (6.23c)
\]

\[
\sigma_{nl}^4 = (\sigma_l^2 + \sigma_l^3 + \sigma_l^4)/3... \quad (6.23d)
\]

In the nonlocal elements with \( k \leq \bar{n} \), the evaluation of nonlocal stress becomes identical with elements away from the boundary. Similar equations are applicable to the nonlocal strains.

Considering the nodal forces, we note that since \( \bar{n} \) nonlocal elements are connected to the node at the boundary \( (k = 0) \), the traction contributed by the nonlocal elements is evaluated by:

\[
T_{nl} = \frac{1}{3} (\sigma_{nl}^1 + \sigma_{nl}^2 + \sigma_{nl}^3)
\]

\[
= \frac{1}{3} \left( \sigma_l^1 + \frac{\sigma_l^2}{2} + \frac{\sigma_l^3}{3} \right) \quad (6.24)
\]

For now consider only the contribution by the nonlocal elements. When the finite element model reaches a stable solution, the total external force at each node converges to zero due to equilibrium. Therefore, according to Eq. 6.13 at node 1 we
have:

\[ \sigma_{nl}^{1} = \sigma_{nl}^{4}, \quad (6.25a) \]

\[ \Rightarrow \sigma_{l}^{1} = \frac{1}{3}(\sigma_{l}^{2} + \sigma_{l}^{3} + \sigma_{l}^{4}), \quad (6.25b) \]

while at node 2:

\[ \sigma_{nl}^{2} = \sigma_{nl}^{5}, \quad (6.26a) \]

\[ \Rightarrow \frac{1}{2}(\sigma_{l}^{1} + \sigma_{l}^{2}) = \frac{1}{3}(\sigma_{l}^{3} + \sigma_{l}^{4} + \sigma_{l}^{5}). \quad (6.26b) \]

Combining Eqs. (6.24) through (6.26), the following relationship between the traction and the local stresses is obtained:

\[ T_{nl} = \frac{1}{9}\left(\sigma_{l}^{1} + 2\sigma_{l}^{2} + 3\sigma_{l}^{3} + 2\sigma_{l}^{4} + \sigma_{l}^{5}\right). \quad (6.27) \]

Eq. (6.27) represents a weighting function with triangular shape, as shown in Fig. 6.5a. Similarly, a weighting function for 2D problem with the size of nonlocal element \( \bar{n} = 25 \) is shown in Fig. 6.5b, in which the weighting function shows a cone shape.

Since force equilibrium is reached for every node inside the model, such a spatial weighting function is applied to all the local parameters inside the structure. The effect of this weighting function will be discussed in Section 6.3.4 as we compare the nonlocal results with those from a local model in a benchmark problem.
For the nodes on the boundaries of a two-dimensional problem, Eqs. 6.14 are no longer applicable, since each node is attached to multiple layers of nonlocal elements. To address this issue, the force vector and stiffness matrix of the nodes on the boundary \( x = 0 \) or \( x = L \) should be modified as:

\[
\begin{align*}
F_{nl}^x &= \sum f_{nl}^x \hat{n}_x, \\
F_{nl}^y &= \sum f_{nl}^y \hat{n}_y, \\
K_{nl}^x &= \sum k_{nl}^x \hat{n}_x, \\
K_{nl}^y &= \sum k_{nl}^y \hat{n}_y,
\end{align*}
\]  

(6.28a)

while the traction should be modified to be:

\[
\begin{align*}
T_{nl}^x &= \sum f_{nl}^x \hat{n}_x, \\
T_{nl}^y &= \sum f_{nl}^y \hat{n}_x.
\end{align*}
\]

(6.29)

Similarly, on the boundaries \( y = 0 \) and \( y = H \), the residual and stiffness matrix
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should be modified as:

\[ F^{nl}_x = \sum \frac{f^{nl}_x}{n_x n_y}, \quad F^{nl}_y = \sum \frac{f^{nl}_y}{n_y}, \quad \] (6.30a)

\[ K^{nl}_x = \sum \frac{k^{nl}_x}{n_x n_y}, \quad K^{nl}_y = \sum \frac{k^{nl}_y}{n_y}, \quad \] (6.30b)

while the traction becomes:

\[ T^{nl}_x = \sum \frac{f^{nl}_x}{n_y}, \quad T^{nl}_y = \sum \frac{f^{nl}_y}{n_y}. \quad \] (6.31)

6.3 Numerical Examples

In this section, we deploy the nonlocal finite element method to several benchmark problems with strain-softening properties, and thus investigate the features of the nonlocal method. The nonlocal finite element models presented here are coded and simulated in Matlab.

6.3.1 One dimensional structure under dynamic load

Here the one-dimensional problem investigated by Bazant (1984) [68] is replicated to provide a verification of our nonlocal finite element model.

The problem is represented in Fig. 6.3: the 1D structure is pulled apart at the
two ends with constant velocity $v = 1$, and the deformations are transmitted towards the center as elastic waves. When the waves from the two ends meet at the center the local strain exceeds the yielding threshold, and strain softening occurs. The simple constitutive relationship shown in Fig. 6.4 is applied to every nonlocal element. The input parameters of the model and the material properties are listed in Table 6.1, in which $L$ denotes the full length of the sample, while $N$ denotes the total number of local elements. The origin lies at the left end of the structure.

<table>
<thead>
<tr>
<th>$E_0$</th>
<th>$E_p$</th>
<th>$\epsilon_y$</th>
<th>$\epsilon_f$</th>
<th>$L$</th>
<th>$N$</th>
<th>$\bar{n}$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.3</td>
<td>1</td>
<td>0.1</td>
<td>1</td>
<td>195</td>
<td>39</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 6.1:** Modeling parameters in 1D nonlocal finite element model
Figure 6.6: Results from the nonlocal model described in Section 6.3.1, corresponding to the example in Bazant et. al (1984) [68]. Parameters referred to Table 6.1. (a) Nodal displacement $u$, (b) Nonlocal strain $\epsilon_{nl}$, (c) Total stress $\sigma_t$, (d) Local strain $\epsilon^l$. 
The results at different time steps are shown in Fig. 6.6, which are almost identical with the original work of [68]. Our results show more fluctuations particularly in the local strain, which is likely caused by the difference in the computational scheme and is not significant to the overall results. The successful reproduction of the 1D strain softening problem verifies the new nonlocal model, and thus we can apply it to investigate other problems.

6.3.2 Statically loaded one-dimensional structure with an imperfection

Consider the example again the one-dimensional structure shown in Fig. 6.3, in this case a semi-static displacement-controlled load is applied; therefore, equilibrium is reached at each time step before moving to the next time step. The nonlocal model is defined by the same parameters in Table 6.1, except for the nonlocal element located at the center, in which the yielding strain is 1% less than other nonlocal elements so that the strain softening first occurs at the center. The origin of the coordinate is set at the center. The modeling results are shown in Fig. 6.7.

For such a one-dimensional strain softening problem, De Borst and Muehlhaus (1992)[91] derived the analytical solution of a gradient type nonlocal method. Inspired by their work, similar approach is presented here for comparison to the nonlocal model.
Figure 6.7: Comparing nonlocal finite element and analytical results for a semi-static one-dimensional strain softening problem. Model and material parameters defined in Table 6.1. (a) Total strain distribution $\epsilon^t$ at the applied load $\Delta U = 1.2$; (b) Stresses distribution by nonlocal finite elements at the applied load $\Delta U = 1.2$; (c) Relationship between the applied load $\Delta U$ and traction $T$. 

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The local strain $\epsilon^l$ in Eq. (6.5) is expanded with a Taylor series at location $x$:

$$\epsilon^l(x + s) = \epsilon^l(x) + s\epsilon'^l(x) + \frac{s^2}{2}\epsilon''^l(x) + \frac{s^3}{6}\epsilon^{(3)}l(x) + \frac{s^4}{24}\epsilon^{(4)}l(x) + O. \quad (6.32)$$

Substituting this series into Eq. (6.1), the nonlocal strain at location $x$ is approximated by:

$$\epsilon^{nl}(x) \approx \epsilon^l(x) + \alpha_f\epsilon'^l(x) + \beta_f\epsilon^{(4)}l(x), \quad (6.33)$$

where

$$\alpha_f = \frac{t^2}{24}, \quad \beta_f = \frac{t^4}{19200}. \quad (6.34)$$

Similarly, the broad-range strain is approximated by:

$$\epsilon^b(x) \approx \epsilon^{nl}(x) + \alpha_f\epsilon'^{nl}(x) + \beta_f\epsilon^{nl(4)}(x), \quad (6.35)$$

Reversely, the local strain $\epsilon^l(x)$ is approximated by:

$$\epsilon^l(x) \approx \epsilon^{nl}(x) - \alpha_f\epsilon'^{nl}(x) - \beta_f\epsilon^{nl(4)}(x), \quad (6.36)$$

As the whole FE model reaches equilibrium, $\epsilon_e^t = T/E_0$; combining with Eq. (6.22)
and Eq. (6.21):

\[
(1 - c_f) \left[ \frac{E_p}{E_0} \epsilon^b + \left( 1 - \frac{E_p}{E_0} \right) \epsilon^y \right] + c_f \epsilon^l = \frac{T}{E_0}. \tag{6.37}
\]

Substituting the Eq. (6.35) and (6.36) into Eq. (6.37):

\[
(1 - c_f) \frac{E_p}{E_0} \left( \epsilon^{nl} + \alpha_f \epsilon^{nl'n} + \beta_f \epsilon^{nl(4)} \right) + c_f \left( \epsilon^{nl} - \alpha_f \epsilon^{nl'n} - \beta_f \epsilon^{nl(4)} \right) = \frac{T}{E_0} - (1 - c_f) \left( 1 - \frac{E_p}{E_0} \right) \epsilon^y. \tag{6.38}
\]

Since \( \epsilon^y \) is constant, differentiating Eq. (6.20) gives:

\[
\epsilon^{nl'n} = \frac{E_0}{E_0 - E_p} \epsilon_p^{nl'n}, \tag{6.39a}
\]

\[
\epsilon^{nl(4)} = \frac{E_0}{E_0 - E_p} \epsilon_p^{nl(4)}. \tag{6.39b}
\]

Substituting (6.39) into (6.38) and reformulating, the following equation is reached:

\[
A \epsilon_p^{nl} + B \epsilon_p^{nl'n} + C \epsilon_p^{nl(4)} = M, \tag{6.40}
\]

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where

\[ A = \frac{E_p}{E_0 - E_p} + c_f, \quad (6.41a) \]
\[ B = \alpha_f \left[ \frac{E_p}{E_0 - E_p} - c_f \frac{E_0 + E_p}{E_0 - E_p} \right], \quad (6.41b) \]
\[ C = \beta_f \left[ \frac{E_p}{E_0 - E_p} - c_f \frac{E_0 + E_p}{E_0 - E_p} \right], \quad (6.41c) \]
\[ M = \frac{T}{E_0} - \epsilon^y. \quad (6.41d) \]

Assuming that the solution of \( \epsilon_{nl}^p(x) \) takes the form:

\[ \epsilon_{nl}^p = H \cos \left( \frac{x}{l_0} \right) + \frac{M}{A}, \quad (6.42) \]

in which \( l_0 \) is a characteristic length. Substituting the above expression of \( \epsilon_{nl}^p \) into Eq. (6.40):

\[ A - \frac{B}{l_0^2} + \frac{C}{l_0^4} = 0. \quad (6.43) \]

Therefore, \( l_0 \) is solved:

\[ l_0 = \sqrt[4]{\frac{B \sqrt{\frac{B}{4A}}}{2A} + \frac{C}{A}}. \quad (6.44) \]
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If we differentiate Eq. (6.42) with respect to time, there is:

$$\dot{\epsilon}_{nl}^p = \dot{H} \cos \left( \frac{x}{l_0} \right) + \frac{\dot{T}}{AE_0}.$$  \hspace{1cm} (6.45)

Assume the region that enters the strain softening stage is measured by a length \(w\). Applying the condition that \(\dot{\epsilon}_{nl}^p = 0\) at the boundary of inelastic region \(x = w/2\):

$$\dot{\epsilon}_{nl}^p = \frac{\dot{T}}{AE_0} \left[ 1 - \frac{\cos(x/l_0)}{\cos(w/2l_0)} \right].$$  \hspace{1cm} (6.46)

The non-trivial solution for \(w\) is \(2\pi l_0\). Therefore, the inelastic strain is simplified as:

$$\dot{\epsilon}_{nl}^p = \frac{\dot{T}}{AE_0} \left[ 1 + \cos(x/l_0) \right].$$ \hspace{1cm} (6.47)

Integrating the above expression over the whole inelastic region, the deformation rate contributed by inelastic strain is obtained; then added by the total elastic strain rate \(\dot{\epsilon}_e^t = \dot{T}/E_0\), the displacement velocity \(\dot{U}\) at the end of the structure can be calculated by:

$$\dot{U} \left( \frac{L}{2} \right) = \frac{\dot{T} L}{2E_0} + (1 - c_f) \frac{\dot{T}}{AE_0} \int_0^{w/2} \left[ 1 + \cos \left( \frac{x}{l_0} \right) \right] dx.$$ \hspace{1cm} (6.48)

Substituting \(w = 2\pi l_0\) into the above integration, the closed-form expression of the
above equation is given by:

\[ \dot{U}(L/2) = \left[ \frac{L}{2E_0} + (1 - c_f) \frac{w}{2AE_0} \right] \dot{T}. \]  \hfill (6.49)

Since the values of \( \dot{U} \) on the two ends are known to be the applied load, the traction rate \( \dot{T} \) is obtained:

\[ \dot{T} = \frac{\Delta \dot{U}}{L} = \frac{\Delta \dot{U}}{E_0 + (1 - c_f) \frac{w}{AE_0}}. \]  \hfill (6.50)

where \( \Delta \dot{U} = \dot{U}(L/2) - \dot{U}(-L/2) \). Substituting the above expression back to Eq. (6.47), the solution for \( \dot{\epsilon}^{\text{nl}} \) is achieved. Furthermore, the slope of the traction-strain relationship can be obtained by reformulating the above equation:

\[ E_p' = \frac{\dot{T} L}{\Delta \dot{U}} = \frac{E_0}{1 + (1 - c_f) \frac{w}{LA}}. \]  \hfill (6.51)

The analytical result of strain distribution is shown in Fig. 6.7a, while the traction-displacement relationship is plotted in Fig. 6.7c. Comparing with the results of nonlocal finite element model, very good agreement is observed.

From the analytical results we can conclude that, when strain softening occurs in the one-dimensional structure as we investigated here, the size of the inelastic region \( w \) depends on the length of the nonlocal element \( l \) as well as the tangent moduli (Eqs. (6.41) and (6.44)), and the relationships are rather complex. The slope of the
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traction-strain of the whole structure should be determined by Eq. (6.51), rather than by Eq. (6.18). Nevertheless, the nonlocal properties such as $l$ and $E_p$ can be determined if the values of the traction-strain slope $E^t_p$ and the length of inelastic region $w$ are provided, which might be estimated from experiments.

6.3.3 Two dimensional model with a local defect

Consider a two-dimensional problem as shown in Fig. 6.8. A pseudo-static uni-axial displacement-controlled load is applied in the $y$-direction. The material follows the constitutive relationship as shown in Fig. 6.4. The yielding strain in the center is 1% less than the rest of the material, so that inelasticity first occurs at the center. The model and material properties follow Table 6.1 except the total number of local elements $N = 55 \times 30$ and the nonlocal element size $\bar{n} = 11$.

![Figure 6.8: Two dimensional material with strain-softening property applied with semi-static uniaxial loads](image)

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Essentially this is the same one-dimensional problem as that discussed in Subsection 6.3.2, although the second dimension is explicitly taken into account. We perform this task for the purpose of verifying the definition of the two-dimensional nonlocal FE model, in particular the definitions of the nonlocal elements on the boundaries. If the nonlocal model is properly defined, the same results (in particular, the strain distribution $\epsilon_{yy}$) in previous subsection should be recorded.

Two sets of modeling results of $\epsilon_{yy}$ are shown in Fig. 6.9. Fig. 6.9a and b show the results with correct boundary definitions of the stiffness matrices, force vectors and tractions (Eqs. (6.28) through (6.31)). To verify the results, the analytical solutions are also plotted in Fig. 6.9b. We can see that the strain distribution along the loading direction agrees well with the analytical results. Therefore, the nonlocal elements on the boundaries are properly defined.

In contrast, modeling results of $\epsilon_{yy}$ using definitions of Eqs. (6.14) for the nonlocal elements on the boundaries are shown in Fig. 6.9c and d, in which spurious results are presented. Comparing with the analytical result, the nonlocal FE results over-estimate the strain on the boundaries. Therefore, it is confirmed that special treatment should be applied to the nonlocal elements on the boundaries instead of using the same definitions as those elements on the interior of the model. Although this implementation is rather straightforward for a rectangular mesh, in domains with irregular shape the correct boundary definition is non-trivial.
Figure 6.9: Nonlocal modeling results with correct and incorrect boundary definitions at load $\Delta U = 1.02$, modeling parameters follows Table 6.1. Analytical results of strain distribution predicted by Eq. 6.42 are plotted as star-marked curves for comparison. (a) and (b): Results with correct boundary definitions (Eqs. (6.28) through (6.31)). (c) and (d): Results with incorrect boundary definitions (Eqs. (6.14)).
6.3.4 Two dimensional model with a line crack

Now consider applying a two dimensional nonlocal model to a solid containing a single line crack at the center, as shown in Fig. 6.10. The normal of the crack is aligned with the y axis, while a uniaxial displacement-controlled load is applied to the sample along the same direction.

To simulate the weakening of the material due to the crack opening, an elasto-plastic constitutive relationship with strain-softening is assigned to the nonlocal elements which contain the crack. An example of this constitutive relationship is shown in Fig. 6.10b.

In order to verify the modeling results by the nonlocal method, the traditional finite element method is also deployed to simulate the same problem. Since the traditional FE method cannot properly handle strain-softening, it is adapted as follows. The crack is assumed to stop developing when the effective material properties...
creased to a specific level, as shown in Fig. 6.10b; after this, the material retains a residual modulus $E_r$. If the external load is removed, the stress follows the path to the origin, following a linear elastic relationship. In the nonlocal FE model, the nonlocal elements associated with the crack (region colored other than blue in Fig. 6.11a) are assigned the elasto-plastic relationship, while the other nonlocal elements and all the local elements are assigned an elastic modulus $E_0$. In the traditional finite element model the residual modulus $E_r$ is assigned to the elements attached to the crack (red region in Fig. 6.11b), and the original elastic modulus $E_0$ is assigned to the rest of the elements. The modeling results at the conjunction point of these two constitutive relationships (Fig. 6.10b) are compared.

![Figure 6.11: Material properties definition. (a) Nonlocal finite element model. (b) Traditional finite element model.](image)

For simplicity, in the traditional FE method, the same residual modulus is assigned to all the “cracked” elements; however, in the nonlocal FE model, different constitutive properties need to be assigned to the nonlocal elements depending on the
size of the crack that each of them cover (illustrated by the different colors around the cracked region). The details regarding the definitions of constitutive properties are not presented here.

In addition, according to the discussion in Section 6.2.3, a triangular / conoid weighting function is applied using the nonlocal elements. To make a fair comparison, the modeling results of the traditional FE model are applied that the weighting function is in accordance with the size of the nonlocal element.

The results of total strain $\varepsilon_{yy}$ and total stress $\sigma_{yy}$ predicted by the nonlocal finite element method are shown in Fig. 6.12, while the weighted results of the corresponding stress and strain predicted by traditional finite elements are shown in Fig. 6.13. Both of the two models contains 81 local elements in $x$ and $y$ directions; in the nonlocal method, the size of the nonlocal element is 5 times the local element. The crack covers 5 local elements. When the strain softening occurs, the nonlocal model reaches stable solutions without any convergence issue.

Comparing the results predicted by these two models, the strain predicted by nonlocal model generally agrees well with the weighted results of the traditional FE method on the spatial distribution, although there is some underestimation of the magnitude. There is a larger discrepancy in the predicted stress as generally the distribution is more concentrated near the crack region.
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Figure 6.12: Modeling results of nonlocal FE method. (a) total strain $\varepsilon_{yy}^t$. (b) total stress $\sigma_{yy}^t$.

Figure 6.13: Modeling results of traditional FE method. (a) Weighted strain $\varepsilon_{yy}^w$. (b) Weighted stress $\sigma_{yy}^w$. 
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6.4 Summary, Conclusion and Discussion

In this chapter, a spatial integration nonlocal finite element method was reviewed and investigated. The nonlocal method is demonstrated through several benchmark problems. The nonlocal method shows advantages in the following aspects:

(a) Reaches a stable, convergent solution under strain-softening.

(b) Overcomes the issue of mesh sensitivity: With the localization limiter, the inelastic strain no longer concentrates abnormally in the smallest element, but is distributed over a prescribed region. The distribution of the inelastic strain, as shown in the example in Section 6.3.2 relates to the modulus of the local and nonlocal element, the local parameter $c_f$, the size of the nonlocal elements, as well as the location in which the strain softening occurs. When the appropriate parameters such as the $c_f$ and the size of nonlocal element are assigned, the response of the material can be simulated in very good agreement with experimental observations.

On the other hand, the technique described here exhibits a number of limitations and difficulties, which include the following.

(a) The appropriate length of the nonlocal element / region of averaging is difficult to determine. Since the length of the nonlocal element is considered a material property, its value is determined so that the results predicted by the nonlocal finite element modeling agree well with experimental observations. However, (1) there’s no common agreement in the field based on which properties should be matched between these two methods; (2) strain softening behavior usually occurs when the material
experiences abrupt failure, condition under which accurate measurements on any properties are highly challenging; (3) even if the required parameters are available from the experiments, analytically calculating the length of the nonlocal element remains a problem. For example, in a 1D strain softening problem such as discussed in Section 6.3.2, there is no straightforward relationship between the length of the nonlocal element $l$ and the region that enters inelastic states $w$; in order to calculate the length of nonlocal element $l$, the traction $T$, the undamaged elastic modulus and the maximum inelastic strain are all required. For more sophisticated problems this is a highly challenging task.

(b) Decomposition of material properties between the local and nonlocal elements is required. In other words, the value of the local parameter $c_f$ is unknown, although in many publications the value 0.1 is empirically presumed. Indeed this is related to the previous question regarding the length of the nonlocal element. Since the total properties decompose into the local and nonlocal parts, and assuming the local properties always elastic, changing the value of $c_f$ requires a change in the nonlocal properties to acquire a consistent material response to those observed in experiments. The changes in $c_f$ and the nonlocal properties lead to a difference in the nonlocal solution. For example, from the analytical solution under pseudo-static 1D strain softening (Section 6.3.2), it is clear that given different values of $c_f$ and $E_p$ (changing in accordance to give a same $E_p^i$), the solutions for the inelastic strain distribution as well as the traction are different, even if the length of the nonlocal element is the
(c) Proper element definition on the boundaries is not well understood. For the imbricate finite element methods, the nonlocal elements on the boundaries are truncated and thus special treatment is required. Section 6.3.3 demonstrates that spurious results occur when the boundary definitions are not properly assigned. When modeling objects with complex shapes, meshing with prismatic elements may not be feasible, and thus the boundary definitions require more effort than Eqs. (6.28) through (6.31). Similar problems also occur in the gradient type nonlocal method, where additional information regarding the gradients of the parameters is required on the boundaries.

(d) Element discretization is limited. For the imbricate finite elements that we presented here, the element discretization is limited; each element should cover the same area/volume and even possess the same shape. Such highly restricted spatial discretization is impractical for most engineering problems. The restriction on discretizations is relieved in the gradient nonlocal method and in spatial averaging with more advanced averaging techniques.

(e) The approach is not effective for heterogeneous materials. As discussed in Section 6.3.4, when modeling materials with heterogeneous properties, a nonlocal element is assigned the effective modulus based on the micro-structure of the material it covers. Since the fraction of nonlocal properties \((1 - c_f)\) is typically 0.9, the heterogeneous feature is therefore smeared by spatial averaging.

(f) Reverse processing is required to resolve the local stress/strain distribution.
CHAPTER 6. NON-LOCAL FINITE ELEMENT MODELING

As shown in Section 6.2.3, a triangular/conoid weighting function is applied to the solutions in a nonlocal model. In Section 6.3.4 it is revealed that the nonlocal modeling results agree well to some extent with the weighted results predicted by the traditional FE method. To obtain the true local distribution of the parameters (stress/strain), reverse processing to the weighting function should be applied to the nonlocal results, although this reverse processing has not been well studied.

Due to limitations and difficulties described above, the nonlocal finite element method has not been widely applied in engineering practice. In recent years new modeling methods which are capable of capturing local failure while providing stable and converging solutions have been developed and applied in fracture and damage simulations, such as XFEM ([98, 99, 52, 100]), cohesive zone element [101, 102, 51], material point method [67], and Voronoi cell methods [103, 104].
Chapter 7

Conclusions and Future Work

7.1 Summary

Applying the concept of Materials By Design, as illustrated in Fig. 1.1, computational models that accurately capture the dynamic failure mechanisms in brittle materials (ceramics, concrete, etc.) enable feedback for material design that will ultimately enhance the material performance under dynamic conditions. This dissertation addresses several key issues in micromechanical models of dynamic brittle failure under compression. Although macroscale modeling has not been performed directly in this work, these studies benefit macroscale modeling by developing a constitutive model that better captures the physical mechanisms of brittle failure, as well as enhancing the feasibility in application by addressing model efficiency.

Comparing with the tensile case, failure of brittle material under compressive loads
CHAPTER 7. CONCLUSIONS AND FUTURE WORK

is featured by the contact on the crack interface, and the consequent crack opening at the non-contact interface due to the frictional sliding. Such an intricate failure mechanism imposes challenges to modeling the behavior of brittle materials under compression.

In Chapter 2, presuming wing-crack damage as the dominant failure mechanism under compression, analytical closed-form solutions for the instantaneous effective compliance (or stiffness) are derived through both kinematic and energetic approaches. These solutions are functions of the density of the pre-existing flaws, geometric measurements of the wing-cracks and the friction coefficient at contact surfaces. Based on the periodic wing-crack RVE, finite element models with periodic boundary conditions are presented in Chapter 3. The strong agreement between the finite element and analytical results verify the effectiveness of the analytical solutions.

In Chapter 4, micromechanical models that address wing-crack growth are investigated. Stress-strain relationships of the brittle materials under dynamic compressive load are established by combining a micromechanics model that predicts crack growth and the analytical anisotropic compliance solution developed in the previous two chapters.

Incorporation of the micromechanics into macro-scale analysis requires that the micromechanics model is implemented at every integration point in the macro-scale model at every time step, which is highly computationally expensive. To address this computational efficiency issue in macroscale simulations, an upscaling technique is
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proposed in Chapter 5. Featuring a storage-retrieve scheme based on a Taylor series expansion of the response, this methodology can be deployed to enhance the efficiency of the macroscale models as well as to enable statistical models that address material heterogeneity.

Given that the experiments and the models both show strain softening behavior at high levels of damage, the macro-scale model must be capable of handling strain softening, which is characterized by a negative tangent modulus. In order to tackle the strain softening effect associated with highly damaged materials, an investigation based on a classical non-local finite element method is presented in Chapter 6.

7.2 Conclusions

The current work addresses issues in modeling dynamic failure in brittle materials, under both tensile and compressive loadings. In particular, a solution for the anisotropic damage-compliance relationship in compression and in tension is proposed, a micromechanics model for calculating the stress-strain behavior is corrected, the proposed anisotropic damage-compliance relationship is incorporated into the micromechanics model, and a non-local finite element model for addressing strain softening at the macro-scale is summarized. Conclusions from all of this work are summarized as follows:

- The micromechanics model based on an isotropic damage model does not give
reasonable results, showing that stress increases indefinitely with strain rather than reaching a peak stress as observed in dynamic compression tests. These results emphasize the necessity of implementing an anisotropic damage model.

- Both analytical and finite element results confirm that the effective compliance tensor of a wing-crack damaged material is highly anisotropic.

- The presence of friction on the flaw surface increases the instantaneous stiffness of the material (i.e., reducing the value of compliance terms), but also leads to asymmetry of the compliance tensor. While this asymmetry is fully justifiable from a physical standpoint, this damage-compliance relationship should be applied carefully in subsequent mechanics models in which this asymmetry may cause numerical difficulties.

- Significant dilatation is observed experimentally in materials under dynamic compression, but many of the existing constitutive models do not capture this dilatation. Volumetric dilatation during the damage process is reflected in the proposed model by a rise in the effective Poisson’s ratio in the direction transverse to maximum compressive loading (noting that there is a corresponding decrease in the effective Poisson’s ratio in the direction of maximum compressive loading).

- Contact (or lack thereof) on the flaw surface leads to the difference between the compressive and tensile solutions for the wing-crack problem. When contact is
lost due to either flaw surfaces fully sliding past each other or to a change in the sign of the applied load, a discontinuous change of the effective compliance occurs.

- The micromechanical model that incorporates the analytical anisotropic compliance solution, as proposed in Chapter 4, is capable of handling a population of flaws with a distribution of sizes and/or orientations.

- When multiple flaw sizes are incorporated into the micromechanics model, the predictions are more physically realistic than a model that assumes a single flaw size. For example, the model that incorporates multiple flaw sizes predicts more of a strain rate effect on strength than a model that assumes a single flaw size.

- Assuming the locations of flaws within a sample are fully random (following a Poisson’s process), statistically-based analyses from the micromechanical model show that the local strength of the brittle material under a dynamic compressive load follows a lognormal distribution. Furthermore, the coefficient of variation of the strength is proportional to the inverse of square root of the sample area in a two-dimensional model.

- The nonlocal finite element method is effective for reaching stable and convergent solutions when modeling strain-softening material; however, numerous difficulties exist that hinder the practical implementation of the nonlocal FE method, suggesting that other approaches such as the material point method
or a different constitutive model implementation may be more reasonable.

7.3 Future Works

This section addresses possible future directions as well as some natural extensions from the present work.

7.3.1 Analytical derivation of the crack compliance tensor

In Chapter 2 solutions of damage-compliance relationship are based on the two dimensional wing-crack model. Almost every expression in these solutions is based on a mechanical analysis without the aid of parameter fitting, except those of the crack compliance tensor $R$ in Eqs. (3.4) through (3.6), which are obtained by numerically analyzing the results of the finite element model. The fitting procedures, although effective, require excessive work in modeling and data processing, and they lack a rigorous physical insight into the problem.

The explicit expressions of crack compliance tensor for regular shaped cracks were extracted by Kachanov [80],[81] based on the work of Budiansky and O’Connell (1976)[82], in which the change of effective compliance is obtained by calculating the potential energies along the edge of the discontinued surfaces (crack). In the
original work there was no restriction on the shape of the discontinued surface; therefore, if we follow the same energetic approach in [40], the mechanics-based analytical expressions of the crack compliance $R$ for the wing-crack model may be accessible.

### 7.3.2 Three-dimensional damage-compliance solutions

Although the solutions of the effective compliance for wing-crack damaged materials presented in Chapter 2 and 3 considers two dimensional problems, the extension to three dimensional solutions are possible through analogous comparisons with the classical solutions for cracked solids. Recall Section 2.8.2 in which the proposed solution is compared with the solutions established by Tonge and Hu, both of which were three dimensional analyses. Such a comparison is only valid after properly reducing the three dimensional solution to two dimensions. The dimension reduction was done by first obtaining the ratio between the classical (Kachanov’s) two dimensional and three dimensional solutions based on the crack compliance concept ([80]). Then this ratio is applied to the solutions of Tonge and Hu’s solutions. The reverse process can be applied to extend the proposed two-dimensional solutions to three dimensions by assuming that the expressions for the crack compliance tensor $R$ remain unchanged. More specifically, the dimensional extension includes three steps: (a) changing the crack density $\eta$ from two dimensional values to comparable three dimensional values;
(b) changing the quadratic expressions for the geometric parameters (flaw size \( s \) and crack length \( l \)) in Eqs (2.89) - (2.93) to cubic terms; (c) multiplying the obtained 2D compliance expressions by the ratio between the 2D-3D solutions in Kachanov’s solution.

Of course, the approach described above is only an approximation to the true three-dimensional behavior. Obtaining a precise mechanics-based analytical solution of a three-dimensional wing-crack damaged solid is a very challenging task. Additional knowledge regarding the geometry and mechanics of the wing-crack in three-dimensional space would be required. The major challenge would be in predicting the shape of the wing-crack as the crack surface grows in three-dimensional space. The 2D wing-crack model assumes slit-shaped flaws and cracks with through-thickness discontinuity surfaces, which is naturally not the case for a three-dimensional material. In practice, for a ceramic sample, pre-existing flaws with high aspect ratios could be treated as circular/elliptical disks. Under an applied compressive load in the \( e_1 \) direction (Fig. 2.2), the wing-cracks propagate not only within the surface of \( e_1 \otimes e_2 \) due to the effective resolved shear stress on the flaw surface (mixed Mode I and II), but also along the \( e_3 \) direction which is primarily in Mode I fracturing; meanwhile the pre-existing flaw may also propagate in the \( e_3 \) direction under Mode III fracture. The three-dimensional geometry of the wing-crack therefore relies on the crack propagation rates of these three modes, which can be determined by the applied load, the critical stress intensity factors of the material as well as the instantaneous
CHAPTER 7. CONCLUSIONS AND FUTURE WORK

geometries of the wing-crack.

In particular, the propagation of the pre-existing flaw surface in Mode III enlarges the sliding surface and further promotes the growth of the wing-cracks. Previous authors (e.g., [8, 82]) tackled the 3D wing-crack model, but they presumed that the geometries of the pre-existing flaws remain unchanged during the damage process, which is not likely to be true. Since both the growth rate of the wing-crack and the damage-compliance relationship increase with the flaw size, the restriction on the growth of pre-existing flaw essentially underestimates the growth rate of damage as well as the instantaneous effective material compliance.

If the three-dimensional geometries and crack growth laws of the wing-crack can be determined, the analytical expressions of the crack compliance tensor for the given geometries could be obtained using the potential integration method proposed by [40], as discussed in the previous section. Once the three-dimensional closed form expression of wing-crack compliance tensor is obtained, the effective material properties in three dimensional space are readily solved following an analysis similar to Chapter 2 of this work.
7.3.3 Constitutive model under unloading/cyclic loading

The analyses performed here mostly assumed a monotonically increasing load. In reality, the stress state in the material at any particular point may follow a complex load path. For example, a loading and unloading is commonly observed in the material under an impact load.

The residual strain under unloading is a well-known phenomenon of damaged solids under cycled load, often referred to as hysteresis. For brittle materials, the hysteresis phenomenon is mainly caused by friction on the discontinued surfaces, which restricts the recovery of the separation on discontinued crack surfaces when the applied load is removed. Consider a brittle solid subjected to a uniaxial compressive load followed by unloading. Following the analysis in Chapter 2, the sliding displacement on the flaw surface $u_s$ increases with increasing load. Upon unloading, the frictional traction $P_{2s}$ (Eq. 2.77) required to move the flaw surface (Fig. 2.3) changes sign from $\mu|P_{2n}|$ to $-\mu|P_{2n}|$, but the force that drives sliding is no longer present. The sliding displacement $u_s$ therefore remains constant until either the resolved shear stress exceeds the static frictional limit (which does not occur in uniaxial unloading) or the flaw surface goes into tension and loses contact. The flaw surface cannot return to its intact position when the applied load is completely removed, therefore resulting in a residual displacement $u_s$. This residual displacement $u_s$ causes a non-zero strain.
at the traction free state, which is the residual inelastic strain.

Using the theoretical framework developed in Chapter 2, the magnitude of the sliding displacement $u_s$ on the flaw surface during the unloading process can be monitored. The magnitude of residual $u_s$ should be affected by the geometric variables $V$ as well as the maximum applied load $\sigma$. Once their relationship is established, the analytical solutions of the residual strain for a given parameter set of the wing-crack geometries are developed. Furthermore, if the fracture mechanism is incorporated to monitor the crack growth process under a given load path, quantitative results for the crack sliding displacement and the effective material properties during the unloading process can be resolved, and then the typical stress-strain curves for loading-unloading can be drawn. Such micromechanics-based analyses on the damaged brittle solids under loading-unloading should also be helpful for understanding progressive fracture (fatigue).

### 7.3.4 Monitoring interface contact under complex load paths

As the components of the stress tensor are perturbed for the purpose of obtaining the instantaneous effective compliance, the magnitude of the perturbations were limited to small values so that the instantaneous contact state was not broken. Sudden jumps of the effective compliance / moduli occur when the instantaneous contact
CHAPTER 7. CONCLUSIONS AND FUTURE WORK

state is not preserved, in other words, when the flaw surfaces lose contact with their counterparts.

As discussed in Chapter 2, the different contact states lead to two distinct solutions of effective material properties: if the flaw surfaces are in contact (state 2 in Fig. 2.6), the compressive solution is presumed and Eqs. (2.89) through (2.94) is applicable; otherwise (state 1 in Fig. 2.6), the tensile solution of Eqs. (2.107), (2.113) and (2.114) should be applied. This is confirmed by the good agreement between the tensile solution and the results of FE models in which excessive amount of perturbation applied and the contact condition failed to be preserved (comparisons not shown).

Contact status can be determined by the normal displacement $u_n$ in state 1 of Fig. 2.6: the flaw surfaces are in contact if $u_n < 0$. As shown in Eq. (2.75), $u_n$ can be evaluated by the applied far-field stress $\sigma$, orientations of the flaw $\phi$, and the crack compliance tensor $R$ which depends on the geometries of the wing-cracks. The presence of friction $\mu$ also affects the contact status by asserting the shear traction $P_s$ on the flaw surface.

The challenge is that abrupt changes in compliance can lead to instability in the implementation of these models. However, if multiple flaw families of different sizes and orientations are considered, the abrupt changes of effective material compliance are decreased in magnitude, since switching the contact states of different flaw families happens at different damage and applied load levels. Therefore, a smoother transition between the compressive and tensile solutions should be observed when
the micromechanical modeling is conducted with multiple flaw families. Using the framework presented here the values of applied load $\sigma$ and length of wing-cracks $l$ can be resolved for each flaw family when contact is lost. Such a solution would be useful, and in some cases necessary, in micro-mechanical modeling of multiaxial and/or complex loading, in order to properly reflect the material behavior.

### 7.3.5 Monitoring interface contact for irregular crack geometries

In Section 7.3.1 we discussed a possible way to obtain the expressions of the compliance tensor ($Z/R$) of a wing-crack, but conceptually this approach should be applicable to any irregularly shaped crack. When such a task is properly done, it becomes possible to monitor the contact status of crack surfaces as well as to predict the sliding distance along the contact faces under any applied loads. Using Hill’s kinematic relationship (Eq. (2.12)), closed form expressions of the effective compliance of the material can be easily solved from this sliding displacements.

Although this kind of task has not been addressed in the current work, it might be helpful for analysing materials with more complex and general damage mechanisms. One example would be the extended scenario of the wing-crack model, when the wing-cracks coalescence with each others and form more complex geometries consisting of several contact faces (pre-existing flaws) and non-contact sections (wing-cracks).
7.3.6 Experiments on brittle materials under compressive loads

The current work has been focused on the theoretical analyses and modeling of brittle materials under compressive loads. Experimental studies on the same topic can provide validation to the theoretical analyses and the model results.

The currently available experimental results can be further exploited to validate the analytical and modeling results. For instance, the Kolsky bar tests conducted by Hogan et al. [31] recorded the strains in both the loading and transverse directions of a brittle material under uniaxial dynamic compressive test; therefore, the effective Poisson’s ratio for the wing-crack damaged stage can be calculated using the data at the time steps around the occurrence of the peak stress. The calculated Poisson’s ratio can be compared against the analytical results presented in current work.

Nemat-Nasser and Horii [10] conducted uniaxial compressive tests on Columbia Resin CR 39 which contained a pre-existing slit crack, and analyzed the development of wing-cracks. The sample was essentially a two-dimensional material, and the whole experimental set-up is similar to the finite element model described in Chapter 3, although it was focused on understanding the contour of the wing-crack. Similar experiments can be conducted again with the goal to measure the material compliance with respect to the wing-crack geometry. The tasks described in Chapter 3 may be applied to define the parameters of such experiments. Advanced techniques such as
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Digital Imaging Correlation (DIC) allow us to observe the important parameters, such as the averaged strain, the sliding distance on the flaw surface, etc.

Experimental study on three-dimensional wing-crack is a more complicated and challenging task. Schulson and colleagues’ conducted dynamic compressive tests on ice \[66, 105, 106\] and successfully captured the formation of wing-crack from the artificially introduced disk-like “flaw” (steel platen). Similar experiments can be extended to further understand the development of wing-crack in 3D space, also observe the effective compliance of the 3D wing-crack-damaged materials.

7.3.7 Models incorporating other damage mechanisms

In the present work we are focused on the micro-cracking mechanism and particularly the wing-crack model, since micro-cracking is the dominant mechanism for inelastic response of brittle materials in the early and moderate damage stages. However, as the wing-cracks continue to develop and grow beyond the RVE boundaries, interaction between nearby cracks imposes more and more influence on the material response, and thus these non-interactive solutions generally underestimate the effective compliance at large damage values. Interaction among the cracks has been tackled by some authors \[80, 18\]; the ideas of these works can be incorporated into this framework to better predict the material response.
In reality the pre-existing flaws are never arranged in a periodic, homogeneous manner, such as shown in Fig. 2.4. In some brittle materials these flaws follow a purely random distribution, while in others they show certain level of clustering [53]. As the wing-cracks continue to grow, the probability of meeting another wing-crack rises. Once two cracks/flaws meet, they coalesce and should be treated as a single crack with a larger size. The probability that any wing-crack coalesces with others may be estimated using statistical methodologies. The degree of crack coalescence depends on the applied load, such as the loading rate. Under pseudo-static load the failure of the brittle material is dominated by a few major cracks nucleated from the largest flaws. On the contrary, under dynamic loads (high strain rate), smaller flaws are also activated and therefore micro-cracks are developed in numerous sites, which raises the probability of crack coalescence. Evidence for this can be found in the comminution zone of ceramic materials under high strain rate impact [108], [109], [39]. The effective material properties at this crack-coalescence stage have not been well investigated. Heavy interactions, local structural instability (such as buckling under axial splitting) and random spatial distributions impose great challenges on these studies.
7.3.8 Incorporating the analytical solutions of wing-crack damage mechanism into macro-scale simulations

Macroscopic simulations on the brittle dynamic failure, such as projectile penetration through a ceramic material or colliding of planetary bodies, generally requires three dimensional modeling and the corresponding constitutive relationships. As discussed in Section 7.3.2, the solutions of wing-crack damaged material properties derived in this work could be extended to three dimensions and provide a constitutive relation representing the damage stage of micro-cracking. The merits of our compliance solution have been summarized in the last sections of Chapter 2 and 3, but challenges rise for our solution to be practically deployed. The challenges may lie in the following aspects in particular:

- Inconsistent results due to the friction / hysteresis under complex load path.
- Asymmetric compliance / stiffness tensor.
- Efficiency in evaluating the components of the compliance / stiffness tensor at a given time step.

Nevertheless, these challenges are not critical to prohibit the implementation of this effective compliance solution.
7.3.9 Extending the implementation of ISAT method

The ISAT method proposed in Chapter 5 tackles the efficiency of the two dimensional wing-crack damage model. This methodology has great potential for improving efficiency in many complex modeling applications. On the other hand, the efficiency gained from this approach is only fully exploited when the number of elements is sufficiently large, so that excessive repeated constitutive modelings would be required.

As discussed in Section 5.1 similar methods have been proposed in different areas of modeling. Although the core ideas are similar (perturbing results with Taylor series, storage-retrieve scheme), as extension of this kind of method generally requires mathematical solutions tailored for the specific problem. An in-depth understanding of the constitutive model is necessary for the derivation of the mathematical solutions, especially the gradient terms.

Since the current three-dimensional constitutive models for wing-crack damaged materials are much different from the two-dimensional model derived in Chapter 5 for the ISAT method (Eqs. (4.45)), some major changes are required. For instance, in Hu’s model the damage parameter is a second rank tensor; therefore, numerical averaging of the flaw parameters with different sizes and orientations is necessary. In Tonge’s model and the current model, the Taylor series expansion for the crack compliance tensor $\mathbf{Z} / \mathbf{R}$ should be obtained. Careful verifications are essential to guarantee the correctness of the Taylor series expansions that are derived.
Appendix A

Finite Element Results of Two-Phase Problem

In this appendix, the finite element models for verifying the erratum of complex variable solution of two-phase problem (Section 4.2.2) are presented. The elastic modulus of the intact material $E_0 = 320$ GPa, Poisson’s ratio $\mu_0 = 0.237$, shear modulus $G_0 = 129.3$ GPa. The far-field stress $\sigma_{11} = -100$ MPa was applied to the model; it should be noted that in Section 4.2.2 the applied load is assumed $-1$ and thus the results in Figs. 4.4 and 4.5 have been normalized. The finite element models were created and calculated in Abaqus, in which plane strain elements were assigned and the properties of matrix material were defined through Engineering Constants.

The first two sections show the FE models of two-dimensional problems with different aspect ratios of the inclusion. By applying the corresponding material property
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

definitions for the matrix into the complex variable solutions derived in Section 4.2.1, the same stress results in the inclusion can be obtained; and four cases from below have been directly compared in Fig. 4.4 and 4.5.

The latter two sections present three-dimensional FE models containing inclusions with two individual aspect ratios. Although not directly related to the analytical solutions, the 3D modelings show that for the same definition of material properties, the results from plane strain 2D models are generally in good agreement with a more realistic 3D ones.

A.1 Two-Dimensional Model, Circular Inclusion

Aspect ratio of the inclusion $a : b = 1$ (circle). The finite element model is shown in Fig. A.1.

Case 1: Isotropic matrix material properties presumed:

$$E_1 = E_2 = E_3 = E_0 \left[ 1 - \frac{\pi^2}{30} (1 + \nu_0)(5 - 4\nu_0)\Omega \right], \quad (A.1a)$$

$$\nu_{12} = \nu_{13} = \nu_{23} = \nu_0, \quad (A.1b)$$

$$G_{12} = G_{13} = G_{23} = \frac{E_0}{2(1 + \nu_0)} \left[ 1 - \frac{\pi^2}{60} (10 - 7\nu_0)\Omega \right]. \quad (A.1c)$$

The results are shown in Fig. 4.4 as red stars. Abaqus result file: Circuler_Gs.odb
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

Figure A.1: Two dimensional finite element model for verifying complex variable method derivation, aspect ratio of inclusion \(a:b=1\).

<table>
<thead>
<tr>
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<th>0.05</th>
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<td>(E_1, E_2, E_3) (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>(G_{12}) (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
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<td>(\sigma^i_{11}) (MPa)</td>
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<td>(\sigma^i_{22}) (MPa)</td>
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<td>-0.9</td>
<td>-1.45</td>
<td>-3.1</td>
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Table A.1: Matrix material properties and stress results in the inclusion, aspect ratio \(a:b=1\), case 1.
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

Case 2: Anisotropic matrix material properties:

\[
E_1 = E_0, \quad E_2 = E_3 = E_0 \left[ 1 - \frac{\pi^2}{30} (1 + \nu_0)(5 - 4\nu_0)\Omega \right], \quad (A.2a)
\]
\[
\nu_{12} = \nu_{13} = \frac{E_1}{E_2}\nu_0, \quad (A.2b)
\]
\[
G_{12} = G_{13} = G_{23} = \frac{E_0}{2(1 + \nu_0)} \left[ 1 - \frac{\pi^2}{60} (10 - 7\nu_0)\Omega \right]. \quad (A.2c)
\]

The results are shown in Fig. 4.5 as red stars. Abaqus result file: *Circuler_E2E3_AG.odb*

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<td>$E_2, E_3$ (GPa)</td>
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<td>240.85</td>
<td>188.08</td>
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<tr>
<td>$G_{12}$ (GPa)</td>
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<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
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<td>$\nu_{12}, \nu_{13}$</td>
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Table A.2: Matrix material properties and stress results in the inclusion, aspect ratio $a : b = 1$, case 2.

Case 3: Anisotropic matrix material properties:

\[
E_1 = E_0 \left[ 1 - \frac{\pi^2}{30} (1 + \nu_0)(5 - 4\nu_0)\Omega \right], \quad E_2 = E_3 = E_0, \quad (A.3a)
\]
\[
\nu_{12} = \nu_{13} = \frac{E_1}{E_2}\nu_0, \quad (A.3b)
\]
\[
G_{12} = \frac{E_0}{2(1 + \nu_0)} \left[ 1 - \frac{\pi^2}{60} (10 - 7\nu_0)\Omega \right], \quad G_{13} = G_{23} = \frac{E_0}{2(1 + \nu_0)}. \quad (A.3c)
\]
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

Abaqus result file: Circuler_E1reduced.odb

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<td>$E_1$ (GPa)</td>
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Table A.3: Matrix material properties and stress results in the inclusion, aspect ratio $a:b = 1$, case 3.

Case 4: Anisotropic matrix material properties:

$$E_1 = E_3 = E_0, \quad E_2 = E_0 \left[ 1 - \frac{\pi^2}{30}(1 + \nu_0)(5 - 4\nu_0)\Omega \right],$$  \hspace{1cm} (A.4a)

$$\nu_{12} = \frac{E_1}{E_2}\nu_0, \quad \nu_{13} = \nu_{23} = \nu_0,$$  \hspace{1cm} (A.4b)

$$G_{12} = G_{23} = \frac{E_0}{2(1 + \nu_0)} \left[ 1 - \frac{\pi^2}{60}(10 - 7\nu_0)\Omega \right], \quad G_{13} = \frac{E_0}{2(1 + \nu_0)}.$$  \hspace{1cm} (A.4c)

Abaqus result file: 2D_circular_E2.odb
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

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<tr>
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<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
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<tr>
<td>$\nu_{12}$</td>
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<td>0.5605</td>
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</tr>
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Table A.4: Matrix material properties and stress results in the inclusion, aspect ratio $a:b = 1$, case 4.

Case 5: Anisotropic matrix material properties:

$$E_1 = E_0, \quad E_2 = E_3 = E_0 \left[ 1 - \frac{\pi^2}{30} (1 + \nu_0)(5 - 4\nu_0)\Omega \right], \quad (A.5a)$$

$$\nu_{12} = \nu_{13} = \frac{E_1}{E_2} \nu_0, \quad (A.5b)$$

$$G_{12} = \frac{E_0}{2(1 + \nu_0)}, \quad G_{13} = G_{23} = \frac{E_0}{2(1 + \nu_0)} \left[ 1 - \frac{\pi^2}{60} (10 - 7\nu_0)\Omega \right]. \quad (A.5c)$$

Abaqus result file: Circuler_weakE2E3.odb
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

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<td>$G_{23}, G_{13}$ (GPa)</td>
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Table A.5: Matrix material properties and stress results in the inclusion, aspect ratio $a : b = 1$, case 5.

A.2 Two-Dimensional Model, Elliptical Inclusion

Aspect ratio of the inclusion $a : b = 3$ (Ellipse). The finite element model was shown in Fig. 4.2.

Case 1: Isotropic matrix material properties follow Eqs. (A.1). The results are shown in Fig. 4.4 as red crosses. Abaqus result file: Ellipse_ac3_G12.odb
### APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

<table>
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</thead>
<tbody>
<tr>
<td>$E_1, E_2, E_3$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{12}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$\sigma_{11}^i$ (MPa)</td>
<td>-105.5</td>
<td>-118.6</td>
<td>-135.9</td>
<td>-158.5</td>
<td>-194.6</td>
</tr>
<tr>
<td>$\sigma_{22}^i$ (MPa)</td>
<td>-0.4</td>
<td>-1.5</td>
<td>-3.2</td>
<td>-5.1</td>
<td>-9.0</td>
</tr>
</tbody>
</table>

**Table A.6:** Matrix material properties and stress results in the inclusion, aspect ratio $a:b = 3$, case 1.

Case 2: Anisotropic matrix material properties follows Eqs. (A.2). The results are shown in Fig. 4.5 as red crosses. Abaqus result file: Ellipse_ac3_E2E3_AG.odb

<table>
<thead>
<tr>
<th>$\Omega$</th>
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<th>0.25</th>
<th>0.35</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2, E_3$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{12}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$\nu_{12}, \nu_{13}$</td>
<td>0.2583</td>
<td>0.3149</td>
<td>0.4032</td>
<td>0.5605</td>
<td>0.9188</td>
</tr>
<tr>
<td>$\sigma_{11}^i$ (MPa)</td>
<td>-99.4</td>
<td>-98.0</td>
<td>-96.2</td>
<td>-93.3</td>
<td>-88.1</td>
</tr>
<tr>
<td>$\sigma_{22}^i$ (MPa)</td>
<td>0.4</td>
<td>1.4</td>
<td>2.4</td>
<td>3.8</td>
<td>5.6</td>
</tr>
</tbody>
</table>

**Table A.7:** Matrix material properties and stress results in the inclusion, aspect ratio $a:b = 3$, case 2.

Case 3: Anisotropic matrix material properties follows Eqs. (A.3). Abaqus result file: Ellipse_ac3_E1reduced.odb
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

<table>
<thead>
<tr>
<th>Ω</th>
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<th>0.35</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{12}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$ν_{12}, ν_{13}$</td>
<td>0.2175</td>
<td>0.1784</td>
<td>0.1393</td>
<td>0.1002</td>
<td>0.0611</td>
</tr>
<tr>
<td>$σ_{i11}$ (MPa)</td>
<td>-106.0</td>
<td>-120.4</td>
<td>-139.5</td>
<td>-166.3</td>
<td>-209.3</td>
</tr>
<tr>
<td>$σ_{i22}$ (MPa)</td>
<td>-0.89</td>
<td>-3.1</td>
<td>-6.3</td>
<td>-11.4</td>
<td>-21.1</td>
</tr>
</tbody>
</table>

Table A.8: Matrix material properties and stress results in the inclusion, aspect ratio $a : b = 3$, case 3.

Case 4: Anisotropic matrix material properties follows (A.4). Abaqus result file: $2D_{ellipse}.E2.odb$

<table>
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<tr>
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<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{12}, G_{23}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$ν_{12}$</td>
<td>0.2583</td>
<td>0.3149</td>
<td>0.4032</td>
<td>0.5605</td>
<td>0.9188</td>
</tr>
<tr>
<td>$σ_{i11}$ (MPa)</td>
<td>-99.7</td>
<td>-99.2</td>
<td>-98.6</td>
<td>-97.9</td>
<td>-97.3</td>
</tr>
<tr>
<td>$σ_{i22}$ (MPa)</td>
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<td>1.3</td>
<td>2.3</td>
<td>3.6</td>
<td>5.7</td>
</tr>
</tbody>
</table>

Table A.9: Matrix material properties and stress results in the inclusion, aspect ratio $a : b = 3$, case 4.

APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

<table>
<thead>
<tr>
<th>$\Omega$</th>
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<th>0.15</th>
<th>0.25</th>
<th>0.35</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2, E_3$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{23}, G_{13}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$\nu_{12}, \nu_{13}$</td>
<td>0.2583</td>
<td>0.3149</td>
<td>0.4032</td>
<td>0.5605</td>
<td>0.9188</td>
</tr>
<tr>
<td>$\sigma_{11}^i$ (MPa)</td>
<td>-99.4</td>
<td>-97.8</td>
<td>-95.6</td>
<td>-91.9</td>
<td>-84.6</td>
</tr>
<tr>
<td>$\sigma_{22}^i$ (MPa)</td>
<td>0.5</td>
<td>1.58</td>
<td>3.0</td>
<td>4.95</td>
<td>8.3</td>
</tr>
</tbody>
</table>

Table A.10: Matrix material properties and stress results in the inclusion, aspect ratio $a:b = 3$, case 5.

A.3 Three-Dimensional Model, Spherical Inclusion

Aspect ratio of the inclusion $a:b:c = 1:1:1$ (spherical). The finite element model is shown in Fig. [A.2]

Case 1: Isotropic matrix material properties follows Eqs. [A.1]. Abaqus result file:

3D_Circular_E1E2E3.odb
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

Figure A.2: Three-dimensional finite element model for verifying complex variable method derivation, spherical inclusion $a : b : c = 1 : 1 : 1$.

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>0.05</th>
<th>0.15</th>
<th>0.25</th>
<th>0.35</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1, E_2, E_3$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{12}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$\sigma_{i1}$ (MPa)</td>
<td>-105</td>
<td>-116.6</td>
<td>-131.4</td>
<td>-151.2</td>
<td>-182.0</td>
</tr>
<tr>
<td>$\sigma_{i2}, \sigma_{i3}$ (MPa)</td>
<td>0.13</td>
<td>0.45</td>
<td>0.89</td>
<td>1.38</td>
<td>1.6</td>
</tr>
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</table>

Table A.11: Matrix material properties and stress results in the inclusion, aspect ratio $a : b : c = 1 : 1 : 1$, case 1.

Case 2: Anisotropic matrix material properties follows Eqs. (A.2), except $G_{23} = E_2/[2(1 + \nu_0)]$. Abaqus result file: 3D_circuler_reducedE2E3_Gc.odb
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

<table>
<thead>
<tr>
<th>Ω</th>
<th>0.05</th>
<th>0.15</th>
<th>0.25</th>
<th>0.35</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2, E_3$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{12}, G_{13}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$G_{23}$ (GPa)</td>
<td>118.68</td>
<td>97.35</td>
<td>76.02</td>
<td>54.69</td>
<td>33.37</td>
</tr>
<tr>
<td>$\nu_{12}, \nu_{13}$</td>
<td>0.2583</td>
<td>0.3149</td>
<td>0.4032</td>
<td>0.5605</td>
<td>0.9188</td>
</tr>
<tr>
<td>$\sigma_{11}^i$ (MPa)</td>
<td>-99.2</td>
<td>-97.6</td>
<td>-95.3</td>
<td>-92.4</td>
<td>-88.6</td>
</tr>
<tr>
<td>$\sigma_{22}^i, \sigma_{33}^i$ (MPa)</td>
<td>1.7</td>
<td>4.6</td>
<td>10.8</td>
<td>17.5</td>
<td>27.1</td>
</tr>
</tbody>
</table>

Table A.12: Matrix material properties and stress results in the inclusion, aspect ratio $a : b : c = 1 : 1 : 1$, case 2.

Case 3: The FE model adopt the same material properties for the matrix as the previous case, but confining the displacement in $e_3$ direction for the boundaries with normal in $e_3$. Abaqus result file: 3D_circular_E2E3_cfz.odb
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>0.05</th>
<th>0.15</th>
<th>0.25</th>
<th>0.35</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2, E_3$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{12}, G_{13}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$G_{23}$ (GPa)</td>
<td>118.68</td>
<td>97.35</td>
<td>76.02</td>
<td>54.69</td>
<td>33.37</td>
</tr>
<tr>
<td>$\nu_{12}, \nu_{13}$</td>
<td>0.2583</td>
<td>0.3149</td>
<td>0.4032</td>
<td>0.5605</td>
<td>0.9188</td>
</tr>
<tr>
<td>$\sigma_{i1}$ (MPa)</td>
<td>-99.2</td>
<td>-97.3</td>
<td>-94.7</td>
<td>-91.0</td>
<td>-84.4</td>
</tr>
<tr>
<td>$\sigma_{i2}$ (MPa)</td>
<td>1.76</td>
<td>5.88</td>
<td>11.1</td>
<td>18.0</td>
<td>28.4</td>
</tr>
<tr>
<td>$\sigma_{i3}$ (MPa)</td>
<td>-23.1</td>
<td>-21.9</td>
<td>-20.3</td>
<td>-18.4</td>
<td>-15.9</td>
</tr>
</tbody>
</table>

Table A.13: Matrix material properties and stress results in the inclusion, aspect ratio $a : b : c = 1 : 1 : 1$, case 3.

Case 4: Anisotropic matrix material properties follow [A.4]. Abaqus result file: 3D_circular_E2.odb

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>0.05</th>
<th>0.15</th>
<th>0.25</th>
<th>0.35</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{12}, G_{23}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.2583</td>
<td>0.3149</td>
<td>0.4032</td>
<td>0.5605</td>
<td>0.9188</td>
</tr>
<tr>
<td>$\sigma_{i1}$ (MPa)</td>
<td>-99.6</td>
<td>-98.7</td>
<td>-97.5</td>
<td>-95.9</td>
<td>-93.2</td>
</tr>
<tr>
<td>$\sigma_{i2}$ (MPa)</td>
<td>1.37</td>
<td>4.56</td>
<td>8.7</td>
<td>14.6</td>
<td>24.6</td>
</tr>
<tr>
<td>$\sigma_{i3}$ (MPa)</td>
<td>0.4</td>
<td>1.28</td>
<td>2.5</td>
<td>4.1</td>
<td>6.8</td>
</tr>
</tbody>
</table>

Table A.14: Matrix material properties and stress results in the inclusion, aspect ratio $a : b : c = 1 : 1 : 1$, case 4.
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

Case 5: Anisotropic matrix material properties follows Eqs. (A.5). Abaqus result file: 3D_circuler_reducedE2E3.odb

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>0.05</th>
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<th>0.35</th>
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</thead>
<tbody>
<tr>
<td>$E_2, E_3$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{23}, G_{13}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$\nu_{12}, \nu_{13}$</td>
<td>0.2583</td>
<td>0.3149</td>
<td>0.4032</td>
<td>0.5605</td>
<td>0.9188</td>
</tr>
<tr>
<td>$\sigma_{11}^i$ (MPa)</td>
<td>-99.5</td>
<td>-98.5</td>
<td>-97.1</td>
<td>-95.3</td>
<td>-92.6</td>
</tr>
<tr>
<td>$\sigma_{22}^i$ (MPa)</td>
<td>1.3</td>
<td>4.3</td>
<td>7.9</td>
<td>12.8</td>
<td>19.8</td>
</tr>
<tr>
<td>$\sigma_{33}^i$ (MPa)</td>
<td>1.6</td>
<td>5.5</td>
<td>10.3</td>
<td>17.0</td>
<td>27.2</td>
</tr>
</tbody>
</table>

Table A.15: Matrix material properties and stress results in the inclusion, aspect ratio $a : b : c = 1 : 1 : 1$, case 5.

A.4 Three-Dimensional Model, Ellipsoidal Inclusion

Aspect ratio of the inclusion $a : b : c = 3 : 1 : 1$ (ellipsoid). The finite element model is shown in Fig. A.3

Case 1: Isotropic matrix material properties follows Eqs. (A.1). Abaqus result file: 3D_Ellipsoid_E1E2E3.odb
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

Figure A.3: Three-dimensional finite element model for verifying complex variable method derivation, ellipsoidal inclusion $a : b : c = 3 : 1 : 1$.

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>0.05</th>
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<th>0.25</th>
<th>0.35</th>
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<tbody>
<tr>
<td>$E_1, E_2, E_3$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{12}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$\sigma_{11}$ (MPa)</td>
<td>-107.5</td>
<td>-126.6</td>
<td>-154.3</td>
<td>-198.4</td>
<td>-281.9</td>
</tr>
<tr>
<td>$\sigma_{22}, \sigma_{33}$ (MPa)</td>
<td>-0.14</td>
<td>-0.51</td>
<td>-1.1</td>
<td>-2.4</td>
<td>-5.3</td>
</tr>
</tbody>
</table>

Table A.16: Matrix material properties and stress results in the inclusion, aspect ratio $a : b : c = 3 : 1 : 1$, case 1.

Case 2: Anisotropic matrix material properties follows Eqs. (A.2), except $G_{23} = E_2/[2(1 + \nu_0)]$. Abaqus result file: 3D_ellipsoid_E2E3_AG.odb
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\Omega & 0.05 & 0.15 & 0.25 & 0.35 & 0.45 \\
\hline
E_2, E_3 (GPa) & 293.6 & 240.85 & 188.08 & 135.3 & 82.55 \\
\hline
G_{12}, G_{13} (GPa) & 120.5 & 102.7 & 85.0 & 67.23 & 49.5 \\
\hline
G_{23} (GPa) & 118.68 & 97.35 & 76.02 & 54.69 & 33.37 \\
\hline
\nu_{12}, \nu_{13} & 0.2583 & 0.3149 & 0.4032 & 0.5605 & 0.9188 \\
\hline
\sigma_{11}^i (MPa) & -99.3 & -97.7 & -95.9 & -93.5 & -90.4 \\
\hline
\sigma_{22}^i, \sigma_{33}^i (MPa) & 1.28 & 4.15 & 7.5 & 11.6 & 16.8 \\
\hline
\end{array}
\]

Table A.17: Matrix material properties and stress results in the inclusion, aspect ratio \(a:b:c = 3:1:1\), case 2.

Case 3: The FE model adopt the same material properties for the matrix as in case 2, but confining the displacement in \(e_3\) direction for the boundaries with normal in \(e_3\). Abaqus result file: 3D_ellipsoid_E2E3_cfz.odb
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>0.05</th>
<th>0.15</th>
<th>0.25</th>
<th>0.35</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2, E_3$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{12}, G_{13}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$G_{23}$ (GPa)</td>
<td>118.68</td>
<td>97.35</td>
<td>76.02</td>
<td>54.69</td>
<td>33.37</td>
</tr>
<tr>
<td>$\nu_{12}, \nu_{13}$</td>
<td>0.2583</td>
<td>0.3149</td>
<td>0.4032</td>
<td>0.5605</td>
<td>0.9188</td>
</tr>
<tr>
<td>$\sigma_{11}^i$ (MPa)</td>
<td>-99.1</td>
<td>-97.0</td>
<td>-94.0</td>
<td>-89.5</td>
<td>-78.0</td>
</tr>
<tr>
<td>$\sigma_{22}^i$ (MPa)</td>
<td>1.3</td>
<td>4.2</td>
<td>7.7</td>
<td>12.0</td>
<td>22.2</td>
</tr>
<tr>
<td>$\sigma_{33}^i$ (MPa)</td>
<td>-23.2</td>
<td>-22.7</td>
<td>-20.9</td>
<td>-19.5</td>
<td>-16.6</td>
</tr>
</tbody>
</table>

Table A.18: Matrix material properties and stress results in the inclusion, aspect ratio $a : b : c = 3 : 1 : 1$, case 3.

Case 4: Anisotropic matrix material properties follows (A.4).

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>0.05</th>
<th>0.15</th>
<th>0.25</th>
<th>0.35</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{12}, G_{23}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.2583</td>
<td>0.3149</td>
<td>0.4032</td>
<td>0.5605</td>
<td>0.9188</td>
</tr>
<tr>
<td>$\sigma_{11}^i$ (MPa)</td>
<td>-99.6</td>
<td>-98.9</td>
<td>-97.9</td>
<td>-96.5</td>
<td>-94.4</td>
</tr>
<tr>
<td>$\sigma_{22}^i$ (MPa)</td>
<td>1.0</td>
<td>3.3</td>
<td>6.1</td>
<td>9.83</td>
<td>15.9</td>
</tr>
<tr>
<td>$\sigma_{33}^i$ (MPa)</td>
<td>0.3</td>
<td>1.0</td>
<td>1.88</td>
<td>2.9</td>
<td>4.1</td>
</tr>
</tbody>
</table>

Table A.19: Matrix material properties and stress results in the inclusion, aspect ratio $a : b : c = 3 : 1 : 1$, case 4.
APPENDIX A. FINITE ELEMENT RESULTS OF TWO-PHASE PROBLEM

Case 5: Anisotropic matrix material properties follows Eqs. (A.5). Abaqus result file: 3D_ellipseinclusion_reducedE2E3.odb

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>0.05</th>
<th>0.15</th>
<th>0.25</th>
<th>0.35</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2, E_3$ (GPa)</td>
<td>293.6</td>
<td>240.85</td>
<td>188.08</td>
<td>135.3</td>
<td>82.55</td>
</tr>
<tr>
<td>$G_{23}, G_{13}$ (GPa)</td>
<td>120.5</td>
<td>102.7</td>
<td>85.0</td>
<td>67.23</td>
<td>49.5</td>
</tr>
<tr>
<td>$\nu_{12}, \nu_{13}$</td>
<td>0.2583</td>
<td>0.3149</td>
<td>0.4032</td>
<td>0.5605</td>
<td>0.9188</td>
</tr>
<tr>
<td>$\sigma_{11}$ (MPa)</td>
<td>-99.3</td>
<td>-97.7</td>
<td>-95.5</td>
<td>-92.7</td>
<td>-87.8</td>
</tr>
<tr>
<td>$\sigma_{22}, \sigma_{33}$ (MPa)</td>
<td>1.28</td>
<td>4.3</td>
<td>7.9</td>
<td>13.5</td>
<td>20.7</td>
</tr>
</tbody>
</table>

**Table A.20:** Matrix material properties and stress results in the inclusion, aspect ratio $a : b : c = 3 : 1 : 1$, case 5.
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Vita

Junwei Liu was born in Guangdong, China on July 22nd, 1984. In July 2007, Junwei graduated with Bachelor's degree from the Automotive Engineering Department in Tsinghua University in Beijing, China. After that, he spent two and a half years in RWTH Aachen University in Germany, where he obtained his Masters degree in Mechanical Engineering with a focus on Computer Aided Production and Conception. In January 2010, Junwei started his Ph.D. career under the advisorship of Lori Graham-Brady in Johns Hopkins University. His research focus during this period was modeling the dynamic failure of brittle materials with the support of the Hopkins Extreme Materials Institute and Army Research Lab. He completed the requirements for the degree of Doctor of Philosophy in February 2015.