Developing the Primordial Inflation Polarization Explorer (PIPER)

Microwave Polarimeter for Constraining Inflation

by

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Abstract

The Inflationary Big Bang model of cosmology generically predicts the existence of a background of gravitational waves due to Inflation, which coupled into the B-mode power spectrum $C_{\ell}^{BB}$ during the epochs of Recombination and Reionization. A measurement of the primordial B-mode spectrum would verify the reality of the Inflationary model and constrain the allowed models of Inflation. In Chapter 1 we describe the background physics of cosmology and Inflation, and the challenges involved with measuring the primordial B-mode spectrum.

In Chapter 2 we describe the Primordial Inflation Polarization Explorer (PIPER), a high-altitude balloon-borne microwave polarimeter optimized to measure the B-mode spectrum on large angular scales. We examine the high level design of PIPER and how it addresses the challenges presented in Chapter 1.

Following the high level design, we examine in detail the electronics developed for PIPER, both for in-flight operations and for laboratory development. In Chapter 3 we describe the Transition Edge Sensor (TES) bolometers that serve as PIPER’s detectors, analyze the Superconducting Quantum Interference Device (SQUID) amplifiers and Mutli-
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channel Electronics (MCE) detector readout chain, and finally present the characterization of both detector parameters and noise of a single pixel device with a PIPER-like (Backshort Under Grid, BUG) architecture to validate the detector design. In Chapter 4 we present a description of the HKE electronics, used to measure all non-detector science timestreams in PIPER, as well as flight housekeeping and laboratory development. In addition to the operation of the HKE electronics, we develop a model to quantify the performance of the HKE thermometry reader (TRead).

A simple simulation pipeline is developed and used to explore the consequences of imperfect foreground removal in Chapter 5. The details of estimating the instrument noise as projected onto a sky map is developed also developed. In particular, we address whether PIPER may be able to get significant science return with only a fraction of its planned flights by optimizing the order that the frequency bands are flown. Additionally, we look at how a spatially varying calibration gain error would affect measurements of the B-mode spectrum.

Finally, a series of appendices presents the physics of SQUIDs, develops techniques for estimating noise of circuits and amplifiers, and introduces techniques from control systems. In addition, a few miscellaneous results used throughout the work are derived.

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Dedication

For Jen.
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Referencing the op amp voltage noise to the amplifier output.

The control loop block diagram of a simple PID loop. The reference (set-point) signal $r(t)$ is compared against the measured output of the process $y(t)$ to form the error $e(t)$. The error $e(t)$ serves as the input to the PID controller $C(s)$ to form the process input $u(t)$. This input drives the process $P(s)$. Additive noise $n(t)$ is added to the output of the process to simulate measurement noise and forms the measured process output ($y$). The measured output is fed back to the reference signal $r(t)$ to close the loop.
Chapter 1

Cosmology

1.1 Cosmological Expansion

The foundation of cosmology is the idea that the laws of physics are the same everywhere and that there are no special locations in the universe. This implies that the universe has translational (i.e. homogeneity) and rotational symmetry (i.e. isotropy). Of course, this is not true on all scales, as it would preclude the existence of astronomical objects such as stars and galaxies. However, for cosmological scales where we average over many such objects, the symmetries are respected.

The spacetime metric that respects translational and rotational symmetry is the maximally symmetric metric, the Robertson-Walker metric

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \]

where \( a(t) \) is the scale factor, which has a value of 1 at present and a smaller value in the
past, and \( K \in \{-1, 0, 1\} \) represents the curvature of the universe.

We model the universe as a fluid, which may be represented using smooth functions of space, for which the stress-energy tensor \( T_{\mu \nu} \) is

\[
T_{\mu \nu} = (\rho + P) u_\mu u_\nu + P g_{\mu \nu}.
\]  

(1.2)
The evolution of the fluid is governed by general relativity, in particular the Einstein field equation

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 8 \pi G T_{\mu \nu}
\]  

(1.3)The 0-0 component of the field equation applied to Eq. (1.2) gives the Friedmann equation,

\[
H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{3 \rho}{8 \pi G} - \frac{K}{a^2}
\]  

(1.4)
where \( H \) is the Hubble constant.

The scaling of \( \rho \) with scale factor \( a \) depends on the source of the energy density, typically the type of matter. For nonrelativistic matter, the number density of particles goes like the volume \( V \sim a^3 \), and the energy per particle is unaffected by the scale factor. For relativistic particles (including photons), the energy per particle additionally scales like \( 1/a \). A final source of energy is a vacuum energy, e.g. a cosmological constant \( \Lambda \), which is independent of the scale factor. Furthermore, with \( \rho(a) \) we may integrate Eq. (1.4) to get \( a(t) \) for each type of matter. We then have

Relativistic (“Radiation”): \( \rho \propto a^{-4} \quad a \propto t^{1/2} \)  
(1.5a)Non-relativistic (“Matter”): \( \rho \propto a^{-3} \quad a \propto t^{2/3} \)  
(1.5b)Vacuum: \( \rho \propto 1 \quad a \propto \exp(\Lambda t) \)  
(1.5c)
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where we have assumed a constant Hubble constant $H$ for the vacuum energy-dominated case. Eq. (1.5) lets us decompose the historical value of $\rho$ in terms of the present day densities (indicated by a subscript $0$)

$$\rho = \rho_{R0}a^{-4} + \rho_{M0}a^{-3} + \rho_{\Lambda0} \equiv \rho_0 \left( \Omega_R a^{-4} + \Omega_M a^{-3} + \Omega_\Lambda \right)$$ (1.6)

where $M$ indicates non-relativistic matter, $R$ indicates relativistic matter (dominated by photons, i.e. radiation, at all relevant times), and we have included the vacuum energy term indicated by $\Lambda$, which does not scale with scale factor. We have also defined the critical density $\rho_0 \equiv 3H_0^2/8\pi G$ and the energy densities $\Omega_X \equiv \rho_X/\rho_0$ at present day in units of the critical density, with $H_0 = 69.6 \pm 0.7 \text{ km s}^{-1} \text{ Mpc}^{-1}$ as the present-day value of the Hubble constant. The current best estimates for the energy distribution of vacuum energy and matter is $\Omega_\Lambda = 0.6911 \pm 0.0062$, $\Omega_M = 0.3089 \pm 0.0062$.[6] The present-day radiation density may be computed from the current temperature of the CMB,

$$\Omega_R = \left[ 1 + 3 \left( \frac{7}{8} \right) \left( \frac{4}{11} \right)^{4/3} \right] \frac{a_B T_0^4}{\rho_0} = (8.6 \pm 0.2) \times 10^{-5}$$ (1.7)

where the non-unity coefficient comes from the contribution of neutrinos and antineutrinos to the relativistic energy density, and the value of $\Omega_R$ is small compared to the other contributors. In analysis, it is typically set to 0.

We note that at small $a$ (early times), the universe was radiation-dominated (relativistic). After that, the universe was matter-dominated (nonrelativistic), and currently the universe is vacuum energy-dominated. From Eq. (1.6), these transitions happened at $a_{RM} = \Omega_R/\Omega_M = (2.78 \pm 0.08) \times 10^{-4}$ and $a_{MA} = (\Omega_M/\Omega_\Lambda)^{1/3} = 0.765 \pm 0.006$. So
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ignoring the epoch of Inflation, we have

\[
\rho \sim \begin{cases} 
    a^{-4} & (\text{radiation dominated}) \quad a < 2.78 \times 10^{-4} \\
    a^{-3} & (\text{matter dominated}) \quad 2.78 \times 10^{-4} < a < 0.765 \\
    a^{-3} & (\text{vacuum energy dominated}) \quad 0.765 < a
\end{cases}
\]  

(1.8)

If the normalized energy density \( \Omega \equiv \rho / \rho_0 \) is 1, then the universe has no curvature, so we may define the following identity from Eq. (1.4) and Eq. (1.6) to include possible curvature of the universe,

\[
\left( \frac{H}{H_0} \right)^2 = \Omega_R a^{-4} + \Omega_M a^{-3} + \Omega_K a^{-2} + \Omega_\Lambda
\]  

(1.9)

where \( \Omega_K \equiv -K/a_0^2 H_0^2 \). There is no evidence for a non-zero curvature, i.e. \( \Omega_K = 0.0008^{+0.0040}_{-0.0039} \) so it is conventionally set to 0 for standard \( \Lambda \)CDM. It is worth noting the scaling with time of \( \Omega_K \) for a non-flat (\( K \neq 0 \)) universe depends on the dominant contributor to the energy density

\[
|\Omega_K| = \frac{|K|}{a^2} \propto |K| \begin{cases} 
    t^{2/3} & \text{Non-relativistic (Matter)} \\
    t & \text{Relativistic (Radiation)} \\
    \exp(-2Ht) & \text{Vacuum}
\end{cases}
\]  

(1.10)

and we see that for a matter- or radiation-dominated epoch the curvature is amplified, and for a vacuum energy-dominated epoch the curvature is suppressed. For a flat (\( K = \Omega_K = 0 \)) universe, the curvature will always stay flat. Henceforth we will assume a flat universe \( K = 0 \).
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The conservation of mass-energy \( \partial_{\mu}T^{\mu\nu} = 0 \) gives us

\[
\dot{\rho} = -\frac{3\dot{a}}{a}(\rho + p)
\]  

(1.11)

which may be combined with the first Friedmann equation Eq. (1.4) to give us the second Friedmann equation,

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \equiv -\frac{4\pi G}{3}\rho(1 + 3w)
\]

(1.12)

where we’ve defined the equation of state \( p = w\rho \). For \( w > -1/3 \), the expansion \( \ddot{a} \) of the universe is slowing (\( \ddot{a} < 0 \)), and for \( w < -1/3 \), the expansion of the universe is accelerating (\( \ddot{a} > 0 \)). We note from Eq. (1.11) that \( \rho \propto a^{-3(1+w)} \) from which we can identify

\[
w = \begin{cases} 
\frac{1}{3} & \text{Relativistic (Radiation)} \\
0 & \text{Non-relativistic (Matter)} \\
-1 & \text{Vacuum}
\end{cases}
\]

(1.13)

and we observe that matter- or radiation-dominated epochs slow the expansion of the universe. For our currently vacuum energy-dominated universe, the expansion is accelerating, consistent with SNIa measurements\(^7\).

The largest distance a particle could have traveled since the beginning of time is called the particle horizon \( \eta \). It should naturally follow a path such that \( ds = 0 \), so we can integrate the metric Eq. (1.1) along this path to gets its value as a comoving distance (equivalently time, since we have set \( c = 1 \)),

\[
\eta = \int_0^r dr = \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da'}{a'^2 H}
\]

(1.14)
where $a(t)$ is determined by Eq. (1.4). For the three scenarios we have been considering, the particle horizon is

$$
\eta = \begin{cases} 
2t^{1/2} + \eta_0 & \text{Relativistic (Radiation)} \\
3t^{1/3} + \eta_0 & \text{Non-relativistic (Matter)} \\
\frac{1}{H} \left(1 - e^{-Ht}\right) + \eta_0 & \text{Vacuum}
\end{cases}
$$

(1.15)

where the constant $\eta_0 > 0$ represents the contribution from an epoch (or epochs) that preceded the one at time $t$. The particle horizon represents the size of the observable universe. Additionally, we note that $\eta = 0$ when $t = 0$, corresponding to a universe of 0 size. This may be used to define the Big Bang. Alternatively, at this point $a = 0$, so the non-comoving characteristic scale of the universe is 0.

The characteristic scale of the universe is given by the so-called Hubble length $H^{-1}$ (the Hubble length and time are the same since we have set $c = 1$), also frequently called simply the horizon. This is the scale that two regions may interact and form an equilibrium. In comoving coordinates it is given by $1/aH$, so for our three scenarios it is given by

$$
1/(aH) = 1/\dot{a} = \begin{cases} 
2t^{1/2} & \text{Relativistic (Radiation)} \\
\frac{3}{2}t^{1/3} & \text{Non-relativistic (Matter)} \\
\frac{1}{H}e^{-Ht} & \text{Vacuum}
\end{cases}
$$

(1.16)

Lastly, we observe that the rate of change of the size of the observable universe com-
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pared to the interaction length is

\[
\frac{d}{dt} \left( \frac{\eta}{1/(aH)} \right) = \begin{cases} 
-\frac{\eta_0}{4} t^{-3/2} & \text{Relativistic (Radiation)} \\
-\frac{2\eta_0}{9} t^{-5/3} & \text{Non-relativistic (Matter)} \\
H(\eta_0 + 1)e^{Ht} & \text{Vacuum.}
\end{cases}
\] (1.17)

So for matter- and radiation-dominated epochs the particle horizon increases more slowly than the interaction scale, and for a vacuum-dominated epoch the particle horizon increases more quickly than the interaction scale. This should not be interpreted that in matter- and radiation-dominated epochs the particle horizon is smaller than the interaction scale (a statement equivalent to \( \eta/(aH)^{-1} < 1 \)), as the derivative does not determine the value. Similarly it is not necessarily true that \( \eta/(aH)^{-1} > 1 \) for a vacuum-dominated epoch.

1.2 Inflation

Inflation is an epoch of accelerating expansion, \( \ddot{a} > 0 \), that is theorized to have taken place shortly after the Big Bang and before the most recent epoch of radiation domination. Prior to Inflation, the ratio of particle horizon to interaction scale was comparable or smaller than 1, i.e. \( \eta/(aH)^{-1} \sim 1 \). In this limit, the observable universe would be uniformly thermalized to a single temperature. Inflation is modeled as the decay of a scalar field pumping energy into the universe in a way that appears as a vacuum energy.
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1.2.1 Phenomenology of Inflation

During the Inflationary epoch, we see from Eq. (1.17) that the observable universe (particle horizon) increased vastly more quickly than the interaction scale. Additionally, the scale factor increased exponentially, \( a \propto \exp(Ht) \). Inflation expanded the scale factor \( a \) by at least 62 \( e \)-foldings\(^1\). This has a number of consequences.

Scale-invariant Gaussian Initial Conditions of Fluctuations

Prior to and during Inflation, the only fluctuations in the energy density were the thermal (quantum) fluctuations. Such fluctuations that couple into the spacetime metric \( g_{\mu\nu} \) along the diagonal are called scalar perturbations (see Sec. 1.5). As the scale factor grew, these fluctuations were expanded. If they were expanded beyond the Hubble length then there would no longer be a mechanism to thermalize them and they would be frozen in. Older fluctuations were expanded more than younger ones and so ended up at larger scales. The fluctuation production was constant throughout Inflation, resulting in a uniform amount of power expanded out to all observable (and possibly non-observable) scales. This set up a scale-invariant background of Gaussian fluctuations from which the evolution of the universe followed, i.e. the initial conditions for the cosmological evolution of energy density fluctuations were determined by Inflation to be scale-invariant and Gaussian.

Sub-unity Spectral Index

We note, however, that only fluctuations that were expanded beyond Hubble scale \( \text{energy scale entering Inflation of } 2 \times 10^{16} \text{ GeV} \) and that the universe entered the latest radiation era immediately after Inflation. See Weinberg\(^8\) or Lyth & Liddle.\(^9\)

\(^1\)The number of \( e \)-foldings depends on the energy scale of the universe at the start of Inflation and the history of the universe prior to the latest radiation epoch discussed in Sec. 1.1. The value of 62 assumes an energy scale entering Inflation of \( 2 \times 10^{16} \text{ GeV} \) and that the universe entered the latest radiation era immediately after Inflation. See Weinberg\(^8\) or Lyth & Liddle.\(^9\)
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would be frozen in. This implies that fluctuations generated at the tail-end of Inflation when there was not enough time to push them out of the horizon would be thermalized away. Thus, we would expect that power at scales comparable to the Hubble length at the end of Inflation would be suppressed, i.e. there should be a slight reduction in power from a pure scale-invariant spectrum. The scale-invariance is quantified by the ΛCDM parameter called the spectral index $n_s$ (see Eq. (1.28)), where $n_s = 1$ corresponds to pure scale-invariance of energy density fluctuations. Thus we expect $n_s < 1$, and the current constraints on the spectral index are $n_s = 0.9667 \pm 0.0040$.[5]

Homogeneity of the Universe Beyond the Hubble Length

From our construction of Inflation, we see that prior to Inflation the entire observable universe was within the Hubble length and so was well thermalized across its entirety and so is well-described by a single temperature. From Eq. (1.17), we see that Inflation increased the size of the universe much beyond the Hubble scale, thus after Inflation the size of the thermalized universe is much larger than the Hubble length.

We note from Eq. (1.17) that if $\eta/(aH)^{-1} > 1$ in a radiation or matter epoch, it must have always been larger than 1 for the entirety of that epoch. This implies that if the observable universe is thermalized and $\eta/(aH)^{-1} > 1$ in a radiation or matter epoch, then the thermalization did not happen in that epoch and there must have been a vacuum-dominated epoch prior to it.

We observe that the entire visible universe is thermalized to 1 part in 100000,[10] though the current Hubble length is approximately 12.6 Mpc.
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**Flatness of the Universe**

Even if the universe prior to Inflation had significant curvature, $|\Omega_K| > 0$, then from Eq. (1.10) we see that the curvature will be driven toward 0 by Inflation. Both a radiation and matter epoch would increase any pre-existing curvature $|\Omega_K| \to \infty$. The vacuum epoch suppression proceeds exponentially, but radiation and matter epochs amplify curvature only as a power law. Thus, even a very brief period of vacuum-dominated expansion could suppress the curvature so that it may never be observable, provided $H$ during Inflation is sufficiently large. As discussed previously, the universe currently appears flat with $\Omega_K = 0.0008^{+0.0040}_{-0.0039}$.

**Dilution of Exotic Particles**

We do not fully understand the physics at very high energies such as were present in the early universe. It is possible that very massive particles with exotic properties, e.g magnetic monopoles, were created before Inflation. The density of any such particles would have been diluted by the expansion of the universe during Inflation such that post-Inflation they are so sparse as to be undetectable.

**Gravitational Waves**

In addition to quantum mechanical fluctuations in the scalar modes, there were also quantum mechanical fluctuations in the tensor modes. All available modes were independently in their quantum vacuum states and so produced fluctuations. The tensor modes coupled into the spacetime metric $g_{\mu\nu}$ to produce metric tensor perturbations (gravitational waves). Then by the same expansion mechanism as for the scalar perturbations, the gravita-
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Coastal waves were rapidly inflated to produce a background of gravitational waves. There is no mechanism by which the gravitational wave background can be destroyed (short of generating even more gravitational waves), so they will persist through all subsequent epochs. We will discuss the effects of a background of gravitational waves on the universe presently.

1.2.2 Mechanics of Inflation

So far we have described Inflation from a phenomenological view, describing its effects on the universe. We now address the physical processes behind how Inflation might have come about. The simplest models of Inflation theorize the existence of a scalar potential field called the inflaton field. Although no scalar field had been found when these models of Inflation were created, the recent detection of the Higgs scalar field\footnote{\textsuperscript{11,12}} lends credence to the idea that Inflation could be driven by a scalar field. Prior to Inflation, the inflaton potential dominated the energy density of the universe. The decay of the inflaton potential pumped energy into the universe which drove a vacuum energy-like expansion. Following Inflation, the inflaton potential energy density was converted in a process called Reheating into more conventional matter and radiation forms, which then evolved as described in Sec.\textsuperscript{[1.1]} Many models of Inflation and the subsequent Reheating have been proposed, but we have little discriminatory power at this point, so we focus on general properties common to the models.

The relativistic action $S$ of a field is is described by the metric $g_{\mu\nu}$ and a Lagrangian
density $\mathcal{L}$

$$S = \int d^4 x \sqrt{-g} \left( \frac{1}{2} M_{Pl}^2 R + \mathcal{L} \right).$$

(1.18)

The simplest $\mathcal{L}$ suitable for Inflation is given by

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi)$$

(1.19)

where $\phi$ is the inflaton field and $V(\phi)$ is some undetermined potential function. The various models of Inflation generally involve choosing $V(\phi)$. The evolution of the inflaton field may be found by extremizing the action, $\delta S = 0$, which results in

$$T^\mu_\nu = \partial^\mu \phi \partial_\nu \phi - \delta^\mu_\nu \left[ \frac{1}{2} \partial^\alpha \phi \partial_\alpha \phi + V(\phi) \right]$$

(1.20)

comparing this to Eq. (1.12) we can identify that the action of the scalar field is to induce an effective energy density and pressure of

$$\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad P = \frac{1}{2} \dot{\phi}^2 - V(\phi)$$

(1.21)

We note that at the minimum of the inflaton potential $V(\phi)$ is typically set to 0. Then for a stationary inflaton at the minimum, the effective energy density and pressure are both 0 and we have returned to an empty vacuum configuration. This fact is utilized to ignore the inflaton field post-Inflation, as the inflaton will naturally decay towards its minimum value and Reheating will dissipate the momentum-like $\dot{\phi}$. Once these conditions are met, the inflaton field has no effect.

The conservation of mass-energy Eq. (1.11) then gives us

$$\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0$$

(1.22)
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where $V'(<\phi>) = \frac{dV}{d\phi}$. The Friedmann equations Eqs. \((1.14)\) and \((1.12)\) are

$$H^2 = \frac{3}{8\pi G} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right]$$  \hspace{1cm} (1.23)

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3} \left[ \dot{\phi}^2 - V(\phi) \right]$$  \hspace{1cm} (1.24)

The most common category of potential $V(\phi)$ is that of slow-roll Inflation. The idea behind this is that the inflaton field must not decay to its minimum too quickly, as this would not allow enough time before the inflaton field decayed away to expand the universe enough to trigger the phenomena described in the previous section. Furthermore, the slope $V''(<\phi>)$ of the inflaton potential must be relatively flat, as a steep $V'(<\phi>)$ would not allow for a relatively constant Hubble parameter during Inflation and an exponential expansion. Equivalently, $V''(<\phi>)$ must be small so that the energy density is dominated by the inflaton potential $V(\phi)$ and not kinetic energy. These conditions are typically codified as conditions on the slow-roll parameters

$$\epsilon \equiv \frac{1}{16\pi G} \left( \frac{V''}{V} \right)^2, \hspace{1cm} \epsilon \ll 1$$  \hspace{1cm} (1.25a)

$$\eta \equiv \frac{1}{8\pi G} \frac{V''}{V}, \hspace{1cm} |\eta| \ll 1$$  \hspace{1cm} (1.25b)

from which we observe that $\dot{\phi}^2 \ll V(\phi)$, and so $\rho \simeq V(\phi) \simeq -p$, which implies an equation of state parameter $w = -1$, consistent with our assertion for an exponential vacuum energy-dominated expansion epoch.
1.3 Recombination and the Cosmic Microwave Background (CMB)

Since Inflation, the universe has consistently expanded and cooled over the course of its lifetime, implying that its temperature in the past was significantly hotter. Prior to an age of \( t \sim 300 \text{kry} \), the universe was filled with a hot plasma of photons, free protons (plus a relatively small fraction of ionized He nuclei), and free electrons, all in thermal equilibrium. The thermal equilibrium temperature was determined by the species with the most entropy, which was the photon due to its significantly larger number density—photons outnumbered every other species by a factor of \( \sim 10^9 \). Electrons exchanged momentum and energy with photons via Thomson scattering and with protons via Coulomb scattering. The abundance of photons with energies larger than the binding energy of Hydrogen and Helium ensured that no neutral atoms would form. Due to the large number of free electrons, the photon mean free path was small \((\ll H^{-1})\).

As the universe cooled, the abundance of ionizing photons decreased sufficiently that bound Hydrogen and Helium could form. This process was delayed significantly beyond a temperature of 13.6 eV \((T = 1.6 \times 10^5 \text{ K})\) to 0.26 eV \((T = 3000 \text{ K}, t = 380000 \text{ yr}, z = 1100)\) due to greatly larger number density of photons \(n_\gamma/n_p \sim 10^9\). At such a large disparity, the photons far into the high-energy tail of the Planck distribution will outnumber the baryons until the temperature drops by orders of magnitude, as seen above. As the electrons were captured, the interaction mechanism between photons and matter van-
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ished, resulting in the decoupling and free-streaming of the photons. This process is called
Recombination\(^2\).

While coupled, the temperature of the photons reflected the equilibrium state of the
total energy density \(\rho\). After the photons decoupled, their distribution was locked in and
their only evolution was with the expansion of the universe \(T \sim 1/a\). By examining the
distribution of the CMB photons today, we can back out what the temperature of each
region of the CMB was at Recombination. The spatial distribution of these temperatures in
the present day tells us about the distribution of energy density of the early universe.

Recombination occurred everywhere throughout the universe at the same time, and the
photons scattered generally isotropically. The photons followed a straight-line path (since
\(K = 0\)) and did not interact for 13.8 Gyr. The origin of photons that originated from
Recombination and are detected by us on Earth today forms a spherical surface called the
Surface of Last Scattering. The signal we receive from the Surface of Last Scattering is the
Cosmic Microwave Background (CMB). It is called the Microwave background since the
3000 K photons have been redshifted by cosmic expansion to the microwave band.

1.4 Reionization

Before addressing the density perturbations, we briefly mention Reionization. As den-
sity perturbations coalesced, the first massive stars formed. The ultraviolet emission from

\(^2\)We have greatly simplified the process of Recombination here. For a more detailed accounting, see
Weinberg\cite{Weinberg} and references therein.
these stars reionized the neutral Hydrogen to again form free protons and electrons. This process completed gradually, beginning at \( z \sim 11 \) and ending (in the sense that nearly all of the Hydrogen was ionized) at \( z \sim 6 \). With the presence of free electrons, the Thomson scattering of photons off of electrons resumed. As with Recombination, the distribution of energy density again couples into the radiation, and continues to couple in to the present day. However, with the density of matter so greatly reduced by the expansion of the universe, the interaction rate is not large enough to affect the overall temperature of the radiation. In the intervening period of time between Recombination and Reionization, the universe had evolved linearly and so the same information couples into the CMB after Reionization as at Recombination.

Since Reionization was a relatively recent event, its effects couple in on larger scales, which our knowledge of is inherently limited by cosmic variance. We therefore do not need a detailed knowledge of Reionization to predict its effects on the CMB. Rather we may model it as a sudden phase transition at \( z_{re} = 8.8^{+1.2}_{-1.1} \) with width \( \Delta z_{re} = 0.5 \). The key feature of Reionization for cosmology is the optical depth due to Reionization \( \tau = 0.066 \pm 0.012 \).

### 1.5 Quantifying Inhomogeneities

Let us sketch how to quantify the inhomogeneities resulting from Inflation. We will add scalar and tensor perturbations to an unperturbed flat homogeneous universe. Vector
perturbations are possible but decay as $1/a^3$, so are generally ignored. The perturbations are added in the conformal Newtonian gauge, for which $t \rightarrow \eta$, where $\eta$ is the conformal time $d\eta = dt/a$ (defined identically to Eq. (1.14) with $c = 1$). The unperturbed metric is given by Eq. (1.1) with $K = 0$,

$$g_{\mu\nu}(\eta) = a^2(\eta) \text{diag}(-1, 1, 1, 1).$$

(1.26)

We add scalar perturbations that couple only into the diagonal elements, and require two perturbation functions $\Phi$ and $\Psi$. The tensor perturbations additionally couple into the spatial cross-terms and for each direction require a perturbation function for each gravitational wave polarization, $+$ and $\times$. The perturbed metric (accounting for only gravitational waves propagating in the $z$-direction) is

$$g_{\mu\nu} = a^2(\eta) \begin{pmatrix}
-(1 + 2\Psi(\vec{x}, \eta)) & 0 & 0 & 0 \\
0 & 1 - 2\Phi(\vec{x}, \eta) & 0 & 0 \\
0 & 0 & 1 - 2\Phi(\vec{x}, \eta) & 0 \\
0 & 0 & 0 & 1 - 2\Phi(\vec{x}, \eta)
\end{pmatrix}$$

$$+ a^2(\eta) \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & h_+(\vec{x}, \eta) & h_\times(\vec{x}, \eta) & 0 \\
0 & h_\times(\vec{x}, \eta) & -h_+(\vec{x}, \eta) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

(1.27)

where the first term includes both the unperturbed metric and the scalar perturbations and the second term includes only tensor perturbations. The $\Psi$ potential represents Newtonian
gravitational perturbations and $\Phi$ represents curvature perturbations while $h_+$ and $h_\times$ represent gravitational waves of the + and $\times$ polarizations.

These metric perturbations can be fed into the Einstein Field Equations Eq. (1.3) and the Boltzmann Equation to derive evolution equations. In Fourier transform space, the expectation value of the resulting fluctuations may then be written

\begin{align}
P_R(k) &= A_s \left( \frac{k}{k_0} \right)^{n_s - 1} \quad (1.28a) \\
P_t(k) &= A_t \left( \frac{k}{k_0} \right)^{n_t} \quad (1.28b)
\end{align}

where $P_R$ is the scalar power spectrum, $P_t$ is the tensor power spectrum, $A_s$ is the scalar amplitude, $A_t$ is the tensor amplitude, $k_0$ is a pivot scale which is typically chosen to be either 0.002 Mpc$^{-1}$ or 0.05 Mpc$^{-1}$, $n_s$ is the scalar spectral index, and $n_t$ is the tensor spectral index. We note that the universe is homogeneous and isotropic, which implies that the power spectra can depend only on the magnitude of the wavenumber $k = |k|$. As discussed in Sec. 1.2.1, we expect both spectra to be nearly scale-invariant, i.e. have no dependence on $k$. The scalar index is slightly smaller than 1, and the tensor index is assumed to be close to 0. The tensor-to-scalar ratio $r$ is defined as the ratio of the tensor and scalar amplitudes,

\[ r = \frac{A_t}{A_s} \quad (1.29) \]

and is a convenient metric to measure the scale of the tensor modes. Note, however, that the value of $r$ is scale-dependent since the scalar and tensor spectral indices are not expected to be the same (see below). The pivot scale in Mpc$^{-1}$ is usually placed into the subscript.
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of \( r \) (e.g. \( r_{0.002} \) or \( r_{0.05} \)) to eliminate any ambiguity.

The slow-roll parameters \( \epsilon, \eta \) can be related to the scalar and tensor spectral indices \( n_s, n_t \) to give\[13\]

\[
\begin{align*}
n_s - 1 &= -6 \epsilon + 2 \eta \quad (1.30a) \\
n_t &= -2 \eta. \quad (1.30b)
\end{align*}
\]

Furthermore, the tensor-to-scalar ratio \( r \) can be related to \( \epsilon \)[\[8\]]

\[
\begin{align*}
\rho &= V^{1/4} = (3.3 \times 10^{16} \text{ GeV}) \cdot r^{1/4} \quad (1.32)
\end{align*}
\]

These predictions are generic for any inflaton potential \( V(\phi) \) that satisfies the slow-roll conditions. We note that all models predict non-zero spectral indices that are not the same. We can take these results a bit further and note that the energy scale of \( V(\phi) \) (and thus the universe during Inflation) is related to \( r \),

\[
\rho = V^{1/4} = (3.3 \times 10^{16} \text{ GeV}) \cdot r^{1/4} \quad (1.32)
\]

Measurements of \( n_s \) and \( r \) allow us to place constraints on the amplitude and shape of the inflaton potential \( V(\phi) \), which we have seen is the key determining function behind the evolution of the universe. Measurement of these parameters, and in turn constraining \( V(\phi) \) has been a key focus for the field this decade. We have a strong detection of a sub-unity value for the spectral index, \( n_s = 0.9667 \pm 0.0040 \)[\[6\]] but no detection of \( r \) has been made. The current upper bound on \( r \) is \( r < 0.0987 \). This limit is based upon the temperature and E-mode auto- and cross-spectra. We note that the temperature spectrum has been measured

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to the cosmic variance limit, so further constrains must come from improved polarization data. Many simpler models of Inflation estimate that $r \sim 0.01$.

In addition to providing information on the origins and history of the early Universe, Inflation can give insight into high energy physics at scales that are unattainable in conventional laboratories. For $r \sim 0.01$, we see that the relevant energy scale is $\rho \sim 10^{16} \text{GeV}$, a factor of $10^{12}$ larger than the 14 TeV that the Large Hadron Collider is capable of. Inflation is the only mechanism by which physics at such large energy scales can be probed.

1.6 Cosmic Microwave Background Fluctuations

We have seen in the previous section a strategy for quantifying the inhomogeneities in space. However, we are only about to see a small subset of these fluctuations: the fluctuations on the surface of last scattering. The spatial fluctuations translate to overdense and underdense regions in space, which in turn translate to cold and hot spots in the CMB. We now analyze how to quantify these hot and cold spots.

1.6.1 Temperature Fluctuations

Photons in the CMB are from a thermal source and have a Planck distribution, in which the spectral intensity at every wavelength is characterized by a single parameter, the tem-
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Temperature $T$,

$$I_\nu(T) = B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/k_BT) - 1}. \quad [\text{MJy/sr} = 10^{-20} \frac{W}{m^2 \text{sr} \text{Hz}}] \quad (1.33)$$

A small variation in the intensity $\delta I_\nu$ can be written as a small variation in the temperature $\delta T$,

$$I_\nu(T_0) + \delta I_\nu(\delta T) = B_\nu(T_0 + \delta T) = B_\nu(T_0) + \frac{\partial B_\nu(T)}{\partial T} \bigg|_{T_0} \delta T$$

$$\delta I_\nu(\delta T) = \frac{\partial B_\nu(T)}{\partial T} \bigg|_{T_0} \delta T \quad (1.34)$$

where we note that the conversion factor is frequency-dependent. Although $\delta T$ has units of K similar to $T$, they have been linearized about $T_0$. We distinguish the linearized units by calling them $K_{\text{CMB}}$. The average temperature $T_0$ is always chosen to be the CMB temperature, $T_0 = T_{\text{CMB}} = 2.72548 \text{ K}$\textsuperscript{16} This gives us a conversion factor

$$\frac{dB_\nu}{dT} = \frac{2h^2\nu^4}{k_B^2 c^2 T_{\text{CMB}}^2} \exp\left(\frac{h\nu}{k_BT_{\text{CMB}}}\right) \left[\exp\left(\frac{h\nu}{k_BT_{\text{CMB}}}\right) - 1\right]^2 \quad [\text{MJy/sr}] \quad (1.35)$$

which at the PIPER frequencies of interest has the values

| $\nu$ [GHz] | $\frac{\partial B_\nu(T)}{\partial T} \bigg|_{T_0}$ [MJy/sr $K_{\text{CMB}}^{-1}$] |
|-------------|-----------------------------------------------|
| 200         | 478                                           |
| 270         | 444                                           |
| 350         | 302                                           |
| 600         | 32                                            |

Although intensity is what we measure, the temperature is more fundamental to the physical processes, and it is more convenient to work in temperature units $K_{\text{CMB}}$. While
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the intensity of the CMB changes with frequency, its temperature remains the same, and so we can ignore the frequency dependence.

The anisotropy of the CMB can be characterized by a direction-dependent variation of the temperature. The temperature of the CMB at present day may be written

\[ T(n) = T_0 + \delta T(n) \equiv T_0[1 + \Theta(n)] \quad [\text{K}_{\text{CMB}}] \]  

where \( n \) is the unit vector indicating direction and \( \Theta \equiv \delta T/T_0 \) is the unitless fractional temperature variation. We are working on the surface of a sphere, so the eigenbasis is that of the spherical harmonics. Thus it is convenient to decompose \( \Theta \) into the spherical harmonics,

\[ \Theta(n) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}^T Y_{\ell m}(n) \]  

where \( a_{\ell m}^T \) are the harmonic coefficients and take the place of the wavenumbers in our spherical basis. The harmonic coefficients may be extracted using the orthonormality of the spherical harmonics

\[ \int d\Omega Y_{\ell m}(n) Y_{\ell' m'}^*(n) = \delta_{\ell \ell'} \delta_{mm'} \]  

Since we have encoded the average temperature in \( T_0 \), the harmonic coefficients must have expectation values of 0,

\[ \langle a_{\ell m}^T \rangle = 0 \]  

where the expectation value is taken over an ensemble of many \( a_{\ell m} \) realizations from many (theoretical) universes with identical properties. Although the mean is zero, the variance is
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\[ \langle a_{\ell m}^T a_{\ell m}^{T*} \rangle = \delta_{\ell \ell'} \delta_{mm'} C_{\ell}^T \quad \text{[unitless]} \quad (1.40) \]

from which we note that the multipole \( \ell \) determines the variance. Since all \( 2\ell + 1 \) harmonic coefficients for a given \( \ell \) are sampled from the same distribution, we can construct an estimator of \( C_{\ell} \),

\[ \hat{C}_{\ell}^{TT} = \frac{1}{2\ell + 1} \sum_m a_{\ell m}^T a_{\ell m}^{T*} \quad \text{[unitless]} \quad (1.41) \]

which can be shown to have a fractional error of

\[ \frac{\Delta \hat{C}_{\ell}^{TT}}{C_{\ell}^{TT}} = \sqrt{\frac{2}{2\ell + 1}} \quad (1.42) \]

which is a reflection of Cosmic Variance, i.e. the degree to which we can constrain the true parameters \( C_{\ell} \) of the underlying distribution is limited since we have only a single universe with a limited number of realizations \( a_{\ell m} \) sampled from the distribution. Cosmic Variance is a more significant limitation on our ability to constrain the \( C_{\ell} \)'s at lower multipole \( \ell \).

The set of coefficients \( C_{\ell}^{TT} \) form the temperature power spectrum of the CMB. The spectra defined in Eq. (1.28) determine the shape of the power spectra, so the power spectrum is a reflection of all of the underlying physics we have discussed so far. To see this, we relate the temperature anisotropy power spectrum \( C_{\ell}^{TT} \) (Eq. (1.40)) to the energy density fluctuation spectrum \( P(k) \) (Eq. (1.27)). We emphasize that \( C_{\ell}^{TT} \) represents the angular fluctuations we see on the surface of last scattering, while \( P(k) \) represents the spatial fluctuations through the volume of the universe. Under the approximation that Recombination
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happened instantaneously, the two quantities are related for temperature by

\[ C_{\ell}^{TT} = \frac{2}{\pi} \int d(\log k) j_{\ell}^2(kD_*) k^3 P(k). \]  

(1.43)

where \( j_{\ell} \) is the spherical Bessel function of the first kind, and \( D_* \) is the comoving distance a photon travels since Recombination. The spatial fluctuation spectrum \( P(k) \) may be the scalar or tensor spectrum, depending on which contribution is to be quantified.

As a final note, the power spectrum is normally presented after being rescaled,

\[ D_{\ell}^{TT} = \frac{\ell(\ell + 1)}{2\pi} C_{\ell}^{TT} \quad \text{[unitless]} \]  

(1.44)

Furthermore, the power spectra is also usually presented with units of \( \mu K^2 \). The conversion follows from expanding \( \delta T \) rather than \( \Theta \) in Eq. (1.36), from which we see that the \( a_{\ell m}^T \) s are multiplied by \( T_0 \) and the \( C_{\ell}^{TT} \) s are multiplied by \( T_0^2 \). The non-unitless \( a_{\ell m} \) s have units of \( K_{\text{CMB}} \) and the non-unitless \( C_{\ell}^{TT} \) s and \( D_{\ell}^{TT} \) s have units of \( K_{\text{CMB}}^2 \).

1.6.2 Polarization Fluctuations

The CMB is not purely unpolarized and so a map of the temperature is not sufficient to encode all of the information in the CMB. We briefly review polarization in general, then examine how polarization is generated in the CMB, and finally look at how to quantify CMB polarization.

A polarized plane wave propagating in the \( z \) direction may be written

\[ E_x = A_x \cos(\omega t - \theta_x), \quad E_y = A_y \cos(\omega t - \theta_y) \]  

(1.45)
Polarization is conventionally quantified by the Stokes parameters,

\[ I = \langle A_x^2 \rangle + \langle A_y^2 \rangle \]  
\[ Q = \langle A_x^2 \rangle - \langle A_y^2 \rangle \]  
\[ U = \langle 2A_x A_y \cos(\theta_x - \theta_y) \rangle \]  
\[ V = \langle 2A_x A_y \sin(\theta_x - \theta_y) \rangle \]

where \( I \) is the intensity, \( Q \) and \( U \) are linear polarization intensities, and \( V \) is the intensity difference between left- and right-handed polarizations. The vector \( \mathbf{S} = (I, Q, U, V) \) is called the Stokes vector. The quantities \( Q \pm iU \) are spin-2 and so a rotation of the coordinate system by \( \phi \) rotates \( Q \) and \( U \) by \( 2\phi \). Clearly the \( Q, U \) basis is not rotationally invariant, which makes it unsuitable for encoding the coordinate-independent physics of the early universe. We will address this issue after discussing mechanisms to generate polarization in the early universe.

Near the time of Recombination, photons were coupled to electrons by Thomson scattering. The plasma can be considered an emission source, so a test point with anisotropic surroundings can have a non-zero polarized scattering distribution. The polarized Thomson scattering cross section is given by

\[ \frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi} |\hat{\epsilon}' \cdot \hat{\epsilon}|^2 \]

where \( \hat{\epsilon}' \) and \( \hat{\epsilon} \) are the incident and scattered polarization angles. The outgoing polarization intensities can be computed from the cross section, and from them we may compute the
resulting Stokes cross sections as a function of incident radiation $I'(\theta, \phi)$,

$$\frac{dI}{d\Omega} = \frac{3\sigma_T}{16\pi} (1 + \cos^2 \theta) I'(\theta, \phi)$$  \hspace{1cm} (1.48a)$$

$$\frac{dQ}{d\Omega} = \frac{3\sigma_T}{16\pi} \sin^2 \theta I'(\theta, \phi)$$ \hspace{1cm} (1.48b)

where we caution that the integral over $d\Omega$ is non-standard since the basis is spin-2 and must be rotated to the working basis before the integration is performed. This fact also allows us to compute $U$ from $Q$ by rotating the coordinate system by $\pi/4$. Integrating over all incoming sources $I'$ gives

$$I = \frac{3\sigma_T}{16\pi} \int d\Omega (1 + \cos^2 \theta) I'(\theta, \phi) = \frac{3\sigma_T}{16\pi} \left[ \frac{8}{3} \sqrt{\frac{\pi}{5}} a'_{00} + \frac{4}{3} \sqrt{\frac{\pi}{5}} a'_{20} \right]$$ \hspace{1cm} (1.49a)$$

$$Q = \frac{3\sigma_T}{16\pi} \int d\Omega \sin^2 \theta \cos(2\phi) I'(\theta, \phi) = \frac{3\sigma_T}{4\pi} \sqrt{\frac{2\pi}{15}} \Re a'_{22}$$ \hspace{1cm} (1.49b)$$

$$U = -\frac{3\sigma_T}{16\pi} \int d\Omega \sin^2 \theta \sin(2\phi) I'(\theta, \phi) = -\frac{3\sigma_T}{4\pi} \sqrt{\frac{2\pi}{15}} \Im a'_{22}$$ \hspace{1cm} (1.49c)$$

where the $\cos(2\phi)$ and $\sin(2\phi)$ in the integrals reflect rotating to the working basis, and the last equalities follow from decomposing the incoming intensity into a spherical harmonic basis $I'(\theta, \phi) = \sum_{\ell m} a'_{\ell m} Y_{\ell m}(\theta, \phi)$, writing the integrands in terms of spherical harmonics $Y_{\ell m}$, and using the orthonormality identity for spherical harmonics. We observe that only quadrupolar distributions about a test point result in linear polarization. Thomson scattering does not produce any circular polarization, though other mechanisms have been proposed that do. No significant circular polarization has been detected.[19]

With no significant $V$, the $Q$ and $U$ parameters totally determine $I$, so the full Stokes vector is determined by $Q$ and $U$. The $Q$, $U$ basis is not rotationally invariant, so we would
like to change to a coordinate-free basis. With these assumptions, the polarization map may be written as a symmetric trace-free $2 \times 2$ tensor:

$$\mathcal{P}_{ab} = \frac{1}{2} \begin{pmatrix} Q & -U \sin \theta \\ -U \sin \theta & -Q \sin^2 \theta \end{pmatrix}$$ (1.50)

where the metric is $g_{ab} = \text{diag}(1, \sin^2 \theta)$. The appropriate orthonormal basis for this construction is that of the spin-2 weighted spherical harmonics $Y_{(\ell m)ab}(n)$. Since we are decomposing a tensor with 2 independent components, it is natural that we would have 2 sets of harmonic coefficients. The decomposition is

$$\frac{\mathcal{P}_{ab}}{T_0} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \left[ a_{\ell m}^E Y_{(\ell m)ab}^E(n) + a_{\ell m}^B Y_{(\ell m)ab}^B(n) \right]$$ (1.51)

where $a_{\ell m}^E$ are the E-mode harmonic coefficients, $a_{\ell m}^B$ are the B-mode harmonic coefficients, and the harmonic functions are given by

$$Y_{(\ell m)ab}^E = N_\ell \left( Y_{(\ell m):ab} - \frac{1}{2} g_{ab} Y_{(\ell m):c^c} \right)$$ (1.52)

$$Y_{(\ell m)ab}^B = \frac{N_\ell}{2} \left( Y_{(\ell m):ac} \epsilon^c_b + Y_{(\ell m):bc} \epsilon^c_a \right)$$ (1.53)

with $^c$ indicating a covariant derivative, $\epsilon_{ab}$ is the antisymmetric tensor, and $N_\ell = \sqrt{\frac{2(\ell-2)!}{(\ell+2)!}}$.

The $E$ and $B$ designations are in analogy to $E$ and $B$ fields in electromagnetism, which are curl-free and gradient-free, respectively. These properties are shared by the $E$ and $B$ maps. Furthermore, we note that this basis is rotationally invariant and suitable for analysis. Characteristic E-mode and B-mode spatial patterns are shown in Fig. 1.1.

We now have 3 sets of harmonic coefficients, $(a_{\ell m}^T, a_{\ell m}^E, a_{\ell m}^B)$. We may use these to
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Figure 1.1: Characteristic E-mode and B-mode spatial patterns. E-modes are maximally symmetric and “curl-free”. B-modes have mirror antisymmetry and are “divergence-free”.

form 6 independent covariances and thus 6 power spectra

\[ \langle a_{\ell m}^X a_{\ell' m'}^{Y^*} \rangle = C^{XY}_{\ell \ell'} \delta_{mm'} \text{ for } XY \in \{TT, EE, BB, TE, TB, EB\}. \] (1.54)

The standard \( Y_{\ell m} \) and the E-mode \( Y^E_{(\ell m)ab} \) spherical harmonics both have parity \((-1)^\ell\), whereas the B-mode \( Y^B_{(\ell m)ab} \) has parity \((-1)^{\ell+1}\), which implies that \( C^{TB}_{\ell} = 0 \) and \( C^{EB}_{\ell} = 0 \).

This remaining 4 spectra can encode interesting physics. These spectra couple to the scalar and tensor perturbation spectra \( P_R(k) \) and \( P_t(k) \) in a process similar to that described for Eq. (1.43). The TT, EE, and TE spectra have been detected and are shown in Fig. 1.2. The BB spectrum will be examined in the next section.
These spectra are well-constrained by current measurements. Note that the units $\mu K^2$ are $\mu K_{\text{CMB}}^2$.
1.6.3 The B-mode Spectrum \((C_\ell^{BB})\)

The primordial B-mode spectrum \((C_\ell^{BB})\) has not yet been detected. Current limits are shown in Fig. 1.3. The B-modes are particularly interesting because scalar perturbations cannot create B-mode fluctuations, so a non-zero B-mode signal is indicative of tensor perturbations. The only tensor perturbations in the early universe were the Inflationary gravitational wave background discussed in Sec. 1.2.1, so a detection of primordial B-modes is a conclusive detection of Inflation.

Intuitively, we can understand the fact that scalar modes cannot contribute to the B-mode spectrum by examining symmetry. There are few physical processes in action in the early universe. Scalar perturbations are described by scalar functions, which are naturally invariant under rotation and reflection. As we can see from Fig. 1.1, this symmetry is matched by E-modes. However, B-modes require some process to break the mirror reflection symmetry, but there is no physical process available to do that for scalar perturbations. Thus, we would not expect scalar perturbations to produce B-mode power because the scalar perturbations have too much symmetry.

We note that the B-modes from Inflation are not caused directly by the scalar inflaton potential \(V(\phi)\), which would contradict the argument above. Rather, the tensor modes were generated by quantum mechanical fluctuations and then the inflaton potential was responsible for expanding the tensor mode fluctuations to cosmological scales. The quantum mechanical mechanisms that produce power in the modes are fundamentally different from the linear general relativistic theories on which we developed our model of cosmological
expansion, and do not respect the underlying symmetries in the same way. The significance of finding cosmological B-modes is not in the presence of B-modes (which we will see is not at all unusual), but rather finding B-modes on cosmological scales from the Recombination and Reionization epochs, which is a uniquely Inflationary prediction.

![Figure 1.3: Limits on the $C_{\ell}^{BB}$ spectrum from a number of recent instruments (modified from Lazear\textsuperscript{3}). No detections have been made. The black line indicates a theoretical B-mode curve with $r = 0.01$. There are additional contributions to the B-mode spectrum (foregrounds not shown) in the form of gravitational lensing of E-mode power into B-mode power (shown in blue). At higher multipoles, the lensing contribution dominates. Note that the units $\mu K^2$ are $\mu K^2_{CMB}$.](image)

The primordial B-mode curve in Fig. 1.3 shows two humps. The hump at $\ell < 10$ is due to the gravitational wave background coupling into the CMB after Reionization. The second hump near $\ell \sim 100$ is from the same gravitational wave background coupling into
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the CMB during Recombination. The signal due to Reionization is at a larger angular scale (lower $\ell$) because Reionization was a more recent event, so the Hubble length was larger at the time. We note that the spectrum falls off rapidly at smaller angular scales (higher $\ell$). However, there are other sources of B-mode power that complicate the measurement of primordial B-modes.

Gravitational Lensing

The complications are always with everything that has happened between when the tensor perturbations imprinted onto the CMB and when we detected the CMB photons today. In the intervening period, the photons had to travel through billions of light years of space, which was not empty. This matter produced a weak gravitational effect on the photons in a process called lensing. The lensing process can transform between E-mode and B-mode power. Since the B-mode power is expected to be a factor of $r \sim 1/100$ smaller, we are only concerned with the conversion of E-mode power into B-mode power. We note that if even a small fraction of E-mode power is converted to B-mode power, it can swamp the primordial signal due to the much larger E-mode signal.

The lensing is done by matter, which was generated from the same set of initial conditions as those that drove the the CMB fluctuations. Given that, one might ask the question why the lensing does not give the same information as from Recombination, similar to how the Reionization signal gives us a second shot at the same information. The answer is that the lensing deflection depends upon the line-of-sight integral of the matter density between the observer and the photon source—essentially, roughly 50 independent over- or under-
densities are summed in the integral. Given that the 50 spots are at different scales and therefore are sampled from different multipole modes, this summation destroys our ability to split out information about the individual terms that contribute to the sum.

The lensing signal is expected to be insignificant on large scales for \( r \sim 0.01 \), since the matter distribution is smoother on larger scales. At smaller scales the lensing spectrum becomes white. Since the primordial B-mode spectrum is decaying at larger \( \ell \), the lensing signal overtakes the primordial signal near the Recombination peak\(^3\) In order to separate out the primordial B-modes from the lensed B-modes, the measured signal must be delensed, a process by which the lensing potential \( \phi \) is estimated and combined with the measured E-mode spectrum to construct an expected lensing signal. This signal is subtracted off to isolate the primordial spectrum. Although this technique has not yet been demonstrated end-to-end, SPTpol\(^{22}\) and ACTpol\(^{23}\) have both detected\(^{24, 25}\) lensed B-modes by using the Cosmic Infrared Background (CIB) as a proxy for \( \phi \) and constructing a lensing template to fit against.

**Polarized Foregrounds**

The matter in the universe is also a source of photons. With the evolution of structure, the physical mechanisms became more complicated and less symmetric. Thus, they are potential sources of B-modes. The only significant sources of polarized foregrounds in the sub-mm range are expected to be synchrotron and polarized thermal dust emission.\(^{26}\)

Both of these foregrounds have a non-blackbody spectrum and so may be distinguished by

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\(^3\)The exact point where the lensing signal overtakes the primordial signal depends on the details of the amplitude and shape of Inflation.
their spectral dependence. The general strategy for eliminating these foregrounds is to fit the amplitude and spectral index of the foreground component (either on a per-pixel basis or using a template) using frequency bands in which the CMB signal is not significant to form a model, and then subtract off the foreground contribution in the science bands where the CMB signal is significant. This strategy relies on the fact that in temperature units the CMB signal has no spectral dependence.

Synchrotron emission comes from the emission from relativistic electrons traveling along a helical path in the galactic magnetic field. For frequencies above 20 GHz, it is well-modeled as a power law $T(\nu) \propto \nu^\beta$ with spectral index $\beta \sim -3$. Synchrotron contributes a signal with linear polarization fraction given by

$$f_p = \frac{p + 1}{p + 7/3} \sim 0.75$$

where $p$ is the power law index of the electron energy number distribution $N(E) \propto E^{-p}$. This is related to the spectral index by $\beta = -(p + 3)/2$, giving a polarization fraction of about 0.75. The superposition of multiple regions of differing magnetic field along the line of sight reduces the observed polarization fraction to $f_p < 0.3$. Even so, we see that synchrotron emission produces a large polarized foreground at lower frequencies.

Interstellar polarized thermal dust emission is the dominant foreground above 100 GHz. It arises from the thermal emission of interstellar dust particles. However, due to our limited understanding of interstellar dust, our ability to model thermal dust emission is limited. A popular model is that of Finkbeiner, et al (FDS Model 8), in which the dust is modeled as a two-component grey-body. Planck uses a single-temperature modified
blackbody model and find that the spectral index $\beta$ varies from 1.3 to 1.8 across the sky and a temperature around 20 K. Constructing the polarization map from the thermal map then requires specifying a polarization fraction and polarization angle in each pixel. The polarization fraction varies significantly over the sky between 0 near the galactic plane and 20% at mid latitudes. The polarization fraction at very high latitudes is not well known. We note that although the thermal dust intensity is smaller away from the galaxy, the polarization fraction can be larger, so even “clean” dust regions (such as the one commonly observed from the southern hemisphere) may have significant polarized dust emission. For the polarization angle, we note that both the synchrotron emission and dust particle alignment from a given region depend on the same underlying magnetic field, so the synchrotron polarization angle may be used as an estimate of the thermal dust polarization angle. The WMAP 23 GHz synchrotron component Q and U maps may be used to estimate the polarization angle, which is then applied to higher frequencies. Note, however, that such maps have resolution limited to $\ell \lesssim 60$ and are of limited use on smaller scales, but the Planck 353 GHz band suggests that there is still a significant amount of power at these small scales. This process is described in more detail in Sec. 5.1.3.

The spectral dependence of the CMB and foregrounds are shown in Fig. 1.4. At all

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4One possible explanation for the decrease in polarization fraction toward the galactic plane is that there are more independent clouds of dust with random polarization directions in the galaxy, which serves to decohere the polarization.
frequencies, the foregrounds dominate the CMB, and so therefore must be subtracted off. At lower frequencies, the dust is insignificant, and at high frequencies the synchrotron is insignificant. Near the foreground minimum, both foregrounds are significant. The B-modes are only a small fraction of the total CMB polarization intensity, so foregrounds are even more significant for them.

Figure 1.4: The polarized foregrounds and polarized CMB intensity. Note that the CMB contains both E-modes and B-modes in this figure, so the B-mode only curve would be significantly lower. Figure modified from Planck 2015 X.
Chapter 2

PIPER Science Goals and Design

The primary aim of PIPER is to constrain Inflation. We have seen in Sec. 1.5 that characterizing the tensor-to-scalar ratio $r$ will provide us information on the inflaton potential $V(\phi)$. The scalar spectrum has already been measured, so we target the tensor spectrum, observationally encoded in the $C^{\text{BB}}_\ell$ power spectrum (Sec. 1.6.3). We have discussed in Sec. 1.6.3 a few aspects of the B-mode spectrum that make measuring the primordial B-mode spectrum, and thus $r$, challenging. In this chapter, we discuss how the PIPER instrument addresses these challenges. A schematic diagram of the PIPER instrument is shown in Fig. 2.1.
Figure 2.1: The PIPER telescope design. One of the twin co-pointed telescopes is depicted. The optical design is simplified in this figure (see Fig. 2.2 for the full design). The entire telescope is enclosed in an open-aperture bucket dewar filled with Liquid Helium at 1.5 K. The first element is the VPM, responsible for modulating the polarization. A vacuum vessel in the dewar houses the analyzer grid, bandpass filters, and detectors. The ADR and SQUID amplifier are also housed in the pressure vessel. The ADR cools the 4 detector arrays to 100 mK. The detectors, SQUID amplifier, and ADRs are magnetically shielded. The SQUID amplifier exits the dewar via a vacuum-sealed trunk directly into the MCE. The ADRs are also controlled by the warm housekeeping electronics (HKE), which are also responsible for measuring the VPM phase and pointing sensors. Data is stored on a flight computer. A communications link is provided by the CIP. Dot-dashed lines indicate fiber optic connections.
CHAPTER 2. PIPER SCIENCE GOALS AND DESIGN

2.1 Sky Coverage

From Fig. 1.3, we see that the dominant features in the Inflationary B-mode spectrum are the Reionization bump at $\ell < 10$ and the Recombination peak near $\ell \sim 100$. The Reionization bump is larger in amplitude and is not contaminated by lensing modes, so PIPER targets larger angular scales. Larger scales require large sky coverage. From Eq. (1.41), we see that our ability to constrain the spectrum depends on the number of modes available to us. For multipole $\ell$, the characteristic scale of a mode is $\theta \sim \pi/\ell$. Then if we observe a fraction $f_s$ of the sky, we will get roughly

$$N \sim \frac{4\pi f_s}{\pi \theta^2/4} = \frac{16 f_s}{\theta^2} \approx 1.62 f_s \ell^2$$

modes from a particular multipole $\ell$. For low $\ell$, the number of available modes on the sky is small, so the marginal improvement in uncertainty with sky fraction $\frac{d}{df_s} \left( \frac{1}{\sqrt{N}} \right) \propto -1/\ell f_s^{3/2}$ is large.

To measure as many modes as possible, PIPER will map about 90% of the sky. For $r \sim 0.01$, the low multipoles will be cosmic variance limited. The higher multipole limit is set by the beam size. Features comparable in scale to and smaller than the beam are smeared out by the beam window function and cannot be resolved. The beam size is chosen to be $\sim 20$ arcmin so that PIPER is able to resolve multipoles safely above the Recombination peak near $\ell \sim 100$. With these choices, PIPER covers the range of multipoles between 2 and 200, covering both the Reionization bump and Recombination peak. We note, how-

\footnote{This feature is called a “peak” because the $D_{\ell}^{BB}$ is frequently plotted instead of $C_{\ell}^{BB}$, in which the $\ell \sim 100$ feature appears peaked.}
ever, that PIPER has limited capability to constrain and eliminate lensing modes that will contaminate the Recombination peak. In order to get a handle on lensing modes, PIPER would have to decrease the beam size considerably, which would make the experiment significantly more complicated. Rather, PIPER will rely on other experiments with smaller beams and sensitivity to large $\ell$ to model the lensing potential. The lensing potential combined with the well-known E-mode spectrum will allow the B-mode spectrum to be cleaned of lensing modes.

2.2 Frequency Bands

Foregrounds are brighter than the primordial B-mode signal at all scales and at all frequencies (Sec. 1.6.3), so the foregrounds must be corrected for. PIPER will fly 4 frequency bands at 200, 270, 350, and 600 GHz, all above the CMB peak frequency (160 GHz). All of these frequency bands are in a regime where the polarized dust emission foreground is dominant. The 200 and 270 GHz bands have significant CMB contributions and constitute the science bands. The 350 and 600 GHz bands are almost entirely dust-dominated and provide maps that allow the dust model to be constrained. As an added bonus, the dust maps are interesting in and of themselves as they provide information on interstellar dust.

Interstellar dust does not have rotational invariance. The correlation length is small, so each pixel is approximately independent. This means that each frequency band allows only a single model parameter to be constrained when using only internal data. With two
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high-frequency bands, PIPER will be able to remove more complicated dust models.

Planck has recently made nearly full-sky polarized thermal dust emission maps using its frequency bands at 353 GHz and higher, combined with external data sources. Planck uses a simplified single-temperature modified blackbody for its thermal emission model. In the range of frequencies between 353 GHz to 3000 GHz, the data is well-matched to this model with variation in the modified blackbody temperature of $\sigma_T = 1.4$ K and spectral index $\sigma_\beta = 0.1$. However, at frequencies below 353 GHz, the spectral index flattens out considerably and has a larger variation across the sky. Furthermore, the Planck dust foreground maps do not have the power to distinguish between various dust models. PIPER maps will be made with greater sensitivity that allow PIPER to discriminate between dust models.

2.3 Environment and Scan Strategy

PIPER will operate above most of the atmosphere at 120,000 feet (36 km) on a high-altitude balloon in order to minimize atmospheric foregrounds. At these altitudes, the atmosphere is only minimally polarized, so the advantage is to the total loading on the detectors. With reduced loading, the detectors may be more weakly coupled to their thermal bath and their thermal bath may be at a lower temperature, both of which reduce the intrinsic noise of the detectors.

PIPER will fly on conventional high-altitude balloon flights with the support of the
CHAPTER 2. PIPER SCIENCE GOALS AND DESIGN

Columbia Scientific Ballooning Facility (CSBF) out of Fort Sumner, NM and Alice Springs, Australia. Each flight allows for about 30 hours of flight time, with a launch at dawn providing 2 days and 1 night of integration time. PIPER will use a constant-elevation azimuthal spin at $\sim 0.5 \text{ deg/s}$ during the night and constant-elevation anti-solar scans for the daytime, allowing a sky coverage of about 55% per flight in the hemisphere of the launch site. A flight out of each of the northern and southern hemispheres would allow PIPER to map about 90% of the full sky.

Each flight will have sensitivity to a single frequency band, so a total of 8 flights will be required to get (nearly) full-sky maps in all 4 frequency bands.

2.4 Dewar and Optics

PIPER will use the ARCADE2 3500 L open-aperture liquid Helium bucket dewar to house the telescopes. An advantage of being in the upper atmosphere is the greatly reduced efficacy of the thermal transport mechanisms. With only about 0.5% the atmosphere, the conduction and convection mechanisms are weak enough that the LHe bath at launch will last throughout the entire 30-hour flight. Although the atmospheric pressure is small, the temperature at float altitude is still about 240 K, so an optical window would be a significant emission source in the sub-mm range. With the open bucket dewar, the window may be

\footnote{A quick estimate of the LHe hold time. The latent heat of evaporation of LHe at 1.4 K is $L (1.4 \text{ K}) \sim 90 \text{ J/mol} = 3375 \text{ J/L}$. Supposing about half of the LHe boils off getting to float (through increased thermal loading at low altitudes and the power necessary to cool the LHe bath), we are left with about 2000 L of LHe. This has a heat capacity of about 6 MJ. The loading at float is dominated by the wall conduction and is about 40 W, which gives us a hold time of 40 hours.}
eliminated.

With the absence of a window, all of the optical elements in the PIPER telescopes may be held at 1.5 K or colder. At these temperatures, they do not thermally emit significant amounts of power, allowing the power loading on the detectors to be further reduced. The optical design is described in detail by Eimer and is shown in Fig. 2.2. All optical elements outside of the vacuum vessel are kept at 1.5 K by superfluid LHe pumps.

The monocrystalline silicon lenses use a metamaterial anti-reflective coating, formed by cutting sub-wavelength grooves into the surface of the silicon. This has the advantage that the AR coating and lens are the same material, so there is no thermal strain associated with the coefficient of thermal expansion (CTE) mismatch between two different materials. Structures suitable for a broadband AR coating for 200 and 270 GHz may be cut into a single lens, but separate lenses with individual AR coatings must be made for 350 GHz and 600 GHz. Between flights, the lenses will be swapped out to match the frequency band of the next flight.
Figure 2.2: The PIPER optical design, from Eimer 2010.
2.5 Front-end Polarization Modulation

The first optical element is a variable-delay polarization modulator (VPM), which consists of a movable flat mirror behind a polarizing free-standing wire grating. The VPM rotates the Stokes vector between linear ($U$) and circular ($V$) polarization as the mirror-grating spacing is changed. By modulating and measuring the mirror-grating spacing, the modulation function may be known precisely. PIPER modulates the polarization signal rapidly at 3 Hz, far faster than the characteristic frequency of the signal. A multipole $\ell$ has a characteristic scale $\theta \sim \pi/\ell$ and will appear in the timestream as a signal at

$$f_{\ell} \sim \frac{\Omega}{\theta} = \frac{\Omega \ell}{\pi} = (0.003 \text{ Hz}) \ell$$

(2.2)

where $\Omega \sim 0.5 \text{ deg/s}$ is the scan rate. For the $\ell \sim 200$ upper limit of PIPER’s multipole range, we get $f_{\ell} \sim 0.5 \text{ Hz}$. The signal is carried on the 3 Hz VPM carrier modulation frequency, so the relevant frequencies will be at $3 \text{ Hz} \pm f_{\ell}$, i.e. between 2.5 Hz and 3.5 Hz.

This technique has the advantage that only the sky signal is modulated, so any instrumental polarization sources will be heavily suppressed by the demodulation step. Essentially, the PIPER VPM is an optical lock-in amplifier with unity gain. The noise properties of the instrument are then dependent primarily on the local noise spectrum around the carrier frequency of 3 Hz. Particularly, the $1/f$ noise below $\sim 1 \text{ Hz}$ is rejected following post-flight software demodulation. This technique has been demonstrated by the ABS instrument. Since the local noise properties around 3 Hz are the dominant contributors to the total noise, PIPER ensures that noise sources in the 1-10 Hz range are minimized.
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An additional advantage is due to the rotation between linear and circular polarization. The total sky circular polarization is expected to be negligible\textsuperscript{[19]} so the VPM modulates with a null signal\textsuperscript{3} We contrast this with the more conventional strategy of using a waveplate, which rotates between linear ($Q$ and $U$) polarizations. In the waveplate case, errors or uncertainty in the $Q$-$U$ rotation can cause $E$-$B$ mixing since the rotation is within the plane formed by the $E$-$B$ basis. The contamination of $E$ into $B$ is problematic since the $B$ signal is expected to be a factor of $r \sim 0.01$ smaller than the $E$ signal. Even a 1% contamination could entirely swamp the $B$ signal. This is not an issue for the VPM since the rotation is in a plane orthogonal to the $E$-$B$ plane.

2.6 Twin Co-pointed Telescopes

A single telescope with a VPM modulates between $U$ and $V$ and therefore only allows precise measurement of $U$ and $V$. A full description of the Stokes vector requires knowledge of $Q$, $U$, and $V$. In order to provide the missing component, PIPER uses twin co-pointed telescopes with VPM gratings rotated relative to each other by 45 degrees. The VPM always modulates the local $U$ and $V$, so by rotating the VPM grating allows one telescope to be sensitive to the sky $U$ and sky $V$ while allowing the other telescope to be sensitive to sky $Q$ and sky $V$. We note that the gratings are rotated by 45 degrees rather than 90 degrees because of the spin-2 property of the $Q$-$U$ basis (Sec.\textsuperscript{1.6.2}). The instrument optics has mirror symmetry about the saggital plane with completely independent optical

\textsuperscript{3}If the V signal is not null then a measurement of it would be a significant discovery in its own right.
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paths for each telescope.

The twin telescope and rapid polarization modulation have a significant advantage in that they allow PIPER to have an “instantaneous” measurement of the full Stokes vector, in the sense that a complete measurement is made roughly every VPM swing, i.e. at 3 Hz. PIPER does not rely on revisiting a spot in the sky with some other instrument configuration (e.g. rotated boresight or different waveplate angle). Cross-linking is only required for instrument calibration, which may be done with a small region since it has comparably few free parameters. The instantaneous measurement mitigates the effect of any slow instrumental drifts and also allows PIPER to use its extremely simple scan strategy.

2.7 Detectors and Sensitivity

PIPER will use 4 arrays, each with a $32 \times 40$ grid of Transition-Edge Sensor (TES) bolometers. Each pixel comprises an absorber strongly thermally coupled to a superconducting thermistor (the TES) with transition temperature $T_c = 140 \text{ mK}$. The absorber and thermistor are weakly thermally coupled ($G = 30 \text{ pW/K}$) to a 100 mK thermal bath. A voltage bias is applied to the TES to hold it on the superconducting transition. On the transition, the temperature sensitivity ($\frac{dR}{dT}$) is enormous, allowing for excellent sensitivity to applied optical power. The negative electrothermal feedback (ETF) of the voltage-biased positive temperature coefficient thermistor expands the dynamic range of the detector. The details of the detectors are discussed in Sec. 3.1.
Phonon (thermal) noise is intrinsic to all detectors and depends on the strength $G$ of the thermal link and temperature of the detectors, $P_{\text{phonon}} \propto \sqrt{GT}$. We have seen in the previous sections that the design of PIPER allows for a smaller $G$, which in turn reduces the phonon noise. The other intrinsic noise source is the inherent variability of the signal from the sky, the photon noise, which we emphasize is a property of the signal and not the instrument. At $T = 140$ mK and $G = 30$ pW/K, the phonon noise is smaller than the photon noise, so the detectors are so-called “background-limited”\textsuperscript{54} Other sources of noise are eliminated by the polarization modulation discussed in the previous section. In the background-limited regime, the total instrument sensitivity is determined only by the number of photons collected from any given region of the sky. As such, further improvements may only be made by collecting more photons: increasing the size of the primary, improving the optical efficiency, or increasing the integration time. PIPER achieves an instrument instantaneous sensitivity of $1.3 \mu K \sqrt{s}$ at 200 GHz and $1.6 \mu K \sqrt{s}$ at 270 GHz.

The detectors are inherently broadband with a reflective backshort that increases the absorptivity at 200, 270, and 350 GHz and attenuates it at 600 GHz. The attenuation at 600 GHz is required to prevent overwhelming the small saturation power $P_{\text{sat}} = 1.2$ pW by the increased power of atmospheric emission at higher frequencies. Since the detectors are broadband, the frequency sensitivity of the telescope is determined by a micromesh bandpass filter\textsuperscript{45} at the opening of the detector package. The filter must be swapped out between flights. Furthermore, the detectors are sensitive only to the total intensity of light incident upon them, so the polarization sensitivity is provided by the combination of the
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VPM and the analyzer grid.

2.8 Cryogenics

An adiabatic demagnetization refrigerator (ADR) is used to cool the detectors from the LHe bath temperature to the 100 mK base temperature. An ADR is a heat pump that exchanges the entropy of a paramagnetic salt for heat using the magnetic field from a superconducting coil. The 4-stage ADR allows for the coldest stage to hold at 100 mK continuously while rejecting up to $10 \mu W$ of power from both a 1.4 K LHe bath as expected at float and a 4.2 K LHe bath as expected on the ground.

2.9 Detector Readout

The detectors are read out by a 2-stage Superconducting Quantum Interference Device (SQUID) pre-amplifier and the Multi-Channel Electronics (MCEs) (discussed in detail in Chapter 3). The SQUID amplifier’s first stage also serves as the detector array’s multiplexer and is built into the backside of the detector array. The SQUID pre-amplifier is cryogenic and low-noise to ensure the total measurement noise is dominated by the photon noise. The warm MCEs use a time-domain multiplexing scheme in which the 40 rows are read out in series, while all 32 columns are read out simultaneously in parallel. Each of the 4 detector arrays requires its own set of MCEs.

The first stage SQUID amplifier uses a nulling feedback loop to ensure stability. The
CHAPTER 2. PIPER SCIENCE GOALS AND DESIGN

MCEs provide a row revisit rate of about 12 kHz, which is used to drive the first stage feedback loop. A final digital filter reduces the signal bandwidth to about 30 Hz, significantly more than the 3 Hz modulation carrier wave generated by the VPM.

All four MCE racks are driven synchronously by a common clock to ensure that data collection is perfectly synchronous. The MCE racks are powered entirely by batteries.

2.10 Electronics

The detector data does not constitute the full set of science data for the instrument. In addition to the power at the detector, we must also know the VPM’s grating-mirror spacing to construct the demodulation function, and the telescope pointing to correctly place the detector signal on the sky. PIPER uses a set of custom low-noise synchronous electronics to perform these tasks.

The PIPER electronics (aka Housekeeping Electronics, or HKE) takes in the MCE clock as its sole clock and so is by construction synchronized up to a constant phase with the detector readout. The VPM phase is measured by a series of capacitive absolute displacement sensors and read out by an HKE fast ADC board. The pointing is updated by gyroscopes, which are also read out by an HKE fast ADC board. This allows the 3 science data streams to be reconciled without a complicated interpolation step. The PIPER pointing actuation uses synchronous low-noise linear motor controllers. Commercial motor controllers that are both synchronous to an external clock and linear are not available.
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Additionally, since there is only a single clock, no contamination at beat frequencies is possible. This ensures that contaminants do not pop up at unexpected frequencies.

The PIPER electronics are also powered entirely by batteries. The short conventional balloon flights allow PIPER to carry enough batteries to power the entire instrument for the entire flight without being excessively heavy.

The PIPER electronics are discussed in great detail in Chapter 4.

2.11 Sensitivity

With this design, PIPER will achieve sufficient sensitivity to detect or constrain the tensor-to-scalar ratio to $r < 0.007$ at the $2\sigma$ level. A successful detection would confirm the epoch of Inflation as a physical reality and give insight into its energy scale. A null detection would reject all of the simple models of Inflation and raise significant questions about its feasibility. Furthermore, PIPER will produce the most sensitive polarized dust maps of its generation of experiments (Fig. 2.3).
Figure 2.3: PIPER will measure the polarized dust emission to a S/N better than 10 even in regions of low dust intensity and for polarization fractions as small as 10%, as seen in this simulation of the polarized dust of the BICEP2 region. Figure from Lazear.2
Chapter 3

Single Pixel Characterization

We characterized a single prototype pixel and measure its noise performance. To understand the detector characterization, we must first understand the detectors and readout chain. We will examine the flight-like detectors and readout chain and detail the differences between the flight-like and laboratory setups. Lastly we will analyze the results of the laboratory test.

3.1 Transition-Edge Sensor (TES) Bolometers

3.1.1 Overview

Each pixel is a suspended transition-edge sensor (TES) bolometers with a superconducting transition at $T_c = 140 \text{ mK}$ thermally coupled to an absorber and backed by a reflective backshort. Both the TES and absorber are suspended on long $350 \mu\text{m}$ thin legs.
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to reduce the thermal conductivity between the absorber and the 100 mK thermal bath to $G = 30 \text{ pW/K}$. With these properties, the saturation power of a pixel is $P_{\text{sat}} = G \Delta T = 1.2 \text{ pW}$. The two lines to bias the TES are carried along 2 of the legs. A reflective back-short sits 238.6 $\mu$m behind the absorber. The backshort serves as a 1/4-wave terminator at 200 GHz and 270 GHz to roughly double absorption but attenuates the signal at higher frequencies, since the unattenuated power would saturate the pixel.

The TES serves as a very sensitive thermistor. The bolometer is voltage-biased such that it always sits on its superconducting transition. As more optical power is incident on the bolometer, it heats up, causing it to move up the superconducting transition and greatly increasing the resistance of the bolometer. The increase in resistance is measured by monitoring the decrease in current. Similarly, a decrease in optical power results in an increase in current. The bolometer relates a change in optical power to a change in electrical current passing through it, allowing the optical power to be monitored by monitoring the current in the bolometer.

We note that if the TES is driven either fully normal or fully superconducting, then it no longer behaves as a thermistor and the current response no longer has any sensitivity to the incident optical power. Furthermore, the transition width is small (order of 1 mK), so a very small change in power ($\Delta P = G \cdot 1 \text{ mK} = 0.03 \text{ pW}$) would knock the TES off its transition. A TES with negligible bias power would have very limited dynamic range. The stability and dynamic range of the bolometer is enhanced by the electrothermal feedback (ETF) of the voltage bias and positive temperature coefficient of the TES. As the
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incident optical power on the detector increases, the temperature of the bolometer increases, and the resistance increases. However, as the resistance increases, the electrical power dissipated by the bias voltage decreases \( \left( \frac{\partial P}{\partial T} \right|_V = -V^2/R^2 \frac{\partial R}{\partial T} \) \), offsetting the increase in optical power. This feedback mechanism allows the TES to sit stably at the bias point partway up the superconducting transition, and results in the power dissipation while the TES is in the transition to be nearly constant. This point is worth reiterating: in order for the temperature of the TES to remain constant and stable, the total power dissipation in the TES \( (P_{\text{tot}} = V^2/R_0 + Q_{\text{opt}}) \) must remain constant and stable. Note that the amount of power that the bias can compensate for is limited from below by \( V = 0 \), and from above by \( V = V_{\text{max}} \) (normally set by the instrumentation). Once those limits are reached, nothing prevents the TES from moving out of its transition.

The TES bias circuit is shown in Figure 3.1. A detailed description of the TES electrothermal circuit may be found in Irwin & Hilton\[43\] and references therein. A lighter overview, in harmonic space, may be found in the appendix of Jones\[52\]. We extract only the final results. As mentioned above, the quantity of interest is the current \( I \) flowing through the TES arm, i.e. through the coupling inductor \( L \). The responsivity of the current to the
Figure 3.1: The TES bias circuit. The TES $R(T, I)$ is voltage-biased by the combination of the load resistor $R_{\text{bias}}$ and the shunt resistor $R_{sh}$. The load resistor $R_{\text{bias}}$ sets the current bias of the two parallel arms, since $R_L \gg R_{sh}$. At the bias point, $R(T, I) \gg R_{sh}$, so almost all of the current flows through the shunt arm, which then sets the voltage bias across the TES. As the resistance of the TES $R(T, I)$ changes, the current flowing through the coupling inductor $L$ changes, which changes the coupling to the SQUID SQ1. The SQUID is read out by an amplifying circuit shown diagrammatically here as an amplifier.

Applied optical power is (from Irwin & Hilton)

$$\frac{\partial I}{\partial P} \equiv s_I(\omega) = -\frac{1}{I_0 R_0} \frac{1}{2 + \beta_I} \frac{1}{1 + i \omega \tau_+} \frac{(1 - \tau_+/\tau_I) (1 - \tau_-/\tau_I)}{(1 + i \omega \tau_-) (1 + i \omega \tau_+)}$$  \hspace{1cm} (3.1)$$

$$\alpha_I \equiv \frac{T_0}{R_0} \frac{\partial R}{\partial T} \bigg|_{T_0} \quad \beta_I \equiv \frac{I_0}{R_0} \frac{\partial R}{\partial I} \bigg|_{T_0} \quad \mathcal{L}_I \equiv \frac{P_0 R_0 \alpha_I}{G T_0}$$

$$\tau = \frac{C}{G} \quad \tau_I = \frac{\tau}{1 - \mathcal{L}_I} \quad \tau_{el} = \frac{L}{R_L + R_0 (1 + \beta_I)}$$

$$\frac{1}{\tau_\pm} = \frac{1}{2 \tau_{el}} + \frac{1}{2 \tau_I} \pm \frac{1}{2} \sqrt{\left(\frac{1}{\tau_{el}} - \frac{1}{\tau_I}\right)^2 - 4 \frac{R_0 \mathcal{L}_I (2 + \beta_I)}{L \tau}}$$  \hspace{1cm} (3.2)$$

where the subscript 0’s indicate the value at the bias point, $\alpha_I$ and $\beta_I$ are the dimensionless
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temperature and current sensitivities respectively, $L_I$ is the low-frequency constant-current loop gain (usually simply called the loop gain), $\tau$ is the natural system thermal time constant, $\tau_I$ is the current-biased thermal time constant, $\tau_{el}$ is the electrical time constant, $R_L = R_{sh} + R_p$ is the Thevenin-equivalent load resistance equal to the sum of the shunt and parasitic resistances, and $\tau_{\pm}$ are the impulse response time constants.

3.1.2 Behavior and Parameter Selection

We note that this system may be underdamped, critically damped, or overdamped. Critical damping occurs when $\tau_+ = \tau_-$, i.e. when the term in the square root of $1/\tau_{\pm}$ is equal to 0,

$$L_{\text{crit}, \pm} = \left\{ L_I \left( 3 + \beta_I - \frac{R_L}{R_0} \right) + \left( 1 + \beta_I + \frac{R_L}{R_0} \right) \right\} \pm 2 \sqrt{L_I (2 + \beta_I) \left[ L_I \left( 1 - \frac{R_L}{R_0} \right) + \left( 1 + \beta_I + \frac{R_L}{R_0} \right) \right]} \frac{R_0 \tau}{(L_I - 1)^2} \quad (3.3)$$

For $L_{\text{crit}, -} < L < L_{\text{crit}, +}$, the response is underdamped. The oscillations in an underdamped system can lead to instability, so the overdamped case is preferred. Furthermore, the $L < L_{\text{crit}, -}$ is preferred over the $L > L_{\text{crit}, +}$ case, since large inductances can increase the electrical response time constant $\tau_{el}$ above the thermal response time constant and limit performance of the detector. In the overdamped case, the impulse response time constants obey $\tau_+ < \tau_-$, so the limiting time constant is $\tau_-$. We note that the current responsivity $s_I(\omega)$ behaves as a filter with poles at $s_+ = -1/\tau_+$ and $s_- = -1/\tau_-$. For $\tau_+ \ll \tau_-$, the $s_+$ pole is much further away from the $i\omega$-axis than the $s_-$ pole, and so the $s_+$ pole does not
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contribute significantly (since the distance from the $s_+$ pole to a point on the $i\omega$-axis does not change as much as the point moves along the axis compared to the distance from the $s_-$ pole to the same point). Thus, for $\tau_+ \ll \tau_-$, the system behaves as a single- pole low-pass with elbow frequency $\omega_+ = 1/\tau_+$.

The inductance is further constrained by the sampling frequency. The MCE revisits each row at a rate of about $f_{\text{sample}} = 12$ kHz, for a Nyquist frequency of about $f_{\text{Ny}} = 6$ kHz. Any noise power at frequencies above the Nyquist frequency will be aliased back into the Nyquist bandwidth, so we should increase $L$ (noting that $\tau_{el} \propto L$) until the dominant impulse response frequency $\omega_+$ is smaller than the Nyquist frequency $\omega_{\text{Ny}} = 2\pi f_{\text{Ny}}$, i.e. $\omega_+ < \omega_{\text{Ny}}$. This condition translates into the constraint on the inductance $L$ of

$$L > L_{min} = \frac{[R_L + R_0(1 + \beta_I)] - \frac{\tau_{\text{Ny}}}{\tau} [\mathcal{L}_I(R_0 - R_L) + R_L + R_0(1 + \beta_I)]}{1/\tau_{\text{Ny}} - 1/\tau_I}$$

(3.4)

A final constraint on the inductance is imposed by the VPM modulation. The science signal is mixed into the 3 Hz VPM modulation frequency, and so the detector response must not roll off frequencies near $f_{\text{VPM}} = 3$ Hz. The limiting frequency is $\omega_+$, so using a safety factor of $F \geq 1$, we require that $F\omega_{\text{VPM}} < \omega_+$. This results in a condition on the inductance $L$ of (noting that the condition is identical to that of Eq. (3.4), with the inequality reversed and $\tau_{\text{Ny}} \rightarrow \tau_{\text{VPM}}/F$)

$$L < L_{\text{VPM}} = \frac{[R_L + R_0(1 + \beta_I)] - \frac{\tau_{\text{VPM}}/F}{\tau} [\mathcal{L}_I(R_0 - R_L) + R_L + R_0(1 + \beta_I)]}{1/\tau_{\text{VPM}}/F - 1/\tau_I}$$

(3.5)

This gives us set of constraints

$$L_{min} < L < \min(L_{\text{VPM}}, L_{\text{crit-}})$$

(3.6)
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where $L_{\text{crit}} = L_{\text{crit}}(\tau, R_0, R_L, \beta_l, L_I)$, $L_{\text{min}} = L_{\text{min}}(\tau, R_0, R_L, \beta_l, L_I, \tau_N)$, and $L_{\text{VPM}} = L_{\text{VPM}}(\tau, R_0, R_L, \beta_l, L_I, \tau_{\text{VPM}}, F)$. Note that $L_{\text{VPM}}$ may not have a physically meaningful value in the overdamped region, in which case it may be ignored (this case is equivalent to $L_{\text{crit}} < L_{\text{VPM}}$, in which $L_{\text{VPM}}$ is irrelevant).

PIPER will operate in a high-loop gain, overdamped, voltage-biased regime. The TES has a target normal resistance $R_N = 20 \, \text{m}\Omega$ and will operate at $R_0 = 8 \, \text{m}\Omega$. The shunt resistor has a resistance of $R_{sh} = 2 \, \text{m}\Omega$, which is expected to dominate the TES parasitic, so $R_L \approx R_{sh} = 2 \, \text{m}\Omega$. The loop gain is expected to be comparable to that of ACT, $L_I \sim 25$. D. Benford estimates $\beta_l \sim 0.3$, based upon measurements from other groups.\[54\]

Furthermore, based upon the similarity of the PIPER detectors with the GISMO detectors, the natural thermal time constant $\tau$ is expected to satisfy $2 \, \text{ms} < \tau < 20 \, \text{ms}$. As discussed above, the Nyquist time constant is approximately $\tau_N = \frac{1}{2\pi f_N} = 1.6 \times 10^{-5} \, \text{s}$, the VPM time constant is $\tau_{\text{VPM}} = 0.05 \, \text{s}$, and we use a safety factor of $F = 10$. These values give us constraints

\[
\begin{align*}
\tau = 2 \, \text{ms} : & \quad L_{\text{min}} = 117 \, \text{nH} \quad L_{\text{VPM}} = \text{N/A} \quad L_{\text{crit}} = 263 \, \text{nH} \\
\tau = 20 \, \text{ms} : & \quad L_{\text{min}} = 188 \, \text{nH} \quad L_{\text{VPM}} = \text{N/A} \quad L_{\text{crit}} = 2630 \, \text{nH}
\end{align*}
\]

which we may combine to get the following constraint, which will work for the full expected range for the natural thermal time constant,

\[188 \, \text{nH} < L < 263 \, \text{nH} \quad (3.7)\]

This is overly restrictive. The detectors will have some spread in their parameters about
the targets. We need only that the large majority of detectors are biased correctly. The consequence for the inductance $L$ falling below $L_{\text{min}}$ is that we alias some extra noise into the system. The consequence for the inductance rising above $L_{\text{crit}}$ is that the detector is operating in the underdamped region and may be unstable. The inductors are located on the NIST 2-D MUX chips and are being designed for 400 nH. The natural thermal time constant strongly influences the upper bound of this inequality. We note from Figure 3.2 that detectors slower than 3 ms will be in the desired region, which includes the vast majority of detectors.

Figure 3.2: The dependence of $L_{\text{min}}$ and $L_{\text{crit}}$ on the natural thermal time constant $\tau$. The overdamped, stable, non-aliasing regime is $L_{\text{min}} < L < L_{\text{crit}}$. The dashed line indicates the actual target inductance, $L = 400$ nH. All detectors slower than 3 ms will be in the desired regime.
3.2 Flight Bias and Readout Chain

The detectors are tiled into arrays of 40 (+1 dark) rows and 32 columns. Each array has 1280 active pixels. Each detector couples to the readout through a coupling inductor, as shown in Fig. 3.1. The detector readout is a combination of a 2-stage SQUID amplifier and multiplexer, provided by NIST, followed by the Multi-Channel Electronics (MCE), provided by University of British Columbia (UBC). The 2-stage SQUID amplifier includes the infrastructure for feeding back on the first stage SQUID, but the MCE handles the feedback loop and is responsible for actuating the feedback. Each MCE can control a single array. Figure 3.3 shows a block diagram of the readout chain.

Optical and electrical power is deposited into the detector. The voltage bias causes the current flowing through the TES arm to change as the amount of incident power changes. This change in current causes a change in magnetic flux coupling from the coupling coil to the first stage SQUID (SQ1). Every pixel in the array has a corresponding SQ1. All of the SQ1s in a column are wired together along with flux-gated switches (Sec. A.4). Only a single switch is turned ON at any given time, so only a single SQ1 is active, thereby selecting which pixel to read out. The SQ1s are voltage-biased, so a change in the flux coupling causes a change in the current through the circuit. A second coupling inductor couples the column’s circuit to a second stage SQUID (SQ2), also commonly called the SQUID series array (SA or SSA). The second stage SQUID is current-biased, so a change in the flux coupling from stage 2 results in a varying voltage response in SQ2. The voltage response of SQ2 is measured with an instrumentation amplifier. A digital feedback loop
in the MCE then takes SQ2 voltage response and feeds it back into the stage 1 feedback (S1FB) loop, which has an inductor that also couples into SQ1 in parallel with the detector coupling inductor. The feedback is tuned so as to null the detector flux. Nulling the total flux into SQ1 enhances its linearity and stability. Since the S1FB signal is opposite the detector response, recording the S1FB signal that the MCE outputs indirectly measures the detector signal. Each column has an identical copy of this readout chain, and the MCE measures and controls all pixels in a row simultaneously. A single column’s readout circuit
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is shown in Fig. 3.4.

![Fig. 3.4: A single column of the analog portion of the readout circuit. Shows only two rows and one column. Additional rows are added by tiling the blocks into the dotted regions of the circuit. Additional columns are added by duplicating this circuit. Note that the RS lines are shared for all columns, with \( N \) inductors in series for \( N \) columns. Each detector bias (DETB) line biases all of the pixels in its row, and may bias multiple rows. Since the DETB line is always biased with a DC value, biasing rows and columns is straight-forward: simply place the biasing block of each pixel to be biased in series. The detectors \( (R_{n TES}) \) are voltage-biased by the shunt resistor \( R_{sh} \). Varying optical loads vary the current through the TES coupling inductor, which varies the flux in the 4-SQUID stage 1 SQUIDs. The stage 1 SQUIDs are voltage-biased by the \( R_{SQ1} \) and the flux-gated switches. Which stage 1 SQUID is biased is determined by the RSn lines (Sec. A.4). As the flux in SQ1 changes, the current in the loop changes. This change in current couples through the SSA inductor into the SQUID series array (SSA). The SSA is current-biased, so the change in SSA flux results in a change in voltage, which is then measured by the UBC AMP and the ADC.](image-url)
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3.2.1 Detector Bias

The detectors are biased by a circuit very similar to Fig. 3.1. In order to reduce the number of wires required, all detectors in a row are biased in series. Furthermore, rows are ganged together into 20 biasing groups. The MCE biases the 20 groups separately to maximize the number of detectors successfully biased onto their superconducting transition. All detectors are biased all the time, and the bias is changed only when the MCEs are retuned.

The bias voltage across the TES arm at $\omega = 0$ from an applied DETB voltage $V_0$ is given by

$$V_{\text{bias}} = \frac{Z_{\text{block}}}{R_{\text{DET}} + nZ_{\text{block}}} V_0 = \frac{1}{n + \frac{R_{\text{DET}}}{Z_{\text{block}}}} V_0$$

$$V_{\text{bias}} = \frac{1}{n + \frac{R_{\text{DET}}}{R_{\text{sh}} \left(1 + \frac{R_{\text{sh}}}{R(T,I)}\right)}} V_0$$

where $n$ is the number of pixels being biased. We note that for $R(T,I) \gg R_{\text{sh}}$, the TES is strongly voltage biased and does not depend on $R(T,I)$. The current flowing through the arm is not simply $V_{\text{bias}}/R(T,I)$ since the TES resistance is current-dependent due to the electrothermal feedback.

Given that the exact resistance of the TES is not a priori known, it is convenient to write the TES bias voltage in terms of quantities that are measured directly. The current through the TES arm is measured from the S1FB coil and the coupling factor $M_1$ between the S1FB and S1IN coils. The total current applied to the circuit is known since the total impedance of the bias circuit is only weakly dependent on $R(T,I)$. Then the TES bias voltage $V_{\text{bias}}$
may be written
\[ V_{\text{bias}} = (I_{\text{total}} - I_{\text{TES}})R_{sh} = \left( \frac{V_0}{R_b} - I_{\text{TES}} \right) R_{sh} \]

where \( R_b \) is the measured total impedance of the bias circuit and \( I_{\text{TES}} \) is the measured (through some other means) current through the TES arm, \( V_0 \) is the voltage applied to the bias circuit, and \( R_{sh} \) is the shunt resistance.

A further consideration for the full bandwidth signal is that the TES resistance \( R(T, I) \) and the coupling inductor form an LR-filter. The electrothermal feedback complicates the analysis of it slightly, but it is included in the analysis of TES in Sec. 3.1.1 as \( \tau_{el} \) in Eq. (3.2). It is approximately \( \tau_{el} \simeq L/R(T, I) \) for \( R(T, I) \gg R_{sh} \) and small current sensitivity \( \beta_I \). This electrical time constant limits the amount of signal bandwidth that may be propagated from the detector to stage 1, as well as the amount of noise power that is propagated to stage 1.

There is also a thermal time constant \( \tau = C/G \) associated with the absorber itself. It factors into the filter at the same point, but it is complicated considerably the by the electrothermal feedback. Its contribution is also accounted for in Sec. 3.1.1. Note that the thermal and electrical time constants combine to form effective detector time constants \( \tau_+ \) and \( \tau_- \). The typical operating regime has \( \tau_+ \ll \tau_- \), for which the effective filter is a single-pole with elbow frequency \( \omega_+ = 1/\tau_+ \). This is discussed in more detail in Sec. 3.1.2. Note, however, that since \( \tau_+ \) is typically constrained most strongly by the L/R time constant, it is common to use the two interchangeably.
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3.2.2 SQUID Readout: Stage 1

The current in the TES arm couples through the stage 1 input inductor SQ1IN to the stage 1 SQUIDs SQ1 to generate a magnetic flux in SQ1. The amount of flux is related to the TES arm current $I_{\text{TES}}$ by the mutual inductance $M_{IN1}$,

$$\Phi_{SQ1} = M_{IN1}I_{\text{TES}}.$$ (3.10)

The stage 1 feedback inductor SQ1FB couples into SQ1 in a similar way, but with a different mutual inductance $M_{FB1}$ and depending on the S1FB arm current $I_{S1FB}$.

$$\Phi_{SQ1} = M_{FB1}I_{S1FB}$$ (3.11)

We note that we can combine these two equations to relate the current through the SQ1FB inductor to the current in the TES arm,

$$I_{\text{TES}} = \frac{M_{FB1}}{M_{IN1}}I_{S1FB} \equiv M_1 I_{S1FB}$$ (3.12)

where we have defined the coupling factor $M_1 \equiv \frac{M_{FB1}}{M_{IN1}}$. This equation implicitly assumes that the S1FB is successfully being used to null the flux from SQ1IN in SQ1, i.e. that $d\Phi_{SQ1}^\text{IN} + d\Phi_{SQ1}^\text{FB} = 0$. We drop the usually-meaningless sign. This is useful since the MCE controls the amount of current it puts through the S1FB arm, and therefore it is always known. Thus, we always know very directly how much current is flowing through the TES arm.

The mutual inductances $M_{IN1}$ and $M_{FB1}$ may be measured directly by locking S2 (i.e. applying feedback to S2FB such that $d\Phi_{SQ2}^\text{IN} + d\Phi_{SQ2}^\text{FB} = 0$) and then sweeping the S1FB
bias or the detector bias. The feedback into stage 2 linearizes the stage 2 response, so the sweeps trace out the $V-\Phi$ curve of SQ1. The curve is periodic in the $\Phi$ axis with period $\Phi_0$, so the current required to change by 1 $\Phi_0$ may be determined by examining the period of the data. Note that knowledge of the voltage gain is not required, since it does not modify the period.

Even though the signal is being nulled by S1FB, we are still interested in how a signal would propagate up the signal chain so that we can understand how the feedback into S1FB works. We will take advantage of the nulling factor again since it implies that the signal value, and thus position on the $I/V-\Phi$ curves, is nearly constant, allowing us to work in the small signal limit.

The stage 1 SQUIDs are voltage-biased by $R_{SQ1}$, so the readout sits at a particular point on the $I-\Phi$ curve (Fig. A.7). The bias point may be adjusted with the DC level of S1FB. The slope of the curve at this point determines the current response,

$$i_1 = \frac{1}{(\partial \phi_1 / \partial I)} \phi_1 = \left[ \frac{M_{IN1}}{(\partial \phi_1 / \partial I)} \right] i_{TES}$$

(3.13)

where we’ve adopted lowercase characters for small signal variations and the derivative is evaluated at the bias point. The term in square brackets is simply a constant. Note that the $I-\Phi$ curve is of the series of 4 SQUIDs in SQ1. The stage 1 input inductor SQ1IN couples equally to each of the 4 SQUIDs, but since the response is a current, the response is the same as for a single SQUID. The advantage of having the 4 SQUIDs in series in this case is to increase the ON dynamic impedance of the SQ1 arm, which makes satisfying the flux-gated switch conditions Eqs. (A.22) and the voltage bias condition easier.
There are $N_{\text{row}} + 1$ SQ1’s in the stage 1 loop. The 1 extra is a dark channel that couples to a TES with no absorber, which allows for monitoring of detector condition independent of the optical loading. Which SQ1 is activated depends on which RS line is energized. When an RS line is energized such that enough current to drive a $\Phi_0/2$ through the corresponding switch SQUID SN, then the voltage drop is primarily across the SQ1 that is turned ON. The RS lines and SNs make up the multiplexer. When the mux switches rows, a step function is injected into the loop. The resulting transient response must be allowed to dampen away before the system reaches equilibrium and is a good measure of the power loading on the detector. This limits the maximum mux rate.

As a final note, the stage 1 mux contributes some additive current noise from the inherent SQUID noise, the bias lines, and thermal noise from the resistors and warm end. This noise contribution is insignificant at low frequencies where the L/R filter has not attenuated the noise, but dominates at higher frequencies. Since the mux noise contributes after this filter, they do not attenuate the mux noise. Furthermore, since we have not yet encountered the first amplifier stage, the additive noise can be significant, as it will be amplified by the amplifier.

3.2.3 SQUID Readout: Stage 2 (Series Array)

The current in the stage 1 loop couples through the stage 2 input inductor SQ2IN (aka SSAIN or SAIN) to the stage 2 SQUID SQ2 (SSA or SA) to generate a magnetic flux in

\footnote{Assuming the RS loops time constants are much faster than the SQ1 time constant.}
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SQ2. The amount of flux is related to the stage 1 loop current $I_1$ by the mutual inductance $M_{IN2}$,

$$\Phi_{SQ2} = M_{IN2}I_1.$$  \hfill (3.14)

The stage 2 feedback inductor SQ2FB (SSAFB or SAFB) couples into SQ2 in a similar way, but with a different mutual inductance $M_{FB2}$ and depending on the S2FB (SSAFB or SAFB) arm current $I_{S2FB}$,

$$\Phi_{SQ2} = M_{FB2}I_{S2FB}.$$  \hfill (3.15)

It is common for diagnostic purposes to lock stage 2, i.e. apply feedback to S2FB such that $d\Phi_{SQ1}^{IN} + d\Phi_{SQ1}^{FB} = 0$. When this is the case, the stage 1 loop current $i_1$ is related to the S2FB current by

$$i_1 = \frac{M_{FB2}}{M_{IN2}}i_{S2FB} \equiv M_2 i_{S2FB}$$  \hfill (3.16)

where we have defined the coupling factor $M_2 \equiv \frac{M_{FB2}}{M_{IN2}}$. With stage 2 locked, the stage 1 readout parameters $M_{FB1}$ and $M_{IN1}$ may be measured, as well as the SQ1 $I$-$V$ curve.

The mutual inductances $M_{IN2}$ and $M_{FB2}$ may be measured by operating the system open loop and accounting for the open loop gain of the amplifier.

In the small signal limit, the voltage response of stage 2 due to current in stage 1 is given by the slope of the SQ2 $V$-$\Phi$ curve. The position of the system on the SQ2 $V$-$\Phi$
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curve may be adjusted using the S2FB line. So the voltage response may be written

\[ v_2 = \frac{1}{(\frac{\partial \Phi_2}{\partial V})} \phi_2 = \left[ M_{IN2} \left( \frac{\partial \Phi_2}{\partial V} \right) \right] i_1 \]

\[ v_2 = \left[ M_{IN2} \left( \frac{\partial \Phi_2}{\partial V} \right) \left( \frac{\partial \Phi_1}{\partial I} \right) \right] i_{TES} \quad (3.17) \]

\[ v_2 \equiv G_{SQ} i_{TES} \quad (3.18) \]

where we’ve adopted lowercase characters for small signal variations and the derivatives are evaluated at the bias points. The term in square brackets in Eq. (3.17) is simply a constant, which we define as \( G_{SQ} \). Note that the \( V-\Phi \) curve for SQ2 is of all 100 of the SQUIDs in the series array combined. The stage 2 input inductor SQ2IN couples (roughly) equally to all SQUIDs in the series array, so the voltage response applies to each of them. Since the SQUIDs are in series, the total voltage response is multiplied by the number of SQUIDs in the series array, providing an effective gain compared to a single stage 2 SQUID. Note, however, that if the coupling of the inductor to the SQUIDs in the series array is not uniform, then the \( V-\Phi \) curves will be out of phase and the total amplitude of the response will not be 100 times as large.

Similarly to the mux, the series array also contributes additive noise to the system. We again do not calculate it here. The series array has a gain of 100 and serves as the pre-amplifier for the readout. This is the last stage where we consider additive noise, since all subsequent stages are attenuated by a factor of 100 when referred to the pre-amplifier input.

The series array is the final cryogenic stage. It is held at the LHe bath temperature (1.5 K in flight, 4.2 K in the laboratory). All stages following the series array are warm (300
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3.2.4 Series Array to MCE Cable

The series array is connected to the next stage (UBC Amplifier) by a 3.5 m long cryogenic cable. The cable must traverse the $\sim 3 \text{ m}$ between the series array in the submarine and the MCE boxes mounted on the top of the dewar. The cable materials are chosen primarily for their thermal characteristics rather than their electrical ones. This results in a potentially significant filter and phase delay due to the cable resistance, inductance, and capacitance. The cable limits the row muxing rate, since it rolls off the step in both the address line and the resulting step response. These effects are discussed further in an upcoming paper by Switzer.\[55\]

3.2.5 UBC Amplifier

The voltage across SQ2 is measured by the UBC amplifier\[4]. The UBC amplifier is a 4-stage amplifier with 5-poles in the transfer function. The gains of the stages are, from stage 1 to stage 4: 6.13, 5.99, 5.52, 1. The total gain is 203. The poles are at 9.7 MHz (multiplicity 2), 15.2 MHz (multiplicity 2), and 7.2 MHz. The 3 dB point for the full amplifier is at 3.2 MHz.

An adjustable DC offset is built into the amplifier that is used to zero out the DC amplitude of the signal. The average optical power and the detector, SQ1, and SQ2 biasing result

\[http://e-mode.phas.ubc.ca/mcewiki/index.php/Readout_Card_Preamp_Chain\]
in a non-zero signal voltage different at the input to the UBC Amplifier. The adjustable DC offset is set to zero this out so that any variation in the signal from zero reflects a change in the optical loading on the detector.

3.2.6 UBC Analog-to-Digital Converter

The signal is digitized at 50 MHz by an ADC. An important consideration is that the mux is switching which row is coupled to the readout. Each switching from the mux induces a step response in the readout signal, which must be allowed to decay away. The mux sits on a single row for row_len samples, discards the first sample_dly of these samples, and accumulates the remaining sample_num = row_len − sample_dly. After that, the mux switches to a different row and does not revisit the original row until the other num_row − 1 rows have been sampled. For PIPER-like parameters (num_row = 41, row_len = 100, sample_num = 10, sample_dly = 90), the MCE has a row dwell period of 2 µs, and a revisit rate of ∼ 12 kHz.

This may be modeled as a 50 MHz sampling step, a FIR, a row dwell period decimation step, and a mux period decimation step. The transfer function referred back to the input of the ADC is shown in Fig. 3.5 and has functional form given by

\[ H_{\text{FIR}}(f) = \text{sinc} \left( \frac{N_{\text{sum}} f}{f_{\text{MCE}}} \right) \]

(3.19)

We note that the row dwell FIR is essentially flat and will be essentially flat for any choice of sample_dly and sample_num due to the large decimation factor. However, the mux
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Figure 3.5: (left) The transfer function in real frequency space of the MCE’s row dwell FIR for \( \text{sampl}_{\text{y}} \text{dly} = 90, \text{sample}_{\text{num}} = 10 \). Note that the FIR has bandwidth out to the Nyquist frequency of 25 MHz, but the single-ended bandwidth after the decimation steps is only \( \sim 6 \text{ kHz} \). If the L/R filter’s cutoff were above 6 kHz, we would have to worry about power from the detector aliasing during the decimation. Otherwise, the decimation resampling picks out the region of the row dwell FIR that is below the post-decimation Nyquist, which is effectively flat for essentially any choice of \( \text{sampl}_{\text{y}} \text{dly} \) and \( \text{sample}_{\text{num}} \). (right) The mux noise transfer function. Note that power from the mux is after the L/R filter and will be aliased by the decimation. In this case, the transfer function of the UBC amplifier limits the aliasing. The dashed line shows the theoretical effect of aliasing if the UBC amplifier had no low pass cutoff.

noise and any pickup in the cable is not filtered by the L/R filter and is limited only by the UBC Amplifier filter (Sec. 3.2.5), and so its power is aliased into the post-decimation signal band and thereby amplified by a factor of 18.
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3.2.7 MCE Feedback

The 12 kHz row revisit rate samples are used to provide feedback to the S1FB line in the form of a PID loop. The current applied to the S1FB coil is given by

\[ i_{S1FB}(s) = H_A(s)H_{FIR}(s) \frac{k_I}{s} v_2(s) = G_{SQ} H_A(s) H_{FIR}(s) \frac{k_I}{s} i_{TES}(s) \]

\[ i_{S1FB}(s) \equiv L(s) i_{TES}(s) \quad (3.20) \]

where we’ve transformed to Laplace space to simplify cascading the transfer functions, \( H_A \) represents the UBC amplifier transfer function (Sec. 3.2.5), \( H_{FIR} \) represents the sampling FIR (Sec. 3.2.6), and we note that \( k_I/s \) represents only an integral term of a PID loop. The MCE admits a full PID loop, but typically only an integral term is used. Qualitatively, this may be understood by noting that the system to be controlled (SQ1) is first order so a D term would over-control the system (see Sec. C.2), i.e. the response to feedback is linear so there should only be a single derivative between the highest and lowest order terms in the controller. The P term is unnecessary because the system has no effective mass (the SQ1 current responds almost instantaneously to a change in applied flux), so there is no need for a proportional term to speed up the response. The I term is always required in order to ensure the system is unbiased.

We identify the open loop gain as \( L(s) = H_A(s) H_{FIR}(s) k_I/s \). The total closed loop transfer function \( H(s) \equiv \frac{i_{S1FB}(s)}{i_{TES}(s)} \) is given by a simple extension of Eq. (C.4). Rather than a \(-1\) feedback signal, the feedback signal is scaled by the \( i_{S1FB} \) to \( i_{TES} \) coupling factor \( M_1 \) (Eq. (3.12)). Note, however, that the MCE does not apply the new feedback value until the
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![Diagram of MCE feedback loop]

Figure 3.6: The MCE feedback loop. The feedback arm has a phase delay $e^{-sT}$, where $T = 1/f_{\text{revisit}} \approx 80 \mu s$ is a single revisit period.

next visit to the row, so there is a phase delay in the feedback arm. The MCE feedback loop is shown in Fig. 3.6 and has a transfer function $H(s)$ of

$$H(s) = \frac{L(s)}{1 + M_1 e^{-sT} L(s)} = \frac{G_{SQ} H_A(s) H_{\text{FIR}}(s) k_I}{s + M_1 G_{SQ} H_A(s) H_{\text{FIR}}(s) k_I e^{-sT}}$$

(3.21)

We note from Fig. 3.5 that $H_A H_{\text{FIR}}$ is essentially flat with amplitude 1 so the effective transfer function is

$$H(s) = \frac{1}{M_1 e^{-sT} + \frac{s}{G_{SQ} k_I}}$$

(3.22)

This gives a frequency space transfer function of (using $s = i\omega$)

$$|H(\omega)| = \frac{1}{\sqrt{M_1^2 + \left(\frac{\omega}{G_{SQ} k_I}\right)^2 - \frac{2M_1 \omega}{G_{SQ} k_I} \sin(\omega T)}} = \frac{1}{M_1 \sqrt{1 + \frac{\Omega^2}{\kappa_I^2} - 2\frac{\Omega}{\kappa_I} \sin(2\pi \Omega)}}$$

(3.23)

where we have defined the normalized I coefficient $\kappa_I \equiv G_{SQ} M_1 k_I / \omega_0$ and the normalized frequency $\Omega \equiv \omega / \omega_0$, where $\omega_0$ is the sampling (row revisit) frequency. We emphasize that the system is digital with an effective sampling rate of $f_0 = \omega_0 / 2\pi = 1/T \sim 12 \text{ kHz}$ so the transfer function is only meaningful to the Nyquist frequency $f_0/2 = 1/2T \sim 6 \text{ kHz}$.

The transfer function is plotted in Fig. 3.7.
Figure 3.7: The MCE feedback loop transfer function, for various values of $\kappa_I$. For values of $\kappa_I$ below $\kappa_I^* \approx 0.0781$, the response is roughly that of a simple low-pass filter. For $\kappa_I > \kappa_I^*$, a super-unity peak develops in the transfer function. The critical $\kappa_I^*$ is defined as $\kappa_I^* \equiv \max \{ \kappa_I : \max_{\omega} M_1 |H(\omega; \kappa_I)| = 1 \}$, i.e. the largest $\kappa_I$ where the maximum of the transfer function is $1/M_1$. 

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The maximum of the transfer function is plotted in Fig. 3.8. We observe that the maximum amplification in the transfer function can be significant (up to 3 orders of magnitude), so we must be careful not to allow the integral coefficient to be too large. We also note that the control loop always reduces the effective bandwidth of the system below the Nyquist limit. Even for the critical integral coefficient $\kappa_I^* \simeq 0.0781$, the effective bandwidth is reduced by a factor of a few.

A theoretical alternative is to use an extremely large integral coefficient, since the amplification factor is reduced for large $\kappa_I$. This is typically impractical because of the nu-
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numerical instability of the real implemented digital feedback algorithm when working with very large coefficients.

Noise from the detector couples in through the same pathway as $i_{\text{TES}}$ and so passes through the full transfer function $H(s)$.

3.2.8 S1FB Readout

Following the nulling feedback loop discussed in the previous section, the S1FB encodes the information about the signal in the TES arm. The S1FB signal is MCE generated and therefore already digitized, but has 6 kHz of bandwidth, far more than is required to encode the signal from the sky and far more than is practical to record. The S1FB signal is passed through a 6-pole Butterworth IIR low-pass digital filter at $\sim 30$ Hz (adjustable) and decimated to $\sim 100$ Hz (adjustable) for storage.

As discussed in Sec. 2.5, the science signal is encoded in frequencies between 2.5 Hz and 3.5 Hz, so the 30 Hz easily covers the signal bandwidth. This is the final data product of the detector readout chain. Each of the 32 columns of all 4 arrays is producing a 100 Hz datasteam, for a total data rate of 12.8 kHz. Each sample is 4 bytes, so we have a lower bound of 51.2 kBps of data.
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3.3 Laboratory Measurements

The laboratory setup had a few differences from the flight setup. The single pixel had a PIPER-like architecture but with non-PIPER-like normal resistance \( R_n \), thermal conductance \( G \), and transition temperature \( T_c \).

The SQ1 mux in the flight configuration is hybridized to the back of the detector array, but the single pixel is an independent device that requires an external mux. For the laboratory tests we used a NIST Mux07a chip with a discrete shunt resistor chip with shunt resistance \( R_{sh} = 0.85 \text{ m}\Omega \). A series inductor (additional to the SQ1 coupling inductor) is intended to be inserted into the circuit to raise the inductance to make the L/R filter useful for anti-aliasing. However, an inductor chip was not available at the time of the measurements, so was not used.

The Mux07a is intended for use with the older generation 3-stage SQUID amplifier. The 3-stage SQUID system adds as its 2nd stage a unity-gain transimpedance amplifier to reduce the output impedance. The 1st stage remains as the mux but directly biases the SQ1 SQUIDs to performing the muxing rather than biasing through flux-gated switches. The 3rd stage is the series array. Both stages 1 and 2 are housed on the mux chip.

The shunt and mux chip were mounted on an alumina board (the Shunt-Mux board), shown in Fig. 3.9. This was in turn mounted in the H-package along with the single pixel chip. The H-package is a magnetically shielded dark test package. It uses an OFHC copper plate with a small neck (the “pizza peel”) enclosed in an aluminum shell with only a small opening to allow the copper plate’s neck through. The aluminum shell is also wrapped in
Figure 3.9: The 0.85 mΩ shunt chip (bottom) and Mux07a chip (top) on the alumina board. Non-functioning SQ1s are skipped. The functional SQ1s are fanned out to use shunt resistors with a uniform spacing. Aluminum wirebonds are used to connect the boards to each other and the gold wirebond pads of the alumina board. Gold wirebonds thermally sink the mux to a gold layer on the alumina board, which is in turn thermally sunk by more gold wirebonds to the pizza peel. Note that the shunt chip was damaged, so extra wirebonds were used to route around the damaged trace. Its functionality was not impaired after the work-around. The single pixel chip and calibration resistor chip are visible at the bottom of the image.
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lead tape for additional shielding. A further high magnetic permeability Amuneal A4K\(^3\) box surrounds the aluminum shell to ensure that the aluminum’s critical field is not exceeded. Aluminum is a type 1 superconductor with a transition temperature of 1.2 K, so as long as the applied magnetic field does not exceed the critical field, the Meissner effect will reject the magnetic flux\(^5\) below the transition temperature. A schematic of the SQUID readout circuit is shown in Fig. [3.10]. The single pixel chip, mux, and shunt are shown in the H-package in Fig. [3.11].

The H-package was placed in the SHINY test cryostat. The installed H-package is shown in Fig. [3.12] in 3 configurations: bare, with lead tape, and with the high-permeability shield. Series arrays were mounted on the SHINY LHe cold plate (held at the LHe bath temperature). NbTi wires were used to connect the H-package to the SSA and to the connectors that lead external to the cryostat. These connectors are thermally sunk to the SHINY LHe cold plate and are manganin from the cold plate to the 300 K.

SHINY uses liquid cryogens to get from 300 K to 2.2 K, in a sequence of LN2, LHe, and finally pumped LHe. An adiabatic demagnetization refrigerator (ADR)\(^46\)\(^47\) cools the test stage from the LHe bath temperature to 100 mK. We note that the ADR uses strong magnetic fields to actuate its salt pills. At the time of the measurements, the ADR leaked \(\sim 40\) gauss into the location of the samples. The magnetic shielding provided by the H-package attenuated the variation in leakage flux to below the sensitivity of our Hall probe (a similar chip to Sec. 4.4.3), i.e. to less than 50 mgauss. Furthermore, the flux leakage was

\(^3\)http://www.amuneal.com
Figure 3.10: A schematic diagram of the 3-stage SQUID amplifier circuit used in the SHINY cryogenic test dewar for the single pixel measurements. Although the Mux 07a can mux up to 32 channels, only 2 are shown for clarity. The board or chip on which each component is shown. Although the intended position of the Nyquist chip is shown, it was not present for the measurements.
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Figure 3.11: The single pixel chip, mux chip, and shunt chip in the H-package. The lid of the H-package has been removed. A 1 kΩ RuOx thermometer and a set of calibration resistors are also present.

Figure 3.12: The H-package installed in the SHINY test cryostat. From left to right: bare, with the lead tape, and with the Amuneal A4K shield.
monitored as the package cooled and a sharp transition from a leakage that depends on ADR current to a constant flux level at the aluminum superconducting transition temperature 1.2 K. This implies that the Meissner effect is responsible for the attenuation, which would give flux stability to orders of magnitude better than the 50 mgauss noise level of our Hall sensor, which is sufficient stability.

An MCE was not available at the time, so the PSquid board (Sec. 4.4.1) was used in its place. The PSquid board is MCE-like in its design and functionality but can operate only a single channel at a time at a sampling rate of 10 kHz on the S1FB feedback loop. Additionally, instead of an IIR filter prior to reporting the S1FB data, it uses an adjustable FIR boxcar filter. The final difference is that the PSquid amplifier has a different gain and does not have the low-pass roll-off of the UBC amplifier, but as we noted above the UBC amplifier’s low-pass properties only had an effect on the mux noise, which we do not expect to be a significant contributor to the noise in this case.

3.4 Results

3.4.1 Parasitic Resistance

The gold bond pads on the Shunt-Mux board contributed a significant amount of resistance to the TES arm. With two calibration resistors we may correct for this parasitic resistance without knowing the mux properties (which were measured later). With S1

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4The Hall chip was not absolutely calibrated, so the absolute flux level could not be measured.
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locked, the slope of the device $I-V$ curve is inversely proportional to the device resistance, $m \propto 1/R$. The ratio of the slopes then gives us the ratio of resistances, independent of the units with which we made the $I-V$ curve.

The amount of gold bond pad in the circuit is approximately the same for all of the channels, so we assume that the parasitic resistance is the same for both calibration channels. Let $m_{40}$ be the slope of the 40 mΩ calibration resistor and $m_{20}$ be the slope of the 20 mΩ calibration resistor. Then the ratio of the slopes is related to the calibration resistances $R_{40}$ and $R_{20}$ by

$$M \equiv \frac{m_{20}}{m_{40}} = \frac{R_{40} + R_p}{R_{20} + R_p}$$

where $R_p$ is the parasitic resistance. The parasitic resistance is then

$$R_p = R_{40} \frac{1 - M \left( \frac{R_{20}}{R_{40}} \right)}{M - 1}$$

so a measurement of the ratio of the slopes and the knowledge of a single calibration resistance and the ratio of the calibration resistances allows us to determine the parasitic resistance $R_p$. We note that the ratio of the slopes is independent of the units in which they are measured, so we do not need to know the mux characteristics to determine it. The measurements of these $I-V$ curves in raw units and a fit of the slopes are given in Fig. [3.13], from which we see that $M = 1.61$.

We then estimate the parasitic resistance using Eq. (3.25) to be

$$R_p = 13 \text{ mΩ}$$

and we note that this is a significant fraction of the calibration resistances and the expected
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Figure 3.13: The calibration resistors $I$-$V$ curves for the 20 mΩ and 40 mΩ calibration resistors in raw units. The fitted slopes are shown.

TES resistance, and so cannot be ignored.

3.4.2 Mux 07a Characterization

We characterized the mux by locking S2 of the SQUID amplifier and driving current through the S1 feedback (S1FB) and S1 input (S1IN) coils. The S1FB coil may be driven directly but the S1IN coil is controlled only through the DETB line. We used one of the calibration resistors to put a known current through the S1IN coil.

The relevant mux parameters are the S1FB and S1IN coupling parameters $M_{FB1}$ and $M_{IN1}$, from which we may derive the S1 feedback-to-input coupling parameter $M_1 \equiv M_{FB1}/M_{IN1}$. With S2 locked, S1 $V$-$\Phi$ curves were made with the 20 mΩ and 40 mΩ calibration resistors. Additionally, the S1 $V$-$\Phi$ curve using the S1FB coil was measured. All 3 of these curves are shown in Fig. 3.14.

We measured the S1 input coil coupling coefficient to be $M_{IN1}^{20\text{m}\Omega} = 5.9 \, \mu\text{A}/\Phi_0$ for the 20 mΩ calibration resistor and $M_{IN1}^{40\text{m}\Omega} = 5.7 \, \mu\text{A}/\Phi_0$ for the 40 mΩ calibration resistor.
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Figure 3.14: S1FB and S1IN \(V-\Phi\) curves. The left and center plots are the S1IN coil coupling to the stage 1 SQUID for the 20 m\(\Omega\) and 40 m\(\Omega\) calibration resistors. The right plot shows the S1FB coil coupling to the stage 1 SQUID. The coupling constants are \(M_{20\ m\Omega}^{\text{IN1}} = 5.9 \mu\text{A}/\Phi_0\), \(M_{40\ m\Omega}^{\text{IN1}} = 5.7 \mu\text{A}/\Phi_0\), and \(M_{\text{FB1}} = 83 \mu\text{A}/\Phi_0\). The S1 \(V-\Phi\) response curve is the same for all 3, but is rescaled and horizontally flipped for the S1FB coil. This is merely a reflection that the coupling coefficient is different and the direction of the field generated by the coil at the SQ1 loop is inverted.

The S1 feedback coil coupling coefficient was measured to be \(M_{\text{FB1}} = 83 \mu\text{A}/\Phi_0\). This gives us a mutual coupling coefficient \(M_1 \sim 14\). The mutual coupling coefficient \(M_1\) is the most interesting parameter of the mux since it allows us to relate the measured S1FB current to the current through the detector, Eq. (3.12).

We note that the locked S2FB axis is shown in raw DAC units. The relevant parameter extracted from the plot is the period of the \(V-\Phi\) curve, so the magnitude of the response in physical units is not useful. One \(\Phi_0\) is one period, so all we require is to know the horizontal axis in physical units. For the S1 feedback coil, this is determined entirely by the biasing resistors (Fig. 3.10), which have known values, and the S1FB DAC. For the S1 input coils, this is determined by the known biasing resistors and calibration resistors, and the DETB DAC.
Figure 3.15: $I$-$V$ curves for the single pixel device at various base temperatures. The dynamic impedance is the inverse of the derivative, so the steeper line on the left is the TES arm resistance while the TES is superconducting (i.e. the parasitic resistance $R_p$), and the shallower line on the right is the normal state resistance ($R_p + R_n$). The inbetween region is an isopower, enforced by the electrothermal feedback.

### 3.4.3 Single Pixel Characteristics

With the mux characterized, $I$-$V$ curves for the devices were collected. Stage 1 was locked and the voltage on the DETB bias line was swept for various base temperatures. The applied DETB value determines the bias voltage across the TES while the nulling S1FB value measures the current response of the TES. Both conversion factors may be computed from Fig. 3.10 and are discussed in Niemack. The $I$-$V$ curves are shown in Fig. 3.15.
At high bias voltages, the TES dissipates enough power to drive it out of the superconducting transition into the normal state. With the series parasitic resistance, the measured normal state resistance is the sum of the TES normal state resistance and the parasitic resistance, $R_n + R_p$. As the bias voltage decreases, the TES falls into the superconducting transition. In the superconducting transition, the electrothermal feedback holds the power roughly constant by adjusting the TES resistance. This produces the $I \propto 1/V$ isopower curve in the transition region. The ETF has no more dynamic range to work with once the TES resistance drops to 0, so at low bias voltages we measure the resistance of the TES arm with the TES in the superconducting state. Note that this is not 0 due to the parasitic resistance, so this slope corresponds to $R_p$.

This data may be represented in the $R$-$P$ coordinate system, as shown in Fig. 3.16. We observe the transition from low resistance to high resistance along an isopower curve, as described above. The first feature we note is the minimum resistance is $13 \text{ m}\Omega$, which is another estimate of the parasitic resistance. This estimate is consistent with that from the calibration resistors made in Sec. 3.4.1. We also see as the base temperature increases, the power dissipation decreases.

We may correct for the parasitic resistance by subtracting the voltage drop across the parasitic resistance $I R_p$ to estimate the voltage drop across the TES $V_{\text{TES}}$,

$$V_{\text{TES}} = V_{\text{TES}}^{\text{measured}} - I_{\text{TES}} R_p. \quad (3.27)$$

We may also correct the power dissipation and measured resistance. The parasitic-corrected plots are shown in Fig. 3.17. The transitions between superconducting and normal are
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Figure 3.16: $R$-$P$ curves for the single pixel device at various base temperatures. The minimum resistance corresponds to the parasitic resistance $R_p$.

The amount of power required to raise the TES temperature from the bath temperature $T_0$ to the temperature $T$ is

$$ P = G(T - T_0) $$

(3.28)

where $G$ is the thermal conductance (units W/K). Generally $G$ may be a power law in temperature, but we assume it is constant here. The TES has a constant transition temperature $T_c$ independent of the bath temperature, which we define as the temperature such that $R = R_n/2$. Then from Fig. 3.17 we compute the amount of power required to raise the temperature of the TES from the bath temperature to the transition temperature and plot it against the bath temperature. The relationship between power dissipation $P$ and bath
Figure 3.17: $R$-$P$ curves for the single pixel device at various base temperatures, corrected for the parasitic resistance $R_p$. 

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Figure 3.18: Powered required to raise the TES temperature from the bath temperature $T_0$ to the transition temperature $T_c$ (defined as the temperature at which $R = R_n/2$) versus the bath temperature $T_0$. The thermal conductance $G$ is estimated by the slope of the linear fit and the transition temperature $T_c$ is estimated by the $x$-intercept of the fit.

The powered required to raise the TES temperature $T_0$ is linear,

$$P = GT_c - GT_0$$  \hspace{1cm} (3.29)

and is plotted in Fig. 3.18. A linear fit estimates the thermal conductance $G$ and the transition temperature $T_c$ as

$$G = 175 \pm 3 \text{ pW/K}$$  \hspace{1cm} (3.30a)

$$T_c = 310 \pm 5 \text{ mK}$$  \hspace{1cm} (3.30b)
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This analysis may be repeated with a power law thermal conductance \( G \equiv \kappa T^\beta \). Using the linear model as the null hypothesis, the difference between the \( \chi^2 \) parameters of the two models is \( \Delta \chi^2 = 1.00 \). This corresponds to a 0.32 chance that we would favor the power law model by chance if the linear model were the true model. Thus, we discard the power law model as not significantly preferred over the linear model.

The key feature of the TES bolometers is that they must be biased onto their transition. The amount of bias power required to put the TES in its transition is \( G(T_c - T_0) \), which we see depends on both \( T_c \) and \( G \). However, since we have 2 adjustable parameters during operation (the base temperature \( T_0 \) may be adjusted with the ADR and the detector bias voltage’s DC level may be adjusted by the MCE), the \( G \) and \( T_c \) parameters do not need to be tuned precisely. A much more important characteristic of the detector array is the \( G \) and \( T_c \) variation across the array. The detector leg geometry is very consistent across the array, so we expect the \( G \) to be consistent across the array. However, the \( T_c \) is determined by the amount of gold deposited in the TES and can easily vary across the array due to spatial inconsistencies in the deposition process. The PIPER arrays target a \( T_c \) variation of less than 5 mK. However, a single pixel test cannot estimate these properties.

3.4.4 Single Pixel Noise

Noise measurements were performed by locking S1, holding the bath temperature fixed, biasing the TES at a constant point on its transition, and recording the raw data at 10 kHz.

\(^5\)This gives us the fit parameters \( \beta = 1.3, \kappa = 900 \text{ pW/K}^{1.3}, T_c = 310 \text{ mK}, \text{ and } G(T_c) = 195 \text{ pW/K.} \)
Figure 3.19: Single pixel power spectrum. The roll-off is due to the S1FB feedback loop. The white noise level exceeds the predicted phonon noise level by a factor of 3 due to aliasing. The $1/f$ noise is most likely due to instability of the bath temperature.

with PSquid’s Raw Data mode. The variance in the raw S1FB data may be converted to NEP in the detector. The resulting power spectrum is plotted in Fig. 3.19.

With no optical loading and at such low temperatures, the phonon noise is the only significant noise source. It contributes a broadband white noise level of

$$\text{NEP}_{\text{phonon}} = \sqrt{4k_B T^2 G} = 30 \text{ aW/}\sqrt{\text{Hz}}$$  \hfill (3.31)$$

where the NEP is evaluated at the single pixel transition temperature and thermal conductance. Note, however, that no Nyquist chip was present in the system so the power was not rolled off below the Nyquist frequency of $f_{Ny} = 5 \text{ kHz}$. Rather, the limiting antialiasing
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filter was an RC filter created by the line impedance from the series array to the PSquid board with a bypass capacitor in PSquid (C64 = 100 nF). Such a filter would admit a total power from the phonon NEP of

\[
P_{\text{total}} = \text{NEP} \sqrt{\int_0^\infty \frac{df}{1 + (2\pi RCf)^2}} = \text{NEP} \frac{1}{\sqrt{4RC}}
\]  

(3.32)

The post-aliasing effective NEP would then be

\[
\text{NEP}_{\text{effective}} = \frac{P_{\text{total}}}{\sqrt{f_{\text{Ny}}}} = \text{NEP} \frac{1}{\sqrt{4RCf_{\text{Ny}}}}
\]  

(3.33)

The wire was manganin and had an impedance of \( R \sim 50 \, \Omega \), which would give an effective NEP of

\[
\text{NEP}_{\text{effective}} \sim 3 \, \text{NEP}
\]  

(3.34)

consistent with the white noise level observed in Fig. 3.19.

The high-frequency roll-off of the power spectrum is due to the feedback loop. The integral coefficient was iteratively tuned such that the spectrum had the maximum amount of bandwidth while still maintaining a flat level, i.e. to the \( \kappa_I^* \) criterion. We see from Fig. 3.7 that the feedback loop transfer function begins to roll off at about \( f_{\text{Ny}}/5 \), consistent with single pixel spectrum.

Finally, we note that the ADR control thermometer was located on the 100 mK cold plate but the primary power dissipation was by the single pixel device located inside of the H-package. These two regions are connected through a thin copper neck and a number of metal-metal joints. Furthermore, the pizza peel has a small heat capacity, being a thin sheet of copper, while the 100 mK cold plate has a significantly larger heat capacity, being
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a large plate of copper. Thus, for variations in power dissipated by the TES and Shunt-Mux chip, we expect a quick response in the bath temperature to which the TES is coupled (the pizza peel) and a comparably slow response in the 100 mK plate temperature. Since the temperature control thermometer was coupled to the 100 mK plate, it would also experience a slow response, and consequently the ADR control loop would respond slowly. This could result in a slow drift in the effective thermal bath temperature of the TES, which would manifest in the power spectrum as $1/f$ noise. We did not have the resources available to quantify this effect.
Chapter 4

Electronics

PIPER uses a number of custom-made printed circuit boards (PCBs), called the HKH\(^1\) electronics, that are ideally suited for high-precision balloon-borne telescopes. Their primary advantages include low cost, a compact form factor, low power consumption, naturally DC powered, low noise, and can accept an external clock. Because of the external clock, all of the PCBs can be made synchronous with the SyncBox (the MCE’s clock). This set of electronics is used to measure or control nearly everything on the payload that is not the detectors. This chapter will describe the PCBs and the decisions that went into their design, as well as quantify their performance.

\(^1\)This originally stood for “House Keeping Electronics”, which was descriptive for its original scope. Given the expanded scope of the electronics, perhaps it should instead stand for “HinderKs Electronics”, after its original creator.
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4.1 Design Overview

The boards may be split into 3 categories:

**Backplane board** These boards are designed to fit into a 3U backplane. In the backplane, all boards share a common clock, a common set of power lines, and a communications bus (unless a dedicated one is required). Most of the boards are backplane boards, including PMaster, TRead, DSPID, PsyncADC (version 2), PMotor, AnalogIn, and AnalogOut.

**Stand-alone board** These boards require their own power and communications bus, and optionally may accept an external clock. They are typically for lab use. PSquid, PSync, and PsyncADC (version 1) are stand-alone boards.

**Auxiliary boards** These boards are intended to be used in conjunction with the other boards to provide extra functionality. These include the various current boost boards and the Fiber-USB board.

Every non-auxiliary board is structured in a similar way. At the heart of the board is one of the family of Microchip dsPIC30F microcontroller chips. The boards are typically split between an analog side and a digital side, each with separate ground planes. Typically a $0 \, \Omega$ surface mount resistor straddling the two ground planes joins the two ground planes, though for grounding reasons sometimes the connection is made off-board instead. The digital side


[^3]: “PIC” historically stood for Peripheral Interface Controller or Programmable Interrupt Controller, but “dsPIC” is just a trade name intended to imply a PIC chip with digital signal processing capabilities.
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requires 0 V and +5 V, and the analog side requires 0 V and ±15 V. These power lines are provided through the backplane for backplane boards or through a connector to some external source for the remaining boards. All non-auxiliary boards have the digital power lines, since the dsPIC chip requires them, but only boards that require the analog powers (e.g. to power an ADC or DAC) have them.

The UART module on the dsPIC controls a pair of half-duplex RS-485 protocol serial buses. The UART module on backplane boards drive a pair of lines in the backplane, which then drive a pair of RS-485 transceivers on the PMaster card, which in turn control a fiber optic transmitter and receiver on the PMaster card. Non-backplane boards have the same structure, but the transceivers, transmitter, and receiver are on the board itself. Auxiliary boards have no need to communicate with a computer and do not implement a serial bus.

The SPI module controls the peripherals that implement the particular functionality of the board. Usually this is some combination of ADCs and DACs, surrounded by filters and amplifiers in different configurations.

The dsPIC microcontroller chip must be supplied with a clock. Backplane boards import the clock from the backplane and do not have a clock directly on the board. The clock transmitted over the backplane is supplied by the PMaster card, which can either use an external or internal clock. Stand-alone boards have both an internal and external clock. The internal clock (on both PMaster and stand-alone boards) is a 5 MHz crystal oscillator. The external clock is piped in through a pair of fiber receivers that receive a raw clock and a synchronization data signal. The external clock for PIPER comes from the UBC SyncBox.
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The dsPIC chip uses a phase-locked loop (PLL) to boost the effective clock speed by a factor of 16 to 80 MHz, then requires 4 clock cycles per instruction for an effective typical instruction rate of 20 Mega-instructions per second (MIPS). The precise instruction rate depends on the precise clock rate, but the boards are designed expecting \( \sim 20 \text{ MIPS} \).

The dsPIC microcontroller is programmed using the gcc-based XC16 compiler, which allows us to code in a specialized version of the C programming language. The dsPIC30F5011 (dsPIC30F6015) has only 4096 (8192) bytes of RAM and 22528 (49152) bytes of flash memory, so the programs are necessarily compact. Each one runs a single main loop that handles interaction with the computer over the serial bus. The functionality of the board is provided primarily through the interrupts, which are allowed to interrupt the current line of execution and run an interrupt handling function. The interrupts are usually triggered after a particular number of clock cycles, thereby providing the precise timing control of the boards. Since interrupts require a fixed number of instructions to process and the functions they trigger require a consistent number of instructions to execute, the boards are able to produce consistent behavior to the stability of 1 clock cycle (determined by the stability of the clock and the PLL, this is typically better than \( \sim 10 \text{ ns} \)). The phase between the interrupt and the execution of the functionality (e.g. sampling an ADC) will not be 0, but it is consistent to a similar level.

\footnote{This is not strictly true for all possible programs that can be put on the board. The program must not allow a second interrupt handler to interrupt the first (i.e. no nested interrupts), since that could introduce an arbitrary phase delay. Our programs that rely on consistent timing do not allow for nested interrupts. Additionally, each interrupt function must have a constant number of instructions before the synchronization-sensitive action. The remaining instructions after the synchronization-sensitive action and the next interrupt may be used without such restrictions.}
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4.2 Overview of Boards

We list here the available boards and a brief summary of their functionality and purpose. Unless otherwise specified, all boards with digital components are fully synchronous with the external clock, as discussed above.

**PMaster** *(Backplane)* Provides clock and communications functionality for a backplane.

Provides an internal clock for the backplane if no external clock is available. Has functionality to receive and interpret the SyncBox signal to provide a consistent and SyncBox-frame-tagged external clock. Distributes the clock to the rest of the cards in the backplane. Also handles communications over the serial bus. Relays signals to/from a pair of lines on the backplane to its fiber transmitter/receiver.

**TRead** *(Backplane)* A low-noise 4-wire resistance bridge card, typically used for reading thermometers. Allows for the muxed reading of 12 (+4 calibration) channels at rate of 16 Hz using a 16 Hz square wave current-biased excitation. Comes in 3 varieties: 1) **TRead Standard** for measuring resistors that are \(~10 \text{k}\Omega\); 2) **TRead LR** for measuring resistors that are \(~100 \text{\Omega}\); 3) **TRead Diode** for measuring diodes by providing a 10 uA DC bias current rather than a square wave.

**DSPID** *(Backplane)* A slow \((\sim 8 \text{\ Hz})\) PID board with a 2-channel (+2 calibration) 4-wire resistance bridge tuned for \(~10 \text{k}\Omega\) that it uses as feedback. Typically used for controlling the adiabatic demagnetization refrigerators (ADRs). Has some specialized control modes that simplify controlling high-inductance cryogenic loads (such as
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ADR coils).

**PSyncADC** *(V2 - Backplane; V1 - Stand-alone)* A fast low-noise ADC card, typically used for measuring science sensors such as VPM position sensors and pointing sensors. This card is not capable of biasing sensors. Can read out up to 32 channels at up to \(\sim 400\) Hz. Although it is housed in a backplane, it has a dedicated serial bus since the backplane serial bus cannot support the data rates required for reporting \(32 \times 400\) samples per second. Version 1 of the board was a stand-alone board that included functionality for parsing the external frame count, and functionality for a 4-channel 12-bit DAC.

**PMotor** *(Backplane)* A linear 3-phase AC motor controller card. Includes a quadrature encoder reader, which is used to commutate the motor controller at 4 kHz. Also implements capability for 4 kHz PID control based on encoder position or encoder velocity. Alternate firmware allows PMotor to drive a 2-input PID loop at 2 kHz using 2 analog in channels.

**AnalogOut** *(Backplane)* A card capable of producing 32 channels of 12-bit analog out values ranging between \(\pm 10.24\) V. It can source 5 mA and has an output impedance of \((100+???)\) \(\Omega\). Updates at \(\sim 1\) Hz.

**AnalogIn** *(Backplane)* A slow ADC card, typically used for measuring non-essential sensors. Can also be configured to bias and read AD590 room temperature temperature transducers. Capable of reading 32 channels of analog input between \(\pm 10\) V, or 32
CHAPTER 4. ELECTRONICS

channels of AD590. Updates at ∼1 Hz.

**PSquid (Stand-alone)** A single-channel SQUID readout board capable of controlling a SQUID readout with up to 3 stages. Uses MCE-like amplifiers. Can PID control any of its DACs at 10 kHz with 32-bit dynamic range on the PID parameters. Also allows reporting of data at up to 10 kHz. Uses a dedicated 417 kbps serial bus over fiber. Used for laboratory testing and as a well-understood reference standard for the MCE.

**Gyro Board (Stand-alone)** A board that biases a pair of 2-axis analog gyroscope chips. The gyros are oriented so as to provide an $x$, a $y$, and 2 $z$ channels, where $z$ is normal to the surface of the board. Each channel is low-passed filtered to have only 9 Hz of bandwidth. The resulting signals are intended to be read out by a PSyncADC board.

**PSync (Stand-alone)** A board that synchronizes the Star Camera by measuring, relative to the external clock, when the shutter on the camera is open. Reports exposure timings over a dedicated 115.2 kbaud serial bus over copper.

**Hall3D (Stand-alone)** A cryogenically compatible 3-D magnetometer board. Uses a trio of 1D magnetometer chips oriented along 3 axes to get 3-D sensitivity. May be read out as a standard 4-terminal device, typically using TRead or a commercial resistance bridge, where it appears as a ∼1 kΩ load.

**Current Boost (Auxiliary)** A series of boards that serve as current amplifiers. Usually also have some voltage gain as well. They provide as much current as necessary to the
load to reach the driving voltage. Currently there are 3 varieties: 1) A power op-amp based design with a current limit of 8 A and a unity voltage gain. 2) A power MOSFET-based design with a current limit of more than 50 A and a unity voltage gain. 3) A 3-channel unipolar power op-amp design for driving all 3 phases of an AC motor with one board. Has a voltage gain of 1.5.

**Fiber-USB (Auxiliary)** A board that allows a computer to interface with our RS-485 over fiber bus using a USB port with a virtual comm port (VCP) interface at up to 2.5 Mbaud. Power is provided to the board by the USB port. Not synchronous with the external clock, but is electrically isolated from its RS-485 partner and carries only digital information.

### 4.3 Backplane Boards

**Piper** utilizes backplane boards\(^5\) as much as possible to take advantage of the benefits afforded by the backplane. The backplane is intended to house a centralized point to measure all non-detector sensors and control as many non-detector systems as possible. All control systems (excluding detectors) requiring high bandwidth (i.e. greater than \(\sim 1 \text{ Hz}\)) or precise timing are controlled by backplane boards, with the control loop run by the microcontroller on each board.

\(^5\)We call PCBs that are connected to a backplane either backplane *boards* or backplane *cards*. The card terminology arose because the backplane boards are typically placed in a 3U standard card rack, where each board takes a card slot.
Figure 4.1: Block diagram showing the architecture of the backplane and the backplane cards. The external clock and frame count is supplied to the PMaster card, which synthesizes a ~1 Hz frame clock. The frame clock and external clock are distributed to all other boards on the backplane. Additionally, serial communications are transported to and from the PMaster card, so there is a single point of contact for the external computer. Lastly, power is provided to the backplane, and the backplane distributes it to all of the cards. Everything except the signals are transported by the B row of the 96-pin connector. The signals are transported on the A and C rows.
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The architecture of the backplane is depicted in Figure 4.1. The backplane transports as much information as possible that is common to all boards in the backplane. This includes the shared clock, a synchronization clock, the serial bus, and power. This information is carried on the B row of the 96-pin connector, which is connected to all boards (i.e. B1 on one board is connected to B1 on every other board). The signals (e.g. voltages to be sampled by the ADC on a board, or voltages to be generated by a DAC and sent to some other device) for any given board are carried through the A and C rows of the 96-pin connector, and passes straight through the backplane to an equivalent 96-pin connector on the back of the backplane. The A and C rows are independent for each card slot (i.e. A1 on one board is not connected to A1 on any other board).

The heart of the backplane is the PMaster board. This is especially true since it is the only board in the backplane that is capable of providing a clock, so if no PMaster is present in the backplane, none of the other boards will function\(^6\). The PMaster also provides a $\sim 1$ Hz synchronization clock that ensures that all of the cards in the backplane have the same phase.

The exact timing parameters are determined by the external clock supplied to PMaster and by the MCE clock. For simplicity, we describe the system using the PIPER configuration, which utilizes a 5 MHz external clock and a 25 MHz MCE clock. Details of how the HKE electronics synchronizes with the MCE and how the various clocks affect this are discussed in Appendix ???. The boards trigger an interrupt every 5000 clock cycles ($= 250 \mu s$)

\(^6\)The exception is the PMotor board, which has an optionally-enabled internal clock for stand-alone use.
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using the OC2 module and Timer2, corresponding to an interrupt rate of 4 kHz. The 250 µs
interrupt period is the smallest effective time division for the boards, and is called a *tick*. A set of 4000 ticks is grouped to make a *frame*, which is the highest level period for the boards and has a period of 1 second. The ticks counter is reset to 0 on every new frame. Data is reported once per frame per board, and the collection of all boards’ data for the frame is called a *data frame*.

Every board keeps track of its own *frame number*, which is the number of frames modulo 256 that have elapsed since the board was powered on. The PMaster board keeps track of and reports both the PMaster frame number and the SyncBox frame count. When a board is powered on, it first runs its initialization routines (e.g. to configure its dsPIC chip pins, initialize its DAC chip, etc.), then it sits and waits for a rising edge on the backplane frame clock. Once a rising edge is found for the first time, the board turns on its interrupt handler and sets its ticks count and frame count to 0, then begins regular operation. Since all boards share a common clock and frame definition, each board’s individual frame count must count up in lock step, and they cannot desynchronize. In the event of a power interruption, once repowered, the board will resynchronize against the backplane frame clock and begin counting from 0 again.

The serial bus is shared for all cards in the backplane, for both transmitting and receiving data, since the 96-pin connector’s B row is shared. Messages transmitted from a computer to the backplane are sent to the PMaster fiber optic receiver, after which the

---

7Except PMaster, which instead waits for a rising edge on the SyncBox frame count, then starts the frame clock.
message passes through the PMaster card to the differential pair B13/14, which is shared by all cards in the backplane. This means that all cards receive every message sent by the computer at the same time. Every card has an adjustable address between 0 and 20 (except PMaster, which is hard-coded at 255), and every message sent to the backplane is prepended by the address of the intended recipient. A board whose address does not match the address specified in the message will ignore the message. Messages transmitted by a board are sent to differential pair B10/11, which is also shared by all boards in the backplane. To prevent more than one board from talking at once, each frame is divided into 21 talk slots, and each board may talk only in the slot assigned, according to its address. Thus, no two boards may share the same address. This limits the number of cards that may be put into a single backplane to 20 (+1 PMaster). Transmissions on B10/11 are passed through the backplane to the PMaster board, which carries it to its fiber optic transmitter, at which point it is sent out over the fiber connection on PMaster.

Each board reports its data once per frame (once per second) during its talk slot in what is called the board’s *data packet*. The full talk slot is not required to transmit the data packet, so the remaining time is used to transmit responses to commands from the user (e.g. a request for a detailed status update, or a request for the software version number). Since each board reports its data once per frame, the data in each data frame corresponds to the same frame for all cards. Thus, the data for every board is frame-tagged with the

---

8. The special address `all` may be used to address all boards at the same time.
9. Note that there are 21 talk slots because PMaster gets a special talk slot at the beginning of the frame. It is shorter than the remaining 20, all of which are the same length. The exact length of the talk slots depend on the HKE-MCE synchronization details. Since PMaster gets a special talk slot, there can be only one PMaster per backplane.
SyncBox frame count by reading the frame count out of the latest PMaster data packet and assigning it to the data packet. An alternative method (and a good cross-check) is to record the SyncBox frame count corresponding to the board’s frame count of 0, and count up from there. Since the frame counts increment in lock step, these two methods must give the same result.

4.3.1 PMaster

The PMaster card controls the synchronization and timing of everything in the backplane. Its purpose is to provide the clock for the backplane, house the fiber transmitter and receiver hardware for the backplane’s serial communications, and interpret the SyncBox’s frame count and frame tag the data packets if the SyncBox if available.

If the SyncBox is present, then the clock line from the SyncBox is piped to the dsPIC chip and to the differential pair B2/3 on the 96-pin connector. If no SyncBox is detected, then a 5 MHz crystal oscillator on the board is powered on and used as the clock in its place. When the board is powered on, PMaster searches for the SyncBox by monitoring the data and clk signals from the SyncBox. The SyncBox transmits the frame count on the data line at 625 Hz, in the form of 40-bit packets (Figure 4.2). The data line is active-low (normally high), so the first bit in the packet is low. Since data bits change on falling edges of the clock, a rising edge on clk will be in the middle of the data bit. Thus, the first rising edge of clk while data is low indicates the presence of a functional

---

10 This method is complicated by the fact that the number of SyncBox frame counts per HKE frame is not 1, and that the individual board’s frame count rolls over every 256 frames.
Figure 4.2: A timing diagram showing the SyncBox clock and data signals, including one 40-bit frame count packet. Note that the data line of the SyncBox is normally high, but when the 40-bit frame count packet is transmitted, the first 8 bits are the Address Zero Sync Bit, followed by the Data Valid Bit, followed by 6 status bits. The AZS bit is always low.

Figure 4.3: A timing diagram of the initialization scheme for PMaster, during which time PMaster is waiting for a signal from the SyncBox. The GATE searches for a rising edge on clk while data is low, which first happens at the Address Zero Sync Bit. When this happens, GATE will latch high and pass clk to SCLK. Note that SData always latches the value of data on the latest rising edge of clk.

SyncBox. Using logic gates, PMaster searches for the above condition and latches GATE high once found (Figure 4.3). A rising edge on GATE indicates to the dsPIC chip that a SyncBox is present, at which point an interrupt handler for decoding the 40-bit frame count packets is enabled. The interrupt handler is triggered on subsequent rising edges of GATE. Once PMaster has finished reading the frame count, it sets the PIC_READY line high, which resets GATE back to low. If the rising edge on GATE is not found within 100
ms (during which time 62 frame count packets should have been sent), then the internal oscillator is used for the clock.

Note that there are many more SyncBox frame counts than there are HKE frames, since the SyncBox frame rate is 625 Hz and the HKE frame rate is 1 Hz. If the PMaster started its FRAME_CLK (the backplane frame clock) on the first SyncBox frame count packet found, there could be an arbitrary phase offset between the SyncBox frames and HKE frames. To eliminate this, PMaster will wait until the SyncBox frame count is an exact integer multiple of the number of SyncBox frames per HKE frame before starting the FRAME_CLK. If the internal oscillator is used, PMaster starts the FRAME_CLK immediately, since there is nothing to phase to.

The FRAME_CLK is sent out to the single-ended B18 on the 96-pin connector and serves as the backplane frame clock. PMaster also provides a SYNC_CLK synchronization line to the backplane on the single-ended B17, but it is currently unused. Lastly, PMaster may trigger a reset of all backplane boards (including itself) by setting the RESET line on B16.

PMaster reports a 26-character packet at 1 Hz to its serial port. It always has the first talk slot in a frame packet, so a packet from PMaster is used to indicate the beginning of a new frame. A description of the PMaster frame packet may be found in Table 4.1 and a typical PMaster frame looks like

*FFM0104FFFFFFFF00000001
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<table>
<thead>
<tr>
<th>Position</th>
<th>Length (chars)</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>‘*’, packet start special character</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>‘FF’, Board address</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>‘M’, Card type</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>Frame counter</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>Unused</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>Status Mask where the bits are</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 (LSB) COM Mode (0 = RS232 or 1 = Fiber)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1 UBC Frame Count present (1 = present)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 CLK Source (0 = External or 1 = Internal)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3 (MSB) Unused</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>UBC Frame Count</td>
</tr>
<tr>
<td>16</td>
<td>8</td>
<td>PIC Frame Count</td>
</tr>
<tr>
<td>24</td>
<td>2</td>
<td>‘\r\n’, end of packet characters</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>26</strong></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1: PMaster packet definition.

#### 4.3.2 TRead

TRead\(^{11}\) is a board capable of biasing and measuring 4-terminal devices, typically used for performing 4-wire measurements on thermometers. The board can measure 16 channels by muxing the excitation and response measurement. It generates a square wave current bias excitation at 16 Hz and measures the resulting voltage response. A measurement can be made on each period of the square wave, for a measurement rate of 16 samples/second, which may be distributed across all 16 channels or parked on a single channel. The bias generation and response measurement circuitry is shared for all 16 channels and 4 channels may be used for on-board calibration resistors, allowing for instrumental biases to be fit out.

A detailed description of TRead can be found in Luke’s TRead document.\(^{27}\) We summarize

\(^{11}\)For *Thermometer Reader.*
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the main results here.

The block diagram in Figure 4.4 describes the board. The board generates a bias current to the test resistor $R_T$ of

$$ I = \left( \frac{V_{\text{ref}}}{R_L} \right) \left( \frac{1}{\alpha} \frac{K_{\text{ADAC}}}{2^{16}} \right) \frac{1}{\left( 1 + \frac{R_T}{2R_L} \right)} $$

(4.1)

which is linear in the ADAC DAC output $K_{\text{ADAC}}$. For $R_T \ll 2R_L$, the current bias is strong (independent of the test resistor). The range of typical nominal (i.e. for $R_T = 0$) current biases for TRead\_Standard is $1 \text{nA} < I < 512 \text{nA}$. The range of typical nominal current biases for TRead\_LR is $30 \text{nA} < I < 2048 \mu\text{A}$. The value of $R_L$ is $110 \text{k}\Omega$ for TRead\_Standard and $1920 \Omega$ for TRead\_LR.

The board reports a demodulated measurement $\Delta$ of the ADC input (units of ADC counts), which is related to the test resistor by

$$ R_T = 2R_L \left( \frac{\Delta}{2\beta - \Delta} \right) $$

(4.2)

where $\beta$ is a gain coefficient describing the excitation current and the amplifier gain (units of ADC counts),

$$ \beta = \left( \frac{2^{17 \text{ counts}}}{4.99} \right) \left( \frac{G_2}{\alpha} \right) \left( \frac{K_{\text{ADAC}}}{K_{\text{GDAC}}} \right) $$

(4.3)

The amount of power dissipated in the test resistor is

$$ P = \frac{1}{16} I^2 R_T = \frac{1}{16} \left( \frac{V_{\text{ref}}}{R_L} \right)^2 \left( \frac{1}{\alpha} \frac{K_{\text{ADAC}}}{2^{16}} \right)^2 \frac{R_T}{\left( 1 + \frac{R_T}{2R_L} \right)^2} $$

(4.4)

where the factor of 1/16 comes from the muxing. Each channel is biased with a duty cycle of 1/16.
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Figure 4.4: Block diagram showing the architecture of the TRead board. A square wave voltage generator creates a current bias through a pair of load resistors $R_L$. The current bias generates a voltage difference across the test device $R_T$. The voltage difference passes through an amplifier and then an analog low-pass filter, and is then digitized. Following digitization, the signal is demodulated, which may be modeled as a finite-impulse response (FIR) digital filter.

The measurement bandwidth of TRead is limited by the challenges of a cryogenic system. Due to the small heat capacities of materials at cryogenic temperatures, the allowed power loading is small. This requires the bias current to be small, which results in small response voltages, necessitating large gain stages. In order to limit the noise bandwidth, a lock-in measurement is performed by mixing the signal with an oscillatory carrier wave and limiting the bandpass to around the carrier wave frequency. The carrier wave frequency limits the available measurement bandwidth. The cryogenic system further requires long, low thermal conductivity wires down to the devices. The Wiedemann-Franz Law then suggests that the wires are high resistance. Long high-resistance wires are susceptible to capacitive effects. In order to mitigate the capacitive impedance, the carrier wave frequency must be kept small. Since the measurement bandwidth is set by the carrier wave frequency, we conclude that cryogenic measurements will have limited bandwidth.
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<th>Position</th>
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<th>Meaning</th>
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</thead>
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<tr>
<td>0</td>
<td>1</td>
<td>‘*’, packet start special character</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>Board address</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>‘T’, Card type</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>Frame counter</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>Thermometer mux (Tmux) setting (‘@’ = all)</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>Status Mask (currently unused)</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>Demod[0]</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>Excitation DAC setting, ADAC[0]</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>Gain DAC setting, GDAC[0]</td>
</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>248</td>
<td>8</td>
<td>Demod[15]</td>
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<tr>
<td>256</td>
<td>4</td>
<td>Excitation DAC setting, ADAC[15]</td>
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<tr>
<td>260</td>
<td>4</td>
<td>Gain DAC setting, GDAC[15]</td>
</tr>
<tr>
<td>264</td>
<td>2</td>
<td>N_SUM, number of samples per half-period</td>
</tr>
<tr>
<td>266</td>
<td>2</td>
<td>‘\r\n’, end of packet characters</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>268</td>
<td>Total</td>
</tr>
</tbody>
</table>

Table 4.2: TRead packet definition.

TRead reports a 268-character packet at 1 Hz through the PMaster serial port. It may occupy any of the 20 address slots between 0 and 0x14. A description of the TRead frame packet may be found in Table 4.2 and a typical TRead frame looks like

```
*02T00@0FFFD9F7C0FF52000FFFD73C0FF52000FFDD0A20FF52000
FFFDCA1D70FF52000FFFD9CD10FF52000FFFD7FD50FF52000FFDDA0CE
0FF52000FFFD7D160FF52000FFFD85850FF52000FFFD83C60FF52000
FFFD7C0D0FF52000FFFD8A080FF5200000029F40FF5200000308F8
0FF520000006BC210FF52000000B62980FF5200042
```
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4.3.2.1 Noise

Let us estimate the noise of TRead measurements. The noise will be a combination of DAC noise, Johnson noise from the biasing circuit and test resistor, amplifier noise, and bit noise in the ADC.

We begin with the DAC. The DAC is a multiplying current DAC, and so acts as a transconductance amplifier to scale a reference voltage $V_{in}$ to some current output. This current is then converted to a voltage using a simple transimpedance amplifier. We note that there are both systematic uncertainties in how well we know the voltage output of the DAC as well as noise on the output of the DAC. The systematic uncertainties may be eliminated by monitoring the calibration resistors (or if not eliminated, reduced such that they are comparable to the measurement uncertainty). The intrinsic noise of the DAC is unavoidable.

We’ll first look at the reference voltage source, Fig. 4.5. Note that the first inverting amplifier is used to generate the negative swing of the square wave source, and so is only used half of the time. It has a gain of -1, so we do not consider it a gain stage. When not in use, it is replaced with a short. The $R_{ADG419}$ is the switch that swaps between the positive and negative swings of the square wave, and so is always in the circuit. The voltage divider $R_{52}/(R_{50} + R_{52})$ is absent for the TRead LR. Finally, the last op amp is simply a buffer. The voltage reference chip has an offset voltage and a noise voltage. The offset voltage is a systematic uncertainty that may be calibrated out, and does not depend on bandwidth. We
Figure 4.5: Simplified circuit showing how TRead generates its reference voltage. Most of the extra circuitry between the voltage reference chip and $V_{in}$ is to accommodate the switching required to generate the square wave.

may model the output of the voltage reference source as

$$V_{ref} = V^{0}_{ref} + \delta V_{ref}$$

(4.5)

The only gain in the system is from the voltage divider, so the output voltage $V_{in}$ is related to the reference voltage by

$$V_{in} = \frac{R_{52}}{R_{50} + R_{52}} V_{ref} \equiv G_{ref} V_{ref}$$

(4.6)

where we’ve defined the reference circuit gain $G_{ref} \equiv R_{52}/(R_{50} + R_{52})$. In addition to the uncertainty of the voltage reference, there is amplifier noise from the op amps and Johnson noise from the resistors. Noise from the first amplifier passes through $G_{ref}$, but noise from the second does not. We compute the noise at each intermediate point A, B, and C.
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The noise at A is simply the voltage reference noise,

\[ \delta V_A = \delta V_{\text{ref}} \]  \hspace{1cm} (4.7)

For the positive swing, \( \delta V_{B+} = \delta V_A \). For the negative swing, we must add in the contribution from the inverting amplifier. \( \delta V_A \) is simply a voltage noise at the input to the amplifier, so we must convert it to the output. The conversion from input to output is the amplifier gain \([B.2]\), which has magnitude 1 and so has no effect. We must additionally add in the noise from the amplifier itself and the Johnson noise from the resistors. This gives us a total noise \( \delta V_B \) at B of

\[
(\delta V_B)^2 = \begin{cases} 
(\delta V_A)^2 & \text{positive} \\
(\delta V_A)^2 + 4(\delta V_{\text{OPA2277}})^2 + R_{45}^2 (\delta I_{\text{OPA2277}})^2 + 4k_B T (R_{45} + R_{47}) & \text{negative}
\end{cases}
\]  \hspace{1cm} (4.8)

Going from A to B involves applying the voltage divider gain to \( \delta V_B \), and adding in the Johnson noise from \( R_{\text{ADG419}} + R_{50} \) and \( R_{52} \). We note that the point C sees a path to ground through \( R_{52} \) in parallel with \( R_{50}, R_{\text{ADG419}} \), and the output impedance of the op amp (Fig. 4.6). The noise \( \delta V_C \) at C is then

\[
(\delta V_C)^2 = G_{\text{ref}}^2 (\delta V_B)^2 + 4k_B T \left[ R_{52} || (R_{50} + R_{\text{ADG419}}) \right] 
\]  \hspace{1cm} (4.9)

Going from C to \( V_{\text{in}} \) simply adds the noise from the second op amp, since it has no gain.

\[
(\delta V_{\text{in}})^2 = (\delta V_C)^2 + (\delta V_{\text{OPA2277}})^2 + \left[ R_{52} || (R_{50} + R_{\text{ADG419}}) \right]^2 (\delta I_{\text{OPA2277}})^2 
\]  \hspace{1cm} (4.10)

The voltage reference is then input into a adjustable transconductance amplifier\[^{12}\] fol-

\[^{12}\] Fancy name for a variable resistor.
Figure 4.6: The voltage divider from the perspective of noise.

This is shown in Fig. 4.7. The MDAC is, all told, a variable-gain voltage amplifier with gain between 0 and 1. Its action is described by

$$V_{DAC} = \frac{DAC}{65535} V_{in} \equiv G_{DAC} V_{in}$$

(4.11)

From a noise perspective, it may be analyzed as a simple inverting amplifier. The voltage noise at $V_{in}$ is treated as a voltage noise at the input to the amplifier. The remaining noise sources are treated as usual. However, note that the left resistor generates a Johnson noise voltage that is bookended by grounds (literal ground on one end and through the output resistance of the second op amp in Fig. 4.5), and so does not contribute. The gain and Johnson noise contribution from the MDAC is code-dependent. We will assume the

13 Fancy name for a variable gain voltage amplifier.
worst case scenario for the Johnson noise \( (R_{\text{right}} || R_{FB} = R_{FB}) \). We will also assume the worst case scenario \( G_{DAC} = 1 \) for all but the input noise \( \delta V_{in} \), and track \( G_{DAC} \) for \( \delta V_{in} \). We use the worst-case for everything else because their noise should not be anywhere near significant for any region of operation of the board, but the DAC noise can become significant for low gain settings. The noise coming out of the DAC is then

\[
(\delta V_{DAC})^2 = G_{DAC}^2 (\delta V_{in})^2 + 4 (\delta V_{OPA2277})^2 + R_{FB}^2 (\delta I_{OPA2277})^2 + 4k_B T R_{FB} \quad (4.12)
\]

The DAC is loaded by the bias resistors and the test resistor. The voltage difference across the test resistor is measured by an instrumentation pre-amplifier. This is shown in Fig. 4.8 We note that an inverting amplifier is used to generate the negative voltage line. The voltage out of the pre-amplifier is given by

\[
V_{\text{pre}} = \frac{2}{1 + \left(\frac{2R_L}{R_T}\right)} G_{\text{pre}} V_{DAC} \quad (4.13)
\]
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Figure 4.8: The bias circuit that loads the DAC. The voltage difference across the test resistor $R_T$ is measured by an instrumentation pre-amplifier. The inputs to the instrumentation amplifier are protected by the protection resistors $R_P$. The test resistor $R_T$ is usually at cryogenic temperatures, $T_0 \sim 1$ K. All other resistors are at room temperature.

where $R_L$ is the total load resistance, which includes the bias resistors, mux impedance, line resistance, and protection resistors. The test resistance can be determined from the amplifier output voltage $V_{pre}$ and the known input voltage $V_{in}$

$$R_T = \frac{2}{(2G_{pre}V_{DAC}/V_{pre}) - 1} R_L. \quad (4.14)$$

From a noise perspective, the circuit looks like Fig. 4.9. There are two DAC voltage noise sources, since both positive and negative voltages relative to ground are generated.

The inverter voltage noise $\delta V_{-1}$ results from the inverter used to create the negative swing and is given by

$$(\delta V_{-1})^2 = 4(\delta V_{OPA2277})^2 + R_{-1}^2 (\delta I_{OPA2277})^2 + 4k_BT(2R_{-1}) \quad (4.15)$$

Each of the noise generators on the main loop ($\delta V_{DAC}, \delta V_{R_L}, \delta V_{-1}$) except for $\delta V_{R_T}$ are scaled similarly. They each generate a current $\delta I = \delta V/(2R_L + R_T)$, which generates a
voltage difference across the inputs of the pre-amp of $\delta V_{\text{in}} = R_T \delta I$. So they are all scaled by $R_T/(2R_L + R_T)$ to get to the amplifier input.

$$\delta V_{\text{in}}^X = \frac{R_T}{2R_L + R_T} \delta V_X \quad \text{for } X \in \{\text{DAC, } R_L, -1\}$$

The $R_T$ Johnson noise generator is handled very similarly, but the current is applied to $2R_L$ instead, so

$$\delta V_{\text{in}}^{R_T} = \frac{2R_L}{2R_L + R_T} \delta V_{R_T}$$

The noise generators on the inputs of the amplifier ($\delta V_{R_P}$, $\delta V_{\text{amp}}$) are referenced to the input of the amplifier already. The current sources $\delta I_{\text{amp}}$ each see a path to ground (or virtual
Figure 4.10: The final stages of the TRed readout chain. Coming out of the pre-amplifier, the signal goes through a second stage (adjustable-gain) amplifier. The signal is then low-pass filtered and digitized. The demodulation of the sampled data is modeled as a unity-gain finite-impulse response digital filter, a digital gain stage, and decimation step.

The total noise at the output of the pre-amplifier is

\[
(\delta V_{\text{pre}})^2 = (\delta V_{\text{ampout}})^2 + G_{\text{pre}}^2 (\delta V_{\text{amp}})^2 + G_{\text{pre}}^2 \left( \frac{2R_L}{2R_L + R_T} \right)^2 (\delta V_{R_T})^2 \\
+ G_{\text{pre}}^2 \left( \frac{R_T}{2R_L + R_T} \right)^2 \left[ 2(\delta V_{\text{DAC}})^2 + (\delta V_{-1})^2 + 2(\delta V_{R_L})^2 \right] \\
+ 2G_{\text{pre}}^2 \left( \frac{2R_L}{2R_L + R_T} \right)^2 (\delta I_{\text{amp}})^2
\] (4.19)

Following the pre-amplifier is adjustable-gain amplifier and then a DAC. We will ignore the noise of the second stage amplifier and all subsequent components, since we assume that their noise is dominated by the pre-amp amplified noise of the previous stages. However, we will include the ADC noise. The noise coming out of the second stage gain is simply

\[
\delta V_G = G \delta V_{\text{pre}}
\] (4.20)

The filter sets the limits of integration of the spectral noise density to get from noise
density \( (V/\sqrt{\text{Hz}}) \) to noise \( (V) \). The filter’s action is described by

\[
\langle V^2 \rangle = \int_{0}^{\infty} |F(f)|^2 (\delta V)^2 \, df
\]

(4.21)

\[
\langle V^2 \rangle = (\delta V)^2 \int_{0}^{\infty} |F(f)|^2 \, df = (\delta V)^2 \int_{0}^{f_{eq}} \, df
\]

\[
\langle V^2 \rangle = (\delta V)^2 f_{eq}
\]

(4.22)

where we’ve assumed the noise is white (i.e. \( \delta V = \text{constant} \)) and we’ve defined the equivalent noise bandwidth \( f_{eq} \equiv \int |F(f)|^2 \, df \). The low-pass filter is a unity-gain 3-pole Bessel filter, which has an unnormalized response of

\[
F(s) = \frac{15}{s^3 + 6s^2 + 15s + 15}
\]

which gives us a equivalent bandwidth \( f_{eq} \) of

\[
f_{eq} = \frac{1}{2\pi} \int_{0}^{\infty} |F(j\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_{0}^{\infty} \frac{225}{\omega^6 + 6\omega^4 + 45\omega^2 + 225} \, d\omega
\]

\[
f_{eq} = 1.0736 f_{3dB}
\]

(4.23)

Let the ADC sample at \( f_s \), so the Nyquist frequency is \( f_{Ny} = f_s/2 \). We require \( f_{Ny} > f_{eq} \) so as not to alias noise. However, this condition results in adjacent samples being correlated. The Wiener-Khinchin theorem states that the autocorrelation \( r_{VV}(\tau) = \langle V(t) V^*(t - \tau) \rangle \) and the power spectral density \( (\delta V)^2 \) are Fourier transform pairs,

\[
r_{VV}(\tau) = \mathcal{F}^{-1} [(\delta V)^2] = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\delta V)^2 e^{i\omega \tau} \, d\omega
\]

(4.24)

For simplicity, we model our filter as a brick-wall filter with cut-off frequency \( f_{eq} \). Then
we see from Eq. (4.24) that the autocorrelation function will be a sinc function:

\[ (\delta V)^2 = (\delta V_0)^2 \text{rect} \left( \frac{\omega}{2\omega_{eq}} \right) \quad \leftrightarrow \quad r_{VV}(\tau) = \frac{\omega_{eq}}{\pi} \text{sinc} \left( \frac{\omega_{eq}\tau}{\pi} \right) \] (4.25)

We observe that the autocorrelation has its first node at \( \tau_0 = \pi/\omega_{eq} = 1/(2f_{eq}) \). The autocorrelation is zero only at the nodes, so all samples are partially correlated to some level, but the most significant correlations are for \( \tau < \tau_0 \). We define the effective correlation time as \( \tau_0 \), i.e. we treat all samples that are separated in time by less than \( \tau_0 \) as perfectly correlated and all samples that are separated in time by more than \( \tau_0 \) as perfectly uncorrelated.

Using this definition, we may define an effective number of independent samples

\[ \frac{N_{\text{eff}}}{N} = \frac{T_s}{\tau_0} = 2\frac{f_{eq}}{f_s} = \frac{f_{eq}}{f_{Ny}} \quad \text{for } f_{eq} < f_{Ny} \text{ only.} \] (4.26)

The ADC is effectively a 16-bit -10.24 V to 10.24 V ADC for a voltage swing of \( \Delta V_{\text{ADC}} = 20.48 \text{ V} \). This leads to a conversion from voltage to ADC counts \( D \) of

\[ D = \frac{(2^{16} - 1)}{\Delta V_{\text{ADC}}} \left( V_G + \frac{\Delta V_{\text{ADC}}}{2} \right) = \frac{(2^{16} - 1)}{\Delta V_{\text{ADC}}} V_G + 2^{15} \] (4.27)

The demodulation function is shown in Fig. 4.11. We note that the CHOP signal sets the sign of the test load bias, so the input to the instrumentation amplifier is inverted when CHOP is negative. Thus, we may write the demodulation filter as

\[ \text{rect } x = \begin{cases} 1 & |x| < 1/2 \\ 0 & |x| > 1/2 \end{cases} \]

\[ \text{sinc } x = \frac{\sin (\pi x)}{\pi x} \].

\[ \text{The rect and sinc functions used here are defined with the engineering convention,} \]

\[ 14 \text{The ADC chip is actually 0 to } V_{\text{ref}} = 4.096 \text{ V with a level-shifted and gained input. This has some advantages in systematic uncertainty rejection, since the DAC and ADC are generated by the same } V_{\text{ref}}, \text{ so systematics in } V_{\text{ref}} \text{ are cancelled out.} \]
Figure 4.11: The TRead demodulation scheme. The CHOP signal biases the circuit with either a positive or negative voltage. For negative biases, the response is inverted, so the DEMOD function subtracts these values. From the perspective of noise, this strategy is effectively equivalent to averaging for a period of $2N_{\text{SUM}}$. However, this system has the advantage of compensating for common mode offsets.

\[ y[n] = \frac{1}{2N_{\text{SUM}}} \left( \sum_{m=0}^{N_{\text{SUM}}-1} x[n - m] - \sum_{m=0}^{N_{\text{SUM}}-1} x[n - N_{\text{SUM}} - N_{\text{GAP}} - m] \right) \quad (4.28) \]

for input samples $x[n]$ and output $y[n]$. Note that there is in reality a zero-padding of length $N_{\text{GAP}}$ at the end of the kernel, but we ignore it since it has no effect on the filter. The nulled regions are present to allow for the step response from flipping the CHOP bias signal to settle, and to provide a node at 60 Hz. In the ideal case $N_{\text{GAP}} = 0$, this filter behaves as a straight $N_{\text{SUM}}$-wide edge filter.

\[ y_0[n] = \frac{1}{2N_{\text{SUM}}} \left( \sum_{m=0}^{N_{\text{SUM}}-1} x[n - m] - \sum_{m=0}^{N_{\text{SUM}}-1} x[n - N_{\text{SUM}} - m] \right) \quad (4.29) \]

We will analyze the effects of both to demonstrate the ideal limit and the effects of realistic parameters. The transfer functions of these two filters in the $z$-domain are given
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by

\[ H(z) = \frac{1}{2N_{\text{SUM}}} \left[ 1 - z^{-(N_{\text{SUM}}+N_{\text{GAP}})} \right] \sum_{m=0}^{N_{\text{SUM}}-1} z^{-m} \]  
(4.30)

\[ H_1(z) = \frac{1}{2N_{\text{SUM}}} \left[ 1 - z^{-N_{\text{SUM}}} \right] \sum_{m=0}^{N_{\text{SUM}}-1} z^{-m} \]  
(4.31)

which with the substitution \( z = e^{i\omega} \), where \( \omega \) is the normalized angular frequency \( \omega = 2\pi fT \), gives us the DTFT of the filter. The standard frequency-domain \((f, \text{not } \omega)\) response of both FIR filters and the anti-aliasing filter are shown in Fig. 4.12 where we have used standard laboratory parameters\(^{16}\). We also include a boxcar filter with \( 2N_{\text{SUM}} \) smaller than the correlation distance to show the effect of the anti-aliasing correlation of the samples.

It is convenient to analyze the filters in the more familiar continuous time domain. The equivalent time domain filters are shown in Fig. 4.13 and may be written

\[ h(t) = \frac{1}{2t_s} \left[ \text{rect} \left( \frac{t-t_s/2}{t_s} \right) - \text{rect} \left( \frac{t-t_G-3t_s/2}{t_s} \right) \right] \]  
(4.32a)

\[ h_1(t) = \frac{1}{2t_s} \left[ \text{rect} \left( \frac{t-t_s/2}{t_s} \right) - \text{rect} \left( \frac{t-3t_s/2}{t_s} \right) \right] \]  
(4.32b)

If we now let \( F_0(\omega) = \mathcal{F} \left[ \frac{1}{t_s} \text{rect} \left( \frac{t}{t_s} \right) \right] = \text{sinc} \left( \frac{\omega t_s}{2\pi} \right) \), then we see that

\[ H(\omega) = \frac{F_0(\omega)}{2} e^{-i\omega(\frac{3}{2}t_G+t_s)} \left[ e^{i\omega(t_G+t_s)/2} - e^{-i\omega(t_G+t_s)/2} \right] \]  
(4.33a)

\[ = ie^{-i\omega(\frac{3}{2}t_G+t_s)} F_0(\omega) \sin \left( \frac{\omega(t_G+t_s)}{2} \right) \]  
(4.33a)

\[ H_1(\omega) = ie^{-i\omega t_s} F_0(\omega) \sin \left( \frac{\omega t_s}{2} \right) \]  
(4.33b)

\(^{16}\)SUM_TICKS = 66, GAP_TICKS = 59, CYCLES_PER_TICK = 5000, CLOCKFREQ = 20000000L.
Figure 4.12: The transfer functions in real frequency space of the demodulation filter (blue), the $N_{\text{SUM}}$ ideal filter (red), the anti-aliasing filter (orange), and an ideal filter that averages only correlated samples (dashed). The sampling frequency in this case is 4 kHz, for a Nyquist frequency $f_{Ny}$ of 2 kHz. The demodulation filter is an $N_{\text{SUM}}$ edge filter (dotted blue) modulated by a sine term. The anti-aliasing filter has a 3 dB point at $f_{3\text{dB}} = 920$ Hz. For filters that average fewer samples than the correlation distance, the anti-aliasing filter limits the total power. For filters that average more samples than the correlation distance, the demodulation filter limits the total power. $N_{\text{GAP}}$ was chosen to put a node in the FIR at 60 Hz. The total powers of the demodulation and $N_{\text{SUM}}$ ideal filters are the same.
Figure 4.13: The equivalent continuous time filters of the demodulation filter (left) and the ideal filter (right). The value $t_S$ is defined as $t_S = N_{\text{SUM}}/f_s$. The offset $t_G$ is defined as $t_G = N_{\text{GAP}}/f_s$.

So the filter functions are

$$|H(\omega)|^2 = \frac{\sin^2 \left( \frac{\omega t_S}{2} \right) \sin^2 \left( \frac{\omega (t_S + t_G)}{2} \right)}{\left( \frac{\omega t_S}{2} \right)^2} \int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega = \frac{\pi}{t_S} \quad (4.34a)$$

$$|H_1(\omega)|^2 = \frac{\sin^4 (\omega t_S)}{(\omega t_S)^2} \int_{-\infty}^{\infty} |H_1(\omega)|^2 \, d\omega = \frac{\pi}{t_S} \quad (4.34b)$$

It can be shown that both filter functions integrate to the same value (which is obvious given Parseval’s Theorem).

The ADC contributes additional noise $\langle D_{\text{ADC}}^2 \rangle$, typically listed in counts$^2$. Note that this is in variance units, not spectral density units, so we must divide by the effective bandwidth to get it into spectral density. We can refer this to the ADC input,

$$\delta V_{\text{ADC}} = \frac{\Delta V_{\text{ADC}}}{(2^{16} - 1)} \sqrt{\frac{\langle D_{\text{ADC}}^2 \rangle}{f_{Ny}}}.$$

where we have divided by $f_{Ny}$, since that is the one-sided noise bandwidth of an $f_s$ sampling rate. We use the one-sided bandwidth and not the two-sided to be consistent with the spectral densities of the electronic components, which are almost always specified as
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one-sided. The one-sided spectral noise density is a factor of $\sqrt{2}$ larger than the two-sided spectral noise density.

We note that the ADC noise does not pass through the analog low-pass filter, so the analog low-pass does not attenuate its noise. However, our system is limited by the digital filter (Fig. 4.12), which mitigates both the ADC and input voltage noise. This does not mean that the analog low-pass filter is useless. If we did not have it, we would have to account for the voltage noise being aliased from out of the sampling band ($|f| > f_{Ny}$) into the sampling band ($|f| < f_{Ny}$), which would significantly increase the effective noise level of the input voltage noise in the sampling band. With the filter, the aliased noise is negligible and we may ignore it. Then we may compute the effective bandwidth of the FIR,

$$f_{\text{FIR}} = \frac{1}{2\pi} \int_0^\infty |H(\omega)|^2 \, d\omega = \int_0^\infty |h(t)|^2 \, dt$$

$$f_{\text{FIR}} = \frac{1}{2t_S}$$ (4.36)

where we have taken advantage of Parseval’s Theorem to convert the integral to the time domain, where it may be evaluated particularly simply. The decimation step does not affect the noise, since it discards only correlated samples. So we may finally compute the noise (variance) of each demodulated measurement, referred to the ADC input

$$\langle V_{\text{ADC}}^2 \rangle = (\delta V_{\text{tot}})^2 f_{\text{FIR}} = \frac{(\delta V_{\text{ADC}})^2 + (\delta V_G)^2}{2t_S}. \quad (4.37)$$

The reference point for this variance may be converted to any desired point by following through the standard linear signal chain.
4.3.2.2 Optimal Excitation

We may compute the optimal excitation waveform and filter. We may compute the noise \( N \) of any filter \( h(t) \) using Parseval’s Theorem,

\[
N[h(t)] = \int_0^T h^2(t) \, dt. \tag{4.38}
\]

where we’ve written the noise as a functional in anticipation of our optimization method. The filter must be matched to the excitation waveform \( c(t) \) such that it correctly has a gain of 1,

\[
C_1[h(t), c(t)] = 0 = \int_0^T h(t)c(t) \, dt - 1. \tag{4.39}
\]

where we’ve converted the convolution to a standard inner product since the filter function’s amplitude can be negated without changing \( N \) and we expect it to be either symmetric or antisymmetric about \( T \). This is still not a fair comparison of different waveforms. We can always reduce noise at this point by increasing the amplitude of the waveform, which in turn reduces the amplitude of the filter. We require an additional constraint on the excitation. Since this is for a cryogenic application, we choose to require that all excitation waves deposit the same amount of power, which gives us the condition

\[
C_2[c(t)] = 0 = \int_0^T c^2(t) \, dt - 1. \tag{4.40}
\]

where we’ve ignored the units and actual value of power dissipation since we are only interested in the shapes of the waveforms at this point. So our task is to minimize the functional \( N[h(t)] \) subject to the constraints \( C_1[h(t), c(t)] = 0 \) and \( C_2[c(t)] = 0 \). This problem
is amenable to analysis by applying Calculus of Variations and Lagrange multipliers. Our extremization conditions then result in

\[
\frac{\delta N}{\delta h} + \lambda_1 \frac{\partial C_1}{\partial h} + \lambda_2 \frac{\partial C_2}{\partial h} = 0 \quad (4.41a)
\]

\[
\frac{\delta N}{\delta c} + \lambda_1 \frac{\partial C_1}{\partial c} + \lambda_2 \frac{\partial C_2}{\partial c} = 0 \quad (4.41b)
\]

where \( \frac{\delta N}{\delta h} \) and \( \frac{\delta N}{\delta c} \) are shorthand for the variational derivative given in general by

\[
\frac{\delta}{\delta y} F \left[ t, y(t), y'(t) \right] = \frac{\partial F}{\partial y} - \frac{d}{dt} \left( \frac{\partial F}{\partial y'} \right).
\]

This gives us

\[
2h(t) + \lambda_1 \int_0^T c(t) \, dt = 0 \quad (4.42a)
\]

\[
\lambda_1 \int_0^T h(t) \, dt + 2\lambda_2 \int_0^T c(t) \, dt = 0 \quad (4.42b)
\]

where we note that the partial derivative may be taken under the integral sign and the time dependence may be ignored (since it is not a total derivative). Using the second to eliminate \( \int_0^T c(t) \, dt \) in the first and differentiating the result with respect to \( t \) we get

\[
2 \frac{dh(t)}{dt} - \frac{\lambda_1^2}{4\lambda_2} h(t) = 0
\]

which is easily solved to find

\[
h(t) = A \exp \left( \frac{\lambda_1^2}{2\lambda_2} t \right) \quad (4.43)
\]

Then using Eq. (4.43) in Eq. (4.41a) and differentiating we can find \( c(t) \),

\[
c(t) = -\frac{A\lambda_1}{2\lambda_2} \exp \left( \frac{\lambda_1^2}{2\lambda_2} t \right) \quad (4.44)
\]
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We require a boundary condition on the excitation waveform. A sensible choice is a periodic boundary condition, \( c(0) = c(T) \). Then we see that

\[
\frac{\lambda_1^2}{2\lambda_2} = \frac{2\pi in}{T}, \quad n \in \mathbb{Z}
\]  

(4.45)

so we see that \( c(t) \) and \( h(t) \) are sinusoids with an integer number of periods in the full period \( T \). There are no further boundary conditions that would restrict \( n \) so any of the solutions would be equally valid. We note that \( n = 0 \) is a valid choice, which corresponds to a constant function. If we had instead chosen an antisymmetric boundary condition we would admit a square wave solution. Thus, a constant function or a square wave are also optimal solutions in addition to the sinusoids.

We could continue with this formalism, but it does not give us more interesting information, so we stop here. It is obvious that the excitation \( c(t) \) and \( h(t) \) will be in phase. From the symmetry of the second constraint with the noise function and the inner product, all optimal solutions will have \( N = 1 \).

4.3.2.3 Sinusoidal Excitation

We consider a minor variation of the optimal sinusoidal excitation waveform. In this mode, the excitation waveform is a sinusoid and the demodulation filter is the sum of sinusoids at the harmonics of the excitation frequency. Although slightly non-optimal, this has the advantage of allowing us to place a node in the transfer function at an arbitrary frequency with minimal penalty. Additionally, non-idealities in the hardware that generates the excitation waveform (e.g. a filter) are exacerbated by the large discontinuities in a
square wave. For time constants much shorter than the sine wave period, these effects may be ignored.

The excitation is a sine wave with frequency $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi N_{ch}}{T_{frame}}$, which goes through one full period for each channel to be measured. Since the mux must change addresses between channels, it is optimal for the excitation to be at a zero at the beginning and end in order to minimize muxing transients. This gives us a normalized excitation of

$$c(t) = \sqrt{\frac{2}{T_0}} \sin(\omega_0 t), \quad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi N_{ch}}{T_{frame}}.$$  (4.46)

For the filter function, we include two terms

$$h(t) = k_0 \sin(\omega_0 t) + k_1 \sin(2\omega_0 t)$$  (4.47)

consisting of the fundamental frequency and first harmonic. We include the second term because it allows us to put a node at an arbitrary frequency, most interestingly 60 Hz. Additional terms could be included to tune the response even more, e.g. by rejecting another frequency or increasing the rejection width of an existing frequency. The analysis method is the same.

We define the frequency we would like to reject as $\omega_r$. Then we require that the (normalized) gain is 1 and that the filter rejects $\omega_r$,

$$c(t) * h(t)\bigg|_{T_0} = -k_0 \sqrt{\frac{2}{T_0}} \int_0^{T_0} \sin^2(\omega_0 t) \, dt = -k_0 \sqrt{\frac{T_0}{2}} = 1$$  (4.48a)

$$\sin(\omega_r t) * h(t)\bigg|_{T_0} = -k_0 \int_0^{T_0} \sin(\omega_r t) \sin(\omega_0 t) \, dt - k_1 \int_0^{T_0} \sin(\omega_r t) \sin(2\omega_0 t) \, dt = 0$$  (4.48b)
so

\[ k_0 = -\sqrt{\frac{2}{T_0}} \]  \hspace{1cm} (4.49a)  

\[ k_1 = \sqrt{\frac{2}{T_0}} \int_0^{T_0} \sin(\omega_r t) \sin(\omega_0 t) \, dt \]  \hspace{1cm} (4.49b)

For \( T_0 = (1 \text{ s})/(16 \text{ channels}) \) and \( \omega_r = 2\pi \cdot 60 \text{ Hz} \), these functions are plotted in Fig. 4.14. Note that these functions are computed in the continuous domain. In the discrete domain (i.e. in the actual implementation), the filter function must be multiplied by \( \Delta t = T_{\text{frame}}/ [2N_{\text{ch}} (N_{\text{SUM}} + N_{\text{GAP}})]^{17} \). The power spectrum of the normalized sine wave filter is plotted in Fig. 4.15. A proxy for the standard deviation of a measurement with a given filter is given by \( \sqrt{\int |H(f)|^2 \, df} \). Thus we may compute the ratio of the standard deviations of a measurement with each filter, which is

\[ \frac{\sigma_{\text{sine}}}{\sigma_{\text{square}}} = 0.778 \]  \hspace{1cm} (4.52a)  

\[ \frac{\sigma_{\text{sine}}}{\sigma_{\text{ideal}}} = 1.072 \]  \hspace{1cm} (4.52b)

where \( \sigma_{\text{sine}} \) is the measurement uncertainty using the 60 Hz rejecting sine wave filter, \( \sigma_{\text{square}} \) is the measurement uncertainty using 60 Hz rejecting square wave filter, and \( \sigma_{\text{ideal}} \) is the measurement uncertainty using any ideal filter. Changing the demodulation filter only

\(^{17}\text{This is because the continuous convolution is defined as} \)

\[ x \ast y = \int_{-\infty}^{\infty} x(\tau) y(t-\tau) \, d\tau \simeq \sum_{m=-\infty}^{\infty} x(m \Delta t) y(n \Delta t - m \Delta t) \Delta t \]  \hspace{1cm} (4.50)

and the discrete convolution is defined as

\[ x \ast y = \sum_{m=-\infty}^{\infty} x[m] y[n-m] \]  \hspace{1cm} (4.51)

so the differential \( d\tau \simeq \Delta t \) is dropped in the discrete case.

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Figure 4.14: The excitation function (left) and filter function (right). The dashed line shows a filter function with no harmonics, $k_1 = 0$.

Figure 4.15: The normalized, constant power, 60 Hz rejecting sine wave and square wave filter transfer functions. The sine wave filter is able to place the 60 Hz node with less of a penalty to the total noise. Both filters reject 60 Hz, but the sine wave node is deeper and broader.
changes the effective bandwidth of the system (Eq. (4.36)), so the total noise may be scaled by these ratios to get estimates of the noise with alternative filters.

A prototype 60 Hz rejecting sinusoidal excitation mode has been implemented on TRead and is available via the demod command.

### 4.3.2.4 Comparison with Test Data

The noise model presented above predicts the noise level as a function of gain, excitation amplitude, and resistance. The model has no tuning parameters. In order to check it, a test harness with 12 resistors with different resistance (+4 calibration resistors) was used. Data was collected by a TRead Standard using the square wave excitation with \( N_{\text{SUM}} = 67 \) and \( N_{\text{GAP}} = 58 \) at 5 different gain settings and 5 different excitation amplitudes. At each setting, 10 minutes worth of data was collected and the noise of all of the 16 channels computed.

For typical parameters \( (G = 8, I = 32 \mu A) \), the noise is dominated at low resistances by the instrumentation amplifier noise voltage (Fig. 4.16). Between \( 9 \, \text{k}\Omega \) and \( 200 \, \text{k}\Omega \), the test resistor’s Johnson noise dominates. Note that this is for a test resistor at 300 K, so for a cryogenic thermometer the test resistor Johnson noise will be negligible. Above \( 200 \, \text{k}\Omega \) the load resistors’ Johnson noise dominates. Standard 1 k\Omega RuOx thermometers have a resistance of \( 30 – 60 \, \text{k}\Omega \) at the cryogenic temperatures of interest, so we see that TRead is limited by the instrumentation amplifier noise. As a final note, this data was collected on an older version of TRead that used an AD620 with an effective voltage noise
Figure 4.16: TRead noise model compared to measured data at $G = 8$, $I = 32 \mu A$, in demodulated counts $\Delta$. The instrumentation amplifier voltage noise dominates the noise power for the interesting region of the voltage range. All terms are included in the model, but only terms that contribute significant amounts of noise are plotted.

of $10 \text{nV/}\sqrt{\text{Hz}}^{18}$. Newer revisions of the board use an AD8221 with $8 \text{nV/}\sqrt{\text{Hz}}$ of voltage noise. The model was compared to the data for parameters across the entire parameter space and shows excellent agreement in all cases (Fig. 4.17). Lastly, the noise model was applied to TRead_LR and shows that the instrumentation amplifier voltage noise dominates for all realistic use cases. It is for this reason that pads and vias for using an arbitrary instrumentation amplifier are included in TRead. As amplifier technology improves, the

\[ (\delta V)_{\text{eff}} = \sqrt{\frac{1}{f_{\text{FIR}}} \int_0^\infty |H(f)|^2 (\delta V)^2_{\text{true}} df } \]  

(4.53)

where $f_{\text{FIR}}$ is the demodulation filter’s effective bandwidth.

\[ \text{Total} \]
\[ \text{InAmp}_V \]
\[ \text{Johnson}_RT \]
\[ \text{Johnson}_RL \]
\[ \text{Data} \]
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Figure 4.17: TRead noise model compared to measured data across the full parameter space. Some data is absent at high $R$ because the chosen parameters result in the response signal saturating the ADC.

board can be improved along with it without having to redesign and turn new boards.

4.3.3 DSPID

The DSPID\textsuperscript{19} board low-bandwidth PID controller suitable for controlling cryogenic actuators. It uses the same 4-wire bridge circuitry from TRead Standard to measure a thermometer at 8 Hz. Although it has a 4-channel mux, the mux is parked on a single channel to maximize the measurement bandwidth up to 8 Hz. The 4-wire measurement is then used to update a PID loop, resulting in a 8 Hz controller. The PID controller is able to update a single 16-bit DAC with values between -2.048 V and +2.048 V. Each DSPID card can control a single actuator. In addition to the 16-bit DAC, four 12-bit DACs with an output voltage range of -10.24 V to +10.24 V are available for manual control only.

\textsuperscript{19}For “Digital Setpoint Proportional Integral Differential”.
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The bandwidth of the controller is limited by the bandwidth of the 4-wire measurement. As discussed in the TRead description, the measurement bandwidth is limited by the cryogenic system. The PID controller has the form of

\[ V[n] = k_P(\Delta[n] - r[n]) + k_I \sum_{i=1}^{\infty}(\Delta[n-i] - r[n-i]) + k_D \frac{\delta \Delta}{\delta n}[n] + c. \tag{4.54} \]

The output voltage is computed immediately after a full measurement \( \Delta[n] \) is completed. We note that the controller is linear in the demodulated measurement, i.e. equivalent to ADC counts. For \( R_T \ll R_L \), then \( \Delta \) is linear in \( R_T \). This means that the PID controller is linear in the resistance of the thermometer, not the temperature. In the limit that the temperature error is small \( (\Delta R \ll 2 \left( \frac{d R}{dT} \right) / \left( \frac{d^2 R}{dT^2} \right)) \), the controller is also linear in temperature. For scenarios where larger error terms are required (e.g. large temperature steps), the controller will not be linear. This is usually not relevant, since for large temperature swings the process parameters (e.g. heat capacity, thermal conductivity) are not constant anyway.

The proportional and integral terms are dependent on the error signal, \( e[n] = \Delta[n] - r[n] \), where \( r[n] \) is the setpoint. We note that the accumulator \( \sum_{i=1}^{\infty}(\Delta[n-i] - r[n-i]) \) does not update until the next step. This is not an issue, since the contribution of the current error term is already included in the proportional term. The differential term uses a low-passed numerical estimate of the demodulated signal derivative, given recursively by

\[ \frac{\delta \Delta}{\delta n}[n] = \frac{1}{2M} \left( (2^M - 1) \frac{\delta \Delta}{\delta n}[n-1] + (\Delta[n] - \Delta[n-1]) \right) \tag{4.55} \]

where \( M \) is an adjustable constant related to \( \log_2 \) of the low pass time constant. We note
that the differential term is more properly a velocity term, since it depends only on the rate of change of the demodulated signal and not the derivative of the error signal. This is typically advantageous and simplifies the control system transfer function.

Each PID parameter has an effective 32-bit range, encoded in two 16-bit parameters \( (x \text{ and } x_{\text{shift}}, \text{ where } x = p, i, d) \). Since the PID loop is implemented in integer arithmetic, there is some subtlety to the implementation. First the measurement (i.e. error, accumulator, or velocity for \( p, i, \) and \( d \), respectively) is multiplied by the coefficient to generate a 32-bit number then the result is right-shifted by the shift value:

\[
x_{\text{term}} = (k_x \cdot m) >> x_{\text{shift}} \quad x \in (p, i, d)
\]

This effectively makes the computations 16-bit fixed point with 32 choices for the position of the decimal point. The 3 individual terms are then summed as 32-bit integers. The resulting output is clipped between -32768 and +32767, since the DAC allows only 16 bits.

The PID control algorithm also has anti-windup, output change rate limiting, and current and voltage limits. If the DAC’s output is near its rails, the integrator term’s accumulator will not update. This prevents the accumulator from grossly overintegrating due to the DAC not being able to produce a sufficiently large output for a reasonable response. The update step is not allowed to change the DAC output by more than 1000 counts per update step. This limits the coupling of an unstable control loop into the system and mitigates the effect of large disturbances. The voltage and current limits are adjustable and have an obvious purpose and effect on the system.

Since it is a backplane board, DSPID may only report back a single frame every sec-
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ond. Since the measurement rate is faster than the reporting rate, the board packs 8 measurements into a single frame. DSPID reports back a 216-character packet at 1 Hz through the PMaster serial port. It may occupy any of the 20 address slots between 0 and 0x14. A description of the DSPID frame packet may be found in Table 4.3 and a typical DSPID frame looks like

*01D010000000000FFFB000000000000FFFB000000000000FFFC0000
0000000FFFC000000000000FFFB000000000000FFFC0000000000
FFFC00000000000FFFC00001900200000000004802041852E830005
0000000000000000000430EF000000000000000101008F020000C6

4.3.4 PSyncADC

The PSyncADC board is a fast synchronous differential analog measurement board capable of measuring up to 32 channels at rates up to the frame rate (∼ 400 Hz). The digitization rate and which channels should be read are both adjustable. The current version (V2) of the board is a pseudo backplane board in the sense that power and the frame clock are provided through the backplane, but the PSyncADC reports data on its own dedicated serial port. The PSyncADC board requires its own serial port because the throughput ($N_{ch} \cdot f_{sample}$) of the board is limited primarily by the bandwidth required to report the data back over the serial port.

The 32 analog in channels are passed into a series of muxes and funneled to a single
### Table 4.3: DSPI D packet definition.

<table>
<thead>
<tr>
<th>Position</th>
<th>Length (chars)</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>‘*’, packet start special character</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>Board address</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>‘D’, Card type</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>Frame counter</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>Thermometer mux (Tmux) setting</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>Status Mask (currently unused)</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>Demod[0] (oldest)</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>Coil Current Measurement[0] (oldest)</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>Applied Coil DAC[0] (oldest)</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>120</td>
<td>8</td>
<td>Demod[7] (newest)</td>
</tr>
<tr>
<td>128</td>
<td>4</td>
<td>Coil Current Measurement[7] (newest)</td>
</tr>
<tr>
<td>132</td>
<td>4</td>
<td>Applied Coil DAC[7] (newest)</td>
</tr>
<tr>
<td>136</td>
<td>4</td>
<td>ADAC, Thermometer excitation amplitude dac</td>
</tr>
<tr>
<td>140</td>
<td>4</td>
<td>GDAC, Thermometer readout gain dac</td>
</tr>
<tr>
<td>144</td>
<td>4</td>
<td>VMon, Coil voltage monitor</td>
</tr>
<tr>
<td>148</td>
<td>4</td>
<td>Analog IN</td>
</tr>
<tr>
<td>152</td>
<td>4</td>
<td>External temp</td>
</tr>
<tr>
<td>156</td>
<td>4</td>
<td>Board temp</td>
</tr>
<tr>
<td>160</td>
<td>4</td>
<td>Vsupply</td>
</tr>
<tr>
<td>164</td>
<td>4</td>
<td>GND</td>
</tr>
<tr>
<td>168</td>
<td>4</td>
<td>AOUT[0]</td>
</tr>
<tr>
<td>172</td>
<td>4</td>
<td>AOUT[1]</td>
</tr>
<tr>
<td>176</td>
<td>4</td>
<td>AOUT[2]</td>
</tr>
<tr>
<td>180</td>
<td>4</td>
<td>AOUT[3]</td>
</tr>
<tr>
<td>184</td>
<td>8</td>
<td>PID setpoint (demod units)</td>
</tr>
<tr>
<td>192</td>
<td>8</td>
<td>PID error (demod units)</td>
</tr>
<tr>
<td>200</td>
<td>8</td>
<td>PID accumulator (demod units)</td>
</tr>
<tr>
<td>208</td>
<td>2</td>
<td>PID P coefficient</td>
</tr>
<tr>
<td>210</td>
<td>2</td>
<td>PID I coefficient</td>
</tr>
<tr>
<td>212</td>
<td>2</td>
<td>N_SUM, number of samples per half-period</td>
</tr>
<tr>
<td>214</td>
<td>2</td>
<td>‘\r\n’, end of packet characters</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Total 216</td>
</tr>
</tbody>
</table>
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readout channel. The readout channel has an adjustable-gain instrumentation amplifier (AD8250) with gains of 1, 2, 5, and 10, followed by a 4.7 $\mu$s 2-pole low-pass Bessel filter and 16-bit 2.048 V bipolar ADC. This gives the board an effective input voltage range of $\pm 2.048 \text{ V}$, $\pm 4.096 \text{ V}$, $\pm 10.24 \text{ V}$, $\pm 20.48 \text{ V}$ for gains 1, 2, 5, and 10, respectively.

The PSyncADC has no provision for biasing a device. It is only capable of measuring the voltage response of some external bias. This also disallows an on-board modulation and demodulation scheme, though one could be implemented externally since the board is fully synchronous with the external clock.

The serial port baudrate required to measure $N_{\text{ch}}$ samples at a sampling rate of $f$ may be computed. Each sample is 16-bits and is encoded in ASCII-Hex, for a length of 4 hex characters. Each hex character requires 10 bits on the serial port (1 start and stop bit). Thus we require $40N_{\text{ch}}$ bits to encode one sample of each channel. However, each packet also reports 12 additional hex characters for formatting and the frame count, so each packet requires $40N_{\text{ch}} + 120$ bits. At a sampling rate of $f$, this gives us

$$B = (40N_{\text{ch}} + 120)f$$

(4.56)

This is plotted in Fig. 4.18. We note that above 560 kbps, the full throughput of the board may be used. Below 560 kbps, there is a tradeoff between the number of channels measured and the sampling rate. Only integer divisors (the $d_{\text{per}}$ parameter) of the maximum sampling rate are allowed to ensure the sampling is fully synchronous with the external clock. 115.2 kbps is the standard backplane baudrate. 417 kbps is the largest baudrate consistent with both OS X and the PIC baudrate generator. The FTDI USB-to-serial chips
Figure 4.18: The maximum sampling frequency $f$ versus number of channels $N_{\text{ch}}$ for common baudrates. (Dashed) The maximum sampling rate is determined by the frame rate, typically $\sim 400$ Hz.
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<table>
<thead>
<tr>
<th>Position</th>
<th>Length (chars)</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>‘*’, packet start special character</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>32-bit frame count</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>‘-’, separator character</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>Sample[0] (counts)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 + 4(Nch - 1)</td>
<td>4</td>
<td>Sample[Nch - 1] (counts)</td>
</tr>
<tr>
<td>10 + 4Nch</td>
<td>2</td>
<td>‘\n’, end of packet characters</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>12 + 4Nch (max 140)</td>
</tr>
</tbody>
</table>

Table 4.4: PSyncADC packet definition.

in the USB-Fiber board are capable of handling baudrates of up to 2.5 Mbps.

PSyncADC reports back a frame of length $4N_{ch} + 12$ (max 140) at 1 Hz through its own serial port. Since it has a dedicated serial port, it does not need an address. A description of the PSyncADC data packet may be found in Table 4.4 and a typical PSyncADC packet looks like

*000036A8-892788BF884287F787E7879B87698703867D862585D185
9485998558853484D785AF85908576856185758553855685008B178B
878BC78C018C9A8CBE834C8003

4.3.5 PMotor

PMotor is a fully synchronous motor controller card with linear output. Whereas most motor controllers pulse-width modulate their outputs to control the amount of power to apply to the motors, PMotor simply varies the voltage level linearly. PMotor uses a
dsPIC30F6015 since it has a built-in Quadrature Encoder Interface (QEI) module for reading the motor controller. The QEI module monitors the encoder QEA/QEB digital pulses for encoder counts. In the standard configuration, an edge on QEA signals an encoder count\(^{20}\). Whether the motor is moving forward or backward is determined by whether the polarity of QEB matches the QEA transition, e.g. a low-to-high transition on QEA and a low QEB state indicates the motor is moving forward. The encoder index pulse zeros the encoder counter to ensure it maintains synchronization\(^{21}\).

PMotor uses the standard 4 kHz interrupt rate. In each interrupt, the encoder counter is compared against that of the previous update and the difference is added to the absolute counter \(\text{abs\_pos}\). This breaks the degeneracy between different revolutions and allows the board to drive geared systems or linear motion stages. The absolute counter \(\text{abs\_pos}\) is a 32-bit integer, so it can encode 858993 distinct revolutions of a 5000 count encoder. There is an additional degeneracy between forward and reverse motion, e.g. advancing \(m\) counts is equivalent to decrementing \(N - m\) counts for an \(N\) count encoder. This means that the motor frequency will alias into the \(-2000\) Hz < \(f_{\text{motor}}\) < 2000 Hz range. The encoder is assumed to be 5000 counts (in \(2\times\) mode).

The board is intended to drive a 3-phase motor and uses a 4-DAC setup to commutate the motor. All 4 DACs are multiplying DACs (MDACs) that scale the output to be a fraction

\(^{20}\)This is \(2\times\) mode, where the number of effective encoder counts in one revolution is double the number of square waves in one motor revolution, which is usually reported in motor data sheets as the “encoder count”. There is also available \(4\times\) mode, where an edge on either QEA or QEB signals an encoder count.

\(^{21}\)Note that the dsPIC30F6015 QEI is supposed to have a symmetric response through the index pulse so that spurious counts are not generated (see dsPIC30F Family Reference section 16.5.3), however it does not. This bug is not reported in the dsPIC30F6010/6015 errata. An additional bug can result in the index pulse not correctly resetting the encoder counter.
of their reference voltage. The first MDAC scales the reference voltage of the other three MDACs in order to control the voltage amplitude of the output. Each of the three remaining MDACs controls one output to match the required phase. The amplitude DAC has 16 bits of range between 0 and 10.24 V, while the phase DACs have 16 bits of range between $\pm V_{\text{amp}}$, where $V_{\text{amp}}$ is the output of the amplitude DAC. The advantage of the amplitude DAC is that it guarantees a full 16 bits of precision on the phase independent of the amplitude.

The commutation assumes that there are two electrical cycles per physical motor revolution, resulting in 2500 counts per electrical revolution. The phase of the stator must lead the phase of the rotor in order for torque to be applied. This phase difference is called the lead angle. Lastly, two phases of the the 3-phase driver follow (equivalently, lead) the reference phase by 120 and 240 degrees. The encoder position is converted to a phase index and advanced by the lead angle and 3-phase phase offset, then indexed into a 625-element sine wave lookup table. The resulting value is used to set the phase DAC for that phase.

The rate of change of the encoder is estimated from the encoder position difference from one interrupt to the next. The estimate is unfiltered. We note that the smallest encoder difference is 1, so the smallest rate of change that can be measured is 1 count/interrupt = 0.8 rev/s. The largest difference is 2500 (due to the degeneracy between positive and negative velocities), corresponding to 2500 counts/interrupt = 2000 rev/s.

Three different modes of operation are available to PMotor. The most basic is FREE_RUN mode, in which case the only parameters are the lead angle and driving amplitude. In this mode, the board does nothing more than commutate the outputs according to the lead angle,
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applying the requested amplitude. In the no-load case, the maximum speed is limited by the back-EMF. As the rotor spins, its magnetic field generates a varying magnetic flux through the stator coils, which in turn generates a back-EMF according to Lenz’s Law. Once the back-EMF is equal to the applied voltage, the motor cannot generate any more torque and the acceleration stops. If a resistive torque is present on the shaft, then the acceleration will stop sooner, when the back-EMF limited torque equals the resistive load.

The motor may also be driven in PID VELOCITY mode, in which the amplitude of the output waveform is controlled by a PI controller on the estimated motor velocity. The quality of this controller is limited primarily by the limitations of the velocity estimate, as discussed above. Since the controller is PI controller, the desired velocity will be achieved so long as it does not exceed the maximum speed for the given resistive torque. The time required to reach the target velocity will further depend on the load on the motor shaft. The board does not attempt to estimate the acceleration, so no differential term is available in PID VELOCITY mode. Furthermore, since the velocity response is a first order system (Newton’s 2nd Law), a differential term serves no purpose.

For large loads, a velocity setpoint ramp was implemented. With this feature, the velocity setpoint is increased linearly from the current value (or 0) to the target. A large load results in a slow response, which could result in the integral term accumulator winding up excessively and result in significant overshoot of the target. While a velocity setpoint ramp results in a slower response, as it should be limited by the ramp speed rather than the response speed, it prevents undesirable response characteristics at the target (such as
overshoot). Furthermore, it allows the response time to be trivially controlled without a model of the system. Since the basis of the velocity setpoint ramp is the velocity PI loop, the controller must still be reasonably tuned for the velocity setpoint to work.

The final mode is `PID_POSITION` mode, in which the board tries to keep the motor at a particular encoder count, i.e. at a particular position. The lead angle is kept constant and the amplitude of the output is actuated by a PID controller of the position. The PID controller has the form

\[ V[n] = k_P (x[n] - r[n]) + k_I \sum_{i=0}^{n} (x[n - i] - r[n - i]) + k_D (x[n] - x[n - 1]) \]  

(4.57)

where \( V \) is the motor amplitude and \( r \) is the target position. The PID controller uses a velocity term rather than a proper differential term (which would look like \( e[n] - e[n - 1] \)) with no filtering on the velocity measurement. The limiting factor in the quality of the controller response is the nonlinear friction at low and zero speeds. Particularly, the transition between static and kinetic friction involves a step discontinuity in the system response function that the PID controller does not have the bandwidth to compensate for. Because of this, for sensible choices of the PID parameters, the steady state position will have a small offset.

The `PID_POSITION` mode also implements a trapezoid move command, in which the setpoint is changed from the current position to the target position with a trapezoidal profile. Much like the velocity ramp, this mode offers known response characteristics largely independent of the actual system characteristics. Note that the trapezoid move is built upon the PID position control loop, so the loop must be reasonably well tuned for the trapezoid
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move to function.

The PID coefficients for both PID VELOCITY and PID POSITION modes are implemented similarly to the DSPID PID loop, with a simple 16-bit coefficient and a shift coefficient for 32 hypothetical bits of range. However, neither PID loops have the same anti-windup and output limiting features of DSPID.

PMotor reports a 61-character packet at 1 Hz through the PMaster serial port. It may occupy any of the 20 address slots between 0 and 0x14. It would be challenging to report a significant amount of the PMotor data in its talk slot on the backplane since its PID loop runs at 4 kHz. Instead primarily the PMotor state is reported, with only a limited sampling of the data. A description of the PMotor frame packet may be found in Table 4.5 and a typical PMotor packet looks like

*00R000000000000000000F020-00C80000000000-00190023030
1C4

For situations where reporting PMotor internal data quickly is required, PMotor may communicate in a raw data mode where it ignores its talk slot and reports data continuously. All other boards in the backplane must have their data reporting disabled so that they do not talk at the same time as PMotor. Depending on the data selected to be reported, the board can report at up to 4 kHz. Changing the selection of data to report requires an update to the function int handler.c: update data raw in the firmware, but switching between backplane mode and raw data reporting may be done on-the-fly.

PMotor has a 16-bit ADC through a 4.7 $\mu$s 3-pole Bessel low-pass filter and an 8-
### Table 4.5: PMotor packet definition.

<table>
<thead>
<tr>
<th>Position</th>
<th>Length (chars)</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>‘*’, packet start special character</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>Board address</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>‘R’, Card type</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>Frame counter</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>Status (unused)</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>Absolute position (abs_pos)</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>Velocity in PIC units (encoder counts difference per interrupt)</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>Output amplitude (counts, 2’s complement format)</td>
</tr>
<tr>
<td>24</td>
<td>1</td>
<td>Mode (FRERUN = ‘F’, PID_POSITION = ‘P’, PID_VELOCITY = ‘V’)</td>
</tr>
<tr>
<td>25</td>
<td>3</td>
<td>Lead angle (degrees, in hex)</td>
</tr>
<tr>
<td>28</td>
<td>1</td>
<td>PID_POSITION sign (‘+’ = positive, ‘-’ = negative)</td>
</tr>
<tr>
<td>29</td>
<td>4</td>
<td>PID_POSITION p coefficient</td>
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<tr>
<td>33</td>
<td>4</td>
<td>PID_POSITION i coefficient</td>
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<tr>
<td>37</td>
<td>4</td>
<td>PID_POSITION d coefficient</td>
</tr>
<tr>
<td>41</td>
<td>1</td>
<td>PID_POSITION p_shift coefficient</td>
</tr>
<tr>
<td>42</td>
<td>1</td>
<td>PID_POSITION i_shift coefficient</td>
</tr>
<tr>
<td>43</td>
<td>1</td>
<td>PID_POSITION d_shift coefficient</td>
</tr>
<tr>
<td>44</td>
<td>1</td>
<td>PID_VELOCITY sign (‘+’ = positive, ‘-’ = negative)</td>
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<tr>
<td>45</td>
<td>4</td>
<td>PID_VELOCITY p coefficient</td>
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<td>49</td>
<td>4</td>
<td>PID_VELOCITY i coefficient</td>
</tr>
<tr>
<td>53</td>
<td>1</td>
<td>PID_VELOCITY p_shift coefficient</td>
</tr>
<tr>
<td>54</td>
<td>1</td>
<td>PID_VELOCITY i_shift coefficient</td>
</tr>
<tr>
<td>55</td>
<td>4</td>
<td>ADC_5 Pot value</td>
</tr>
<tr>
<td>59</td>
<td>2</td>
<td>‘\r\n’, end of packet characters</td>
</tr>
</tbody>
</table>

Total: 268
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channel mux. For a step function of the full ADC range to settle to less than 1 bit requires \(16 \log(2) \tau \simeq 11 \tau = 52 \mu s\). Each interrupt has \(1/(4 \text{kHz}) = 250 \mu s\) of time to operate, so each interrupt could sample no more than 4 of the ADC channels.

**External Input MIMO Firmware**

An alternative firmware may be loaded onto PMotor that uses one or more external inputs as the input to the control loop rather than the encoder. In this mode, the encoder is not measured. The bandwidth is limited by the filter settling time on the external measurements, and the PID calculation times are comparatively small. The number of outputs are limited by the number of independent DACs at 3. Thus, with this firmware, PMotor may implement a 4-input 3-output MIMO control loop at 4 kHz. A limitation of this mode is that the update equation is hard-coded into the firmware, so changing the control loop requires a firmware update.

### 4.4 Stand-alone Boards

Some applications are not compatible with the limitations of being in a backplane or gain no advantage by being in a backplane. For these, we use a number of stand-alone boards. Many of the boards are capable of synchronizing with an external clock and decoding the SyncBox frame count, so they may be operated synchronously despite not residing in a backplane.
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4.4.1 PSquid

PSquid is a single-channel SQUID readout board capable of controlling a SQUID readout with up to 3 stages, typical of SQUID readouts in use at GSFC prior to the adoption of the 2-stage readout. PSquid is equally capable of reading out a 2-stage system. PSquid can operate a PID loop at up to 20 kHz\(^\text{22}\) to null the response and measure the feedback signal.

PSquid accepts an external clock signal over a fiber optic line, but not a SyncBox data line. There is no facility to decode the SyncBox frame counts, so while the board may be driven with a common clock, it is not possible to synchronize the PSquid data stream with an external datastream. If no external clock is present, the board may be driven by an internal 5 MHz oscillator. Similar to other boards that use these dsPIC chips, the clock is up-scaled using a \(16 \times\) PLL, and each instruction takes 4 cycles for an instruction rate of 20 MIPS in internal clock mode.

PSquid has 9 16-bit DACs available, labeled DETB, S1B, S1FB, S2B, S2FB, S3B, S3FB, Offset, and Gain. Only the Offset and Gain DACs are fixed in their use, the remaining 7 come out of the board on connectors and may be connected as desired. The Offset and Gain DACs control the DC offset and adjustable gain of the sole ADC channel on the board and so are not general purpose. Furthermore, the S1B (also called Row Select or Address) DAC is fed through an 8-channel mux to allow for control of multiple channels without adjusting wiring. Note, however, that all 8 channels cannot be controlled with 20

\(^{22}\)As of this writing, only a more conservative 10 kHz mode is implemented. The maximum update rate is limited primarily by the number of instruction cycles required to perform the PID update step. After measuring this during use, it is projected that a 20 kHz update rate is readily attainable.
kHZ of bandwidth, rather the 20 kHZ of bandwidth is fixed and distributed among the 8 channels. The DETB DAC output also passes through an inverter. The un-inverted and inverted signals come out on a pair of 2-pin headers DETB+ and DETB-, both referenced to AGND. With this configuration, a differential signal with double the amplitude may be made by connecting the DETB+ signal line to the signal and the DETB- signal line to the return. Where a single-ended signal is acceptable, either DETB+ or DETB- may be used on their own.

PSquid has a single ADC with a readout chain shown in Fig. 4.19. One of 3 different pre-amplifiers may be selected by changing the jumper JP18. The first (Int Amp) is a UBC-like adjustable-offset amplifier. The second (Diff Amp) pre-amplifier is a standard instrumentation amplifier such as used in TRead. Note that the instrumentation amplifier uses a 2-pin molex header for its input. The final (Ext Amp) pre-amplifier is not on-board but is instead an 8-pin header that allows an external pre-amplifier to be used.

The standard configuration uses the internal amplifier. This configuration is suited for a 2-wire measurement of a SQUID series array on the input SAIN. The bias current is supplied by the S3B DAC biasing the resistor $R_{bias}$ and the voltage response is measured by an adjustable-offset internal pre-amplifier with gain and offset given by

$$V_{out} = -10 \left[ \left(1 + \frac{R_1 R_3 + R_1 R_2}{R_2 R_3}\right) \frac{V_{in}}{R_3} - \frac{R_1}{R_3} V_{off} \right] = -115V_{in} + 5V_{off} \quad (4.58)$$

where $V_{in}$ is the pre-amplifier input voltage and $V_{off}$ is the Offset DAC voltage, $0 \leq V_{off} \leq 1.024$ V. The input to the amplifier is single-ended and referenced against AGND\footnote{An optional simple RC filter may be included on the output. Also, the ground may be lifted by a jumper}. This configuration
Figure 4.19: PSquid ADC signal pathway for the standard (Int Amp) configuration. The gain of internal amplifier is given by Eq. (4.58). The standard configuration is for a 2-wire measurement of a SQUID series array, so S3B provides a bias current through the bias resistor $R_{bias}$. The dashed box indicates the internal amplifier. Alternative amplifiers may be used in place of the internal amplifier using the Ext Amp 8-pin header (J19), which would replace the dashed box. A 500 Ω output impedance current source can be switched in to “zap” the SQUID to heat it up.
lowing the pre-amplifier is an adjustable-gain amplifier with gain between -1 and -16384 and a 3-pole Bessel low-pass filter. In the standard configuration, the filter is not populated. Finally the 16-bit -2.048 V to 2.048 V ADC digitizes the signal at up to 20 kHz.

The SAIN line is clamped by diodes to less than about 1 V. Additionally, a current source (ZAP) may be switched in to SAIN. The purpose of the current source is to rapidly heat the load, which is useful in the case of magnetic flux becoming trapped in the SQUID series array’s superconducting magnetic shielding.

The Diff Amp choice of pre-amplifier uses an onboard instrumentation amplifier similar to the one used by TRead. The jumper JP14 switches the pre-amplifier gain between 1 and 100. It is the only choice of pre-amplifier that provides a differential input, however it does not connect to the S3B DAC and so is not suitable for SQUID measurements. It allows an external signal to be measured for implementation of a SISO control loop at 20 kHz. The pre-amplifier input comes in on a separate 2-pin header (J20). While the instrumentation amplifier is in use, the S3B DAC does not come out in a natural location, though it can still be taken from the external amplifier header J19 (see below).

The Ext Amp choice of pre-amplifier allows an arbitrary pre-amplifier to be implemented externally by connecting it to an 8-pin header with all of the relevant signals. The 8-pin header pinout may be found in Table 4.6. All signals necessary to replace the internal amplifier are present on J19. The purpose of this is to allow PSquid to be modified to have amplifiers that match the MCEs in the event that the MCE design changes, or to allow for if an alternative current return path has already been provided externally.
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<table>
<thead>
<tr>
<th>Pin</th>
<th>Signal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>AGND</td>
</tr>
<tr>
<td>2</td>
<td>V+</td>
</tr>
<tr>
<td>3</td>
<td>V-</td>
</tr>
<tr>
<td>4</td>
<td>AGND</td>
</tr>
<tr>
<td>5</td>
<td>S3B</td>
</tr>
<tr>
<td>6</td>
<td>Offset</td>
</tr>
<tr>
<td>7</td>
<td>Ext Amp In</td>
</tr>
<tr>
<td>8</td>
<td>Ext Amp Out</td>
</tr>
</tbody>
</table>

Table 4.6: PSquid External Amp 8-pin header (J19) pinout. Input and output are single-ended and referenced against AGND. S3B and Offset are controlled by their corresponding DACs and are single-ended referenced against AGND.

Improved amplifiers to be used as improved commercial amplifiers are developed. The remaining DACs come out of the board on 2-pin connectors. They also have optional 1-pole RC filters, optional series resistors, and may optionally have their ground lifted.

PSquid computes a new PID loop iteration every interrupt, so the interrupt rate determines the PID rate. The standard interrupt rate is 10 kHz, but this could be adjusted up to 20 kHz without consequence. Adjusting the interrupt rate requires updating the firmware (change `psquid.c`). Each interrupt performs the following steps:

1. Read ADC

2. *(If PID enabled)* Compute new PID value and update corresponding DAC

3. *(If sweep enabled)* Update sweep DAC

4. *(Data/Sweep/Raw modes)* Output data to serial port

5. Update asynchronous DAC

The ADC is read out at the beginning of every interrupt. The PID loop may be operated independent of any other operation of the board. If the PID loop is enabled, a new DAC
value will be computed from the measured ADC value and this value will be written to the PID DAC. Any of the 9 DACs (including Offset and Gain) may be used as the PID DAC. If the PID loop is disabled, this step is simply skipped.

Following that, PSquid will operate one of 4 exclusive modes. The simplest is the Do-Nothing mode. In this mode, no actuation is performed and no data is reported. This is the default mode, and so PSquid does not by default report data. The next is Data mode, in which no actuation is performed, but the values of the ADC and a user-chosen DAC are written to the serial port. Because of the volume of data involved, PSquid uses a dedicated serial port over fiber operating at 417 kbaud\(^{24}\). Even still, reporting this data every interrupt would overwhelm the serial port, so the data is accumulated over a user-selected number of interrupts and only the accumulated (summed, not averaged) value is reported. Data packets are reported back in pseudo-binary with the form

\[ \hat{D}D\hat{D}D \ A\hat{A}44\backslash r\n \]

where the \( \hat{\ } \) is a control symbol for synchronization, \( D\hat{D}\hat{D}D \) is the 32-bit accumulated DAC value in pure binary, and \( A\hat{A}44 \) is the 32-bit accumulated ADC value in pure binary. The DAC value is accumulated even though the DACs are controlled exactly by the PIC because the PID loop may be changing the DAC value during the accumulation period, so it is not

\[ \text{Baud Rate} = \frac{F_{CY}}{16(BRG + 1)} \quad (4.59) \]

where \( BRG \) is a non-zero integer and \( F_{CY} \) is the instruction rate. For the standard \( F_{CY} = 20 \text{ MHz} \), the PIC chip is capable of generating up to 1.25 Mbaud. However, we could not make this baud rate work on OS X.
guaranteed to be a constant value.

The third mode is Sweep mode in which a triangle wave output is applied to a user-selected DAC. The swept DAC, a user-selected DAC, and the ADC are reported back over the serial port with the form

```
^SS DDDD AAAA\r\n```

where the `^` is a control symbol, `SSSS` is the 16-bit swept DAC value in pure binary, `DDDDDDDD` is the 32-bit accumulated DAC value, and `AAAAAAAA` is the 32-bit accumulated ADC value. Note that the swept DAC value is only 16-bits because it is set exactly by the PIC and will not be changed until after the accumulation period, so each value during the accumulation period would be identical. The accumulation period is determined by the sweep rate. The 32-bit accumulated ADC and DAC values function identically to those of Data mode.

To specify a sweep, the user supplies the swept DAC start value, swept DAC end value, swept DAC step size, a settle period \( N_{\text{Settle}} \), an accumulation period \( N_{\text{Avg}} \), and a number of sweep periods. The swept DAC is immediately set to the start value. Every \( N_{\text{Settle}} + N_{\text{Avg}} \) interrupts, the step size will be added (or subtracted, for downward legs of the sweep) to the swept DAC value. To let the transient response settle out, the first \( N_{\text{Settle}} \) interrupts have their data discarded. The next \( N_{\text{Avg}} \) data points are accumulated and reported. If the next step would exceed the end point, the sweep reverses.

\[25\] We call it \( N_{\text{Avg}} \) rather than \( N_{\text{Accum}} \) because we wish to be consistent with the notation in the PSquid GUI, discussed below. The GUI divides by the accumulation period automatically to form the average.
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This continues until the number of sweep periods has been reached. The number of sweep periods may be set to 0 to trigger an indefinite triangle wave.

The final mode is Raw Data mode, which is a specialized version of Data mode. This mode is designed to output data at the full interrupt rate, so it limits its output to only a single ADC or DAC. If the PID loop is enabled, a user-specified (usually the locked) DAC is reported. Otherwise, the ADC is reported. The format of the reported packet in this mode is

\[ ^\mathrm{D}D \]

No accumulation is done in this mode, so only a 16-bit value $DD$ is reported, and packets are sent out at the full interrupt rate.

Note that in every data reporting mode, data is reported in binary. A character $^\mathrm{~}$ is a valid binary data value ( = 0x5e), so there is a degeneracy between the synchronization control character and the data value 0x5e. The data value is expected to change, so the data stream may be synchronized by matching repeated $^\mathrm{~}$ characters every $\text{FRAME}_\text{LEN}$ bytes. A Consistent-Overhead Byte Stuffing (COBS) scheme could be used to break this degeneracy, but we do not implement one here in order to simplify the PIC code.

These 4 modes (Do-Nothing, Data, Sweep, and Raw Data) are mutually exclusive, so changing from one mode to another will disable all other modes.

The final step in the interrupt handler is to update a DAC asynchronously. An array of the intended DAC values for the 9 DACs is stored. The user may only request that the DAC value be changed, which will update the array of intended DAC values. The board iterates
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through the DACs at a rate of one per interrupt and updates the DAC with the intended value. The user may not directly change a DAC value and the DAC is only guaranteed to be updated within 9 interrupts of the request.

PSquid requires $+5\,\text{V}$, $\pm15\,\text{V}$, and ground. It has separate analog ground (AGND) and digital ground (DGND) planes which may be connected onboard by populating R79. For low-noise operation, this is usually undesirable since the grounding scheme likely involves an external star point at which AGND and DGND are already connected. For laboratory use, the user must be careful to ensure that both AGND and DGND are connected appropriately.

4.4.1.1 PSquid Software

PSquid has been designed to be used with an external Python wrapper that translates higher level commands into the low level commands used by the board. Because of this, no convenience commands have been programmed into the firmware, so interacting with the board directly is challenging for an end user. Furthermore, directly interpreting and plotting the data reported by the board is essentially impossible for a user. We describe here the psquid.py Python wrapper and the psquidgui.py GUI. The architecture of the PSquid software is shown in Fig. 4.20. PSquid Wrapper

The PSquid wrapper comprises two main classes, PSquid and PSquidSerial. Since data is being transmitted at a high bandwidth, we construct a class dedicated to interacting with the serial port. All serial port drivers have a hardware-limited buffer size.
Figure 4.20: PSquid software block diagram. The PSquid board interacts to the computer over its serial port. The realtime class PSquidSerial collects data from the serial port and saves it to a Datafile, and passes on commands from the asynchronous PSquid wrapper class. The datafile is read by the figure generator PyOscope. The GUI windows are constructed by the PSquidApp class. GUI actions are bound to the PSquid wrapper and the figure generator by a Binder class. The psquidgui.py script initializes all of the components, binds the actions, and starts the event handling loop.
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Most operating systems implement a software buffer to expand the buffer size, but this buffer may be only 4 kB large. At a data rate of up to 60 kBps, this buffer could fill up in 67 ms. If the buffer is not emptied in this time, data will be discarded. To avoid this, the PSquidSerial class sets up a separate thread and empties the serial buffer to memory every 10 ms. A separate thread is useful because its responsivity depends only on the OS scheduler and is independent of the load on the main thread.

PSquidSerial is a state machine that keeps track of which mode the board is in (default, Data, Sweep, Raw Data) and if in a data reporting mode, dumps the data to a file on the hard disk. Any other process may then read the data from the file, e.g. to plot it. Transition functions clean up the file headers and open a new file as necessary. A realtime flag is included in PSquidSerial which determines if data is written to the hard drive immediately or if it should wait until a full block (where the size of a full block is determined by the Python write buffer size) before writing out to the disk. Since a write-to-disk operation is expensive, realtime mode should not be used in high data rate modes (e.g. Raw Data mode), but it is useful for low data rate modes (e.g. a Data mode with large $N_{Avg}$) to make the data recording more responsive.

A lock on the PSquidSerial data buffer ensures that the data cannot be corrupted. It acquires its own lock when it transfers data from the serial buffer to its internal buffer. PSquidSerial also exposes a set of minimally blocking read methods to transfer data from its internal buffer to a processing thread. The methods acquire the lock only for the

---

26 A good modification would be to write the data to an HD5 array.
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duration required to transfer out the data and then immediately give it up. Since only a single thread may hold the lock at any given time, the lock guarantees the thread-safety of the reads and writes.

The user is not expected to interact with PSquidSerial directly. For user interaction, the class PSquid is supplied. The PSquid class supplies high level commands to all of the PSquid board functionality. It handles the conversion between voltage units and DAC units, looks up DAC channel indices by name, selects the locked DAC by default for data reporting modes, organizes named or temporary files for data recording, and converts natural gain units to Gain DAC units. For all of its commands it also verifies that the arguments are valid. It also monitors the PSquidSerial state for reporting purposes.

The PSquid operates asynchronously in the primary thread. Its commands to the PSQuid board are made through PSquidSerial using the thread-safe blocking methods.

PyOscope

The data file is read, interpreted and plotted by a separate plotting module called pyoscope. The module pyoscope is an interactive matplotlib figure generator that is designed to read data files from disk and generate figures with standard plotting features that is trivially compatible with embedding into an external application.

The pyoscope module implements two main classes, PyOscopeStatic and PyOscopeRealtime. The first facilitates the interactive figure generation features of pyoscope. Particularly, it streamlines the creation of figures from a particular static dataset. The more useful class is PyOscopeRealtime, which subclasses PyOscopeStatic, and pro-
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provides additional functionality to generate dynamic figures from a changing dataset. The

PyOscopeRealtime class is aliased as PyOscope.

`pyoscope` may use either an external event handling loop or the built-in `matplotlib` event handling loop. To update a plot, the event handling loop simply needs to call the `PyOscope.update` method. The class will then efficiently update the figure and serve the updated figure in the attribute `PyOscope.fig`. Note that `pyoscope` is designed to determine which plotting backend (e.g. wxagg, qt, macosx, etc.) is used and use an efficient update scheme, however few schemes have been implemented. A slower backend-independent scheme is used as backup for cases where the backend-specific update has not been implemented.

The datafile is read in using a reader class. A `pyoscope`-compatible reader class must be created for every datafile format that is to be read. Each reader class must implement a common set of methods that will be used by `pyoscope`. Details may be found in `ReaderInterface` class docstring.

**PSquidApp**

The `PSquidApp` is a set of GUI windows written in wxPython. It comprises a plot panel, a manual DAC control panel, and a tabbed tool panel. The plot panel displays figures made by `pyoscope`. The DAC control panel houses 9 sliders that control the values of each of the 9 DACs available on PSquid. It additionally has corresponding text boxes that allow the DAC values to be controlled more precisely or to display the numerical value of the current setting.
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The tabbed tool panel is the primary way for the user to command the PSquid. It houses a serial of tool panels in separate tabs corresponding to the various functions of the board. The Communications (Comm) panel configures the program to connect to the board. The Monitor (Mon) panel configures and enables/disables the Data mode of PSquid. The Sweep panel configures and enables/disables the Sweep mode of PSquid. Note that the Sweep and Data modes are mutually exclusive, so turning on one will necessarily turn off the other. The Lock panels configure the PID loop. Lock (M) is a manual lock tool in which the user manually inputs the PID parameters. Lock (G) is an experimental graphical lock tool in which the user interacts with the plot panel to determine what the PID parameters should be.

Note that PSquidApp only creates the windows and controls, but does not assign actions to those controls. This is done to maximize the modularity of the code and increase its robustness. The binding of actions to the controls is performed by the Binder class.

Binder

The Binder class assigns all of the logic to the windows and controls. It specifies how each control should interact with the PSquid wrapper and in turn how the windows should respond to data from PSquid. It also coordinates the creation of data files by PSquid with the reading and plotting of data by pyoscope.

The bindings are pushed out to a separate class rather than integrated into PSquidApp because it allows for most robust code. One may construct the GUI windows without binding the controls to verify that they are being constructed correctly. Additionally, one
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may modify the bindings to change the behavior without worrying about corrupting the
window creation. Lastly, the windows and logic are in separate locations so one knows
where to go to modify one or the other.

psquidgui.py

The psquidgui.py script creates all of the GUI windows, initializes the pyoscope
and PSwid instances, and binds them all together. In OS X it should be invoked as

pythonw psquidgui.py

but in other systems it may be invoked with the standard python binary.

4.4.2 Gyro Board

The Gyro Board biases a pair of LPY403AL evaluation boards, which are 2-axis analog
MEMs gyro chips. The evaluation boards were chosen over the bare chips because they
had a more convenient package. A 3 V regulated power line supplies a stable source of
power to the chips.

Each chip measures the rotation rate about axes normal to the chip and across the chip
(from sockets to sockets). The two chips are mounted rotated 90 degrees relative to each
other about the board’s normal axis. In this configuration, the rotation about the axis normal
to the board (designated \( z \)) is measured by both LP403AL chips, and each chip then takes
one of \( x \) and \( y \). In total, \( x \) and \( y \) are measured once while \( z \) is measured twice. The resulting
analog signals are put out on an 8-pin connector (J2) and must be read by an external device
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(assumed to be PSyncADC).

The 4× output modes of the LPY403AL are used, which give a rate sensitivity of 33.3 mV/dps. The output of the chip ranges from 0.5 V to 2.5 V, with zero rotation corresponding to 1.5 V. This gives a measurement range of ±30 dps. The chip output is passed through a $G = 10$ differential amplifier with the negative terminal referenced to the reference voltage $V_{\text{ref}} = 1.5$ V. The differential amplifier has a tunable offset that can ensure that 0 V output corresponds to 0 dps. The output of the amplifier has a sensitivity of 333 mV/dps and an output range of -10 V to +10 V. The trim resistances are controlled by trim pots on the front of the board.

Following the differential amplifier is a 2-pole butterworth low-pass filter with $f_{\text{3dB}} = 9$ Hz. The LPY403AL chip has an internal bandwidth of 140 Hz and a rate noise density of 0.01 dps/√Hz, so the low-pass filter significantly cuts the signal and noise bandwidths. The Gyro Boards are expected to measure the rotation of the payload, which is not expected to change faster than 1 Hz, so this reduction is advantageous.

The LPY403AL chip was characterized on a prototype board. The spectrum (Fig. 4.21) is white with a $1/f$ knee near $10^{-2}$ Hz. The noise density rolls off around 11 Hz due to the 11 Hz low-pass filter on the prototype board. The white noise level is measured to be 0.015 dps/√Hz, slightly larger than the value expected for the chip.
Figure 4.21: LPY403AL rate noise density spectrum in units of degrees per second per root Hertz. The top plot shows a 1024-point binned spectrum. The spectrum is rolled off by the low-pass filter (11 Hz). The bottom plot shows the same spectrum with no binning and shows a $1/f$ knee near $10^{-2}$ Hz. The white noise level is $0.015 \text{ dps}/\sqrt{\text{Hz}}$. This data was collected using a prototype of the Gyro Board which had a different amplifier and filter.
4.4.3 Hall3D

The Hall3D board is a cryogenic 3-axis magnetometer board that combines 3 nominally room-temperature Lakeshore HGT-2101 single-axis Hall effect magnetometers to make a 3-axis magnetometer board. The HGT-2101 chips are simple 4-wire devices\footnote{Note that HGT-2101 is a true 4-wire devices, unlike a resistor. The I- and V- must not be shorted together, and the I+ and V+ must not be shorted together, or else the device will not function.} that may be read out by any bridge. They have a nominal input impedance of $450 - 900 \Omega$, a nominal bias current of 1 mA, and a nominal minimum operating temperature of $-40 \degree C$. The chips are magnetically sensitive normal to the surface of the chip.

3 chips are mounted on each board in 3 different orientations. Since the chips are sensitive to the normal direction, this means that we must stand up 2 of the chips 90 degrees on edge to measure the non-normal directions (Fig.4.22). Each chip is accompanied by a 4-pin microdot connector that serves as the only connection to the chip. In addition to the magnetometer chips, a temperature diode and a heater resistor are included on the board. They may be used to monitor or control the temperature of the board. A thermally isolated G-10 bridge is included on the board between the mounting screws and the magnetometer region in order to allow a large temperature gradient to be formed. If desired, the operational section of the board could be heated to the nominal temperature range of the magnetometers with a manageable power loading on the cold bath. Vias for brass jumper wires are included on either end of the bridge to control the thermal conductivity. However, we have found that this is unnecessary. The magnetometers have been tested to be functional down to 100 mK.
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Figure 4.22: A populated Hall3D board. The $x$-axis and $y$-axis chips are stood up on edge to provide sensitivity to the two planar axes. All devices (magnetometers, diode, resistor) are individually output on 4-pin microdot connectors.

All 3 chips on a single Hall3D board were calibrated at both 300 K and 77 K with a 39.9 Gauss/Amp Helmholtz coil with 28.8 cm diameter. The Helmholtz coil was initially calibrated using an integrated COTS Lakeshore magnetometer and matches the computed value. The chips were biased and read out using a TRead LR bridge with a gain of 16 and nominal excitation current of $400 \mu A$. The raw demodulated output $D$ of the TRead will be linear in applied magnetic field for a linear magnetometer. We used the raw demodulated output since the resistance measurement is nonlinear in the voltage response (Sec. 4.3.2) and thus nonlinear in the applied magnetic field. The sensitivity at 300 K was -4700 counts/Gauss and at 77 K was -5900 counts/Gauss (Fig. 4.23). The measurement noise was 340 counts, corresponding to 72 mgauss at 300 K and 57 mgauss at 77 K. The Hall3D board was cooled to 77 K by immersing it in a LN2 bath. The sensitivities could be
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Figure 4.23: Calibration of a single HGT-2101 magnetometer chip at 300 K and 77 K. The vertical axis is raw demodulated counts from the TRead.LR. The TRead.LR board was set to have a gain of 16 and a nominal excitation current of 400 \( \mu \text{A} \).

related to resistance measurements, but we do not do that here. Lastly we note that the sensitivity depends on the bias current, so if more sensitivity is required a larger bias current may be used, at the cost of additional power dissipation. We note that the sensitivity of the magnetometers increases with decreasing temperature. However, the sensors have a transient response as the temperature changes that must be allowed to settle (Fig. 4.24). The board requires about 2 hours to fully settle. Anecdotally we do not see significant transients below 77 K, so we theorize that the transients are caused by internal mechanical stresses in the film. For a standard cooldown in a dewar via cryocooler or liquid cryogens, the thermal time constant of the cryogenic system will dominate and the HGT-2101 transient will be unnoticeable. The variation in the response across the 3 chips is insignificant, so we use a single chip as the representative.
Figure 4.24: Transient response of the HGT-2101 magnetometers as the chip is cooled from 300 K to 77 K. The transient response of the chip thermalizations over a period of about 2 hours. The steps at 15:30 and 18:30 form the basis of the 300 K and 77 K calibrations. The coil current is not shown.
Chapter 5

Simulations

Simulation is a powerful tool for answering questions about the instrument. We will make simulations of the measurements in the context of 2 questions.

Frequency Band Optimization

PIPER will measure the full sky in 4 frequency bands over a series of 8 flights. Each flight will be sensitive to a single frequency band. With a flight cadence of 1 flight every 6 months, it will take a full 4 years to collect data in all frequency bands and achieve the full science goals of PIPER. However, a subset of the data could be sufficient to make a detection of $r$. We will analyze the ability of different subsets of PIPER data to constrain the B-mode spectrum. Such knowledge may inform the order with which PIPER flies its frequency bands, i.e. which subsets of data are collected first.

Calibration Gain Errors

We have previously discussed how the VPM mitigates cross-polarization errors. How-
ever, there is still a possibility of long time-scale drifts in the gain of the instrument. Locally
the Stokes parameters will be measured correctly. However, for regions farther away the
slow drifts may incorrectly bias the relative amplitudes of the regions. This may appear
as B modes where there are none. We will analyze the effect of a spatial pattern of gain
miscalibrations on the measured spectrum. This facilitates developing a specification on
the gain stability.

5.1 Simulation Strategy

The general strategy for performing these simulations is the same regardless of the
question to be answered. An underlying cosmology is chosen and the statistical properties
of the CMB are computed. These statistical properties are used to generate an ensemble
of simulated CMB maps. To each map a foreground model is added (in map space). In
addition to a foreground model, a random realization of the instrument noise is added to
the map to generate a simulated instrument map. A foreground removal algorithm is used
to clean the simulated instrument map of the foregrounds. The post-cleaning simulated
CMB sky is used to estimate the statistics properties of the cosmology, i.e. a simulated
measurement of the power spectra. The ensemble of these measurements is combined to
provide a statistical estimate of the power spectrum that would be measured. Finally, this
estimate is compared against the input spectrum used to generate the simulated CMB maps
to get a sense of the biases and constraining power of the instrument.
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Figure 5.1: The simulation strategy.
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The effects of the phenomenon we are interested in simulating the effect of are injected
into appropriate spots in this chain. This allows the comparison of the final simulated CMB
maps to the input maps to quantify the effect of the phenomenon. The simulation strategy
is shown in Fig. 5.1.

5.1.1 Cosmology Simulation

The power spectra $C_{\ell}^{XX}$ are generated from an underlying cosmology by the CLASS\textsuperscript{58} simulation code\textsuperscript{1}. For the questions we are interested in answering, the particular underly-
ing cosmology is not significant. We only require a cosmology that is similar to our real
cosmology. We choose a generic $\Lambda$CDM cosmology with

$$
\begin{align*}
  h & = 0.7 \\
  T_{\text{CMB}} & = 2.726 \text{ K} \\
  \Omega_b & = 0.05 \\
  N_{\text{ur}} & = 3.046 \\
  \Omega_{\text{CDM}} & = 0.25 \\
  \Omega_K & = 0 \\
  z_{\text{re}} & = 10 \\
  A_s & = 2.3 \times 10^{-9} \\
  n_s & = 1.0 \\
  r & = 0
\end{align*}
$$

where we note that our cosmology has $C_{\ell}^{\text{BB}} = 0$ if we exclude lensing. Thus any measure-
ment of B-modes from our simulations are created by the effect we are studying.

CLASS will produce $C_{\ell}^{XX}$ with and without lensing contributions. We are primarily
interested in large scales at which lensing is small, so we use the unlensed spectra. Only a

\textsuperscript{1}http://class-code.net
single set of $C_{\ell}^{XX}$ are required since they describe the statistical properties of a CMB sky. Realizations of a CMB may be sampled from this set of $C_{\ell}^{XX}$.

### 5.1.2 CMB Simulation

The simulated skies are represented in the HealPix\(^{[59]}\) coordinate system and generated by the healpy module, a Python wrapper around the HealPix Fortran code. We produce $M$ (order of 100s) realizations of a CMB sky in map space from the power spectra with the synfast algorithm. The resulting skies are in CMB temperature units $K_{\text{CMB}}$ discussed in Sec. 1.6.1.

### 5.1.3 Foreground Simulation

At high frequencies and large angular scales, dust dominates the foreground (Fig. 1.4) in intensity. With this in mind, we will begin by constructing a foreground map that comprises only dust.

Let us model the thermal dust emission as a power law,

$$I_\nu(p) = I_{\nu_0}(p) \left( \frac{\nu}{\nu_0} \right)^\beta$$

(5.1)

where $I_\nu$ is the spectral intensity in MJy/sr, $I_{\nu_0}$ is some reference amplitude map at the reference frequency $\nu_0$, and $\beta$ is the spectral index. So given a map $I_{\nu_0}(p)$ at frequency $\nu_0$, we may construct a map of the thermal dust emission at an arbitrary frequency by scaling it

\[^{[59]}\text{http://healpy.readthedocs.org/}\]
by \( \left( \frac{\nu}{\nu_0} \right)^\beta \). We ignore more sophisticated models that involve modeling the dust as particles at a particular temperature with a particular emissivity. We use the Planck thermal dust component map\(^3\) at \( \nu_0 = 353 \) GHz as our reference map (Fig. 5.2).

Figure 5.2: Planck thermal dust emission component map (from Commander-Ruler algorithm) at 353 GHz. From COM.CompMap_dust-commrul_0256_R1.00.fits intensity field. Histogram is equalized. Pixels that had a negative value in the original map have had their value replaced with 0.

Some pixels in the Planck dust emission map have a negative intensity. These can potentially cause problems and are not physically realizable, so we replace the value of such pixels with 0.

\(^3\)This map is produced by the Planck team by using a parameterized CMB + Foreground model and an MCMC solver to minimize the \( \chi^2 \) of the model given the data, the Commander-Ruler algorithm. The map generated by only one component of the model, using the optimal parameters, is the component map. In particular, we use only the intensity (I) component of the \( N_{\text{side}} = 256 \) map.

The map is available from the Planck Legacy Archive under Maps → Foreground maps → Dust → COM.CompMap_dust-commrul_0256_R1.00.fits.
we note that our power law is not in thermodynamic temperature units, but is rather in MJy/sr. This is incompatible with our sky maps, which are typically in CMB temperature units ($K_{CMB}$). The conversion between spectral intensity $I_\nu$ and thermodynamic temperature $T$ is

$$I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/k_BT) - 1} \tag{5.2}$$

However, we already have a temperature reference, the CMB at $T = 2.72548 \text{ K}$\textsuperscript{16}. So rather than treat the foregrounds as a separate temperature, we treat it as a perturbation of the CMB temperature, i.e.

$$I_{CMB} + I_{fg} = B_\nu(T_{CMB} + \Delta T) = B_\nu(T_{CMB}) + \left(\frac{\partial B_\nu}{\partial T}\right)_{T_{CMB}} \Delta T \tag{5.3}$$

to first order. We identify $I_{CMB} = B_\nu(T_{CMB})$ as the relation describing the thermodynamic temperature of the CMB, which leaves us with the relationship between the foreground spectral intensity in spectral intensity units and CMB temperature units,

$$I_{fg} = \left(\frac{\partial B_\nu}{\partial T}\right)_{T_{CMB}} \Delta T = \left(\frac{2h^2\nu^4}{k_Bc^2T_{CMB}^2} \frac{1}{\exp \left(\frac{h\nu}{k_BT_{CMB}}\right)} \left[\exp \left(\frac{h\nu}{k_BT_{CMB}}\right) - 1\right]^2\right) \Delta T \tag{5.4}$$

We note that $\left(\frac{\partial B_\nu}{\partial T}\right)_{T_{CMB}}$ is still a function of $\nu$, so the conversion will still depend on the frequency band of interest. We also note that the power law scaling of thermal dust intensity (Eq. (5.1)) cannot easily be written in CMB temperature units, since the conversion factor scales with frequency. The power law scaling should be done in spectral intensity units and then converted to CMB temperature units.
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Some conversion factors for our frequencies of interest,

\[
\left( \frac{\partial B_\nu}{\partial T} \right)_{T_{\text{CMB}}} \bigg| (200 \, \text{GHz}) = 478 \, \frac{\text{MJy/sr}}{\text{K}_{\text{CMB}}}
\]

\[
\left( \frac{\partial B_\nu}{\partial T} \right)_{T_{\text{CMB}}} \bigg| (270 \, \text{GHz}) = 444 \, \frac{\text{MJy/sr}}{\text{K}_{\text{CMB}}}
\]

\[
\left( \frac{\partial B_\nu}{\partial T} \right)_{T_{\text{CMB}}} \bigg| (350 \, \text{GHz}) = 302 \, \frac{\text{MJy/sr}}{\text{K}_{\text{CMB}}}
\]

\[
\left( \frac{\partial B_\nu}{\partial T} \right)_{T_{\text{CMB}}} \bigg| (600 \, \text{GHz}) = 32 \, \frac{\text{MJy/sr}}{\text{K}_{\text{CMB}}}
\]

The CMB polarization is converted into CMB temperature units in a similar way (Sec. 1.6.1), so CMB temperature units are a common basis for CMB polarization fluctuations and foregrounds.

5.1.3.1 Polarized Dust Intensity

As a simple estimate of the polarized dust intensity, let us suppose that there is a constant polarization fraction of the thermal dust intensity,

\[
p = \frac{I_p}{I} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}
\]

(5.5)

where \( I_p \) is the polarized intensity, \( Q, U, \) and \( V \) are the polarized components of the Stokes vector, and \( I \) is the total intensity. Planck estimates that maximum polarization fraction is \( p_{\text{max}} = 20\% \), so we will use that as a pessimistic limit across the whole sky. In reality, we expect the polarization fraction to be smaller than this, especially in the galactic
plane where Planck reports the polarization fraction to typically be closer to 5%.

We may then construct a polarized intensity map

\[ I_p(p) = p_{\text{max}} I_{\nu}(p) \]  

(5.6)

where \( p \) here is the pixel index and \( p_{\text{max}} = 20\% \) is the maximum polarization fraction.

Figure 5.3: Naive polarized dust intensity \( I_p = pI \). Uses a constant polarization fraction \( p = p_{\text{max}} = 0.2 \) to construct a map from the thermal dust intensity map.

5.1.3.2 Polarized Dust Components

The polarized dust intensity map does not contain all of the information about the polarized radiation. We note from the definition of \( I_p \),
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\[
I_p = \sqrt{Q^2 + U^2 + V^2}
\]  

(5.7)

that at each frequency and in each pixel, we must specify 3 numbers \(Q, U,\) and \(V\) (equivalently, the E-field vector components in some coordinate system, \(E_x\) and \(E_y,\) and the phase between the E-field components, \(\phi \equiv \theta_x - \theta_y\)) to fully specify the polarized light.

We assume that the circularly polarized component \(V\) is small so we will set it to 0 and ignore it. This leaves us 2 numbers that we need to specify.

**Synchrotron-tracking Polarization Direction Model**

Both synchrotron\(^{27}\) and dust\(^{60}\) depend on the interstellar magnetic field in which their photons originate. If both phenomena produce radiation from the same region then they are coupled to the same magnetic field, and hence we would expect some correlation between the synchrotron and dust components. In particular, the synchrotron polarization angle should be a tracer of the dust polarization angle.\(^{33}\)

To this end, we use the WMAP 23 GHz synchrotron component \(Q\) and \(U\) maps and estimate from them the polarization angle,

\[
\gamma = \frac{1}{2} \arctan (-U, Q)
\]

(5.8)

where we follow the convention of Delabrouille et al. for the orientation of the Stokes parameters. Following Delabrouille et al., we smooth the \(Q\) and \(U\) maps to 3° before
computing the polarization angle map $\gamma(p)$. We note that the polarization angle map is independent of frequency.

These angle maps may then be applied to the polarization intensity maps of Section 5.1.3.1 to get the $Q$ and $U$ components of the polarized dust,

$$Q(p) = I_p(p) \cos 2\gamma(p) = p_{\text{max}} I_\nu(p) \cos 2\gamma(p)$$ (5.9a)

$$U(p) = I_p(p) \sin 2\gamma(p) = p_{\text{max}} I_\nu(p) \sin 2\gamma(p).$$ (5.9b)

The resulting polarization angle map $\gamma(p)$ and example $Q$ and $U$ dust foreground maps at 353 GHz are shown in Figure 5.4.

We note that this smoothing procedure suppresses power at angular scales smaller than $3^\circ$, corresponding to supressing power at $\ell \gtrsim 60$. A Planck analysis of mid-latitude dust at 353 GHz suggests that there is still a significant amount of power on these scales. One possible method of incorporating this would be to simply extend the power spectrum of our dust maps using the power law scaling found by Planck,

$$D_\ell^{XX} = A^{XX} (\ell/80)^{\alpha_{XX}+2}$$ (5.10)

where $\alpha_{XX} = -2.42 \pm 0.02$, $X \in \{E, B\}$, and then reconstructing the map from the power spectra. We do not use this technique for these simulations.
Figure 5.4: *Top*: The polarization angle map $\gamma(p)$. The map is smoothed to $3^\circ$. *Middle*: The resulting polarized dust foreground $Q$ map at 353 GHz. *Bottom*: The resulting polarized dust foreground $U$ map at 353 GHz.
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5.1.4 Instrument Noise

We describe here how to estimate the contribution of instrument noise to the maps. Our ultimate goal is to start with the noise properties of the instrument and end with the instrument noise contribution to each HealPIX pixel.

5.1.4.1 NEP

Detector noise is most conveniently quoted in noise-equivalent power (NEP), which is defined as the amount of optical power that must be incident such that the signal-to-noise ratio (S/N) is 1. It is a measure of noise in optical power units. Note that NEP may be referenced at various different points in the instrument, e.g. inside the detector, incident on the detector, incident at the aperture of the telescope. The noise (N) component is intrinsic to the detector, and so is always referenced to inside the detector. Thus, the S/N = 1 condition must always be enforced inside the detector. For NEPs referenced at some other point, the signal (S) component must first be transferred to inside the detector, and then compared. Explicitly,

**NEP (strict definition)**

The amount of optical signal power incident at the reference point that creates a signal-to-noise ratio of 1 (S/N = 1) as measured inside of the detector.

From Richards,[44] we note that the noise in a bolometer is

\[
\frac{P_N^2}{B} = 2 \int d\nu h^2 \nu^2 2N (n + n^2) = 2 \int d\nu \nu P_\nu h \nu + \int d\nu P_\nu \frac{c^2}{A \Omega \nu^2} \quad \left[ \frac{W^2}{Hz} \right] \quad (5.11)
\]
where the integral is taken over the passband and the first term on the right hand side corresponds to \( n \) and the second term corresponds to \( n^2 \). \( B \) is the detector bandwidth, \( N \) is the number of modes \( (N = A\Omega/\lambda^2, \text{from the Antenna Equation}) \), \( n \) is the number of photons per mode, the 2 in front of the first integral comes from the conversion between integration time and bandwidth (see below and Appendix D), and the 2 inside the integral comes from the 2 polarization states per photon. The energy per photon is \( h\nu \). The term \( n + n^2 \) is the thermal expectation value for the variation in the number of photons per mode, \( \langle (\Delta n)^2 \rangle = n + n^2 \).

We note that the number of photons per mode is

\[
n = \frac{1}{\exp \left( \frac{h\nu}{k_B T} \right) - 1}
\]

where \( T \) references the signal (sky) temperature, which is the temperature of the CMB in this case, \( T = 2.726 \text{ K} \). For the 4 PIPER frequency bands (200, 270, 350, 600 GHz), this gives photon numbers \( n \)

\[
\begin{align*}
n &= 3 \times 10^{-2} \text{ for } \nu = 200 \text{ GHz} \\
9 \times 10^{-3} \text{ for } \nu = 270 \text{ GHz} \\
2 \times 10^{-3} \text{ for } \nu = 350 \text{ GHz} \\
3 \times 10^{-5} \text{ for } \nu = 600 \text{ GHz}
\end{align*}
\]

We note that \( n \ll 1 \) for all frequency bands, so \( n^2 \ll n \), and we may ignore the second term in Eq. (5.11). So we use for the noise power,

\[
\frac{P^2_N}{B} = 2 \int d\nu P_\nu h\nu \left[ \frac{W^2}{\text{Hz}} \right]
\]
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where \( P_\nu \) is the power spectral density.

Let us compute the NEP inside the detector using this expression. The NEP is defined as the amount of power required to give a signal to noise ratio of 1 when there is 1 Hz of noise bandwidth. Note that the signal bandwidth is already implicitly integrated out of this expression. So we have,

\[
\frac{S}{N} = \frac{\text{NEP}}{P_N(1\text{ Hz})} = 1
\]

\[
\text{NEP} = P_N(1\text{ Hz})
\]

\[
\text{NEP}^2 = (1\text{ Hz}) \cdot \frac{P_N^2}{B}
\]

Note that in this scenario, NEP has units of W, which strictly matches the definition given above. However, it is conventional to fold the 1 Hz of noise bandwidth back into the definition of NEP to remind what the scaling with bandwidth is and so that a factor of 1 Hz does not need to be carried around,

\[
\text{NEP}_{\text{conventional}}^2 = \frac{\text{NEP}^2}{1\text{ Hz}} = \frac{P_N^2}{B}
\]

where the explicit definition of conventional NEP is

**NEP (conventional)**

The amount of optical signal power incident at the reference point that creates a signal-to-noise ratio of 1 (\( S/N = 1 \)) as measured inside of the detector, all divided by \( 1\text{ Hz}^{1/2} \).

Henceforth, we will use only the conventional NEP and drop the subscript.
Let us now consider a system with detector absorption efficiency $\eta$ and optical efficiency $\tau$, where $0 < \eta, \varepsilon, \tau \leq 1$. These parameters make no difference to the NEP referenced inside the detector.

We will compute the NEP referenced to power incident on the detector. The signal power incident to the detector will produce an amount of power inside the detector that is reduced by the factor $\eta$. So the NEP condition is then

$$\frac{\eta \cdot \text{NEP}_D \sqrt{B}}{P_N} = 1$$

and we see that the NEP at the detector input is

$$\text{NEP}_D^2 = \frac{1}{\eta^2} \frac{P_N^2}{B} = \frac{2}{\eta^2} \int d\nu P_\nu h\nu$$  \hspace{1cm} (5.13)

Similarly, the NEP referenced to power incident on the primary of the telescope is determined by noting that the power at the primary must pass through the optical system, with losses according to the optical efficiency $\tau$, and then be absorbed by the detector. So our NEP condition is

$$\frac{\eta \tau \cdot \text{NEP}_P \sqrt{B}}{P_N} = 1$$

and the NEP at the telescope input is

$$\text{NEP}_P^2 = \frac{1}{\eta^2 \tau^2} \frac{P_N^2}{B} = \frac{2}{\eta^2 \tau^2} \int d\nu P_\nu h\nu$$  \hspace{1cm} (5.14)

We will assume that the NEP inside the detector is known, as it is typically measured independently.

$^4$Alternatively called detector absorptivity.
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As a final note, most of the transformations in this section do not tell us how to calculate the noise power of the bolometer, which is an intrinsic aspect of the bolometer itself and should be computed from the perspective of heat inside the detector. It is a property of the detector that is independent of the optical loading, since optical power is converted to heat, and is interchangeable with all other sources of heat in the detector (such as electrical). Rather, what these transformations tell us is how to translate the intrinsic noise in the detector (encoded as NEP inside the detector) to reference points outside of the detector. Since the power is transported as photons coming out of the detector (in the time-reversed sense), we must understand the properties of the photons and how they transport power in order to understand how the detector NEP translates to noise at other points. That is the purpose of this section.

However, in addition to intrinsic detector noise, there is also intrinsic noise in the signal from the sky in what is called photon noise. Photon noise can be modeled by using $P_\nu = A\Omega \eta \tau B_\nu(T)$,

$$\text{NEP}_{\text{photon}}^2 = \left(\frac{2}{1 \text{ Hz}}\right) A\Omega \eta \tau \left(\frac{2h^2}{c^2}\right) \int_{\Delta\nu} d\nu \frac{\nu^4}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} \left[\frac{W^2}{\text{Hz}}\right]$$  \hspace{1cm} (5.15)

$$\text{NEP}_{\text{photon}}^2 = \left(\frac{2}{1 \text{ Hz}}\right) A\Omega \eta \tau \left(\frac{2h^2}{c^2}\right) \left(\frac{k_BT}{h}\right)^5 \int_{x_1}^{x_2} dx \frac{x^4}{e^x - 1} \left[\frac{W^2}{\text{Hz}}\right]$$  \hspace{1cm} (5.16)

where $x = \frac{h\nu}{k_B T}$. See below for details of the transformation.

Since it originates from the sky, it must be treated slightly differently. For a single detector there is no difference, but for many detectors the optical system can correlate pixels. This is impossible for noise intrinsic to the detector since each detector pixel is an
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independent device. Readout noise could correlate different pixels to each other, but we do not consider this case here.

5.1.4.2 NET, NEQ, NEU, NEV

The first quantity we will examine is the noise-equivalent temperature (NET). This is the same as the NEP, except in CMB temperature units ($K_{CMB}$). It is defined as the change in thermodynamic temperature of a reference $T = T_{CMB}$ blackbody at the reference point that would generate an amount of optical power in the detector such that the S/N is 1. Note that this definition implicitly includes a fair amount of information about the instrument, such as the etendue, the optical efficiency, and the spectral bandwidth. This information is required since we must know how a change in the temperature of the sky’s photons propagate to the detector, and the path of propagation is through the instrument. Also note that we must assume a base sky temperature since the spectral distribution of the sky is temperature-dependent.

The NET is defined relative to the NEP by

$$\text{NET} = \text{NEP} \left( \frac{dP}{dT} \right) \left[ \frac{K_{CMB}}{\sqrt{Hz}} \right]$$ (5.17)

where $\frac{dP}{dT}$ is the change in optical power incident on the detector due to a change in sky temperature. This is straight-forward to compute if we first find the power incident on the detector, which is (assuming a thermal source)

$$P(T) = \int_{\Delta\nu} d\nu \eta(\nu)\tau(\nu)P_\nu(T) = A\Omega \int_{\Delta\nu} d\nu \eta(\nu)\tau(\nu)B_\nu(T) \quad \text{[W]}$$ (5.18)
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where we have implicitly defined \( P_\nu = \eta \tau A \Omega B_\nu(T) \) as the spectral power density, \( \Delta \nu \) is the spectral passband, and \( B_\nu(T) \) is the Planck distribution, which we note already includes both polarization modes,

\[
B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}.
\]

Then \( \frac{dP}{dT} \) is given by

\[
\frac{dP}{dT} = A \Omega \int_{\Delta \nu} d\nu \eta(\nu) \tau(\nu) \frac{dB_\nu(T)}{dT}
\]

The derivative of the Planck distribution can be shown to be

\[
\frac{dB_\nu(T)}{dT} = \frac{2h^2}{c^2 k_B T^2} \frac{\nu^4 \exp\left(\frac{h\nu}{k_B T}\right)}{\left[\exp\left(\frac{h\nu}{k_B T}\right) - 1\right]^2}
\]

so

\[
\frac{dP}{dT} = A \Omega \int_{\Delta \nu} d\nu \eta(\nu) \frac{2h^2}{c^2 k_B T^2} \frac{\nu^4 \exp\left(\frac{h\nu}{k_B T}\right)}{\left[\exp\left(\frac{h\nu}{k_B T}\right) - 1\right]^2}
\]

which may be put in the more numerically convenient form by using the substitution \( x = \frac{h}{k_B T} \nu \),

\[
\frac{dP}{dT} = A \Omega \frac{2 k_B}{c^2} \left(\frac{k_B T}{h}\right)^3 \int_{x_1}^{x_2} dx \eta(x) \tau(x) \frac{x^4 e^x}{(e^x - 1)^2}
\]

The integral does not have a nice algebraic solution even if \( \eta(x) \tau(x) \) is constant, but is amenable to quadrature for realistic bandpass functions \( \eta(x) \). We note, as discussed above, that the etendue \( (A \Omega) \), the band-pass, the efficiency, and the sky temperature are involved in this conversion factor.

The units of \( P \) is \( \text{W} \), so \( \left[ \frac{dP}{dT} \right] = \frac{\text{W}}{\sqrt{\text{Hz}}} \), and

\[
[\text{NET}] = \left[ \frac{\text{NEP}}{\left( \frac{dP}{dT} \right)} \right] = \frac{\text{W}}{\sqrt{\text{Hz}}} \frac{\text{K}}{\text{W}} = \frac{\text{K}}{\sqrt{\text{Hz}}}
\]
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However the conventional unit for NET is $K_{\text{CMB}}\sqrt{s}$, for which the procedure to get the noise figure in K is to divide by the square root of the integration time, i.e. more integration time results in less noise. This is conceptually intuitive, since one would expect that the measurement from each period of time would be independent, so the number of independent samples would go like $f_{\text{sample}} \cdot T_{\text{integration}}$, and the noise would go down by the square root of the number of independent samples.

We note that formally the units $K_{\text{CMB}}/\sqrt{Hz}$ and $K_{\text{CMB}}\sqrt{s}$ are equivalent. However, the conversion between the two is a bit more subtle. We wish to convert from units of bandwidth to units of integration time, but due to the Nyquist sampling theorem, we must integrate for 2 seconds to get 1 Hz of bandwidth (see Appendix D). The correct conversion factor is

$$1 = \frac{2s}{1 \text{ Hz}^{-1}}$$

Then the NET may be written in conventional units as

$$\text{NET} = \left(\sqrt{2} \frac{\sqrt{s}}{\text{Hz}^{-1/2}}\right) \frac{\text{NEP}}{\frac{dP}{dT}} \left[\frac{K_{\text{CMB}}\sqrt{s}}{}\right] \quad (5.21)$$

Next let us examine the noise-equivalent Q-parameter (NEQ). Again this is simply a unit transformation, this time from temperature intensity units ($K_{\text{CMB}}$) to polarization intensity units (also in $K_{\text{CMB}}$). NEQ is a noise-equivalent measure of the amount of noise in the measurement of the Stokes Q parameter. There is also a noise-equivalent U-parameter (NEU), but it is usually identical to NEQ, so typically only NEQ is listed. The conventional units of NEQ are the same as NET, $K_{\text{CMB}}\sqrt{s}$. There is also a noise-equivalent V-parameter (NEV), which has the same units and a similar interpretation. It is rarely listed, since the
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V-parameter is rarely of interest.

To convert from NET to NEQ, we note that in order to measure the temperature intensity, we need only to measure 1 number. However, to measure the polarization, we must measure 3 numbers ($Q$, $U$, and $V$, or equivalently $E_x$, $E_y$, and $\phi$). Because of this, we must split our observation time to measuring 3 different things, so for a single parameter we get only a fraction of the observation time. If the observation time is split between $Q$, $U$, and $V$ according to the ratios $f_Q$, $f_U$, and $f_V$ (where $0 \leq f_X \leq 1$ and $f_Q + f_U + f_V = 1$), then we will reduce the observation time for $Q$ according to $T_{\text{integration}} \rightarrow f_Q T_{\text{integration}}$ (and similarly for $U$ and $V$). Then since the number of independent measurements is linear in the integration time, $N_{\text{obs}} = f_{\text{sample}} T_{\text{integration}}$, we get only a fraction of the number of independent samples, $N_{\text{obs}} \rightarrow f_Q N_{\text{obs}}$. This change results in a $1/\sqrt{f_Q}$ increase in noise, since noise goes like $1/\sqrt{N_{\text{obs}}}$. Thus, the generic conversions from NET to NEQ, NEU, and NEV are

$$
\begin{align*}
\text{NEQ} & = \left( \sqrt{\frac{2}{f_Q \text{ Hz}^{-1/2}}} \right) \frac{\text{NEP}}{\frac{\text{d}P}{\text{d}T}} \left[ K_{\text{CMB}} \sqrt{s} \right] \\
\text{NEU} & = \left( \sqrt{\frac{2}{f_U \text{ Hz}^{-1/2}}} \right) \frac{\text{NEP}}{\frac{\text{d}P}{\text{d}T}} \left[ K_{\text{CMB}} \sqrt{s} \right] \\
\text{NEV} & = \left( \sqrt{\frac{2}{f_V \text{ Hz}^{-1/2}}} \right) \frac{\text{NEP}}{\frac{\text{d}P}{\text{d}T}} \left[ K_{\text{CMB}} \sqrt{s} \right]
\end{align*}
$$

For PIPER, the VPM modulation strategy is to drive the VPM flat with a truncated sinusoid\(^5\) which results in $\sqrt{f_Q} = \sqrt{f_U} = \frac{1}{2} \cdot \frac{4}{3} \sqrt{f_V}$\(^6\). Combining this with the normalization

\(^5\)Truncated in the phase domain, not the amplitude domain.

\(^6\)See PIPER proposal, if available to you. Particularly, each telescope has 0.8 sensitivity to local $Q$ and 0.6 sensitivity to $V$. Since sensitivity goes like $\sqrt{f}$, this gives us $\sqrt{f_Q} = \frac{0.8}{0.6} \sqrt{f_V} = \frac{4}{3} \sqrt{f_V}$. But since both telescopes measure $V$, and instrument $Q$ and $U$ are measured by only one telescope each, the $V$ weight is doubled.
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condition, \( f_Q + f_U + f_V = 1 \), we see that

\[
f_Q = f_U = \frac{4}{17} \approx 0.2353 \quad \text{and} \quad f_V = \frac{9}{17} \approx 0.5294.
\]  

(5.22)

which gives us the PIPER-specific conversions,

\[
\text{NEQ} = \text{NEU} = \left( \frac{\sqrt{34}}{2} \frac{\sqrt{s}}{\text{Hz}^{-1/2}} \right) \text{NEP} \left( \frac{dP}{dT} \right) = \left( \frac{2.915\sqrt{s}}{\text{Hz}^{-1/2}} \right) \text{NEP} \left( \frac{dP}{dT} \right) \left[ K_{\text{CMB}} \sqrt{s} \right]
\]

(5.23)

\[
\text{NEV} = \left( \frac{\sqrt{34}}{3} \frac{\sqrt{s}}{\text{Hz}^{-1/2}} \right) \text{NEP} \left( \frac{dP}{dT} \right) = \left( \frac{1.943\sqrt{s}}{\text{Hz}^{-1/2}} \right) \text{NEP} \left( \frac{dP}{dT} \right) \left[ K_{\text{CMB}} \sqrt{s} \right]
\]

(5.24)

Lastly, note that we have not specified the reference point for the NEP in any of these calculations. All of these conversions are independent of the efficiencies of the telescope and detector, so moving the reference point of the NEP will change the reference point of the derived quantity (e.g. NEQ or NEU) in the same way. Thus, the derived quantities (NEQ, NEU, NEV) have the same reference point as the NEP that was used to construct them. Transforming a derived quantity to move the reference point is done in the same way that transforming the NEP is done.

5.1.4.3 Map Sensitivity

Now that we have in hand the NEQ, which describes the noise in a given detector pixel and how it depends on integration time, we turn to estimating the amount of noise in a region on the sky. We have already accounted for the noise properties of the detector (encoded in the NEP), so the frame of the detectors is no longer convenient. We are really
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interested in the noise in each sky pixel, not in the noise in each detector pixel, so we must project the detector noise onto the sky. Then we can accumulate the integration time per region on the sky and get the noise in each region on the sky.

Let us write the NEQ referenced to the sky, $\text{NEQ}_{\text{sky}}$. We will assume that there is perfect transfer from the sky to the telescope primary, i.e. there are no atmospheric effects or CMB secondaries. These would factor in at the transfer from the primary to the sky and would require some other efficiency parameter in addition to $\eta$ and $\tau$. Thus, the sky NEQ is equal to the NEQ at the primary, $\text{NEQ}_{\text{sky}} = \text{NEQ}_P$, and we have

$$\text{NEQ}_{\text{sky}} = \left( \frac{2}{\Delta f} \frac{s}{\text{Hz}^{-1}} \right) \text{NEP}_P^2 = \left( \frac{2}{\Delta f} \frac{s}{\text{Hz}^{-1}} \right) \frac{1}{\eta^2 \tau^2} \text{NEP}_P^2.$$  

Over the course of an experiment, the detector spends some period of time looking at each unit of solid angle on the sky. Supposing we have some kind of idealized experiment that uniformly sampled a region of the sky with an overlap factor $f_{OL}$, then the integration time per unit solid angle is

$$t = \frac{f_{OL} T_e}{\Omega_e} = \frac{f_{OL} T_e}{4 \pi f_e} \left[ \frac{s}{\text{sT}} \right]$$  (5.25)

where $T_e$ is the total integration time of the experiment and $\Omega_e$ is the solid angle observed by the experiment.

To get a sense of the meaning of $f_{OL}$, we can imagine two different experiments with the same etendue $A \Omega$. The first experiment has a small beam $\Omega_1$, and so in the experiment period $T_e$ can only cover the experimental region $\Omega_e$ by raster scanning. Each beam spot gets an integration time of only $T_e/(\Omega_e/\Omega_1)$. The second experiment has a large beam
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$\Omega_2 > \Omega_1$, and so can cover the experimental region more quickly, and so cover it more times in the given experimental period. In particular, experiment 2 has an integration time of $T_e/(\Omega_e/\Omega_2)$. If experiment 2 followed a similar raster scan strategy, it would complete $f_{OL} = \frac{T_e/(\Omega_e/\Omega_2)}{T_e/(\Omega_2/\Omega_1)} = \frac{\Omega_2}{\Omega_1}$ scans in the time experiment 2 took to complete a single scan.

Then as we noted before, the integration period and the number of independent samples are related, so we should divide $\text{NEQ}_p$ by $\sqrt{t}$ to get the map sensitivity $m$,

$$m^2 = \frac{\text{NEQ}_{sky}^2}{t} = \frac{1}{t} \left( \frac{2}{f_Q \text{ Hz}^{-1}} \right) \frac{1}{\eta^2 \tau^2} \frac{\text{NEP}^2}{(\frac{dP}{dT})^2} = \frac{4\pi f_e e}{f_{OL} T_e} \left( \frac{2}{f_Q \text{ Hz}^{-1}} \right) \frac{1}{\eta^2 \tau^2} \frac{\text{NEP}^2}{(\frac{dP}{dT})^2}$$  \hspace{1cm} (5.26)

$$m = \frac{1}{\sqrt{t}} \left( \sqrt{\frac{2}{f_Q \text{ Hz}^{-1/2}}} \right) \frac{1}{\eta^2 \tau} \frac{\text{NEP}}{(\frac{dP}{dT})} \left[ \text{K}_\text{CMB} \sqrt{\text{sr}} \right]$$  \hspace{1cm} (5.27)

We note that $m$ has units of $[m] = \text{K}_\text{CMB} \sqrt{\text{sr}}$. It represents the amount of noise one would get in a region that subtends a particular solid angle. Larger regions require more integration time, and so their noise is reduced. Put another way, 1 second of integration time can measure a region of a particular size (in solid angle) to some fixed amount of precision.

To measure a larger region (in solid angle), one must put together many of these fixed size regions. Since each subregion is uncorrelated, this involves combining the fluctuations of many random variables, which reduces the total variance of the whole region.

As a final note, the map sensitivity can be trivially extended to include experiments that map the sky in a non-uniform way. Simply let the integration time per solid angle depend on direction, $t = t(n)$, then

$$m^2(n) = \frac{\text{NEQ}_{sky}^2}{t(n)} = \frac{1}{t(n)} \left( \frac{2}{f_Q \text{ Hz}^{-1}} \right) \frac{1}{\eta^2 \tau^2} \frac{\text{NEP}^2}{(\frac{dP}{dT})^2} \left[ \text{K}_\text{CMB}^2 \cdot \text{sr} \right]$$  \hspace{1cm} (5.28)

This expression contains the full directional dependence for our purposes because the NEQ
depends solely on the properties telescope and detector. For a space-based telescope, this is valid. For a balloon- or ground-based experiment, it would be wise to include the effects of the atmosphere. Since the atmosphere is most conveniently represented in the coordinate system of the experiment and not in the CMB coordinate system, it is preferable to include atmospheric effects in the integration time and to construct an effective integration time, \( t(n) \rightarrow \tilde{t}(n) \). Using the effective integration time \( \tilde{t} \), the above equation would then encode all of the directional dependencies.

### 5.1.4.4 Multiple Detectors

Suppose instead of a single detector we have an array of detectors. We would like to understand how to estimate the sensitivity of the full array from the properties of the single pixel and the experiment properties. We first assume that all pixels in an array are identical. The map sensitivity is computed from the NEQ via the integration time per solid angle,

\[
m^2 = \frac{\text{NEQ}_{\text{sky}}^2}{t}, \quad t = \frac{f_e T_e}{\Omega_e}.
\]

Having many detectors factors into the \( t \), but not the \( \text{NEQ}_{\text{sky}} \). We examine how having many detectors influences \( t \) by examining how it influences the components of \( t \), i.e. \( T_e \) and \( \Omega_e \). In all cases, changing from 1 detector to \( N \) detectors changes the total integration time from \( T_e \xrightarrow{N} NT_e \), since each of the \( N \) detectors is integrating. Additionally, \( N \) detectors can map out the sky \( N \) times as quickly, so as long as the regions never overlap, \( \Omega_e \xrightarrow{N} N\Omega_e \).
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In this case, the integration time per steradian scales like

\[
t \xrightarrow{N} \frac{f_e N T_e}{N \Omega_e} = \frac{f_e T_e}{\Omega_e}
\]

\[
t \xrightarrow{N} t \quad \text{for detectors tiled normal to the travel direction}
\]

While the map sensitivity is not improved, the experiment covers a larger area in the same period of time. This scenario is relevant for detectors that are tiled in a direction normal to the direction of travel along the sky.

For the scenario where the \( N - 1 \) additional detectors are integrating a region that the experiment has already covered, the behavior is different. This scenario is relevant for detectors that are tiled in a direction parallel to the direction of travel along the sky. In this case, the additional detectors are not measuring new solid angles, so the experimental area does not scale with number of detectors, \( \Omega_e \xrightarrow{N} \Omega_e \). However, the integration time always scales up, \( T_e \xrightarrow{N} N T_e \). Thus, in this case, the integration time per steradian scales like

\[
t \xrightarrow{N} \frac{f_e N T_e}{\Omega_e} = N \frac{f_e T_e}{\Omega_e}
\]

\[
t \xrightarrow{N} N t \quad \text{for detectors tiled parallel to the travel direction}
\]

Note that the scaling, and thus the integration time per solid angle, depends on the scan strategy.

Next we consider the case where the beams of some adjacent detectors overlap on the sky. The effects of this depends on where the noise originates from. When the beams do not overlap, there is no need to distinguish the source of noise, and noise from any origin may be treated similarly. We will first consider noise that originates from the detector (e.g.
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phonon noise). Following that we will consider noise that originates from the sky (e.g. photon noise).

Intrinsic Detector Noise

For noise that originates in the detector, the noise is independent of the signal. The noise in each pixel is independent from every other pixel. In this case, if there is overlap in the beams, then the same signal is measured in more than one pixel. Since we are ignoring photon noise (natural variations in the signal), the signal is always equal to its mean value. In this case, the overlaps are essentially multiple measurements of the same signal, each measurement with uncorrelated noise (since the noise originates from inside each independent pixel). Thus, this scenario is no different from the overlapping region being visited twice at two different time periods, a scenario that was discussed above. In this case, the integration time scales with the number of detectors, regardless of the beam overlap.

\[ t \overset{N}{\rightarrow} Nt \]

Sky Noise

For noise that originates in the sky, the noise and signal are correlated. We again model the signal as the mean, and now the noise is the variance. For regions of the sky that are not overlapping, each instance of noise is a realization of the 0-mean Gaussian with variance equal to the sky variance. For regions that are overlapping, both detector pixels will sample the same region and thus the same noise realization. Thus we have only 1 sample of the noise in the overlap region and we cannot integrate down the noise, even though we have
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multiple measurements, because the measurements are correlated. This implies that \( t \xrightarrow{N} t \) for regions of overlap. For the regions without overlap, everything works as described in the above general case. So we have

\[
t \xrightarrow{N} t \quad \text{for overlapping regions}
\]

\[
t \xrightarrow{N} Nt \quad \text{for non-overlapping regions.}
\]

A convenient way of approximating this is by replacing the actual number of pixels with an effective number of pixels. We simply take the width of the beam of the full array and divide by the beam width of a single pixel to get the effective number of independent pixels along that direction \( N_{\text{eff}} \). We note that this is only necessary for detectors tiled parallel to the direction of the scan, since for perpendicularly tiled detectors the integration time per solid angle does not scale with number of detectors, so the transformation \( N \rightarrow N_{\text{eff}} \) makes no difference.

5.1.4.5 Pixel Noise

Let us use our map sensitivity to estimate the noise in each pixel of the sky, which depends on our pixelization scheme. Equal-area pixelization schemes are the most tractable, including HealPIX. In such schemes, each pixel has a fixed angular size, \( \Omega_p \). The map sensitivity tells us how much noise is in a region of some angular size, so we simply divide the map sensitivity by the pixel size to get the pixel noise \( N_p \),

\[
N_p = \frac{m}{\sqrt{\Omega_p}} = \frac{1}{\sqrt{\Omega_p} t} \left( \sqrt{\frac{2}{f_Q \text{ Hz}^{-1/2}}} \right) \frac{1}{\eta} \frac{\text{NEP}}{\frac{dP}{dT}} \left[ K_{\text{CMB}} \sqrt{\text{pixel}} \right] (5.29)
\]
where we have included the unitless “pixel” to indicate the scaling of the pixel noise with the number of pixels. Note that the units justify this, since the units of \( \Omega_p \) may be written \([\Omega_p] = \text{sr/pixel}\). To get the noise in a single pixel, simply divide by \( \sqrt{\text{pixel}} \). To get the average noise level in a region \( N_{\text{pix}} \) pixels large, simply divide by \( \sqrt{N_{\text{pix}}} \).

5.1.4.6 Map Sensitivity in Angular Units

Frequently features are measured in units of radians (or degrees, arcminutes, etc.) rather than steradians. We would like to know what the map sensitivity for such features are. We suppose that a feature described by some angular size \( \theta \) is actually a spherical cap with angular diameter \( \theta \). Note that the opening angle for a spherical cap with angular diameter \( \theta \) is \( \theta/2 \), i.e. suppose the spherical cap is centered on the \( z \)-axis, then the polar angle between the \( z \)-axis and the edge of the spherical cap is \( \theta/2 \).

The solid angle of this spherical cap is

\[
\Omega(\theta) = \int d\Omega = \int_{0}^{\theta/2} \sin \theta' \, d\theta' \int d\phi
\]

\[
\Omega(\theta) = 2\pi \left[ 1 - \cos \left( \frac{\theta}{2} \right) \right]
\]

We note that in the small angle limit,

\[
\Omega(\theta) \approx \frac{\pi \theta^2}{4}
\]

as expected, since it reduces to a flat circle in this limit. To get the NEQ in radians, we
simply use this expression for $\Omega_e$ in Eq. (5.26),

$$m = \sqrt{\frac{2\pi [1 - \cos (\theta/2)]}{f_{OL} T_e}} \left( \sqrt{\frac{2}{f_Q \text{Hz}^{-1/2}}} \right) \frac{1}{\eta T} \left( \frac{\text{d}P}{\text{d}T} \right) \text{[K}_\text{CMB} \cdot \text{rad]} (5.31)$$

$$m \approx \frac{\sqrt{\pi \theta}}{2\sqrt{f_{OL} T_e}} \left( \sqrt{\frac{2}{f_Q \text{Hz}^{-1/2}}} \right) \frac{1}{\eta T} \left( \frac{\text{d}P}{\text{d}T} \right) \text{[K}_\text{CMB} \cdot \text{rad]} (5.32)$$

### 5.1.4.7 Instrument Noise Map

We have described the statistical properties of the instrument noise in each pixel. A realization of the instrument noise map is generated by sampling a number from the Normal distribution for each pixel, then multiplying by the pixel noise $N_p$.

$$N_{\text{instr}}(p) = N_p x_p, \quad x_p \in \mathcal{N}(0, 1) (5.33)$$

### 5.1.5 Internal Linear Combination (ILC) Foreground Removal

We use an Internal Linear Combination\cite{62,64} method for foreground removal. The ILC method has the advantage that it does not require external knowledge, e.g. a foreground template, and is comparably simple to other common methods. For the questions posed at the beginning of this chapter, we do not need to make the most ideal maps and extract real cosmological parameters from them. Rather, we are interested in comparing the effects of different simulated experiments. In this sense, a foreground removal method that works fairly well that is applied consistently is more important to us.
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We describe the ILC algorithm and quantify its performance. Consider a measurement of the full CMB sky in \( k \) different frequency bands. The product of such a measurement is \( k \) temperature maps

\[
T_i(p) = \text{Temperature map of frequency band } \nu_i \text{ with pixel index } p \quad (5.34)
\]

where \( i \) is the frequency band index with \( i = 1, \ldots, k \) and \( p \) is the pixel index with \( p = 1, \ldots, N \). The map is represented in CMB temperature units \( K_{\text{CMB}} \). Care must be taken to ensure that all maps that are used have the same number of pixels and that all maps have been smoothed to the same resolution.

The temperature maps can (theoretically) be decomposed into a CMB component and a residual component

\[
T_i(p) = T_{\text{CMB}}(p) + R_i(p), \quad (5.35)
\]

where the \( T_{\text{CMB}}(p) \) component does not dependent on frequency (since we have chosen thermodynamic temperature units), and the \( R_i(p) \) component encodes all sources of signal that are not from the CMB. Since the CMB component is independent of frequency, we expect that we can combine the various maps in different frequencies to construct an estimator of the CMB map \( T_{\text{CMB}}(p) \). We construct an estimator using a linear combination of
the maps in different frequency bands with to-be-determined weights $w_i(p)$,

$$ \hat{T}(p) = \sum_{i=1}^{k} w_i(p) T_i(p) $$

$$ = \sum_i w_i(p) [T_{CMB}(p) + R_i(p)] $$

$$ \hat{T}(p) = T_{CMB}(p) \sum_i w_i(p) + \sum_i w_i(p) R_i(p). $$

We note that there are $\text{num} (w_i(p)) = Nk$ unknown parameters in our estimator. In order for the estimator $\hat{T}(p)$ to have unity gain in $T_{CMB}(p)$, we must have

$$ \sum_i w_i(p) = 1 \quad \forall \ p $$

which provides $N$ constraint equations. This results in

$$ \hat{T}(p) = T_{CMB}(p) + \sum_{i=1}^{k} w_i(p) R_i(p). $$

Further properties of the estimator $\hat{T}(p)$ will be determined by how we choose to constrain the remaining $(N - 1)k$ degrees of freedom.

We may represent the system of equations in a matrix formalism. Define vectors as column vectors of the frequency bands, so

$$ \mathbf{T}^T = (T_1(p), \ldots, T_k(p)) \quad \text{and} \quad \mathbf{w}^T = (w_1(p), \ldots, w_k(p)), $$

then we may write the decomposition, the estimator, and unity gain constraint as

$$ \mathbf{T} = T_{CMB} \mathbf{1} + \mathbf{R} $$

$$ \hat{T} = \mathbf{w}^T \mathbf{T} $$

$$ \mathbf{1}^T \mathbf{w} = 1 $$
Lastly we note that the input to the ILC algorithm is a set of maps in $k$ different frequency bands, while the output is a single map.

### 5.1.5.1 Constant Weighting Factors

Suppose the weight factors $w(p)$ were uniform across the entire map, so $w(p) = w$. This provides the remaining $(N - 1)k$ constraints. We note in this case that our estimator $\hat{T}$ is a linear function with regards to $T$, so then the variance of $\hat{T}$ has a particularly simple form

$$\text{var } \hat{T} = w^T \text{cov}(T, T) w$$  \hspace{1cm} (5.41)

where $(\text{cov}(T, T))_{ij} = \text{cov}(T_i, T_j)$ is the covariance matrix of $T$.

We choose to optimize $w$ so that $\text{var } \hat{T}$ is minimized, i.e.

$$w^* = \arg \min_w \left( \text{var } \hat{T}(w; T) \right)$$  \hspace{1cm} (5.42)

We will do this in two spaces: the raw temperature map space $\{T\}$, and the foreground-background space $\{T_{\text{CMB}}, R\}$. The first solution will be used to actually perform the foreground cleaning, while the second will give us a sense of what the algorithm is doing.

---

7 Expectation values are defined over the pixels, so

$$\langle T_i \rangle = \frac{1}{N} \sum_{p=1}^{N} T_i(p)$$

then

$$\text{cov}(T_i, T_j) = \langle T_i T_j \rangle - \langle T_i \rangle \langle T_j \rangle.$$ 

It is worth noting that $\text{cov}(T, T) = \text{cov}(T, T)^T$ and $\text{cov}(T, T) \geq 0$, i.e. the covariance matrix is symmetric and positive semi-definite.
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Raw Temperature Map Space

Let $C = \text{cov}(T, T)$, i.e. $C_{ij} = \text{cov}(T_i, T_j)$. Then we may write our minimization problem as

$$\text{var} \hat{T} = w^T C w \quad \text{subject to} \quad 1^T w = 1$$

which has solution (see Appendix E)

$$w^* = \frac{C^{-1} 1}{(1^T C^{-1} 1)}$$

We note that written out in element form we have

$$w^*_i = \frac{\sum_j C^{-1}_{ij}}{\sum_{jk} C^{-1}_{jk}}$$

which matches Eriksen et al. 63

Eq. (5.44) describes how the foreground removal step is implemented. We quickly estimate how expensive the algorithm is. $C$ is a $k \times k$ matrix, where $k$ is the number of frequency bands, which is typically order unity. Thus, computing $w^*$ from $C$ requires the inversion of a small matrix and the sum of its components and may be done cheaply. Far more expensive is the computation of each element of the matrix $C_{ij} = \frac{1}{N} \sum_{p=1}^{N} T_i(p)T_j(p) - \frac{1}{N^2} \sum_{p=1}^{N} T_i(p) \sum_{p=1}^{N} T_j(p)$, which is $O(4N)$. Given that matrix inversion of $C$ is pessimistically $O(k^3)$ and $N = 12N_{\text{side}}^2$, our total complexity is $O(48 N_{\text{side}}^2 + k^3)$.

Foreground-background Space

Let us now perform the same optimization supposing we already know the decomposition between residual (foreground) and CMB (background). With $T = T_{\text{CMB}} 1 + R$, the
covariance matrix \( \mathbf{C} \) may be decomposed into foreground, background, and cross components,

\[
\mathbf{C} = \text{cov} (T_{\text{CMB}} \mathbf{1} + \mathbf{R}, T_{\text{CMB}} \mathbf{1} + \mathbf{R})
\]

\[
= \var (T_{\text{CMB}}) \mathbf{1} \mathbf{1}^T + \text{cov} (T_{\text{CMB}}, \mathbf{R}) \mathbf{1}^T + \mathbf{1} \text{cov} (T_{\text{CMB}}, \mathbf{R})^T + \text{cov} (\mathbf{R}, \mathbf{R})
\]

\[
\mathbf{C} = \sigma_B^2 \mathbf{1} \mathbf{1}^T + \mathbf{X} \mathbf{1}^T + \mathbf{1} \mathbf{X}^T + \mathbf{F}
\]

where we’ve defined \( \sigma_B^2 \equiv \var (T_{\text{CMB}}) \), \( \mathbf{X} \equiv \text{cov} (T_{\text{CMB}}, \mathbf{R}) \), and \( \mathbf{F} \equiv \text{cov} (\mathbf{R}, \mathbf{R}) \). This is also solved by Lagrange multipliers as described in Appendix E to give

\[
w^* = \frac{1 + (\mathbf{1}^T \mathbf{G}^{-1} \mathbf{X})}{(\mathbf{1}^T \mathbf{G}^{-1} \mathbf{1})} \mathbf{G}^{-1} \mathbf{1} - \mathbf{G}^{-1} \mathbf{X} \tag{5.47}
\]

where we’ve defined \( \mathbf{G} \equiv \mathbf{F} + \mathbf{1} \mathbf{X}^T \). In component form, this is

\[
w_i^* = \frac{1 + \sum_{jk} (\mathbf{F} + \mathbf{1} \mathbf{X}^T)^{-1}_{jk} X_j}{\sum_{jk} (\mathbf{F} + \mathbf{1} \mathbf{X}^T)^{-1}_{jk}} \sum_j (\mathbf{F} + \mathbf{1} \mathbf{X}^T)^{-1}_{ij} X_j - \sum_j (\mathbf{F} + \mathbf{1} \mathbf{X}^T)^{-1}_{ij} X_j \tag{5.48}
\]

which differs slightly from the result of Efstathiou et al\(^\text{65}\) in that \( \mathbf{F} \rightarrow \mathbf{F} + \mathbf{1} \mathbf{X}^T \).

By inserting Eq. (5.47) into Eq. (5.40b) we can estimate the bias from the algorithm.

\[
\hat{T} = T_{\text{CMB}} + \left[ \frac{1 + (\mathbf{1}^T \mathbf{G}^{-1} \mathbf{X})}{(\mathbf{1}^T \mathbf{G}^{-1} \mathbf{1})} \right] (\mathbf{1}^T \mathbf{G}^{-1} \mathbf{R}) - (\mathbf{X}^T \mathbf{G}^{-1} \mathbf{R}) \tag{5.49}
\]

and we see that the estimator \( \hat{T} \) is biased by spurious correlations between the foreground and background, quantified by \( \mathbf{X} \).

### 5.1.5.2 Piecewise Constant Weighting Factors

In this scheme, the map is divided into \( r \) different regions \( \mathcal{A}_r \) and the Constant Weighting Factor scheme is applied to each region independently. The regions are not necessarily

\(^8\)Explicitly, \( \mathbf{X}_i = \text{cov} (T_{\text{CMB}}, R_i) \).
This requires the definition of expectation value to be redefined to

\[ \langle T_i \rangle = \frac{1}{N_r} \sum_{p \in \mathcal{N}_r} T_i(p) \tag{5.50} \]

where \( \mathcal{N}_r \) is the set of pixels in region \( r \) and \( N_r = \text{num}(\mathcal{N}_r) \) is the number of pixels in region \( r \), and \( r \) is merely an index to label each region. The results of Sec. 5.1.5.1 hold for each region independently.

This strategy may be used to split up the sky into regions with different expected foreground properties. As an example, we do not expect the foreground properties of the Galactic plane to be the same as the foreground properties at high Galactic latitudes. Similarly, we would not expect the foreground properties looking directly at the center of the galaxy to be the same as looking directly away from the center of the galaxy. Thus, we may choose to separate these regions out.

### 5.1.5.3 Polarization ILC

So far we have discussed the ILC algorithm in the context of temperature maps, but we must clean polarization maps, particularly \( Q \) and \( U \) maps. We note that there is no reason that the \( Q \) and \( U \) maps should be correlated since the \( Q \) and \( U \) components are themselves independent. Thus we may perform ILC on the \( Q \) and \( U \) maps independently.
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5.1.6 Power Spectrum Estimation

Power spectra \((C_{\ell}^{XX})\) from the foreground-cleaned maps are generated using HealPIX’s \texttt{anafast} algorithm. Each map produces a full set of power spectra, so with \(M\) map realizations we will have \(M\) sets of power spectra. The set of \(M\) spectra allows the effects of the simulation to be put on a statistical footing, i.e. it is possible estimate the distribution of possible power spectra. This is important because while we can make arbitrarily many sky realizations in simulation, in the real world have only one, so we must understand how well we can constrain the true underlying power spectra from our one sky realization.

5.1.7 Parallelization

Every sky realization is totally independent of every other sky realization. The process of adding foregrounds and noise, estimating the foreground-cleaned map, and computing a power spectrum from the map is embarrassingly parallel. Furthermore, it is by far the most expensive part of the simulation pipeline. We may accelerate the simulation process by parallelizing this section of the pipeline, running an independent realization on each core. The \texttt{gs66-kappa} cluster at NASA-GSFC was used for processing.

5.2 Frequency Band Optimization

We simulate having different subsets of the full PIPER data set by limiting which frequency bands we use in the pipeline. The number of sky realizations was set to \(M = 100\)
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with maps of size $N_{\text{side}} = 512$ (i.e., 3145728 pixels of area $4 \times 10^{-6}$ sr). A PIPER-like value for instrumentation noise of $N_p = (4, 6, 21, 347) \mu\text{K}/\text{sr}$ for $(200, 270, 350, 600)$ GHz. The ILC was performed on the entire maps as a single region. The results of the simulation are shown in Fig. 5.5.

![Figure 5.5: The $C_{\ell}^{\text{BB}}$ power spectra from $M = 100$ realizations of a cosmology with a null input BB spectrum for simulated experiments with frequency bands (left) 200, 270 GHz, (middle) 200, 600 GHz, and (right) 200, 270, 350, 600 GHz.](image)

It is clear that for all of our simulated experiments our foreground cleaning methodology is insufficient for recovery of the CMB map for the purpose of constraining cosmology. However, we observe that the simulation with only the lowest 2 bands is significantly more contaminated than the experiments with experiments with both 200 and 600 GHz. Furthermore, the noise level from the experiment with all 4 frequency bands is not significantly improved from the noise level of the experiment with just the 200 and 600 GHz bands.

This result suggests that it may be valuable to prioritize a higher frequency band flight following the first science band flight.
5.3 Calibration Gain Noise

We would like to investigate the effect of a calibration gain error in a generic way. To this end, we consider a multiplicative gain error characterized by a small amount of power in a single harmonic bin. Such a gain map is described in harmonic space by

\[ G(\ell) = \delta(\ell) + g\delta(\ell - \ell_0) \]  

(5.51)

where the first term is equivalent to a constant unity-gain map, and the second term adds an error in the \( \ell_0 \) bin. The \( g \) parameters characterizes the scale of the distortion, though we will renormalize it in map space, so it is not of great interest. We multiply the sky map \( S(p) \) by the gain map \( G(p) \) to produce the post-gain map \( M(p) \). In harmonic space, this is a convolution,

\[ M(\ell) = S(\ell) * G(\ell) = S(\ell) + gS(\ell - \ell_0) \]  

(5.52)

so we expect the effect of the gain calibration error is to mix the signal back into itself at a higher multipole.

The gain map is constructed by using \texttt{anafast} on a power spectrum \( \delta(\ell - \ell_0) \). We note that the input power spectrum only describes the statistical properties of the gain map. The particular phase is arbitrary. The \texttt{anafast} routine constructs a realization of the gain error map with the correct statistical properties. This map is then normalized to have the desired standard deviation characterized by \( g \). Such a map has the correct standard deviation but a mean of 0, so we add it to a uniform map with value 1 to produce the gain
map $G(p)$.

\[ G(p) = 1 + g \frac{G_{\ell_0}}{\text{std}(G_{\ell_0})} \]  \hspace{1cm} (5.53)

where $G_{\ell_0} = \text{anafast}(\delta(\ell - \ell_0))$. Such a gain map is generated for each of the $M$ sky realizations and is multiplied after the full sky map is formed, as indicated by the dashed nodes in Fig. 5.1.

We note that the gain map realization and CMB realization are independent, so there is no need to generate an ensemble of gain map realizations for each CMB realization. Each element in such an ensemble is equivalent to a new CMB realization. We use $M = 400$ CMB realizations with a single gain map realization that is shared by both the $Q$ and $U$ maps. A 5% ($g = 0.05$) calibration error is used. This simulation is repeated for 125 choices of $\ell_0$ ranging from 2 to 400 with logarithmic spacing. We use an instrument noise of $1/10$th the PIPER instrument noise, $N_p = (0.4, 0.6, 2.1, 34.7) \mu K$/sr, to ensure the calibration gain is the dominant effect. A sampling of the results for a few choices of $\ell_0$ is shown in Fig. 5.6.

We observe that the resulting spectra are significantly contaminated at $\ell_0$, even with a 5% gain error. The magnitude of the contamination is huge, most likely due to the inadequacy of the foreground removal strategy. A better foreground removal algorithm would allow us to quantify the precise level of gain calibration we require. From this simulation it is still evident that we must control the gain on scales that we care about. For PIPER to have significant constraining power at low $\ell$, we must control the gain on large scales. This implies that we must control against long term drifts in the gain.
Figure 5.6: The $C_{\ell}^{\text{BB}}$ power spectra from $M = 400$ realizations of a cosmology with a null input BB spectrum for simulated experiments with 5% gain calibration errors in the specified $\ell_0$ bins.
Appendix A

Superconducting Quantum Interference Devices (SQUIDs)

The detector multiplexer and readout both rely heavily on Superconducting Quantum Interference Devices (SQUIDs). We provide an overview of the relevant pieces of the physics of SQUIDs so that we may better understand the mux and readout.

A.1 Josephson Junctions

The foundation of SQUIDs is the Josephson Junction. A Josephson Junction is a sandwich of superconductor-insulator-superconductor. Our analysis follows that of Feynman.

\[\text{The middle part does not strictly need to be an insulator. It may be any material that inhibits the coherence of the superconducting phase parameter. A normal metal is most commonly used in SQUIDs, though a weakened superconductor (e.g. by making the superconductor thinner, or via the proximity effect of a normal metal) works just as well.}\]
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The macroscopic wavefunction of each of the superconductors may be written

\[ \psi_1 = \sqrt{\rho_1} e^{i\phi_1} \]  (A.1a)
\[ \psi_2 = \sqrt{\rho_2} e^{i\phi_2} \]  (A.1b)

The (real-valued) coefficients \( \rho_i \) are the charge density of the Cooper pairs (units of charge/volume), with charge \( q = 2e \), in superconductor \( i \), and the phases \( \phi_i \) are quantum mechanical parameters describing the coherence of the superconductivity in superconductor \( i \). For a strongly coupled superconductor, the phase parameter is continuous and varies with the magnetic potential. We will see shortly that the insulator causes a discontinuity in the phase parameter across the barrier between superconductors 1 and 2.

We apply a voltage \( V \) across the junction (one lead connected to superconductor 1 and the other lead connected to superconductor 2). Then the system may be described by a system of coupled Schrödinger Equations,

\[ i\hbar \frac{\partial \psi_1}{\partial t} = eV \psi_1 + K \psi_2 \]  (A.2a)
\[ i\hbar \frac{\partial \psi_2}{\partial t} = -eV \psi_2 + K \psi_1 \]  (A.2b)

where we have defined the zero energy level to be half of the energy shift as a Cooper pair crosses the insulator, \( \Delta E = 2eV \), and implicitly defined both superconductors as having
identical properties. Inserting Eqs. (A.1) into this system gives us the equations

\[ \dot{\rho}_1 = -\dot{\rho}_2 = \frac{2K}{\hbar} \sqrt{\rho_1 \rho_2} \sin \phi \] (A.3a)

\[ \dot{\phi}_1 = -\frac{eV}{\hbar} - \frac{K}{\hbar} \sqrt{\frac{\rho_2}{\rho_1}} \cos \phi \] (A.3b)

\[ \dot{\phi}_2 = \frac{eV}{\hbar} - \frac{K}{\hbar} \sqrt{\frac{\rho_1}{\rho_2}} \cos \phi \] (A.3c)

where we have defined \( \phi = \phi_2 - \phi_1 \). We note that \( \dot{\rho}_1 \) and \( -\dot{\rho}_2 \) are the rates at which Cooper pairs enter superconductor 1 and leave superconductor 2, respectively. Since the two ends are connected by a wire (i.e. whatever circuit is generating \( V \)), electrons that leave superconductor 2 will eventually find their way back to superconductor 1. Thus, charge is conserved. Furthermore, since the superconductors have identical properties, the Cooper pair density should be the same in both, so \( \rho_1 = \rho_2 = \rho_0 \). The interpretation of \( \dot{\rho}_i \) is then the amount of charge passing through some region in a period of time, forming a current density

\[ J = \dot{\rho} = \frac{2K}{\hbar} \rho_0 \sin \phi \]

\[ J = J_0 \sin \phi \] (A.4)

Note that this is the amount of charge passing through the volume, not the change in the amount of charge contained in the volume (which is essentially constant). The way to convert from \( J \) to proper current \( I \) in Amps depends on the type and geometry of the junction. The conversion does not depend on any of the values comprising \( J \), so there is simply a constant of proportionality. We may then write the current,

\[ I_c = I_0 \sin \phi \] (A.5)
where the maximum critical current $I_0$ is simply some value that is typically measured rather than computed. Eq. (A.5) is the first Josephson Equation. We can interpret this equation by noting that when $V = 0$, the phase difference $\phi$ is constant. Since the allowed current is non-zero but the voltage is 0, this current is a supercurrent, which is allowed up to $I_c$. Thus we identify $I_c$ as the critical supercurrent allowed across the junction, which is attenuated by the two superconductors interfering with each other.

Next we note that $\dot{\phi} = \dot{\phi}_2 - \dot{\phi}_1$, which combined with our equations for $\dot{\phi}_i$ gives us the second Josephson equation,

$$V = \frac{\hbar}{2e} \dot{\phi}.$$  

(A.6)

The second Josephson Equation shows that a constant voltage will cause the phase difference to oscillate, which in turn will cause the critical current to oscillate.

We can model the Josephson Junction as a non-linear inductor by applying the Josephson equations Eqs. (A.5) and (A.6) to the voltage drop across an inductor

$$V = L \frac{dI}{dt}$$

$$\implies L = \frac{\hbar}{2eI_0} \frac{1}{\cos \phi} \equiv \frac{L_J}{\cos \phi}$$  

(A.7)

where $L_J \equiv \frac{\hbar}{2eI_0} = \frac{\Phi_0}{2\pi I_0} = \frac{(0.3 \text{ nH} \cdot \mu\text{A})}{I_0}$.  

(A.8)

Since the superconducting phase parameter varies with the magnetic potential, it can be shown that the phase difference across the Josephson Junction is

$$\phi = \frac{2\pi \Phi}{\Phi_0}$$  

(A.9)
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where $\Phi$ is the flux enclosed by the junction. Since $\phi$ is periodic, changes in $\Phi$ of $\Phi_0$ do not result in a change in the behavior of the system.

As a final note, Josephson Junctions are inherently quantum devices that rely on the interference between the superconducting phase parameters between superconductors 1 and 2. Thermal fluctuations can wash out the interference pattern and destroy the quantum properties of the Josephson Junction, including all supercurrent effects. If the thermal fluctuations result in a variation in the flux of $\delta\Phi \gtrsim \Phi_0/2$, then the phase difference will be essentially randomized. We may estimate the scale of the thermal fluctuations with the Equipartition Theorem and use it to constrain the inductance,

$$\frac{1}{2} \left( \frac{\delta\Phi}{L} \right)^2 = \frac{1}{2} kT, \quad \frac{1}{8} \frac{\Phi_0^2}{L} \gtrsim \frac{1}{2} kT \quad \Rightarrow \quad L \lesssim \frac{\Phi_0^2}{4kT} = \frac{(80 \text{ nH} \cdot \text{K})}{T} \quad (A.10)$$

A.2 DC SQUIDs

A DC SQUID is a loop of superconducting metal with Josephson Junctions at opposite ends (Figure A.1). The superconductors on the same side of both Josephson Junctions is shared, so there are only two total superconductors. Leads on each side of the Josephson Junctions apply voltage and transport current to the SQUID. The total inductance of the SQUID is the sum of both the Josephson Junction inductance $L_J$ and the loop inductance $L$. We will first examine the behavior of our SQUID with no voltage drop, then see what
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happens in the presence of a non-zero voltage. Much of this discussion follows that of Tinkham.56

Figure A.1: A DC SQUID is a superconducting loop with Josephson Junctions at each end. The superconducting loop can intercept magnetic flux $\Phi$, which changes the behavior of the SQUID.

Since there are Josephson Junctions on each arm of the SQUID, it may admit current even in the absence of a voltage. Each Josephson Junction has a phase difference, which we write as $\phi_A$ and $\phi_B$ for junctions A and B, respectively. We redefine $\phi \equiv \phi_B - \phi_A$. We may write the current through the SQUID using the first Josephson Equation,

$$I = I_0 \sin \phi_A + \sin \phi_B$$

$$I = 2I_0 \sin \left( \frac{\phi_A + \phi_B}{2} \right) \cos \left( \frac{\phi_B - \phi_A}{2} \right)$$

$$I = 2I_0 \sin \left( \frac{\phi_A + \phi_B}{2} \right) \cos \left( \frac{\pi \Phi}{\Phi_0} \right)$$

where we have utilized the result $\phi = 2\pi \Phi/\Phi_0$, which is the 2-junction form of Eq. (A.9).
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The current is maximized when $\phi_A + \phi_B = \pi \implies \phi_A = \pi/2 - \pi \Phi/\Phi_0$ and $\phi_B = \pi/2 + \pi \Phi/\Phi_0$, with maximum value (Fig A.2)

$$I_c = 2I_0 \left| \cos \left( \frac{\pi \Phi}{\Phi_0} \right) \right|,$$

which describes the maximum amount of supercurrent that the SQUID will admit without inducing a voltage drop.

![Figure A.2](image)

Figure A.2: The maximum critical current $I_c$ of a DC SQUID with no voltage across it depends on the magnetic flux $\Phi$ intercepted by the loop, and forms a double-slit interference pattern with spacing $\Phi/\Phi_0$. (Solid) A symmetric SQUID with negligible screening currents ($\beta L \ll 1$) has a maximum value of twice the individual Josephson Junction critical current $I_0$ and a minimum of 0. (Dashed) When screening currents are significant ($\beta L \gtrsim 1$), the troughs are not as deep and reach a minimum at $I_{\text{min}}$. In the strong screening limit ($\beta L \gg 1$), the peak-to-trough distance is $\Delta I_c \sim \Phi_0/L$, independent of $I_0$.

If the applied current $I$ exceeds this maximum value $I_c(\Phi)$, then the excess current will be admitted through the impedance $R = R_{JJ}|R_{JJ} = R_{JJ}/2$ of the SQUID, which is half of the impedance of a single Josephson Junction $R_{JJ}$. Note, however, that once a voltage is applied to the SQUID, the phase differences across the Josephson Junctions $\phi_A$ and $\phi_B$
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will evolve, so we cannot simply use \( I_c \) (since \( I_c \) relies on a particular choice of \( \phi_A + \phi_B \)) and must instead use the more generic Josephson currents, resulting in

\[
I = I_0 \left( \sin \phi_A + \sin \phi_B \right) + \frac{V}{R}, \\
V = R \left[ I - 2I_0 \cos \left( \frac{\pi \Phi}{\Phi_0} \right) \sin \bar{\phi} \right],
\]

(A.12)

where we have defined \( \bar{\phi} \equiv \frac{1}{2} (\phi_A + \phi_B) \) to be the average phase difference. The effects of Junction capacitance could also be included in Eq. (A.12) to make what is called the RCSJ model, but the capacitances of the point junctions we use are negligible, so we do not include it here. Since the voltage varies rapidly in time (assuming it is not \( \Phi = \Phi_0 / 2 \)), but we measure the voltage only slowly, we are interested in the time-averaged voltage \( \langle V \rangle \).

Finally, we note that \( \dot{\bar{\phi}} = \frac{1}{2} \left( \dot{\phi}_A + \dot{\phi}_B \right) = 2eV/h \), from the second Josephson Equation.

So,

\[
\langle V \rangle = \frac{h}{2e} \left( \langle \dot{\bar{\phi}} \rangle \right) = \frac{h}{2e} \oint \dot{\bar{\phi}} \, dt = \frac{2\pi}{\oint \frac{d\bar{\phi}}{V(\bar{\phi})}},
\]

(A.13)

\[
\langle V \rangle = 2\pi \int_{0}^{2\pi} \frac{d\bar{\phi}}{R \left[ I - 2I_0 \cos \left( \frac{\pi \Phi}{\Phi_0} \right) \sin \bar{\phi} \right]}.
\]

(A.14)

Note that Eq. (A.13) is generally true and can be used to find (most likely numerically) the response of a generic SQUID, e.g. using the RCSJ model, but Eq. (A.14) is valid only for our simpler case. It may be integrated to give

\[
\langle V \rangle = \begin{cases} 
0 & \text{for } |I| \leq I_S \cos \left( \frac{\pi \Phi}{\Phi_0} \right) \\
I_S R \sqrt{\left( \frac{I}{I_S} \right)^2 - \cos^2 \left( \frac{\pi \Phi}{\Phi_0} \right)} & \text{for } |I| > I_S \cos \left( \frac{\pi \Phi}{\Phi_0} \right)
\end{cases}
\]

(A.15)
where $I_S \equiv 2I_0$ is the total maximum critical current of the SQUID, which is twice that of an individual Josephson Junction. We note that it has a maximum value when $\Phi = (n + 1/2)\Phi_0$ and a minimum value when $\Phi = n\Phi_0$. For $|I| < I_S \cos \left(\frac{\pi \Phi}{\Phi_0}\right)$, the SQUID can admit the entire bias current as supercurrent, and so the voltage is $\langle V \rangle = 0$.

Figures A.3, A.4, and A.5 show the $I-\langle V \rangle$, $I-R_{\text{eff}}$, and $\langle V \rangle-\Phi$ curves, respectively.

When a flux $\Phi$ is applied to the SQUID loop, Lenz’s Law dictates that a screening current $I_L$ will be generated in the loop. The screening current will generate a screening flux with magnitude $\Phi_L = LI_L$. This screening flux prevents the total flux in the loop from reaching $(n + 1/2)\Phi_0$, so the critical current is never fully suppressed (see Fig A.2). Note that once the applied flux exceeds $\Phi_0/2$, it becomes more energetically favorable (in the sense that the screening current required is smaller) to push the total flux towards $\Phi_0$, so in that case the screening flux will switch signs. The screening strength is parameterized by the screening parameter

$$\beta_L \equiv 2\pi LI_0/\Phi_0 = L/L_J. \quad (A.16)$$

For $\beta_L \gg 1$, the screening dominates and the total flux is always near $n\Phi_0$. For $\beta_L \ll 1$, the screening is negligible. The screening factor determines how large the minimum critical current $I_{\text{min}}$ is, below which the SQUID is fully off. The voltage response and effective resistance are also affected by $\beta_L$ (see Figs. A.3 and A.4). The $\langle V \rangle-\Phi$ response is largely unchanged except for the fully off state. Note that SQUIDs with large loops will have a large inductance $L$ and consequently a large screening parameter $\beta_L$. This is common for the coupling SQUIDs in the detector readout. The flux-gated switches in the multiplexer...
Figure A.3: The $I$-$\langle V \rangle$ curve for a DC SQUID. The true voltage is oscillating at a higher frequency than we measure, so we instead plot the time-averaged value $\langle V \rangle$. The uppermost curve corresponds to $\Phi = (n + \frac{1}{2})\Phi_0$, for which the SQUID critical current $I_c$ has been fully suppressed. This curve is valid only for the weak screening ($\beta_L \ll 1$) limit. The SQUID behaves as a resistor with resistance $R = R_{JJ}||R_{JJ} = R_{JJ}/2$, where $R_{JJ}$ is the impedance of an individual Josephson Junction. The lowermost curve corresponds to $\Phi = n\Phi_0$, for which the SQUID critical current is not suppressed at all. The SQUID admits supercurrent up to its critical current $I_S = 2I_0$, twice the critical current of an individual Josephson Junction. Any excess current is shunted through the impedance of the SQUID. (Dashed) The dashed line shows the effect of a screening current ($\beta_L \gtrsim 1$) on the maximum voltage response, $\Phi = (n + \frac{1}{2})\Phi_0$. As the screening strength increases (increasing $\beta_L$), the line moves more towards the minimum curve, but can get no closer than $\Delta I \sim \Phi_0/L$ at $V = 0$. The minimum response is unchanged.
Figure A.4: The $I-R_{\text{eff}}$ curve for a DC SQUID. The uppermost curve corresponds to $\Phi = (n + \frac{1}{2}) \Phi_0$, for which the SQUID is totally resistive, valid only in the weak-screening ($\beta_L \ll 1$) limit. The lowermost curve corresponds to $\Phi = n \Phi_0$, for which the SQUID critical current is not suppressed at all. The SQUID admits supercurrent up to its critical current then passes the remaining current resistively. (Dashed) The dashed line shows the effect of a screening current ($\beta_L \gtrsim 1$) on the maximum effective resistance. As the screening strength increases (increasing $\beta_L$), the line moves more towards the minimum curve, but can get no closer than $\Delta I \sim \Phi_0/L$ at $R_{\text{eff}} = 0$. 
Figure A.5: The $\langle V \rangle$-$\Phi$ curve for a DC SQUID at a few representative bias current values $I$. (Solid) The SQUID is partially on when $I < I_S$. For fluxes $\Phi$ near $n\Phi_0$, the critical current is large enough to admit the full bias current as supercurrent, but must shunt some current through the impedance for fluxes near $(n + \frac{1}{2})\Phi_0$. (Dashed) The SQUID has just fully turned on when $I = I_S$. It can admit the full bias current as supercurrent only when $\Phi = n\Phi_0$. The peak-to-peak amplitude is maximized here. (Dash-dotted) The SQUID is fully on when $I > I_S$. It must always shunt current through its impedance. The peak-to-peak amplitude decreases as $I$ increases above $I_S$. Note that the qualitative behavior is not significantly different between the weak-screening and non-weak-screening cases. All that changes is the SQUID will not turn partially on until the bias current exceeds the minimum critical current, $I_{\text{min}} < I < I_S$. The SQUID is fully off (and has no voltage response) when $I < I_{\text{min}}$. 
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must have $\beta L \ll 1$ so that there is a large change in effective resistance between the on and off states. As a final note, the condition in Eq. (A.10) must also be satisfied for the SQUID loop with the SQUID loop inductance $L$, in addition to for the Josephson Junctions themselves. If the condition is not satisfied, the double-slit interference pattern will be washed out.

### A.3 Voltage and Current Biases

DC SQUIDs may be current biased or voltage biased\(^2\). We briefly examine these two cases and the SQUID response in each.

**Current Bias**

In the current biased case, $I = I_{\text{bias}} = \text{constant}$, and we are interested in the voltage $|V|$ response to changing flux $\Phi$. When constructing the equations, we were implicitly working in this regime, and we simply reproduce the response derived above (Eq. (A.15)) and plotted in Fig[A.5]

\[
\langle V \rangle = \begin{cases} 
0 & \text{for } \cos \left( \frac{\pi \Phi}{\Phi_0} \right) \geq \frac{|I_{\text{bias}}|}{I_S} \\
I_S R \sqrt{\left( \frac{I_{\text{bias}}}{I_S} \right)^2 - \cos^2 \left( \frac{\pi \Phi}{\Phi_0} \right)} & \text{for } \cos \left( \frac{\pi \Phi}{\Phi_0} \right) \leq \frac{|I_{\text{bias}}|}{I_S}
\end{cases}
\]  

(A.17)

**Voltage Bias**

In the voltage biased case, $\langle V \rangle = V_{\text{bias}} = \text{constant}$, and we are interested in the current $I$ response to changing flux $\Phi$. We simply invert Eq. (A.15) to get the response and plot it

\[^2\text{Or even weakly biased with some arbitrary } I = f(V) \text{ curve, though we do not consider this case.}\]
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We also plot the effective resistance $R_{\text{eff}}$ in Fig A.7.

\[ I = I_S \sqrt{\left( \frac{V_{\text{bias}}}{I_S R} \right)^2 + \cos^2 \left( \frac{\pi \Phi}{\Phi_0} \right)} \]  

(A.18)

Figure A.6: The $I$-$\Phi$ curve for a DC SQUID at a few representative bias voltage values $\langle V \rangle$. (Solid) The $V_{\text{bias}} = 0$ case. This is simply the zero-voltage critical current curve shown in Fig A.2. Note that as before, this curve is valid for the weak-screening limit ($\beta_L \ll 1$). In other screening strength regimes ($\beta_L \gtrsim 1$), the minimum current does not reach zero. (Dashed) The $V_{\text{bias}} = I_S R$ case. The peak-to-peak amplitude decreases with increasing $V_{\text{bias}}$, though there is no particular critical value where interesting behavior occurs. For non-weak-screening regimes, the troughs are not as low, resulting in a smaller peak-to-peak amplitude. (Dash-dotted) The $V_{\text{bias}} = 2I_S R$ case.

Note that the current can go to zero only in the weak screening ($\beta_L \ll 0$) regime with zero bias voltage ($V_{\text{bias}} = 0$). The peak-to-peak amplitude of the response decreases with increasing bias voltage and with increasing screening strength $\beta_L$. 
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Figure A.7: The $\langle V \rangle$-$R_{\text{eff}}$ curve for a DC SQUID. The uppermost curve corresponds to $\Phi = (n + \frac{1}{2})\Phi_0$, for which the SQUID is totally resistive, valid only in the weak-screening ($\beta_L \ll 1$) limit. The lowermost curve corresponds to $\Phi = n\Phi_0$, for which the critical current is not suppressed and a portion of the current is passed through the SQUID as supercurrent, thus lowering the effective resistance. (Dashed) The dashed line shows the effect of a screening current ($\beta_L \gtrsim 1$) on the maximum effective resistance. As the screening strength increases (increasing $\beta_L$), the top curve moves toward the minimum curve.

A.4 Flux-gated Switches

We observe in Figs. A.4 and A.7 that the resistance of a DC SQUID can be controlled effectively by changing the flux applied to the loop. This is the basis of a switch: by changing the impedance of one path from low to high diverts current to alternative pathways. The basic circuit for using a SQUID as a flux-gated switch is shown in Fig. A.8.

The bias may be either a voltage or current bias. The current bias case is simpler to understand, so we cover it first. The voltage bias case is similar in principle, but is less obviously extensible.
Suppose the bias is a current bias $I_{\text{bias}} \lesssim I_S$ compared to the switch SQUID’s (SN) max critical current $I_S$. We may apply flux through the coil coupling into the switch SQUID. If the applied flux is near $\Phi = n\Phi_0$ (the OFF or CLOSED state), then the critical current in SN is not suppressed, and SN can pass the full bias current with no resistance (see Fig. A.4). Thus, no current is diverted to the load arm and no voltage drop is generated across the load.

If now the applied flux is near $\Phi = (n + 1/2)\Phi_0$ (the ON or OPEN state), then the resistance of SN is increased. If $R_{SN} \gg R_1 + Z_{\text{load}}$, then the bias current is now diverted away from the SQUID and into the load, thereby current-biasing the load.

We require a few conditions for this scheme to work. We must have $R_{SN}(\Phi = n\Phi_0) \ll$
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\( R_1 + Z_{\text{load}} \ll R_{\text{SN}}(\Phi = (n + 1/2)\Phi_0) \), so that the current goes to the correct spots in each \( \Phi \) state. This is more easily accomplished if the switch SQUID is in the weak-screening \((\beta_L \ll 1)\) limit, since this maximizes the resistance change between \( \Phi \) states. The resistor \( R_1 \) is required if the load is a superconductor, e.g. if it is another SQUID.

This scheme is fairly obviously extended. A new block of switch + load may be appended in the dashed region. Each block requires its own coupling inductor and only a single switch SQUID should be ON at a time. Thus we require \( 2N + 2 \) wires to drive \( N \) loads. We note that a significant advantage of this scheme is that only 2 lines directly couple into the bias circuit, compared to the \( 2N + 2 \) of a standard multiplexing circuit. This reduces the noise power coupling into the load. This scheme was described by Beyer & Drung.\(^{68}\)

Now let us consider the case where we voltage bias the bias circuit. The general idea will be the same: only a single switch SQUID will be ON, and the voltage drop will be across that SQUID, thereby voltage biasing the load. We note that the coupling coil switches SN between the two \( \Phi \) states of Fig. A.7. The resistance of a single block is \( R_{\text{block}} = R_{\text{SN}}|(R_1 + Z_{\text{load}})\). We should have \( R_{\text{SN}} \ll R_1 + Z_{\text{load}} \) in the \( \Phi = n\Phi_0 \) state and \( R_{\text{SN}} \gg R_1 + Z_{\text{load}} \) in the \( \Phi = (n + 1/2)\Phi_0 \) state.

We will assume that all \( N \) blocks are identical. The \( N \) blocks make a \( N \)-resistor voltage divider. The total impedance of the bias circuit is \( R_{\text{bias}} = \sum_{i=0}^{N - 1} R_{\text{block},i} = (N - 1)(R_1 + Z_{\text{load}}) + R_{\text{SN}}^{\text{max}} \), where the last equality follows if only a single switch SQUID is ON. Then the voltage drop across the ON SQUID is \( V_{\text{ON}} = \frac{R_{\text{SN}}^{\text{max}}}{R_{\text{bias}}} \) and across all of the OFF SQUIDs
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is \( V_{\text{OFF}} = \frac{R_1 + Z_{\text{load}}}{R_{\text{bias}}} \), so

\[
\frac{V_{\text{ON}}}{V_{\text{OFF}}} = \frac{R_{\text{SN}}^{\text{max}}}{R_1 + Z_{\text{load}}}. \tag{A.19}
\]

I.e., the voltage bias across the ON SQUID is significantly larger than the voltage bias across the OFF SQUIDs. Relative to the bias voltage, we get

\[
V_{\text{ON}} = \frac{R_{\text{SN}}^{\text{max}}}{(N - 1)(R_1 + Z_{\text{load}}) + R_{\text{SN}}^{\text{max}}} V_{\text{bias}} \approx \left[ 1 - (N - 1) \frac{R_1 + Z_{\text{load}}}{R_{\text{SN}}^{\text{max}}} \right] V_{\text{bias}} \tag{A.20}
\]

\[
V_{\text{OFF}} = \frac{R_1 + Z_{\text{load}}}{(N - 1)(R_1 + Z_{\text{load}}) + R_{\text{SN}}^{\text{max}}} V_{\text{bias}} \approx \left( \frac{R_1 + Z_{\text{load}}}{R_{\text{SN}}^{\text{max}}} \right) V_{\text{bias}} \tag{A.21}
\]

where the approximations hold if \( R_{\text{SN}}^{\text{max}} \gg (N - 1)(R_1 + Z_{\text{load}}) \). As a further condition, we require that every switch SQUID can comfortably support as supercurrent the full range of current that will be passing through the load when it is ON. This is to ensure that we do not inadvertently turn ON a switch SQUID that is supposed to be OFF.

We must limit cross-coupling between loads. For incoherent loads, we require \( V_{\text{ON}} \gg \sqrt{NV_{\text{OFF}}} \). For coherent loads, we require \( V_{\text{ON}} \gg NV_{\text{OFF}} \). This results in the conditions

\[
R_{\text{SN}}^{\text{max}} \gg \sqrt{N}(R_1 + Z_{\text{load}}) \quad \text{for incoherent loads} \tag{A.22a}
\]

\[
R_{\text{SN}}^{\text{max}} \gg N(R_1 + Z_{\text{load}}) \quad \text{for incoherent loads.} \tag{A.22b}
\]

The conditions for the switch are most easily satisfied for a switch SQUID in the weak-screening \((\beta_L \ll 1)\) limit. Furthermore, it is possible to use a series array \((M \text{ DC SQUIDs wired in series})\) as the switch SQUID. This distributes the voltage drop across the \(M\) SQUIDs in the series array. Since the \(V-R\) curve (Fig. A.7) for a SQUID is

\[
R_{\text{SN}}^{\text{OFF}} = \frac{V}{I_{S\,R}} - \frac{1}{2} \left( \frac{V}{I_{S\,R}} \right)^3 + \mathcal{O} \left( \left( \frac{V}{I_{S\,R}} \right)^5 \right) \]

in the OFF state, the total resistance of the
APPENDIX A. SUPERCONDUCTING QUANTUM INTERFERENCE DEVICES (SQUIDS)

$M$ SQUIDs is $MR(V/M) < R(V)$. Additionally, since the $V-R$ curve is roughly constant in the ON state, the resistance of the series array is $MR_{SN}^{\text{max}}$. 
Appendix B

Electronic Noise Sources

Methods for dealing with noise sources in equilibrium at constant temperatures are well known and commonly taught in elementary electronics courses. However, when dealing with cryogenic systems, these conditions are not always met, so we must expand our methodology to account for this. Finally, we analyze the noise of a simple amplifier circuit to demonstrate the technique.

B.1 Power Exchange between Resistors at Different Temperatures

Nyquist’s original explanation of Johnson noise set up two resistors both at temperature $T$ separated by a matched transmission line (Fig. B.1).

Power generated in $R_1$ travels down the transmission line to $R_2$, where it is deposited.
APPENDIX B. ELECTRONIC NOISE SOURCES

Figure B.1: Power generated in resistor $R_1$ is deposited in resistor $R_2$, and power generated in resistor $R_2$ is deposited in resistor $R_1$. In thermal equilibrium where both resistors are at the same temperature, the noise voltage generated in each resistor is $\langle V_i^2 \rangle \, df = 4k_B T R_i$, and the power deposited in $R_2$ by $R_1$ is $P_2 = \frac{R_2}{(R_1 + R_2)^2} \cdot 4k_B T R_1$, and vice versa. For resistors at the same temperature, we note that $P_1 = P_2$, so the resistors are in thermal equilibrium.

Nyquist used the thermal equilibrium condition to show that the noise voltage in each resistor is given by

$$\langle (\delta V_i)^2 \rangle \, df = 4k_B T R_i \, df \tag{B.1}$$

which we recognize as the standard Johnson-Nyquist noise formula. These voltage fluctuations are converted to current fluctuations in the loop, $\delta I$:

$$\delta I = \frac{\delta V}{R_1 + R_2}$$

so the power dissipated in resistor $R_i$ that is generated by the other resistor is given by $(\delta I)^2 R_i$,

$$dP_{1 \rightarrow 2} = dP_{2 \rightarrow 1} = \frac{R_1 R_2}{(R_1 + R_2)^2} 4k_B T \, df. \tag{B.2}$$

So in thermal equilibrium, when the temperatures of resistors 1 and 2 are the same, the power exchange is equal in each direction and no net power flows from one resistor to the
APPENDIX B. ELECTRONIC NOISE SOURCES

other.

If we now consider the case where the resistors are at different temperatures, \( T_1 \) and \( T_2 \), then we note that the power exchange is no longer balanced:

\[
dP_{1 \to 2} = \frac{R_1 R_2}{(R_1 + R_2)^2} 4k_B T_1 df \\
\]

\[
dP_{2 \to 1} = \frac{R_1 R_2}{(R_1 + R_2)^2} 4k_B T_2 df
\]

and the net power into each resistor is

\[
dP_1 = dP_{2 \to 1} - dP_{1 \to 2} = -dP_2 = \frac{R_1 R_2}{(R_1 + R_2)^2} 4k_B (T_2 - T_1) df
\]

The resistors will exchange thermal energy until they come to equilibrium at the same temperature unless some external source is supplying the power differential (e.g. a heater or refrigerator). We have not accounted for any filtering of the noise due to electrical filters in the system. These will factor in when the power is integrated over the frequency to form the total power.

### B.2 Equivalent Noise Sources

Johnson noise sources may be modeled as an ideal voltage noise generator in series with a noiseless \((T = 0)\) version of the original component. Which order the voltage noise generator and noiseless component are replaced in is irrelevant (Fig. [B.2]).
APPENDIX B. ELECTRONIC NOISE SOURCES

\[ \sqrt{4k_B T_1 \Delta f} \]

\[ \sqrt{4k_B T_1 \Delta f} \]

Figure B.2: The voltage noise source may be placed in either orientation.

This trivially gives the correct results when there are no voltage or current sources in the circuit, as in Fig. B.2. This is also true for more complicated circuits, as we shall show below.

\[ V - IR_1 - IR_2 = 0 \]  \hspace{1cm} (B.6)

and the voltage drops across resistors 1 and 2 are

\[ V_1 = \frac{R_1}{R_1 + R_2} V \]  \hspace{1cm} \text{and} \hspace{1cm} \[ V_2 = \frac{R_2}{R_1 + R_2} V, \]

since the circuit forms a voltage divider. If we include Johnson noise for \( R_1 \), we get the
APPENDIX B. ELECTRONIC NOISE SOURCES

circuit in Fig. B.4 with the addition of a noise voltage generator $\delta V_1$. The noise voltage generator induces a perturbation current $\delta I_1$. Kirchoff’s laws then give

$$V + \delta V_1 - (I + \delta I_1)R_1 - (I + \delta I_1)R_2 = 0 \quad (B.7)$$

We note that the perturbations $\delta V_1$ and $\delta I_1$ are zero-mean but the normal terms are not. This means we can split this equation into zero-mean and non-zero-mean parts,

$$V - IR_1 - IR_2 = 0 \quad \delta V_1 - \delta I_1 R_1 - \delta I_1 R_2 = 0 \quad (B.8)$$

where we recognize the first equation as the equation for the base circuit. The second equation describes the Johnson noise contribution, with the equivalent circuit shown in Fig. B.5. We note that the original voltage source $V$ does not contribute to it. The noise voltage generated across $R_2$ is then the voltage difference between B and C,

$$\Delta V_2 = R_2\delta I_1 = \frac{R_2}{R_1 + R_2} \delta V_1 = \frac{R_2}{R_1 + R_2} \sqrt{4k_B T_1 R_1 \Delta f} \quad (B.9)$$
APPENDIX B. ELECTRONIC NOISE SOURCES

More interestingly, the noise voltage generated across $R_1$ is not simply $\delta V_1$! It is the voltage difference between A and B, which includes the Johnson noise generator voltage source $\delta V_1$, since that is internal to the resistor $R_1$ normally. This gives us

$$\Delta V_1 = -\delta V_1 + \delta I_1 R_1 = \left(-1 + \frac{R_1}{R_1 + R_2}\right) \delta V_1 = \frac{R_2}{R_1 + R_2} \sqrt{4k_B T_1 R_1 \Delta f} \quad (B.10)$$

from which we note that $\Delta V_1 = \Delta V_2$, which makes sense since the nodes A and C are at the same potential. A word of caution on the sign of $\delta V_1$ and $\delta I_1$: both distributions are symmetric and centered around 0, so the sign is physically meaningless. However, the signs for $\delta V_1$ and $\delta I_1$ must be consistent with each other. Thus, the contributions to $\Delta V_1$ from the $R_1 \delta I_1$ and $\delta V_1$ are opposite in sign.

An identical analysis with $\delta V_1$ to the right of $R_1$ yields the same results, justifying our assertion that which side of the resistor we place the noise generator voltage source is irrelevant. More complicated circuits are amenable to these techniques combined with standard circuit analysis techniques.

Figure B.5: Equivalent circuit for noise terms.
APPENDIX B. ELECTRONIC NOISE SOURCES

We can codify the procedure as follows:

1. Add a single ideal noise voltage generator source to the resistor being analyzed.

2. Replace ideal voltage and current sources with shorts (their non-ideal impedances may be left in place).

3. Analyze the resulting equivalent circuit for noise terms.

4. Repeat for all resistors of interest. Add resulting noise terms in quadrature.

Note that the noise from a single resistor at a time may be analyzed. The resulting noise voltages will then add in quadrature, since the noise from different resistors is independent and noise is a 2nd-order quantity.

B.3 Noise of a Basic Inverting Amplifier

We demonstrate the techniques presented in the previous section to analyze the noise of the basic inverting amplifier circuit in Fig. B.6.

The noises associated with the amplifier are Johnson noise for resistors $R_1$, $R_2$, and $R_3$, voltage noise across the op amp inputs, and current noise in each op amp input. These are shown in Fig. B.7 The relevant points to reference our noise are at $V_{in}$, across the inputs to the op amp, and at $V_{out}$. We will see how to convert between them shortly.

Op Amp Voltage Noise
Figure B.6: A basic inverting amplifier. We model the differential input impedance of the op amp as $R_{\text{in}}$ and ignore the common mode input impedance (treat it as $\infty$). We let $R_o$ be the internal output impedance of the op amp.

We first consider the op amp voltage noise $\delta V_{\text{OA}}$. We note that it is already referenced to the op amp input. Let us examine how to reference it to the circuit output and input. There is no convenient resistor across which to calculate the voltage at the new reference points. To account for this, we simply insert a target noise voltage source at our reference point (Fig. B.8) before analyzing the circuit.

The circuit is straight-forward to analyze. The op amp is treated as ideal now, so both inputs are at ground. Thus, the voltage at A is simply $\delta V_{\text{OA}}$. The current through $R_1$ is then $\delta I_1 = \delta V_{\text{OA}}/R_1$. Since the op amp is ideal, no current flows into the terminals, and so the entirety of that current must flow through $R_2$. We may close the circuit through the output noise voltage $\Delta V^\text{out}_{\text{OA}}$ and the output impedance $R_o$. Thus, we have the voltage loop equation
Figure B.7: The noise associated with the inverting amplifier. Each noise source is treated independently and summed in quadrature at the chosen reference point. The input impedance of the load on $V_{out}$ is assumed to be large compared to $R_o$. The parentheses around $V_{in}$ and $V_{out}$ indicate that we are simply identifying those node locations, not specifying their voltages. We note that $(V_{in})$ is connected to ground because we assume that the output impedance of the previous stage is small, but $(V_{out})$ is not, since the next stage’s input impedance is expected to be large. The op amp output has a pathway to ground through its output impedance, where the internal voltage supply is absent, since we are examining noise voltages only.

$$\delta I_{OA}(R_1 + R_2 + R_o) - \Delta V_{OA} = 0$$

which gives us

$$\Delta V_{OA}^{out} = \frac{R_1 + R_2 + R_o}{R_1} \delta V_{OA} \equiv \left( G_N + \frac{R_o}{R_1} \right) \delta V_{OA}. \quad (B.11)$$

We note that the quantity $G_N \equiv 1 + R_2/R_1$ is commonly called the “noise gain”. It is called this because for op amps with a well-defined gain-bandwidth product and systems where $G_N \gg R_o/R_1$, this is the gain that determines the noise bandwidth, not the amplifier
APPENDIX B. ELECTRONIC NOISE SOURCES

![Diagram](image)

Figure B.8: Referencing the op amp voltage noise to the amplifier output.

$$\text{gain}\ (G = -\frac{R_2}{R_1})$$

Since $\delta V_{OA}$ is referenced to the input of the op amp by definition, we observe that Eq. (B.11) is generally true for any noise that has been referenced to the op amp input. The conversion factor is always $G_N + \frac{R_o}{R_1} \simeq G_N = 1 + \frac{R_2}{R_1}$, where we’re assuming $R_o$ is small from here on out. Furthermore, since the amplifier output is referenced to ground and the amplifier input is referenced to ground, the conversion between the two is simply the amplifier gain $G$. This gives us the conversion table (in = amplifier input, 1)

$\text{The justification for this statement is this derivation. The effective gain of this amplifier on the op amp noise is } G_N, \text{ not } G.$
APPENDIX B. ELECTRONIC NOISE SOURCES

OAI = op amp input, out = amplifier output

\[ \Delta V_{in}^{\text{OAI}} = \frac{G}{G_N} \Delta V_{\text{OAI}} = \left(1 + \frac{R_1}{R_2}\right) \Delta V_{\text{OAI}} \quad (B.12a) \]

\[ \Delta V_{out}^{\text{OAI}} = G_N \Delta V_{\text{OAI}} = \left(1 + \frac{R_2}{R_1}\right) \Delta V_{\text{OAI}} \quad (B.12b) \]

\[ \Delta V_{out}^{\text{in}} = G \Delta V_{in} = R_2 R_1 \Delta V_{in} \quad (B.12c) \]

**Op Amp Current Noise**

We next treat the current noise of the op amp. We will first do the current noise on the non-inverting terminal \( \delta I_{OP+} \). The current noise generator on the non-inverting terminal must return to ground. The only path to ground (since we are ignoring the input impedance) is through \( R_3 \), so the current noise generator \( \delta I_{OA} \) generates a voltage across the op amp input of\(^2\)

\[ \Delta V_{IAO+}^{\text{OAI}} = \delta I_{OA+} R_3 \quad (B.13) \]

which when referenced to the amplifier output is

\[ \Delta V_{IAO+}^{\text{out}} = G_N \delta I_{OA+} R_3 = \left(1 + \frac{R_2}{R_1}\right) R_3 \delta I_{OA+} \quad (B.14) \]

The current noise on the inverting terminal \( \delta I_{OP-} \) is handled similarly. The current must return to ground. The generator sees a path to ground with impedance \( R_1 || (R_2 + R_o) \),

\(^2\)The sign of the noise generators is irrelevant, since they are mean 0. We drop the negative sign on the amplifier gain for convenience.

\(^3\)The fact that the resulting voltage generator is on the non-inverting terminal is irrelevant, since the op amp input-referenced noise is of the potential difference between the inverting and non-inverting terminals.
APPENDIX B. ELECTRONIC NOISE SOURCES

so the voltage at the inverting terminal is

\[ \Delta V_{\text{IOA}}^{\text{OA}} = \delta I_{\text{OA}} \frac{R_1(R_2 + R_o)}{R_1 + R_2 + R_o} \]  \hspace{1cm} (B.15)

which when referenced to the amplifier output is

\[ \Delta V_{\text{IOA}}^{\text{out}} \approx \delta I_{\text{OA}} - R_2, \hspace{1cm} R_o \ll R_2. \]  \hspace{1cm} (B.16)

**Johnson Noise**

We now turn our attention to the Johnson noise of the resistors. We begin with \( R_1 \). The voltage noise generator generates a current \( \delta I_{R_1} = \delta V_{R_1} / (R_1 + R_2 + R_o) \). The voltage at the inverting input is then

\[ \Delta V_{\text{IOA}}^{\text{OA}} = \delta I_{R_1} (R_2 + R_o) = \frac{R_2 + R_o}{R_1 + R_2 + R_o} \delta V_{R_1}. \]  \hspace{1cm} (B.17)

It is not simply \( R_1 \delta I_{R_1} \) because the Johnson noise generator is in the path from ground through \( R_1 \) to the inverting terminal. An equally valid way of reaching the result, however, is \( \Delta V_{R_1}^{\text{OA}} = \delta V_{R_1} - \delta I_{R_1} R_1 \). Referenced to the output, this is

\[ \Delta V_{R_1}^{\text{out}} = G_N \Delta V_{R_1}^{\text{OA}} \simeq \frac{R_2}{R_1} \delta V_{R_1} = G \delta V_{R_1}. \]  \hspace{1cm} (B.18)

We may obtain this result more quickly by noting that we may equally well put \( \delta V_{R_1} \) to the left of \( R_1 \), in which case it is a voltage noise that is referenced to input. Then to reference it to output, we simply multiply by the amplifier gain \( G \).
APPENDIX B. ELECTRONIC NOISE SOURCES

Next we consider $R_2$. The resulting current is, as before, \( \delta I_{R_2} = \delta V_{R_2}/(R_1 + R_2 + R_o) \).

The voltage at the inverting terminal is then

\[
\Delta V_{R_2}^{OAi} = \delta I_{R_2} R_1 = \frac{R_1}{R_1 + R_2 + R_o} \delta V_{R_2} \tag{B.19}
\]

which when referenced to the amplifier output is

\[
\Delta V_{R_2}^{out} = G_n \Delta V_{R_2}^{OAi} \simeq \delta V_{R_2} \tag{B.20}
\]

Finally, we consider $R_3$. We note simply that the voltage noise generator may be placed above $R_3$, in which case it is already referenced to the op amp input.

\[
\Delta V_{R_3}^{OAi} = \delta V_{R_3} \tag{B.21}
\]

Referenced to the amplifier output, this is

\[
\Delta V_{R_3}^{out} = G_n \delta V_{R_3} = \left(1 + \frac{R_2}{R_1}\right) \delta V_{R_3}. \tag{B.22}
\]

**Total Noise**

Summed together, the total noise referenced to the amplifier output is

\[
(\Delta V_{out}^{total})^2 = (\Delta V_{OA})^2 + (\Delta V_{1OA+})^2 + (\Delta V_{1OA-})^2 + (\Delta V_{R_1}^{out})^2 + (\Delta V_{R_2}^{out})^2 + (\Delta V_{R_3}^{out})^2
\]

\[
(\Delta V_{out}^{total})^2 = G_N^2 (\delta V_{OA})^2 + \left[R_2^2 + G_N^2 R_3^2\right] (\delta I_{OA})^2
\]

\[
+ G^2 (4k_B T_1 R_1) + (4k_B T_2 R_2) + G_N^2 (4k_B T_3 R_3) \tag{B.23}
\]
APPENDIX B. ELECTRONIC NOISE SOURCES
Appendix C

Continuous-time PID Control Loops

Consider the control system in Fig. C.1, which describes a simple PID controller. The transfer functions of the PID controller and process are given by $C(s)$ and $P(s)$ in Laplace space. We want to analyze how well the controller can actuate the system so its output $y$ matches the reference $r$, i.e. determine the full system transfer function $H(s) = Y(s)/R(s)$.

The open loop transfer function $L(s)$ is determined by the response of the system in the absence of feedback,

$$L(s) = C(s)P(s). \quad \text{(C.1)}$$

Upon closing the feedback loop we find that the output depends on the error, $Y(s) = E(s)L(s) + N(s)$. Combining this with the definition of the error $E(s) = R(s) - Y(s)$, we find that
Figure C.1: The control loop block diagram of a simple PID loop. The reference (setpoint) signal \( r(t) \) is compared against the measured output of the process \( y(t) \) to form the error \( e(t) \). The error \( e(t) \) serves as the input to the PID controller \( C(s) \) to form the process input \( u(t) \). This input drives the process \( P(s) \). Additive noise \( n(t) \) is added to the output of the process to simulate measurement noise and forms the measured process output \( y \). The measured output is fed back to the reference signal \( r(t) \) to close the loop.

\[
Y(s) = \frac{L(s)}{1 + L(s)} R(s) + \frac{1}{1 + L(s)} N(s) \tag{C.2}
\]

\[
Y(s) = H(s) R(s) + H_N(s) N(s). \tag{C.3}
\]

Since there are two inputs (the reference signal \( r \) and the noise \( n \)), we have a pair of transfer functions. The system transfer function is

\[
H(s) = \frac{Y(s)}{R(s)} = \frac{L(s)}{1 + L(s)} = \frac{C(s) P(s)}{1 + C'(s) P(s)} \tag{C.4}
\]

The noise figures into the control system differently and has a noise transfer function

\[
H_N(s) = \frac{1}{1 + L(s)} = \frac{1}{1 + C(s) P(s)} \tag{C.5}
\]
These expressions are generically true for any controller $C(s)$ and process $P(s)$. The PID controller’s transfer function is given by

$$C(s) = k_P + \frac{k_I}{s} + k_D s$$  \hspace{1cm} (C.6)

where $k_P$, $k_I$, and $k_D$ are the proportional, integral, and differential coefficients.

Consider a process whose response $x(t)$ to an input signal $u(t)$ described by the first order differential equation

$$\frac{dx(t)}{dt} + ax(t) = bu(t)$$  \hspace{1cm} (C.7)

with transfer function

$$P(s) = \frac{X(s)}{U(s)} = \frac{b}{s + a}.$$  \hspace{1cm} (C.8)

This describes a generic first order process.

A second order process\footnote{Note this is not fully generic. The fully generic form additionally allows a driving term proportional to $\frac{du(t)}{dt}$.} and its transfer function are given by

$$\frac{d^2 x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_0 x(t) = bu(t)$$  \hspace{1cm} (C.9a)

$$P(s) = \frac{b}{s^2 + a_1 s + a_0}.$$  \hspace{1cm} (C.9b)
APPENDIX C. PID CONTROL LOOPS

Since second order processes describe harmonic oscillators, they are frequently recast in the standard notation

\[ \frac{d^2 x(t)}{dt^2} + 2\zeta \omega_0 \frac{dx(t)}{dt} + \omega_0^2 x(t) = K \omega_0^2 u(t) \]  

(C.10a)

\[ P_0(s) \equiv P(s) = \frac{\omega_0^2}{s^2 + 2\zeta \omega_0 s + \omega_0^2} \]  

(C.10b)

where \( \omega_0 \) is the natural frequency, \( \zeta \) is the dissipation constant, and \( K \) is the gain.

Our basic strategy for understanding most control systems will be to match their transfer functions to a 2nd order system and then use the well-known solutions of the harmonic oscillator equations to compute the response of the control system.

The frequency space transfer function and phase of the harmonic oscillator system are

\[
|P_0(\omega)| = |P(\omega)| = \frac{K \omega_0^2}{\sqrt{(\omega^2 - \omega_0^2)^2 + (2\zeta \omega_0 \omega)^2}} \]  

(C.11a)

\[
\arg P_0(\omega) = \arg P(\omega) = \arctan \left( \frac{2\zeta \omega_0 \omega}{\omega^2 - \omega_0^2} \right) \]  

(C.11b)

and the transient (homogeneous) solutions are

\[
x_0(t) \equiv x(t) = \begin{cases} 
  e^{-\zeta \omega_0 t} (c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t}) & \zeta \neq 1 \text{ (underdamped/overdamped)} \\
  (c_1 + c_2 t) e^{-\omega_0 t} & \zeta = 1 \text{ (critically damped)} 
\end{cases} \]  

(C.12)

where \( \bar{\omega} = \omega_0 \sqrt{1 - \zeta^2} \). We note for \( \zeta < 1 \), \( \bar{\omega} \) is real and the response is oscillatory, corresponding to the underdamped case. For \( \zeta > 1 \), \( \bar{\omega} \) is imaginary and the response is
purely exponential, corresponding to the overdamped case.

Next considering the fully generic second order system

\[ P(s) = K \frac{\beta \omega_0 s + \omega_0^2}{s^2 + 2\zeta \omega_0 s + \omega_0^2} = \frac{\beta s}{\omega_0^2} \frac{K\omega_0^2}{s^2 + 2\zeta \omega_0 s + \omega_0^2} + \frac{K\omega_0^2}{s^2 + 2\zeta \omega_0 s + \omega_0^2} \]  

we observe that the frequency response and phase are

\[ |P(\omega)| = |P_0(\omega)| \sqrt{1 + \beta^2 \omega^2} \] (C.14a)

\[ \arg P(\omega) = \left( \frac{\omega}{\omega_0} \right) \left( \frac{2\zeta \omega_0^2 + \beta (\omega^2 - \omega_0^2)}{2\beta \zeta \omega_0^2 + (\omega^2 - \omega_0^2)} \right) \] (C.14b)

and the transient response is

\[ x(t) = x_0(t) + \frac{\beta x_0(t)}{\omega_0} \frac{dx_0(t)}{dt} \] (C.15)

It is usually desireable for \( \beta \) to be small so that the response is close to the ideal sinusoid response.

\section{State-space Representation of Systems}

For complicated systems it may be more convenient to work with the systems in state-space. The state-space representation utilizes a set of intermediate variables such that the output is linear with respect to the state variables and the input, and such that the state
APPENDIX C. PID CONTROL LOOPS

variable evolution is a first order equation of the state variables and input. Using $y$ for the output, $x$ for the state variables, and $u$ for the input, we have

\[
\frac{dx}{dt} = Ax + Bu \tag{C.16a}
\]
\[y = Cx + Du. \tag{C.16b}
\]

Note that $x$, $y$, and $u$ may be vectors and $A$, $B$, $C$, and $D$ are matrices. The transfer function is straightforward to compute by Laplace transforming the system directly,

\[
sX(s) = AX(s) + BU(s)
\]
\[Y(s) = CX(s) + DU(s)
\]

and eliminating $X(s)$,

\[
H(s) = \frac{Y(s)}{U(s)} = C (sI - A)^{-1} B + D \tag{C.17}
\]

C.2 First Order Systems

We may now expand the transfer function of the control system’s transfer function using the expressions for $C(s)$ and $P(s)$. We first consider a generic first order process.

\[
H(s) = \frac{b (k_D s^2 + k_P s + k_I)}{(1 + bk_D)s^2 + (a + bk_P)s + bk_I}
\]
We observe that the differential term overcontrols the system and is unnecessary. With $k_D = 0$, we find

$$H(s) = \frac{b(ks + k_I)}{s^2 + (a + bk_P)s + bk_I}$$ \hspace{1cm} (C.18)

which is a second order system. The DC response $H(0) = 1$ as long as $k_I \neq 0$, reflecting the well-known fact that a pure proportional controller has an offset in the response, but an integral controller does not. If $k_I = 0$, then $H(0) = 1$ iff $a = 0$, which is a very uninteresting process system.

This system may be compared against the harmonic oscillator system Eq. (C.13) to get the correspondence

$$bk_I = \omega_0^2 \hspace{1cm} (C.19a)$$

$$a + bk_P = 2\zeta\omega_0 \hspace{1cm} (C.19b)$$

$$bk_P = \beta\omega_0 \hspace{1cm} (C.19c)$$

## C.3 Second Order Systems

Next we compute the transfer function of a generic second order process.

$$H(s) = \frac{b_1k_D}{s^3 + \left(\frac{b_1k_P + b_2k_D}{1+b_1k_D}\right)s^2 + \left(\frac{b_1k_I + b_2k_P}{1+b_1k_D}\right)s + \frac{b_2k_I}{1+b_1k_D}}$$ \hspace{1cm} (C.20)
APPENDIX C. PID CONTROL LOOPS

This is typically compared against a harmonic oscillator system with an extra pole, for which the homogeneous system is \((s + \alpha \omega_0)(s^2 + 2\zeta \omega_0 s + \omega_0^2)\). This gives the correspondence

\[
\frac{a_1 + b_1 k_P + b_2 k_D}{1 + b_1 k_D} = (\alpha + 2\zeta) \omega_0 \quad \text{(C.21a)}
\]

\[
\frac{a_2 + b_1 k_I + b_2 k_P}{1 + b_1 k_D} = (1 + 2\alpha \zeta) \omega_0^2 \quad \text{(C.21b)}
\]

\[
\frac{b_2 k_I}{1 + b_1 k_D} = \alpha \omega_0^3 \quad \text{(C.21c)}
\]
Appendix D

Relationship between Integration Time and Bandwidth

Suppose we integrate a signal $x(t)$ for a period of time $T$. This is equivalent to filtering the signal with a rect filter,

$$h(t) = \text{rect}(t/T) = u(t + T/2) - u(t - T/2), \quad (D.1)$$

where $u(t)$ is the Heaviside step function. In harmonic space, this filter is

$$H(f) = F\{h(t)\} = T \text{sinc} (fT) = T \frac{\sin(\pi fT)}{\pi fT} \quad (D.2)$$

where we define our fourier transform as

$$F\{x(t)\} = \int_{-\infty}^{\infty} df \ x(t) \exp(-2\pi if t)$$

The bandwidth of a signal is conventionally defined as the distance in frequency space between the first positive and first negative node. We note that the sinc function is symmet-
APPENDIX D. INVERSE SECONDS TO HZ

ric and has its first node at \( f = 1/T \), so the bandwidth is \( B = 2/T \). Thus the relationship between integration time and bandwidth is

\[
T \text{ seconds integration time} \leftrightarrow \frac{2}{T} \text{ Hz bandwidth}
\]

1 second integration time \( \leftrightarrow \) 2 Hz bandwidth

and so the conversion between seconds of integration time and Hz of bandwidth is

\[
1 = \frac{1 \text{ s}}{\frac{1}{2} \text{ Hz}^{-1}} = \frac{2 \text{ s}}{1 \text{ Hz}^{-1}}
\]  \hspace{1cm} (D.3)

As a final note, the relationship between bandwidth and integration time is inversely proportional. As the integration time increases, we integrate a smaller bandwidth. This is intuitively correct, since as the bandwidth decreases, the noise will decrease, and as the integration time increases, the noise will decrease.
Appendix E

Optimization of ILC Weights

The ILC solution can be posed as the minimization problem

\[
\begin{align*}
\text{Minimize} & \quad \text{var } \hat{T} = w^T C w \\
\text{subject to} & \quad 1^T w = 1
\end{align*}
\]

with \( w \) as the independent variable. This is amenable to solution by Lagrange multipliers.

Define the function

\[
L = \text{var } \hat{T} + \lambda (1^T w - 1) .
\] (E.1)

Then the solution to our optimization problem is given by the set of equations

\[
\begin{align*}
\frac{\partial L}{\partial w^T} &= 2Cw + \lambda 1 = 0 & \text{(E.2a)} \\
1^T w &= 1 & \text{(E.2b)}
\end{align*}
\]

Solving Eq. (E.2a) for \( w \)

\[
w = -\frac{\lambda}{2} C^{-1} 1
\]

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APPENDIX E. OPTIMIZATION OF ILC WEIGHTS

and plugging it into Eq. (E.2b) allows us to solve for \( \lambda \),

\[
1^T w = -\frac{\lambda}{2} 1^T C^{-1} 1 = 1
\]

\[
\lambda = -2 (1^T C^{-1} 1)^{-1} \tag{E.3}
\]

and then we may substitute into \( w \),

\[
w^* = \frac{C^{-1} 1}{(1^T C^{-1} 1)} \tag{E.4}
\]
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Vita

Justin Lazear was born in Northern California to Yvonne and Ed Lazear. He attended the California Institute of Technology in Pasadena, CA as an undergraduate and received a BS in Physics in 2008. He worked in the OBSCOS group while at Caltech and continued that work into early 2009. Following this, he joined the Johns Hopkins University Department of Physics and Astronomy in Baltimore, MD in 2009 as a graduate student.