A Coupled Multi-physics Analysis Model for Integrating Transient Electro-Magnetics and Structural Dynamic Fields with Damage

by

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Abstract

The development of advanced material processing and the appearance of novel materials introduce a broad and promising area of multi-functional structures. The improvements of these structures over the traditional ones are the various capabilities to perform multiple tasks. Multi-physics phenomena, such as mechanical (ME) and electromagnetic (EM) coupling, are fundamental to study such structures. Some examples of these structures may be components of small unmanned airborne vehicles (UAVs), active skins of aircraft, or meta-materials for optical and communication systems. There is a need for a robust, coupled multi-physics computational model and codes supporting meaningful design of multi-functional structures and devices.

In this dissertation, a generalized framework is developed for coupling EM and dynamic ME fields under finite deformation. To achieve a versatile and robust coupling scheme between fields, the problems are solved in a staggered way using a time-domain finite element (FE) method by a high performance parallel program. The deformation information from solved ME field is used to obtain the EM field in the same configuration. To account for finite deformation and its effects on the EM fields, a Lagrangian description is invoked for both ME field and EM field. Unlike tra-
ditional scheme to simulate EM field, the coupling scheme maps Maxwell’s equations from spatial to material coordinates in the reference configuration. For an efficient solution, a scalar potential and vector potentials are chosen as independent solution variables in lieu of EM field variables to reduce the degree of freedom. Non-uniqueness in the solution of the reduced set of equations is overcome through the introduction of a gauge condition in the FE formulations. The boundary conditions are appropriately represented in terms of the potentials that do not represent physical variables.

A high performance, parallel code in FE method is developed to solve the multi-physics problems. The computational domain is decomposed and distributed to multiple processors using the ParMETIS library. Subsequently, the Portable, Extensible Toolkit for Scientific Computation or PETSc library, which is a Message Passing Interface (MPI) based library, is employed to accomplish the parallelization of the code for assembling and solving both the ME and the EM problems. Selected features of the code are validated using existing solutions in the literature, as well as comparison with results of simulations with commercial software. Convergence and accuracy of the code are examined.

In view of functionality of different devices, two sets of multi-physics phenomena are studied thoroughly using the framework. The load-bearing antenna application requires the coupling between transient EM and dynamic ME fields. The simulations
predict the evolution of electrical and magnetic fields and their fluxes in a vibrating substrate undergoing finite deformation. In this application, the Lorentz force generated in the coupling is negligible compared with the external applied mechanical load. Thus, the coupling is only one-way from the ME field to the EM fields. The effects of mechanical load frequency, amplitudes and direction on EM fields are investigated.

Furthermore, a novel self-sensing piezoelectric sensor is introduced by implementing a three-dimensional isotropic damage model with piezoelectric material. In the contrast to the load-bearing antenna application, two-way coupling of electric field and ME field is necessary through piezoelectricity. The piezoelectric coupling is achieved in the reference configuration to accommodate the coupling under finite deformation. The damage criterion is established from the maximum deviatoric strain energy throughout the mechanical loading history. The degradation impacts on both mechanical stiffness and piezoelectric coupling constant. The damage developed in the structure can be predicted by the electric field generated from piezoelectricity. The difference of the electric field between damaged and undamaged structure is employed to correlate with the damage parameter and its time derivative. A rigorous functional form is proposed for the correlation function. The model is calibrated and validated by different simulation cases.
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Chapter 1

Introduction

Multifunctional structures are experiencing significant improvement and increasingly gaining importance [1, 2] as the development of novel material processing. They are of great interest in aerospace industry, consumer electronics and structural health monitoring (SHM) applications. These structures can be components of small unmanned airborne vehicles (UAVs), active skins of an aircraft[3, 4], vibration control devices [5, 6], meta-materials [7, 8] for optical and communication systems, flexible devices in energy, sensing, memory such as high mobility and stretchable electronics[9–12], elastomer-based EM devices, optical dielectric resonator antennas etc.. The multifunctional structures that are governed by multi-physics principles such as mechanical and electromagnetic relations. Different coupling effects between mechanical and electromagnetic relations result in two sets of applications.

The first type coupling scheme requires analysis of EM fields under structural deformation or complex material evolution. However, the EM fields generate negligible effect on the mechanical field. These devices can have multiple phases in the form of embedded conductors and/or reinforcing fibers in the matrix or substrates
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made of dielectrics or organic materials. Conductors can also serve as reinforcing phases for structural integrity and stability. Standard EM devices such as antennae have been traditionally based on stiff structures and are typically designed for transmission of EM waves alone without deformation considerations. In most of these devices, the mechanical field in the structure has significant effect on the EM signals. Discussions on utility and requirements of load-bearing antennae have been made in [13, 14]. Load-bearing antennae are subjected to mechanical vibrations, which have frequencies that are considerably different from those for the EM fields. Modeling the evolution of EM fields in a moving deformable media to analyze load-bearing antennae is a challenging problem. Only a limited number of coupled computational models are available in the literature. FE analysis of the EM problems for signal transmission in antennas has been mostly carried out in the frequency domain [15]. However, frequency domain computations are not suitable when coupled with finite deformation analysis. Novel descriptions of hp-adaptive FE methods for EM problems have been developed in [16, 17]. Various generalizations of the hp-adaptive FE methods have been made for coupled multi-physics problems with ME fields in [18]. For multifunctionality, electromagnetic equations need to be solved in the reference configuration, requiring a strong coupling with the deformation field.

The other type coupling scheme asks for a two-way coupling between EM fields and ME field. One example of such structure constituent is the piezoelectric material. The coupling between the fields happens naturally since the piezoelectric material is
able to convert mechanical energy to electrical energy and vice-versa. Piezoelectricity rises from the absence of inversion symmetry of crystalline materials. Traditionally, the behavior of piezoelectric materials is governed in small elastic deformation regime [19]. By integrating advanced composite materials in structural components, novel intelligent multi-functional structures improve the functionality of traditional materials. In these materials and structures, piezoelectric phenomenon can happen under finite deformation with both flexibility and functionality provided. To observe piezoelectric phenomena under finite deformation, it is proposed that piezoelectric material is bonded with elastic substrate or embedded in a laminated composite structure [6, 20, 21]. However, in real application, delamination and impact damage can develop when the composite structure endures finite deformation. SHM of such structures reduces inspection/maintenance costs within life-cycle and improves the reliability by predicting failure and detecting damage. The SHM methods include strain gauge[22], optical fibers[23], eddy current[24], acoustic emission[25], model analysis and lamb waves[26]. Most existing methods are qualitative other than quantitative. It is reported that although repeated mechanical and electrical cyclic loading of piezoelectric ceramics results in a progressive degradation [27], the performance of piezoelectric ceramics is durable and efficient for millions of cycles [28]. Thus early warning may not be necessary in every application. In this work, a damage detector based on piezoelectricity is proposed. The change of electric field signal between the damaged and undamaged structures under the same mechanical loading is employed to identify
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the degradations of stiffness and piezoelectric coupling term. A quantitative damage sensing is achieved in real-time.

As stated, both applications and coupling schemes require to analyze electromagnetic/electric field with finite deformation and/or complex material evolution. Thus, traditional method to solve EM field in current configuration without considering material form formulation is not suitable for the purpose to couple with ME field. There is a need for a robust, coupled multi-physics computational model and codes for meaningful design of multifunctional structures and devices.

Earlier works [29–38] discuss the coupling between mechanical field and electromagnetic fields, invoking a Lagrangian description to couple the EM fields and finite deformation in the same configuration. Based on the proposed solving scheme, the electric field coupled with finite deformation are carried out in electromagnetic forming (EMF) processing, where EM forces are utilized to deform a solid body in the forming process in[39, 40]. An analytical treatment is discussed in [29] for transformations of the Maxwell’s equations in a moving medium, where effective EM fields due to deformation are derived. Multi-physics problems of charged particles due to EM forces and the near-field effects on particulate dynamics is modeled in [41–44]. Piezoelectric material coupled with finite deformation is studied in [45], however the constitutive equations are in the current configuration. The constitutive relationship of piezoelectric materials with finite deformation analysis is studied in [32, 46, 47]. [48] introduces the topic of fracture for piezoelectric ceramic. Among the review lit-
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eratures, the study of coupling of piezoelectricity with phenomenological damage at continuum level is limited. Experimentally, [27] reports that the piezoelectric material properties are changed under high electrical and mechanical stress. Piezoelectric material property $d_{33}$ in soft ceramics (PZT-5A) decreases under cyclic mechanical loading while hard piezoelectric material (PZT-4D) is more sensitive to constant load with extended periods other than cyclic loading condition. [49, 50] studied the damage of lead zirconate titanate piezoelectric ceramic with cyclic loading. It is reported that the piezoelectric constant $d_{33}$ of the experiment sample decreases with increasing cycle number when the applied stress is at critical level. However, the electromechanical coupling coefficient does not have the similar trend and is not significantly affected by cyclic loading. As for the change of dielectric properties, the experiments cannot give consistent result from different loading types. [51] introduces a piezoelectric damage model based on energy equivalence. The derivation reveals that the piezoelectric material property is affected by both mechanical damage and electric damage. Although it is well received that the mechanical damage will alter the piezoelectric coupling performance, the assumption that piezoelectric damage need to taken electric damage into account is not reported in other literatures.

Recently a few of the multi-physics commercial codes have included coupled mechanical-electromagnetic or ME-EM analysis capabilities. Table 1.1 summarizes capabilities and limitations of some of these codes.
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<table>
<thead>
<tr>
<th>Software</th>
<th>Capability</th>
<th>Limitations</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABAQUS</td>
<td>Piezoelectric material analysis with small deformation, low frequency transient EM field</td>
<td>Does not couple high frequency EM fields with ME fields. Lack of piezoelectric material simulation in finite deformation</td>
</tr>
<tr>
<td>COMSOL</td>
<td>Coupled low frequency EM field, radio frequency EM field using vector potentials with ME field</td>
<td>Does not fully couple transient EM and ME fields in time domain</td>
</tr>
<tr>
<td>ANSYS</td>
<td>Couples EM (HFSS) and structural problems with updated configuration</td>
<td>Does not have two way coupling for EM-ME problems</td>
</tr>
</tbody>
</table>

Table 1.1: Comparison of a few commercial codes for multi-physics ME-EM simulations

In the present dissertation, a generalized framework to couple dynamical mechanical field and transient EM fields using FE method is developed as published in [52]. This model is extended in [53] to overcome the disparate time scale of multi-physics coupling in time domain using wavelet transformation induced multi-time scaling (WATMUS) method. This framework suits the need to couple transient electromagnetic and dynamic mechanical fields to predict the evolution of electrical and magnetic fields and their fluxes in a vibrating substrate undergoing finite deformation. A catalog of EM problems in deforming media can be solved with this framework, e.g. load-bearing antennae. To achieve coupling between fields with disparate frequency ranges, the governing equations are solved in the time domain, as opposed to the frequency domain. Following methods developed in [30–32], a Lagrangian description is invoked, in which the coupling scheme maps Maxwell’s equations from spatial to material coordinates in the reference configuration. Weak forms of the coupled tran-
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sient EM and dynamic ME equations are generated in the reference configuration by applying the Euler-Lagrange stationary conditions [30], and the Galerkin’s method is used for developing the FE equations. For an efficient solution process, physical variables in the Maxwell’s equations are written in terms of a scalar potential and vector potentials. The process reduces the number of Maxwell’s equations and associated field variables. Non-uniqueness in the solution of the reduced set of equations is overcome through the introduction of a gauge condition in the FE formulations. The boundary conditions also need to be appropriately represented in terms of the potentials that do not represent physical variables.

To further extend the study to the proposed damage detector based on piezoelectric material, a theoretic and numerical tool which can couple mechanical field, damage mechanics and electric field under finite deformation at continuum level is studied based on the coupling platform developed. Piezoelectric constitutive relations and phenomenological damage mechanics coupling with finite deformation in the reference configuration are examined. The reference configuration establishes a platform to analyze the electrical effects undergo the complex material behaviors including damage. It benefits the study of material inelasticity by formulating the problem in material form. The numerical analysis is conducted by implementing piezoelectric material and continuum damage in the framework developed in [52]. The coupled scheme is robust and versatile achieved using the staggered way.
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1.1 Organization of the Thesis

The present dissertation starts with the discussion on the governing equations for the transient EM problems in a finite deformation dynamical setting in Section 2.1 and Section 2.2. The weak forms are derived in Section 2.3. FE implementation using updated Lagrangian formulation for the mechanical problem and total Lagrangian formulation for the EM fields are developed in Section 2.4. Aspects of parallel implementation of the code are discussed in this section. To further bringing piezoelectric material into the framework, the piezoelectric material constitutive equations in the reference configuration are obtained in Section 3.1. Objectivity of the coupled system is examined. To numerically study the correlation between the electric field and damage parameter in damage detector application, the continuum damage model is introduced into the platform. Both the damage criteria and evolution law are included in Section 3.2 coupled with piezoelectricity. Weak forms of the implementation of piezoelectric material and continuum damage model are updated in Section 3.3. FE method to analyze the damage detector application is illustrated in Section 3.4.

Section 4.1 introduces numerical examples that are simulated with the coupled dynamic electromagnetic (ME-EM) model and code. Selected features of the code are validated using existing solutions in the literature, as well as comparison with results of simulations by commercial software and codes. Piezoelectric model in the finite deformation is validated with analytical solution in small deformation regime.
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Subsequently two coupled simulations, featuring EM fields excited by steady-state and time varying electric current source in dynamically vibrating conducting media, are conducted and studied. Results demonstrate the effectiveness of the proposed method. An example of piezoelectric material problem is carried out. Results from developed code featuring piezoelectricity coupling with finite deformation and from Abaqus limited with piezoelectricity coupling with small deformation are compared. Last but not least, the idea of damage detector is shown by calibrating the correlation function between variables of electric field and damage parameter in Section 4.3. A numerical validation of the correlation function is carried out by studying the stretchable electronics using the obtained correlation function. The thesis is concluded in Chapter 5.
Chapter 2

Finite Element Model for Coupled Transient Electromagnetic and Dynamical Mechanical Field

Evaluation of transient EM fields variables in a deformable, conducting medium undergoing finite dynamic deformation requires a comprehensive solution approach that couples the EM and ME fields through their governing equations. The evolution of the EM fields variables depends on the deformed configuration, as well as on the point-wise velocity distribution of the vibrating medium. The mechanical field variables, in turn, are subject to be affected by electromagnetically induced forces, e.g. the Lorentz force. In many practical applications, such as load-bearing antennae, the magnitude of the Lorentz force is negligible in comparison with the externally applied mechanical forces on the structure. In these cases, the explicit effect of EM fields variables on the mechanical field, e.g. the deformation of the structure may be ignored. This reduces to a one-way coupling between the two fields, i.e. only the EM variables are affected by the deformation of the structure.
CHAPTER 2. FE MODEL FOR TRANSIENT EM AND DYNAMIC ME COUPLING

However, in other applications, such as piezoelectric material, the coupling between ME and EM fields happens at the constitutive relation level. Rather than solve the coupled problems one-way, fully coupling approach is necessary, i.e. the mechanical field is also affected by the electric field in the structure. In this case, the coupling between electric field and mechanical field need to be implemented as a two-way coupling scheme.

To accommodate both scenarios under one unified framework for coupled electromagnetic and mechanical problems, a staggered solution approach providing a versatile and robust coupling scheme is pursued. Firstly, the ME field is solved. Deformation, velocity and acceleration obtained from the solved configuration are transferred to evaluate the EM fields in the same configuration. If the one way coupling is applicable, the ME field solution will proceed without the information from EM fields. Otherwise, the information from EM fields, such as electric field and voltage will be recorded for the coupled ME field.

In this chapter, the governing equations for the mechanical and electromagnetic problems under the reference configuration are discussed. Weak forms and boundary conditions are examined in the reference configuration and potentials for the EM fields. The FE implementation is conducted for both ME and EM fields.
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2.1 Governing Equations for the Finite Deformation Dynamics Problem

The mechanical response of the conducting medium is modeled using governing equations for a hyperelastic material undergoing finite deformation under dynamic loading conditions. In a Lagrangian formulation, the reference configuration \( \Omega_0(= \Omega(t_0)) \) at time \( t_0 \) is expressed in terms of the material coordinates \( X_I \), \( I = 1, 2, 3 \), while the current configuration \( \Omega(t) \) at time \( t \) is represented by the current coordinates \( x_i \), \( i = 1, 2, 3 \). The deformation of the body is expressed using the single-valued mapping function \( x_i = \varphi_i(X_J, t) \). Correspondingly, the Cartesian components of the displacement vector in the material coordinates are expressed as: \( u_i(X_J, t) = x_i - \delta_{iJ} X_J \).

2.1.1 Some Hyperelastic Material Models

2.1.1.1 Neo-Hookean Material Model

Several constitutive relations for the hyperelastic material at finite strains are implemented. One of them is neo-Hookean material model, for which the strain energy density function \( \Psi \) is expressed in terms of kinematics variables as:

\[
\Psi = \frac{1}{2} \lambda (ln J)^2 - \mu \ln J + \frac{1}{2} \mu (C_{II} - 3) \quad (2.1)
\]
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where $\lambda$ and $\mu$ are Lamé constants, and

$$C_{IJ} = \left( \frac{\partial x_k}{\partial X_I} \frac{\partial x_k}{\partial X_J} \right)$$ \hfill (2.2)

is the right Cauchy-Green deformation tensor. $C_{II}$ is the first invariant of $C_{IJ}$. The positive-valued Jacobian $J$, which determines admissible deformation mapping, is defined in terms of the deformation gradient tensor $F_{iJ}$ as:

$$J = \text{det}(F_{iJ}) > 0 \quad \text{where} \quad F_{iJ} = \frac{\partial x_i}{\partial X_J}$$ \hfill (2.3)

The stress-strain relation for finite strains is derived from the energy density expression in (2.1) as:

$$S_{IJ} = 2 \frac{\partial \Psi}{\partial C_{IJ}} = \lambda \ln J C^{-1}_{IJ} + \mu (\delta_{IJ} - C^{-1}_{IJ})$$ \hfill (2.4)

where $S_{IJ}$ and $\sigma_{ij}$ are components of the second Piola-Kirchhoff and Cauchy stress tensors respectively. The relationship between them is:

$$S_{IJ} = J \frac{\partial X_I}{\partial x_m} \frac{\partial X_J}{\partial x_n} \sigma_{mn}$$ \hfill (2.5)

Assume a uniform load in $x$-direction is applied on a infinite structure, the stress-strain curve in the $x$-direction of neo-Hookean material is shown in Fig. 2.1. Both second Piola-Kirchhoff stress and Cauchy stress are plotted. The material properties of the structure are $\lambda = 4.0385 \times 10^7$ and $\mu = 2.6923 \times 10^7$. 

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Figure 2.1: The stress-strain curve of neo-Hookean material under uniform load in x-direction. x-axis is the stretch in x-direction, y-axis is the stress value.

2.1.1.2 Modified Neo-Hookean Material Model

A second hyperelastic model implemented is modified neo-Hookean model. The strain energy density function is decomposed into volumetric part $\Psi^{vol}(J)$ and deviatoric part $\Psi^{dev}(\overline{C}_{IJ})$ as:

$$\Psi^{ME}(J, \overline{C}_{IJ}) = \Psi^{vol}(J) + \Psi^{dev}(\overline{C}_{IJ}) = \frac{1}{2}(\lambda + \frac{2}{3}\mu) \ln J^2 + \frac{1}{2}\mu (\overline{C}_{II} - 3) \quad (2.6)$$
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Here $\overline{C}_{IJ} = \overline{F}_{kI} \overline{F}_{kJ}$, representing the volume preserving deformation by

$$\overline{F}_{kI} = J^{-\frac{2}{3}} F_{kI} \quad s.t. \quad det(\overline{F}_{kI}) = 1 \quad (2.7)$$

The stress-strain relation for this model is also decomposed into volumetric and deviatoric part as:

$$S_{IJ}^{\text{vol}} = 2 \frac{\partial \Psi^{\text{vol}}(J)}{\partial C_{IJ}} = \frac{\partial \Psi^{\text{vol}}(J)}{\partial J} \frac{\partial J}{\partial C_{IJ}} = (\lambda + \frac{2}{3} \mu)lnJ \ C_{IJ}^{-1} \quad (2.8)$$

$$S_{IJ}^{\text{dev}} = 2 \frac{\partial \Psi^{\text{dev}}}{\partial C_{IJ}} = \mu J^{-2/3} \left( \delta_{IJ} - \frac{1}{3} C_{IJ}^{-1} C_{KK} \right) \quad (2.9)$$

and

$$S_{IJ} = S_{IJ}^{\text{vol}} + S_{IJ}^{\text{dev}} = (\lambda + \frac{2}{3} \mu)lnJ \ C_{IJ}^{-1} + \mu J^{-2/3} \left( \delta_{IJ} - \frac{1}{3} C_{IJ}^{-1} C_{KK} \right) \quad (2.10)$$

The detailed derivation is obtained using the relationship in the Appendix.

Using the same boundary conditions and material properties as neo-Hookean case for stress-strain illustration, the stress-strain curve of modified neo-Hookean model is shown in Fig. 2.2.
Figure 2.2: The stress-strain curve of modified neo-Hookean material under uniform load in \( x \)-direction. \( x \)-axis is the stretch in \( x \)-direction, \( y \)-axis is the stress value.

### 2.1.1.3 Logarithmic Stretch Model

Another hyperelastic logarithmic stretch model [54] is also implemented in this work, where the strain energy density function is defined in terms of principal strain \( \epsilon_m \) as:

\[
\Psi = \frac{1}{2} \lambda \sum_{m=1}^{3} (\epsilon_m)^2 + \mu \sum_{m=1}^{3} (\epsilon_m^2) 
\]  

(2.11)
where $\varepsilon_m = \log(\lambda_m)$, $\lambda_m$ being the square root of the principal values of $C_{ij}$. The principal values $\tau_m$ of Kirchhoff stress $\tau_{ij}$ are expressed as:

$$\tau_m = \frac{\partial \Psi}{\partial \varepsilon_m} = \lambda \left( \sum_{m=1}^{3} \varepsilon_m \right) + 2\mu \varepsilon_m$$

Finally from Eq. (2.12), the second Piola-Kirchhoff is obtained as

$$S_{IJ} = \frac{\partial x_I}{\partial x_k} \tau_{kl} \frac{\partial x_J}{\partial x_l} = \frac{\partial x_I}{\partial x_k} \sum_m \left[ \lambda \left( \sum_{m=1}^{3} \varepsilon_m \right) + 2\mu \varepsilon_m \right] q_i^{(m)} q_l^{(m)} \frac{\partial x_J}{\partial x_l}$$

where $q_i^{(m)}$ are direction cosines of principal direction $m$ and component direction $i$.

Adopting the same boundary conditions and material properties, the stress-strain curve for the logarithmic stretch model is shown in Fig. 2.3.

Above are three types of hyperelastic materials implemented in the in-house code. The stress-strain curve of these materials are shown in Fig. 2.4. Significant nonlinearity is observed from the stress-strain curve distinguished with the linear elastic behavior.

2.1.2 Strong Form for the Finite Deformation Dynamics Problem

In finite deformation, the undeformed and deformed configurations are different. As shown in Fig. 2.5, the map between deformed current configuration to the undeformed reference configuration is achieved with deformation gradient $F_{iJ}$. The first
and second Piola-Kirchhoff stresses are also related to the map between the current configuration and the reference configuration. To obtain the formulation for finite deformation in the reference configuration, we start from the equilibrium of a continuum body in the current deformed configuration:

\[
\frac{\partial {^t \sigma}_{ij}}{\partial {^t x}_j} + {^t p}b_i = {^t \rho}a_i \]

(2.14)
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Figure 2.4: Hyperelastic material stress-strain curve. (a) Second Piola-Kirchhoff stress $S_{11}$ vs. stretch $\lambda_1$. (b) Cauchy stress $\sigma_{11}$ vs. stretch $\lambda_1$.

Figure 2.5: Deformation of a continuum body in finite deformation theory: undeformed and deformed configurations.
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where \( t^\sigma_{ij} \) is the Cauchy stress in the current configuration, \( b_i \) is the body force per unit mass, \( t^\rho \) is the density of the deformed material, and the density of undeformed material is conducted as \( 0^\rho = J^\rho t^\rho \).

Use the principle of virtual work at \( t = t \) and apply the divergence theorem, the weak form is obtained as:

\[
\int_{tV} t^\sigma_{ij} \partial \delta u_i \partial x_j \, dt \, V + \int_{tV} t^\rho \ddot{u}_i \delta u_i \, dt \, V = \int_{tS} t^\sigma_{ij} \dot{u}_i n_j \, dt \, S + \int_{tV} t^\rho b_i \delta u_i \, dt \, V
\]  

(2.15)

The integration domain is the current configuration \( t^1V \) or the surface \( t^1S \) of the current configuration at \( t \). In order to solve it, the formulation can be mapped to the reference configuration. Choose the configuration \( t = 0 \) as the reference, the mapping can be achieved using the relationship from mass conservation:

\[
0^\rho = J^\rho t^\rho \quad d^1V = J^0V \quad s.t. \quad 0^\rho d^0V = t^\rho d^1V
\]  

(2.16)

Eq. (2.15) is then:

\[
\int_{0V} tP_{ij} \frac{\partial \delta u_i}{\partial x_j} \, d^0V + \int_{0V} 0^\rho \ddot{u}_i \delta u_i \, d^0V
\]

\[
= \int_{0S} tP_{ij} \dot{u}_i N_j \, d^0S + \int_{0V} 0^\rho b_i \delta u_i \, d^0V
\]  

(2.17)

The equilibrium equation at \( t = 0 \) in the reference configuration in finite deor-
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motion theory is expressed as:

$$\frac{\partial t_0 P_{iJ}}{\partial X_J} + ^0 \rho b_i = ^0 \rho \ddot{u}_i$$

where $t_0 P_{iJ}(= J^t \sigma_{ik} F_{jk}^{-1} = \frac{\partial x_i}{\partial X_K} t_0 S_{KJ})$ is the first Piola-Kirchhoff stress. Detailed derivation is listed in the Appendix.

2.2 Governing Equations for the Electromagnetic Problem in Current and Reference Configurations

The governing equations for the EM problem is based on the conventional Maxwell’s equations for a conducting medium. In the current configuration, these equations are expressed in terms of the current coordinates $x_i(t)$ as:

$$d_{i,i} = q_e \quad \text{Gauss’s law of electricity} \quad (2.19a)$$

$$b_{i,i} = 0 \quad \text{Gauss’s law of magnetism} \quad (2.19b)$$

$$\varepsilon_{ij} e_{k,j} = -\frac{\partial b_i}{\partial t} \quad \text{Faraday’s law of magnetism} \quad (2.19c)$$

$$\varepsilon_{ijk} h_{k,j} = \frac{\partial d_i}{\partial t} + \dot{j}_i^f \quad \text{Ampere’s law} \quad (2.19d)$$

Here $d_i$ represents the Cartesian components of the electrical displacement field vector, $q_e$ is the free charge density, $b_i$ are the components of the magnetic induction field vector, $e_i$ is the electric field, $h_i$ is the magnetic field strength, and $j_i^f$ is the free
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charge current defined as:

\[ j_i^f \triangleq j_i^c + q_e \dot{x}_i. \tag{2.20} \]

where \( j_i^c \) is conducting current and \( \varepsilon_{ijk} \) is the Levi-Civita permutation symbol. The constitutive laws for an isotropic material in the current configuration, in the absence of magnetization and polarization, are given as:

\[ d_i = \varepsilon e_i, \quad h_i = \frac{1}{\mu} b_i, \quad j_i^c = \sigma (e_i + \varepsilon_{ijk} \dot{x}_j b_k) \tag{2.21} \]

where the permittivity \( \varepsilon \), permeability \( \mu \) and conductivity \( \sigma \) are material constants.

For coupling with the set of mechanical field equations for the deforming medium, it is necessary to represent the Maxwell’s equations in a reference material configuration \( \Omega_0 \). The Lagrangian description in section 2.1, provides a consistent platform for solving the coupled dynamic-EM problem. The transformation from current to reference configuration requires the flux derivative of any field vector \( V_i \) in terms of the local time derivatives as [55]:

\[ \dot{V}_i = \frac{\partial V_i}{\partial t} + V_{j,j} \dot{x}_i + (V_i \dot{x}_k - V_k \dot{x}_i)_k \tag{2.22} \]
satisfying the integral form of the flux derivative relation:

\[
\frac{d}{dt} \int_{\partial \Omega} V_i n_i ds = \int_{\partial \Omega} V_j n_j ds. \quad (2.23)
\]

Here \( \dot{x}_i \) corresponds to the velocity field and \( (\cdot)_i \) corresponds to the spatial derivative. Derivation of the relation (2.22) utilizes the Nanson’s formula of surface area transformation between the current and the reference configurations, expressed as:

\[
n_i ds = J X_{j,i} N_f dS \quad \text{or} \quad n ds = J F^{-T} N \quad (2.24)
\]

where \( n_i ds \) and \( N_f dS \) are differential area vectors, \( ds \) and \( dS \) are surface areas, and \( n_i \) and \( N_f \) are surface normals in the current and reference configurations respectively. The flux derivative of the electric displacement \( d_i \) is obtained from Eq.(2.22) as:

\[
\frac{\partial}{\partial t} \frac{\partial d_i}{\partial t} = (d_i \dot{x}_k - d_k \dot{x}_i)_k - d_{j,j} \dot{x}_i \quad (2.25)
\]

Now

\[
\varepsilon_{ijk} \frac{\partial}{\partial x_j} h_k = \varepsilon_{ikj} \frac{\partial}{\partial x_k} h_j = -\varepsilon_{ijk} \frac{\partial}{\partial x_k} h_j \quad \text{and} \quad (d_i \dot{x}_k - d_k \dot{x}_i)_k = (\delta_{im} \delta_{kn} - \delta_{km} \delta_{in})(d_m \dot{x}_n)_k = \varepsilon_{jik} \varepsilon_{jmn} (d_m \dot{x}_n)_k
\]

Substituting Eq. (2.19a) and Eq. (2.25) in Eq. (2.19d) and rearranging indices yields
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the relation:

\[
\varepsilon_{ikj}[h_j + \varepsilon_{jmn}(d_m \delta_{jn})]_{,k} = \delta^* - \delta^* - q_e \dot{x}_i \tag{2.27}
\]

where, the identity \( \sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{imn} = \delta_{im} \delta_{kn} - \delta_{km} \delta_{in} \) is used. Eq. (2.27) is transformed to the reference configuration by integrating it over an arbitrary surface area in the current configuration, applying the Kelvin-Stokes’ theorem, and mapping to the reference configuration using Eq. (2.24) to yield:

\[
\oint_C [h_i + \varepsilon_{imn}(d_m \delta_{jn})]_{,i,j} dX_J = \int_{\partial \Omega_0} (\delta_j + j^f_j - q_e \dot{x}_j) JX_{I,j} N_I dS_0 \tag{2.28}
\]

where \( C \) corresponds to the contour line around the surface. The reference configuration electric displacement, magnetic field and free charge current in the conductor are defined in terms of those in the current configuration as:

\[
D_I \triangleq JX_{I,j} d_j \tag{2.29a}
\]

\[
H_J \triangleq [h_i + \varepsilon_{imn}(d_m \delta_{jn})]_{,i,j} \tag{2.29b}
\]

\[
J^c_I \triangleq JX_{I,j} j^c_j \tag{2.29c}
\]

With these reference configuration field variables, Eq. (2.28) is reduced to the configuration-invariant relation:

\[
\varepsilon_{IJK} H_{K,j} = \frac{d}{dt} D_I + J^c_I \tag{2.30}
\]
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The permutation operator $\varepsilon_{IJK}$ in the reference configuration is related to $\varepsilon_{ijk}$ as:

$$
\varepsilon_{IJK} = J^{-1} \varepsilon_{ijk} x_{i,I} x_{j,J} x_{k,K} \quad \text{or inversely} \quad \varepsilon_{ijk} = J \varepsilon_{IJK} X_{I,i} X_{J,j} X_{K,k} \quad (2.31)
$$

In a similar manner, substituting $b_i$ in Eq. (2.22) and using Eqs. (2.19b) and (2.19c) results in a relation between the magnetic induction field $b_i$ and the electric field $e_i$ as:

$$
\varepsilon_{ijk} e_{k,j} = (b_i \dot{x}_k - b_k \dot{x}_i),k - \ast b_i \quad \text{or} \quad (2.32a)
$$

$$
\varepsilon_{ijk}[e_j - \varepsilon_{jmn}(b_m \dot{x}_n)],k = - \ast b_i \quad (2.32b)
$$

Integrating Eq. (2.32b) over a surface area in the current configuration, applying the Kelvin-Stokes’ theorem, and subsequently transforming into the reference configuration yields:

$$
\oint_C [e_j - \varepsilon_{jmn}(b_m \dot{x}_n)] x_{j,I} dX_I = - \int_{\partial \Omega_0} \ast b_j J X_{J,i} N_J dS_0 \quad (2.33)
$$

The electric and magnetic induction fields in the reference configuration are defined as:

$$
E_I \triangleq [e_j - \varepsilon_{jmn}(b_m \dot{x}_n)] x_{j,I} \quad (2.34a)
$$

$$
B_J \triangleq J X_{J,i} b_i \quad (2.34b)
$$
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The Faraday’s Eq. (2.19c) in the reference configuration is then expressed as:

\[ \varepsilon_{IJK} E_{K,J} = -\frac{d}{dt} B_I \]  \hspace{1cm} (2.35)

For the EM fields law of electricity in Eq. (2.19a), the transformation is achieved by integrating over the volume and applying the divergence theorem, along with the Nanson’s formula of surface transformation, to yield:

\[ \int_{\partial\Omega} d_i n_i ds = \int_{\Omega} q_e dv \Rightarrow \int_{\partial\Omega_0} d_i J X_{J,i} N_J dS = \int_{\Omega_0} q_e J dV \]  \hspace{1cm} (2.36)

By defining the charge density in the reference configuration as \( Q_e = J q_e \), Eq. (2.19a) in the reference configuration becomes

\[ D_{I,I} = Q_e \]  \hspace{1cm} (2.37)

Following the same mapping procedure, the EM fields law of magnetism of Eq. (2.19b) is transformed to:

\[ B_{J,J} = 0 \]  \hspace{1cm} (2.38)

In summary, the four Maxwell’s equations in the reference configuration are written
in the indicial and vector forms as:

<table>
<thead>
<tr>
<th>Indicial Notations</th>
<th>Vector Notations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_{I, I} = Q_e )</td>
<td>( \nabla_X \cdot \vec{D} = Q_e ) (2.39)</td>
</tr>
<tr>
<td>( B_{J, I} = 0 )</td>
<td>( \nabla_X \cdot \vec{B} = 0 ) (2.40)</td>
</tr>
<tr>
<td>( \varepsilon_{IJK} \frac{\partial}{\partial X_J} E_K = -\frac{d}{dt} B_I )</td>
<td>( \nabla_X \times \vec{E} = -\frac{d}{dt} \vec{B} ) (2.41)</td>
</tr>
<tr>
<td>( \varepsilon_{IJK} \frac{\partial}{\partial X_J} H_K = \frac{d}{dt} D_I + \vec{J}_I^c )</td>
<td>( \nabla_X \times \vec{H} = \frac{d}{dt} \vec{D} + \vec{J}_I^c ) (2.42)</td>
</tr>
</tbody>
</table>

Constitutive relations in the reference configuration, relating \( D_I \) and \( E_I \), \( B_I \) and \( H_I \), as well as \( \vec{J}_I^c \) and \( \vec{E}_I \) and \( \vec{B}_I \), are nonlinear due to coupling with the deformation fields. The following steps are used to determine the set of constitutive relations.

Using the inverse of Eq. (2.34b), i.e.

\[
\vec{b}_j = J^{-1} \vec{x}_{j, I} B_I \tag{2.43}
\]

and the velocity relations

\[
\dot{x}_i = -x_{i, K} \frac{\partial X_K}{\partial t} \quad \text{or inversely} \quad \frac{\partial X_K}{\partial t} = -X_{K, i} \dot{x}_i \tag{2.44}
\]
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in Eq. (2.34a), its inverse is obtained as:

\[ e_i = E_J X_{J,i} + \varepsilon_{imn} (b_m \dot{x}_n) = E_J X_{J,i} - X_{J,i} (\varepsilon_{imn} J^{-1} x_{i,j} x_{m,L} x_{n,K}) \left( \frac{\partial X_K}{\partial t} B_L \right) \]

\[ = X_{J,i} [E_J + (\varepsilon_{JKL} \frac{\partial X_K}{\partial t} B_L)] \quad (2.45) \]

Substituting Eqs. (2.29a) and (2.21) in Eq. (2.45) yields the constitutive relation for the electrical displacement field in the reference configuration as:

\[ D_I = \varepsilon J X_{I,j} X_{J,i} [E_J + \varepsilon_{JKL} \left( \frac{\partial X_K}{\partial t} B_L \right)] = \varepsilon J C_{1,j}^{-1} [E_J + \varepsilon_{JKL} \left( \frac{\partial X_K}{\partial t} B_L \right)] \quad (2.46) \]

In a similar manner, the magnetic field strength in both configurations are related as:

\[ h_i = H_J X_{J,i} - \varepsilon_{imn} (d_m \dot{x}_n) = H_J X_{J,i} + X_{J,i} (\varepsilon_{imn} J^{-1} x_{i,j} x_{m,L} x_{n,K}) \left( \frac{\partial X_K}{\partial t} D_L \right) \]

\[ = X_{J,i} [H_J - (\varepsilon_{JKL} \frac{\partial X_K}{\partial t} D_L)] \quad (2.47) \]

Substituting Eqs. (2.43) and (2.46) in Eq. (2.47) results in the constitutive relation for magnetic field strength in the reference configuration as:

\[ H_J = h_{i,i} J + \varepsilon_{JKL} \frac{\partial X_K}{\partial t} D_L = \frac{1}{\mu} J^{-1} x_{i,M} x_{i,J} B_M + \varepsilon_{JKL} \left( \varepsilon_{JC}^{-1} L_N [E_N + \varepsilon_{NPQ} \left( \frac{\partial X_P}{\partial t} B_Q \right)] \right) \]

\[ = \frac{1}{\mu} J^{-1} C_{MJ} B_M + \varepsilon_{JKL} \left( \varepsilon_{JC}^{-1} L_N [E_N + \varepsilon_{NPQ} \left( \frac{\partial X_P}{\partial t} B_Q \right)] \right) \quad (2.48) \]

Finally, incorporating Eq. (2.29c) and Eq. (2.45) in Eq. (2.21) provides the consti-
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tutive relation for the current in the reference configuration as:

\[ J_i^c = \sigma J C_{i,j}^{-1} E_j \]  (2.49)

2.2.1 Scalar and Vector Potentials in Current and Reference Configurations

For effective solution of the Maxwell’s equations, a scalar potential \( \phi \) for the electric field and a vector potential \( \mathbf{a} \) for the magnetic field in the current configuration have been proposed as primary variables, e.g. in \([55]\). This representation in terms of the potential functions reduce the Maxwell’s equations from four to two independent equations. Since the divergence of the magnetic induction vector \( \mathbf{b} \) (in Eq. (2.19c)) is zero, the magnetic vector potential \( \mathbf{a} \) can be derived using a vector identity, i.e.

\[ \nabla \cdot \mathbf{b} = \nabla \cdot (\nabla \times \mathbf{a}) = 0 \quad \Rightarrow \quad b_i = \varepsilon_{ijk} a_{k,j} \]  (2.50)

The Faraday’s law in Eq. (2.19c) may be rewritten in terms of the vector potential \( a_i \) by substituting the expression for the magnetic field \( b_i \) in Eq. (2.50), i.e.

\[ \varepsilon_{ijk} (e_k + a_k)_j = 0 \]  (2.51)
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Since the curl of the gradient of any scalar field is a null vector, the gradient of $\varphi$ may be added to the LHS of Eq. (2.51) without changing the RHS, i.e.

$$\varepsilon_{ijk}(e_k + \dot{a}_k + \varphi, k), j = 0 \quad (2.52)$$

The electric field may be then be expressed using mixed potentials as:

$$e_k = -\varphi, k - \dot{a}_k \quad (2.53)$$

For electrostatic problems, the electric potential $\varphi$ corresponds to the ratio of potential energy to charge. Introduction of these two potentials in the Gauss’s law for magnetism in Eq. (2.19b) and the Faraday’s law in Eq. (2.19c) results in identities. The other two Maxwell’s equations are reformulated using the mixed potentials as:

$$\nabla^2 \varphi + \frac{\partial}{\partial t}(a_{i,i}) = -q_e \quad (2.54a)$$

$$\nabla^2 a_i - \mu \varepsilon \frac{\partial^2 a_i}{\partial t^2} - (a_{k,k} + \mu \varepsilon \frac{\partial \varphi}{\partial t}), i = -\mu j_i \quad (2.54b)$$

The corresponding reduced forms of the Maxwell’s equations in the reference configuration requires consistent forms of both the potentials in this configuration. Defin-
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ing transformation functions as:

\[ A_K = a_i x_{i,K} \iff a_i = A_K X_{K,i} \quad \text{and} \quad (2.55a) \]

\[ \Phi = \phi - \frac{dx_i}{dt} a_i \iff \phi = \Phi - \frac{\partial X_I}{\partial t} A_I \quad (2.55b) \]

Scalar and vector potentials defined in the reference configuration follow relations similar to those in the current configuration, i.e.

\[ E_I = -\Phi_{,I} - \frac{\partial A_I}{\partial t} \quad (2.56a) \]

\[ B_I = \varepsilon_{IJK} A_{K,J} \quad (2.56b) \]

Details of the derivation of Eqs. (2.55a), (2.55b) and proofs of Eqs. (2.56a) and (2.56b) are given in [55]. Substituting Eqs. (2.56a) and (2.56b) in Eq. (2.39) and Eq. (2.42) results in the two governing equations

\[ \left( \varepsilon J C^{-1}_{I,J} \tilde{E}_J \right)_{,I} = Q_e \quad (2.57a) \]

\[ \varepsilon_{IJK} \left( \frac{1}{\mu_j} C_{KL} \varepsilon_{LMN} A_{N,M} + \varepsilon_{KPQ} \frac{\partial X_P}{\partial t} \varepsilon J C^{-1}_{Q,R} \tilde{E}_R \right)_{,J} = \frac{d}{dt} \varepsilon J C^{-1}_{IP} \tilde{E}_P \\
+ \sigma J C^{-1}_{IQ} \left( -\Phi_{,Q} - \dot{A}_Q \right) \quad (2.57b) \]
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where \( \tilde{E}_I \) is defined as:

\[
\tilde{E}_I = E_I + \varepsilon_{IJK} \frac{\partial X_J}{\partial t} B_K = -\Phi_{,I} - \dot{A}_I + \varepsilon_{IJK} \frac{\partial X_J}{\partial t} \varepsilon_{KMN} A_{N,M} \tag{2.58}
\]

2.2.2 Constraint Gauge Condition for Solution Uniqueness

While the reduced representation of Maxwell’s equations in terms of potentials is computationally advantageous, the solution leads to non-uniqueness of the potentials and consequent singular tangent matrices for certain state of the EM fields. For example, the vector potential relation in Eq. (2.56b), can admit multiple solutions of the magnetic field due to the condition:

\[
\nabla \cdot B = \nabla \cdot (\nabla \times A + \nabla \Psi) = 0 \implies B_I = \varepsilon_{IJK} A_{K,J} + \psi_{,I} \tag{2.59}
\]

where \( A_I \) is the vector potential component and \( \Psi \) is any arbitrary scalar potential in the reference configuration. This non-uniqueness will lead to a singular tangent stiffness matrix and consequent instability in the finite element model. This has been averted through the introduction of a variety of constraint gauge conditions in the literature, viz. [56–60]. The Coulomb gauge condition proposed in [57, 58, 60] is utilized in the present work. It is stated as:

\[
A_{I,I} = 0 \tag{2.60}
\]
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This constraint forces the divergence of the vector potential \( \mathbf{A} \) to be zero. From the Helmholtz theorem [58], the vector potential \( \mathbf{A} \) can be decomposed into a divergence free rotational term and a curl-free irrotational term. Due to the vanishing irrotational part in Eq. (2.56b), the rotational term contributes to the magnetic field \( \mathbf{B}_I \) and hence the Coulomb gauge implies a restriction to the irrotational term. The constraint gauge condition is implemented in the finite element in a weak sense, using the penalty formulation following [57, 58, 60].

2.3 Weak Forms of Coupled ME-EM Problem

Weak forms of the Lagrangian FE formulation in the reference configuration and their algorithmic implementation are discussed in this section. The principle of virtual work is used to obtain the weak form of the finite deformation dynamics problem. The Hamilton’s principle of stationary action, in conjunction with the penalty formulation of the gauge constraint condition, is used to develop the weak form of the time dependent EM fields. Appropriate boundary conditions are developed for both the mechanical and electromagnetic problems in the reference configuration. Based on the nature of the problem, the solving scheme is robust and versatile due to the staggered coupling scheme. For multi-physics analysis in the loading bearing multi-functional devices, the EM fields in the conducting medium are affected by the deformation and other mechanical field variables. However, the coupling from EM fields to ME field, will be treated differently according to application. For example, in load-bearing
antenna, the EM fields have negligible effect on the mechanical fields in comparison with the external loading. Hence, the Lorentz force is assumed to be negligible in this study. While for piezoelectric application, the piezoelectricity phenomenon affects the deformation spontaneously. Thus the two-way coupling is necessary in the piezoelectric cases. This is be discussed in Chapter 3.

A robust coupling between the mechanical field and the EM fields is thus considered, and a staggered solution scheme is implemented. A high performance parallel implementation of the coupled code is also summarized in this section.

2.3.1 Weak Form and Boundary Conditions of the Finite Deformation Dynamics Problem

In small deformation problems, the deformed configuration is the same with undeformed configuration. In contrast to this case, the deformed configuration is unknown in finite deformation problems. Since the configuration is unknown, it is impossible to carry out calculations or to evaluate deformation and kinematic variables based on the configuration. To overcome this problem, either total Lagrangian (TL) formulation or updated Lagrangian (UL) formulation is employed. For both methods, the deformation and kinematics variables in the unknown, current configuration are mapped to known configuration. In TL, all static and kinematic variables are referred to the virgin configuration at time $t = 0$. While in UL, all static and kinematic variables are referred to the configuration at time $t = t$. In present work, UL is used to
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solve the mechanical problem.

Due to the path dependency, finite deformation problems are solved using time incremental scheme. As shown in Fig. 2.6, there are three configurations involved, virgin reference configuration at $t = 0$, updated reference configuration at $t = t$ and the current unknown configuration at $t = t + \Delta t$. As introduced in Sec.2.1.2, the deformed reference configuration at $t = t$ and the undeformed configuration at $t = 0$ in Fig. 2.6 can both serve as the reference configuration, which is the undeformed configuration in Fig. 2.5.

![Figure 2.6: Deformation of a continuum body: undeformed, updated deformed and unknown configurations.](image)

The strong form of the mechanical problem in finite deformation at $t = t + \Delta t$ using updated Lagrangian formulation is:

$$
\frac{\partial t^{t+\Delta t}P_{ij}}{\partial t^iX_j} + t^i\rho b_i = t^i\rho \ddot{u}_i 
$$

(2.61)
where \( t^{+\Delta t} P_{iJ} = (t^{+\Delta t} J t^{+\Delta t} \sigma_{ik} t^{+\Delta t} F_{kj}^{-1} \frac{\partial t^{+\Delta t} S_{ij}}{\partial x_k}) \) is the first Piola-Kirchhoff stress, \( \rho \) is the density in the updated reference configuration and \( b_i \) is the body force per unit mass.

The principle of virtual work is developed by taking the inner product of Eq. (2.61) with the virtual displacement \( \delta u_i = \delta U_I \) and integrating over volume in the reference configuration in updated Lagrangian method \( \Omega_t \) as:

\[
\int_{tV} \frac{\partial t^{+\Delta t} P_{iJ}}{\partial t x_J} \delta u_i d^4V + \int_{tV} t^{+\Delta t} \rho b_i \delta u_i d^4V = \int_{tV} t^{+\Delta t} \rho \ddot{u}_i \delta u_i d^4V
\]  

(2.62)

By applying the divergence theorem together with integration by parts, it renders to:

\[
\int_{tV} t^{+\Delta t} P_{iJ} \frac{\partial \delta u_i}{\partial t x_J} d^4V + \int_{tV} t^{+\Delta t} \rho \ddot{u}_i \delta u_i d^4V = \int_{tS} t^{+\Delta t} P_{iJ} \delta u_i \bar{N}_J d^3S + \int_{tV} t^{+\Delta t} \rho b_i \delta u_i d^4V
\]  

(2.63)

Displacement and traction conditions on respective boundaries in the reference configuration are expressed as:

\[
U_I = \delta t_i u_i = \bar{U}_I^0 \quad \text{on} \quad \Gamma_u \in \partial \Omega_0
\]

\[
T_I = P_{iJ} N_J = \bar{T}_I^0 \quad \text{on} \quad \Gamma_t \in \partial \Omega_0
\]  

(2.64)

Implement the traction term of Eq. (2.64) into Eq. (2.63), it becomes to
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\[
\int_{tV}^{t+\Delta t} P_{ij} \frac{\partial \delta u_i}{\partial x_j} d^4V = \int_{tS}^{t+\Delta t} \bar{T}^0_{ij} \delta u_i d^4S + \int_{tV}^{t+\Delta t} \rho b_i \delta u_i d^4V - \int_{tV}^{t+\Delta t} \rho \ddot{u}_i \delta u_i d^4V \tag{2.65}
\]

The left hand side of Eq. (2.65) is the internal virtual work and the right hand side is the external virtual work and inertia.

The left hand side of Eq. (2.65) was derived from using Eq. (A.7)

\[
\int_{tV}^{t+\Delta t} P_{ij} \frac{\partial \delta u_i}{\partial x_j} d^4V = \int_{tV}^{t+\Delta t} J \sigma_{ij} \frac{\partial \delta u_i}{\partial x_j} d^4V \tag{2.66}
\]

Since the Cauchy stress \( t+\Delta \sigma_{ij} \) is symmetric, The left hand side of Eq. (2.65) is further derived to:

\[
\int_{tV}^{t+\Delta t} J \sigma_{ij} \frac{\partial \delta u_i}{\partial x_j} d^4V = \int_{tV}^{t+\Delta t} \frac{1}{2} \left( \frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right) d^4V = \int_{tV}^{t+\Delta t} J \sigma_{ij} \delta \varepsilon_{ij} d^4V \tag{2.67}
\]

where \( \varepsilon_{ij} \) is the infinitesimal strain.

Introduce the Green-Lagrange strain tensor or simply Green strain tensor \( E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}) \). From the relationship between \( \delta \frac{t+\Delta t}{t} E_{IJ} \) and variation of infinitesimal strain \( \delta \frac{t+\Delta t}{t} \varepsilon_{mn} \)

\[
\delta \frac{t+\Delta t}{t} E_{IJ} = \frac{t+\Delta t}{t} x_{m,J} \frac{t+\Delta t}{t} x_{n,I} \delta \frac{t+\Delta t}{t} \varepsilon_{mn} \tag{2.68}
\]
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And the relation between second Piola-Kirchhoff stress \( t^+\Delta t S_{KJ} \) and Cauchy stress \( t^+\Delta t \sigma_{ij} \) in Eq. (2.5) is inverted as:

\[
\hat{\sigma}_{ij} = J^{-1} \hat{F} \hat{S} \hat{F}^T
\]  

(2.69)

The left hand side of Eq. (2.65) becomes to

\[
\int_{tV} t^+\Delta t \hat{F} t^+\Delta t \sigma_{ij} \delta \varepsilon_{ij} d^t V = \int_{tV} t^+\Delta t \hat{F} t^+\Delta t J^{-1} \frac{\partial t^+\Delta t x_i}{\partial X_M} t^+\Delta t S_{MN} \frac{\partial t^+\Delta t x_j}{\partial X_N} \delta \varepsilon_{ij} d^t V
\]

\[
= \int_{tV} t^+\Delta t S_{MN} \delta t^+\Delta t E_{MN} d^t V
\]  

(2.70)

Eq. (2.65) becomes:

\[
\int_{tV} t^+\Delta t S_{MN} \delta t^+\Delta t E_{MN} d^t V
\]

\[
= \int_{tS} T_I^0 \delta u_i d^t S + \int_{tV} t^\rho b_i \delta u_i d^t V - \int_{tV} t^\rho \ddot{u}_i \delta u_i d^t V
\]  

(2.71)

If in load-bearing antennae case where the EM fields are assumed to have negligible effect on the mechanical fields, the discrete form of Eq. (2.71), together with the boundary conditions Eq. (2.64), are sufficient for solving the deformation related fields of the substrate.

If in piezoelectric sensor case where the mechanical field is affected by the electric field, the constitutive equations, body force may be altered due to the effect from
another physics domain. However, the weak form of the mechanical field in finite
deformation dynamical problem remains the same as Eq. (2.71). Further implement-
tion and alteration of the weak form due to piezoelectricity and damage will be
discussed in future context.

2.3.2 Weak Form of the Electromagnetic Problem in the Reference Configuration

The weak form of the transient EM problem is derived using the Hamilton’s principle
[30, 55]. It involves minimization of the action functional $S$ over the time range $t_1 - t_2$,
defined in terms of the time-dependent Lagrangian density $\mathcal{L}$ in the reference domain,
expressed as:

$$\delta S = \delta \int_{t_1}^{t_2} \int_{\Omega_0} \mathcal{L} d\Omega_0 dt = 0$$  \hspace{1cm} (2.72)$$

where the Lagrangian density in the reference and current configurations, $\mathcal{L}$ and $l$
respectively, are given as:

$$\mathcal{L} = Jl = \left( \frac{\epsilon}{2} e_i e_i - \frac{1}{2\mu} b_j b_j + j_k a_k - q\varphi \right)$$  \hspace{1cm} (2.73)$$

$\mathcal{L}$ may be expressed in terms of the scalar and vector potentials in the reference config-
uration using the expressions for $e_i$ and $b_i$ in Eqs. (2.45) and (2.43), and substituting
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Eqs. (2.55a) and (2.55b) as:

\[ \mathcal{L} = \frac{\varepsilon}{2} J_{JK} \tilde{E}_J \tilde{E}_K - \frac{J^{-1}}{2\mu} (C_{LM} B_L B_M) + J_N A_N - Q\Phi \]  

(2.74)

Here \( E_I, B_I \) and \( \tilde{E}_I \) are in terms of the scalar and vector potentials as given in Eqs. (2.56a), (2.56b) and (2.58).

Furthermore, the gauge condition in Eq. (2.60) is implemented using the penalty method to constrain the vector potentials following [57, 58, 60]. In this process, an additional term \( \frac{1}{p}(\nabla \cdot A)^2 \) is added to the Lagrangian density function as:

\[ \mathcal{L} = \frac{\varepsilon}{2} J_{JK} \tilde{E}_J \tilde{E}_K - \frac{J^{-1}}{2\mu} (C_{LM} B_L B_M) + J_N A_N - Q\Phi + \frac{1}{p} (A_{F,P})^2 \]  

(2.75)

The term \( \frac{1}{p} \) is the penalty coefficient, for which \( p \) is generally of the order of the electric permittivity \( \varepsilon \).

From Eq. (2.72), setting the variation of the action functional \( S \) with respect to \( \Phi \) and \( A \) to zero yields:

\[ S_{,\Phi} [\delta \Phi] = \int_{t_1}^{t_2} \int_{\Omega_0} \mathcal{L}_{,\Phi} [\delta \Phi] \Omega_0 dt = 0 \]  

(2.76a)

\[ S_{,A} [\delta A] = \int_{t_1}^{t_2} \int_{\Omega_0} \mathcal{L}_{,A} [\delta A] \Omega_0 dt = 0 \]  

(2.76b)
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Since \( t_1 \) and \( t_2 \) are arbitrary, Eqs. (2.76a) and (2.76b) lead to:

\[
\int_{\Omega_0} L_{\Phi} [\delta \Phi] dV_0 = \int_{\partial \Omega_0} N_I \left( \varepsilon J C_{IJ}^{-1} \tilde{E}_J \delta \Phi \right) dS_0 - \int_{\Omega_0} \left( \varepsilon J C_{IJ}^{-1} \tilde{E}_J - Q \right) \delta \Phi_I dV_0 = 0
\]

(2.77a)

\[
\int_{\Omega_0} L_{A} [\delta A] dV_0 = \int_{\partial \Omega_0} N_L (\varepsilon_{KLM} Q_M \delta A_K) dS_0 - \frac{2}{p} \int_{\partial \Omega_0} A_{R,R} N_K \delta A_K dS_0 + \\
\int_{\Omega_0} \varepsilon_{SPR} Q_P \frac{\partial}{\partial X_R} \delta A_S dV_0 - \int_{\Omega_0} \frac{d}{dt} \varepsilon J C_{KJ}^{-1} \tilde{E}_J \delta A_K dV_0 - \int_{\Omega_0} \sigma J C_{KI}^{-1} E_I \delta A_K dV_0 + \\
\frac{2}{p} \int_{\Omega_0} A_{R,R} \delta A_{K,K} dV_0 = 0
\]

(2.77b)

where the condensed term is expressed as:

\[
Q_M = \frac{1}{\mu_j} C_{MN} B_N + \varepsilon_{MNP} \frac{\partial X_N}{\partial t} \varepsilon J C_{PQ}^{-1} \tilde{E}_Q
\]

(2.78)

The weak forms in Eqs. (2.77a) and (2.77b) reduce to the strong forms in Eqs. (2.57a) and (2.57b). The scalar potential \( \Phi \) and vector potentials \( A \) are the solutions of Eqs. (2.77a) and (2.77b).

2.3.2.1 Boundary Conditions in Terms of the Potentials

Since potential functions are chosen as primary variables in the solution of the EM fields, the actual physical boundary conditions should be represented in terms of the scalar and vector potentials in the reference configuration. This is obtained by replacing the field variables with the potential expressions in the boundary term of
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the weak form Eq. (2.77a), i.e.,

\[
\int_{\partial \Omega_0} N_I \left( \varepsilon_0 J C_{I,j}^{-1} \tilde{E}_j \delta \Phi \right) dS_0 = \\
\int_{\partial \Omega_0} N_I \left[ \varepsilon_0 J C_{I,j}^{-1} (-\Phi_{,j} - \dot{A}_j + \varepsilon_{JLK} \frac{\partial X_L}{\partial t} \varepsilon_{KMN} A_{N,M}) \right] \delta \Phi dS_0
\]

where \( \delta \Phi \) is the variation of \( \Phi \). For Dirichlet boundary conditions, e.g. applied voltage \( V = \Phi = \Phi_0 \) in electrostatic problem or for grounded boundary \( \Phi = 0, \delta \Phi = 0 \). More investigation is needed for parts of the boundary that are not governed by the Dirichlet boundary condition. For example, the boundary equation for the reference electric displacement field \( D_I \) in Eq. (2.46) is \( N_I D_I = \rho_s \), where \( \rho_s \) is the surface charge density. This boundary condition can be implemented by substituting the value of surface charge density. Another boundary condition that can be expressed using Eq. (2.79) is current injection, which is governed by the reference configuration Ohm’s law Eq. (2.49). Assuming a fixed boundary in Eq. (2.44) i.e. \( \frac{\partial X}{\partial t} = 0 \), the boundary condition reduces to:

\[
\int_{\partial \Omega_0} N_I \left[ \varepsilon J C_{I,j}^{-1} (-\Phi_{,j} - \dot{A}_j) \right] \delta \Phi dS_0 = \int_{\partial \Omega_0} N_I \left[ \varepsilon \frac{J_I}{\sigma} \right] \delta \Phi dS_0
\]

This mixed form may be used to introduce the injected current along the surface normal.

The next step is to examine the boundary condition associated with the weak
form in Eq. (2.77b). Following the same procedure as before, the boundary terms corresponding to the Dirichlet type condition reduce to:

$$\int_{\partial \Omega_0} N_L \left[ \varepsilon_{KLM} \left( \frac{1}{\mu J} C_{MN} B_N + \varepsilon_{MRP} \frac{\partial X_R}{\partial t} \varepsilon J C^{-1}_{PQ} \tilde{E}_Q \right) \delta A_K \right] dS_0 - \frac{2}{p} \int_{\partial \Omega_0} A_{R,R} N_K \delta A_k dS_0$$

(2.81)

This includes boundary condition terms from both the Maxwell’s equation and the gauge condition. When values of $A_I$ are prescribed on the boundary, both terms in Eq. (2.81) go to zero. This simple boundary condition is associated with the magnetic field $B = 0$ at the boundary. From Eq. (2.48) it is evident that for conditions other than the Dirichlet boundary condition, Eq. (2.81) will hold when $N_I \times H_I$ is assigned to the boundary. This result has the physical manifestation of the surface current.

To understand the effect of the first term in Eq. (2.81), a fixed and undeformed boundary is assumed, thus yielding the relation:

$$\int_{\partial \Omega_0} N_L \varepsilon_{KLM} \left( \frac{1}{\mu J} C_{MN} B_N \right) \delta A_K dS_0 = \int_{\partial \Omega_0} N \times \left[ \frac{1}{\mu J} (\nabla \times A) \right] \delta A dS_0$$

(2.82)

Using a vector identity this may be rewritten as:

$$\int_{\partial \Omega_0} N \times \left[ \frac{1}{\mu J} (\nabla \times A) \right] \delta A dS_0 = \int_{\partial \Omega_0} \frac{1}{\mu J} [\nabla_A (N \cdot A) - (N \cdot \nabla) A] \delta A dS_0$$

(2.83)

Here $N \cdot A = A^a$ is the scalar component of the vector potential $A$ normal to the
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\( \nabla A \) is the Feynman subscript notation, which implies that the subscripted gradient operates only on \( A \). The other part is expressed as \((N \cdot \nabla)A = \frac{\partial A}{\partial N}\), which corresponds to the flux, and hence the Neumann boundary condition in terms of \( A \). If a tangential magnetic field is imposed on the boundary, there is no normal component of \( A^n \) and the gauge condition term vanishes. If the boundary is considered as a magnetic wall, the Neumann boundary condition in terms of \( A \) should be considered for a fixed and undeformed boundary.

2.4 Finite Element Implementation of Coupled ME-EM Problem

Numerical implementation of the weak forms and boundary conditions is conducted for the coupled problem using a staggered approach, where the dynamic displacement field is solved first followed by the electromagnetic problem.

For the finite deformation dynamical problem, the updated Lagrangian method is adopted. As stated in Sec. 2.3.1 the formulation is solved incrementally to accommodate path dependency. Starting from the weak form in Eq. (2.71),

\[
\int_{t_V}^{t+\Delta t} S_{MN} \delta^{t+\Delta t} E_{MN} d^1V = t^{+\Delta t} R - \int_{t_V}^{t+\Delta t} \rho \ddot{u}_i \delta u_i d^1V
\]

(2.84)

where \( t^{+\Delta t} R \) is the external work. It includes the traction applied on the surface \( \int_S T_i^0 \delta u_i d^1S \) and body force which direction remains unchanged during the loading.
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\[ \int_{t_V}^{t} \rho_b \delta u_i d^t V. \]

As for the increment of second Piola-Kirchhoff stress, since the reference configuration is at \( t = t \) in the updated Lagrangian formulation,

\[ t^+ \Delta_t S_{IJ} = t^t S_{IJ} + \Delta S_{IJ} = t^t \sigma_{ij} + \Delta S_{IJ} \quad (2.85) \]

where \( t^t \sigma_{ij} \) is the Cauchy stress at \( t = t \), and it is equivalent to the second Piola-Kirchhoff stress in the same configuration.

Recall the Green-strain definition:

\[ t^+ \Delta_t E_{IJ} = \frac{1}{2} \left( t^{t+\Delta t} C_{IJ} - \delta_{IJ} \right) \]

\[ = \frac{1}{2} \left( \frac{\partial t^{t+\Delta t} u_I}{\partial t^t X_J} + \frac{\partial t^{t+\Delta t} u_J}{\partial t^t X_I} + \frac{\partial t^{t+\Delta t} u_I}{\partial t^t X_J} \frac{\partial t^{t+\Delta t} u_J}{\partial t^t X_I} \right) \quad (2.86) \]

Since \( t^+ \Delta_t u_I = t^t u_I + \Delta u_I \), the \( t^+ \Delta_t E_{IJ} \) is further expanded as:

\[ t^+ \Delta_t E_{IJ} = \frac{1}{2} \left( \frac{\partial (t^t u_I + \Delta u_I)}{\partial t^t X_J} + \frac{\partial (t^t u_J + \Delta u_J)}{\partial t^t X_I} + \frac{\partial (t^t u_I + \Delta u_I)}{\partial t^t X_J} \frac{\partial (t^t u_J + \Delta u_J)}{\partial t^t X_I} \right) \]

\[ = \frac{1}{2} \left( \frac{\partial t^t u_I}{\partial t^t X_J} + \frac{\partial t^t u_J}{\partial t^t X_I} + \frac{\partial t^t u_I}{\partial t^t X_J} \frac{\partial t^t u_J}{\partial t^t X_I} \right) \]

\[ + \frac{1}{2} \left( \frac{\partial \Delta u_I}{\partial t^t X_J} + \frac{\partial \Delta u_J}{\partial t^t X_I} + \frac{\partial t^t u_I}{\partial t^t X_J} \frac{\partial \Delta u_J}{\partial t^t X_I} + \frac{\partial t^t u_J}{\partial t^t X_I} \frac{\partial \Delta u_I}{\partial t^t X_J} \right) + \frac{1}{2} \frac{\partial \Delta u_I}{\partial t^t X_J} \frac{\partial \Delta u_J}{\partial t^t X_I} \quad (2.87) \]

Here

\[ t^t E_{IJ} = \frac{1}{2} \left( \frac{\partial t^t u_I}{\partial t^t X_J} + \frac{\partial t^t u_J}{\partial t^t X_I} + \frac{\partial t^t u_I}{\partial t^t X_J} \frac{\partial t^t u_J}{\partial t^t X_I} \right) \quad (2.88) \]
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is Green-Lagrange strain at $t = t$, and

$$
\Delta e_{IJ} = \frac{1}{2} \left( \frac{\partial \Delta u_I}{\partial t} X_J + \frac{\partial \Delta u_J}{\partial t} X_I + \frac{\partial \Delta u_I}{\partial t} X_I \frac{\partial \Delta u_J}{\partial X_J} \right) \\
\Delta \eta_{IJ} = \frac{1}{2} \frac{\partial \Delta u_I}{\partial X_J} \frac{\partial \Delta u_J}{\partial X_I}
$$

(2.89)

are linear and nonlinear increments of Green-Lagrange strain from $t = t$ to $t + \Delta t$ respectively.

Since $\delta^t_t E_{IJ} = 0$, from the decomposition in Eq. (2.87), the variation of Green-Lagrange strain is:

$$
\delta^{t+\Delta t}_t E_{IJ} = \delta \Delta e_{IJ} + \delta \Delta \eta_{IJ}
$$

(2.90)

Use the relationship of Eq. (2.85) and Eq. (2.90), the left hand side of Eq. (2.84) is updated as

$$
\int_{t_V} \int^{t+\Delta t}_t S_{MN} \delta^{t+\Delta t}_t E_{MN} d^t V = \int_{t_V} \int^{t+\Delta t}_t (t \sigma_{mn} + \Delta S_{MN}) \delta \Delta e_{MN} + \delta \Delta \eta_{MN} d^t V \\
= \int_{t_V} t \sigma_{mn} \delta \Delta e_{MN} d^t V + \int_{t_V} t \sigma_{mn} \Delta \eta_{MN} d^t V \\
+ \int_{t_V} \Delta S_{MN} \delta \Delta e_{MN} d^t V + \int_{t_V} \Delta S_{MN} \Delta \eta_{MN} d^t V
$$

(2.91)

To linearize the nonlinear relationship in Eq. (2.91), term $\int_{t_V} \Delta S_{MN} \Delta \eta_{MN} d^t V$ is negligible due to the higher order concern. And the constitutive relationship is
linearized as:

\[ \Delta S_{MN} = t_{C_{MNPQ}} \Delta E_{IJ} \approx t_{C_{MNPQ}} \Delta e_{MN} \] (2.92)

where \( t_{C_{MNPQ}} \) is the incremental stress-strain tensors at time \( t \) referred to the configuration at time \( t \). This linearization treatment eliminates the higher order terms in \( \int_V \Delta S_{MN} \delta \Delta e_{MN} dt \).

With the linearization in Eq. (2.92), Eq. (2.91) is further derived as:

\[
\int_V t_{C_{MNPQ}} \Delta e_{MN} \delta \Delta e_{MN} dt_V = \int_V (t_{\sigma_{mn}} + \Delta S_{MN}) (\delta \Delta e_{MN} + \delta \Delta \eta_{MN}) dt_V \\
= \int_V t_{\sigma_{mn}} \delta \Delta e_{MN} dt_V + \int_V t_{\sigma_{mn}} \delta \Delta \eta_{MN} dt_V (2.93) \\
+ \int_V t_{C_{MNPQ}} \Delta e_{MN} \delta \Delta e_{MN} dt_V
\]

Insert Eq. (2.93) back to Eq. (2.84), the weak form of updated Lagrangian formulation is:

\[
\int_V t_{C_{MNPQ}} \Delta e_{MN} \delta \Delta e_{MN} dt_V + \int_V t_{\sigma_{mn}} \delta \Delta \eta_{MN} dt_V \\
= t + \Delta t R - \int_V t \rho \dot{u}_i \delta u_i dt_V - \int_V t_{\sigma_{mn}} \delta \Delta e_{MN} dt_V (2.94)
\]

The semidiscretization method involving separation of variables is used to represent the variables as discrete functions of space, yet continuous functions of time.

For 8-noded brick elements, independent variables (displacements, scalar and vector
potentials) of the coupled problem in each element $e$ are interpolated as:

$$u^e(X,t) \approx \sum_{\alpha=1}^{8} u^e_{\alpha}(t) N^e_{\alpha}(X),$$
$$\Phi^e(X,t) \approx \sum_{\alpha=1}^{8} \Phi^e_{\alpha}(t) N^e_{\alpha}(X),$$
$$A^e(X,t) \approx \sum_{\alpha=1}^{8} A^e_{\alpha}(t) N^e_{\alpha}(X)$$

(2.95)

where $N(X)$ are trilinear isoparametric shape functions of the brick element [61]. Note that both the mechanical and electromagnetic field variables adopt the same finite element mesh. The implicit time integration Newmark-beta method, which assumes constant average acceleration in each time step [62], is implemented for integrating the dynamic problems as:

$$u^e(X,t_n) \approx u^e(X,t_{n-1}) + \Delta t \dot{u}^e(X,t_{n-1}) + \frac{1}{4} \Delta t^2 \left( \ddot{u}^e(X,t_n) + \ddot{u}^e(X,t_{n-1}) \right)$$
$$\dot{u}^e(X,t_n) \approx \dot{u}^e(X,t_{n-1}) + \frac{1}{2} \Delta t \left( \dddot{u}^e(X,t_n) + \dddot{u}^e(X,t_{n-1}) \right)$$

(2.96)

This constant-average-acceleration (or trapezoidal rule) scheme is unconditionally stable. Both the initial displacement and velocity are set to be zero, i.e. $u(X,0) = 0, \dot{u}(X,0) = 0$.

The mass matrix $[M]$ can be treated either as consistent mass matrix or lumped mass matrix. For the consistent mass matrix, it is implemented as:
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\[
[M] = \int_{\Omega_0} [N]^T \rho [N] \tag{2.97}
\]

since the integration is performed in the reference configuration. \([N]\) is are trilinear isoparametric shape functions of the brick element in matrix form. As for the consistent mass matrix, the matrix is not diagonal. To speed up the inversion of mass matrix, lumped mass matrix is employed. HRZ (Hinton-Rock-Zienkiewicz) method [54] is used to calculate the lumped matrix, it follows the rules as:

1. Use only diagonal terms of the consistent mass matrix,

2. Scale the diagonal terms in such a way that total element mass is preserved.

Although the inversion calculation of lumped mass matrix is faster, it is not recommend to replace consistent mass matrix for higher-order elements. It brings in less accurate results comparing with the consistent mass matrix.

For the EM fields, total Lagrangian method is employed to map the current configuration formulation at time \(t = t + \Delta t\) back to the original configuration at time \(t = 0\). The weak form Eq. (2.77) is:

\[
\int_{\partial \Omega_0} 0 N_I \left( \varepsilon^t J^t + \Delta t C^{-1} J^t E J \delta \Phi \right) dS_0 - \int_{\Omega_0} \left( \varepsilon^t J^t + \Delta t C^{-1} J^t E - Q \right) \delta \Phi I dV_0 = 0 \tag{2.98a}
\]
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\[ \int_{\partial \Omega} \varepsilon_{KLM} Q_M \delta A_K dS_0 - \frac{2}{p} \int_{\partial \Omega} A_{R,R} \varepsilon_{KLM} \delta A_K dS_0 + \]

\[ \int_{\Omega} \varepsilon_{S_{PR}} Q_P \frac{\partial}{\partial X_R} \delta A_S dV_0 - \int_{\Omega} \frac{d}{dt} \varepsilon_{t+\Delta t_0} J^{t+\Delta t_0} C_{KJ}^{-1} \tilde{E}_J \delta A_K dV_0 \]

\[ - \int_{\Omega} \sigma^{t+\Delta t_0} J^{t+\Delta t_0} C_{KI} E_I \delta A_K dV_0 + \frac{2}{P} \int_{\Omega} A_{R,R} \delta A_{K,K} dV_0 = 0 \]

where \( Q_M \) is calculated by:

\[ Q_M = \frac{1}{\mu^{t+\Delta t_0} J^{t+\Delta t_0} C_{MN} B_{N} + \varepsilon_{MNP} \frac{\partial X_N}{\partial t} \varepsilon^{t+\Delta t_0} J^{t+\Delta t_0} C_{PQ}^{-1} \tilde{E}_Q} \]

The backward Euler method is implemented for integrating the EM fields as:

\[ \hat{\Phi}_e(X, t_n) \approx \frac{\Phi_e(t_n) - \Phi_e(t_{n-1})}{\Delta t} N^e_{\alpha}(X) \]

\[ \hat{A}_e(X, t_n) \approx \frac{A^e_{\alpha}(t_n) - A^e_{\alpha}(t_{n-1})}{\Delta t} N^e_{\alpha}(X) \]

where \( \Delta t \) is time step that is determined from the higher frequency response in the coupled problem. For the electromagnetic field, \( \Phi(X, 0) \) is set to be zero throughout domain except for certain boundary conditions. The vector potential \( \mathbf{A} \) and its rate \( \dot{\mathbf{A}} \) are obtained from the solution of the problem with \( \Phi(X, 0) = 0 \).

For efficient computing of the large degrees of freedom (DOF) coupled ME-EM model, the finite element code is parallelized using available libraries. The ParMETIS [63] library is utilized to decompose the computational domain and distribute to multiple processors. Subsequently, the Portable, Extensible Toolkit for Scientific
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Computation or PETSc [64] library, which is a Message Passing Interface (MPI) based library, is employed to accomplish parallelization of the code. Both the ME and EM codes are developed using PETSc to guarantee the same structure. A linear solver, SuperLU [65] is used to solve the matrix equations. Details of the solution methodology for the coupled problem are shown in the flowchart of Fig. 2.7.

2.5 Conclusion

In this chapter, a platform that can couple finite deformation dynamic problem with both nonlinear geometric and material and transient electromagnetic problem with geometric nonlinear is developed. To analyze the effect of deformation and material evolution on the EM fields, material form formulation is adopted to solve EM fields in the reference configuration. Weak forms for both physics descriptions are derived. Boundary conditions for EM fields are particularly studied when expressed using scalar and vector potentials. FE implementations for both fields are elaborated in the chapter. This chapter is concluded with a flowchart to describe the key steps of the EM-ME code.
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Figure 2.7: Flow chart of the coupled mechanical and EM fields problem
Chapter 3

Piezoelectric Constitutive Equations with Continuum Damage Model in the Reference Configuration

3.1 Piezoelectric Material Model in Coupled Electric Field with Finite Deformation Setting

3.1.1 Piezoelectric Material Constitutive Equations in the Current Configuration

The piezoelectric effect is generally analyzed and studied under small deformation setting due to its natural properties. The piezoelectric effect exists in the material which has no inversion symmetric. Some common examples of piezoelectric material are crystals and some ceramics. In such materials, elastic constitutive equations in the small deformation are adequate for mechanical material properties study. Meanwhile, the electric field is calculated in the same configuration. This set of constitutive relation is well studied for decades. Simple derivation is present here to introduce the
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piezoelectricity and compare with the constitutive equations in the reference configuration under finite deformation setting.

As shown in [19, 33], the canonical electro-mechanical power per unit volume of the continuum is,

\[ \dot{e} = \sigma_{ij} \dot{\varepsilon}_{ij} + e_k \dot{d}_k \quad (3.1) \]

where \( e \) is the internal energy. \( \sigma_{ij} \) is the Cauchy stress, \( \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \) is the strain tensor from displacement \( u_i \), \( e_k = \phi, k \) is the electric field derived from electric potential \( \phi \) and \( d_k \) is the electric displacement field.

Refer to [66], for small deformation, define electric enthalpy density \( h \) as,

\[ h = e - e_i d_i \quad (3.2) \]

From Eq. (3.2) and Eq. (3.1), it is obtained that

\[ \dot{h} = \sigma_{ij} \dot{\varepsilon}_{ij} - d_i \dot{e}_i \quad (3.3) \]

This implies \( h(\varepsilon_{ij}, e_k) \). Propose the form of electric enthalpy \( h \) in small deformation,

\[ h = \frac{1}{2} C^{E}_{ijkl} \varepsilon_{kl} \varepsilon_{ij} - \frac{1}{2} \varepsilon^s_{ij} e_j e_i - e_{imn} \varepsilon_{mn} e_i \quad (3.4) \]
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Cauchy stress $\sigma_{ij}$ and electric displacement field can be obtained from Eq. (3.4) and as

$$\sigma_{ij} = \frac{\partial h}{\partial \varepsilon_{ij}} = C_{ijkl}^{E} \varepsilon_{kl} - e_{kij} e_k$$  \hspace{1cm} (3.5)

and

$$d_i = \frac{\partial h}{\partial e_i} = e_{imn} \varepsilon_{mn} + e_{ij}^s e_j$$ \hspace{1cm} (3.6)

The stored energy can be obtained from Eq. (3.2) and Eq. (3.4)

$$e = \frac{1}{2} C_{ijkl}^{E} \varepsilon_{kl} \varepsilon_{ij} + \frac{1}{2} e_{ij}^s e_j e_i$$ \hspace{1cm} (3.7)

Although there are other forms of constitutive equations of piezoelectric material in terms of $(\varepsilon_{ij}, d_k), (\sigma_{ij}, e_k), (\sigma_{ij}, d_k)$ as independent variables in [66], the chosen form (3.5) and (3.6) are widely employed. It also follows the in house EM code developed in [52] in the same manner with reference to the constitutive relationship in mechanical field and electric field.

3.1.2 Piezoelectric Material Constitutive Equations in the Reference Configuration

A Lagrangian description for piezoelectricity is invoked to study the coupling under finite deformation. The Maxwell’s equations in the reference configuration is derived
in [30–32, 52] as Eq. (2.39), (2.40),(2.41) and (2.42).

For piezoelectric application, free charge $Q_e$ and conducting current $J^c_I$ do not exist in the material. And the magnetic field has negligible effect on the electric field. Based on the above assumptions, the governing equations are

\begin{align}
D_{I,I} &= 0 	ag{3.8a} \\
B_{J,J} &= 0 \tag{3.8b} \\
\varepsilon_{IJK} \frac{\partial}{\partial X_J} E_K &= 0 \tag{3.8c} \\
\varepsilon_{IJK} \frac{\partial}{\partial X_J} H_K &= \frac{d}{dt} D_I \tag{3.8d}
\end{align}

By Poynting theorem, multiply $E_I$ on both sides of (3.8d) and subtract the result of (3.8c) multiplying $H_I$ on both sides, using the identity of $-(\varepsilon_{IJK} E_J H_K)_{,I} = E_I \varepsilon_{IJK} \frac{\partial}{\partial X_J} H_K - H_I \varepsilon_{IJK} \frac{\partial}{\partial X_J} E_K$,

\begin{equation}
-(\varepsilon_{IJK} E_J H_K)_{,I} = E_I \dot{D}_I \tag{3.9}
\end{equation}

Integrate (3.9) over the reference configuration, by using the divergence theorem,

\begin{equation}
\int_{\partial \Omega_0} -\varepsilon_{IJK} E_J H_K N_I dS_0 = \int_{\Omega_0} E_I \dot{D}_I dV_0 \tag{3.10}
\end{equation}

The LHS of (3.10) is the flux of EM energy leaving the system from the boundary, and the RHS representing the change of the EM energy of the system with respect
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to time. Eq. (3.10) introduces the EM energy balance of the system.

Following the derivation framework in [67], for a deformation due to the external mechanical loading and EM field, the first law of thermodynamics is

\[
\frac{d}{dt} \mathcal{E}(t) = P_{int}^{EM}(t) + P_{int}^{ME}(t) + Q(t) \tag{3.11}
\]

where \( P_{int}^{EM}(t) \) is the internal EM work, \( P_{int}^{ME}(t) \) is the internal mechanical work, \( Q(t) \) is the rate of the thermal work and \( \mathcal{E}(t) \) is the internal energy. The operate \( \frac{d}{dt} \) is the material time derivate. It is equivalent to \( \frac{d}{dt} \Phi = \dot{\Phi} \).

From the derivation above, \( P_{int}^{EM} = \int_{\Omega_0} E_I \dot{D}_I dV_0 \) and \( P_{int}^{ME} = \int_{\Omega_0} S_{IJ} \dot{E}_{IJ} dV_0 \).

Since the internal energy in material description is

\[
\mathcal{E} = \int_{\Omega_0} edV_0 \tag{3.12}
\]

Thus (3.11) becomes

\[
\frac{d}{dt} \int_{\Omega_0} edV_0 = \int_{\Omega_0} E_I \dot{D}_I dV_0 + \int_{\Omega_0} S_{IJ} \dot{E}_{IJ} dV_0 + \int_{\Omega_0} RdV_0 - \int_{\partial\Omega_0} Q_N dS_0 \tag{3.13}
\]

And the local form is

\[
\dot{\varepsilon} = E_I \dot{D}_I + S_{IJ} \dot{E}_{IJ} + R - Q_{K,K} \tag{3.14}
\]

As for the second law of thermodynamics, the Clausius-Duhem law,
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\[ \Gamma (t) = \frac{d}{dt} \int_{\Omega_0} \eta dV_0 + \int_{\partial \Omega_0} \frac{1}{\Theta} Q_K N_k dV_0 - \int_{\Omega_0} \frac{1}{\Theta} R dV_0 \geq 0 \quad (3.15) \]

The pointwise local form in the reference configuration for this is

\[ \dot{\eta} + \frac{1}{\Theta} Q_{K,K} - \frac{1}{\Theta^2} Q_L \Theta_{,L} - \frac{1}{\Theta} R \geq 0 \quad (3.16) \]

Substitute the entropy source \( R \) in (3.16) with relationship in (3.14), the Clausius-Duhem inequality for the coupled system in the reference configuration is expressed

\[ E_I \dot{D}_I + S_{IJ} \dot{E}_{IJ} - \dot{\varepsilon} + \Theta \dot{\eta} - \frac{1}{\Theta} Q_L \Theta_{,L} \geq 0 \quad (3.17) \]

And it leads to a stronger form Clausius-Planck inequality,

\[ D_{int} = E_I \dot{D}_I + S_{IJ} \dot{E}_{IJ} - \dot{\varepsilon} + \Theta \dot{\eta} \geq 0 \quad (3.18) \]

indicating the internal dissipation \( D_{int} \) is non-negative.

Define Helmholtz free energy \( \psi \)

\[ \psi = e - \Theta \eta \quad (3.19) \]

Take the time derivative of \( \psi \),

\[ \dot{\psi} = \dot{e} - \Theta \dot{\eta} - \Theta \dot{\eta} \quad (3.20) \]
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Express the \( \dot{e} \) from (3.20) and substitute back to (3.18), it is obtained as,

\[
D_{int} = E_I \dot{D}_I + S_{IJ} \dot{E}_{IJ} - \dot{\Psi} - \dot{\Theta} \eta \geq 0 \tag{3.21}
\]

When there is not internal dissipation, i.e. \( D_{int} = 0 \), and thermal effect is ignored, it is obtained that,

\[
E_I \dot{D}_I + S_{IJ} \dot{E}_{IJ} - \dot{\Psi} = 0 \tag{3.22}
\]

However, the internal dissipation will be semi-definite if any degradation or failure happen within the material.

Using Legendre transformation, define the electric enthalpy density in the reference configuration

\[
H = \Psi - E_I D_I \tag{3.23}
\]

Take the time derivative on both sides and rearrange the equation,

\[
\dot{\Psi} = \dot{\mathcal{H}} + \dot{E}_I D_I + E_I \dot{D}_I \tag{3.24}
\]

Replace the \( \dot{\Psi} \) in (3.22) with (3.24),

\[
\dot{\mathcal{H}} = S_{JK} \dot{E}_{JK} - \dot{E}_I D_I \tag{3.25}
\]

which give the constitutive relationship for piezoelectric material as
In the piezoelectric system, the mechanical field and electric field are coupled by a piezoelectric constant $\varepsilon_{KIJ}$. By using hyperelastic material model for the deformation constitutive relations, adopting the idea from the electric enthalpy density in the current configuration as in Eq. (3.4), an explicit form for the electric enthalpy density $\mathcal{H}$ in the reference configuration is proposed:

$$\mathcal{H} = \Psi^{ME}(C_{IJ}) - \varepsilon_{KIJ}E_{IJ}E_K - \frac{1}{2} \varepsilon J C^{-1}_{IJ} E_I E_J$$

(3.27)

where $\Psi^{ME}(C_{IJ})$ is the stored energy density for hyperelastic material. The positive-valued Jacobian $J$, which determines admissible deformation mapping, is defined in terms of the deformation gradient tensor $F_{iJ}$ as in Eq. (2.3). And $C_{IJ}$ is the right Cauchy-Green deformation tensor defined in Eq. (2.2).

Eq. (3.27) indicates that $\mathcal{H}$ is not convex. This conclusion is also included in [48]. $\mathcal{H}$ has a saddle point when $(E_{IJ}, E_K) = (0, 0)$. Substitute $\mathcal{H}$ of (3.27) into (3.26a) and (3.26b) respectively, it is obtained,
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\[ S_{IJ} = \frac{\partial \mathcal{H}}{\partial E_{IJ}} = \frac{\partial \Psi^{ME}}{\partial E_{IJ}} s_{IJ}^{\text{ME}} - \frac{1}{2} \varepsilon J \left[ \left( E_P C_{PQ}^{-1} E_Q \right) C_{IJ}^{-1} - E_P E_Q \left( C_{P1}^{-1} C_{JQ}^{-1} + C_{Pj}^{-1} C_{IQ}^{-1} \right) \right] \]

(3.28a)

\[ D_K = -\frac{\partial \mathcal{H}}{\partial E_K} = \varepsilon_{KIJ} E_{IJ} + \varepsilon J C_{Lk}^{-1} E_L \]

(3.28b)

The last term of Eq. (3.28a) \( S_{IJ}^{\text{Maxwell}} \) is defined as Maxwell stress in the reference configuration as in [47]. It represents as the mapping between configurations. This coupling term was examined carefully by early pioneer such as [29, 68, 69], and revisited by [36] recently. It can be proved that the Maxwell stress in the reference configuration obey the Piola transformation with the Maxwell stress defined in the current configuration [34, 37, 70].

Based on the proposed form of electric enthalpy density of Eq. (3.27), the Helmholtz free energy of the coupled system can be obtained from Eq. (3.23) and Eq. (3.28b) as:

\[ \Psi = \Psi^{ME} (C_{IJ}) + \frac{1}{2} \varepsilon J C_{IJ}^{-1} E_I E_J \]

(3.29)

The Helmholtz free energy of this coupled system in the reference configuration can be obtained from Eq. (3.7) using Piola transformation. The detailed derivation
can be found in [30] and [52]. The expression decouples the energy of mechanical energy and electric energy. Since $\Psi^{ME} \geq 0$ and $\frac{1}{2} \varepsilon C^{-1}_{I J} E_I E_J \geq 0$, the proposed $\mathcal{H}$ will render non-negative Helmholtz free energy. This indicates that the coupled system is stable.

### 3.1.3 Objectivity of the Piezoelectric Material Constitutive Equations in the Reference Configuration

Since the free energy of the coupled system is defined by right Green tensor $C_{I J}$ as for mechanical fragment and electric field in the reference configuration $E_I$ as for electric fragment. It is important to ensure these variables are objective in order to solve the problem in the reference configuration. Impose a rigid body rotation $Q_{ki}$, the deformation gradient in the updated configuration $F^*_{kI}$ is:

$$F^*_{kI} = Q_{ki} F_{iJ} \tag{3.30}$$

it is straightforward to show that $C^*_PQ$ is objective as:

$$C^*_PQ = F^*_k F^*_k = Q_{ki} F_{iP} Q_{kl} F_{lQ} = C_{PQ}. \tag{3.31}$$

Recall the stored energy for hyperelastic material in Eq. (2.1) and Eq. (2.6), it is defined in terms of deformation variables as Jacobian $J$, right Green tensor $C_{I J}$ or the deviatoric part of $C_{I J}$. In [67], the stored energy defined by $\Psi^{ME}(C_{I J})$ is proved
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to be not only objective but also isotropic.

As for the electric field in the reference configuration, as in [52], the Piola transform with absence of the magnetic field is

$$E_J = F_{iJ}e_i$$  \hspace{1cm} (3.32)

A rigid body rotation $Q_{ki}$ is applied on both deformation gradient $F_{iJ}$ and the electric field in the current configuration $e_i$. The electric field in the reference configuration after the transformation is conducted as

$$E_J^* = F_{kJ}^*e_k^*.$$  \hspace{1cm} (3.33)

Combing the Eq. (3.30) and $e_k^* = Q_{ki}e_i$, It is concluded that $E_J^* = E_J$, which guarantee the objective of the free energy of the coupled system.

3.2 Continuum Damage Model in Piezoelectric Material Model under Finite Deformation

Piezoelectric material is a sensor by its nature. It converts the mechanical energy to the electric energy and vice versa. Due to the rigid properties of traditional piezoelectric materials, the sensors made by piezoelectricity is incompatible under finite deformation. However, smart structures contain piezoelectric material which can undergo finite deformation are extensively studied and implement in applications of
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control vibrations, energy harvesting [71, 72], etc.. Damage of the constituents within the smart structures, mostly composites, initial at microstructure scale in the form of fiber-matrix or particle-matrix interfacial de-bonding. The evolution of the damage represented by crack growth propagate under large deformation, leading to failure. Using the altered electric field generated by piezoelectricity in the damaged structure, it is anticipated that the damage can be captured in real-time and in-situ. A homogenized continuum damage model is developed based on the damage detection application implemented with piezoelectricity.

3.2.1 Continuum Damage Model of Hyperelastic Material Coupled with Piezoelectricity

There are many literatures focus on the continuum damage model in pure mechanical problem. Starting from [73], the macroscopic damage was proposed by introducing the phenomenological damage parameter $D$. The damage model is assumed to be isotropic and evaluated by a scalar damage parameter $D \in [0,1)$. It means the defects and voids are spatially distributed uniformly in all directions.

As shown in Fig. 3.1, Fig. 3.1a illustrates a uniform stress applied on the top and bottom surface of a structure in undamaged state. If the same stress applied on a damaged structure shown in Fig. 3.1b developed from Fig. 3.1a, a uniform strain $\varepsilon$ is distributed in the damaged structure. This deformation state of Fig. 3.1b is also interpreted in Fig. 3.1c, where the equivalent strain $\varepsilon$ as Fig. 3.1(b) is measured and
corresponding effective stress $\tilde{\sigma}$ is applied.

![Figure 3.1: The illustration of principle of strain-equivalence and effective stress. (a) The virgin state with applied stress $\sigma$. (b) Stress $\sigma$ applied on damage state. (c) The effective stress $\tilde{\sigma}$ applied on virgin state with equivalent strain of damage state.](image)

In the strain-equivalence principle [74, 75], the structure with isotropic damage parameter $D$ has smaller effective area to resist stress built up internally due to deformation. The effective resistance area is evaluated by $D$ as:

$$\tilde{S} = S - S_D = S(1 - D) \tag{3.34}$$

In the uniaxial case shown in Fig. 3.1 $\sigma = \frac{F}{S}$ where $F$ is the force applied on a
section of RVE, the effective stress $\tilde{\sigma}$ is then:

$$\sigma S = F = \tilde{\sigma} \tilde{S} \quad (3.35)$$

With Eq.(3.34), the effective stress is

$$\tilde{\sigma} = \frac{\sigma}{1 - D} \quad (3.36)$$

This principle can be further extended into finite strain regime as [76, 77] for hyperelastic materials.

According to various research work [27, 28], it is reported that the coupling performance of piezoelectric material will decrease with mechanical loading. In addition, interfacial de-bounding or delamination will take place in composite materials containing piezoelectric fiber/particle as reinforcement. The damage caused by mechanical loading will also affect piezoelectric property. Thus, the piezoelectric constants are assumed to be affected by the damage parameter. However, the damage effect on dielectric property permittivity has not been reported in literature consistently. Assume the damage model is isotropic, evaluated by a scalar damage parameter $D \in [0, 1)$. Based on these assumptions from literature reviews, the piezoelectric enthalpy in reference configuration with damage can be obtained as

$$\mathcal{H} = \Psi^{ME}(E_{IJ}, D) - \mathcal{E}_{KIJ}(D)E_{IJ}E_{K} - \frac{1}{2}\varepsilon\mathcal{C}_{IJ}^{-1}E_{I}E_{J} \quad (3.37)$$
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Assume the mechanical response of the coupled system behaviors as compressible neo-Hookean material. In [76], the author introduced a damage model implemented into hyperelastic material featuring viscoelastic response. In the present dissertation, the viscoelastic part is eliminated from the formulation in [76]. Similar formulation can be found in [77]. The damage model in [77] is also extended with discontinuous damage model to represent Mullins effect and continuum damage featuring cyclic loading effect.

Following similar idea of the damage model in [76] and [77], the mechanical energy is decomposed into volume preserving $\Psi_{vol}$ and deviatoric energy $\Psi_{dev}$, assuming the uncoupling between each other in compressible neo-Hookean material, i.e. modified neo-Hookean material in Sec. 2.1.1.2.

\[
\Psi^{ME} = \Psi_{vol} + \Psi_{dev} \tag{3.38}
\]

The functional form of $\Psi_{vol}$ and $\Psi_{dev}$ are:

\[
\Psi_{vol} = \frac{1}{2}(\lambda + \frac{2}{3}\mu) \left[ \frac{1}{2} (J^2 - 1) - \ln J \right] \tag{3.39a}
\]

\[
\Psi_{dev} = \frac{1}{2}\mu(\bar{I}_C - 3) \tag{3.39b}
\]

respectively. Here $\lambda$ and $\mu$ are Lamé constants. $\bar{I}_C$ is the first invariant of $\bar{C}_{IJ} = F_{ki}F_{kj}$, representing the volume preserving deformation by $F_{ki}$ defined in Eq. (2.7).
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Correspondingly, using the relationship in Eq. (A.1), Eq. (A.3) and Eq. (A.4), the stress term $S_{IJ}^{ME}$ is then decomposed into volumetric part $S_{ij}^{vol}(J)$ and deviatoric part $S_{ij}^{dev}(\mathbf{C}_{ij})$ accordingly as Eq. (2.8) and Eq. (2.9).

Assume the damage model is isotropic, evaluated by a scalar damage parameter $D \in [0, 1)$, which means the defects and voids are spatially distributed uniformly in all directions. The damage parameter is characterized by the reduction of the Young’s modulus and the degradation of piezoelectric material properties. However, the damage is merely from mechanical loading aspect other than the electric field aspect.

As stated in [76] and [77], for materials which behave distinguishable differently in volumetric and deviatoric response, the damage is confined for deviatoric deformation. By using this treatment, nonsymmetric tangent moduli is avoided as described in [76]. For simplicity, piezoelectric constant is also multiply by $1 - D$ representing the material degradation for the electromechanical coupling. The mechanical energy containing damage is expressed as:

$$\Psi^{ME}(J, \mathbf{C}_{ij}, D) = \Psi^{vol}(J) + (1 - D) \Psi^{dev}(\mathbf{C}_{ij})$$

(3.40)

And the electric enthalpy is

$$\mathcal{H} = \Psi^{vol}(J) + (1 - D) \Psi^{dev}(\mathbf{C}_{ij}) - (1 - D)\mathbf{e}_{Kl}E_{lj}E_{k} - \frac{1}{2}\varepsilon J C_{lj}^{-1}E_{l}E_{j}$$

(3.41)
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According to Eq. (3.21), since the existence of damage, the internal dissipation $D_{int} \geq 0$. For the coupled system, replace Helmholtz free energy $\Psi$ with relationship in Eq. (3.24), it is obtained,

$$D_{int} = S_{IJ} : \dot{E}_{IJ} - \dot{H} - \dot{E}_I D_I \geq 0 \quad (3.42)$$

with ignorance of thermal effect.

Since $\mathcal{H}$ with damage model is the function of damage parameter $D$, Jacobian $J$, modified right Cauchy-Green tensor $\overline{C}_{IJ}$, and electric field in reference configuration $E_K$, the time derivative of electric enthalpy with damage model becomes

$$\dot{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial J} \dot{J} + \frac{\partial \mathcal{H}}{\partial \overline{C}_{IJ}} : \dot{\overline{C}}_{IJ} + \frac{\partial \mathcal{H}}{\partial E_K} \dot{E}_K + \frac{\partial \mathcal{H}}{\partial D} \dot{D} \quad (3.43)$$

Since

$$\dot{J} = \frac{\partial J}{\partial \overline{C}_{IJ}} : \dot{\overline{C}}_{IJ} \quad (3.44)$$

Eq. (3.41) and Eq. (3.43) render to the constitutive equations of the damaged coupled system from
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\[ S_{IJ} = \frac{\partial H}{\partial E_{IJ}} = 2 \frac{\partial H}{\partial C_{IJ}} = 2 \frac{\partial \Psi_{vol}}{\partial C_{IJ}} + 2(1 - D) \frac{\partial \Psi_{dev}}{\partial C_{IJ}} - (1 - D) \mathbf{C}_{KK} - S_{IJ}^{Maxwell} \] (3.45a)

\[ D_K = -\frac{\partial H}{\partial E_K} = \varepsilon J C_{KK}^{-1} E_L + (1 - D) \mathbf{C}_{KK} E_{IJ} \] (3.45b)

From Eq. (2.8) and Eq. (2.9), \( S_{IJ} \) of Eq. (3.45a) is further expanded as,

\[ S_{IJ} = (\lambda + 2\frac{2}{3} \mu) \ln J C_{IJ}^{-1} + (1 - D) \mu J^{-2/3} \left( \delta_{IJ} - \frac{1}{3} C_{IJ}^{-1} C_{KK} \right) \]

\[ - (1 - D) \mathbf{C}_{KK} E_{IJ} - S_{IJ}^{Maxwell} \] (3.46)

where \( S_{IJ}^{Maxwell} \) is the same in (3.28a).

And the internal dissipation is

\[ D_{int} = (\Psi_{dev} - \mathbf{C}_{KK} E_{IJ} E_K) \dot{D} \geq 0 \] (3.47)

The damage conjugate force \( \Psi_{dev} - \mathbf{C}_{KK} E_{IJ} E_K \) needs to be semi-positive definite to guarantee the monotonic increase of damage parameter. This is a constraint of the damage model coupled with piezoelectric material. The simulation will check the value of \( D_{int} \) to guarantee the damage propagation is thermodynamically admissible.
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3.2.2 Damage Evolution of Continuum Damage Model of Hyperelastic Material Coupled with Piezoelectricity

From Eq. (3.47), the damage evolution law is required to evaluate the propagation of damage with respect to the mechanical loading. It is assume that the damage parameter $D(\Xi_{\text{max}})$ is defined as a function of the maximum equivalent strain throughout the deformation history, where

$$\Xi_{\text{max}} = \max_{s \in (-\infty, t]} \sqrt{2\Psi_0^{\text{dev}} (\bar{C}_{IJ})}$$

(3.48)

Define a damage criterion in strain-space throughout deformation history as,

$$\varphi (E_{IJ}, \Xi_{\text{max}}) = \sqrt{2\Psi_0^{\text{dev}} (\bar{C}_{IJ})} - \Xi_{\text{max}} \leq 0$$

(3.49)

The damage surface is defined as $\varphi = 0$. The normal to the damage surface can be obtained as

$$N = \frac{\partial \varphi}{\partial E} = \frac{1}{\Xi_{\text{max}}} \frac{\partial \Psi_0^{\text{dev}}}{\partial E} \quad \text{or} \quad N_{IJ} = \frac{\partial \varphi}{\partial E_{IJ}} = \frac{1}{\Xi_{\text{max}}} \frac{\partial \Psi_0^{\text{dev}}}{\partial E_{IJ}}$$

(3.50)

The damage will increase irreversibly upon the loading from the damage state. The evolution law is represented by a nonlinear function of the history maximum equivalent strain $\Xi_{\text{max}}$ as
\begin{align*}
\dot{D} = \begin{cases} 
\dot{D}(\Xi_{\text{max}}) & \text{if } \varphi = 0 \text{ and } \mathbf{N} : \mathbf{E} > 0 \\
0 & \text{otherwise}
\end{cases} 
\end{align*}

(3.51)

employing the similar saturation function for damage parameter 

\[ D = 1 - g(\Xi_{\text{max}}) \]

from [76], where

\[ g(\Xi_{\text{max}}) = \beta + (1 - \beta) \frac{1 - e^{-x/\alpha}}{x/\alpha} \]

(3.52)

Here \( \alpha \in (0, \infty) \) is the saturation rate and \( \beta \in [0, 1] \) represents the damage limit.

When the damage criterion is met, the evolution of \( D \) is represented as

\[ \dot{D} = -\frac{g'}{\Xi_{\text{max}}} \frac{\partial \Psi_{\text{dev}}}{\partial \mathbf{E}_{IJ}} : \dot{\mathbf{E}}_{IJ} \]

(3.53)

Denoting that \( g' = \frac{1 - \beta e^{-x/\alpha}}{\alpha} \frac{1 - e^{-x/\alpha}}{x/\alpha} \leq 0 \) as long as \( \alpha \) and \( \beta \) are in their intervals. In this case, it is guarantee that \( \dot{D} \geq 0 \).

3.3 Weak Forms for Piezoelectric Material in Finite Deformation Coupled with Continuum Damage Model

As introduced in Sec. 2.3, the coupled system is solved in a staggered way. The mechanical field is solved using the weak form of Eq. (2.71) and implemented into FE method using Eq. (2.94). The EM fields are solved subsequently using Eq. (2.98).
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By bringing in the piezoelectric material constitutive relation and continuum damage model, the weak forms and their implementation need alteration accordingly.

3.3.1 Weak Form and Boundary Conditions of the Finite Deformation Dynamics Problem with Piezoelectric Material and Continuum Damage Model

By implementing piezoelectric material, the constitutive relations of modified neo-Hookean in Sec. 2.1.1.2 is enriched to:

\[ S_{IJ} = \left( \lambda + \frac{2}{3} \mu \right) ln J C_{IJ}^{-1} + \mu J^{-2/3} \left( \delta_{IJ} - \frac{1}{3} C_{IJ}^{-1} C_{KK} \right) \]

\[- \varepsilon_{KIJ} E_K - S_{IJ\text{Maxwell}} \]

\[ = S_{ME}^{IE} - S_{IJ}^{\text{Piezo}} - S_{IJ}^{\text{Maxwell}} \]  

(3.54)

by eliminating damage effect from Eq. (3.46). The second Piola Kirchhoff stress \( t^{+}_{t} S_{MN} \) of the weak form in Eq. (2.71) is then updated with Eq. (3.54). The piezoelectric coupling term \( S_{ij}^{\text{Piezo}} = \varepsilon_{KIJ} E_K \) and Maxwell stress \( S_{ij}^{\text{Maxwell}} \) requires the information from the electric field. Since the system is solved in a staggered way, the electric field variables are known when the calculation stepping into the
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mechanical field problem. Thus, Eq. (2.71) is altered as

\[
\int_{tV}^{t+\Delta t} t S_{MN}^{ME} \delta_{t}^{+\Delta t} E_{MN} d^t V = \int_{tS} \bar{T}_I^0 \delta_{t}^{+\Delta t} S + \int_{tV} \rho b_i \delta_{t}^{+\Delta t} u_i d^t V - \int_{tV} \rho \ddot{u}_i \delta_{t}^{+\Delta t} u_i d^t V
\]

\[
+ \int_{tV}^{t+\Delta t} t S_{MN}^{Piezo} \delta_{t}^{+\Delta t} E_{MN} d^t V + \int_{tV}^{t+\Delta t} t S_{MN}^{Maxwell} \delta_{t}^{+\Delta t} E_{MN} d^t V
\]

(3.55)

The mechanical boundary conditions are:

\[
U_I = \delta_{t}^{+\Delta t} u_i = \bar{U}_I^0 \quad \text{on} \quad \Gamma_u \in \partial \Omega_0
\]

\[
T_I = P_{ij}^{ME} N_j = \bar{T}_I^0 \quad \text{on} \quad \Gamma_t \in \partial \Omega_0
\]

(3.56)

for displacement and traction applied.

If continuum damage model is adopted in analysis, the deviatoric part of mechanical stress \( S_{MN}^{dev} \) and piezoelectric coupling constant \( E_{IMN} \) are affected by the damage parameter by multiply \( 1 - D \) in front. The weak form is then:

\[
\int_{tV}^{t+\Delta t} t S_{MN}^{dev} \delta_{t}^{+\Delta t} E_{MN} d^t V + \int_{tV} (1 - D) t S_{MN}^{dev} \delta_{t}^{+\Delta t} E_{MN} d^t V
\]

\[
= \int_{tS} \bar{T}_I^0 \delta_{t}^{+\Delta t} S + \int_{tV} \rho b_i \delta_{t}^{+\Delta t} u_i d^t V - \int_{tV} \rho \ddot{u}_i \delta_{t}^{+\Delta t} u_i d^t V
\]

\[
+ \int_{tV} (1 - D) t S_{MN}^{Piezo} \delta_{t}^{+\Delta t} E_{MN} d^t V + \int_{tV} t S_{MN}^{Maxwell} \delta_{t}^{+\Delta t} E_{MN} d^t V
\]

(3.57)

The damage model will not affect the density since the mass is conservative. The term representing body force remain unchanged. Moreover, the boundary conditions
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are

\[ U_I = \delta_I u_i = \bar{U}_I^0 \quad \text{on} \quad \Gamma_u \in \partial \Omega_0 \]

\[ T_I = \left[ P_{vol}^{ij} + (1 - D)P_{dev}^{ij} \right] N_J = \bar{T}_I^0 \quad \text{on} \quad \Gamma_t \in \partial \Omega_0 \quad (3.58) \]

It is obvious that if the same traction is applied, the stress evaluated at \( \Gamma_t \) will increase if the damage is developed on-site.

3.3.2 Weak Form and Boundary Conditions of the Electric Field with Piezoelectric Material and Continuum Damage Model

Magnetic field is not produced from piezoelectricity and piezoelectricity does not affected by magnetic field. The weak form of magnetic field in Eq. (2.77b) can be removed from EM problem in Eq. (2.77). Eliminating the magnetic effect and implementing piezoelectric constitutive relation in Eq. (2.77a), the weak form of the electric field is:

\[
\int_{\partial \Omega_0} N_I \left( \varepsilon J C_{ij}^{-1} E_J - \mathcal{E}_{IPQ} E_{PQ} \right) \delta \Phi dS_0 - \int_{\Omega_0} (\varepsilon J C_{ij}^{-1} E_J - \mathcal{E}_{IPQ} E_{PQ}) \delta \Phi_{x_i} dV_0 = 0
\]

(3.59)

The boundary condition which is the first term in Eq. (3.59) is modified from Eq. (2.79) as
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\[
\int_{\partial\Omega_0} N_I \left( \varepsilon_0 JC^{-1}_{I,J} E_J - \mathcal{E}_{IPQ} E_{PQ} \right) \delta \Phi dS_0 = \\
\int_{\partial\Omega_0} N_I \left[ \varepsilon_0 JC^{-1}_{I,J} (-\Phi,J) - \mathcal{E}_{IPQ} E_{PQ} \right] \delta \Phi dS_0
\]

(3.60)

For Dirichlet boundary conditions, as \( V = \Phi = \Phi_0 \) or for grounded boundary \( \Phi = 0 \), Eq. (3.60) is vanished as \( \delta \Phi = 0 \).

Another common boundary condition which leads to drop Eq. (3.60) is \( N_I D_I = 0 \), where \( D_I \) is defined in Eq. (3.45b). This boundary condition is applied at the interface of dielectric-dielectric when one of the dielectric constant is large compared with the other one, such as air-dielectric interface.

If a current governed by the reference configuration Ohm’s law Eq. (2.49) is injected into the structure at a free boundary, the boundary condition is implemented as:

\[
\int_{\partial\Omega_0} N_I \left[ \varepsilon J C^{-1}_{I,J} (-\Phi,J) \right] \delta \Phi dS_0 = \int_{\partial\Omega_0} N_I \left[ \frac{J_I}{\sigma} \right] \delta \Phi dS_0
\]

(3.61)

Otherwise, if the boundary is not free, i.e. Green strain tensor \( E_{PQ} \neq 0 \), the injection current is introduced as:

\[
\int_{\partial\Omega_0} N_I \left[ \varepsilon J C^{-1}_{I,J} (-\Phi,J) \right] \delta \Phi dS_0 = \int_{\partial\Omega_0} N_I \left[ \varepsilon \frac{J_I}{\sigma} + \mathcal{E}_{IPQ} E_{PQ} \right] \delta \Phi dS_0
\]

(3.62)

In the case of continuum damage model is implemented, piezoelectric material
constant will be affected. The weak form becomes:

\[
\int_{\partial\Omega_0} N_I \left[ \varepsilon J C_{IJ}^{-1} E_J - (1 - D) \mathbf{e}_{IPQ} E_{PQ} \right] \delta \Phi dS_0 \\
- \int_{\Omega_0} \left[ \varepsilon J C_{IJ}^{-1} E_J - (1 - D) \mathbf{e}_{IPQ} E_{PQ} \right] \delta \Phi, I dV_0 = 0
\]  

(3.63)

As for the boundary conditions, the damage will not affect Dirichlet boundary condition when \( \delta \Phi = 0 \). Homogeneous Neumann boundary condition gives same description as undamaged case. Furthermore, if a current injects through a free boundary, damage parameter has no effect on it. The only scenario needs to be considered is a current injects through a constraint boundary. The boundary condition in this situation is:

\[
\int_{\partial\Omega_0} N_I \left[ \varepsilon J C_{IJ}^{-1} (-\Phi, J) \right] \delta \Phi dS_0 = \int_{\partial\Omega_0} N_I \left[ \varepsilon \frac{J_I}{\sigma} + (1 - D) \mathbf{e}_{IPQ} E_{PQ} \right] \delta \Phi dS_0
\]  

(3.64)

### 3.4 FE Implementation for Piezoelectric Material in Finite Deformation Coupled with Continuum Damage Model

Utilizing the same framework developed for EM-ME coupling in Sec. 2.4, extra work is necessary to incorporate piezoelectric material in the reference configuration accommodating finite deformation setting using FE method. In this section, the continuum damage model implementation is introduced firstly. The damage effect
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on mechanical field with piezoelectric field and electric field is illustrated as follows. The constitutive relation of piezoelectric material renders alteration in treatment of increment scheme for second Piola-Kirchhoff stress.

3.4.1 FE Implementation of the Continuum Damage Model

The implementation of the continuum damage model is done through interfacing the model with existing material constitutive relation. The stresses and all the state variables are calculated at each integration point of the finite element model for the given strain increment.

With all state variables including stress \( t^i S_{ij} = t^i \sigma_{ij} \), damage parameter \( t^D \) from mechanical field obtained from the \( n^{th} \) converged step at time \( t = t \), the mechanical field variables are updated with given displacement increment \( \Delta U_I \). The formulation of updated Lagrangian method to solve weak form of mechanical field is based on the incremental scheme. However, damage evolves as a history variable. The reference configuration in such scenario refers to the original configuration at \( t = t \). Essential steps in implementing algorithm for the damage model are given below.

1. Assuming no damage evolution at the \((n+1)^{th}\) step at time \( t = t + \Delta t \), calculate incremental values for the strain driven process.

\[
(3.65)
\]
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From Eq. (3.65), the strain decomposition for volume preserved part is:

\[
\begin{align*}
\tau_{t+\Delta t}^J &= \tau_{t+\Delta t}^J \cdot J^{-\frac{1}{3}} \tau_{t+\Delta t}^F, \\
\tau_{t+\Delta t}^C &= \tau_{t+\Delta t}^C \\
\end{align*}
\]

(3.66)

2. Predictor algorithm for damage evolution.

Calculate the history variables for the damage model:

\[
t_{t+\Delta t}^{\Xi_{max}} = \max \{t_{t+\Delta t}^{\Xi_{trial}}, t_{t+\Delta t}^{\Xi_{max}}\}
\quad \text{where} \quad t_{t+\Delta t}^{\Xi_{trial}} = \sqrt{2 \tau_{t+\Delta t}^{\Psi_{dev}}}
\]

(3.67)

where

\[
t_{t+\Delta t}^{\Psi_{dev}} = \Psi_{dev} \left(\tau_{t+\Delta t}^{t+\Delta t \xi_{IJ}}\right)
\]

(3.68)

If \(t_{t+\Delta t}^{\Xi_{max}} = t_{t+\Delta t}^{\Xi_{max}}\), damage evolution does not take place in current increment, proceed to step 3 with:

\[
t_{t+\Delta t}^D = t^D
\]

(3.69)

Proceed to step 3 with no damage evolution option.

If \(t_{t+\Delta t}^{\Xi_{max}} = t_{t+\Delta t}^{\Xi_{trial}}\), damage will evolve in current increment. Damage parameter is updated as

\[
t_{t+\Delta t}^D = t^D - \frac{g'}{t_{t+\Delta t}^{\Xi_{max}}} \frac{\partial \Psi_{dev}}{\partial E_{I,J}} : \Delta E_{I,J}
\]

(3.70)

Proceed to step 3 with damage evolution.
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3. Update tangent elastic stiffness matrix stress deviator and piezoelectric coupling constant based on damage evolution

The tangent modulus of stress deviator is calculated with updated damage parameter if the damage is evolving as:

\[ t_{C_{IJKL}}^{dev} = g(t + \Delta t \Xi_{max}) \frac{\partial \Psi_0^{dev}}{\partial E_{IJ} \partial E_{KL}} + g' \frac{\partial \Psi_0^{dev}}{t + \Delta t \Xi_{max}} \frac{\partial \Psi_0^{dev}}{\partial E_{IJ} \partial E_{KL}} \] (3.71)

and

\[ \mathbf{E}_{IPQ} = g(t + \Delta t \Xi_{max}) \mathbf{E}_{IPQ} \] (3.72)

Otherwise, if the damage evolution criterion is not met, the tangent modulus of stress deviator is

\[ t_{C_{IJKL}}^{dev} = g(t + \Delta t \Xi_{max}) \frac{\partial \Psi_0^{dev}}{\partial E_{IJ} \partial E_{KL}} \] (3.73)

and the piezoelectric coupling constant:

\[ \mathbf{E}_{IPQ} = g(t + \Delta t \Xi_{max}) \mathbf{E}_{IPQ} \] (3.74)

If the damage model is not activated, the deviatoric fragment of second Piola Kirchhoff stress and the piezoelectric coupling constant will not be affected. This step for implementing damage will be bypassed.
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3.4.2 FE Implementation of the Finite Deformation Dynamics and Electric Field with Piezoelectric Material

Enriched piezoelectric material relation in the finite deformation, the second Piola-Kirchhoff stress adopts additional terms of piezoelectric effect and Maxwell stress. From the reference configuration at $t = t$, the stress at $t = t + \Delta t$ builds up on the increment as

$$
{}^{t+\Delta t}S_{IJ} = {}^{t}S_{IJ} + \Delta S_{IJ} = {}^{t}\sigma_{ij} + \Delta S_{IJ}
$$

(3.75)

where $^{t}\sigma_{ij}$ is obtained from previous step, and the increment $\Delta S_{IJ}$ is

$$
\Delta S_{IJ} = \Delta S_{IJ}^{vol} + \Delta S_{IJ}^{dev} + \Delta S_{IJ}^{Piezo} + \Delta S_{IJ}^{Maxwell}
$$

(3.76)

Express Eq. (3.75) in terms of increment of strain $\Delta E_{IJ}$ and electric field $\Delta E_{K}$:

$$
\Delta S_{IJ} = {}_{t}C_{IJKL}\Delta E_{KL} + {}_{t}P_{SIJ}\Delta E_{S}
$$

(3.77)

where

$$
{}_{t}C_{IJKL} = {}_{t}C_{IJKL}^{vol} + {}_{t}C_{IJKL}^{dev} + {}_{t}C_{IJKL}^{Maxwell}
$$

(3.78)

The tangent moduli are defined as,
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\[ tC_{IJKLM}^{\text{vol}} = 2 \frac{\partial S_{IJ}^{\text{vol}}}{\partial C_{KL}} = 4 \frac{(\partial \Psi_{\text{vol}})^2}{\partial C_{IJKL}} \]  
(3.79a)

\[ tC_{IJKLM}^{\text{dev}} = 2 \frac{\partial S_{IJ}^{\text{dev}}}{\partial C_{KL}} = 4 \frac{(\partial \Psi_{\text{dev}})^2}{\partial C_{IJKL}} \]  
(3.79b)

\[ tC_{IJKLM}^{\text{Maxwell}} = 2 \frac{\partial S_{IJ}^{\text{Maxwell}}}{\partial C_{KL}} \]  
(3.79c)

Moreover, \( P_{SII} = \mathcal{H}_{SII}^{\text{Maxwell}} - E_{SII} \), where

\[ \mathcal{H}_{SII}^{\text{Maxwell}} = \frac{\partial S_{IJ}^{\text{Maxwell}}}{\partial E_S} \]  
(3.80)

The derivation and expression of \( C_{IJKLM}^{\text{Maxwell}} \) and \( \mathcal{H}_{SII}^{\text{Maxwell}} \) is listed in the Appendix.

As for the linearization, the increment of electric field is evaluated by:

\[ \Delta E_S = t^+ \Delta t E_S - t E_S = (t \Phi_S, S - t^+ \Delta t \Phi_S, S) \]  
(3.81)

By using trapezoidal rule

\[ t^+ \Delta t \Phi = t \Phi + \Delta t \frac{t^+ \Delta t}{2} \Phi + H.O.T \]  
(3.82)

where Eq. (3.81) is further derived as

\[ \Delta E_S = -\Delta t t \dot{\Phi}_S \]  
(3.83)
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Employing the strain increment in Eq. (2.90), second Piola-Kirchhoff stress is calculated as:

\[
\Delta S_{IJ} \approx tC_{IJKL} \Delta e_{KL} + tP_{SIJ} \Delta t^t \Phi_s
\]  

(3.84)

for linearization.

The weak form of mechanical field with piezoelectric implemented is expanded from Eq. (3.57):

\[
\int_{tV} t^t C_{MNPQ} \Delta e_{MN} \delta \Delta e_{MN} d^t V + \int_{tV} t^t \sigma_{mn} \delta \Delta \eta_{MN} d^t V - \int_{tV} tP_{SMN} \Delta \eta_{MN} d^t V = \int_{tS} t^t \bar{T}_I^i \delta u_i d^t S + \int_{tV} t^t \rho b_i \delta u_i d^t V - \int_{tV} t^t \rho \ddot{u}_i \delta u_i d^t V - \int_{tV} t^t \sigma_{mn} \delta \Delta e_{MN} d^t V + \int_{tV} tP_{SMN} \Delta e_{MN} d^t V
\]  

(3.85)

If the damage model is triggered, the tangent stiffness matrix of deviatoric second Piola-Kirchhoff stress \( tC_{IJKL}^{dev} \) and the piezoelectric coupling term \( tP_{SIJ} \) will be affected by Eq. (3.71) and Eq. (3.72) is the damage is evolving or by Eq. (3.73) and Eq. (3.74) is the damage is not evolving to accordingly.

As for the electric field, using the total Lagrangian formulation, the weak form Eq. (3.59) is implemented as
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\[
\int_{\partial \Omega_0} N_I (\varepsilon^{t+\Delta t}_0 J^{t+\Delta t}_0 C_{IJ}^{-1} \varepsilon^{t+\Delta t}_0 E_J - \mathcal{E}_{IPQ}^{t+\Delta t}_0 E_{PQ}) \delta \Phi dS_0 \\
- \int_{\Omega_0} (\varepsilon^{t+\Delta t}_0 J^{t+\Delta t}_0 C_{IJ}^{-1} \varepsilon^{t+\Delta t}_0 E_J - \mathcal{E}_{IPQ}^{t+\Delta t}_0 E_{PQ}) \delta \Phi, I dV_0 = 0 
\]

(3.86)

Express the electric field in terms of electric scalar potential, Eq. (3.86) is

\[
\int_{\partial \Omega_0} N_I (\varepsilon^{t+\Delta t}_0 J^{t+\Delta t}_0 C_{IJ}^{-1} \varepsilon^{t+\Delta t}_0 \Phi)_{,J} + \mathcal{E}_{IPQ}^{t+\Delta t}_0 E_{PQ} \delta \Phi dS_0 \\
- \int_{\Omega_0} (\varepsilon^{t+\Delta t}_0 J^{t+\Delta t}_0 C_{IJ}^{-1} \varepsilon^{t+\Delta t}_0 \Phi)_{,J} + \mathcal{E}_{IPQ}^{t+\Delta t}_0 E_{PQ} \delta \Phi, I dV_0 = 0 
\]

(3.87)

The mechanical field related variables at time \( t + \Delta t \) is known from the staggered solution before calculate the electric field. The piezoelectric coupling term will be changed according to Eq. (3.72) or Eq. (3.74) when the damage model is activated.

Details of the solution methodology for the coupled problem with damage and piezoelectricity are shown in the flowchart of Fig. 3.2.
Figure 3.2: Flow chart of the coupled mechanical and EM field problem
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3.5 Conclusion

Based on the framework developed in Chapter 2, piezoelectric material relation is incorporated under finite deformation. The constitutive equations in the reference configuration are studied. Based on the constitutive equations, weak forms and boundary conditions are examined to accommodate the change brought in by two-way coupling. Moreover, continuum damage model is implemented in the numerical platform. With this tool, the effect of material evolution on the electric field can be studied. FE implementation is refined according to the contents.
Chapter 4

Validations of the EM-ME Code and Numerical Analysis for the Coupled Problems

Numerical examples, simulated using one-way coupled dynamic, electromagnetic (ME-EM) code, are divided into two categories. The first category corresponds to validation examples, where the results of simulation with the code are compared with existing analytical solutions or results from commercial software like COMSOL [78] and ANSYS [79]. These packages generally have limited capacity with respect to solving coupled, dynamic electromagnetic problems. Correspondingly, selected features of the ME-EM code are tested against these codes. The comparisons are respectively made for (a) an electro-static problem, (b) a magneto-static problem, (c) a transient magnetic problem, (d) a transient electromagnetic problem, all above cases are without mechanical excitation, and (e) a piezoelectric bimorph beam, in small deformation regime. Subsequent to the validation tests, the code is used to simulate two examples of coupled ME-EM problems to investigate the EM fields in deforming substrate as load-bearing antennae application. The first example is for a steady-state electromagnetic field in a dynamically loaded structure. The second
example solves a transient electromagnetic field problem in a structure undergoing finite deformation under dynamic condition. These examples demonstrate the effectiveness of the coupled mechanical-electromagnetic code and model. Moreover, the piezoelectric material model with finite deformation is carried out comparing with piezoelectric material problem in small deformation by Abaqus. It demonstrates the effectiveness of the coupled electromechanical coupling of piezoelectricity in finite deformation. Based on it, a self-sensing piezoelectric damage detector is proposed. Numerical calibration and validation of this virtual sensor are studied.

4.1 Validation Examples

4.1.1 Electrostatic Problem for a Micro-strip

Electrostatic problems of a shielded micro-strip have been discussed in [57] and also in COMSOL AC/DC module tutorial model [78] primarily in 2D. The corresponding 3D model, solved using the ME-EM code, is illustrated in Fig.4.1a. A homogenous Neumann boundary condition is applied on the lateral boundary in $z$-direction, which keeps any field from transmitting across the interface. This boundary condition also helps in reducing the model dimensions due to half-symmetry. In the ME-EM code, the mechanical response is frozen for this problem and hence the current and reference configurations are identical. Only the scalar potential $\varphi$ is relevant for this electrostatic problem. In this setting, the governing equations Eq. (2.54a) and Eq.
(2.54b) are reduced to
\[ \nabla^2 \varphi = -q_e \]  \quad (4.1)

Fig. 4.1 shows the dimensions of the 10mm x 10.5mm x 6mm computational domain with boundary conditions. For the inner conductor, the potential is \( \varphi = 1 \) V, while the shielding conductor has a scalar potential value of \( \varphi = 0 \) V assigned to the outer boundary.

Figure 4.1: Microstrip model and boundary conditions. (a) 3D model with homogeneous Neumann boundary condition on lateral boundaries in the z direction; (b) mesh of ME-EM FE code containing 158976 brick elements for the half-symmetry model.

Results of simulation with the ME-EM code containing 158976 8-noded brick elements are shown in Fig. 4.2a. The permittivity of the dielectric is chosen to be \( \varepsilon = 1.0 \) F/m for simplicity in post-processing. The contour plot is for the distribution of the electric potential \( \varphi \), while the arrows indicate the electric field. The length and direction of the arrow correspond to the magnitude and direction of the electric field at each nodal point. Comparison of the results of the ME-EM code with two different 3D mesh densities, and the COMSOL code with a 2D mesh are shown in Fig. 4.2b.
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The results demonstrate good agreement in the distribution of the scalar potential, even with a relatively coarse ME-EM mesh. The electric potential decays from 1V at the inner boundary to 0 at the outer boundary. To study convergence of the ME-

![Figure 4.2: (a) Scalar potential and electric field distribution in the 3D microstrip model using the ME-EM code; (b) comparison of scalar potential distribution along x-axis generated by the ME-EM code and COMSOL.](image)

EM model and code, seven additional mesh densities are simulated. The finest mesh shown in Fig. 4.1 contains 159876 brick elements, whose solution is referred to as the reference solution. The error with respect to the reference solution is evaluated using the $L_2$ norm of $\varphi$ in the domain, defined as:

$$
\|e\|_{L_2} = \left( \frac{\int_{\Omega} (\varphi - \hat{\varphi})^T (\varphi - \hat{\varphi}) d\Omega}{\left( \int_{\Omega} \varphi^T \varphi d\Omega \right)^{1/2}} \right)^{1/2},
$$

(4.2)

where $\varphi$ is the reference electric potential solution in the finest mesh model. The
corresponding error plot is shown in Fig. 4.3a, exhibiting convergence with increasing degrees of freedom (DOF). The rate of convergence is shown in the log-log plot 
\(\log \| e \|_{L^2} \approx \log k - \beta \log N\) where \(N\) is the DOF) of Fig. 4.3b, with a slope of \(\beta = 0.77\).

![Figure 4.3: (a) Plot of the error \(\| e \|_{L^2}\) with increasing DOF; (b) convergence rate of results with the ME-EM code](image)

4.1.2 Magnetostatic Problem with Current Injection

This example simulates the response of a rectangular conductor, shown in Fig. 4.4, with an injected steady-state direct current, and compares it with results using the COMSOL code. For steady-state electromagnetic fields or in cases where the transients are not significant enough to produce induction terms, the electric and magnetic fields are decoupled and can be solved independently. When the current inside the conductor reaches steady-state, the magnetic field in this magneto-static problem is
expressed as:

$$\nabla^2 \mathbf{a} = -\mu_0 \mathbf{j}, \quad (4.3)$$

which is reduced from Eq. (2.54b) by neglecting the time-dependent contributions of

the scalar and vector potentials. Additionally, the gauge condition reduces to $\nabla \cdot \mathbf{a} = 0$. Instead of assigning the point-wise current density inside conductor, this problem

is solved by using the ME-EM code, for which the governing equations are obtained

by substituting the gauge condition and Ohm’s law $\mathbf{j} = \sigma \mathbf{e} = \sigma (\nabla \varphi - \frac{\partial \mathbf{a}}{\partial t})$ in Eqs. (2.54a) and (2.54b) in the absence of deformation. The corresponding equations are:

$$\nabla^2 \varphi = -q_e \quad (4.4a)$$

$$(\nabla^2 a_i - \mu \varepsilon \frac{\partial^2 a_i}{\partial t^2}) - (\mu \varepsilon \frac{\partial \varphi}{\partial t})_i = \mu \sigma (\varphi_i + \frac{\partial a_i}{\partial t}) \quad (4.4b)$$

The current density in the conductor, in the right hand side of Eq. (4.3), is imposed

by the mixed boundary condition given in Eq. (2.80), which assigns direct current injected from one end. The conductor is grounded at the other end that is manifested

by setting the scalar potential to $\varphi = 0$. The lateral sides are kept from any current flow by imposing a Dirichlet boundary condition for the vector potential, i.e. $\mathbf{a} = \mathbf{0}$. After the solution reaches the steady-state, the time dependent terms in Eq. (4.4a) and Eq. (4.4b) drop off. Both scalar potential and vector potential are obtained as solutions. The results of ME-EM code are compared with those of COMSOL simulation in Fig.4.5, which shows $\varphi$ at the middle point of the surface where current
Figure 4.4: FE model and mesh of the rectangular conductor showing current injection as well as electric potential distribution.

is injected. The conducting material is considered to be aluminum, for which the permittivity is $\varepsilon = 7.0832 \times 10^{-11} \, F/m$, the permeability is $\mu = 1.2567 \times 10^{-6} \, H/m$ and the conductivity is $\sigma = 37.8 \times 10^6 \, S/m$. Since the ME-EM code has transient effects, results of the magnetostatic problem are achieved in the steady-state limit. Fig.4.5 shows that the results approach the steady-state value faster with smaller time steps. This convergence trend is an indicator of stability of the ME-EM code.

4.1.3 Transient Magnetic Field Problem

This example, discussed as case VM167 in the ANSYS verification manual [79], deals with the development of a transient magnetic field in a static semi-infinite conductor. At start, the vector potential $a$ in the conductor is zero, while at $t = 0^+$ a pulse in $a$ is imposed on the boundary that generates a transient magnetic field through the
conductor. An eddy current field is rendered in the governing equations by removing the time-dependent electric displacement field in Eq. (2.19d). With the constitutive relation in Eq. (2.21) and zero scalar potential, the governing equation after applying the gauge condition $\nabla \cdot \mathbf{a} = 0$, reduces to:

$$
\nabla^2 \mathbf{a} - \mu \sigma \frac{\partial \mathbf{a}}{\partial t} = 0,
$$  \hspace{1cm} (4.5)

In ANSYS [79], the semi-infinite conductor is modeled in 2D as a thin wire with a graded mesh to better capture the signal near the boundary where the pulse vector is applied. On the other end, a homogenous Dirichlet boundary condition $\mathbf{a} = \mathbf{A} = \mathbf{0}$ is
applied. The corresponding computational domain with graded mesh and boundary
conditions for the 3D ME-EM model are depicted in Fig.4.6. The material perme-
ability is \( \mu = 1.2567 \times 10^{-6}\ H/m \) and the conductivity is \( \sigma = 2.5 \times 10^6\ S/m \). The

\[
\begin{align*}
\mu & = 1.2567 \times 10^{-6}\ H/m \\
\sigma & = 2.5 \times 10^6\ S/m 
\end{align*}
\]

Figure 4.6: Model of transient magnetic field problem in ME-EM code, the gradient
mesh in y direction is kept same as that of in ANSYS
transient vector potential and induced eddy current are monitored for the domain
and reported for a point near the boundary where the pulse is imposed and tran-
sient results are sensitive. A minimum time step of \( \Delta t = 0.0002\ sec \) is used in the
simulations. Comparison of results by the ME-EM code and ANSYS are shown in
Fig.4.7a. As observed, the ME-EM code is able to capture and reproduce the tran-
sients simulated in ANSYS. An time-dependent error measure between the ME-EM
and ANSYS results is defined as 
\[
e(t) = \left( \frac{A^{\text{ANSYS}}_z(t)-A^{\text{EM}}_z(t)}{A^{\text{ANSYS}}_z(t)} \right)^2 \times 100\%
\]
and plotted in Fig. 4.7b. While initial transient effects give rise to a few spikes in error, it quickly
subsides to zero with time.
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Figure 4.7: Transient magnetic field simulation results: (a) Comparison of vector potential in the $z$ direction between results generated by the ME-EM code and ANSYS; (b) Local error with evolving time.
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4.1.4 Transient Electromagnetic Field Problem

This final validation example explores a coupled, transient electric and magnetic field problem using the ME-EM code by simulating the transient response due to alternating current (AC) in a slot embedded conductor. The problem is introduced as a case study VM186 in the ANSYS verification manual [79], as well as in [80] for studying skin effect. The geometric model contains three parts, viz. air, conductor and steel as shown in Fig.4.8a. The conductor carries a current \( I \) in the axial direction, perpendicular to its cross section. The permeability of the conductor is \( \mu = 1.0 \, \text{H/m} \) while its conductivity is \( \sigma = 1.0 \, \text{S/m} \). In the governing equations (4.6), the total measure-

Figure 4.8: (a) A slot embedded conductor simulated in the transient electromagnetic model; (b) equivalent boundary conditions on the conductor domain boundaries, thus avoiding free-space; (c) 3D FE model of the conductor implemented in the ME-EM code.

able current is comprised of two parts, viz. (a) a source current \( I_s \) determined by the
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scalar potential and (b) an eddy current $I_e$ which is induced by the time derivative of
the vector potential. Since the scalar and vector potentials are independent variables
to be solved from Eq. (4.6), $I_s$ and $I_e$ are a-priori unknown.

$$\nabla^2 \varphi = -q_e$$

$$\nabla^2 a_i - (\mu \varepsilon \frac{\partial \varphi}{\partial t})_i = \mu \left( \sigma \underbrace{\nabla \varphi}_{I_s} + \sigma \frac{\partial a}{\partial t} \right)$$

The values of permeability for steel and the conductor material are significantly
different, which makes it difficult to transmit magnetic flux across the interface. This
assumption restricts the magnetic field to be within the conducting slot and hence
only the conductor is simulated. A homogenous Neumann boundary condition, in
terms of the gradient of the vector potential, is imposed on the interface between
conductor and steel to generate a magnetic wall as shown in Fig.4.8b. The effect of
EM fields in the free space or air is incorporated through an equivalent boundary
condition as given in [81]. The computational domain is reduced to the conductor
only by adding a Neumann boundary condition on the interface between conductor
and air as:

$$\frac{\partial a_z}{\partial n} = -\frac{\mu I}{w}$$

Here $w$ is the width of the cross-section as shown in Fig. 4.8b. The 3D computational
model with boundary conditions are depicted in Figs.4.8b and 4.8c.
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Results of simulations, in terms of the total current and the eddy current, by the ME-EM code and ANSYS are compared in Fig. 4.9a. While the results are generally in good agreement, a small difference in the early time steps arises from the initial conditions. To comprehend convergence of the ME-EM results with decreasing time-steps, simulations are conducted with different time steps $\Delta t$ and an error measure $e$ is defined as:

$$e = \left\| \frac{I_{ME-AM}(\Delta t) - I_{ANSYS}}{I_{ANSYS}} \right\|$$

(4.8)

The error is plotted as a function of the inverse of the time step, i.e. $\frac{1}{\Delta t}$. It reduces rapidly with decreasing time-steps as shown in Fig. 4.9b.
Figure 4.9: Comparison of the results of transient electromagnetic field simulations by the ME-EM code and ANSYS: (a) total and eddy current results simulation; (b) rate of convergence with decreasing time steps.
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4.1.5 Piezoelectric Bimorph Beam

The piezoelectric model is validated with piezoelectric bimorph beam case. Many researchers such as [47, 82] refer to the model. The analytical solution of the bimorph beam is studied extensively in [83]. It can be carried out for comparison when deformation is small.

![Figure 4.10: Piezoelectric bimorph beam example. (a) Geometry and boundary condition of the bimorph beam. (b) The electric field in z-direction obtained from simulation](image)

A piezoelectric bimorph beam is constituted by upper and lower part with parallel or anti-parallel polarization as shown in Fig. 4.10a. The mechanical boundary condition of the validation model is clamped at the end $S_2$ and set free at the other end. A voltage $V = 1V$ is applied on the top and bottom surface. The interface of the upper and lower part is grounded. The material is neo-hookean, with Lamé constant set as $\lambda = 0$, $\mu = 10GPa$. The piezoelectric coupling constant is $\mathcal{E}_{31} = \mathcal{E}_{32} = 0.0460C/m^2$. The permittivity is $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 0.1062 \times 10^{-9} F/m$. The contour of the electric field in $z$-direction is shown in Fig. 4.10b. The electric field induced stress deforms...
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the beam. If the deformation is small, i.e. the geometric and material nonlinearity is not significant, the analytic solution of the deflection along the $x$-direction is

$$\omega = 3 \frac{c_{31} V}{E h^2} x^2$$  \hspace{1cm} (4.9)

where $E$ is the Young’s modulus for the small deformation region. If the deformation is not significant, the mechanical properties for linear elastic and hyperelastic material is comparable. The deflection of the beam along $x$-direction is plotted in Fig. 4.11. The simulation result matches the analytical solution very well.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{Figure4_11.png}
\caption{The deflection along $x$-direction in the bimorph beam from analytic solution and simulation}
\end{figure}
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4.2 Coupled Mechanical and Electromagnetic Field Simulations

Subsequent to validation of the electromagnetic part of the model and code, this section examines its abilities in predicting the evolution of coupled, dynamic and transient electromagnetic fields in multi-physics problems. Two examples are specifically considered in this section. The first examines the effect of dynamic deformation on steady-state electromagnetic fields in a conductor, while the second example explores its effect on transient electromagnetic field variables. For both examples, the conducting substrate is a rectangular plate as shown in Fig. 4.12. The logarithmic stretch model is employed. The mechanical properties in Eq. (2.13) are given in terms of the Lamé constants as $\lambda = 40.4 \text{ GPa}$ and $\mu = 26.9 \text{ GPa}$ respectively, and the electromagnetic properties in Eqs. (2.21) are: $\varepsilon = 7.0832 \times 10^{-11} \text{ F/m}$, $\mu = 1.2567 \times 10^{-6} \text{ H/m}$ and $\sigma = 37.8 \times 10^6 \text{ S/m}$. The deforming conductor is fixed at one end, i.e. $u = 0$ on $S_2$. The other end $S_1$ is subjected to a $z$-direction sinusoidal excitation $u_z(t) = u_0 \sin(\omega_{me}t)$. For the EM problem, the top, bottom and side surfaces, designated as $S_L$, are assigned homogeneous Dirichlet boundary conditions for the vector potential i.e. $A = 0$ in the reference configuration. A current $J_y(t) = J_0 \sin(\omega_{em}t)$ is injected into the fixed end $S_2$ of the conductor. $S_1$ is grounded by setting scalar potential $\Phi = 0$ in the reference configuration. Values of the displacement parameters $(u_0, \omega_{me})$ and current parameters $(J_0, \omega_{em})$ are varied
for the different examples in this section.

Figure 4.12: Schematic model of a vibrating conductor with injected current. The mechanical loading is in z-direction, and the current injects through y-direction

4.2.1 Effect of Dynamically Deforming Substrate on Steady-State Electromagnetic Field

This example represents a coupled multi-physics problem, in which the steady-state electromagnetic field is unilaterally affected by deformation fields in a dynamically loaded conducting substrate. The EM field is affected by various characteristics of the mechanical field, viz. (i) loading direction, (ii) displacement amplitude, (iii) velocity amplitude and (iv) loading frequency. The effect of direction is first examined with excitations perpendicular and parallel to the EM source, followed by the other characteristics.

In the first simulation, the face $S_1$ is subjected to a z-direction sinusoidal excitation $u_z(t) = u_0 \sin(\omega_{me} t)$, where $u_0 = 0.2 \, m$ and $\omega_{me} = 100 \, Hz$. Here the mechanical
excitation direction is perpendicular to the EM source as shown in Fig. 4.12. A constant direct current \( J_0 = 200 \, \text{A} \), \( \omega_{em} = 0 \) is injected into the fixed end \( S_2 \). For each time step, the EM field variables \( E, B, D \) and \( H \) in the reference configuration are obtained by Eqs. (2.56a), (2.56b), (2.46) and (2.48) respectively, following which the current configuration field variables \( e, b, d \) and \( h \) are calculated using transformations in Eqs. (2.45), (2.43), (2.29a) and (2.47). For a fixed conductor, the injected current will induce a steady-state \( y \)-direction electric field \( e_y \). However with sinusoidal excitation, the electric field \( e_y \) at a point \((x = 0.025 \, \text{m}, y = 0.125 \, \text{m}, z = 0.015 \, \text{m})\) evolves following the orthogonal, time-harmonic velocity \( v_z \) as shown in Fig. 4.13a. The frequency of \( e_y \) is double the frequency of \( v_z \), which is perpendicular to \( e_y \) arising from the cross product of velocity and EM fields in Eqs. (2.77a) and Eq. (2.77a). To comprehend the effect of load direction on EM frequency, the mechanical excitation is applied in the \( y \) direction instead of the \( z \) direction, i.e. \( u_y(t) = u_0 \sin(\omega_{me} t) \) in Fig. 4.12 with \( u_0 = 0.04 \, \text{m} \) and \( \omega_{me} = 100 \, \text{Hz} \). Here the mechanical and EM loadings are in the same direction. The cross product vanishes, yielding the same frequency for \( e_y \) and \( v_z \) as shown in the plot of Fig. 4.13b. Thus, while the amplitude of oscillation of \( e_y \) is not significantly affected by the direction of mechanical fields, it has a more profound effect on the EM frequency.
Figure 4.13: Plot of the velocity velocity and electric field components as functions of time: (a) $v_z$ and $e_y$; (b) $v_y$ and $e_y$. 
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Next, the influence of the amplitude and frequency of mechanical excitation
\[ u_z(t) = u_0 \sin(\omega_{me} t) \]
in Fig. 4.12, on the EM field is considered. Three specific
cases are studied, viz. (i) \( u_0 = 0.2 \) m, \( \omega_{me} = 100 \) Hz, (ii) \( u_0 = 0.2 \) m, \( \omega_{me} = 200 \) Hz
and (iii) \( u_0 = 0.1 \) m, \( \omega_{me} = 200 \) Hz. The effect of the excitation frequency and am-
plitude of velocity compared for the results from (i) and (ii). Fig. 4.14a plots \( e_y \) for
the corresponding two frequencies. Both the amplitude and frequency of the electric
field \( e_y \) are affected by changes in \( \omega_{me} \). The frequency \( \omega_{em} \) of \( e_y \) nearly doubles, con-
sistent with the change in \( \omega_{me} \). The amplitude of \( e_y \) also changes, even though the
displacement amplitude is unchanged. This is due to the increased (nearly doubled)
amplitude of velocity with doubled \( \omega_{me} \). The maximum amplitude of \( e_y \) increases from
\( 1.023 \times 10^{-5} \) V/m to \( 1.821 \times 10^{-5} \) V/m for frequency change from \( \omega_{me} = 100 \) Hz to
\( \omega_{me} = 200 \) Hz. The corresponding change in the minimum amplitude is small from
\( 3.437 \times 10^{-6} \) V/m to \( 2.928 \times 10^{-6} \) V/m. This infers that the velocity field \( v_z \) affects
the amplitude of \( e_y \) in a nonlinear fashion. The excitation displacement amplitude
for case (iii) is half of the other two, while its velocity amplitude is the same as case
(ii). Results of cases (i) and (iii) reveal the influence of magnitude of the displace-
ment field on the EM field. Fig. 4.14b shows that the maximum amplitude of \( e_y \)
\( 1.012 \times 10^{-5} \) V/m for (iii) is almost the same as the amplitude \( 1.023 \times 10^{-5} \) V/m
for (i). However, the minimum amplitude increases significantly by reducing \( u_0 \) from
\( 3.437 \times 10^{-6} \) V/m for (i) to \( 5.137 \times 10^{-6} \) V/m for (iii). From Figs. 4.14a and 4.14b, it
may be concluded that the magnitude of velocity has a significant effect on the peak
value of $e_y$, while the displacement affects the lowest values. Also the mechanical loading frequency directly determines the frequency of the EM field.
Figure 4.14: Comparison of $e_y$ for: (a) different mechanical frequencies $\omega_{me} = 100 \ Hz$ and $\omega_{me} = 200 \ Hz$ with $u_0 = 0.2 \ m$; (b) different mechanical amplitudes and frequencies $u_0 = 0.2 \ m$, $\omega_{me} = 100 \ Hz$ and $u_0 = 0.1 \ m$, $\omega_{me} = 200 \ Hz$. 

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A few observations may be made from this simple example as summarized below:

• For dynamic loads parallel to a steady-state EM source, the EM field will evolve with a frequency that is similar to that of the loading. For loads orthogonal to the EM source, the EM frequency can be significantly higher due to the coupling.

• The amplitude of the applied displacement affects the lower values of the oscillatory EM field, while the amplitude of the velocity field affects the peak EM values.

• The frequency of the mechanical excitation affects the frequency of the EM field to an extent that depends on the loading direction. The importance of coupling the two fields in a multi-physics analysis is realized though this example.

4.2.2 Effect of Dynamically Deforming Substrate on Transient Electromagnetic Field

This example extends the study in the previous section to a time-dependent electromagnetic field caused by an alternating current in a dynamically vibrating rectangular conductor. The current injected at the fixed end \( S_2 \) in Fig. 4.12 is \( J_y(t) = J_0 \sin(\omega_{em} t) \), where \( J_0 = 200 \ A \) and \( \omega_{em} = 500 \ Hz \). The effect of the AC current frequency \( \omega_{em} \), as well as the applied displacement and velocity amplitudes \((u_0, v_0)\) on the EM field are investigated. For the examples considered, the \( z \)-direction displacement excitation has the parameters \( u_0 = 0.2 \ m, \omega_{me} = 100 \ Hz \). Thus \( \frac{\omega_{em}}{\omega_{me}} = \frac{T_{me}}{T_{em}} = 5 \), where \( T_{me} \) and \( T_{em} \) are the velocity and current time-periods respectively. Fig. 4.15 is a plot of the oscillating electric field component \( e_y \) at the centroid of the conductor. Clearly
the frequency is affected by that of both the current and mechanical excitations. The oscillations exhibit two types of response periods, viz. (i) one characterized by the shorter period $T_e$ in Fig. 4.15 that follows the high frequency pattern of the imposed electric current, and (ii) a longer period $T_c$ corresponding to the distance between two identical maximum peaks, which follows the frequency of the mechanical excitations. To study this behavior further, simulations are conducted with $\omega_{me} = 200 \text{ Hz}$ and $\omega_{me} = 300 \text{ Hz}$ respectively. The corresponding results are shown in Figs. 4.16a and 4.16b. It is generally observed that the longer period $T_c$ scales with the ratio $\frac{\omega_{em}}{\omega_{me}}$, in proportion to electric current period $T_{em}$.

![Plot of $e_y$ and $v_y$ with time for current and mechanical frequencies $\omega_{em} = 500 \text{ Hz}, \omega_{me} = 100 \text{ Hz}$](image)

Figure 4.15: Plot of $e_y$ and $v_y$ with time for current and mechanical frequencies $\omega_{em} = 500 \text{ Hz}, \omega_{me} = 100 \text{ Hz}$. 
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Figure 4.16: Plot of $e_y$ and $v_y$ with time for different ratios of $T_{em}$ and $T_{me}$ with: (a) $\omega_{me} = 200 \text{ Hz}$, $\omega_{em} = 500 \text{ Hz}$ and (b) $\omega_{me} = 300 \text{ Hz}$, $\omega_{em} = 500 \text{ Hz}$.

Finally, to study the effects of displacement and velocity amplitude on EM fields, three simulations with the following conditions are conducted: (i) $u_0 = 0.2 \text{ m}, \omega_{me} =$
100 Hz, $\omega_{em} = 500$ Hz, (ii) $u_0 = 0.2 \, m, \omega_{me} = 200 \, Hz, \omega_{em} = 500 \, Hz$, and (iii) 
$u_0 = 0.1 \, m, \omega_{me} = 200 \, Hz, \omega_{em} = 500 \, Hz$. Fig. 4.17a compares the result of $e_y$
for (i) and (ii) with doubled velocity, while the comparison for the same velocity in 
(i) and (iii) is shown in Fig. 4.17b. The overall maximum peaks increase with time 
for both figures. However the minimum value of the signal decreases with increasing
velocity thus widening the gap between the maximum and minimum values. While
an increase in displacement amplitude leads to larger peak values of $e_y$, it reduces
the minimum value thus shifting both the minimum and maximum EM amplitudes
upwards. In conclusion, the resulting EM signals in the coupled system have a com-
plicated dependence on the mechanical and EM fields that require a robust analysis
capability.
Figure 4.17: Comparison of $e_y$ for (a) varying mechanical frequency and velocity amplitude $u_0 = 0.2 \text{ m, } \omega_{me} = 100 \text{ Hz, } \omega_{em} = 500 \text{ Hz}$ and $u_0 = 0.2 \text{ m, } \omega_{me} = 200 \text{ Hz, } \omega_{em} = 500 \text{ Hz}$; and (b) varying displacement amplitude $e_y$ in $u_0 = 0.2 \text{ m, } \omega_{me} = 100 \text{ Hz, } \omega_{em} = 500 \text{ Hz}$ and $u_0 = 0.1 \text{ m, } \omega_{me} = 200 \text{ Hz, } \omega_{em} = 500 \text{ Hz}$. 
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4.2.3 Comparison of Piezoelectric Model in Finite Deformation and Small Deformation

The piezoelectric material relations are incorporated in Abaqus with small deformation setting. In this section, a comparison between finite deformation piezoelectric problem carried out by in-house code and small deformation piezoelectric problem carried out by Abaqus is illustrated. Using this example, the convergence of the staggered coupling scheme for piezoelectricity is studied.

The model analyzed is taken from [84]. As shown in Fig. 4.18, Kagome-cut and square-cut of certain auxetic structure can achieve negative Poisson’s ratio.

Figure 4.18: Different cut scheme to design negative Poisson’s ratio a) The homogenous structure with cut, b) Kagome-cut pattern and angles, c) Square-cut pattern and angles

The analysis of the cut angles leads to homogenous Poisson’s ratio shown in Fig. 4.19. It is observed that with finite deformation, the minimum Poisson’s ratio is obtained at angle $\theta = 90^\circ$. To validate the piezoelectric material constitutive relations under finite deformation and compare the results with small deformation piezoelectric
problem, the square-cut pattern is adopted to study.

Figure 4.19: Poisson’s ratio with respect to cut angle in a) Kagome-cut pattern and b) square-cut pattern. Small deformation and large deformation analysis are compared.

The simulating model is shown in Fig. 4.20. The model is three-dimensional with thickness \( t = 100\text{mm} \). \( L_x = L_y = 525\text{mm} \) in the \( x \)- and \( y \)-directions. It contains \( 10 \times 10 \) undeformed RVE as shown in Fig. 4.19b. The simulating angle \( \theta = 90^\circ \) and the length of the cut is \( l_{cut} = 40\text{mm} \). The boundary condition for mechanical field is \( u_x \) and \( y_y \) fixed at \( x = 0, y = 0 \). \( u_y \) is constrained at other locations where \( y = 0 \). A pressure of \( 100\text{MPa} \) is applied at \( y = L_y \). For the electric field, ground the structure at \( x = 0, y = 0 \). The material properties are chosen as \( \lambda = 1.907 \times 10^9\text{Pa}, \mu = 3.390 \times 10^9\text{Pa} \), permittivity \( \varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 1.06 \times 10^{-10}\text{F/m} \) and the piezoelectric coupling effect \( \mathcal{E}_{11} = 0.2034\text{ C/m}^2, \mathcal{E}_{12} = -0.2034\text{ C/m}^2 \). As for the model in ABAQUS, only linear elastic material for mechanical field is provided.
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Thus the effect Young’s modulus is \( E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = 8 \text{GPa} \) and Poisson’s ratio \( \nu = \frac{\lambda}{2(\lambda+\mu)} = 0.18 \). All the other material properties remain the same.

\[
T_y = T_0
\]

\[
L_y
\]

\[
L_x
\]

\[
\theta = 90^\circ
\]

Figure 4.20: 10 × 10 mesh and zoom-in view for the square-cut pattern of auxetic structure with cut angle \( \theta = 90^\circ \).

The results are listed in Tab. 4.1. The discrepancy is from the linear and nonlinear material from ABAQUS and EM-ME code and the formulation for piezoelectric material from two programs.

<table>
<thead>
<tr>
<th>parameter</th>
<th>ABAQUS Model</th>
<th>EM-ME Code</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>min</td>
<td>max</td>
</tr>
<tr>
<td>( \Phi(MV) )</td>
<td>-2.8</td>
<td>10.4</td>
</tr>
<tr>
<td>( \varepsilon_{11} )</td>
<td>-0.166</td>
<td>0.133</td>
</tr>
<tr>
<td>( \varepsilon_{22} )</td>
<td>-0.057</td>
<td>0.191</td>
</tr>
<tr>
<td>( y_{max}(mm) )</td>
<td>594.5</td>
<td>579.91</td>
</tr>
</tbody>
</table>

Table 4.1: Comparison of piezoelectric material model between ABAQUS solution in small deformation and ME-EM simulation in finite deformation

Using this model, convergence of the staggered method is performed. Pick the
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node labeled as red dot in Fig. 4.20, where $x = 235\text{mm}$ and $y = 290\text{mm}$. The results of this node are studied under different loading conditions. In the default solving scheme, mechanical field under finite deformation is solved first. The configuration obtained from the mechanical regime is frozen when the electric field analysis is preformed. Fig. 4.21 shows the mechanical loading boundary conditions. To achieve same magnitude of applied force, time step $t_s = 10, t_s = 20$ and $t_s = 40$ are carried out. The loading with larger number of time step has smaller increments leading to more accurate results. Moreover, a step scheme which load one step and hold 5 step is executed as an iterative method. Time step $t_s = 5, t_s = 10$ are carried out for the iterative method.

![Figure 4.21: Different loading scheme to study convergence](image)

Figure 4.21: Different loading scheme to study convergence
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The results are plotted in Fig. 4.22. Both the electric potential $\Phi$ and Green strain in y-direction $E_{22}$ are given same convergent results with both iterative loading and direct loading with 20 and 40 time steps. The results prove that the implemented staggered method can achieve accurate result as long as the time step is well controlled.
Figure 4.22: (a) The electric potential $\Phi$ results under different loading scheme; (b) The Green strain $E_{22}$ under different loading scheme.
4.3 Self-sensing Piezoelectric Damage Detector

4.3.1 Calibration of the Correlation Function between Damage and the Electric Field

Since the damage evolution law depends on the deviatoric strain energy $\Psi^{\text{dev}}$, the relationship between $\Psi^{\text{dev}}$ and damage parameter $d$ can be obtained. However, it is not feasible to measure the deviatoric strain energy $\Psi^{\text{dev}}$ within a deformed structure. As introduced in 3.2, the damage arisen from mechanical loading will degrade both the stiffness and piezoelectric coupling of the coupled system. This will alter the electric field under certain mechanical load. In this case, if the difference of electric field between damaged and undamaged sample $\Delta E_I$ can identify the $\Psi^{\text{dev}}$ under the same mechanical and electric boundary condition, a functional form of the correlation function can be established as $\Delta E_I = f(d, \dot{d}, \cdots, \text{etc.})$ consequently.

In order to accomplish this goal, two assumptions need to be proved in numerical studies.

- The electric field in the damaged structure is different than the electric field in the undamaged structure under same mechanical and electric boundary condition.

- The deviatoric strain energy $\Psi^{\text{dev}}$ can be correlated with damage parameter $d$.

Numerical model with the capacity to simulate finite deformation and piezoelec-
tric coupling implemented with continuum damage model is used for the purpose of sensitivity studies, without the knowledge of the actual functional form and independent variables of the correlation function. A thin plate model with a narrow section in the middle is employed to explore the connection between electric field generated from piezoelectric coupling and the damage from mechanical loading. The dimensions of flat specimen are $H = 0.01m$, $L = 0.05m$, $l = 0.019m$, $r = 0.003m$, $R = 0.007m$ and $\theta = 0.6435rad$. The thickness of the model is $t = 0.002m$. The sketch and the mesh of the extruded 3D model is shown in Fig. 4.23.

![Figure 4.23: (a) Dimension of 2D sketch of the thin plate model with narrow section; (b) 3D model and the mesh the thin plate.](image)

This geometry provide spatial distribution of the damage in $X$-$Y$ plane. The coupon shape can generate nonuniform stress field when a uniformed load is applied.
in $y$-direction. The stress concentration helps to study the correlation function by providing different damage value within the structure.

The mechanical properties are given as $\lambda = 0$, $\mu = 35 GPa$ and the Poisson ratio is $\nu = 0$. The piezoelectric constant is $\varepsilon_{22} = 0.001$. The permittivity is $\varepsilon = 1.0$.

The thin plate is fixed at one end, i.e. $u = 0$ on the surface of $S_2$. The other end $S_1$ is subjected to a $y$-directional uniaxial strain load $u_y = 0.0005t$ incrementally. For the piezoelectric problem, the boundary condition is to set $\Phi = 0$ where $y = 0.025$.

A sensitivity study of damage parameter $d$ is designed to find the relationship between $\Delta E_I$ and $\Psi^{dev}$. However, damage model is suppressed. Instead, the damage is set to be constant at all the integration points within each model. No further damage propagation is permitted. Thus $\dot{d} = 0$ as a consequence. The damage parameters are set to be 0.05, 0.1, 0.15 and 0.2, respectively. Additionally, an undamaged model is carried out as the reference case.

Since the plate is thin in $z$ direction, variation of solutions along $z$-direction is not significant, the middle layer where $z = 0.001$ is focused on for analysis. The variables, such as electric field, Green strain, deviatoric strain energy are recorded at the nodal points in the interval of $0.0048 \leq x \leq 0.0052$, $0.022 \leq y \leq 0.028$ and $z = 0.001$.

Damage parameter $d$ in the numerical cases soften the stiffness and decrease the piezoelectric material constant. As a result, the electric field in these cases is different with the electric field captured in the undamaged case. The differences of electric field between four damaged case and the undamaged case are calculated at each
incremental time step. The corresponding deviatoric strain energies are obtained.

The relationship is plotted as Fig. 4.24.

![Diagram of electric field differences between undamaged model and damaged models with constant damage parameter.](image)

Figure 4.24: The electric field differences between undamaged model and the damaged models with constant damage parameter

It is shown that, as the damage softening the material both from mechanical and piezoelectric aspects, the difference of the electric field in each scenario with a constant damage parameter has linear relationship with the strain energy. The relationship can be simply obtained as $\Delta E_y = f(d) \Psi^{dev}$, where $f(d)$ value returns to the slope of the line in Fig. 4.24. $f(d)$ can be fitted as

$$f(d) = 3.408e^{-12}d^2 + 12.39e^{-12}d$$  \quad (4.10)
Furthermore, the relationship between $\Psi^{\text{dev}}$ and damage parameter is studied. A numerical model with damage evolution is carried out. The parameter of the damage model Eq. (3.52) is chosen as $\alpha = 600$ and $\beta = 0.5$. This parameter will stay unchanged for the calibration and validation of the correlation function to detect damage in this paper. The numerical analysis employ the same geometry, boundary condition as the previous case. Numerical analysis with and without damage model is simulated.

![Graph showing the relationship between damage parameter $d$ and deviatoric energy $\Psi^{\text{dev}}$ in the uniaxial tension case.](image)

Figure 4.25: Relationship between damage parameter $d$ and deviatoric energy $\Psi^{\text{dev}}$ in the uniaxial tension case.

In the case with damage model, the deviatoric strain energy $\Psi^{\text{dev}}$ at each integration point and the damage are well correlated as shown in Fig. 4.25. It is anticipated since the damage evolution law is based on the history of $\Psi^{\text{dev}}$ value. The curve can
be fitted by a polynomial function of damage parameter $d$ as

$$\Psi^{\text{dev}} = g(d) = 7.146e^7d^4 + 3.587e^6d^2 - 1.873e^4d. \quad (4.11)$$

Without considering the time derivative of damage parameter $\dot{d}$, the correlation function is obtained by combining Eq. (4.10) and Eq. (4.11) together:

$$\Delta E_y = (3.408e^{-12}d^2 + 12.39e^{-12}d) \times (7.146e^7d^4 + 3.587e^6d^2 - 1.873e^4d) \quad (4.12)$$

However, it is insufficient to serve as correlation function. The numerical cases carried out can only prove that the electric field difference $\Delta E_y$ can be an indicator of damage state. But $d$ and $\dot{d}$ are related in such a way that it is impossible to study the sensitivity of $\dot{d}$ by freezing $d$. It is also observe from Fig. 4.26 that Eq. (4.12) cannot agree with the simulation result. Besides the difference in the value, the simulation results of $\Delta E_y$ is not a injective function from $d$ as shown in Fig. 4.26, $\dot{d}$ needs to introduced as another argument.
Propose a polynomial form of \( d \) and \( \dot{d} \) to correlate \( \Delta E_y \),

\[
\Delta E_y = a_1 d^4 + a_2 d^3 \dot{d} + a_3 d^2 \dot{d}^2 + a_4 d \ddot{d}^3 + a_5 \dot{d}^4 + a_6 \dot{d}^2 + a_7 d^2
\]
\[
+ a_8 \dot{d} \ddot{d} + a_9 d + a_{10} d^3 + a_{11} \dot{d}^2 d + a_{12} \dot{d} \ddot{d}^2
\]  

(4.13)

Fit the equation with the set of data from the model, the updated correlation function form is obtained as,
\[ \Delta E_{y}^{\text{ref}} = 2.2e^{-3} d^4 + 1.576e^{-4} d^3 \dot{d} + 4.946e^{-6} d^2 \dot{d}^2 + 7.572e^{-8} d \ddot{d}^3 \]
\[ - 6.565e^{-11} \dddot{d}^4 + 1.087e^{-9} \dddot{d}^2 + 7.285e^{-5} d^2 + 2.399e^{-6} d \dot{d} \]
\[ - 2.883e^{-6} d - 5.714e^{-4} d^3 - 7.285e^{-7} \ddot{d}^2 d - 3.749e^{-5} \dot{d} \ddot{d}^2 \]

(4.14)

Plot the fitted correlation function with the simulation results in Fig. 4.27. The correlation function agrees with the simulation result perfectly.

Figure 4.27: \( \Delta E_{y} \) and \( d \) plot from simulation and fitted correlation function of \( \Delta E_{y}(d, \dot{d}) \)

A closer examination which is shown in Fig. 4.28a demonstrate the correlation matches well with the scatter points. And the error of the correlation function is well bounded within 3%. The fit results agrees with the simulation perfectly in this
updated functional form for correlation function as shown in Fig. 4.27. It is concluded that a significant improvement is achieved by using both $d$ and $\dot{d}$ as independent variables.
Figure 4.28: (a) Zoom view of comparison between simulation and fitted correlation function; (b) Relative error of the fitted correlation function from the simulation result.
CHAPTER 4. VALIDATIONS AND NUMERICAL CASES

Another factor needs to be addressed is the effect of material properties. For material setting, two properties are studied. One is from mechanical regime the other one is from piezoelectric effect. As for the mechanical regime, the Lamé constant $\lambda$ and $\mu$ are given for hyperelastic material. $E = \frac{\mu (3\lambda + 2\mu)}{\lambda + \mu}$ is chosen as the variable to affect the correlation function. In the previous simulation, $E = 70\, MPa$ is assigned. To study the sensitivity of the objective function with respect to $E$, $E = 7\, MPa$ and $E = 35\, MPa$ are chosen with all the other material properties and boundary conditions remaining the same. Name three cases with:

- Modulus I: $E = 70\, MPa$
- Modulus II: $E = 35\, MPa$
- Modulus III: $E = 7\, MPa$

Both damaged and undamaged cases are carried out. Plot the electric field difference $\Delta E_y$ and damage parameter $d$ at same nodes locating at the area of $0.0048 \leq x \leq 0.0052$, $0.022 \leq y \leq 0.028$ and $z = 0.001$, the results are shown in Fig. 4.29.
Multiply $\Delta E_y$ obtained from Modulus II and Modulus III by 0.5 and 0.1 respectively, Fig. 4.30 shows all results return to the same correlation function obtained in case Modulus I. The relations between $\Delta E_y$ and $E$ is linear.
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Since the effect modulus are different from case to case, by applying same boundary condition, the larger effect modulus case gives smaller damage. However, the electric fields between damaged and undamaged simulation $\Delta E_y$s remain at similar level.

As for the electric regime, simulations with different value of piezoelectric coupling constant $\varepsilon_{S1J}$ are carried out to study the sensitivity of $\Delta E_S$ with respect to $\varepsilon_{S1J}$. In the previous calibration simulation $\varepsilon_{22} = 0.001$ is set to represent the piezoelectric coupling effect. On top of it, $\varepsilon_{22} = 0.002$ and $\varepsilon_{22} = 0.004$ are used with all the other material properties and boundary conditions remaining the same. Name three cases
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with:

- Piezo I: $\varepsilon_{22} = 0.001$
- Piezo II: $\varepsilon_{22} = 0.002$
- Piezo III: $\varepsilon_{22} = 0.004$

Perform simulations on both damaged and undamaged cases, the $\Delta E_y - d$ plot in the three simulating cases are shown in Fig. 4.31.

![Figure 4.31: The $\Delta E_y - d$ with different piezoelectric coupling constant](image)

Since the damage is controlled by mechanical response, with same effect moduli, all simulations render to similar magnitude of damage at the same location. However,
different $\Delta E_y$s are recorded since the piezoelectric coupling terms $\mathcal{E}_{22}$ are different. The smaller value of $\mathcal{E}_{22}$ means weaker coupling between the mechanical regime and electric field and leads to smaller $\Delta E_y$.

Multiply $\Delta E_y$ obtained from Piezo II and Piezo III by 0.5 and 0.25 respectively, Fig. 4.32 shows these altered values agrees with the correlation function obtained in case Piezo I. The local relationship between $\Delta E_y$ and $\mathcal{E}_{22}$ is obviously linear dependent.

Combining the sensitivity study of the material properties, it is concluded that $\Delta E_y \propto \mathcal{E}_{22}$ for the piezoelectric constant. And $\Delta E_y \propto \frac{1}{E}$ for the mechanical effect modulus. Chosen the calibration material properties as the reference $E^{ref}$ and $\mathcal{E}_{22}^{ref}$, the relationship can be expressed as

$$\Delta E_y = \frac{E^{ref}}{\mathcal{E}_{22}^{ref}} \times \frac{\mathcal{E}_{22}}{E} \Delta E_y^{ref} = 10^9 \frac{\mathcal{E}_{22}}{E} \Delta E_y^{ref}$$  \hspace{1cm} (4.15)

where $\Delta E_y^{ref}$ returns to the correlation function obtained in Eq. (4.14).
4.3.2 Numerical Validation of the Piezoelectric Damage Detector: Stretchable Electronics

Stretchable electronics is a promising application combining mechanics and electric engineering to achieve flexibility for electronics. The current stretchable electronics studies focus either in mechanics area or electric engineering area. The lack of the coupling of mechanical field and electric field can be well studied using the proposed numerical tool for the multi-physics problems. Furthermore, the piezoelectric damage detector can help to capture the damage of the structure in real-time fashion.
Adopting the structure in [12], the numerical tool of piezoelectric damage detector can be used to study the coupling problem between damage, deformation and electric field. The sketch of the structure and the three-dimensional mesh is shown in Fig. 4.33. The dimension of the structure is set as $w = 0.4$, $\alpha = 15^\circ$, $R = 2$, $l = 2$.

The mechanical properties are assigned as $\lambda = 0$ and $\mu = 35GPa$ for the compressible neo-Hookean material. The piezoelectric constant is $\mathcal{E}_{21} = -0.001$, which couples the deformation in $x$-direction and the electric field in $y$-direction. The permittivity is $\varepsilon = 1.0$.

A displacement of $u_0 = \sin 2\pi t$ is applied on the ends of the structure as shown in Fig. 4.33a from $t = 0$ to $t = 0.25$ incrementally. The Green strain $E_{11}$ of the deformed configuration is shown in Fig. 4.34a. The elbow location of the serpentina undergoes maximum Green strain in $x$-direction. According to the spatial distribution
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of damage as shown in Fig. 4.34b, the maximum damage is developed at the same location despite the boundary condition applied on the ends. When a tension load is applied on the structure, both tension and compression deformation exist inside the structure. Since the damage can only be initiated and evolved from tension, the damage is suppressed when the local deformation is compression.
Figure 4.34: (a) Green strain in $x$-direction in stretchable electronics serpentine; (b) Damage developed in stretchable electronics serpentine.
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To detect the damage developed in the serpentes structure, employing the correlation function obtained from the calibration case. Since the elbow area in the serpentes is where most damage occurs a priori, the bottom surface with the coordinates of $-2.0 \leq x \leq -1.6, 0.18 \leq z \leq 0.22$ are analyzed. Fig. 4.35 is the result of the simulation and the correlation function. The correlation function captures the damaged evolution as the result agrees with the simulation.

Figure 4.35: $\Delta E_y - d$ plot from simulation and fitted correlation function of $\Delta E_y(d, \dot{d})$ in stretchable electronics structure

The error is plotted as
Figure 4.36: Relative error of the fitted correlation function from the simulation result in the stretchable electronics structure.

The error is well bounded in this validation case. It shows the robustness of the correlation function. With the correlation function, local damage can be sensed in real-time. The self-sensing damage detector is proved to accomplish the duty using the numerical tool developed.

4.4 Conclusion

Validations of EM-ME code are carried out with respect to analysis in existing literature. Each validation case is focus on a particular function that the EM-ME code delivering. With the validated case, EM-ME code is used to study the deformation effects on the electromagnetic field in deformable conductor. Furthermore, the
self-sensing piezoelectric damage detector is studied using the code with piezoelectric material and damage model implemented in the finite deformation setting. A correlation function is calibrated to predict the damage and its evolution quantitively in-situ. The chapter ends with extend the obtained correlation function to a stretchable electronic model and good agreement between correlation function prediction and simulation result is achieved.
Chapter 5

Conclusions

This paper develops a finite element model for multi-physics analysis, coupling transient electromagnetic and dynamic mechanical fields in the time-domain. The model framework is able to predict the evolution of electrical and magnetic fields and their fluxes in a vibrating media undergoing finite deformation. To account for finite deformation and its effects on the electromagnetic fields, a Lagrangian description is invoked to develop the finite element formulation. In this formulation, the coupling scheme maps Maxwell’s equations from spatial to material coordinates in the reference configuration. Weak forms of the coupled transient EM and dynamic ME equations are generated in the reference configuration. For efficient solution with reduced degrees of freedom, a scalar potential and vector potential are chosen as independent solution variables in lieu of electromagnetic field variables. The introduction of the potential function can result in non-uniqueness of the electromagnetic solution, which is overcome by introducing a Coulomb gauge condition. The resulting finite element ME-EM code with large degrees of freedom is parallelized using the ParMETIS library for domain decomposition and the MPI-based PETSc library for solving.
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developed platform is further enriched with piezoelectric material constitutive in the finite deformation. To accommodate piezoelectricity coupling, the piezoelectric material relation is implemented in the Lagrangian description as previous treatment. Moreover, continuum damage model is implemented. The damage model expand the horizon of the EM-ME code to study the material evolution effect on the EM field. It also provides a tool to use the EM field identifying the material evolution. Damage detector idea is risen with this added functions.

Two categories of numerical examples are solved using the ME-EM model and code. First the electromagnetic part of the code is validated by comparing with results from commercial software like COMSOL and ANSYS for electro-static, magneto-static, transient magnetic, and transient electromagnetic problems, without mechanical excitation. Convergence studies are made to examine the stability and accuracy of the model with satisfactory results. The piezoelectric material model is validated using the piezo bimorph beam. Analytical solution is used to compare with the reference configuration formulation in small deformation regime.

Subsequent to the validation tests, the ME-EM code is used to simulate two coupled ME-EM problems. The first example examines the effect of dynamic harmonic excitation of the substrate on a steady-state electromagnetic field. The effects of mechanical load frequency, amplitudes and direction on EM fields are investigated. In the second example, effects are explored for an additional transient electrical current on the boundary of the dynamically deforming conductor.. Characteristics of the
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coupled solutions clearly demonstrate the complexity brought about by coupling the two physical phenomena. As for the piezoelectric material model, an auxetic model is used to study the difference between the proposed formulation and the formulation implemented in ABAQUS with small deformation. Convergence study of the two-way direct coupling is carried out, the time step is constrained for accurate results. Self-sensing damage detector is proposed based on the numerical tool developed featuring piezoelectric material, continuum damage model under finite deformation. A correlation function is calibrated by study the sensitivity of $\Delta E_I$, the electric field difference between undamaged and damaged model with respect to damage parameter $d$ and material properties. The calibrated correlation function is validate using a stretchable electronics example to conclude the disseration.
Chapter A

Appendix A

A.1 Derivation of Second Piola-Kirchhoff stress for modified neo-Hookean material model

To obtain Eq. (2.8), using

\[ \frac{\partial J}{\partial C_{IJ}} = \frac{J}{2} C_{IJ}^{-1} \]  \hspace{1cm} \text{(A.1)}

\[ S_{IJ}^{vol} = 2 \frac{\partial W^{vol}(J)}{\partial C_{IJ}} = \frac{\partial W^{vol}(J)}{\partial J} \frac{\partial J}{\partial C_{IJ}} = (\lambda + \frac{2}{3} \mu) \ln J C_{IJ}^{-1} \]  \hspace{1cm} \text{(A.2)}

To obtain Eq. (2.9), using the relationship of

\[ \frac{\partial J^{-2/3}}{\partial C_{IJ}} = -\frac{1}{3} J^{-2/3} C_{IJ}^{-1} \]  \hspace{1cm} \text{(A.3)}

\[ \frac{\partial \tilde{C}_{IJ}}{\partial C_{KL}} = \frac{\partial J^{-2/3}}{\partial C_{IJ}} = J^{-2/3} \left( \mathbb{I}_{IJKL} - \frac{1}{3} C_{IJ} \otimes C_{KL}^{-1} \right) \]  \hspace{1cm} \text{(A.4)}
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\[
S_{IJ}^{dev} = 2 \frac{\partial W^{dev}}{\partial C_{IJ}} = 2 \frac{\partial W^{dev}}{\partial C_{II}} \frac{\partial C_{IJ}}{\partial C_{IJ}} = \mu J^{-2/3} \left( \delta_{IJ} - \frac{1}{3} C_{IJ}^{-1} \right) (A.5)
\]

A.2 Derivation of weak form for updated Lagrangian method

From Eq. (2.16), the first term on the left hand side in Eq. (2.15) is

\[
\int_{t+\Delta t}^{t+\Delta t} \sigma_{ij} \frac{\partial \delta u_i}{\partial t} d^{t+\Delta t} V = \int_{t}^{t+\Delta t} \sigma_{ij} \frac{\partial \delta u_i}{\partial t} d^{t+\Delta t} V = \int_{t}^{t+\Delta t} \sigma_{ij} \frac{\partial \delta u_i}{\partial t} d^{t+\Delta t} V
\]

The first Piola-Kirchhoff (PK) stress is defined as

\[
t+\Delta t P_{iJ} = t+\Delta t J t+\Delta t \sigma_{ij} t+\Delta t F_{j}^{-1} \quad s.t. \quad P = J \sigma F^{-T} \quad (A.7)
\]

Adopting the relationship between 1st PK stress and Cauchy stress Eq. (A.7) and Nanson’s formula Eq. (2.24), the first term on the right hand side in Eq. (2.15) associated with surface traction is

\[
\int_{t+\Delta t}^{t+\Delta t} \sigma_{ij} \delta u_i t+\Delta t n_j d^{t+\Delta t} S = \int_{t}^{t+\Delta t} \sigma_{ij} \delta u_i t+\Delta t J t+\Delta t F_{j}^{-1} t N_j d^{t+\Delta t} S = \int_{t}^{t+\Delta t} P_{iJ} t N_j d^{t+\Delta t} S
\]
APPENDIX A.

The rest terms in Eq. (2.15) can be transferred into the updated reference configuration using Eq. (2.16) as

\[
\int_{t+\Delta t}^{t+\Delta t} \rho \dddot{u}_i \delta u_i d^{t+\Delta t}V = \int_{t}^{t+\Delta t} \rho \dddot{u}_i \delta u_i d^{t}V
\]

(A.9a)

\[
\int_{t+\Delta t}^{t+\Delta t} \rho b_i \delta u_i d^{t+\Delta t}V = \int_{t}^{t+\Delta t} \rho b_i \delta u_i d^{t}V
\]

(A.9b)

The finite deformation formulation in the reference configuration is then obtained as Eq. (2.62).

A.3 Derivation of tangent moduli of Maxwell stress

The expression of \( tC_{IJKL}^{\text{Maxwell}} \) can be obtained from Eq. (3.28a) and Eq. (3.79c)

\[
tC_{IJKL}^{\text{Maxwell}} = 2 \left. \frac{\partial S_{IJ}^{\text{Maxwell}}}{\partial C_{KL}} \right|_{C_s} = -\frac{1}{2} E_P E_Q \left[ \frac{J}{2} C_{KL} \left( C_{PQ}^{-1} C_{IJ}^{-1} - C_{PI}^{-1} C_{IQ}^{-1} - C_{PJ}^{-1} C_{IQ}^{-1} \right) \right.

+ J C_{IJ}^{-1} \left( C^{-1} \odot C^{-1} \right)_{PQKL} + J C_{PQ}^{-1} \left( C^{-1} \odot C^{-1} \right)_{IJKL}

- J C_{Q}^{-1} \left( C^{-1} \odot C^{-1} \right)_{PJKL} - J C_{PI}^{-1} \left( C^{-1} \odot C^{-1} \right)_{JQKL}

- J C_{IQ}^{-1} \left( C^{-1} \odot C^{-1} \right)_{PJKL} - J C_{PQ}^{-1} \left( C^{-1} \odot C^{-1} \right)_{IQKL} \right]

(A.10)

where
APPENDIX A.

\[ [[\bullet] \circledast [\bullet]]_{IKL} = \frac{1}{2} [[\bullet]_{IK} [\bullet]_{LJ} + [\bullet]_{IL} [\bullet]_{JK}] \quad (A.11) \]

The expression of \( H_{SIJ}^{\text{Maxwell}} \) is derived from Eq. (3.28a) and Eq. (3.80) as

\[ H_{SIJ}^{\text{Maxwell}} = -\varepsilon J P \left[ C^{-1}_{P} C^{-1}_{I} - (C^{-1} \circledast C^{-1})_{PSIJ} \right] \quad (A.12) \]
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