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ON EXISTENCE IN THE LARGE OF SOLUTIONS OF HYPERBOLIC
PARTIAL DIFFERENTIAL EQUATIONS

by

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ABSTRACT

This note is concerned with existence in the large of solutions of 
\[ z_{xy} = f(x,y,z,p,q), \] 
where \( z(x,0) \) and \( z(0,y) \) are prescribed. Theorems proved 
are analogues of theorems in ordinary differential equations. For example, 
a condition on \( f \) sufficient to ensure existence in the large is 
\[ |f(x,y,z,p,q)| \leq \varphi(|z| + |p| + |q|), \] 
where \( \varphi(t) \) is a positive, non-decreasing, 
continuous function, defined for \( t \geq 0 \), satisfying \( \int_0^\infty dt/\varphi(t) = \infty \). This is 
an analogue of a theorem of Wintner in ordinary differential equations. 
An analogue of another theorem of Wintner on the asymptotic behavior of 
solutions is also proved.
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1. Introduction. Many authors have discussed the existence of solutions of the initial value problem

\[ z_{xy} = f(x,y,z,p,q), \ z(x,0) = \sigma(x), \ z(0,y) = \Upsilon(y), \]

where \( \sigma(0) = \Upsilon(0) = z_0 \), on a rectangle \( R_{ab} : 0 \leq x \leq a, \ 0 \leq y \leq b \). A \( C^1 \) solution of (1.1), which possesses a continuous second mixed derivative, will be said to be of class \( C^* \). If \( \sigma \) and \( \Upsilon \) are \( C^1 \) functions, then \( C^* \) solutions of (1.1) are equivalent to \( C^1 \) solutions of

\[ z(x,y) = \sigma(x) + \Upsilon(y) - z_0 + \int_0^x \int_0^y f(s,t,z(s,t),z_x(s,t),z_y(s,t))dsdt. \]

In [1], for example, the vector analogue of the following theorem is proved.

\([\ast]\) Let \( D_{ab} = \{(x,y,z,p,q) : (x,y) \in R_{ab}, \ z,p,q \text{ arbitrary}\} \). Let \( f(x,y,z,p,q) \) be continuous and bounded on \( D_{ab} \), and let \( f \) satisfy a uniform Lipschitz condition with respect to \( p \) and \( q \) on \( D_{ab} \). Let \( \sigma(x), \ \Upsilon(y) \) be of class \( C^1 \) on the respective intervals \( 0 \leq x \leq a, \ 0 \leq y \leq b \), and satisfy \( \sigma(0) = \Upsilon(0) \). Then there exists on \( D_{ab} \) a function \( z = z(x,y) \) of class \( C^* \) satisfying (1.1).

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For purposes below, the following consequence (cf. [1], p. 837), of the proof of (*) is most important: If \( f \) is not bounded, then it is possible to find a number \( d \), where \( 0 < d \leq \min(a, b) \), such that a solution \( z = z(x, y) \) exists on the square \( R_{dd} \). If the absolute values of \( \sigma'(x), \sigma'(y), \sigma'(x), \sigma'(y) \) are bounded by \( M \) and if \( C \) is a bound for \( |f| \) on the product space of \( R_{ab} \times \mathbb{E}(z, p, q): |z|, |p|, |q| \leq \frac{4M}{3} \), then \( d \) can be chosen to be \( \min(a, b, \frac{3M}{C}, \frac{1}{M\sqrt{C^{1/3}}}) \).

Hence in the above theorem, in order to prove the existence of a solution on the entire rectangle \( R_{ab} \), one assumes the boundedness of \( |f| \) on \( D_{ab} \). For unbounded \( f \), the proof provides the existence of a solution on a sub-rectangle of \( R_{ab} \). (In what follows, a subrectangle of \( R_{ab} \) shall always mean a rectangle in \( R_{ab} \), which has two sides along \( x = 0 \) and \( y = 0 \), respectively.)

One might consider that a local uniform Lipschitz condition on \( f \) provides the existence of a solution on a subrectangle of \( R_{ab} \), (i.e., a "local solution") and that the boundedness of \( |f| \) is a condition which assures that the solution can be continued to the entire rectangle \( R_{ab} \).

In papers, published more recently, the Lipschitz condition of (*) is generalized, being replaced by analogues of conditions used in ordinary differential equations (cf. [3], for references). The conclusions are generally similar to those of (*); if \( |f| \) is bounded on \( D_{ab} \), then there exists a solution on the entire rectangle \( R_{ab} \), but existence is proved only on a subrectangle in the case where \( f \) is unbounded. Exceptions are, for example, [2] and [4] where conditions are investigated which assure the existence of solutions on the whole of \( R_{ab} \) and which replace the assumption of the boundedness of \( |f| \) essentially by the condition that \( |f| \) has a majorant linear in \( |z|, |p|, |q| \). It seems of interest, therefore, to consider
in some detail conditions on \( f \) which will insure that "local solutions" can be continued to the entire rectangle.

In the corollary to Theorem 1, we make use of an analogue of a condition Wintner used in ordinary differential equations (cf. [5], Appendix). Theorem 2 is also an analogue of a theorem of Winter [6] on the asymptotic behavior of solutions of ordinary differential equations.

In what follows, we shall assume the existence of a "local solution", without specifying particular smoothness assumption on \( f \) to assure this. Further, we assume the existence of a number "\( d \)" as in the remark following (*), without requiring that \( d \geq \min (a,b,3M/C,M^2/c^2) \). The meaning of the term "local solution" is made precise below (cf. Hypothesis A).

2. Statement of results: Let \( R_{ab} = \mathcal{Z}(x,y): 0 \leq x \leq a, 0 \leq y \leq b/2 \) and \( R = \mathcal{Z}(x,y): x \geq 0, y \geq 0/2 \). Let \( D_{ab} \) be the product set \( R_{ab} \times \mathcal{Z}_{\text{entire (z,p,q) - space}^1} \) and \( D = R \times \mathcal{Z}_{\text{entire (z,p,q) - space}^1} \).

**Hypothesis A.** \( f(x,y,z,p,q) \) is a continuous function of five variables on the set \( D_{ab} \). For any point \( (\alpha, \beta) \), \( 0 \leq \alpha < a, 0 \leq \beta < b \), \( \sigma(x) \) and \( \mathcal{T}(y) \) are any two \( C^1 \) - functions defined respectively on \( \alpha \leq x \leq \gamma \), \( \beta \leq y \leq \delta \), where \( \gamma \leq \alpha \) and \( \delta \leq \beta \), and \( \sigma(\alpha) = \mathcal{T}(\beta) = z_0 \). \( M \) is a bound for the functions \( |\sigma(x)|, |\mathcal{T}(y)|, |\sigma'(x)|, |\mathcal{T}'(y)| \) and \( C \) is a bound for \( |f| \) on the product space \( R_{ab} \times \mathcal{Z}_{(z,p,q)}: |z|, |p|, |q| \leq M^2/2 \). Then there exists a positive number \( d \) with the property that \( d = d(M,C) \) depends only on \( M \) and \( C \), but not on \( \sigma(x) \) and \( \mathcal{T}(y) \), and that a \( C^1 \) - solution \( z = z(x,y) \) of

\[
(2.1) \quad z(x,y) = \sigma(x) + \mathcal{T}(y) - z_0 + \int_{\alpha}^{\gamma} \int_{\beta}^{\delta} f(s,t,z(s,t),z_x(s,t),z_y(s,t)) \, ds \, dt.
\]

exists on the square \( \alpha \leq x \leq \alpha + d_0, \beta \leq y \leq \beta + d_0 \), where \( d_0 = \min (d, \gamma - \alpha, \delta - \beta) \).

**Hypothesis B.** For \( (x,y,z,p,q) \in D_{ab} \),
where $\varphi(x, y, z, p, q)$ is a continuous, non-negative function defined for $(x, y) \in R_{ab}$ and $z, p, q \geq 0$, non-decreasing in each of the variables $z, p$ and $q$. Furthermore, $\varphi$ has the property that if $\sigma(x)$ and $\tau(y)$ are any two $C^1$-functions, defined respectively on $0 \leq x \leq a, 0 \leq y \leq b$, non-decreasing in $x$ and $y$, and satisfying $\sigma(0) = \tau(0) = z_0$, then all $C^1$-solutions $z = z(x, y)$ of

$$ z(x, y) = \sigma(x) + \tau(y) - z_0 + \int_0^x \int_0^y \varphi(s, t, z(s, t), z_x(s, t), z_y(s, t)) \, ds \, dt, $$

which exist on subrectangles of $R_{ab}$, can be continued in at least one way to the entire rectangle $R_{ab}$.

**Theorem 1.** Let $f$ satisfy Hypotheses A and B. Let $\sigma(x)$ and $\tau(y)$ be two $C^1$-functions defined respectively on $0 \leq x \leq a, 0 \leq y \leq b$, and let $\sigma(0) = \tau(0) = z_0$. Then all $C^1$-solutions of (1.1), defined on subrectangles of $R_{ab}$, can be continued in at least one way to the entire rectangle $R_{ab}$.

**Hypothesis B'.** For $(x, y, z, p, q) \in D_{ab}$,

$$ |f(x, y, z, p, q)| \leq \varphi(|z| + |p| + |q|), $$

where $\varphi(t)$ is a positive, non-decreasing, continuous function defined for $t \geq 0$ satisfying

$$ \int_0^\infty dt / \varphi(t) = \infty. $$

We then have the following corollary to Theorem 1.

**Corollary.** If, in Theorem 1, Hypothesis B is replaced by Hypothesis B', then the assertion remains valid.

The hypothesis and theorem to follow concern the case where $R_{ab}$ is replaced by $R$.

**Hypothesis A'.** The same as Hypothesis A except that $f$ is defined on $D$ and $(a, b)$ is any point in $R$.

**Hypothesis B''.** For $(x, y, z, p, q) \in D$,
\[(2.6) \quad |f(x,y,z,p,q)| \leq \lambda(x+y)\varphi(|z| + |p| + |q|),\]

where \(\lambda(t)\) is a non-negative, continuous function defined for \(t \geq 0\), satisfying

\[(2.7) \quad \int_{0}^{\infty} t \lambda(t)dt < \infty,\]

and \(\varphi(t)\) is as in Hypothesis \(B\).

Theorem 2. Let \(f\) satisfy Hypotheses \(A\) and \(B\). Let \(\varpi(x)\) and \(\Upsilon(y)\) be two \(C^1\) functions defined for \(x \geq 0\) and \(y \geq 0\), respectively, satisfying

\(\varpi(0) = \Upsilon(0) = z_0\). Then any \(C^*\) solution \(z = z(x,y)\) of (1.1), defined on a subrectangle of \(R_{ab}\), can be continued, in at least one way to the quarter plane \(R: x \geq 0, y \geq 0\). Moreover if \(|\varpi(x)|, |\Upsilon(y)|, |\varpi'(x)|\) and \(|\Upsilon'(y)|\)

are bounded for \(x \geq 0, y \geq 0\), then there exist three \(C^1\) functions \(\varepsilon(x,y), \varepsilon_1(x)\) and \(\varepsilon_2(y)\) defined for \(x \geq 0, y \geq 0\), and a constant \(K\) such that

\[(2.8) \quad z(x,y) = \varpi(x) + \Upsilon(y) - z_0 + K + \varepsilon_1(x) + \varepsilon_2(y) + \varepsilon(x,y),\]

and also

\[(2.9) \quad z_x(x,y) = \varpi'(x) + \varepsilon_{1x}(x) + \varepsilon_x(x,y),\]

\[(2.10) \quad z_y(x,y) = \Upsilon'(y) + \varepsilon_{2y}(y) + \varepsilon_y(x,y),\]

where \(\varepsilon_1(x), \varepsilon_{1x}(x) \rightarrow 0\) as \(x \rightarrow \infty\); \(\varepsilon_2(y), \varepsilon_{2y}(y) \rightarrow 0\) as \(y \rightarrow \infty\) and \(\varepsilon(x,y), \varepsilon_x(x,y), \varepsilon_y(x,y) \rightarrow 0\) as \(x\) or \(y \rightarrow \infty\).

Remark 1. Analogues of Theorem 1 and its Corollary can be proved for the following Cauchy problem: Let \(\varpi(x)\) and \(\Upsilon(y)\) be functions of class \(C^1\) defined on \(0 \leq x \leq a\) and \(0 \leq y \leq b\) respectively. Let \(\Gamma: x = x(u), y = y(u)\), where \(0 \leq u \leq 1\), be an arc of class \(C^1\) joining \((0,b)\) and \((a,0)\). Let \(x'(u) > 0, y'(u) < 0\). Such a curve has no tangent parallel to either axis and is therefore non-characteristic. Consider solutions of the problem

\[(2.11) \quad z_{xy} = f(x,y,z,p,q), \quad z(x(u),y(u)) = \varpi(x(u)) + \Upsilon(y(u)).\]

For this problem an existence theorem, corresponding to (*) , is proved in [1] (cf. p. 840). As in (*), \(f\) is assumed bounded on \(D_{ab}\). Consider the
case where \( f \) is unbounded. Let \( M \) be a bound for the functions \( |\sigma(x)|, 
|\mathcal{U}(y)|, |\sigma'(x)|, |\mathcal{U}'(y)| \), and let \( C \) be a bound for \(|f|\) on the set \( R_{ab} \times \mathbb{S}(z,p,q) : |z|, |p|, |q| \leq M^2 \). Then the theorem provides the existence of a \( C^* \) - solution of (2.11), on the common part of the two \((x,y)\) - sets:

\[
\begin{align*}
(2.12) & \quad \max (0, x(u) - d) \leq x \leq \min (a, x(u) + d), 
 y = y(u), \ 0 \leq u \leq 1, \\
(2.13) & \quad x = x(u), \ \max (0, y(u) - d) \leq y \leq \min (b, y(u) + d), \ 0 \leq u \leq 1,
\end{align*}
\]

where \( d = \min \left( \frac{M^2}{C^2}, 2M/C \right) \). Let \( L \) be the boundary of the common part of

(2.12) and (2.13). Let \( d_0 \) be the minimum distance between \( L \) and the curve \( \Gamma \). Then it is seen that the solution exists on

\[
(2.14) \quad \max (0, x(u) - d_0) \leq x \leq \min (a, x(u) + d_0), \ y = y(u), \ 0 \leq u \leq 1.
\]

The part of the boundary, that is disjoint from the boundary of \( R_{ab} \), consists of two non-characteristic arcs of class \( C^1 \). Sets like (2.14) will be called neighborhoods of \( \Gamma \).

The following hypotheses are analogous to Hypotheses A and B.

**Hypothesis A**. \( f(x,y,z,p,q) \) is a continuous function of five variables, on the set \( D_{ab} \). \( \Gamma : x = x(u), \ y = y(u), \ 0 \leq u \leq 1 \), is a non-characteristic arc of class \( C^1 \), joining \((\alpha,\beta)\) and \((\gamma,\delta)\), where either \( \alpha \) and \( \delta \) are zero, or \( \beta = \beta \) and \( \gamma = \gamma \). \( \sigma(x) \) and \( \mathcal{U}(y) \) are two \( C^1 \) - functions defined for \( \alpha \leq x \leq \gamma, \ \beta \leq y \leq \delta \) respectively. \( M \) is a bound for the function \( |\sigma(x)|, 
|\mathcal{U}(y)|, |\sigma'(x)| \) and \(|\mathcal{U}'(y)|\) and \( C \) is a bound for \(|f|\) on the product space \( R_{ab} \times \mathbb{S}(z,p,q) : |z|, |p|, |q| \leq M^2 \). There exists a positive number \( d \), with the property that \( d = d(M, C) \) depends only on \( M \) and \( C \), but not on \( \sigma(x) \) and \( \mathcal{U}(y) \), and that a \( C^* \) - solution of (2.11) exists on the set

\[
\max (0, x(u) - d) \leq x \leq \min (a, x(u) + d), \ y = y(u), \ 0 \leq u \leq 1.
\]

**Hypothesis B**. For \((x, y, z, p, q) \in D_{ab},

\[
(2.15) \quad |f(x,y,z,p,q)| \leq \varphi(x,y,|z|,|p|,|q|),
\]
where \( \varphi(x,y,z,p,q) \) is a continuous, non-negative function defined for \((x,y) \in R_{ab}\) and \(z,p,q \geq 0\), non-decreasing in each of the variables \(z, p\) and \(q\). \( \Gamma' \):
\[ x = x(u), \quad y = y(u), \quad 0 \leq u \leq 1, \]
is a non-characteristic arc of class \(C^1\), joining \((0,b)\) and \((a,0)\). \( \sigma(x) \) and \( \tau(y) \) are two \(C^1\) functions defined, respectively, on \(0 \leq x \leq a, 0 \leq y \leq b\), and non-decreasing in \(x\) and \(y\). Furthermore \(\varphi\) has the property that all \(C^*\) solutions of (2.11), existing on a neighborhood of \( \Gamma' \), can be continued to the entire rectangle \(R_{ab}\).

The following theorem and its corollary are analogues of Theorem 1 and its Corollary.

**Theorem 3.** Let \( f \) satisfy Hypotheses A" and B"." Let \( \sigma(x) \) and \( \tau(y) \) be two \(C^1\) functions defined on \(0 \leq x \leq a, 0 \leq y \leq b\) respectively. Let \( \Gamma' \):
\[ x = x(u), \quad y = y(u), \quad 0 \leq u \leq 1, \]
be a non-characteristic arc of class \(C^1\), joining \((0,b)\) and \((a,0)\). Then all \(C^*\) solutions of (2.11) can be continued in at least one way to the entire rectangle \(R_{ab}\).

**Corollary.** If, in Theorem 3, Hypothesis B"" is replaced by Hypothesis B', the assertion remains valid.

The proofs are similar to those of Theorem 1 and its Corollary and are omitted.

**Remark 2.** It will be clear, from the proofs, that the above theorems remain valid if \(f, \varphi, \sigma, \tau, u\) are \(n\)-vectors (say with norm \(|z| = \sum_{k=1}^{n} |z^k| \) or \(|z| = \max(|z^1|, \ldots, |z^n|)\) if \(z = (z^1, \ldots, z^n)\)). Of course \(\varphi\) will still be a function of five variables in Theorems 1 and 3, and of one variable in the corollaries. \(f\) will be a function of \((3n + 2)\) variables. In Theorem 2, \(\epsilon_1(x)\), \(\epsilon_2(y)\) and \(\epsilon(x, y)\) will be \(n\)-vectors.

These results answer some questions suggested by Professor P. Hartman.

I also wish to acknowledge helpful discussions with him.
3. Proof of Theorem 1. (i) First, let \( \sigma(x) \) and \( \tau(y) \) be two non-negative, non-decreasing \( C^1 \) - functions defined respectively on \( 0 \leq x \leq a \), and \( 0 \leq y \leq b \), with \( \sigma(0) = \tau(0) = z_0 \). Then there exists a solution of (2.3) on \( R_{ab} \). To see this, let \( M \) be a bound for the functions \( \sigma(x), \tau(y), \sigma'(x) \) and \( \tau'(y) \) and let \( C \) be a bound for \( \varphi \) on the product space \( R_{ab} \times \mathfrak{C}(z,p,q) : |z|, |p|, |q| \leq 4M \). Define \( \varphi^*(x,y,z,p,q) \) to be \( \varphi(x,y,z,p,q) \) or \( C \) according as \( \varphi \) does not or does exceed \( C \). Then a solution \( z = z(x,y) \) of

\[
z(x,y) = \sigma(x) + \tau(y) - z_0 + \int_0^x \int_0^y \varphi^*(s,t,z(s,t),z_x(s,t),z_y(s,t)) ds dt
\]

exists on \( R_{ab} \), and is the limit of the successive approximations defined by \( z_0(x,y) = \sigma(x) + \tau(y) - z_0 \),

\[
z_n(x,y) = z_0(x,y) + \int_0^x \int_0^y \varphi^*(s,t,z_{n-1}(s,t),z_n(s,t),z_{n-1}(s,t),z_n(s,t)) ds dt
\]

(cf. [3], p. 4). For \( (x,y) \in R_{a\alpha} \), where \( \alpha = \min(a/b, M/C, M^2/C^2) \), it is easily seen that the non-negative functions \( z(x,y), z_x(x,y), z_y(x,y) \), do not exceed \( 4M \). Therefore if \( (x,y) \in R_{a\alpha} \), we may replace \( \varphi^* \) in (3.1) by \( \varphi \).

So a solution of (3.1) is a solution of (2.3) on \( R_{a\alpha} \). Under Hypothesis B, this solution can be continued, as a solution of (2.3), to the entire rectangle \( R_{ab} \).

(ii) Let \( \sigma(x) \) and \( \tau(y) \) be as in the statement of the theorem and let \( z = z(x,y) \) be a solution of (1.1), defined on a subrectangle \( R_{a\beta} \) of \( R_{ab} \).

Let \( M \) be a bound for \( |\sigma'(x)|, |\tau'(y)|, |\sigma''(x)| \) and \( |\tau''(y)| \) on \( 0 \leq x \leq a \) and \( 0 \leq y \leq b \). Let \( z = u(x,y) \) be a \( C^1 \) - solution of (2.3) on \( R_{ab} \), with

\[
\sigma(x) = 3Me^x \quad \text{and} \quad \tau(y) = 3Me^y.
\]

Then for all \( (x,y) \in R_{a\beta} \),

\[
u(x,y) > |z(x,y)|, u_x(x,y) > |z_x(x,y)|, u_y(x,y) > |z_y(x,y)|.
\]

Obviously these inequalities hold for \( x = 0 \) and \( y = 0 \). Suppose they do not hold for all \( (x,y) \in R_{a\beta} \). Let \( (x_0,y_0) \), \( x_0 \neq 0 \), \( y_0 \neq 0 \) be the nearest point to \( (0,0) \) for which at least one inequality in (3.2) becomes an equality.
First suppose \( u(x_0, y_0) = |z(x_0, y_0)| \). Then (3.2) holds, for all \( (x, y) \in \mathbb{R} \setminus \{0\} \) with the exception \( (x, y) = (x_0, y_0) \). Then

\[
u(x_0, y_0) = |z(x_0, y_0)| \leq |\nabla(x_0) + \nabla_y(y_0) - z_0| + \int_0^{x_0} \int_0^{y_0} |f(s, t, z(s, t), z_x(s, t), z_y(s, t))| \, ds \, dt \]

\[
\leq 3M + \int_0^{x_0} \int_0^{y_0} \phi(s, t, u(s, t), u_x(s, t), u_y(s, t)) \, ds \, dt.
\]

This last inequality follows from (2.2), (3.2) and the monotony of \( \phi \). Finally

\[
u(x_0, y_0) < 3Me^x + 3Me^y - 3M + \int_0^{x_0} \int_0^{y_0} \phi(s, t, u(s, t), u_x(s, t), u_y(s, t)) \, ds \, dt.
\]

Strict inequality holds in this case because \( x_0, y_0 \neq 0 \) and hence \( e^{x_0}, e^{y_0} > 1 \).

But this inequality contradicts the fact that \( u(x, y) \) is a solution of (2.3) on \( \mathbb{R}^{ab} \) with \( \nabla(x) = 3Me^x \) and \( \nabla_y(y) = 3Me^y \). Hence \( u(x_0, y_0) \neq |z(x_0, y_0)| \). An analogous argument establishes the fact that \( u_x(x_0, y_0) \neq |z_x(x_0, y_0)| \) and \( u_y(x_0, y_0) \neq |z_y(x_0, y_0)| \). Hence (3.2) holds for all \( (x, y) \in \mathbb{R}^{ab} \).

This means that \( z(x, y) \), its derivatives, any of its continuations and their derivatives, are majorized respectively by \( u(x, y) \), \( u_x(x, y) \) and \( u_y(x, y) \).

Let \( N \) be a bound for \( u(x, y) \), \( u_x(x, y) \), \( u_y(x, y) \) and let \( M_0 = \max(M, N) \). Let \( C_0 \) be a bound for \( |z| \) on the product space \( \mathbb{R}^{ab} \times \mathcal{F}(z, p, q) \): \( |z|, |p|, |q| \leq 4M_0 \).

We may consider \( \nabla(x) \) and the value of \( z \) on \( x = a \) as the initial data for a solution of \( z_{xy} = f(x, y, z, p, q) \), on the rectangle \( a \leq x \leq a + d_0, 0 \leq y \leq \beta \). Under Hypothesis A, and because of the choice of \( M_0 \), a solution exists on a square of side \( d_0 = \min(d(M_0, C_0), a - \alpha, \beta) \). This solution together with the given one serves to define \( z(x, y) \) on the square \( a \leq x \leq a + d_0, 0 \leq y \leq d_0 \). If \( a + d_0 \neq a \), then the value of \( z \) on the side \( x = a + d_0 \) together with \( \nabla(x) \) are initial data for a further solution of \( z_{xy} = f(x, y, z, p, q) \). As has been shown above, this initial data and its derivatives are majorized by \( M_0 \).

Hence under Hypothesis A, we get a solution on another square with specified side. We can repeat the argument, until \( \mathbb{R}^{ab} \) has been covered by a finite
number of squares on each of which \( z(x,y) \) is defined to be a solution of 
\[ z_{xy} = f(x,y,z,p,q). \]
Obviously the function \( z(x,y) \), so defined on \( R_{ab} \), is a solution of (1.1), and this is a continuation of the solution as stated in the theorem.

4. Proof of Corollary. We wish to show that Hypothesis B is a particular case of Hypothesis B. To do this, it only remains to show that if \( z(x,y) \) is a \( C^1 \)-solution of
\[ (4.1) \quad z(x,y) = \sigma(x) + \mathcal{U}(y) - z_0 + \int_0^x \int_0^y \varphi(z(s,t) + z_x(s,t) + z_y(s,t)) \, ds \, dt, \]
existing on a subrectangle of \( R_{ab} \), it can be continued, in at least one way, to the whole of \( R_{ab} \). In (4.1) it is assumed that \( \sigma(x) \) and \( \mathcal{U}(y) \) are non-negative, non-decreasing \( C^1 \)-functions as in Hypothesis B. The method of proof is similar to that used in Theorem 1. First, we obtain a "local existence" statement for (4.1), and then the existence of a priori bounds for the solution and its derivatives.

Let \( \sigma(x) \), \( \mathcal{U}(y) \), \( M \) and \( C \) be as in part (i) of the proof of Theorem 1. It was pointed out there, that there exists a solution of (4.1) on \( R_{dd} \), where \( d = \min \left( a, b, \frac{3M}{C}, \frac{3M}{C^2} \right) \).

We now show that there exists a constant \( K = K(M, a, b) \), depending only on \( M, a \) and \( b \), and not on \( \sigma(x) \) and \( \mathcal{U}(y) \), such that if \( z(x,y) \) is a \( C^1 \)-solution of (4.1) defined on \( R_{ab} \), or on a subrectangle of it, then the non-negative functions \( z(x,y), z_x(x,y) \) and \( z_y(x,y) \) are bounded by \( K \). To see this, let \( r(x,y) = z(x,y) + \int_0^x \int_0^y \varphi(r(s,t)) \, ds \, dt \).

By differentiating (4.1), with respect to \( x \) and \( y \) respectively, we get
\[ (4.3) \quad z_x(x,y) \leq M + \int_0^y \varphi(r(x,t)) \, dt, \]
\[ (4.4) \quad z_y(x,y) \leq M + \int_0^x \varphi(r(s,y)) \, ds. \]
The addition of (4.2), (4.3) and (4.4) results in

\[(4.5) \quad r(x, y) \leq 5M + \int_0^x \int_0^y \varphi(r(s, t))dsdt + \int_0^x \varphi(r(s, y))ds + \int_0^y \varphi(r(x, t))dt.\]

Let \( R(t) = \max \{r(u, v) \mid (u, v) \in R_{ab}\} \), where the maximum is taken over all \((u, v)\) such that \((u, v) \in R_{ab}\), \(z(u, v)\) exists and \(u + v \leq t\). \(R(t)\) is a continuous, non-negative, non-decreasing function defined for \(0 \leq t \leq a + b\). Also \(r(x, y) \leq R(x + y)\).

From (4.5) and the monotony of \(\varphi\) it is seen that

\[(4.6) \quad r(x, y) \leq 5M + \int_0^x \int_0^y \varphi(R(s + t))dsdt + \int_0^x \varphi(R(s + y))ds + \int_0^y \varphi(R(x + t))dt.\]

The double integral over \(R_{xy}\), is not greater than the double integral over the triangle bounded by \(x = 0\), \(y = 0\) and the line through \((x, y)\) with slope \(-1\).

Hence on making the change of variables \(s + t = u\), \(s - t = v\), it is seen that the double integral on the right side of (4.6) is not greater than \(\int_0^{x+y} u \varphi(R(u))du\). An obvious change of variable, in the single integrals, reduces them to \(\int_0^{x+y} \varphi(R(u))du\) and \(\int_0^{x+y} \varphi(R(u))du\). Replacing the lower limits, in the single integrals, by zero does not decrease their values, and so (4.6) implies

\[(4.7) \quad r(x, y) \leq 5M + \int_0^{x+y} (u + 2) \varphi(R(u))du.\]

Hence

\[(4.8) \quad R(t) \leq 5M + \int_0^t (u + 2) \varphi(R(u))du.\]

Let

\[H(t) = 5M + \int_0^t (u + 2) \varphi(R(u))du, \text{ so that } R(t) \leq H(t).\]

Then \(H'(t) = (t + 2) \varphi(R(t))\). Since \(\varphi\) is non-decreasing \(H'(t) \leq (t + 2) \varphi(H(t))\), and so

\[(4.9) \quad \int_0^{a+b} H'(t)dt/\varphi(H(t)) \leq \int_0^{a+b} (t + 2)dt = \frac{3}{2}(a + b)^2 + 2(a + b).\]

Letting \(H(t) = u\), (4.9) becomes

\[(4.10) \quad \int_{5M}^{H(a + b)} du/\varphi(u) \leq \frac{3}{2}(a + b)^2 + 2(a + b).\]

In view of (2.5), it is clear that there is a constant \(K = K(M, a, b)\) such that
\( H(a + b) \leq K. \)

The continuation can now be proved in a manner exactly analogous to the proof of Theorem 1. One can cover \( R_{ab} \), successively, by squares of side \( d \), on each of which a solution exists. The details are omitted.

5. **Proof of Theorem 2.** Suppose that the \( C^1 \) - solution \( z = z(x,y) \) of (1.1) exists on \( R_{ab} \), and consider \( R_{ab} \), where \( R_{ab} \supset R_{ab}^\prime \). We may replace \( \lambda(x+y) \) in (2.6) by the constant which is the max \( \lambda(x+y) \) for all \( (x,y) \in R_{ab} \). Absorbing this constant in the function \( \varphi \), we get condition (2.4). Hence the proof of a continuation of the solution to \( R_{ab} \) is contained in the Corollary to Theorem 1. Since \( R_{ab} \) is any rectangle, the continuation assertion of Theorem 2 is proved.

Let \( r(x,y) = |z(x,y)| + |z_x(x,y)| + |z_y(x,y)| \). Let \( |\sigma(x)|, |\mathcal{T}(y)|, |\mathcal{U}'(x)| \) and \( |\mathcal{U}'(y)| \) not exceed \( M \), for \( x \geq 0 \) and \( y \geq 0 \), where \( M \) is a constant.

From (2.6), it is seen that

\[
|z(x,y)| \leq 3M + \int_0^x \int_0^y \lambda(s + t) \varphi(r(s,t)) ds dt.
\]

Differentiating (1.2), with respect to \( x \) and \( y \), respectively, we get

\[
|z_x(x,y)| \leq M + \int_0^y \lambda(x + t) \varphi(r(x,t)) dt,
\]

\[
|z_y(x,y)| \leq M + \int_0^x \lambda(s + y) \varphi(r(s,y)) ds.
\]

Adding (5.1), (5.2) and (5.3),

\[
|z(x,y)| \leq 5M + \int_0^x \int_0^y \lambda(s + t) \varphi(r(s,t)) ds dt + \int_0^x \lambda(s + y) \varphi(r(s,y)) ds + \int_0^y \lambda(x + t) \varphi(r(x,t)) dt.
\]

Let \( R(t) = \max r(u,v) \), where the maximum is taken over all \( (u,v) \in R \) satisfying \( u + v \leq t \). \( R(t) \) is a continuous, non-decreasing function defined for \( t \geq 0 \).

Also \( r(x,y) \leq R(x+y) \). Proceeding exactly as in the argument following (4.6) above, one easily derives the inequality

\[
R(t) \leq 5M + \int_0^t (u + 2) \lambda(u) \varphi(R(u)) du.
\]
Letting $H(t)$ be the expression on the right of this inequality, and proceeding as above one easily sees that

$$\int_{\mathbb{S}^2}^{H(t)} \frac{du}{\varphi(u)} \leq \int_0^t (u + 2) \lambda(u) du. \quad (5.6)$$

Because of (2.7), the right side of (5.6) is bounded as $t \to \infty$, and therefore

$$\lim_{t \to \infty} \int_{\mathbb{S}^2}^{H(t)} \frac{du}{\varphi(u)} < \infty. \quad (5.7)$$

Hence, noting (2.5), we conclude that $\lim_{t \to \infty} H(t) < \infty$ and therefore $\lim_{t \to \infty} R(t) < \infty$.

Hence $r(x,y)$ is uniformly bounded in the quarter plane $\mathbb{R}$ by a constant $N$, say.

This, together with (2.6), implies

$$\int_0^x \int_0^y |f(s,t,z(s,t),z_x(s,t),z_y(s,t))| ds dt \leq N \int_0^x \int_0^y \lambda(s + t) ds dt. \quad (5.8)$$

The double integral on the right of (5.8) is not greater than the corresponding integral taken over the triangle bounded by $x = 0$, $y = 0$ and the line through $(x,y)$ of slope $-1$. On making the change of variables $s + t = u$, $s - t = v$, we see that the integral on the right of (5.8) is not greater than $\int_0^{x+y} \lambda(u) du$, and so (2.7) implies that this integral converges absolutely as $x$ or $y \to \infty$. Hence $f(x,y,z(x,y),z_x(x,y),z_y(x,y))$ is absolutely integrable over the quarter-plane $\mathbb{R}$. Write

$$\int_0^x \int_0^y f(s,t,z(s,t),z_x(s,t),z_y(s,t)) ds dt =$$

$$\left\{ \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty f(s,t,z(s,t),z_x(s,t),z_y(s,t)) ds dt \right\}$$

Let the integrals on the right side of (5.9) be $K$, $\varepsilon_1(x)$, $\varepsilon_2(y)$ and $\varepsilon(x,y)$ respectively. Then (5.9) and (2.2) imply (2.8). The limit assertions of Theorem 2 for $\varepsilon_1(x)$, $\varepsilon_2(y)$ and $\varepsilon(x,y)$, are obvious since the variables tending to infinity occur as limits in the integrals. Differentiation of (2.8), with respect to $x$ and $y$ respectively, gives (2.9) and (2.10). In these cases the variable which tends to infinity may occur in the integrand. Then we proceed
as follows: Consider the case of

\[ \varepsilon_x(x,y) = - \int_y^\infty f(x,t,z(x,t),z_x(x,t),z_y(x,t))dt. \]

Because of the uniform boundedness of \( r(x,y) \) in \( R \) and (2.6), it follows that

\[ \int_y^\infty \left| f(x,t,z(x,t),z_x(x,t),z_y(x,t))dt \right| \leq \varphi(N) \int_y^\infty \lambda(x+t)dt. \]

By a simple change of variable, this last integral becomes \( \int_x^\infty \lambda(t)dt \)

which \( \to 0 \) as \( x \) or \( y \to \infty \). Hence \( \varepsilon_x(x,y) \to 0 \) as \( x \) or \( y \to \infty \). Similar considerations apply to the other limits. This concludes the proof of Theorem 2.

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References


