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EQUIVARIANT EMBEDDINGS IN EUCLIDEAN SPACE

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Equivariant Embeddings in Euclidean Space

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Section 1. Introduction.

Let G be a group of transformations on a topological space E. If $p \varepsilon E$ we denote by G_p the set of transformations in G which keep p fixed. If H is a subgroup of G, we denote by (H) the totality of subgroups of G of the form xHx^{-1} with x in G. We denote by L(G,E) the totality of (G_p) as p varies over E. The orbits Gp and Gq through points p and q of E are called equivalent if $(G_p) = (G_q)$. Thus G has but a finite number of inequivalent orbits in E if and only if L(G,E) is a finite set. This is the case for example if E is a compact differentiable manifold and G is a compact group of differentiable transformations (cf. section 7).

The main results are the following.

THEOREM 6.1. Let G be a compact Lie group operating faithfully on a separable metric n-dimensional space E. Assume G has only a finite number of inequivalent orbits in E. Then there exists a homeomorphism φ of E into a Euclidean space E^m and an isomorphism Θ of G into the unitary group on E^m such that φ is equivariant with respect to Θ, i.e., $\varphi(gp) = \Theta(g)\,\varphi(p)$ for all $p \varepsilon E$, $g \varepsilon G$. Furthermore, if G has no fixed points on E, then Θ, φ may be so chosen that $\Theta(G)$ has no fixed points on E^n except the origin.

THEOREM 2.1. Let G be a compact Lie group of transformations on a completely regular space E. Then at each point p of E there exists a pseudo-section to the orbit through p. (See section 3 for definitions).

THEOREM 4.2. Let G be a compact Lie group of transformations a separable metric finite dimensional space E. Assume all the ~~orbits~~ are equivalent. Then there exists a finite set of local cross-sections whose orbits cover E.

Theorem 2.1 on pseudo-sections is a more general version of a theorem first proved by Montgomery and Yang for spaces satisfying suitable connectivity conditions. The proof of Montgomery and Yang is strictly topological; in contrast, our proof hinges essentially on producing a suitable representation of the transformation group.

From the point of view of transformation groups, one can obtain quite directly some information about the conjugacy of subgroups of a compact Lie group. Thus we can obtain the result:

THEOREM 7.1. In a compact Lie group, any set of (connected) analytic subgroups whose normalizers are mutually non-conjugate (under an inner automorphism) is finite. Any set of semi-simple analytic subgroups which are mutually non-conjugate is finite.

This result is useful in finding conditions under which L(G,E) is finite. This question will be taken up in a future paper.

It is of interest to note that Theorem 2.1 yields as a consequence the result of Montgomery and Zippin that nearly closed subgroups of a compact Lie group are conjugate (see Corollary 3.2 in section 3).

Section 2. Faithful representations of orbits.

LEMMA 2.1. Let H be a closed subgroup of the compact Lie group G. Then there is representation α of G by unitary transformations on the finite dimensional complex Euclidean space E^n and a point $p \varepsilon E^n$ such that $\alpha^{-1}(\alpha(G)_p) = H$. If $H \neq G$, α can be so chosen that $\alpha(G)$ keeps only the origin fixed.

Proof. If G = H, the lemma is obviously true. We assume therefore $G \neq H$. For any compact Lie group F containing the closed subgroup H, there exists an irreducible representation β_F whose restriction to H contains the trivial unit representation (cf. Chevalley "Theory of Lie Groups", vol. 1, Prop. 5, p. 192, p. 211). Taking F to be a closed subgroup of G properly containing H, the representation β_F is contained in the restriction to F of some representation of G (loc. cit. Prop. 4, p. 191). We denote this representation of G by α_F. Let V_F, E_F denote the representation spaces of β_F, α_F respectively. Select any point q other than the origin in each V_F which is fixed under $\beta_F(H)$ and set $H_F = \alpha_F^{-1}(\alpha_F(G)_q)$. Set $K = \bigcap_F H_F$ (all $F \supset H$ properly). Then K is a compact subgroup of G containing H. If K contains H properly, then β_K is not the unit representation and thus $\beta_K(K)_q \neq \beta_K(K)$. Now $H_K \cap K = \alpha_K^{-1}(\alpha_K(G)_q) \cap K$ is the totality of elements x of K with $\alpha_K(x)q = q$ and, since $q \varepsilon V_K$, coincides with $\beta_K^{-1}(\beta_K(K)_q)$. Thus $K \neq H$ implies $H_K \cap K \neq \beta_K^{-1}(\beta_K(K)) = K$; that is, K is not contained in H_K -- a contradiction. Thus $H = K = \bigcap_F H_F$ (all $F \supset H$).

It is next to be observed that any (well-ordered) descending chain of compact subgroups of a compact Lie group is finite; for in a descending chain, only a finite number of subgroups of the same dimension can occur, and only a finite number of dimensions can occur. On the other hand, we can clearly well order a subset of the subgroups F containing H -- say $F_1, F_2, \cdots, F_\alpha \cdots$ (α an ordinal less than γ) so as to obtain a strictly descending chain $H_1, H_2, \cdots H_\alpha, \cdots$ (all $\alpha < \gamma$) with the property $H = \bigcap_\alpha H\alpha$ (all $\alpha < \gamma$.) Hence

γ is a finite ordinal $n + 1$ and $H = H_{F_1} \cap \cdots \cap H_{F_n}$. Set $\alpha = \alpha_{F_1} + \cdots + \alpha_{F_n}$, $E^N = E_{F_1} + \cdots E_{F_n}$ (direct sum), and $p = (q_1, \cdots, q_n) \in E$ where q_i is a non-zero element of V_{F_i} $(i = 1, \cdots, n)$. Then $\alpha^{-1}(\alpha(G)_p) = \bigcap_i \alpha^{-1}{}_F(\alpha_F(G)q_i) = H_{F_1} \cap \cdots \cap H_{F_n} = H$, as asserted in the lemma.

If $H \neq G$, we could have selected the α_F in the construction above so as to not contain the trivial unit representation of G. For G being compact α_F is a direct sum of irreducible representations; upon omitting from the sum the trivial representations, we obtain a representation whose restriction to F contains β_F but which does not contain the trivial representation. Selecting for each F such an α_F, we obtain an α which does not contain the trivial representation of G. Hence the only fixed point of $\alpha(G)$ in E^H is the origin.

DEFINITION. Let G be a compact group operating on a topological space E. A G-equivariant map of G into a finite dimensional complex or real Euclidean space E^N is a continuous map φ of E into E^N together with a continuous homomorphism into the unitary group of E^N such that $\Theta(g)\varphi(p) = \varphi(gp)$ for all $p \in E$, $g \in G$. A G-equivariant map is called a G-equivariant homeomorphism if the associated φ is a homeomorphism.

The associated Θ of a G-equivariant homeomorphism is an isomorphism if the group G operates faithfully on E.

We remark that a complex Euclidean space E^N can be identified in a natural way with a real Euclidean E^{2N}, and that real Euclidean E^N can be extended naturally to a complex Eculidean E^N. These natural isomorphisms convert G-equivariant maps into complex Euclidean space to G-equivariant maps into real Euclidean space and vice-versa.

The following is a fundamental result about extensions of G-equivariant maps due to A. Gleason (Proc. Amer. Math. Soc. v. 1 (1950), pp. 35-43).

GLEASON'S LEMMA: Let G be a compact group operating on a completely regular (resp. normal) space E and let F be a compact (resp. closed) subset invariant under G. Then any G-equivariant map of F into E^N can be extended to a G-equivariant map of E into E^N.

For the sake of completeness, we repeat the proof of this lemma. Let φ be a continuous map of F into E^N, and let θ be a homomorphism of G into the unitary group of E^N with $\varphi(gp) = \theta(g)\,\varphi(p)$ for all $p \varepsilon F$ and $g \varepsilon G$. Extend φ to a continuous map ψ of E into E^N (Tietje Extension Lemma). Set

$\Phi(p) = \int_G \theta(g)^{-1}\,\psi(gp)\,dg$, for all $p \varepsilon E$. Then Φ gives the desired extension, since $\int_G \theta(g)^{-1}\,\psi'(gp)dg = \int_G \varphi(p)dg = \varphi(p)$ for $p \varepsilon F$ and

$$\Phi(g_1 p) = \int_G \theta(g)^{-1}\,\psi(gg_1 p)dg) = \int_G \theta(gg_1^{-1})^{-1}\,\psi(gg^{-1}g_1 p)pdg$$

$$= \int_G \theta(g_1)\,\theta(g)^{-1}\,\psi(gp)dg = \theta(g_1)\,\Phi(p)$$

for all $p \varepsilon E$.

THEOREM 2.1. Let G be a compact Lie group of transformations on a completely regular space E. Let $p_1, \cdots, p_n$ be any finite set of points of E. Then there is a G-equivariant map (φ, θ) of E with φ a homeomorphism on the orbits through $p_1, \cdots, p_n$ and $\theta(G)$ keeping only the origin fixed if G has no fixed point on E.

<u>Proof</u>. Set $H_i = G_{p_i}$. By Lemma 2.1, there is a representation α_i of G by unitary transformations on Euclidean space E^{N_i} and a point $q_i \;\varepsilon\; E^{N_i}$ such that $\alpha_i^{-1}(\alpha_i(G)_{q_i}) = H_i$ and $\alpha_i(G)$ keeps only the origin fixed if $H_i \neq G$, $i = 1, \cdots, n$. Let $\varphi_i(gp_i) = \alpha_i(g)p_i$, $i = 1, \cdots, n$. Then φ_i is a homeomorphism of the orbit through p_i. Set $E^N = E^{N_1} + \cdots + E^{N_n}$ (direct), and identify E^{N_i} with a sub-

space of E^N in the natural way. Let $\varphi(p) = \varphi_i(p)$ for p in the orbit G_{p_i} through p_i $(i = 1, \cdots, n)$, set $\theta = \alpha_1 + \cdots + \alpha_n$ and set $F = G_{p_1} \cdots G_{p_n}$. Then (φ, θ) is clearly a G-equivariant homeomorphism of F into E^N. By Gleason's lemma, (φ, θ) can be extended to a G-equivariant map of E into E^N, which we denote by (φ, θ) also. If G has no fixed points on E, then $H_i \neq G$ for all i and hence $\theta(G)$ keeps only the origin of E^N fixed.

Section 3. Existence of pseudo-sections.

DEFINITION. Let G be a compact Lie group of transformations on a topological space E, and let $p \in E$. A pseudosection to the orbit Gp at p is a closed subset K containing p satisfying: 1) E is invariant under the isotropy group G_p; 2) there exists a continuous corss-section map f into G of a neighborhood U of the coset G_p in G/G_p such that the mapping $(u,q) \longrightarrow f(u)q$ is a homeomorphism of the product space $U \times K$ onto a neighborhood of p; 3) $gK \cap K$ is empty if $g \in G - G_p$.

A pseudo-section is closely related to a notion employed by Koszul (Colloques Intern. de CNRS, Strassbourg, 1953, pp. 137-41) and its existence has been proved by him in the case of compact groups of differentiable transformations on a differentiable manifold.

In case all the orbits in E are equivalent, we call a pseudo-section at p a "local cross-section at p".

If K is a local cross-section at p to the orbit through p, then gK is disjoint from K for element g of G which is not in the isotropy subgroup G_p. Thus $G_q \subseteq G_p$ for all q in K. Since no compact Lie group is conjugate to a proper subgroup of itself, $G_q = G_p$ for all q in K. As a result distinct points of K belong to distinct orbits. A local cross-section at p can be characterized as a closed subset K suth that; 1) distinct points of K lie in distinct orbits; 2) $G_p = G_q$ for each q in k; 3) GK is a neighborhood of p. Thus a local cross-section K at p is a local cross-section at all of its points which are interior to the set GK. We define a "local cross-section" to be a local cross-section at <u>some</u> point.

If K is a pseudo-section at p and $g \in G$, then gK is a pseudo-section at gp.

LEMMA 3.1. Let G, G' be compact Lie groups of transformations on the space E, E' respectively. Let φ be a continuous map of E into E', let Θ be a homomorphism of G into G', and assume $\varphi(gq) = \Theta(g)\varphi(q)$ for all $g \varepsilon G$, $q \varepsilon E$. Let $p \varepsilon E$, let K' be a pseudo-section to the orbit $G'\varphi(p)$ at $\varphi(p)$ and assume φ is 1-1 on the orbit Gp. Then $K = \varphi^{-1}(K')$ is a pseudo-section to the orbit Gp at p.

Proof. By hypothesis, $G'_{\varphi(p)}K = K$ and there exists a continuous cross-section map f of an open neighborhood U of $G'_{\varphi(p)}$ in $G'/G_{\varphi(p)}$ such that $F':(u,q) \to f(u)q$ is a homeomorphism of $U \times K'$ onto a neighborhood N' of $\varphi(p)$. Inasmuch as φ is one-to-one on the orbit Gp, $\Theta^{-1}(G'_{\varphi(p)}) = G_p$ and Θ induces a homeomorphism of G/G_p onto $G'/G'_{\varphi(p)}$; we identify G/G_p with $G'/G'_{\varphi(p)}$. To verify that K is a pseudo-section, we observe first that $G_pK = \Theta^{-1}(G'_{\varphi(p)})\varphi^{-1}(K') = \varphi^{-1}(G'_{\varphi(p)}K') = \varphi^{-1}(K') = K'$. Next, $F:(u,q) \to f(u)q$ is clearly a continuous map of $U \times K$ into E and it is one-to-one also; for if u_1, u_2 and q_1, q_2 are distinct elements in U and K respectively, then $\varphi(f(u_i)q_i) = (f(u_i))\varphi(q_i)$ are distinct elements of N since $\Theta(f(u_i))$ and $\varphi(q_i)$ $(i = 1,2)$ are distinct elements of G' and K' respectively. Finally, F is an open mapping of $U \times K$ onto the neighborhood $\varphi^{-1}(N')$ of p since $F = \varphi^{-1}F'$. Thus K is a pseudo-section.

LEMMA 3.2. Let G be a compact group of linear transformations of the real or complex finite dimensional linear space V. For any $v \varepsilon V$, there exists a pseudo-section at v to the orbit through v.

Proof. By the well-known unitary trick, an inner product may be introduced on V which is preserved by the elements of G. For any $v \varepsilon V$, the orbit Gv is a submanifold of V. Let L denote the affine subspace perpendicular to the tangent plane to Gv at v. Clearly L is invariant under G_v. Let $\underline{G}$ and $\underline{G}_v$ denote the Lie algebra of G and the Lie subalgebra of G_v respectively. Let $X_1, \cdots, X_n$ be a base for the Lie algebra G with $X_{s+1}, \cdots, X_n$ a base for G_v.

Let C denote the linear subspace spanned by $X_1, \cdots, X_s$, let γ denote the map $t_1X_1 + \cdots + t_nX_n \longrightarrow \exp t_1X_1 \exp t_2X_2 \cdots \exp t_nX_n$ of $\underline{G}$ into G, and let $\underline{W}$ be a neighborhood of zero in $\underline{G}$ on which the map γ is one-to-one and regular. Since G_v is a closed subgroup of G, $\gamma(\underline{W}) \cap \underline{G_v} = \gamma(\underline{W} \cap \underline{G_v})$ for $\underline{W}$ suitably small. Selecting such a small $\underline{W}$, we deduce that the projection π of G onto G/G_v maps $\gamma(\underline{W} \cap \underline{G})$ homeomorphically and bi-differentiably onto a neighborhood U_1 of the coset G_v in G/G_v. Set $f(\pi(g)) = g$ for $g \in \gamma(\underline{W} \cap \underline{G})$, and set $F(u,q) = f(u)q$ for $u \in U$, $q \in L$. The map F is differentiable and regular at the point (G_v,v) of $(G/G_v) \times L$ and hence by the implicit function theorem F is a homeomorphism of a neighborhood $U \times K_1$ of (G_v,v) onto a neighborhood of v in V. Since $G_v v = v$ and G_v preserves distance, there is a neighborhood K_2 of v in K_1 which is invariant under G_v. Since F is a homeomorphism, $gK_2 \cap K_2$ is empty for $g \in \pi^{-1}(U) - G_v$. Let e denote the minimum distance between v and gv for $g \in G - \pi^{-1}(U)$, and let K be closed ball in K_2 with center v and radius $e/4$. Then $gK \cap K$ is empty for all $g \in G - G_v$. Hence K is a pseudo-section at v.

THEOREM 3.1. Let G be a compact Lie group of transformations on a completely regular space E. Then at each point p of E, there exists a pseudo-section to the orbit through p.

<u>Proof.</u> Let $p \in E$. By Theorem 2.1, there is a G-equivariant map (φ,θ) of E into some E^N with φ one-to-one on the orbit through p. By Lemma 3.2, there exists a pseudo-section K' at $\varphi(p)$ to the orbit through $\varphi(p)$. Set $K = \varphi^{-1}(K')$. By Lemma 3.1, K is a pseudo-section at p to the orbit through p.

<u>Note.</u> The hypothesis that G be compact is not superfluous. It is easy to find examples of groups of linear transformations which do not admit pseudo-sections.

COROLLARY 3.1. Let G be a compact Lie group of transformations on a completely regular space E, let U be a neighborhood of the identity in G, and let $p \varepsilon E$. Then there is a neighborhood N of p such that for each $q \varepsilon N$, $G_q \subset gG_pg^{-1}$ with $g \varepsilon U$.

Proof. Let K be a pseudo-section to the orbit Gp at p. Then gK is disjoint from K for g not in G_p. Hence $G_q \subset G_p$ for $q \varepsilon K$. Since $G_{gq} = gG_qg^{-1}$, the neighborhood UK has the desired property.

Note. We could have taken for N the neighborhood UL where L is the set of all $q \varepsilon E$ with $G_q \subset G_p$.

From the foregoing we deduce the following result of Montgomery and Zippin ("A Theorem on Lie Groups", Bull. Amer. Math. Soc., v. 48 (1942), pp. 448-452).

COROLLARY 3.2. Let G be a compact Lie group, let U be a neighborhood of the identity in G, and let H be a closed subgroup of G. There exists a neighborhood V of the identity such that any subgroup in the subset VH is conjugate to a subgroup of H by an element in U.

Proof. Let E be the set of all closed subsets of G topologized by the metric $d(A,B) = \sup C(p,B) + \sup c(A,q)$, where $A \varepsilon E$, $B \varepsilon E$, and $c(p,q)$ is a right invariant metric on the compact group G. The group G operates on E by left translation and the map $(g,A) \longrightarrow gA$ of $G \times E$ into E is continuous. Clearly the isotropy subgroup of the point $H \varepsilon E$ is the subgroup H, i.e., $G_H = H$. Let L be the set of all points A in E with $G_A \subset G_H$ and set $N = UL$. Then as remarked above, N is a neighborhood of H in E and therefore contains a ball with center H and radius d_o.

Let V be a closed ball with center at the identity of G and radius d_o. If F is a closed subset of VH which meets H, then

$$d(FH,H) = \sup_{\substack{f \in F \\ h \in H}} c(fh,H) + \sup_{h \in H} c(FH,h) = \sup_{f \in F} c(f,H) \leq \sup_{g \in VH} c(g,H)$$

$$\leq \sup_{g \in V} c'g,H) \leq d_o$$

and consequently $FH \in N$.

Suppose now that F is a subgroup in VH. Since F is a closed subgroup in VH, no generality is lost when we assume that F is closed. Then $FH \in UL$. Obviously $F \subset G_{FH}$. It follows immediately that $gFg^{-1} \subset G_H \subset H$ with $g \in U$. Proof of the Corollary is now complete.

<u>Note</u>. In their result, Montgomery and Zippin do not impose the hypothesis that G is compact; i.e., they assume that G is a Lie group and H a compact subgroup.

It can be proved that Corollary 3.2 implies Corollary 3.1 and hence the two are equivalent.

Section 4. Finite spanning set of cross-sections.

A covering of a topological space is called <u>star-finite</u> if each set of the covering meets at most a finite number of others; the covering is called <u>star-bounded</u> if there is a finite number b such that each set of the covering meets at most b others. Such a number b is called a <u>bound</u> of the covering.

We require the following fact.

THEOREM 4.1. Any open covering of a finite dimensional separable regular space admits a star-bounded open refinement.

Inasmuch as an n-dimensional separable regular (and hence metric) space can be embedded in a bounded portion of Euclidean 2n + 1 space, Theorem 4.1 will follow immediately from

THEOREM 4.1'. Let O be a bounded open set in Euclidean r-space E^r. Let S be an open covering of O. Then there exists a star-bounded open refinement of O.

<u>Proof</u>. Inasmuch as any open set in E^r is a union of disjoint <u>connected</u> <u>ope</u> sets, there is no generality lost if we add the hypothesis that O is connected. Assume therefore that O is connected as well as bounded and open.

Let $B = \bar{O} - O$, and let $c(p) = \frac{1}{2}d(p,B)$ where $d(p,q)$ is the Euclidean metric in E^r. The function $c(p)$ is continuous on the compact set O and therefore attains its maximum at a point $p_o \varepsilon O$. Set $a = C(p_o)$, and we denote the set consisting of p_o by H_o. Inductively, we define $H_{n+1} = \Sigma_p S(p,C(p)) \ (p \varepsilon H_n)$ where $S(p,C)$ is the closed ball with center p and radius c. We next define the family of sets $H(t)$, $0 \leq t < \infty$ as follows: $H(t + na) = \Sigma_p S(p,tc(p))/a)(p \varepsilon H$ for $0 \leq t \leq a$. Clearly $H(na) = H_n$ $(n = 0,1,\cdots)$. The proof of Theorem 4.1' is arranged in a series of remarks.

1. $H(t)$ is compact. We prove this for t between na and (n + 1)a by induction on n. The assertion is true for n = 0. Assuming by induction that $H_n = H(na)$

is compact, let $q \varepsilon \overline{H(t)}$, $na < t \leq (n + 1)a$. Then $q = \lim q_k$ with $q_k \varepsilon S(p_k, (t - na)\, c(p_k)/a)$ where each p_k is in H_n. H_n being compact, we can assume without loss of generality that $\lim p_k = p$ where $p \varepsilon H_n$. Hence $c(p) = \lim c(p_k)$ and therefore $d(q,p) = \lim d(q_k, p_k) \leq (t - na)\, c(p)/a$. Consequently $q \varepsilon H(t)$, $H(t)$ is closed and therfore compact for $na \leq t \leq (n + 1)a$. Hence $H(t)$ is compact for all t.

2. If $t < t'$, then $H(t) \subset \operatorname{int} H(t')$. This follows at once from the observation that if $c < c'$ then $S(p,c)$ is in the interior of $S(p,c')$.

3. $\Sigma_t H(t) = O$ $(0 \leq t < \infty)$. By the preceding remark, $\Sigma_t H(t)$ is open. We now prove that it is closed in O. Clearly it equals $\Sigma_n H_n$. Suppose therefore that q is in the closure of $\Sigma_n H_n$. Then there is a point $p \varepsilon \Sigma_n H_n$ with $d(q,p) < c(q)$. Say for definiteness $p \varepsilon H_n$. Then $q \varepsilon S(p, c(p) \subset H_{n+1}$, and therefore $\Sigma_t H(t)$ is closed in O. But O being connected, we infer $\Sigma_t H(t) = O$.

4. If $s < t$ and $q \varepsilon H(t)$, then $d(q, H(s)) \leq t - s$. Suppose first that $na \leq s \leq (n + 1)a$ for some n. Then there is a point $p \varepsilon H_n$ with $d(p,q(\leq (t - na)c($ Let q_1 be the point on the line segment pq at the distance $(s - na)c(p)/a$ fro p. Then $q_1 \varepsilon H(s)$ and $d(q_1, q) = (t - s)c(p)/a \leq t - s$. Now let s and t be arbitrary with $0 \leq s < t$. Then there are integers k and h such that $ka \leq s \leq (k + 1)a \leq \cdots \leq ha \leq t \leq (h + 1)a$. By the foregoing result, there is a point q_1 in $H(ha)$ with $d(q,q_1) \leq t - ha$. Inductively we get a point q_n in $H(na)$ such that $d(q_n, q_{n-1}) \leq a$ $(n = 1, \cdots h - k)$. We then have $d(q,H_s) \leq d(q,q_1) + \cdots + d(q_{h-k}\, H_s) \leq (t - ha) + (ha - (h-1)a + \cdots + ((k + 1)a - s) = t - s$. Proof is now complete.

Let $\underline{S}$ be an open covering of O. For each integer n, we select a finite covering $\underline{R}_n$ of the compact set $H_n - \operatorname{int} H_{n-1}$ by open sets in O each of which lies in some set of S and in the open set $\operatorname{int} H_{n+1} - H_{n-2}$. Let $\underline{R}$ denote the union of $\underline{R}_n$ for all n. Any set of $\underline{R}_n$ meets at most the sets of $\underline{R}_{n+k}$, $k = -2$,

-1, 0, 1, 2. Hence $\underline{R}$ is a star-finite open refinement of $\underline{S}$.

5. Given a positive number t, there exists a positive number L satisfying the condition: if A is a set in 0 of diameter less than L and A meets $H(t)$, then A lies in a set of $\underline{R}$. For let L_1 be the Lebesgue number of the finite open covering of $H(t + 2a)$ by $\underline{R}$. Let $L_2 = d(H(t), 0 - H(t + 2a))$ and set $L = \min (L_1, L_2)$. Clearly L satisfies the required condition. We <u>define</u> the number $L(t)$ to be the maximum of the numbers satisfying the condition. Clearly $L(t)$ decreases to zero as t increases to infinity.

6. $L(t + s) \geq L(t) - s$. For let A be a set in 0 of diameter less than $L(t) - s$ and meeting $H(t + s)$. Then there is a point q in A with $d(q, H(t)) \leq s$ by Remark 4 above. Let A_1 be a ball of diameter s meeting both $H(t)$ and A, and set $A_2 = A + A_1$. A_2 has a diameter less than $L(t)$ and meets $H(t)$; therefore it lies in some set of $\underline{R}$. Hence A lies in a set of $\underline{R}$, and thus $L(t + s) \geq L(t) - s$.

It follows directly from Remark 6 that $| L(t + s) - L(t)| < |s|$ and hence $L(t)$ is a continuous positive function of t, $0 \leq t < \infty$. Moreover $L(t + s) \div L(t) \geq 1/2$ if $s \leq L(t)/2$.

7. We denote by D_u, $u > 0$, the decomposition of E^r formed by planes $x_i = nu/($ $(n = 0, \pm 1, \cdots)$, where $x_1, \cdots x_r$ form an orthonormal base of linear function on E^r. Each cube of the decomposition has diameter u. We define the sequence of numbers t_n and u_n as follows: $t_o = 0$, $t_{n+1} = t_n + \frac{1}{2}L(t_n)$; $u_n = L(t_n)/2^{[t_n]}$, where $[t_n]$ is the largest integer less than or equal to t_n. In proving Theorem 4.1' no generality is lost in assuming $a = 1$ and we <u>henceforth</u> assume $a = 1$. Then $L(t) \leq 1$ and $1/4 \leq u_{n+1}/u_n \leq 1$.

8. $d(H_{n+1}, B) \geq \frac{1}{2}d(H_n, B)$. For given $q \in H_{n+1}$, there is a point $p(q)$ in H_n with $d(p(q), q) \leq \frac{1}{2}d(p(q), B)$. Therefore $d(q, B) \geq \frac{1}{2}d(p(q), B) \geq \frac{1}{2}d(H_n, B)$ so that $d(H_{n+1}, B) \geq \frac{1}{2}d(H_n, B)$. Since $d(H_o, B) = 1$, we conclude $d(H_n, B) \geq 1/2^n$.

-1, 0, 1, 2. Hence $\underline{R}$ is a star-finite open refinement of $\underline{S}$.

5. Given a positive number t, there exists a positive number L satisfying the condition: if A is a set in O of diameter less than L and A meets $H(t)$, then A lies in a set of $\underline{R}$. For let L_1 be the Lebesgue number of the finite open covering of $H(t + 2a)$ by $\underline{R}$. Let $L_2 = d(H(t), O - H(t + 2a))$ and set $L = \min (L_1, L_2)$. Clearly L satisfies the required condition. We define the number $L(t)$ to be the maximum of the numbers satisfying the condition. Clearly $L(t)$ decreases to zero as t increases to infinity.

6. $L(t + s) \geq L(t) - s$. For let A be a set in O of diameter less than $L(t) - s$ and meeting $H(t + s)$. Then there is a point q in A with $d(q, H(t)) \leq s$ by Remark 4 above. Let A_1 be a ball of diameter s meeting both $H(t)$ and A, and set $A_2 = A + A_1$. A_2 has a diameter less than $L(t)$ and meets $H(t)$; therefore it lies in some set of $\underline{R}$. Hence A lies in a set of $\underline{R}$, and thus $L(t + s) \geq L(t) - s$.

It follows directly from Remark 6 that $|L(t + s) - L(t)| < |s|$ and hence $L(t)$ is a continuous positive function of t, $0 \leq t < \infty$. Moreover $L(t + s) \mid L(t) \geq 1/2$ if $s \leq L(t)/2$.

7. We denote by D_u, $u > 0$, the decomposition of E^r formed by planes $x_i = nu/$ $(n = 0, \pm 1, \cdots)$, where $x_1, \cdots x_r$ form an orthonormal base of linear functions on E^r. Each cube of the decomposition has diameter u. We define the sequence of numbers t_n and u_n as follows: $t_o = 0$, $t_{n+1} = t_n + \frac{1}{2}L(t_n)$; $u_n = L(t_n)/2^{[t_n]}$, where $[t_n]$ is the largest integer less than or equal to t_n. In proving Theorem 4.1' no generality is lost in assuming $a = 1$ and we henceforth assume $a = 1$. Then $L(t) \leq 1$ and $1/4 \leq u_{n+1}/u_n \leq 1$.

8. $d(H_{n+1}, B) \geq \frac{1}{2}d(H_n, B)$. For given $q \in H_{n+1}$, there is a point $p(q)$ in H_n with $d(p(q), q) \leq \frac{1}{2}d(p(q), B)$. Therefore $d(q, B) \geq \frac{1}{2}d(p(q), B) \geq \frac{1}{2}d(H_n, B)$ so that $d(H_{n+1}, B) \geq \frac{1}{2}d(H_n, B)$. Since $d(H_o, B) = 1$, we conclude $d(H_n, B) \geq 1/2^n$.

9. $d(H(t), O - H(s)) \geq (s - t)/2^{n+2}$ if $n \leq t \leq s \leq n + 2$. For any t, $H(t)$ contains all points within the distance $1/2\ (t - [t]\ d(H_{[t]}, B)$ of $H_{[t]}$. If $[s] = [t]$ then $H(s)$ contains ll points within $1/2(s - t)\ d(H_{[t]}, B)$ of $H(t)$. If $[s] = [t] + 1$, then $H_{[s]}$ contains all points within $1/2([s] - t)d(H_{[t]},B)$ of $H(t)$ and $H(s)$ contains all points within $1/2\ (s - [s])d(H_{[s]},B)$ of $H_{[s]}$. Since $\Sigma_q S(q,a)$ (all $q \varepsilon S(p,b) = S(p, a + b)$ for balls in E^r, we infer that $H(s)$ contains all points within $1/2(s - [s])d(H_{[s]},B) + 1/2([s] - t)d(H_{[t]},B)$ of $H(t)$. Hence $d(H(t), O - H(s)) \geq \frac{1}{2}(s - t)d(H_{[s]},B) \geq (s - t)/2^{n+2}$.

10. Let $\underline{G}_n$ denote the collection of closed cubes from the decomposition D_{u_n} which meet $H(t_n) - H(t_{n-1})$. $\underline{G}_n$ is a finite collection and the set $G_n = \Sigma Q$ (all $Q \varepsilon \underline{G}_n$) is compact. $G_n \subset H(t_n + 1/2\ L(t_n)$ by Remark 9, and hence $G_n \subset H(t_{n+1})$. On the other hand, $d(H(t_{n-2}), O - H(t_{n-1})) \geq (t_{n-1} - t_{n-2})/ 2^{[t_{n-2}] + 2} \geq L(t_n)/2^{[t_n] + 3}$ and therefore G_n does not meet $H(t_{n-2})$, that is $G_n \subset O - H(t_{n-2})$. As a result $G_n \cap G_{n+k}$ is empty if $|k| \geq 3$.

11. Let $\underline{F}_n$ be the collection of open cubes obtained by enlarging each cube of $\underline{G}_n$ to an open cube with same center and side $(1 + e_n)u_n$ where e_n is positiv and satisfies

$$e_n < \min (1, e_{n-1}, 1/2d(G_{n-3}, G_n), 1/2\ d(G_n, G_{n+3})).$$

Then $(1 + e_n)u_n/(1 + e_{n+k})\ u_{n+k} \leq 1$ if $k = 0, -1, -2$ and ≤ 6, if $k = 1,2$ respectively. Hence each set of $\underline{F}_n$ meets sets from only $\underline{F}_{n+k}$ $(k = -2, -1, 0, 1, 2)$. If $Q \varepsilon \underline{F}_n$, then Q meets no more than 3^r sets from each of F_n, $\underline{F}_{n-1}$, $\underline{F}_{n-2}$ and no more than 10^r sets from $\underline{F}_{n+1}$ and 34^r sets from $\underline{F}_{n+2}$. Set $b = 3.3^r + 10^r + 34^r$, and set $\underline{F} = \Sigma_n \underline{F}_n$ (all $n = 0,1,\cdots$). Then $\underline{F}$ is a star bounded open refinement of $\underline{S}$ with bound b. Proof of Theorem 4.1' is now complete.

THEOREM 4.2. Let G be a compact Lie group of transformations on a separa metric finite-dimensional space E. Assume all the orbits are equivalent. Th there exists a finite set of local cross-sections whose orbits cover E.

Proof. Let X denote the space of orbits of G in E, and let π denote the continuous map of E onto X which send each point of E into its orbit under G. Clearly π is a homeomorphism on local cross-sections and therefore X is a finit dimensional separable regular space. A subset of X is called "liftable" if it is the image under π of a subset of a local cross-section in E. Let $\underline{S}$ denote the collection of open liftable subsets of X. Clearly $\underline{S}$ is an open covering o X. Let $\underline{F}_1$ be a star-bounded open refinement of $\underline{S}$ with bound b. The space X i normal and therefore the covering $\underline{F}_1$ is shrinkable to a covering F by closed sets whose interiors cover X; $\underline{F}$ is a fortieri star-bounded with bound b.

Now select from $\underline{F}$ a maximal subcollection of disjoint sets $\underline{M}_1$. Inductively, select in $\underline{F} - (\underline{M}_1 + \cdots + \underline{M}_n)$ a maximal subcollection of disjoint clos sets and denote it by $\underline{M}_{n+1}$. Then $\underline{F} = \underline{M}_1 + \cdots + \underline{M}_k$ with $k \leq b + 1$. For otherwise, there is a set $V \varepsilon \underline{M}_{b+2}$ which meets some set of M_i $(i = 1, \cdots, b + 1)$. Since no set of $\underline{M}_i$ is in $\underline{M}_j$ for $i \neq j$, V meets more than b set — a contradiction.

Now set $L_i = \Sigma V$ (all $V \varepsilon \underline{M}_i$) $i = 1, \cdots, k$. Each point in L_i has a neighborhood meeting only a finite number of sets of M_i and thus L_i is closed, $i = 1, \cdots, k$.

We assert that each L_i is liftable $i = 1, \cdots, k$. In proving this, assume for definiteness that $i = 1$. For each $V \varepsilon M_1$, there corresponds a homeomorphism φ_V of V into E such that $\pi \cdot \varphi_V$ = identity and $\varphi_V(V)$ is a local cross-section in E.

Let H denote the isotropy subgroup G_p for some definite point p in E. For eac $V \varepsilon \underline{M}_1$ select an element $p_V \varepsilon V$ and an element g_V in G such that $G_{g_V p_V} = g_V G_{p_V} g^{-1}_V = H$. Then set $K_1 = \Sigma_V \, g_V \varphi_V(V)$ (all $V \varepsilon \underline{M}_1$). It is easily verified that K_1 is closed, that

$G_q = H$ for all $q \in K_1$, and that distinct points of K_1 lie on distinct orbits. It follows at once that K_1 is a local cross-section, and hence L_i is liftable, $i = 1, \cdots, k$. Let $K_1, \cdots, K_k$ denote local cross-sections mapping onto $L_1, \cdots, L_k$ by π. Then $GK_1 + \cdots + GK_L = E$.

Section 5. Union of homeomorphisms.

Let G be a compact Lie group of transformations of a space E having no fixed points, and let φ be a G-equivariant map of E into Euclidean space with associated homomorphism θ. The map ε is called an <u>n.t. map</u> if the representation θ does not contain the trivial representation, i.e., if the origin is only point fixed under $\theta(G)$.

LEMMA 5.1. Let G be a compact Lie group of transformations of a space E, and let φ be a G-equivariant homeomorphism of E into E^N. Then there is a G-equivariant homeomorphism φ_1 of E into E^{2N} with $|\varphi_1(p)| = 1$ for all $p \varepsilon E$. If φ is an n.t. map, then φ_1 can be chosen so as to be an n.t. map.

<u>Proof</u>. We introduce the functions

$$\alpha(r) = ((1 + r^2)/(4 + r^2))^{\frac{1}{2}} \text{ and } \beta(r) = (1 - \alpha^2(r))^{\frac{1}{2}} \text{ on } 0 \leq r < \infty ;$$

we define maps A and B of Euclidean space minus the origin into the ball of radius 1/2 as follows: $A(v) = \alpha(|v|)|v|^{-1}v$ and $B(v) = \beta(v)|v|^{-1}v$ for $v \varepsilon E^N$.

We form $E^N \times E^1$, and set $\psi(p) = (\varphi(=),w)$ where w is a fixed non-zero vector in E^1. Set $\varphi_1(p) = \psi(p)/|\psi(p)|$ and set $\theta_1 = \theta + \theta_0$ (direct) where $\theta_0(G)$ consists only of the identity transformation of E^1. Then φ_1 is G-equivariant.

If G has no fixed points on E, then $\varphi(E)$ does not contain the origin of E^N. The map $\varphi_1(v) = (A(v), B(v))$ of E into the unit sphere of $E^N \times E^N$ is equivariant with respect to $\theta + \theta$ (direct) and is a homeomorphism. Clearly it is an n.t. map if φ is.

LEMMA 5.2. Let G be a compact Lie group of transformations on a metric space E, and let T_1, T_2 be invariant subsets with $E = T_1 + T_2$ and T_2 closed. Assume there exists a G-equivariant homeomorphism φ_i of E into E^{n_i} $(i = 1,2)$. Then there exists a G-equivariant homeomorphism φ of E into Euclidean space E^N, which is an n.t. map if each φ_i is an n.t. map.

Proof. By Lemma 5.1 we may assume that $|\varphi_1(p)| = 1$ for all $p \varepsilon T_1$. By Gleason's lemma, φ_2 can be extended to a G-equivariant map of E into E^{n_2}, which we denote by φ_2 also. Let $d_1(x,y)$ denote the metric on E. Then $d_1(gx,gy)$ regard as a function on $G \times E \times E$ is continuous . Consequently $d(x,y) = \sup_g d_1(g$ (all $g \varepsilon G$) is continuous on $E \times E$. Moreover $d(x,y)$ is a metric on E; it is equivalent to $d_1(x,y)$ since every d_1 ball contains a concentric d ball by definition of d, and every d ball contains a concentric d_1 ball by the continuity of the function d. It is clear too that $d(gx,gy) = d(x,y)$.

Set $d(x) = \inf_t (d(x,t) + |\varphi_2(x) - \varphi_2(t)|) - \varphi_2(t)|)$ (all $t \varepsilon T_2$). The function $d(x)$ is continuous on E, zero on T_2, and non-zero on $T_1 - T_2$. In addition $d(gx) = d(x)$ for all $g \varepsilon G$. Define φ as the map of E into $E^{n_1} \times E^{n_2} = E^{n_1 + n_2}$ given by:

$$\varphi(x) = (d(x)\varphi_1(x), (1 + d(x)\varphi_2(x))) \qquad \text{for } x \varepsilon T_1$$
$$= (\quad 0 \quad , \varphi_2(x)) \qquad \text{for } x \varepsilon T_2$$

The map φ is clearly continuous, G-equivariant, and is n.t. if φ_1 and φ_2 are n.t. It is clear too that φ is one-to-one, that it is a homeomorphism on T_2 and on T_1 also. To complete the proof that φ is a homeomorphism, it suffices to demonstrate that if $x_n \varepsilon T_1 - T_2$, $\bar{x} \varepsilon T_2$, and $\varphi(x_n) \to \varphi(\bar{x})$, then $x_n \to \bar{x}$. To this end, we observe that $d(x_n) \to d(x)$ and $\varphi_2(x_n) \to \varphi_2(x)$. Let t_n be a point of T_2 with $d(x_n, t_n) < 2d(x_n)$ and $|\varphi_2(x_n) - \varphi_2(t_n)| < 2d(x_n)$. Since $d(x) = 0$; $\lim \varphi_2(t_n) = \lim \varphi_2(x_n) = \varphi_2(x)$. Since φ_2 is a homeomorphism on T_2, $\lim t_n = x$ and hence $\lim x_n = x$. Proof of the Lemma is now complete.

Section 6. The embedding theorem.

Throughout this section E denotes a finite dimensional separable metric space and G a compact Lie group of transformations on E with $L(G,E)$ finite, i.e., with at most a finite manner of inequivalent orbits. By "Euclidean space we understand finite dimensional real or complex Euclidean space with a distinguished origin. If H_1 and H_2 are closed subgroups of G, we mean by $H_1 (\leq) H_2$ that H_1 is conjugate in G to a subgroup of H_2, and by $H_1 (<) H_2$ that H_1 is conjugate to a proper subgroup of H_2. If $H_1 (\leq) H_2$ then $H_1' (\leq) H_2'$ for any H_1' in (H_1) and H_2' in (H_2). The relation $(\leq)$ is clearly transitive. Furthermore if $H_1 (\leq) H_2$ and $H_2 (\leq) H_1$ then H_1 is conjugate to H_2; for H_1 and H_2 must have the same dimension and the same number of connected components. Upon carrying H_1 into a subgroup H_1' of H_2 by an inner automorphism, we find that H_1' and H_2 have the same Lie algebra, and therefore the same connected component of the identity. Since they have the same number of connected components, $H_1' = H_2$ and therefore H_1 and H_2 are conjugate.

In the set $L(G,E)$ we define $(H_1) < (H_2)$ if $H_1 (<) H_2$. This relation is well defined and is a partial ordering. We set E_p = the set of all $q \varepsilon E$ with $(G_p) = (G_q)$, T_p = the set of all $q \varepsilon E$ with $(G_p) \leq (G_q)$ and S_p = the set of all q with $(G_p) < (G_q)$; that is $T_p = E_p + S_p$. According to a theorem of Montgomery and Zippin (Bull. Amer. Math. Soc., v. 48 (1942), pp. 448-452) (cf. also COROLLARY 3.1 above), $G_{q_1} (\geq) G_q$ for all points q_1 is some neighborhood of q. It follows immediately that S_p and T_p are closed sets of E. It is to be noticed that E_p, S_p, and T_p are invariant under G for any $p \varepsilon E$. Also, all orbits in E_p are equivalent.

LEMMA 6.1. Let $p \varepsilon E$. Then there is a G-equivariant homeomorphism of of E_p into Euclidean space, which is n.t. if $G_p \neq G$.

Proof. By Theorem 4.2 there exists in E_p a finite set of local cross-sections to the orbits $K_1, \cdots, K_k$ such that $E_p = GK_1 + \cdots + GK_k$. By Lemma 2.1, there exists a representation α of G into the unitary group on the Euclidean space E^n and a point v other than the origin of E^n such that 1) $\alpha^{-1}(\alpha(0)_v) = G_p$ and 2) α does not contain the trivial representation of G if $G_p \neq G$. Let V denote the one-dimensional subspace spanned by v and the origin. Let r_i be an integer such that K_i can be embedded homeomorphically in E^{r_i} $(i = 1, \cdots, k)$. We identify E^{r_i} with the subspace $V + \cdots + V$ of $E^n + \cdots + E^n = E^{nr_i}$, and obtain thereby a homeomorphism ψ_i of K_i into E^{nr_i} with the property that $\beta_i(G_p)\varphi_i(q) = \varphi_i(q)$ for all $q \in K_i$ where $\beta_i = \alpha + \cdots + \alpha$ (r_i times) $(i = 1, \cdots, k)$. As a result the map $\bar{\varphi}_i : (gG_p, q) \longrightarrow (\beta_i(g) \psi(q), \alpha(g)v)$, where $g \in G$, $q \in K_i$ is a well-defined continuous one-to-one map of $(G/G_p) \times K_i$ into $E^{n(r_i+1)}$ $(i = 1, \cdots, k)$. It is clear too that the inverse mapping is continuous.

Let π_i denote the map $(gG_p, q) \longrightarrow gq$ of $(G/G_p) \times K_i$ onto GK_i. Each π_i is well-defined since $G_q = G_p$ for all $q \in K_i$. π_i is a homeomorphism in a set $U \times K_i$ where U is a neighborhood in G/G_p by definition of a pseudo-section and hence π_i is a homeomorphism throughout $(G/G_p) \times K_i$ $(i = 1, \cdots, k)$. Set $\varphi_i = \bar{\varphi}_i \cdot \pi^{-1}$. Then φ_i is a G-equivariant homeomorphism of GK_i which is n.t. if $G_p \neq G$. Since each GK_i is closed in E, we can construct a G-equivariant homeomorphism φ of E_p in Euclidean space by repeated applications of Lemma 5.2. The map φ is n.t. if $G_p \neq G$.

THEOREM 6.1'. Let G be a compact Lie group operating on a separable metric finite dimensional space E. Assume L(G,E) is finite. Then there exists a G-equivariant homeomorphism of E into a Euclidean space E^n which is n.t. if G has no fixed points in E.

Proof. The set of conjugacy classes L(G,E) is partially ordered by the relation $\leq$ introduced above. We define the length of $L(G,E)$ as the maximum number of elements appearing in a linearly ordered subset. The theorem is proved by induction on the length of $L(G,E)$.

If the length of $L(G,E)$ is 1, then $E_p = T_p$ for any $p \in E$, and therefore E_p is closed in E. Now there exists a finite set of points $p_1, \cdots, p_r$ in E such that $E = E_{p_1} + \cdots + E_{p_r}$. By Lemma 6.1 there is a G-equivariant homeomorphism of E_{p_i} into Euclidean space which is n.t. if $G_{p_i} \neq G$, $i = 1, \cdots, r$. By repeated applications of Lemma 5.2, there exists a G-equivariant homeomorphism of E into Euclidean space which is n.t. if $G_{p_i} \neq G$ for all $p \in G$, that is, if G has no fixed points in E.

Assume inductively that the theorem is true whenever the length if less than $L(G,E)$. There obviously exists in E a finite set of points $p_1, \cdots, p_r$ such that $E = T_{p_1} + \cdots + T_{p_r}$. Each $T_{p_i} = E_{p_i} + S_{p_i}$ and hence length $L(G, S_{p_i}) \leq$ length $L(G,E) - 1$, $i = 1, \cdots, r$. By the induction hypothesis there is a G-equivariant homeomorphism of S_{p_i} which is n.t. if G has no fixed point on S_{p_i} and a similar assertion holds for E_{p_i}, $i = 1, \cdots, r$. By Lemma 5.2, a similar assertion holds for each T_{p_i} and also for $T_{p_i} + \cdots + T_{p_r} = E$. Proof of the theorem is now complete.

Theorem 6.1 mentioned in the introduction is simply a restatement of Theorem 6.1' coupled with the observation that the unitary representation which is associated with a G-equivariant map is faithful if G operated faithfully on E.

If G is a compact group operating faithfully on a space E and there is a G-equivariant homeomorphism of E into Eculidean space, then E is separable, metric, and finite dimensional; also G is a Lie group. We show in Section 7 that $L(G,E)$ is finite. Thus the hypotheses on E of Theorem 6.1 are necessary and sufficient for the existence of a G-equivariant homeomorphism into Euclidean space.

Section 7. Groups acting differentiably. Applications.

We collect first several remarks about compact Lie groups of differentiable transformations. Numbers 1, 2, and 3 below are noted independently by Montgomery and Yang. We include them here for the sake of completeness.

Throughout this section G denotes a compact Lie group of differentiable transformations, M denotes a differentiable manifold, and E^n denotes a real Euclidean n-space with distinguished origin and n finite.

1. Let G operate on M, and let $p \in M$. There is a pseudo-section to the orbit through p which is a closed ball submanifold (of lower dimension in general).

Proof. The isotropy subgroup G_p is a compact group of differentiable transformations keeping the point p fixed. Hence by a result of Bochner admissible coordinates may be introduced in a neighborhood of p with respect to which G_p is a group of orthogonal transformations. Since G_p keeps invariant the tangent space at p to the orbit Gp, it keeps invariant a complementary subspace K in the new coordinates. With the help of the implicit function theorem one can see that the mapping $(g,q) \to gq$ is a homeomorphism of $U \times K_1$ onto a neighborhood of p, where U is a differentiable local cross-section to the coset G_p in G and K_1 is a ball neighborhood of p in K. Select a ball K_2 in K_1 with center p so that $gK_2 \cap K_2$ is empty for $g \in G - G_p$ (see Lemma 3.2). It follows that the ball submanifold K_2 is a pseudo-section.

2. If M is compact, then L(G,M) is finite.

Proof. We use induction on dim M. Let P(n) denote the assertion that L(G,M) is finite if dim $M \leq n$. Let Q(n) denote the assertion that $L(G,E^n)$ is finite if G is a compact group of linear transformations of E^n. The well-known "unitary trick" tells us that a compact group of linear transformations of E^n is equivalent to a compact group of orthogonal transformations. Since the latter keeps the unit sphere S^{n-1} invariant and sends rays into rays, we see that $Q(n)$ is equivalent to $P(n-1)$ if $M = S^{n-1}$. Also, no generality is lost in assuming G operates faithfully for the subgroup of G operating trivially is in every isotropy subgroup.

The assertion $P(0)$ is true, for then G is simply a finite group of permutations of a finite set.

Assume now dim $M = n$ and $P(n-1)$ is true. Hence $Q(n)$ is true. Now since M is compact, there is a finite number of ball-submanifold pseudo-sections $K_1, \cdots, K_s$ through points $p_1, \cdots, p_s$ respectively such that $M = GK_1 + \cdots + GK_s$ and G_{p_i} is equivalent to a linear group on K_i. If g is not in G_{p_i}, then gK_i does not meet K_i so that $G_q \subset G_{p_i}$ for all $q \in K_i$. Hence $(G_q) \leq (G_{p_i})$ for all $q \in GK_i$, and therefore the number of elements in $L(G, GK_i)$ is no greater than the number of elements in $L(G_p, K_i)$ the latter being finite by $Q(n)$. Hence $L(G,E)$, which has no more elements than $\Sigma_i L(G, GK_i)$ is a finite set.

In view of the equivalence between $Q(n)$ and $P(n-1)$ when $M = S^{n-1}$, we conclude

3. $L(G, E^n)$ is finite if G is a compact group of linear transformations on E^n.

4. If $L(G, M)$ is finite, one can follow through our construction of the G-equivariant embedding of M in Euclidean space and obtain after slight modifications a differentiable G-equivariant embedding. If M is a compact differentiable manifold, a short proof can be given based on the following method.

Let B denote the set of differentiable functions on M. Let $\{U_\alpha\}$ be a finite covering of M by coordinate neighborhoods and let $\{V_\alpha\}$ be an open covering with each $V_\alpha \subset U_\alpha$. For each $f \in B$ define $\|f\| = \sup_p(|f(p)|+$

$|\partial f/\partial x^i_\alpha (p)|)$ (all α with $p \varepsilon V_\alpha$, all $p \varepsilon M$). B is a Banach space with $||f||$ as norm. If $g \varepsilon G$ and f is a function on M (resp. on G) we define gf to the function $f \cdot g^{-1}$. We say a function f on M (resp. on G) is a <u>representation function</u> if the linear span of the set of functions Gf is finite dimensional. The representation functions on G are continuous and by the Peter-Wey theorem approximate uniformly any continuous function on G.

We assert now that the representation functions in B form a dense subset of B. For given any $f \varepsilon B$ and any positive number e, there is a neighborhood U of the identity in G such that $||gf - f|| < e/2$ for all $g \varepsilon U$. Let $s = \sup_g ||gf||$ (all $g \varepsilon G$). Let v be a non-negative continuous function on G vanishing outside U with $\int_G v(g)dg = 1$, the Haar measure of G being one. For any continuous function w on G, we set $f_w = \int_G w(g)gf\, dy$; the functi f_w is in B. Now $||f_v - f|| = ||\int_G v(g)gf\, dg - f|| = ||\int_G v(g)(gf - f)dg|| \leq \int_G v(g)\, e/2\, dg \leq e/2$. Next select a representation function u on G such that $|v(g) - u(g)| < e/2s$ for all $g \varepsilon G$. Then $||f - f_u|| \leq ||f - f_v|| + ||f_v - f_u|| \leq e$. Moreover f_u is a representation function on M for

$$g_1(f_u) = g_1 \int_G u(g)gf\, dg = \int_G u(g)g_1 gf\, dg = \int_G u(g_1^{-1}g_1 g)g_1 gf\, dg$$

$$= \int_G u(g_1^{-1}g)gf\, dg = f_{g_1 u}.$$

Since f_u depends linearly on u, it follows that Gf_u lies in a finite dimensional subspace of B. Thus f_u is a representation function on M lying on an e-neighborhood of f, and therefore the representation functions in B are dense in B.

Let $f_1, \cdots, f_n$ be the component functions of a differentiable embedding φ of M into E^n. We can assume that M is a metric space. Then select approximating representation functions $h_1, \cdots, h_n$ whose functional matrix has the same rank as the functional matrix of $f_1, \cdots, f_n$ i.e., dim M. Each point lies in a neighborhood on which the mapping $\varphi_1 : p \to (h_1(p), \cdots, h_n(p))$ is one-to-one and regular. Take a finite covering by such neighborhoods and let b denote the Lebesgue number of this covering. Then we select representation functions $k_1, \cdots, k_n$ which are so close to $f_1, \cdots, f_n$ respectively, that if $k_i(p) = k_i(q)$, $i = 1, \cdots, n$ then $d(p,q) < b$. Select from the linear span in D of each Gh_i and Gk_j a base with first base vector h_i and h_j respectively, and with respect to which the operations of G are orthogonal. Let $h_{i,1}, \cdots h_{is_i}$ and $k_{j,1}, \cdots, k_{j,t_j}$ denote the bases for the linear spans of Gh_i and Gk_j respectively. Then $p \to (h_{1,1}(p), \cdots, k_{n,t_n}(p))$ is a differentiable, regular G-equivariant homeomorphism of M into a Euclidean space.

The foregoing proof of the existence of a G-equivariant embedding in Euclidean space applies with a slight modification to compact subsets of a differentiable manifold. However it cannot be generalized to arbitrary differentiable manifolds for a compact Lie group of transformations can have an infinite number of inequivalent orbits.

5. If the transformation group G is not compact, then L(G, E) can be infinite even if E is Euclidean space and G is an algebraic Lie group of linear transformations. For let G be the algebraic linear group in E^3 whose Lie algebra is the set A of matrices M(a,b) of the form

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The Lie algebra A is abelian. Let B(u) be the set of all M(a,b) with $a + bu = 0$, and let H(u) be the analytic subgroup corresponding to B(u). Then H(u) is the isotropy subgroup of the vector (0,1,u). Thus G has an infinite number of distinct isotropy subgroups and being abelian, $L(G,E^3)$ is infinite.

THEOREM 7.1. Let G be a compact Lie group. Then there exist at most a finite number of mutually non-conjugate subgroups which are normalizers of analytic subgroups. Moreover, there exist at most a finite number of mutually non-conjugate semi-simple analytic subgroups.

Proof. Let A denote the Lie algebra of G, let E denote the exterior algebra of A, and let P denote the projective space of one dimensional linear subspaces of E. Each linear subspace B of A determines a point in P by the Grassman correspondance; this point we denote by B^*. The adjoint representation of G on A induces a representation π of G had by projective transformations of P and clearly a subgroup N of G keeps a linear subspace B invariant if and only if $\pi(N)$ keeps the point B^* fixed. If H is an analytic subgroup of G and B is its Lie algebra, then $xHx^{-1} = H$ if and only if $Adx(B) = B$, and therefore if and only if $\pi(x)(B^*) = B^*$. Cohsequently a subgroup N is a normalizer of some analytic subgroup of G if and only if $N = \pi^{-1}(\pi(G)_{B^*})$ with B a Lie subalgebra of A. Since $L(\pi(G), P)$ is finite, G has at most a finite number of mutually non-conjugate normalizers of analytic subgroups.

In order to prove the second part of the theorem, it suffices to prove that there are only a finite number of distinct semi-simple analytic subgroups which have the same normalizer. Upon considering the corresponding Lie algebra, it suffices to prove that a Lie algebra contains only a finite number of distinct semi-simple ideals. This follows in turn from the fact that (1) the linear span of the semi-simple ideals in a Lie algebra is semi-simple and (2) a semi-simple Lie algebra is the direct sum of all is minimal ideals and therefore has but a finite number of ideals.

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FOOTNOTE

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