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On the Fundamental Group of a Homogeneous Space

G. D. Mostow *

§1. It is well known that the fundamental group $F$ of a group manifold $G$ is abelian. P. A. Smith proved in [10] that the rank of $F$ cannot exceed the dimension of $G$. In this paper, it is proved that more generally if the fundamental group of a homogeneous space (see §2 for definition) $M$ is solvable, then it is finitely generated and its rank cannot exceed the dimension of $M$. Our method consists of reducing to the case in which the group acting transitively on the homogeneous space is itself solvable and then applying results of the author on solvable Lie groups.

§2. We collect here some elementary observations about discrete solvable groups.

Let $G$ be a solvable group, and inductively let $G^{(k+1)}$ denote the commutator subgroup $[G^{(k)}, G^{(k)}]$, $(k = 0, 1, 2, \ldots)$ where $G^{(0)} = G$. By definition $G^{(r)} = \{\text{identity}\}$ for some finite $n$. The smallest such $n$ is called the length of $G$. We say that $G$ has finite rank $r$ and only if the abelian $G^{(k)}/G^{(k+1)}$ is finitely generated for each $k$ and $r = \sum_k \text{rank } G^{(k)}/G^{(k+1)}$, the "rank" of a finitely generated abelian group having the usual meaning of betti number.

Solvable groups of finite rank behave like finitely generated abelian groups with respect to the rank function. Throughout this section $G$ denotes a solvable group of rank $r$.

2.1. $G$ is finitely generated. Explicitly, if $x_1^{(i)}, \ldots, x_s^{(i)}$ is a base for $G^{(i)} \mod G^{(i+1)}$, then $x_j^{(i)}$ all $i, j$ generates $G$.

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2.2. If $H$ is a normal subgroup of $G$, then $H$ as well as $G/H$ is finitely generated, and rank $G = \text{rank } G/H + \text{rank } H$.

Proof. We prove the assertion by induction on the length of $G$.

If the length is one, $G$ is abelian and then the result is true as is well known. Set $N_i = H \cap G^{(i)}$, $i = 0, \ldots, n$, where $G^{(n)} = \text{(identity)}$, and set $M_i = H \cap G^{(i)}$, $i = 0, \ldots, n$. Then $N_i/N_{i+1} = H \cap G^{(i)}/H \cap G^{(i+1)} = G^{(i)}/G^{(i)} \cap H \cap G^{(i+1)}$ and is a finitely generated abelian group, since $G^{(i)}/G^{(i+1)}$ is by definition of rank $G$. Also $M_i/M_{i+1} = H \cap G^{(i)}/H \cap G^{(i+1)} = (H \cap G^{(i)})G^{(i+1)}/G^{(i+1)} = H \cap G^{(i+1)} \cap G^{(i)}(i+1)$ is a finitely generated abelian group. Now $(G^{(i)}/G^{(i+1)})/(H \cap G^{(i+1)} \cap G^{(i+1)}) = G^{i}/G^{(i+1)} \cap G^{i}$ implies at once that rank $G^{(i)}/G^{(i+1)} = \text{rank } N_i/N_{i+1} + \text{rank } M_i/M_{i+1}$. Furthermore $N_i/H$ can be identified with $(G/H)^{(i)}$ so that summing over $i$, we obtain

$$\text{rank } G = \text{rank } G/H + \sum_i \text{rank } N_i/M_{i+1}.$$ 

It remains to prove therefore that rank $H = \sum_i \text{rank } M_i/M_{i+1}$. We note that $H = M_0 \supset M_1 \supset H^{(1)}$. Since $M_1 \subset G^{(1)}$, the length of $M_1$ is less than the length of $G$ and therefore by the induction hypothesis rank $M_1 = \text{rank } M_1/H^{(1)} + \text{rank } H^{(1)}$. Starting with the definition of rank, we get

$$\text{rank } H = \text{rank } H/H^{(1)} + \text{rank } H^{(1)}$$
$$= \text{rank } H/M_1 + \text{rank } M_1/H^{(1)} + \text{rank } H^{(1)}$$
$$= \text{rank } H/M_1 + \text{rank } M_1.$$ 

By applying the induction hypothesis in turn to $M_1, M_2, \ldots$ we obtain
rank $M_1 = \sum_{i=1}^{n} \text{rank} \left( M_i / M_{i+1} \right)$, so that rank $H = \sum_{i=0}^{n} M_i / M_{i+1}$. Consequently rank $G = \text{rank} \ G / H + \text{rank} \ H$. Proof is now complete.

2.3. If $H \subset G$, then rank $H \leq \text{rank} \ G$.

Proof. By 2.2, rank $H = \sum_{i=1}^{n} M_i / M_{i+1}$, where $M_i = H \cap G^{(i)}$. Since $M_i / M_{i+1} = (H \cap G^{(i)}) G^{(i+1)} / G^{(i+1)}$, we have rank $M_i / M_{i+1} \leq \text{rank} \ G^{(i)} / G^{(i+1)}$ after each $i$ and hence rank $H \leq \text{rank} \ G$.

2.4. Suppose that $G$ is a topological group, and that as a discrete solvable group, it has finite length $n$. Then $G$ contains a closed normal abelian subgroup other than (identity), namely the topological closure of $G^{(n-1)}$. More generally, $\bar{G}^{(k)}/\bar{G}^{(k+1)}$ is abelian for $k = 0, 1, \ldots$, where $\bar{G}^{(k)}$ denotes the topological closure of $G^{(k)}$. It follows at once that $H^{(n)} = (\text{identity})$ where $H^{(0)} = G$ and $H^{(k+1)}$ is the topological closure of $[H^{(k)}, H^{(k)}]$ so that $G$ is solvable in the sense of topological groups. The converse is also true.

§3. By a homogeneous space we mean a connected space $M$ on which a Lie group $G$ operates transitively. If $G$ is a topological group, we denote by $G^0$ the connected component of its identity and by German capital $G$ the Lie algebra of the analytic group $G^0$. It is easily seen that if a Lie group $G$ is transitive on $M$, then so is $G^0$. If $G$ is a Lie group operating transitively on $M$ and $G_p$ denotes the subgroup of $G$ keeps fixed a point $p$ in $M$, then $M$ can be identified with the space of cosets $G/G_p$ under the correspondence $g p \rightarrow g G_p$. In turn, when $G$ is a connected Lie group, the fundamental group of a space of cosets $G/H$ can be identified with $\hat{G}/\hat{G^0}$ ([7]) where $\hat{G}$
is the simply connected covering group of $G$ and $\hat{H}$ is the complete inverse
image of $H$ in $G$ under the covering homomorphism of $\hat{G}$ onto $G$. Consequently,
the theorem announced in §1 can be restated as:

**THEOREM 1':** Let $G$ be a (connected) simply connected Lie group and $H$ a closed subgroup. If $H/H^C$ is solvable, then its rank cannot exceed $\dim G/H$.

**REMARK.** The hypothesis that $G$ is simply connected can be dropped from Theorem 1'. For if $G$ is merely connected, we let $\hat{G}$ denote its simply connected covering group and $\pi$ the covering homomorphism of $\hat{G}$ onto $G$. If $H$ is a closed subgroup of $G$ with $H/H^C$ solvable then $\pi^{-1}(H)/(\pi^{-1}(H))^C$ is solvable and $\pi$ induces a homomorphism of this group onto $H/H^C$. Hence the rank of $H/H^C$ cannot exceed $\rank \pi^{-1}(H)/(\pi^{-1}(H))^C \leq \dim \hat{G}/\pi^{-1}(H) = \dim G/H$.

§4. If $F$ is a closed subgroup of $G$ with $F^C \subset G^C$, we denote by $P(F, G)$ the assertion that $F/F^C$ is solvable and $\rank F \cap G^C/F^C \leq \dim G/F$.

Let $H$ and $F$ be closed subgroups of the Lie group $G$ with $H/H^C$ and $F/F^C$ solvable and with $H \subset F$. Then $\rank H/H^C = \rank H \cap F^C/H^C + \rank H/H \cap F^C = \rank H \cap F^C/H^C + \rank HF^C/F^C \leq \rank H \cap F^C/H^C + \rank F/F^C$. Consequently $P(H, F)$ and $P(F, G)$ imply $P(H, G)$.

§5. We first consider Theorem 1' in the case that $G$ is a solvable simply connected Lie group and $H$ is an arbitrary closed subgroup. We appeal to the theorem in [6], Sec. 8 on the existence of a "simplifying sequence" $G = G_1, F_1, G_2, F_2, \ldots G_n, F_n$. By Property 5 (ibid.) of such a sequence, there is a closed subgroup $H_n$ of finite index and normal in $H$ such that $H_n$ can be imbedded in the simply connected solvable Lie group $F_n$ as a closed
uniform subgroup, i.e. $F_n^n/H_n^n$ is compact. By properties 1 and 3 of a
simplifying sequence, $\dim F_n^n \leq \dim G$. Since $\text{rank } H_n^c/H_n^c = \text{rank } H_n^c/H_n^c$ and
$\dim F_n^n \leq \dim G/H_n^n$, it suffices to prove $P(H_n^n, F_n^n)$. In other words, in
proving Theorem 1', when $G$ is solvable, we can assume that $H$ is a closed
uniform subgroup of $G$.

Let $A$ denote the maximum normal analytic subgroup of $G$ that is
contained in $H$. Now $H^c \supset A$, rank $H^c/H^c = \text{rank } H^c/A^c/A^c$ and $\dim G/H = \dim G/A/H/A$. Consequently, $P(H, G)$ is implied by $p(H/A, G/A)$ and it
suffices to prove the latter. That is, we are reduced to the case that
$G$ is a simply connected solvable Lie group and $H$ is a closed uniform
subgroup containing no analytic subgroup normal in $G$ other than (identity).
Under these hypotheses, we can apply the theorem in §5 of [6] and con-
clude that 1) $HN$ is closed in $G$, $N$ being the maximum normal analytic
nilpotent subgroup of $G$, and 2) $N^c \subset N$.

$HN$ is uniformly compact. For the factor space $N/H \cap N$ is homeomorphic
to $HN/H$, a closed subset of the compact space $G/H$ and hence compact. The
normalizer $N(H^c)$ of $H^c$ in $N$ is therefore a closed uniform subgroup of $N$.
Finally, $N$ is simply connected because it is an analytic subgroup of a
simply connected solvable group. (See [3].)

§6. We continue the convention of §3 of denoting Lie subgroups and
their Lie algebras by the same Roman and German capitals respectively. We
denote by $\text{Ad}$ and $\text{ad}$ the adjoint representation of Lie groups and Lie algebras
respectively.

Let $N(H^c)$ denote the normalizer of $H^c$ in $N$. Clearly
$N(H^C) = \{g | g \in N, \text{Ad } g(H) = H \}$. Also, for all $X$ in $N$, $\text{ad } X = \log \exp \text{ad } X$, since $\text{ad } X$ is nilpotent and there is no question about the convergence of the power series $\log U = \sum_{k=1}^{\infty} (-1)^{k+1} (U-I)^k / k$ when $(U-I)$ is a nilpotent endomorphism. As a result, $\text{ad } X = \log \exp \text{ad } X = \log \exp X$, and $\text{Ad } \exp X(H) = H$ implies $\text{ad } X(H) \subset H$. Consequently, $\text{Ad } \exp tX(H) = \exp \text{ad } tX(H) = H$ for all real $t$ whenever $\text{Ad } \exp X(H) = H$. Inasmuch as each element of $N$ lies on a one parameter subgroup (see [1] Theorem 6, cf. also [6] §8), it follows at once that $N(H^C)$ is connected. Indeed $N(H^C)$ is a connected uniform subgroup and thus $N/N(H^C)$ is compact. Since $N$ is simply connected, $N/N(H^C)$ is simply connected and in fact homeomorphic to a euclidean space (see [4], [5], or [6]). It follows that it must be a point and $N = N(H^C)$, that is $H^C$ is normal in $N$.

Set $N_1 = N/H^C$, $H_1 = H/H^C$. Then $H_1$ is a discrete uniform subgroup in $N_1$. Now the commutator subgroup $H_1^{(1)} = [H_1, H_1]$ is uniform in $N_1^{(1)} = [N_1, N_1]$ and more generally $H_1^{(h)}$ is uniform in $N_1^{(h)}$ $(h = 1, 2, ...)$ ([4], [5]). Hence $H_1 N_1^{(1)}/H_1 = N_1^{(1)}/H \cap N_1^{(1)}$ is compact and thus $H_1 N_1^{(1)}$ is closed. More generally $(H_1 \cap N_1^{(k)})/N_1^{(k+1)}$ is closed in $N_1^{(k)}$ and thus $H_1 \cap N_1^{(k)}/H_1 \cap N_1^{(k+1)} = (H_1 \cap N_1^{(k)})/N_1^{(k+1)}$ is a closed subgroup of the simply connected abelian Lie group $N_1^{(k)}/N_1^{(k+1)}$. Being discrete, rank $H_1 \cap N_1^{(k)}/H_1 \cap N_1^{(k+1)} \leq \dim N_1^{(k)}/N_1^{(k+1)}$. By repeated application of 2,2, we have rank $H_1 \cap N_1 = \sum_k \text{rank } H_1 \cap N_1^{(k)}/H_1 \cap N_1^{(k+1)} \leq \sum_k \dim N_1^{(k)}/N_1^{(k+1)} = \dim N_1$. Consequently $\text{rank } H \cap N/H^C = \text{rank } H_1 \cap N_1 \leq \dim N/H^C$. On the other hand, $\text{rank } H/H \cap N = \text{rank } HN/N \leq \dim G/N/HN/N = \dim G/HN = \dim G/N$ since $HN/N$ is a closed and therefore discrete subgroup of the abelian Lie group $G/N$. 
Hence $\text{rank } H/H^C = \text{rank } H \cap N/H^C + \text{rank } H/H \cap N \leq \dim N/H^C + \dim G/N = \dim G/H^C = \dim G/H$. Proof of Theorem 1' is now complete for the case that $G$ is solvable.

§7. We now prove Theorem 1' by induction on $\dim G$. If $\dim G = 1$, $G$ is abelian and the theorem holds. Suppose the theorem holds when the dimension is less than $\dim G$. Let $L \cdot R$ be a Levi-decomposition for $G$, $R$ being the radical and $L$ any maximal semi-simple analytic subgroup. Since $G$ is simply connected, the product $LR$ is a semi-direct product so that $L$ and $R$ are each simply connected ([7]). Let $Z$ denote the center of $G$ and set $F = \overline{ZHR}$. Clearly normalizer of the analytic subgroup $H^C R$ contains $F$. Hence $H^C R$ is a closed analytic normal subgroup of $F$. We consider separately the possibilities $F = G$ and $F \neq G$.

Case 1. $G = F = \overline{ZHR}$. Here $\overline{ZHR}$ is a dense solvable subgroup of $G/H^C R$ and thus $L/H^C R = \overline{LR/H^C R} = \overline{G/H^C R}$ is solvable. However it is also a homomorphic image of a semi-simple $L$. Hence it reduces to a simple element and $G = H^C R$. Since $H^C R$ is a normal analytic subgroup of the simply connected $G$, it is closed ([9], p. 279) and therefore $G = H^C R$. Then $H/H^C = (H \cap R) H^C = H \cap R/H^C \cap R$ and $\text{rank } H/H^C = \text{rank } H \cap R/H^C \cap R \leq \text{rank } H \cap R/(H \cap R)^C \leq \dim R/(H \cap R)^C \leq \dim R/H^C \cap R = \dim H^C R/H^C = \dim G/H^C = \dim G/H$.

§8. Case 2. $F \neq G$. Then $\dim F < \dim G$ so that $p(H, F)$ holds by the induction hypothesis. Furthermore $F/F^C$ is solvable so that it remains only to prove $P(F, G)$. Since $R \subset F^C$, this is equivalent to $P(F/R, G/R)$. Set $G_1 = G/R$ and $F_1 = F/R$. $G_1$ can be identified with the simply connected semi-
simple group \( L \) and \( F_1 \) is a closed subgroup containing the center \( Z \) of \( G \).

We let \( G_2 \) denote \( \text{Ad} G_1 \), \( \text{Ad} \) being the adjoint representation. The kernel of \( \text{Ad} \) is the center \( Z \) of \( G_1 \). Now \( G_2 \) is topologically the direct of a product \( K_2 \times \mathbb{R}/\text{compact subgroup} \ K_2 \) and a euclidean space \( \mathbb{R} \). (See [8] Lemma 2.6.) Consequently \( G_1 \), which is the simply connected covering space of \( G_2 \) can be identified with \( K_1 \times \mathbb{R} \) where \( K_1 \) is the simply connected covering group of \( K_2 \) and can be identified moreover with \( \text{Ad}^{-1}(K_2) \) (see [8] Lemma 2.7). Let \( A_2 \) denote the connected component of the identity of the center of \( K_2 \), and let \( A_1 \) denote \( (\text{Ad}^{-1}(A_2))^\mathbb{C} \). Then \( K_1 = K_1 / A_1 \times A_1 \) as a direct product of topological groups and \( K_1 / A_1 \) is compact ([8] p. 980). Hence \( Z \cap A_1 \) is of finite index in \( Z \), the kernel of \( \text{Ad} \). Also, \( A_1 \) is a simply connected abelian analytic group whose dimension equals the rank of the discrete uniform subgroup \( Z \cap A_1 \).

We will have to know below that \( \dim G_1 / N_1 \geq \text{rank } Z / Z \cap N_1 \) if \( N_1 \) is an analytic subgroup of \( G_1 \) with \( \text{Ad} N_1 \) closed. To prove this, we observe first that \( \dim G_1 / N_1 \geq \dim A_1 / A_1 \cap N_1 \), the dimensions being computed from the Lie algebras. Next, \( ZN_1 = \text{Ad}^{-1}(\text{Ad} N_1) \) is a closed subgroup and thus \( Z(A_1 \cap N_1)^\mathbb{C} = A_1 \cap ZN_1 \) is closed subgroup of the abelian group \( A_1 \). Hence \( \text{rank } Z(A_1 \cap N_1)^\mathbb{C} \leq \text{dim } A_1 / Z(A_1 \cap N_1) = \text{dim } A_1 / A_1 \cap N_1 \). Now \( Z(A_1 \cap N_1)^\mathbb{C} = Z(A_1 \cap N_1) / (A_1 \cap N_1)^\mathbb{C} \) contains the subgroup \( Z / Z \cap (A_1 \cap N_1)^\mathbb{C} \) and this in turn has \( Z / Z \cap N_1 \) as a homomorphic image. Therefore

\[
\text{rank } Z / Z \cap N \leq \text{rank } Z / Z \cap (A_1 \cap N_1)^\mathbb{C} \leq \text{rank } Z(A_1 \cap N_1) / (A(A_1 \cap N_1))^\mathbb{C} \\
\leq \text{dim } A_1 / A_1 \cap N_1 \leq \text{dim } G / N_1.
\]
§9. We complete the discussion of Case 2 of §3 by proving \( P(F_1, G_1) \). Ad \( G_1 \) can be described as the totality of automorphisms of the Lie algebra \( G_1 \) of \( G_1 \), which can be generated by inner derivations. Since all the derivations of a semi-simple Lie algebra are inner, \( \text{Ad} \; G \) is the connected component of the identity of \( \text{Aut} \; G_1 \), the group of automorphisms of \( G_1 \). Clearly \( \text{Aut} \; G_1 \) is an algebraic group and \( \text{Ad} \; G_1 \) is a normal subgroup of finite index in it. Let \( N_2 \) be the smallest algebraic subgroup of \( \text{Aut} \; G \) containing \( F_2 = \text{Ad} \; F_1 \). Then as is well known, \( N_2 \) has only a finite number of topologically connected components. Consequently, \( N_2 \) is of finite index in \( F_2 \).

We assert \( N_2^C \not\subset G_2 \). For the normalizer of \( F_2^C \) in \( \text{Aut} \; G \) can be characterized as \( g \mid \text{Aut} \; G_2, g(F_2^C) = F_2^C \) and is therefore algebraic. Hence \( F_2^C \) is normal in \( N_2^C \). Suppose for the moment that \( N_2^C \subset G_2 \). Then \( F_2^C \) is normal in \( G_2 \). Hence \( G_3 = G_2/F_2^C \) is semi-simple -- in fact, it is a factor in a direct product decomposition of \( G_2 \) and therefore has a trivial center. Thus it can be identified with its own adjoint group and may be regarded as the topologically connected of the identity of an algebraic group.

It follows readily that \( N_3 = N_2^C/F_2^C = G_2/F_2^C = G_3 \) is the connected component of the identity of the smallest algebraic group containing the solvable group \( F_2/F_2^C \). Hence \( G_3 \) is solvable -- a contradiction. Consequently \( N_2^C \not\subset G_2 \).

Let \( N_1 = \text{Ad}^{-1}(N_2^C) \). Since \( \text{Ad}^{-1}(F_2 \cap N_2^C) = \text{Ad}^{-1}F_2 \cap \text{Ad}^{-1}N_2^C = F_1 \cap N_1 \), we have \( F_1/F_1 \cap N_1 = F_2/F_2 \cap N_2 \). Thus rank \( F_1 \cap N_1/F_1^C = \text{rank} \; F_1/F_1^C \)-rank \( F_1/F_1 \cap N_1 = \text{rank} \; F_1 \). Consequently, in order to prove \( P(F_1, G_1) \), it suffices
to prove $P(F_1 \cap N_1, G_1)$. Since $\dim N_1^C < \dim G_1$, we assert $P(F_1 \cap N_1, N_1)$ by
the induction hypothesis. By the result in §8, we obtain
$\dim G_1/N_1^C \geq \text{rank } Z/N_1^C = \text{rank } ZN_1^C/N_1^C = \text{rank } N_1/N_1^C$ thus establishing
$P(N_1, G_1)$. Combining these assertions, we conclude $P(F_1 \cap N_1, G_1)$. Proof
of Theorem 1' is now complete.

**REMARK.** It is to be noted that if the fundamental group of a
homogeneous space is not solvable, the first betti number of the space
may be infinite. For example if $H$ is the group of covering transforma-
tions of a Riemann surface of infinite genus, then $H$ can be identified
with a discrete subgroup of the group $G$ of conformal mappings of the
interior of the unit circle. The homogeneous space $G/H$ has dimension
three and first betti number is infinite.
Errata for "Factor Spaces of Solvable Groups", Ann. of Math., v. 60, July 1954 (pp. 1-27).

p. 1. Theorem A Read "homeomorphic" for "isomorphic"

p. 11. $\lambda$. 16 Read "$I^n - \phi \{1\}$" for "$I^n - \{i\}$"

$p$. 26 should read "Hence $S \subset CT$, $S = (C \cap \bar{S})T$, $T = \bar{S} \cap P$,
$\bar{S} = (\bar{C} \cap \bar{S})(P \cap \bar{S})$, and ..."

$p$. 28 Insert "in the" at the end of the line

p. 12. Omit lines 9-20. Instead prove the assertion on line 21 directly by induction on the subspace annihilated by $(\text{Ad} n - \lambda)^k$.

p. 15. $\lambda$. 15b, $\lambda$. 9b: "$\phi" for "$\Theta"

p. 17. $\lambda$. 15, $\lambda$. 24 "$\sim$" for $J$

$p$. 19 "$\bar{S}$" for $S$'

$p$. 19 (part d) $\lambda$. 1) Read "$M + [\bar{F}^{\infty}, \bar{F}^{\infty}] \supseteq \bar{F}^{\infty}$" for
"$M + [F^{\infty}, F^{\infty}] \supseteq F^{\infty}$".

p. 22. $\lambda$. 17b "solvable group $G$" for "group $G$"

p. 24. $\lambda$. 1b. Insert "For let $K$ denote the kernel of the epimorphism
of the semi-direct product $\Delta \cdot \Gamma$ onto $\Delta \cdot \Gamma$. Then $K$ consists
of all elements $d \in \Delta \cdot \Gamma$. The isomorphisms between $\Delta_i$ and $\Gamma_i(i=1,2)$
extend to an isomorphism between $\Delta_i \cdot \Gamma_i$ and $\Delta_i \cdot \Gamma_i$ under
which $K_i$ corresponds to $K_2$. The induced isomorphism of
$\Delta_1 \cdot \Gamma_1$ onto $\Delta_2 \cdot \Gamma_2$ is the desired extension of $\Theta$".

$\lambda$. 3b. Should read "be extended uniquely to an isomorphism be-
tween the nilpotent groups $\Gamma_1$ and $\Gamma_2$ which coincides with $\Theta$
on $\Delta_1 \cap \Gamma_1$ (cf. [5])"

$\lambda$. 4b. Insert sentences: "$\Delta_1 \cap \Gamma_1 = T_1 \cap N_1 / S_1^c$. For
$\Delta_1 \cap \Gamma_1 \supseteq T_1 \cap N_1 / S_1^c$ and each is uniform in the same nilpotent
simply connected subgroup $\Gamma_1$. Hence $(\Delta_1 \cap \Gamma_1)/(T_1 \cap N_1 / S_1^c)$ is
finite -- for it is a covering group of a covering of one compact
factor space by another. However $T_1 \cap N_1 / S_1^c = \Theta^{-1}((\Delta_2 \cap N_2 / S_2^c) \cap N_1 / S_1^c)$
is the intersection of the kernels of $\rho_2$ and $\rho_2$ where
$\rho_1: \Delta_1 \longrightarrow \Delta_1 \cap \Gamma_1 \cap N_1 / S_1^c$ (i=1, 2). Hence $\Delta_1 / (T_1 \cap N_1 / S_1^c)$ has an
isomorphic image in the free abelian group $\rho_1(\Delta_1) + \rho_2(\Delta_1)$
(direct); therefore it is trivial and
$\Delta_1 \cap \Gamma_1 = T_1 \cap N_1 / S_1^c$".

$\lambda$. 5b. Read "$\Theta(T_1 \cap N_1 / S_1^c)$" for "$\Theta(T_1)$"
Bibliography


Footnote

We take this opportunity to call attention to some errata of [6]. See last section of this paper.
Appendix

We take this opportunity to prove a result used above whose proof does not appear in the literature.

Theorem. A real algebraic group has only a finite number of topologically connected components.

Proof. Let $G$ be a real algebraic group. Then by [11], $G$ is a semi-direct product $MN$ where $M$ is a maximal fully reducible subgroup and $N$ is the maximum normal algebraic subgroup of unipotent elements; moreover, $N$ is algebraically connected and each element in it is the exponential of an element in its Lie algebra. It follows that $N$ is topologically connected and that $G$ and $M$ have the same number of connected components.

We now assert that $M$ has a finite number of topologically connected components. A proof of this is implicit in the proof of Theorem 6.2 of [12], but since no explicit mention is made of this observation, we sketch the proof here.

Let $M_X$ denote the complex algebraic group obtained from $M$ upon extending the ground field to the field of complex numbers, and let $M^0_X$ denote the algebraically connected component of the identity. We identify $M$ with a subset of $M_X$ in the natural way. Then $M \cap M^0_X$ is the algebraically connected component of the identity in $M$ and $M^0_X$ is topologically connected (see [2] v. 2). Now $M^0_X$ has a compact real form $M^0_X$ invariant under complex conjugation ([12], p. 54, Theorem 6.2, second paragraph). By definition, it is an invariant compact form of $M \cap M^0_X$. Hence by Lemma 2.2 of [12],
\[ M \cap M^0_X = (M \cap M^0_X \cap M^0_K) \cdot \exp(M \cap \sqrt{-1} M_K) \]
as a topological direct product, where \( M \) and \( M_K \) denote the Lie algebras of
\( M \) and \( M_K^0 \) respectively. Consequently, \( M \cap M^0_X \) and \( M \cap M^0_X \cap M^0_K = M \cap M^0_K \) have
the same number of connected components. Since \( M \cap M^0_K \) is a closed subgroup
of a compact group it has but a finite number of connected components. Since
\( M \cap M^0_X \) is of finite index in \( M \), it follows that \( M \) has but a finite number of
connected components.
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Member  A.M.S. Summer Math. Inst.  1953
Visiting Professor  Conselho Nacional de Pesquisas, Brazil  1953-54
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