ON UNIQUENESS QUESTIONS FOR HYPERBOLIC DIFFERENTIAL EQUATIONS

by

John P. Shanahan

May 1959

Technical Note No. 11
prepared under
Contract No. AF 18(603)-41

Qualified requestors may obtain copies of this report from ASTIA Document Center, Arlington Hall Station, Arlington 12, Virginia. Department of Defense contractors must be established for ASTIA services or have their "need-to-know" certified by the cognizant military agency of their project or contract.
ABSTRACT

This note is concerned with questions of uniqueness, existence and convergence of successive approximations for a solution of an initial value problem, where \( z_{xy} = f(x, y, z, z_x, z_y) \) and \( z(x, 0), z(0, y) \) are assigned. There are obtained analogues of the Nagumo and Kamke criteria in the theory of ordinary differential equations. The method employed is related to the arguments used by Viswanatham to prove the convergence of successive approximations for ordinary differential equations under conditions similar to those in Kamke's general uniqueness theorem.
ON UNIQUENESS QUESTIONS FOR HYPERBOLIC DIFFERENTIAL EQUATIONS

by

John P. Shanahan

1. Statement of results. This note is concerned with the existence and uniqueness of solutions of the initial value problem

\[ z_{xy} = f(x,y,z,p,q), z(x,0) = \sigma(x), z(0,y) = \tau(y), \]

where \( \sigma(0) = \tau(0) = z_0 \)

on a rectangle \( R : 0 \leq x \leq a, 0 \leq y \leq b \). By a solution is meant a continuous function having partial derivatives almost everywhere and satisfying the integral equation

\[ z(x,y) = \sigma(x) + \tau(y) - z_0 + \int_0^x \int_0^y f(s,t,z(s,t),z_x(s,t),z_y(s,t)) \, ds \, dt. \]

(1)

Actually it will be clear from the conditions imposed on \( \sigma, \tau \) and \( f \) that any solution of (1) is uniformly Lipschitz continuous. Let \( D \) be the five-dimensional set \( D = \{ (x,y,z,p,q) : (x,y) \in R \text{ and } z,p,q \text{ arbitrary } \} \). Let \( f(x,y,z,p,q) \) be defined and continuous on \( D \), such that \( |f(x,y,z,p,q)| < N = \text{const. for } (x,y,z,p,q) \in D \). Let \( \sigma(x), \tau(y) \) be defined and uniformly Lipschitz continuous on \( 0 \leq x \leq a, 0 \leq y \leq b \), respectively (so that \( |\sigma(x) - \sigma(x')| \leq K|x - x'|, |\tau(y) - \tau(y')| \leq K|y - y'| \) for some

This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract No. AF 18(605)-41. Reproduction in whole or in part is permitted for any purpose of the United States Government.
constant K) and let $\sigma(0) = \mathcal{T}(0) = z_0$. In addition, for $(x,y) \in \mathbb{R}$ and arbitrary $z,p,q,\bar{z},\bar{p},\bar{q}$ assume that

\begin{equation}
|f(x,y,z,p,q) - f(x,y,\bar{z},\bar{p},\bar{q})| \leq \varphi(x,y,|z - \bar{z}|,|p - \bar{p}|,|q - \bar{q}|),
\end{equation}

where $\varphi(x,y,z,p,q)$ is a continuous, non-negative function defined for $(x,y) \in \mathbb{R}$ and non-negative $z,p,q$, non-decreasing in each of the variables $z,p,q$ with the property that for every $(a,b)$, where $0 < a \leq \alpha$, $0 < b \leq \beta$, the only solution of

\begin{equation}
z(x,y) = \int_0^x \int_0^y \varphi(s,t,z(s,t),z_x(s,t),z_y(s,t))dsdt
\end{equation}

in the rectangle $R_{\alpha\beta} : 0 \leq x \leq \alpha, 0 \leq y \leq \beta$ is $z = 0$.

Theorem (*). Under the above assumptions on $\sigma$, $\mathcal{T}$, $f$ and $\varphi$, (1) possesses one and only one solution on $\mathbb{R}$. This solution is the uniform limit of the successive approximations defined by

\begin{equation}
z_0(x,y) = \sigma(x) + \mathcal{T}(y) - z_0
\end{equation}

and, for $n = 1, 2, 3, \ldots$, by

\begin{equation}
z_n(x,y) = z_0(x,y) + \int_0^x \int_0^y f(x,y,z_{n-1}(s,t),z_x(s,t),z_y(s,t))dsdt.
\end{equation}

The existence assertion of (*) neither implies nor is implied by that in Hartman-Wintner [3] and its generalizations due to Conti, Szmydt, Ciliberto, Kisynski (for references, see [6] and [2]). The uniqueness assertion of (*) can be considered as a crude analogue of Kamke’s uniqueness theorem (cf. [5], p. 139) in the theory of ordinary differential equations.

Finally, the assertion concerning the convergence of successive approximations is an analogue of a result on ordinary differential equations (cf. Viswanatham [8] and references there to van Kampen, to Wintner and to Dieudonné, and Coddington and Levinson [1]).

A theorem similar to (*), in which $f$ and $\varphi$ do not depend on $p,q$ is
proved by Guglielmino [2]. The proof of (*) below will be a generalization of that of [2]. A uniqueness theorem for (1) involving a majorant function of the form $\varphi(z, p, q) = \varphi(|z| + |p| + |q|)$ is given in [6].

Remark. It will be clear from the proofs that (*) remains valid if $f, z, p, q, \sigma, \tau$ are n-vectors (say, with the norm $|z| = \sum_{k=1}^{n} |z^k|$ or $|z| = \max(|z^1|, \ldots, |z^n|)$ if $z = (z^1, \ldots, z^n)$).

A theorem suggested by Nagumo's uniqueness theorem (cf. [4], p. 97) for ordinary differential equations is the following:

Theorem (**). Let $f(x, y, z, p, q)$ be defined, continuous and bounded on $D$, and satisfy, for $xy > 0$ and arbitrary $z, p, q, \bar{z}, \bar{p}, \bar{q},$

\begin{equation}
|f(x, y, z, p, q) - f(x, y, \bar{z}, \bar{p}, \bar{q})| \leq c_1(x, y)|z - \bar{z}|/xy + c_2(x, y)|p - \bar{p}|/y + c_3(x, y)|q - \bar{q}|/x,
\end{equation}

where $c_i(x, y), i = 1, 2, 3$, are non-negative, continuous functions such that $c_1 + c_2 + c_3 = 1$.

Let $\sigma(x)$, $\tau(y)$ be as in (*), and, in addition, let

\begin{equation}
\sigma_x(0) = \lim_{x \to +0} \sigma(x), \quad \tau_y(0) = \lim_{y \to +0} \tau(y)
\end{equation}

exist. Then (1) has at most one solution $z = z(x, y)$. Furthermore, if

\begin{equation}
c_1(0, 0) > 0,
\end{equation}

then the solution is the uniform limit of the successive approximations (4).

In (6), $x$ [or $y$] tends to $+0$ through the set of values on which $\sigma_x$ [or $\tau_y$] exists.

Remark 1. (***) is valid if $f, z, p, q, \sigma, \tau$ are n-vectors (say $z = (z^1, \ldots, z^n)$ and either $|z| = \sum_{k=1}^{n} |z^k|$ or $|z| = \max(|z^1|, \ldots, |z^n|)$).

Remark 2. A modification of an example of Perron [7] in the theory of ordinary differential equations will show that (***) is false if
$c_1 = \text{const.} > 1$, $c_2 = c_3 = 0$ (so that $f$ does not depend on $p,q$). Also, a modification of an example of Haviland [4] shows that successive approximations need not converge if $c_1 = \text{const.} > 1$, $c_2 = c_3 = 0$.

The proof of (*) will be given in Sections 2-4 below; that of (**) in Sections 5-6; finally, the proof of the last remark will be indicated in Section 7.

The results above answer some questions suggested by Professor P. Hartman. I also wish to acknowledge helpful discussions with him.

2. Proof of (*). Preliminaries. In the proof of (*) below, there is no loss of generality in supposing that $\varphi$ is bounded, say $0 \leq \varphi(x,y,z,p,q) \leq 2N$ on $D$. For otherwise $\varphi$ can be replaced by $\overline{\varphi}$, where $\overline{\varphi}(x,y,z,p,q)$ equals $\varphi(x,y,z,p,q)$ or $2N$ according as $\varphi(x,y,z,p,q)$ does or does not exceed $2N$.

It is clear that $\overline{\varphi}$ is continuous and non-decreasing in each of the variables $z,p,q$. Furthermore, the only solution $z(x,y)$ of

\[
(3') \quad z(x,y) = \int_0^x \int_0^y \overline{\varphi}(s,t,x(s,t),z_x(s,t),z_y(s,t)) \, ds \, dt
\]

on any rectangle $R_{\alpha\beta} : 0 \leq x \leq \alpha (\leq a)$, $0 \leq y \leq \beta (\leq b)$ is $z \equiv 0$.

In order to see this, note that $\varphi(x,y,0,0,0) \equiv 0$, so that there exists an $\varepsilon > 0$ such that $0 \leq \varphi(x,y,z,p,q) \leq 2N$ if $|z|, |p|, |q| < \varepsilon$.

Suppose that $z(x,y) \neq 0$ is a solution of (3') on $R_{\alpha\beta}$. Let $d, 0 \leq d \leq (a^2 + b^2)^{\frac{1}{2}}$, be the largest value of $r$ for which $z(x,y) \equiv 0$ in the intersection $S_r$ of $x^2 + y^2 \leq r^2$ and $R_{\alpha\beta}$. If $U$ is any neighborhood of $S_d$ (relative to $R_{\alpha\beta}$), there exists a rectangle $R_{\gamma_6}$ in $U$ on which $z \neq 0$.

Since $z \equiv 0$ on $S_d$, it is clear that if $U$ is "sufficiently small", then, on $U$ (hence on $R_{\gamma_6}$), $|z| < \varepsilon$ and, almost everywhere, $|z_x| + |z_y| < \varepsilon$.

But then $z \neq 0$ is a solution of (3) on $R_{\gamma_6}$. Since this is impossible, the only solution of (3') on $R_{\alpha\beta}$ is $z \equiv 0$. 
It will be convenient to have the following notation. \( R_1 \) denotes a subset (not always the same) of \( R \) of the form \( E \times [0,b] \), where \( E \) is a (Lebesgue) measurable subset of \( [0,a] \) with \( \text{meas } E = a \). Similarly, \( R_2 \) is a subset (not always the same) of the form \( [0,a] \times E \), where \( E \) is a measurable subset of \( [0,b] \) and \( \text{meas } E = b \). Partial derivatives \( z_x, z_y \) of a function \( z \) will be denoted by \( p, q \).

3. Lemma for (\(*\)). The proof of (\(*\)) will depend on the following lemma.

**Lemma 1.** Let \( \alpha(x,y), \beta(x,y), \gamma(x,y) \) be non-negative, measurable functions defined on \( R, R_1, R_2 \), respectively, such that \( \alpha \) is continuous, \( \beta \) is uniformly Lipschitz continuous with respect to \( y \) and \( \gamma \) is uniformly Lipschitz continuous with respect to \( x \). In addition, let

\[
\begin{align*}
\alpha(x,y) & \leq \int_0^x \int_0^y \varphi(s,t,\alpha(s,t),\beta(s,t),\gamma(s,t))dsdt, \\
\beta(x,y) & \leq \int_0^y \varphi(s,t,\alpha(s,t),\beta(s,t),\gamma(x,t))dt, \\
\gamma(x,y) & \leq \int_0^x \varphi(s,y,\alpha(s,y),\beta(s,y),\gamma(s,y))ds,
\end{align*}
\]

where \( \varphi \) satisfies the conditions of (\(*\)) and is bounded. Then \( \alpha = \beta = \gamma = 0 \).

Note that the Lipschitz continuity of \( \beta \) [or \( \alpha \)] with respect to \( y \) [or \( x \)] is assumed to be uniform with respect to \( x \) and \( y \).

The proof of the lemma below follows a suggestion made by R. Sackssteder. My original proof, which will be omitted, depended on two results. The first result is an existence theorem for

\[
\begin{align*}
z(x,y) = \psi(x,y) + \int_0^x \int_0^y \varphi(s,t,z(s,t),p(s,t),q(s,t))dsdt,
\end{align*}
\]

where \( \psi \) is a non-negative, uniformly Lipschitz continuous function which is non-decreasing in \( x \) and in \( y \). This existence theorem is proved by using the successive approximations \( z_0 = \psi(x,y) \) and

\[
\begin{align*}
z_n(x,y) = z_0(x,y) + \int_0^x \int_0^y \varphi(s,t,z_{n-1},p_{n-1},q_{n-1})dsdt
\end{align*}
\]
which satisfy

\[(12) \quad z_n \leq z_{n+1}, \quad p_n \leq p_{n+1}, \quad q_n \leq q_{n+1}\]

The second result is the fact that if $\Psi$ is replaced by another function $\tilde{\Psi}$ with similar properties and

\[(13) \quad \Psi \leq \tilde{\Psi}, \quad \Psi_x \leq \tilde{\Psi}_x, \quad \Psi_y \leq \tilde{\Psi}_y,
\]

then the corresponding solution $\tilde{z}$ satisfies

\[(14) \quad z \leq \tilde{z}, \quad p \leq \tilde{p}, \quad q \leq \tilde{q}.
\]

**Proof.** Define sequences of successive approximations as follows: Let

\[(15) \quad z_0(x,y) = \alpha(x,y), \quad u_0(x,y) = \beta(x,y), \quad v_0(x,y) = \gamma(x,y)
\]

and, for $n \geq 1$,

\[(16) \quad z_n(x,y) = \int_0^x \int_0^y \varphi(s,t,z_{n-1}(s,t),u_{n-1}(s,t),v_{n-1}(s,t)) \, ds \, dt,
\]

\[(17) \quad u_n(x,y) = \int_0^y \varphi(x,t,z_{n-1}(x,t),u_{n-1}(x,t),v_{n-1}(x,t)) \, dt,
\]

\[(18) \quad v_n(x,y) = \int_0^x \varphi(s,y,z_{n-1}(s,y),u_{n-1}(s,y),v_{n-1}(s,y)) \, ds.
\]

The functions $z_n$, $u_n$, $v_n$ are defined on sets $R$, $R_1$, $R_2$, respectively, which can be taken independent of $n$. The inequalities (7), (8), (9) give the case $n = 0$ of

\[(19) \quad z_0 \leq z_1, \quad u_0 \leq u_1, \quad v_0 \leq v_1
\]

The cases $n > 0$ of these inequalities follow by induction by virtue of the monotonicity of $\varphi$.

The boundedness of $\varphi$ implies the uniform boundedness of the functions $z_n$, $u_n$, $v_n$. Hence, as $n \to \infty$

\[(20) \quad z = \lim z_n, \quad u = \lim u_n, \quad v = \lim v_n,
\]

exist on $R$, $R_1$, $R_2$, respectively. It is clear from (15) and (19), (20) that

\[(21) \quad 0 \leq \alpha \leq z, \quad 0 \leq \beta \leq u, \quad 0 \leq \gamma \leq v.
\]

Lebesgue's theorem on term-by-term integration under bounded convergence implies
(22) \[ z(x,y) = \int_0^x \int_0^y \varphi(s,t,z(s,t),u(s,t),v(s,t))dsdt, \]
(23) \[ u(x,y) = \int_0^y \varphi(x,t,z(x,t),u(x,t),v(x,t))dt, \]
(24) \[ v(x,y) = \int_0^x \varphi(s,y,z(s,y),u(s,y),v(s,y))ds. \]

It is clear that \( z_x = u, z_y = v \) almost everywhere. Thus the assumption on \( \varphi \) concerning (6) shows that \( z = u = v = 0 \). Lemma 1 follows from (21).

4. Proof of (*). (i). Let \( z(x,y) \) be a solution of (1). There exist functions \( u(x,y), v(x,y) \) defined on sets \( R_1, R_2 \), respectively, such that

(25) \[ z(x,y) = \sigma(x) + \tau(y) - z_0 + \int_0^x \int_0^y f(s,t,z(s,t),u(s,t),v(s,t))dsdt, \]
(26) \[ u(x,y) = \sigma_x(x) + \int_0^y f(x,t,z(x,t),u(x,t),v(x,t))dt, \]
(27) \[ v(x,y) = \tau_y(y) + \int_0^x f(s,y,z(s,y),u(s,y),v(s,y))ds, \]

and the relations \( u = z_n, v = z_y \) hold almost everywhere. In order to see this, note that almost everywhere on \( R \),

\[ z_x(x,y) = \sigma_x(x) + \int_0^y f(x,t,z(x,t),u(x,t),v(x,t))dt, \]
\[ z_y(x,y) = \sigma_y(y) + \int_0^x f(s,y,z(s,y),u(s,y),v(s,y))ds. \]

The expressions on the right side of these equations are defined for \( (x,y) \) on sets \( R_1, R_2 \), respectively. Define \( u(x,y), v(x,y) \) to be these expressions on \( R_1, R_2 \). In particular \( z_x = u \) and \( z_y = v \) almost everywhere. Hence (26), (27) hold on (possibly different) sets \( R_1, R_2 \). Clearly (25) is valid for all \( (x,y) \) on \( R \).

(ii) Uniqueness in (*). Suppose that (1) possesses two solutions \( z = z_1(x,y), z_2(x,y) \) on \( R \). Let \( u_1(x,y), v_1(x,y) \) and \( u_2(x,y), v_2(x,y) \) be the functions associated with \( z_1, z_2 \) by (i). Let \( \alpha = |z_1 - z_2|, \beta = |u_1 - u_2|, \gamma = |v_1 - v_2| \). If the relations (25) for \( z = z_1, z_2 \) are subtracted, it is seen that the inequality (2) for \( f \) implies (7). Similarly
(26), (27) imply (8), (9) respectively.

The functions $a$, $b$, $c$ satisfy the assumptions of Lemma 1. Hence the uniqueness assertion in (*) follows from Lemma 1.

(iii). **Existence and successive approximations.** Let $z_0(x,y)$, $z_1(x,y), \ldots$ be the successive approximations defined by (4). Corresponding to each $z_n(x,y)$, it is possible to introduce functions $u_n(x,y)$, $v_n(x,y)$ defined on sets $R_1$, $R_2$, respectively, and satisfying $u_0 = \sigma_x(x)$, $v_0 = \sigma_y(y)$,

$$z_n(x,y) = \sigma(x) + \tau(y) - z_0 + \int_0^y \int_0^x f(s,t,z_{n-1}(s,t)) ds dt,$$

$$u_n(x,y) = \sigma(x) + \int_0^y f(x,t,z_{n-1}(x,t),u_{n-1}(x,t),v_{n-1}(x,t)) dt,$$

$$v_n(x,y) = \tau(y) + \int_0^x f(s,y,z_{n-1}(x,t),u_{n-1}(s,y),v_{n-1}(x,t)) ds.$$

The sets $R_1$, $R_2$ can be assumed to be independent of $n$.

Let $z_m = |z_m - z_n|$, $U_{mn} = |u_m - u_n|$, $V_{mn} = |v_m - v_n|$ and

$$a_k(x,y) = \frac{1}{m,n \geq k} z_{mn}, \quad b_k(x,y) = \frac{1}{m,n \geq k} U_{mn}, \quad c_k(x,y) = \frac{1}{m,n \geq k} V_{mn}.$$

It is clear that $z_{mn}$, $U_{mn}$, $V_{mn}$ are uniformly Lipschitz continuous with respect to $(x,y), (s,t)$, respectively, and that a corresponding statement holds for $a_k$, $b_k$, $c_k$.

By subtracting the relation $(26)$ from $(28) - 1$ and using the inequality (2) for $f$, it is seen that

$$z_{mn}(x,y) \leq \int_0^x \int_0^y \varphi(s,t,z_{m-1}(s,t),u_{m-1}(s,t),v_{m-1}(s,t)) ds dt.$$

Thus, if $m,n \geq k$, the monotony of $\varphi$ shows that

$$z_{mn}(x,y) \leq \int_0^x \int_0^y \varphi(s,t,a_{k-1}(s,t),b_{k-1}(s,t),c_{k-1}(s,t)) ds dt.$$

Hence

$$a_k(x,y) \leq \int_0^x \int_0^y \varphi(s,t,a_{k-1}(s,t),b_{k-1}(s,t),c_{k-1}(s,t)) ds dt.$$
Similarly
\[ \beta_k(x,y) \leq \int_0^y \varphi(x,t) \alpha_k^{-1}(x,t) \beta_k^{-1}(x,t) \gamma_k^{-1}(x,t) \, dt, \]
\[ \gamma_k(x,y) \leq \int_0^x \varphi(s,y) \alpha_k^{-1}(s,y) \beta_k^{-1}(s,y) \gamma_k^{-1}(s,y) \, ds. \]
By (31), the sequences \( \{ \alpha_k(x,y) \} \), \( \{ \beta_k(x,y) \} \), \( \{ \gamma_k(x,y) \} \) are non-increasing (and non-negative). Let \( \alpha(x,y) \), \( \beta(x,y) \), \( \gamma(x,y) \) denote the respective limits of these sequences. The Lipschitz continuity of \( \alpha_k \), \( \beta_k \), \( \gamma_k \) is preserved under the limiting process. Lebesgue's theorem on term-by-term integration under bounded convergence gives the inequalities (7), (8), (9). Hence Lemma 1 shows that \( \alpha = 0 \), \( \beta = 0 \), \( \gamma = 0 \) on \( R, R_1, R_2 \), respectively. This implies the existence of the functions \( z = \lim z_n \), \( u = \lim u_n \), \( v = \lim v_n \) on \( R, R_1, R_2 \), as \( n \to \infty \), satisfying (25), (26), (27). It is clear that the limit function \( z(x,y) \) is a solution of (1).

Finally, the equicontinuity of the functions \( z_n(x,y) \) (implied by their uniform Lipschitz continuity) shows that \( z(x,y) \) is the uniform limit of the \( z_n(x,y) \). This proves (*)

5. Lemma for (**). The proof of (**) will depend on the following lemma:

Lemma 2. Let \( \alpha(x,y) \), \( \beta(x,y) \), \( \gamma(x,y) \) be non-negative, measurable functions defined on \( R, R_1, R_2 \), respectively, so that \( \alpha \) is continuous, \( \beta \) is uniformly Lipschitz continuous with respect to \( y \) and \( \gamma \) is uniformly Lipschitz continuous with respect to \( x \). Furthermore, assume that

\[ \alpha(x,y)/xy > 0 \text{ as } 0 < xy \to 0 \]

and that, uniformly with respect to \( x \) and \( y \), respectively,

\[ \beta(x,y)/y > 0 \text{ as } y \to 0 \text{ and } \gamma(x,y)/x \to 0 \text{ as } x \to 0. \]

Finally, suppose that
(34) \[ \alpha(x,y) \leq \int_0^x \int_0^y \left\{ c_1(s,t) \alpha(s,t)/st + c_2(s,t) \beta(s,t)/t + c_3(s,t) \gamma(s,t)/s \right\} ds dt, \]

(35) \[ \beta(x,y) \leq \int_0^y \left\{ c_1(x,t) \alpha(x,t)/xt + c_2(x,t) \beta(x,t)/t + c_3(x,t) \gamma(x,t)/x \right\} dt, \]

(36) \[ \gamma(x,y) \leq \int_0^x \left\{ c_1(s,y) \alpha(s,y)/sy + c_2(s,y) \beta(s,y)/y + c_3(s,y) \gamma(s,y)/s \right\} ds, \]

where \( c_1, c_2, c_3 \) are as in the first part of (**) Then \( \alpha = \beta = \gamma = 0. \)

Proof. By (32), if \( \alpha(x,y)/xy \) is defined as 0 when \( xy = 0 \), it becomes a continuous function on \( R \). Hence, it assumes its maximum \( M_1 \) at some point \( (x^1, y^1) \in R \). Let \( M_2 = \max_{x \in R} \beta(x,y)/y \) and \( M_3 = \max_{x \in R} \gamma(x,y)/x \) for \( (x,y) \in R \).

Note that there exist numbers \( M_{j,k} \), where \( j, k = 1, 2, 3 \), satisfying

(37) \[ M_{j,k} > 0 \text{ and } \sum_{k=1}^{3} M_{j,k} = 1 \text{ for } j = 1, 2, 3, \]

and

(38) \[ M_j \leq \sum_{k=1}^{3} M_{j,k} M_k. \]

If \( M_1 \neq 0 \), then \( M_1 = \alpha(x^1, y^1)/x^1 y^1 \) holds for some point \( (x^1, y^1) \) of \( R \) with \( x^1 y^1 > 0 \). In this case, (38) follows from (34) with \( (x,y) = (x^1, y^1) \) if

(39) \[ M_{1,k} = (x^1 y^1)^{-1} \int_0^1 \int_0^1 c_k(s,t) ds dt. \]

If \( M_1 = 0 \), let \( M_{1,k} = c_k(0,0). \)

In order to obtain (38), let \( (x_j, y_j) \), where \( j = 1, 2, \ldots \), be points of \( R \) such that \( \lim (x_j, y_j) = (x^2, y^2) \) exists, \( \lim \beta(x_j, y_j)/y_j = M_2 \) and \( \lim \beta(x_j, y) = \beta(y) \) exists uniformly for \( 0 \leq y \leq b. \) Then (35) leads to (38) with

(40) \[ M_{2,k} = (y^2)^{-1} \int_0^y c_k(x^2, t) dt \text{ or } M_{2,k} = c_k(x^2, 0) \]

according as \( y^2 > 0 \) or \( y^2 = 0 \). A relation of the type \( (38\_3) \) is obtained
similarly.

Let \( M_j = \max (M_1, M_2, M_3) \). Suppose, if possible, that \( M_j > 0 \). Assume, for the moment, that \( M_j > M_j \) if \( j \neq J \). Then, by (37) and (38), \( M_{ij} = 1 \) and \( M_{jk} = 0 \) for \( k \neq J \). But the derivation of (38) can then be modified to obtain \( M_j < M_j \). For example, if \( J = 1 \), then \( c_1(s,t) = 1 \) and \( c_2(s,t) = c_3(s,t) = 0 \) in (34) when \( (x,y) = (x^1, y^1) \), while \( \alpha(s,t)/st \) is nearly zero for small \( st \), so that one obtains \( M_1 < M_1 \). Or if \( J = 2 \), then \( y^2 > 0 \) and \( c_1(x^2,t) = 1 \), \( c_2(x^2,t) = c_3(x^2,t) = 0 \) for \( 0 \leq t \leq y^2 \), while the relations

\[
\beta(y) \leq \int_0^y \beta(t)dt/t, \quad \beta(y^2)/y^2 = M_2
\]
give \( M_2 < M_2 \) since \( \beta(t)/t \) is nearly 0 for small \( t \) by the uniformity of the first limit relation in (33).

Similar arguments show that if two or three of the numbers \( M_1, M_2, M_3 \) are equal to \( M_j > 0 \), one is led to a contradiction. Hence \( M_j = 0 \). This proves the lemma.

6. Proof of (**). (i). Uniqueness in (**). Let \( z = z_1(x,y), z_2(x,y) \) be two solutions of (1) on \( R \). Let \( u_1(x,y), v_1(x,y) \) and \( u_2(x,y), v_2(x,y) \) be the functions associated with them as in the proof of (*).

Let \( \alpha = |z_1 - z_2|, \beta = |u_1 - u_2|, \gamma = |v_1 - v_2| \). It will be verified that, as \( x \) (or \( y \)) \( \to 0 \), then except for sets of measure zero,

\[
(41) \quad \alpha(x,y), \beta(x,y), \gamma(x,y) \to 0.
\]

Consider the case \( x \to 0 \). The assertions (41) concerning \( \alpha \) and \( \gamma \) are clear. In order to verify assertion (41) for the function \( \beta \), it will first be shown that if \( z = z(x,y) \) is any solution of (1) (say, \( z = z_1 \) or \( z = z_2 \)) and if \( u(x,y), v(x,y) \) are its associated functions, then

\[
(42) \quad \lim u(x,y) = \rho(y), \text{ as } x \to 0, \text{ exists uniformly in } y.
\]
To see this, let $x_j$, where $j = 1, 2, 3, \ldots$ be a sequence of $x$ values such that $\lim x_j = 0$ and $\lim u(x_j, y) = \rho(y)$ exists uniformly as $j \to \infty$. Putting $x = x_j$ in (26) and letting $j \to \infty$, it is seen that

$$\rho(y) = \sigma_x(pos) + \int_0^y f(0, t, \tau(t), \rho(t), \tau_y(t)) \, dt.$$  

We note that $\rho(y)$ is continuous. Furthermore, $\rho(y)$ does not depend on the sequence $x_1, x_2, \ldots$. Suppose that another sequence leads to a different limit $\overline{\rho}(y) \neq \rho(y)$. By substituting $\overline{\rho}$ for $\rho$ in (43), and subtracting, we get

$$|\overline{\rho}(y) - \rho(y)| \leq \int_0^y |f(0, t, \tau(t), \overline{\rho}(t), \tau_y(t)) - f(0, t, \tau(t), \rho(t), \tau_y(t))| \, dt.$$  

Since $f, \rho, \overline{\rho}$ are continuous and $\rho(0) = \overline{\rho}(0) = \sigma_x(pos)$, the integrand of (44) can be made small by making $y$ small. Hence

$$|\overline{\rho}(y) - \rho(y)|/y \to 0, \text{ as } y \to 0.$$  

By relation (5),

$$|\overline{\rho}(y) - \rho(y)|/y \leq y^{-1} \int_0^y c_2(0, t)|\overline{\rho}(t) - \rho(t)| \, dt/t,$$

Using (45) as before, this leads to a contradiction. Hence $\overline{\rho} = \rho$.

Therefore every sequence, for which the limit in (42) exists, leads to the same limit. Hence (42) holds.

If $\lim u_1(x, y) = \rho_1(y)$ and $\lim u_2(x, y) = \rho_2(y)$, as $x \to 0$, we can repeat the above argument and obtain $\rho_1 = \rho_2$. This completes the verification of (41).

We now verify assumptions (32) and (33) of Lemma 2. Consider, for example, the assertion

$$\beta(x, y)/y \to 0 \text{ as } y \to 0.$$  

By putting $u = u_1, u_2$ in (26) and subtracting we get

$$\beta(x, y) \leq \int_0^y |f(x, t, z_1(x, t), u_1(x, t), v_1(x, t)) - f(x, t, z_2(x, t), u_2(x, t), v_2(x, t))| \, dt.$$
Now the integrand of (47) can be made small, by making \( \gamma \) small, and using (41). This proves (46). The other limits in (32) and (33) are verified similarly. The other assumptions of Lemma 2 are quite straightforward. Therefore \( \alpha = \beta = \gamma = 0 \). This proves "uniqueness".

(ii). **Existence and successive approximations in (**)**. Let \( z_0(x,y), z_1(x,y), \ldots \), be the successive approximations defined by (4). Corresponding to \( z_n(x,y) \) it is possible to introduce, as in the proof of (**), functions \( u_n(x,y), v_n(x,y) \) defined on sets \( R_1, R_2 \) (independent of \( n \)) and satisfying \( u_0 = \sigma_x(x), v_0 = \tau_y(y), (28_n), (29_n) \) and \( (30_n) \). Let \( Z_{mn}, U_{mn}, V_{mn} \) be defined as in the existence proof of (*) above. It will be verified that, given \( \varepsilon \), there exists a \( \Phi(\varepsilon) \) and an \( N(\varepsilon) \), such that

\[
Z_{mn}(x,y), U_{mn}(x,y), V_{mn}(x,y) < \varepsilon
\]

for \( x < \Phi(\varepsilon) \) and for all \( m,n > N(\varepsilon) \). A similar statement will be seen to hold when \( x \) is replaced by \( y \). The assertion (48) concerning \( Z_{mn} \) and \( V_{mn} \) is clear. In order to verify (48) for the function \( U_{mn} \) it will first be shown that

\[
\lim_{n} u_n(x,y) = h_n(y), \text{ as } x \to 0, \text{ exists uniformly in } y \text{ and } n.
\]

It is easily verified, by induction, that \( h_n(y) \) exists uniformly in \( y \) for fixed \( n \), where

\[
h_n(y) = \sigma_x(y) + \int_0^y f(0,t,\tau(t), h_{n-1}(y), \tau_y(t)) dt.
\]

To see the uniformity in \( n \), define

\[
\bar{u}_n(x,y) = z_n(x,y) - \sigma(x) - \tau(y) + z_0; \quad \bar{u}_n(x,y) = u_n(x,y) - \sigma_x(x);
\]

\[
\bar{v}_n(x,y) = v_n(x,y) - \tau_y(y);
\]

\[
g(x,t,z,p,q) = f(x,y,z + \sigma(x) + \tau(y) - z_0, p + \sigma_x(x), q + \tau_y(y)).
\]

For \( \bar{u}_n \) we define \( h_n \) corresponding to \( h \). Clearly \( g \) satisfies a condition
analogous to (5), \( \overline{u}_0(x,y) = \overline{u}_0(y) = 0 \), and

\[
(53_n) \quad \overline{u}_n(x,y) = \int_0^y g(x,t,\overline{u}_{n-1}(x,t),\overline{v}_{n-1}(x,t))dt, \quad n \geq 1
\]

\[
(54_n) \quad \overline{v}_n(y) = \int_0^y g(0,t,0,\overline{u}_{n-1}(t),0)dt, \quad n \geq 1.
\]

To prove (49) it suffices to verify that

\[
(55) \quad \lim_{x \to 0} \overline{u}_n(x,y) = \overline{u}_n(y), \quad \text{as } x \to 0, \text{ exists uniformly in } y \text{ and } n.
\]

By subtracting (54n) from (53n), it is seen that

\[
(56) \quad |\overline{u}_n(x,y) - \overline{u}_n(y)| \leq \int_0^y \left\{ |g_1 - g_2| + |g_2 - g_3| \right\} dt
\]

where \( g_1 = g(x,t,\overline{z}_{n-1}(x,t),\overline{y}_{n-1}(x,t)), \quad g_2 = g(0,t,0, \overline{u}_{n-1}(x,t), \overline{v}_{n-1}(x,t)), \quad g_3 = g(0,t,0, \overline{u}_{n-1}(t),0) \). We note that, given \( \epsilon > 0 \), there exists a \( \delta(\epsilon) \) such that \( |g_1 - g_2| < \epsilon \) if \( x < \delta \) for all \( y \) and \( n \).

Hence, noting (5),

\[
(57_n) \quad |\overline{u}_n(x,y) - \overline{u}_n(y)| \leq \int_0^y \left\{ \epsilon + t^{-1} c_2(0,t)|\overline{u}_{n-1}(x,t) - \overline{u}_{n-1}(t)| \right\} dt.
\]

By continuity, because of (6*), \( c_2(0,t) < 1 \) for small \( t > 0 \). Hence there exists a number \( \delta, 0 < \delta < 1 \), such that

\[
\int_0^y c_2(0,t)dt \leq \delta y \text{ for } 0 < y \leq b.
\]

A simple induction shows that

\[
(58) \quad |\overline{u}_n(x,y) - \overline{u}_n(y)| \leq (1 - \delta^n) \epsilon y / (1 - \delta) \leq b \epsilon / (1 - \delta).
\]

This proves (55). Hence (49) is established.

Next we note that \( \overline{u}_n(y), \quad n = 0,1,2, \ldots, \) are the successive approximations for the initial value problem

\[
(59) \quad dw/dt = F(t,w), \quad w(0) = \sigma_x(+0),
\]

where \( F(t,w) = f(0,t,\overline{u}_y(t),\overline{v}_y(t)), \quad F(t,w) = f(0,0,\overline{u}(0),\overline{w}, \overline{v}(+0)) \) as \( t \to +0 \). The proof of the main theorem in [8] shows that these successive approximations converge uniformly, (60) being is bounded, measurable and continuous in \( w \) (for almost all fixed \( t \)).
Nagumo's uniqueness condition (cf. [4], p. 97). Hence

\[ \lim_{n \to \infty} h_n(y) = h(y), \text{ exists uniformly in } y \text{ as } n \to \infty. \]

Now (61) and (49) together verify (48) for \( U_{mn}(x,y) \). Hence (48) is established.

By an argument similar to that used in verifying (46) it is seen that, given \( \epsilon > 0 \), there exists \( \delta(\epsilon) \) such that

\[ (xy)^{-1} Z_{mn}(x,y) < \epsilon, \text{ for } xy < \delta(\epsilon), \text{ for } m,n > N(\epsilon) \]

(62)

\[ x^{-1} U_{mn}(x,y) < \epsilon, \text{ for } x < \delta(\epsilon), \text{ for } m,n > N(\epsilon) \]

\[ y^{-1} V_{mn}(x,y) < \epsilon, \text{ for } y < \delta(\epsilon), \text{ for } m,n > N(\epsilon). \]

Now defining \( \alpha_k, \beta_k, \gamma_k \) as in (31), we note that we can substitute them for \( Z_{mn}, U_{mn}, V_{mn} \), respectively, in (62) changing \( m,n > N(\epsilon) \) to \( k > N(\epsilon) \). Proceeding as in the analogous section of the proof of theorem (*), we conclude that \( \alpha, \beta, \gamma \) satisfy (34), (35) and (36) and also (32) and (33). Therefore, by Lemma 2, the successive approximations converge uniformly to a solution of (1).

7. Counter-examples. (a). Let \( a = b = 1, 1 + \epsilon = \delta^2, \epsilon > 0, \delta > 1 \).

Let \( f(x,y,z,p,q) \) be independent of \( p, q \) and defined by

\[
f(x,y,z,p,q) = \begin{cases} 
0 & \text{if } (x,y) \in R, z \leq 0, \\
(1 + \epsilon) z/xy & \text{if } (x,y) \in R, 0 < z < (xy)^\delta, \\
(1 + \epsilon) (xy)^{\delta - 1} & \text{if } (x,y) \in R, (xy)^\delta \leq z.
\end{cases}
\]

Then \( f(x,y,z,p,q) \) is continuous and satisfies (5) for \( c_1(x,y) = 1 + \epsilon, \) (and \( c_2 = c_3 = 0 \)). Let \( \sigma(x) = U(y) = 0 \). Then (1) has an infinity of solutions, namely, \( z = c(xy)^\delta \), where \( 0 < c < 1 \). (b). Let \( a = b = 1, \)

\[ R^0 = \{(x,y) : 0 < x, y \leq 1 \}, 1 + \epsilon = \delta^2, \epsilon > 0, \quad R^0 \]

\[ f(x,y,z,p,q) = \begin{cases} 
0 & \text{if } x = 0, y = 0, \\
(xy)^\delta - 1 & \text{if } (x,y) \in R^0, z < 0, \\
(xy)^\delta - 1 - (1 + \epsilon) z/xy & \text{if } (x,y) \in R^0, 0 \leq z \leq (xy)^\delta, \\
- \epsilon (xy)^\delta - 1 & \text{if } (x,y) \in R^0, (xy)^\delta < z.
\end{cases}
\]
Then $f(x, y, z, p, q)$ satisfies the same relation (5) as in example (a). However, in (4), $z_{2n} = 0$, $z_{2n + 1} = (xy)^{3/8^2}$, so that the successive approximations (4) do not converge.

THE JOHNS HOPKINS UNIVERSITY

References


Mathematics Genealogy Project

John P. Shanahan
MathSciNet

Ph.D. The Johns Hopkins University 1962

Dissertation:

Advisor 1: Philip Hartman

No students known.

If you have additional information or corrections regarding this mathematician, please use the update form. To submit students of this mathematician, please use the new data form, noting this mathematician's MGP ID of 151444 for the advisor ID.

The Mathematics Genealogy Project is in need of funds to help pay for student help and other associated costs. If you would like to contribute, please donate online using credit card or bank transfer or mail your tax-deductible contribution to:

Mathematics Genealogy Project
Department of Mathematics
North Dakota State University
P. O. Box 6050
Fargo, North Dakota 58108-6050