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PART I

Recent Work on Families of Periodic Solutions of
Differential Equations.

PART II

Singular Solutions of Systems of Real Analytic Equations.

PART III

The Strömgren-Wintner Principle on the Termination
of Families of Periodic Solutions.

by

Daniel C. Lewis, Jr.

Contract Number: AF 18(600)-665

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This report gives a comprehensive survey of an attempted global theory of perturbation of periodic solutions of systems of ordinary differential equations containing a parameter. Various other topics of independent interest related to this theory were also treated in the papers prepared under this contract but can not be abstracted here.

Qualified requestors may obtain copies of this report from the ASTIA Document Service Center, Arlington Hall Station, Arlington 12, Virginia. Department of Defense contractors must be established for ASTIA services, or have their "need-to-know" certified by the cognizant military agency of their project or contract.
All the definite results obtained under The United States Air Force Contract Number AF 18 (600)-665, Project Number R-354-10-35, can be subsumed under the general title "Recent Work on Families of Periodic Solutions of Ordinary Differential Equations." This title has therefore been adopted for Part I of this Technical Report. In this Part I, we give a general summarizing account of all the results obtained under this contract which appear to be worth publishing, with suitable references to all work that actually has been published as well as to the more recent results which are now presented in detail under Part II of this Report, entitled "Singular Solutions of Systems of Real Analytic Equations", and under Part III, entitled "The Strömgren-Wintner Principle on the Termination of Families of Periodic Solutions." It is planned eventually to submit Parts II and III, possibly with some revisions, for publication elsewhere.
PART I

RECENT WORK ON FAMILIES OF PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

BY

Daniel C. Lewis, Jr.

We are concerned with systems of differential equations of the form

\[ \frac{dx}{dt} = f(x, t, \mu) \]

where \( x \) and \( f \) are vectors, while \( t \) and \( \mu \) are real variables. \( f \) is always assumed to be at least of class \( C^1 \) in \( x, t, \) and \( \mu, \) although, of course, the more complete results are obtained under more stringent restrictions, such as analyticity.

The case where \( f(x, t, \mu) \) does not depend explicitly on \( t \) is known as the autonomous case. It is then possible to have periodic solutions of various periods.

If, however, \( f(x, t, \mu) \) does depend explicitly on \( t, \) it is assumed to be periodic in \( t \) with a prescribed period \( T \) and the system in such circumstances is called non-autonomous. Since it is always possible to carry out the transformation \( t' = 2\pi T^{-1}t \) on the independent variable, we usually, as here, assume without loss of generality that \( T = 2\pi. \) The only periodic solutions of (1) to be expected in the non-autonomous case would then have period \( 2\pi \) or an integral multiple of \( 2\pi. \)

This cleavage into the two cases, autonomous and non-autonomous, is traditional and rather fundamental, although two of the minor results of the present work are to the effect that, on the one hand, the local theory of the autonomous case can be reduced to the non-autonomous case with reduction by one unit in the order of the system (cf. Paper 2, p. 812) and,
on the other hand, the non-autonomous case can always be reduced to the autonomous case with augmentation by one unit in the order of the system (cf. Paper 6, pp. 12-14).

Poincaré considered the problem of proving the existence of periodic solutions of (1) when the parameter $\mu \neq 0$ from a known periodic solution with $\mu = 0$. His fundamental result can be most simply stated by first defining the degeneracy of the given periodic solution $S$ to be the number of linearly independent periodic solutions of the variational equations corresponding to $S$. Poincaré showed that if the periodic solution for $\mu = 0$ was non-degenerate (i.e. had degeneracy 0), there must also be periodic solutions for $|\mu| < k$, where $k$ is a sufficiently small positive number. He also obtained the same result for certain cases of degeneracy $> 0$; but in no case did he give any explicit estimate as to how large $k$ could be taken. Our main results are related to just this question of estimating $k$. In Paper 1, in which we were concerned entirely with the non-degenerate case, we gave an explicit estimation of $k$, which was shown to be, in a certain sense, the best possible estimate. In Paper 2, we also obtained an explicit estimate for what might be considered the general degenerate case, but we were unable in this case to prove that our estimate was the best possible one. Finally in Paper 3, we did the same thing for degenerate cases of a more restricted class, which, however, is sufficiently extensive to be of the first order of importance. In fact Paper 3 is concerned with systems, like the equations of celestial mechanics, with a certain number $p$ of independent first integrals. If $S$ is a periodic solution of (1) with $\mu = 0$, its degeneracy is known to be $\leq p$. Our primary purpose was to estimate $k$ when the degeneracy was exactly $p$. This paper stands in close relationship to some work by Ernst Hölder and affords considerable clarification of the meaning of

These results on the estimation of \( k \) turned out, however, to be very disappointing when applied to practical problems. For example in the case of van der Pol's equation \( x'' + \mu(1 - x^2)x' + x = 0 \) the best value to be obtained by these methods for \( k \) was apparently less than .01. It appeared equally futile to consider the asymptotic situation for large \( |\mu| \). Hence in the later stages of our work we considered the global perturbation of periodic solutions without regard to degeneracy for a particular value of \( \mu \). The theory is closely related to the Strömgren-Wintner termination principle. Although it gives no specific estimate for \( k \), it provides a theorem which (in the analytic cases) yields a value for \( k \) under favorable particular circumstances. For instance, we can obtain for the van der Pol equation by this method that \( k = \infty \). See Paper 6, pp. 16 and 17.

Paper 5 is a general discussion of systems of real analytic equations of the form \( F_i(x_1, \ldots, x_m) = 0, \ i = 1, \ldots, m - 1 \). The theory is in a sense of a more general scope than the theory of periodic solutions of the system (1). Yet, in another sense, it may be shown that the problem of the global perturbation of periodic solutions is co-extensive with a global implicit function theory. To this end it may be remarked that the global perturbation of the equilibrium points of the system (1) in the autonomous case is exactly the same as the global implicit function theory of the equations \( f(x, \mu) = 0 \). Since an equilibrium point is a special case of a periodic solution, even though for some purposes it may be desirable to exclude it, the global theory of perturbation of periodic solutions is at least as general as the implicit function theory. On the other hand it can not be more general, because the perturbation theory
of periodic solutions is always reducible to a problem about implicit functions, as for example by way of the so-called bifurcation equations (cf. Paper 3, bottom of page 541 and top of next page). In any event the theory of Paper 5 is a prerequisite to the results given in Paper 6 on the generalized Strömgren-Wintner termination principle.

Paper 4 is off the beaten track of most other parts of our work. It is concerned with the manner in which the period of a family of periodic solutions varies from member to member in the autonomous case. It is here assumed that (1) is of the special form (like the equations of celestial mechanics) which admits a relatively invariant line integral and therefore possesses a first integral. It then turns out that the period is related to this first integral in a very interesting manner, for details of which we must refer to Paper 4 itself.


1. Periodic Solutions of Differential Equations Containing a Parameter.

2. On the Perturbation of a Periodic Solution when the Variational System has Non-trivial Periodic Solutions.

3. On the Role of First integrals in the Perturbation of Periodic Solutions.

   Part II of this Technical Report.

   Part III of this Technical Report.
PART II

SINGULAR SOLUTIONS OF SYSTEMS OF REAL ANALYTIC EQUATIONS

by

Daniel C. Lewis, Jr.

Our problem is primarily to study the solutions of a system of real analytic equations of the form \( G_i(x_1, \ldots, x_n) = 0 \), with \( i = 1, 2, \ldots, m \), in the neighborhood of a known solution, which, without loss of generality, is assumed to be at the origin. The word, "singular," in the title is intended to suggest that no hypotheses will be made on the rank of the jacobian matrix of the \( G \)'s with respect to the \( x \)'s at the origin.

The problem is of fundamental importance in connection with the Strömgren-Wintner principle of natural termination of families of periodic solutions in the equations of celestial mechanics. It was also encountered in the author's attempts to generalize the principle so as to apply to any analytic system of differential equations admitting families of periodic solutions. The main result needed for this purpose is the assertion that, if the given solution at the origin is not isolated (i.e. it is a limit point of other solutions), then there always exist at least two distinct analytic curves with end points at the origin, such that every point on both curves satisfies the system \( G_i(x_1, \ldots, x_n) = 0 \), \( i = 1, \ldots, m \). This assertion is an immediate corollary of Theorems 8 and 9 of this paper.

The fact that this theorem is intuitively evident perhaps can not be denied. For example, in connection with the Strömgren-Wintner termination principle, G. D. Birkhoff, speaks rather vaguely about certain "theoremes
bien connus des fonctions implicites" [Birkhoff 1]. In oral conversation with the author more than a quarter of a century ago, he spoke even more emphatically about the triviality of the matter. Yet there is no known reference in the literature which clears up the matter completely and rigorously. The theorem is actually not true if modified so that analyticity is replaced in hypothesis and conclusion by $C^\infty$. This may be shown by simple examples. Even the treatment of Wintner [Wintner 1] seems to be based on work of Poincaré [Poincaré 1 and 2], which has been criticized in two separate respects by Bliss [Bliss 1] and Osgood [Osgood 1].

The treatment in this paper is based on material in the second volume of Osgood's Lehrbuch der Funktionentheorie on functions of several complex variables. But Osgood, and his predecessor, Weierstrass, were principally concerned with a complex theory; while, for the applications in mind, we must emphasize those matters bearing on the characterization of real solutions. Accordingly, in connection with the obvious fact that a polynomial irreducible in a certain real field is not necessarily irreducible in a corresponding complex field, we use a different concept of reducibility and the whole treatment has to be modified and re-examined.

Our treatment also differs from the work of Bliss, not only in its emphasis on the real theory, but also because of our unwillingness to make a hypothesis on the non-vanishing of certain resultants. We pay for this unwillingness (as Osgood does also) in that most of our theorems are true only for so-called non-specialized variables (cf. Definition 6).

In this paper we are principally concerned with real analytic functions of several real variables. Nevertheless, methods from the classical theory of analytic functions of several complex variables will be introduced. Hence we
must often consider the analytic extension of real analytic functions into
the complex domain, as can always be done with the help of power series
expansions. For this reason, we shall say that a function $f(z_1, \ldots, z_n)$
is real if it takes on only real values when $z_1, \ldots, z_n$ are real, even
though we may often consider this function also for complex values of
$z_1, \ldots, z_n$. If $f(z_1, \ldots, z_n)$ is real (in this sense) and also analytic,
we see from a consideration of power series that the complex domain of
definition contains a domain $R$ symmetric in the real domain and that in $R$
\[
f(z_1, \ldots, z_n) = \overline{f(\overline{z}_1, \ldots, \overline{z}_n)},
\]
where the superscript dash replaces the underneath quantity by its conjugate
complex. A function having this property, common to all real analytic
functions, will be said to be self conjugate. We shall have frequent
occasion to refer to this property.

We also have to consider more general analytic functions of several
complex variables. If $f(z_1, \ldots, z_n)$ is such a function defined, say, in a
(complex) neighborhood of a real point, we can show from power series develop-
ments that there always exist two real analytic functions $\varphi(z_1, \ldots, z_n)$ and
$\psi(z_1, \ldots, z_n)$ such that $f(z_1, \ldots, z_n) = \varphi(z_1, \ldots, z_n) + i\psi(z_1, \ldots, z_n)$. We
now set $\overline{f}(z_1, \ldots, z_n) = \varphi(z_1, \ldots, z_n) - i(z_1, \ldots, z_n)$. This $\overline{f}$ will be
called the conjugate of $f$. It also is an analytic function of $z_1, \ldots, z_n$.
From the reality and self conjugacy of $\varphi$ and $\psi$, it is readily verified that
\[
\overline{f}(\overline{z}_1, \ldots, \overline{z}_n) = \overline{f(z_1, \ldots, z_n)}.
\]
THEOREM 1. Let \( G(x_1, \cdots, x_n) \) be analytic in a neighborhood of the origin, but suppose it does not vanish identically. Then there exists a non-singular linear transformation, \( x_i = \sum_{j=1}^{\infty} c_{ij} z_j^{\lambda_j} \), with real coefficients, and a positive integer \( m \), such that

\[
G(x_1, \cdots, x_n) = \prod_{1}^{n} (z_1, \cdots, z_n) \left[ z^m + A_1(z_1, \cdots, z_{n-1}) z^{m-1} + \cdots + A_m(z_1, \cdots, z_{n-1}) \right]
\]

where \( A_i(z_1, \cdots, z_{n-1}) \), for \( i = 1, 2, \cdots, m \), is analytic in a neighborhood of the origin and vanishes at the origin itself, while \( \Phi \) is also analytic in a neighborhood of the origin but does not vanish there.

Moreover, if \( G(x_1, \cdots, x_n) \) is real (when the \( x \)'s are real), then \( \Phi \) and the \( A_i \)'s are also real (when the \( z \)'s are real).

The first paragraph of this theorem is essentially the famous "preparation theorem" of Weierstrass, which in its most elementary form is concerned with the case where \( G(x_1, \cdots, x_{n-1}, 0) \neq 0 \) and where the above mentioned linear transformation can always then be chosen as the identity transformation. The generalization considered here is also essentially well known (cf. Osgood [1, p. 73] or Weierstrass [1, pp. 135-142]) since in the choice of the coefficients \( c_{ij} \) given by Osgood, or, more originally, by Weierstrass, one is always obviously permitted to make them all real.

The real case mentioned in the second paragraph of the theorem is discussed by Osgood (loc. cit. p. 76), at least for the elementary form of the Weierstrass preparation theorem. The extension to the presently desired form is trivial.
Definition 1. In the sequel we shall discuss polynomials in $w$ of the form

$$w^m + A_1(z_1, \ldots, z_n)w^{m-1} + \cdots + A_n(z_1, \ldots, z_n)$$

in which the coefficients $A_1, A_2, \ldots, A_m$ are analytic functions of $z_1, \ldots, z_n$ in a neighborhood $N$ of the origin, vanish at the origin, and are real (when $z_1, \ldots, z_n$ are real). A polynomial of this type will be called a "real distinguished polynomial." It will be said to be irreducible in the $R$-sense if it can not be written as the product of two other real distinguished polynomials of lower degree.

Notice that the definition of $R$-irreducibility introduced here is different from the ordinary irreducibility of the usual complex theory. Thus the distinguished polynomial $w^2 + z^2$ is $R$-irreducible but is reducible in the ordinary sense, with factors $(w + iz)$ and $(w - iz)$, since in the complex case we do not require all coefficients to be real.

It is important to emphasize that, according to our definition, a real distinguished polynomial always has its leading coefficient equal to 1.

Lemma 1. If $F(w, z)$ is $R$-irreducible, then there exists a real point $(a) = (a_1, \ldots, a_n)$, near the origin where the discriminant of $F$ does not vanish.

Here $z$ stands for the set $(z_1, \ldots, z_n)$.

Proof: The case where the degree $m$ of $F$ is 1 is trivial and is hereby excluded in this proof. Therefore we take $m \geq 2$. 

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The discriminant is an integral rational function of $A_1, A_2, \ldots, A_m$, the coefficients of $F$. If it vanished for all real points (a) near the origin, it would (by analytic continuation) also have to vanish in the whole complex neighborhood of the origin. Thus $F$ and $\frac{\partial F}{\partial w}$ considered as polynomials in $w$ would have a common factor at each point near the origin. This common factor (of degree $\geq 1$) can be found by a well known rational process (the Euclidean algorithm) and thus we are led to a factorization of $F$ into some such form as

$$F(w, z) = Q_1(w, z)Q_2(w, z),$$

where $Q_1$ and $Q_2$ are polynomials in $w$ of degree $\geq 1$ with coefficients which are real rational functions of those coefficients of $F$ which do not vanish identically in $(z)$. We also may assume the coefficients of the leading terms of $Q_1$ and $Q_2$ to be 1. Now, since $F$ is a distinguished polynomial, its coefficients are bounded and tend to 0 as $(z) \to 0$. Assuming $|A_1| < a$, it is easy to prove that the absolute value of even the largest root of $F = 0$ cannot exceed $\max (mt, \sqrt[mt]{ma})$. Thus the roots are all bounded and, in fact $\to 0$, as $(z) \to 0$. The coefficients of $Q_1$ (other than the leading coefficient), being elementary symmetric functions of a subset of the roots of $F$, must also be bounded. Hence their singularities, when considered as functions of $(z)$, must be removable. Moreover they tend to 0 as $(z) \to 0$. Hence the $Q_1$ are also real distinguished polynomials, and (1) now shows that $F$ is not $R$-irreducible, which is contrary to hypothesis. Thus the assumption that the discriminant of $F$ vanishes for all real values of $(z)$ near the origin leads to a contradiction, and this establishes Lemma 1.
At such a point (a), where the discriminant does not vanish, the
equation \( F(w, a) = 0 \) has \( m \) distinct roots \( b_1, b_2, \cdots, b_m \). Moreover the
implicit function theorem asserts that in a neighborhood of (a) there are
defined \( m \) analytic functions \( w_1(z), \cdots, w_m(z) \) such that \( w_i(a) = b_i \) and
\( F(w_i(z), z) = 0 \), \( i = 1, 2, \cdots, m \).

Lemma 2. Either \( w_1(x) \) can be continued analytically into each of the other
\( m-1 \) "branches," \( w_2(z), \cdots, w_m(z) \) by following suitable paths through the
neighborhood \( N \) of the origin (and this is what must happen if \( b_1 \) is real and
also if \( m \) is odd) or else it can be continued analytically only into \( \frac{1}{2}m - 1 \)
of the other branches, in which case the conjugate, say \( \overline{w}_1(z) \), of \( w_1(z) \), is
not attained by continuation of \( w_1(z) \). In this second alternative, all of
the \( \frac{1}{2}m \) branches not obtained by analytic continuation of \( w_1(z) \) may be obtained
by taking the conjugates of \( w_1(z) \) and those branches obtained by analytic con-
tinuation from \( w_1(z) \).

Proof: Suppose that \( w_1(z) \) can be continued into \( w_2(z), \cdots, w_k(z) \) but into
no further branches. Then the symmetric functions

\[
S_1(z) = - \left[ w_1(z) + \cdots + w_k(z) \right]
\]

\[
S_2(z) = + \left[ w_1(z)w_2(z) + \cdots + w_{k-1}(z)w_k(z) \right]
\]

\[
\vdots
\]

\[
S_k(z) = (-1)^{k} w_1(z)w_2(z) \cdots w_k(z)
\]
must be analytic and single valued functions of $z$ with, at worst, removable
singularities. Moreover these symmetric functions (with their singularities
thus removed) must vanish at the origin. Thus $G(w, z) = w^k + S_1(z)w^{k-1} + \cdots + S_k(z)$ is a polynomial with analytic coefficients which, except for the leading
coefficient, $\to 0$ as $(z) \to 0$. Evidently also $G(w_1(z), z) = 0$, for $i = 1, \cdots, k$. Hence $F(w, z)$ must be divisible by $G(w, z)$. Hence if the $S$'s are
real, the R-irreducibility of $F$ implies that $F = G$ and $k = m$. If, however,
the $S$'s are not real, the real $F$ must also be divisible by the conjugate $\overline{G}$ of
$G$ and hence by the real distinguished polynomial $\overline{G}$. In this case $F = \overline{G}$, be-
cause of the given R-irreducibility of $F$ and $m = 2k$. The various assertions
of Lemma 2 are now obvious.

**Definition 2.** Let $F(w, z_1, \cdots, z_r)$ be an R-irreducible real distinguished
polynomial in $w$ with coefficients analytic in $z_1, \cdots, z_r$. The equation
$F = 0$ then defines $w$ as a (multiply valued) function of $z_1, \cdots, z_r$ in a
neighborhood $N$ of the origin. Such a function is called an R-pseudo-algebraic
function of $z_1, \cdots, z_r$. The set of values $(w, z_1, \cdots, z_r)$ which satisfy
$F = 0$ in $N$ is called an R-Riemann configuration of dimension $r$ imbedded in
$(r + 1)$ (complex) dimensional space. It will be denoted by $\mathbb{R}_F(r, r + 1)$.

**Definition 3.** Let $w_1, \cdots, w_k$ be $k$ R-pseudo-algebraic functions of $z_1, \cdots, z_r$,
satisfying equations $F_1 = 0, \cdots, F_k = 0$ respectively. The closure of the
set of $(k + r)$-tuples $(w_1, \cdots, w_k, z_1, \cdots, z_r)$ and their conjugates
$(\bar{w}_1, \cdots, \bar{w}_k, \bar{z}_1, \cdots, \bar{z}_r)$, where $w_1, \cdots, w_k$ at $z_1, \cdots, z_r$ are obtained by
simultaneous analytic continuation from suitably assigned initial values at
a particular real point $P(z_1, \cdots, z_r)$, and $\bar{w}_1, \cdots, \bar{w}_k$ are obtained from
the corresponding conjugate complex initial values, is called an $R$-Riemann configuration of dimension $r$ imbedded in $(r + k)$ dimensional space. It is denoted by $R_{F_1} \ldots F_k (r, r + k)$. Closure is taken relative to a neighborhood of the origin, and it is clear that $R_{F_1} \ldots F_k (r, r + k)$ is perfect as well as closed.

Clearly $R_{F}(r, r + 1)$ is a special case of $R_{F_1} \ldots F_k (r, r + k)$ with $k = 1$. The case $k > 1$ is, however, more complicated because the configuration may depend upon the choice of the initial value of each $w$ at the point $P$. Thus, suppose $w_1$ and $w_2$ satisfy the equations $w_1^2 - z = 0$ and $w_2^2 - z = 0$. If we take $P$ at the point $z = +1$ and assume for $w_1$ and $w_2$ the initial values $+1$ and $-1$ respectively, we find that $w_1 = w_2$ everywhere on $R(1, 3)$. If, however, we assume for $w_1$ and $w_2$ the initial values $+1$ and $-1$ respectively, we find that $w_1 = -w_2$ everywhere on $R(1, 3)$.

**Definition 4.** Points where the discriminants of the equations $F_i = 0$, $i = 1, \ldots, k$, do not vanish, are said to be ordinary points of $R_{F_1} \ldots F_k (r, r + k)$. Points $(w, z_1, \ldots, z_r)$ such that each equation $F_i(w, z_1, \ldots, z_r) = 0$ is identically satisfied by an analytic function $w_i(z_1, \ldots, z_r)$ defined near $(z_1, \ldots, z_r)$ with $w_i = w_i(z_1, \ldots, z_r)$ are called regular points of $R_{F_1} \ldots F_k (r, r + k)$.

It is clear that ordinary points are always regular. But regular points need not be ordinary. Thus, in the case $R_F(2, 3)$, defined by $F = w^2 - z_1 z_2$, the point $(z_1 = 0, z_2 = 1, w = 0)$ is regular, since the equation is satisfied by $w = z_1^{1/2}$ or by $-z_1^{1/2}$ which are analytic and single valued near $(0, 1)$. 


But it is not an ordinary point, since the discriminant, which is \(4z_1^2z_2\), vanishes if \(z_1 = 0\).

**Definition 5.** A (multiple valued) function \(W\) is said to be single valued on the \(R\)-Riemann configuration \(R(r, r + k)\) if, to each point \((w_1, \ldots, w_k; z_1, \ldots, z_r)\) of the latter, there corresponds just one value of \(W\), say, \(W(w_1, \ldots, w_k; z_1, \ldots, z_r)\).

It is continuous at \((w'_1, \ldots, w'_k; z'_1, \ldots, z'_r)\) if \(W(w_1, \ldots, w_k; z_1, \ldots, z_r) \rightarrow W(w'_1, \ldots, w'_k; z'_1, \ldots, z'_r)\) whenever \((w_1, \ldots, w_k; z_1, \ldots, z_r) \rightarrow (w'_1, \ldots, w'_k; z'_1, \ldots, z'_r)\). \(W\) will be called self conjugate if it satisfies the condition

\[
W(\bar{w_1}, \ldots, \bar{w_k}; \bar{z_1}, \ldots, \bar{z_r}) = \overline{W(w_1, \ldots, w_k; z_1, \ldots, z_r)}
\]

Evidently, in the immediate neighborhood of any regular point of \(R(r, r + k)\) the values of a continuous \(W\) are those of one or more continuous single valued functions of \(z_1, \ldots, z_r\). This accounts for the usual practice of omitting the arguments \(w_1, \ldots, w_k\) in the notation \(W(w_1, \ldots, w_k; z_1, \ldots, z_r)\). If the point is an ordinary point the practice is especially convenient, since in this case only one such continuous single valued function makes its appearance. We speak of \(W\) as being analytic at any regular point of \(R(r, r + k)\) if each of the single valued functions of \(z_1, \ldots, z_r\) mentioned above is analytic at the point in question.

**Theorem 2.** Let \(P(w, z)\) be an \(R\)-irreducible real distinguished polynomial of degree \(m\). Let \(W(z) = W(z_1, \ldots, z_n)\) be single valued, self conjugate, and continuous on \(R_r(n, n + 1)\). Let \(W\) be analytic at ordinary points of the
configuration and let it vanish at the origin. Then $W$ satisfies an equation $G(W, z) = 0$, where $G$ is an $R$-irreducible real distinguished polynomial of degree $p$, where $p$ is a divisor of $m$. Furthermore to every root $\bar{W}$ of $G(W, z) = 0$ for a particular $(z)$ there correspond $mp^{-1}$ roots, $\bar{W}_1', \ldots, \bar{W}_m'$, in general distinct, such that $\bar{W} = W(\bar{W}_1', z_1', \ldots, z_n')$, $i = 1, 2, \ldots, mp^{-1}$.

Proof: Let $(w_i', z')$, $i = 1, \ldots, m$, be $m$ "superimposed" ordinary points of $R_F(n, n + 1)$, so that the $w_i$ are distinct numbers. Let $W_k(z')$ correspond to $(w_k', z_1', \ldots, z_n')$. Near $(z_1', \ldots, z_n')$, $W$ therefore breaks up into $m$ branches, (not necessarily distinct), $W_1(z), \ldots, W_m(z)$, each branch analytic near $(z_1', \ldots, z_n')$. The symmetric expressions,

$$
\varphi_1(z) = -\left[\frac{W_1(z) + \cdots + W_m(z)}{W_1(z) + \cdots + W_m(z)}\right]
$$

$$
\varphi_2(z) = +\left[\frac{W_1(z)W_2(z) + W_1(z)W_3(z) + \cdots + W_{m-1}(z)W_m(z)}{W_1(z) + \cdots + W_m(z)}\right]
$$

$$
\cdots
$$

$$
\varphi_m(z) = (-1)^mW_1(z)W_2(z) \cdots W_m(z),
$$

are therefore also analytic in the neighborhood of $(z')$. By analytic continuation over any path in $N$ through ordinary points of $R_F$, the $\varphi$'s as thus defined, are single-valued, even though the individual $W$'s may not be. This is because of the fact that $W$ being single valued on $R_F$, an arbitrary circuit on $N$ has only the effect of permuting the $W_1, \ldots, W_m$ among themselves. Moreover the $\varphi$'s have only removable singularities and vanish at the origin. Also the $\varphi$'s must be real when $(z)$ is real, since in the above symmetric expressions,
each $W_k(z)$ is either real or accompanied by its conjugate complex, as we see from the assumed self conjugacy of $W(w;z)$. Hence $W$ satisfies the equation

$$(2) \quad W^m + \varphi_1(z)W^{m-1} + \cdots + \varphi_m(z) = 0$$

whose left member is a real distinguished polynomial, but is not necessarily R-irreducible. Let now $G(w,z)$ of degree $p$ be an R-irreducible factor of the left member. Then some particular branch of $W$ would satisfy the equation

$$(3) \quad G(W,z) = 0.$$ 

But since all branches of $W$ are connected with either a single branch or a pair of conjugate complex branches by paths through $R_P$ (cf. Lemma 2 and Definition 5), it is clear that all branches of $W$ must satisfy (3). Hence, either $p = m$, or, in case $p < m$, the roots of (2) must all be multiple and $p$ must be a divisor of $m$. The other assertions of the theorem are obvious.

**Theorem 3:** The function $W$ of Theorem 2 can be represented in the form

$$(4) \quad W = \frac{H(w,z_1, \ldots, z_n) - W^m}{F^1(w,z_1, \ldots, z_n)} \quad F^1 = \frac{\partial F}{\partial w},$$

at all points of $R_P(n, n+1)$ where $F^1 \neq 0$. Here $H$ is a real distinguished polynomial of the same degree $m$ as $F$ and reduces to $W^m$ at those points of $R_P(n, n+1)$ at which $F^1 = 0$. 

Proof: Confining attention initially to a neighborhood of an ordinary point, we denote the branches (not necessarily distinct) of $W$ by $W_1, W_2, \ldots, W_m$, corresponding respectively with the branches $W_{1}', \ldots, W_{m}'$ of the function $w$ defined by $F(w, z) = 0$, which of course serves to specify the Riemann configuration $K_F(n, n + 1)$.

Then the expressions $P_h(z_1, \ldots, z_n) = \sum_{s = 1}^{m} W_s h_s (h = 0, 1, 2, \ldots, m-1)$, together with their analytic continuations, are single valued and analytic at all ordinary points of the neighborhood $N$ of the origin. In fact, a circuit drawn on $R_F$ near the origin can at most permute the terms in the above indicated sum, since $W$ is single valued on $K_F(n, n + 1)$. Moreover $P_h$ is real when $(z)$ is real, because each complex term $T$ in the sum corresponds to another term $\overline{T}$ of the sum where $\overline{T}$ is the conjugate of $T$. Since $P_h$ is obviously bounded in $N$, it has no unremovable singularities. Finally the $P$'s all vanish at the origin, since the $W$'s do.

Evidently \[ F(w, z) = \sum_{s = 1}^{m} (w - w_s) = w^m + A_1 w^{m-1} + \cdots + A_m \]

Let \[ \Phi(w, z) = \sum_{s = 2}^{m} (w - w_s) = w^{m-1} + B_1 w^{m-2} + \cdots + B_{m-1}. \]

Then $\Phi = \frac{F}{w - w_1}$. Hence, from the division algorithm for polynomials, it is seen that the $B$'s are distinguished real polynomials of degree $\leq m - 1$ in $w_1$, their non-leading coefficients, in fact, in each case, being some of the $A$'s.

It is also clear that $\Phi'(w_1, z) = F'(w_1, z)$ and that $\Phi(w_s, z) = 0$ for $s = 2, 3, \ldots, m$. Hence (with $B_0 = 1$), we find that
\[
\sum_{h=0}^{m-1} P_h B_{m-1-h} = \sum_{h=0}^{m-1} B_{m-1-h} \left( \sum_{s=1}^{m} w_s h_s \right) = \sum_{s=1}^{m} \left( \sum_{h=0}^{m-1} w_s h_s B_{m-1-h} \right) w_s = \sum_{s=1}^{m} \Phi(w_s, z) w_s = F'(w_1, z) w_1.
\]

In other words, at points where \( F'(w_1, z) \neq 0 \), we have

\[
W_1(z) = \frac{H(w_1, z) - w_1^m}{F'(w_1, z)}, \text{ where } H(w_1, z) = w_1^m + \sum_{h=0}^{m-1} P_h(z) B_{m-1-h}(w_1, z).
\]

The formula (4) follows by analytic continuation of the other \( w \)'s and \( W \)'s out of \( w_1 \) and \( W_1 \) (in accordance with Lemma 2 and Theorem 2) or/and replacement of \( w_1 \) and \( W_1 \) by their conjugate complexes. The other statements of Theorem 3 are trivial.

**THEOREM 4.** Let \( k \) functions \( w_1, \ldots, w_k \) be defined near the origin by \( k \) equations \( F_i(w_1, z) = 0, \ i = 1, \ldots, k \), where \( F_i(w_1, z) = F_i(w_1, z_1, \ldots, z_n) \) is a real distinguished R–irreducible polynomial of degree \( m_i \) in \( w_1 \). Then there exists an R–irreducible real distinguished polynomial \( F(w, z) \) of some degree \( m \) in \( w \) such that each of the \( k \) functions \( w_1 \) is single valued and self conjugate on the R–Riemann configuration \( R_F(n, n+1) \) and \( w \) is a suitably chosen linear combination of \( w_1, \ldots, w_k \) with real coefficients.

**COROLLARY to THEOREM 4 and Definition 3.** An alternative representation of an R–Riemann configuration of dimension \( r \) imbedded in \( (r + k) \) dimensions may be given in the form of \( k \) self conjugate functions \( w_1(w_1, z_1, \ldots, z_r), \ldots, w_k(w_1, z_1, \ldots, z_r) \)
single valued on a suitably chosen \( R_F(r, r + 1) \), and analytic at every
regular point of \( R_F(r, r + 1) \).

**COROLLARY of THEOREMS 3 and 4.** The \( w_i \) of the previous theorem admit near the
origin (except where \( F' = 0 \)) simultaneous representations of the form

\[
(5) \quad w_i(z) = \frac{H_i(w(z_1, \ldots, z_n), z_1, \ldots, z_n) - w^m}{F'(w(z_1, \ldots, z_n), z_1, \ldots, z_n)}
\]

where \( F' = \partial F/\partial w \) and where \( H_i \) is a real distinguished polynomial in \( w \) of
the same degree \( m \) as \( F \). \( H_i \) reduces to \( w^m \) where \( F' = 0 \).

**Proof of Theorem 4.** Let \( (z') \) be any real point near the origin where none
of the discriminants of the \( k \) equations \( F_1 = 0 \) are 0. We may prove the
existence of such a point \( (z') \) by induction. **Lemma 1** takes care of the case
\( k = 1 \). We assume inductively that it is true when \( k = \lambda \) and show that it is
then true when \( k = \lambda + 1 \). Let \( (z'') \) be a real point where none of the first
\( \lambda \) discriminants are zero. From the continuity of the discriminants it
follows that there is a real neighborhood \( U \) of \( (z'') \) where none of the first
\( \lambda \) discriminants are 0. If the \((\lambda + 1)^{th}\) discriminant were equal to 0 every-
where in \( U \), it would, by analytic continuation, have to vanish identically.
The argument used in the proof of **Lemma 1** shows then that \( F_{\lambda + 1} \) would be \( R-
reducible. Hence there is a point \( (z') \) in \( U \) where the discriminant of \( F_{\lambda + 1} = 0 \)
is not 0; and the other discriminants are also not 0 at \( (z') \) as we know from
the construction of \( U \) which contains \( (z') \).

Let \( w_i(1), w_i(2), \ldots, w_i(m_1) \) be the \( m_1 \) branches of the function \( w_i(z) \),
ordered in an arbitrary manner. Evidently their values at \((z')\) are distinct.

Now consider the function

\[
    w^{(1)}(z) = \sum_{i=1}^{k} a_i w_i^{(1)}(z)
\]

where the \(a_i\)'s are real constants. Starting at \((z')\) we perform analytic continuation along all possible closed paths which avoid all but ordinary points. Upon completion of such a path we arrive back at the point \((z')\), with \(w_i^{(1)}(z)\) replaced by \(w_i^{(k_i)}(z)\). If the path is non-trivial \(k_i \neq 1\) for at least one value of \(i\). Correspondingly \(w^{(1)}(z)\) will be replaced by, say \(w^*(z)\); and \(w^*(z)\) will be distinct from \(w^{(1)}(z)\) in the non-trivial case mentioned above if the \(a_i\)'s are chosen so that

\[
    \sum_{i=1}^{k} a_i \left[ w_i^{(1)}(z') - w_i^{(k_i)}(z') \right] \neq 0.
\]

(6)

Since each of the functions \(w_i(z)\) has but a finite number of branches, there are only a finite number of inequalities of the form (6) to be satisfied in order to insure that the set of all possible \(w^*\)'s should be distinct from \(w^{(1)}\) and from each other. Since the coefficient of \(a_i\) for at least one value of \(i\) is different from 0 in (6), we can choose the \(a_i\)'s so that all conditions of the form (6) are satisfied.

Denote the resulting analytic continuations of \(w^{(1)}(z)\) by use of subscripts as follows: \(w^{(2)}(z), w^{(3)}(z), \ldots, w^{(m)}(z)\). The polynomial

\[
    \mathcal{P}(w,z) = \prod_{h=1}^{m} (w - w^{(h)}(z)) = \sum_{p=0}^{m} A_p(z) w^{m-p}
\]

of degree \(m\) in \(w\) has coefficients
$A_p(z_1, \ldots, z_n)$ which are single valued analytic functions of $z_1, \ldots, z_n$, with, at worst, removable singularities. This is because the $A_i$'s (except for $A_0 = 1$) are elementary symmetric functions of all possible analytic continuations of $w^{(1)}(z)$.

Case 1. If the $A_i$'s are all real (when the $z$'s are real), which will occur, if all possible branches of each $w_i(z)$ can be obtained by analytic continuation of a single branch, we need only take $F = \overline{F}$ with $m = \overline{m}$, and the proof is essentially complete. To conjugate branches of $w(z)$ will correspond sets of conjugate branches of $w_i(z)$. The R-irreducibility of $F$ is obvious from the fact that every branch of $w$ can be obtained by analytic continuation from a single branch.

Case 2. If the $A_i$'s are not all real, which will occur, if for at least one value of $i$ the function $w_i(z)$ is such that not every branch can be derived from $w_i^{(1)}(z)$ by analytic continuation, we take $F$ to be the product of $\overline{F}$ and its conjugate with $m = 2\overline{m}$. Again the proof is essentially complete. The R-irreducibility of $F$ follows from the fact that every branch of $w$ can be obtained by analytic continuation from a single pair of conjugate complex branches.

One further remark should be made about the above proof in connection with the Corollary of Theorem 4 and Definition 3. According to Definition 3, an $R_{T_1} \cdots R_{T_k} F(r, r + k)$ is not uniquely determined simply by $F_{T_1}, \ldots, F_{T_k}$ but also on the initial values of $w_{T_1}, \ldots, w_k$ at the point $P(z')$, whereas an $R_{T_1} \cdots R_{T_k} F(r, r + k)$ is uniquely determined, once the $F$ is given. But the above proof shows a corresponding lack of uniqueness in the construction of $F$. 

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because of the fact that the $m_i$ branches of $w_i(z)$ can be ordered in an arbitrary manner. The initial value of $w_i$ at $P(z')$ is, in the notation of the above proof, $w_i^{(1)}(z')$.

**Theorem 5.** Let $F(w, z_1, \ldots, z_n)$ be an $R$-irreducible real distinguished polynomial of degree $m$ in $w$. Let $\Phi(z_1, \ldots, z_{n-1}, v)$ be an $R$-irreducible real distinguished polynomial of degree $k$ in $v$. Then the points of $R_{n-1} n + 1$ such that $\Phi(z_1, \ldots, z_{n-1}, z_n) = 0$ coincide with the points of a finite number of $R$-Riemann configurations of the form $R_{\Phi}(n-1, n+1)$.

**First part of proof.** Choose a real point $(a_1, \ldots, a_{n-1})$ such that $\Phi = 0$ has distinct roots $b_1, \ldots, b_k$ (cf. Lemma 1). Let $v_i = v_i(z_1, \ldots, z_{n-1})$ be the analytic function defined near $(a_1, \ldots, a_{n-1})$ such that $v_i(a_1, \ldots, a_{n-1}) = b_i (i = 1, 2, \ldots, k)$ and such that $\Phi(z_1, \ldots, z_{n-1}, v_i(z_1, \ldots, z_{n-1})) \equiv 0$. Then $F(w, z_1, \ldots, z_{n-1}, v_i(z_1, \ldots, z_{n-1}))$ is a polynomial in $w$ with coefficients analytic in $z_1, \ldots, z_{n-1}$ near $(a_1, \ldots, a_{n-1})$. Such a polynomial, that can not be written as the product of several polynomials of lower degree with coefficients analytic near $(a_1, \ldots, a_{n-1})$ is said to be irreducible at the point $(a_1, \ldots, a_{n-1})$. (This is not the $R$-irreducibility which is elsewhere in this paper of primary importance). The product of all distinct factors of $F(w, z_1, \ldots, z_{n-1}, v_i(z_1, \ldots, z_{n-1}))$ irreducible at the point $(a_1, \ldots, a_{n-1})$ (that is, with each one taken just once) is a polynomial $P_i$ of degree $\leq m$. The discriminant of $P_i$, since it has no multiple factors, is not identically zero near $(a_1, \ldots, a_{n-1})$ even though it may vanish at $(a_1, \ldots, a_{n-1})$. The discriminant of $\Phi$ is also not identically zero near
(a₁', ⋯, a_n-1'), since it is not even zero at this point. Since these discriminants are analytic at (a₁', ⋯, a_n-1'), there is a real point (a₁', ⋯, a_n-1') near (a₁', ⋯, a_n-1') where none of the k + 1 discriminants are zero. For, of course, otherwise the product of the k + 1 discriminants would vanish identically for real values of (z₁', ⋯, z_n-1'), and hence, because of their analyticity, also for complex values of (z₁', ⋯, z_n-1'). Thus at least one of the discriminants would vanish identically, contrary to fact.

Those factors of P₁, irreducible at (a₁', ⋯, a_n-1'), are therefore no longer irreducible at (a₁', ⋯, a_n-1') if they are of degree > 1. Hence, at the point (a₁', ⋯, a_n-1'), P₁ and hence also F(w, z₁', ⋯, z_n-1', v₁(z₁', ⋯, z_n-1')) can be factored into linear factors with coefficients analytic at (a₁', ⋯, a_n-1'), although in the case of F some of these factors may be repeated. Thus, although the algebraic equation in w,

\[ F(w, a₁', ⋯, a_n-1', v₁(a₁', ⋯, a_n-1')) = 0, \]

may indeed have multiple roots, the corresponding branches of the function w(z₁', ⋯, z_n-1') defined by the equation,

\[ F(w, z₁', ⋯, z_n-1', v₁(z₁', ⋯, z_n-1')) = 0, \]

would be identically equal near (a₁', ⋯, a_n-1') because they would each equal identically a single branch of the function defined by \( P₁ = 0 \).

We now fix attention on a particular analytic solution, say \( \hat{w} \), of (7), defined near (a₁', ⋯, a_n-1'). In addition to satisfying (7), it must also satisfy the equation \( R(w, z₁', ⋯, z_n-1') = 0 \), where
\[ R = \prod_{i=1}^{k} F(w, z_1, \ldots, z_{n-1}, v_i(z_1, \ldots, z_{n-1})). \]

Since those \( v \)'s which are not real, must occur in conjugate imaginary pairs when \( z_1, \ldots, z_{n-1} \) are real, it is clear from the reality of \( F(w, z_1, \ldots, z_n) \) that the factors \( F(w, z_1, \ldots, z_{n-1}, v_i) \) must also occur in conjugate complex pairs. Hence \( R \) is a real polynomial in \( w \).

Because of the symmetry in \( v_1, \ldots, v_k \), the coefficients of \( R \), considered as analytic functions of \( z_1, \ldots, z_{n-1} \), when continued analytically from the vicinity of \( (a'_1, \ldots, a'_{n-1}) \) throughout a whole neighborhood of the origin will be single valued and, except for removable singularities, analytic everywhere. Finally all the coefficients of \( R \), except the coefficient of the leading power of \( w \), which is 1, are easily seen to vanish at the origin, since the same is true of the coefficients of \( \hat{\Phi} \) and \( F \). Thus \( R \) is a real distinguished polynomial in \( w \) such that \( \hat{w}(z_1, \ldots, z_{n-1}) \) satisfies the equation \( R = 0 \) near \( (a'_1, \ldots, a'_{n-1}) \).

Let \( S \) be that \( R \)-irreducible real distinguished polynomial in \( w \) (of degree \( \leq \) degree of \( R \)) with coefficients analytic in \( z_1, \ldots, z_{n-1} \) such that \( \hat{w} \) satisfies the equation \( S = 0 \). Then all analytic continuations of \( \hat{w} \) also satisfy the equation \( S = 0 \). The two equations \( \hat{\Phi} = 0 \) and \( S = 0 \) with appropriate initial values for \( w \) and \( v \) at \( (a') \) evidently determine an \( R \)-Riemann configuration of the form \( R_{\hat{\Phi}}(n-1, n+1) \); and from the definition of \( S \) it is clear that every point of \( R_{\hat{\Phi}}(n-1, n+1) \) also lies on \( R_{\hat{\Phi}}(n, n+1) \) as well as satisfying \( \hat{\Phi} = 0 \).

We can, of course, get, in general, more than one configuration of the form \( R_{\hat{\Phi}}(n-1, n+1) \) having the property just stated. This is because of the
freedom in the choice of \( w \) from among the roots of \( F(w, z_1', ..., z_{n-1}', v_1') = 0 \) near \( (a_1', ..., a_{n-1}') \). Thus, with \( m = 2, k = 1, F = w^2 - 5z_1w + 6z_2', \)
\( \Phi = v - z^2 \), we get \( S = w - 3z \), for one choice of \( \Phi \) and \( S = w - 2z \), for the other. There is also an apparent freedom in the choice of \( v_1 \) from among the roots of \( \Phi = 0 \), but the effect of this freedom is unimportant because of the irreducibility of \( \Phi \). There are in any case but a finite number of possibilities corresponding to each assignment of \( (a_1', ..., a_{n-1}') \).

We must still show that any point \( (w', z_1', ..., z_n') \) of \( R_F(n, n+1) \), in a sufficiently small neighborhood \( N \) of the origin and such that
\( \Phi(z_1', ..., z_{n-1}', z_n') = 0 \), lies on a suitably chosen \( R_S \Phi(n-1, n+1) \), corresponding to an assigned \( (a_1', ..., a_{n-1}') \). This is what we are concerned with in the following.

**Second part of proof.** We suppose first that \( (z_1', ..., z_{n-1}') \) is a point where the discriminant of \( \Phi \) does not vanish. The \( z \)'s here, as well as \( w \), are not necessarily real.

The equation \( \Phi(z_1', ..., z_{n-1}, v) = 0 \) considered as an algebraic equation in \( v \) has \( k \) distinct analytic solutions \( v_i(z_1', ..., z_{n-1}', i = 1, ..., k \) defined near \( (z_1', ..., z_{n-1}') \). One of these, say \( v_1', \) takes on the value \( z_n' \) at the point \( (z_1', ..., z_{n-1}') \). The expression
\( F(w, z_1, ..., z_{n-1}, v_1') \) is analytic in \( z_1, ..., z_{n-1} \) near this point and is a polynomial in \( w \) which vanishes when \( w = w', z_i = z_i', i = 1, ..., n-1 \). Let \( F(w, z_1, ..., z_{n-1}) \) be the product of all the distinct factors of
\( F(w, z_1, ..., z_{n-1}, v_1') \) irreducible at \( (z_1', ..., z_{n-1}') \). Evidently
\( F(w, z_1', ..., z_{n-1}') = 0 \) has a root \( w = w' \). Either \( w' \) is a simple root, in
which case \( w_l(z_1', \cdots, z_{n-1}') \) will be defined locally as that analytic solution of \( F(w, z_1, \cdots, z_{n-1}) = 0 \) which takes on the value \( w' \) at \( (z_1', \cdots, z_{n-1}') \); or else the point \( (z_1', \cdots, z_{n-1}') \) may be considered as an end point of a short curve \( C \) near \( (z_1', \cdots, z_{n-1}') \) every non-end point of which is a point where the discriminant of \( F \), as well as the discriminant of \( \overline{F} \) does not vanish. At such a point the roots of \( \overline{F} = 0 \) are distinct, and it will be possible to define in some neighborhood of every non-end point of \( C \) at least two distinct analytic functions \( w_1(z_1, \cdots, z_{n-1}) \) and \( w_2(z_1, \cdots, z_{n-1}) \) such that \( \overline{F}(w_1(z_1, \cdots, z_{n-1}), z_1, \cdots, z_{n-1}) = 0 \) while both \( w_1 \) and \( w_2 \to w' \) as \( (z_1, \cdots, z_{n-1}) \to (z_1', \cdots, z_{n-1}') \) and as \( w_1 \) and \( w_2 \) are continued analytically along \( C \). We need fix attention only upon one of these, say \( w_1(z_1', \cdots, z_{n-1}') \).

The curve \( C \) is now extended so that one end point is at \( (a_1', \cdots, a_{n-1}') \) while the other is still at \( (z_1', \cdots, z_{n-1}') \). This extension can clearly be made so as to avoid points where the discriminant of \( \overline{F} \) vanishes and so as to remain within the neighborhood \( N \) of the origin. Hence the \( \overline{v}_i(z_1, \cdots, z_{n-1}) \) can be continued analytically along \( C \) all the way to \( (a_1', \cdots, a_{n-1}') \). The same is therefore true also of \( F(w, z_1, \cdots, z_{n-1}, \overline{v}_i) \) considered as functions of \( z_1, \cdots, z_{n-1} \).

By a familiar argument \( \overline{F} = \prod_{i=1}^{k} F(w, z_1, \cdots, z_{n-1}, \overline{v}_i) \) may be considered a polynomial in \( w \) with coefficients analytic in the whole neighborhood \( N \) of the origin. Let \( \overline{S} \) be the polynomial factor of \( \overline{F} \), irreducible in \( N \), the vanishing of which on \( C \) near \( (z_1', \cdots, z_{n-1}') \) is effected by equating \( w \) to the root \( w_l(z_1', \cdots, z_{n-1}') \) of the equation \( F(w, z_1, \cdots, z_{n-1}, \overline{v}_i) = 0 \). The path \( C \) can be slightly modified, if necessary, so that the discriminant of \( \overline{S} \) (as well as
that of \( \Phi \) does not vanish at any non-end point of \( C \). Although this discriminant might conceivably vanish at \( (a_1', \ldots, a_{n-1}') \), it is already known from the way in which this point was chosen that every solution of the \( k \) equations \( F(w, z_1', \ldots, z_{n-1}', \overline{v}_1(z_1', \ldots, z_{n-1}')) = 0, \ i = 1, 2, \ldots, k \), (taken individually and not simultaneously, of course) for \( w \) as a function of \( z_1', \ldots, z_{n-1}' \) is single valued and analytic near \( (a_1', \ldots, a_{n-1}') \). This is because the solutions of \( F(w, z_1', \ldots, z_{n-1}', \overline{v}_1) = 0 \) are the same as the solutions of \( F(w, z_1', \ldots, z_{n-1}', v_1) = 0 \) of the first part of the proof, while the solutions of these latter equations are equivalent, near \( (a_1', \ldots, a_{n-1}') \), to the solutions of \( P_i(w, z_1', \ldots, z_{n-1}') = 0 \) in connection with which the point \( (a_1', \ldots, z_{n-1}') \) was purposely chosen so that the discriminant of each of these \( P_i \)'s would not vanish there. Hence \( w_1(z_1', \ldots, z_{n-1}') \) can be continued analytically all the way to \( (a_1', \ldots, a_{n-1}') \). Thus it has been shown that the point \( (z_1', \ldots, z_n', w') \) is a point of \( R_{S \Phi}(n-1, n+1) \) as introduced in the first part of the proof merely by setting \( v_1 = \overline{v}_1 \) and \( w = w_1 \) near \( (a_1', \ldots, a_{n-1}') \). It may also be remarked that \( S \) is then either the same as \( S \) or is \( \overline{S} \) multiplied by its conjugate complex, but this is not essential to the proof.

Finally, since an arbitrary point of \( R_{\Phi}(n, n+1) \) such that \( \Phi = 0 \) may be considered as a limit point of other such points where the discriminant does not vanish, the complete theorem follows from the definition of \( R_{S \Phi}(n-1, n+1) \) which requires it to be closed in \( N \).

**Theorem 6.** Let \( F(w, z_1, \ldots, z_n) \) be an \( R \)-irreducible real distinguished polynomial in \( w \). Let \( \Phi(z_1, \ldots, z_{n-1}, v) \) be an \( R \)-irreducible real
distinguished polynomial in \(v\). If \(W(z_1, \ldots, z_n)\) is self-conjugate, continuous, and single valued on \(R_F(n, n+1)\), is analytic at all regular points of \(R_F(n, n+1)\), and vanishes at the origin, then \(W\) is analytic at all regular points of any \(R\)-Riemann configuration \(R_S\Phi(n-1, n+1)\) chosen as in the previous theorem so that it consists only of points on \(R_F(n, n+1)\) for which \(\Phi(z_1, \ldots, z_{n-1}, z_n) = 0\).

Proof. By Theorem 2, there exists an \(R\)-irreducible real distinguished polynomial in \(W, L = WP + A_1(z_1, \ldots, z_n)WP^{-1} + \cdots + A_{p-1}(z_1, \ldots, z_n)W + A_p(z_1, \ldots, z_n),\) such that the function \(W(z_1, \ldots, z_n)\) defined in the neighborhood \(N\) of the origin by \(L = 0\) is identical with the given \(W\). The fact that \(W\) is continuous and single valued on \(R_F(n, n+1)\) means, by definition 5, the following:

There is a continuous single-valued mapping of points \((w, z_1, \ldots, z_n)\) such that \(F(w, z_1, \ldots, z_n) = 0\) onto points \((\bar{w}, \bar{z}_1, \ldots, \bar{z}_n)\) such that \(L(\bar{w}, \bar{z}_1, \ldots, \bar{z}_n) = 0\). Hence, if \(z_n = Z(z_1, \ldots, z_{n-1})\), where \(Z\) is continuous, we have a continuous single-valued mapping of points \((w, z_1, \ldots, z_{n-1}, Z(z_1, \ldots, z_{n-1}))\) such that \(F(w, z_1, \ldots, z_{n-1}, Z(z_1, \ldots, z_{n-1})) = 0\) onto points \((\bar{w}, \bar{z}_1, \ldots, \bar{z}_{n-1}, Z(\bar{z}_1, \ldots, \bar{z}_{n-1}))\) such that \(L(\bar{w}, \bar{z}_1, \ldots, \bar{z}_{n-1}, Z(\bar{z}_1, \ldots, \bar{z}_{n-1})) = 0\).

Let \((w', z'_1, \ldots, z'_n)\) be any regular point of \(R_S\Phi(n-1, n+1)\). Then, by definition of a regular point, there exist analytic functions \(v(z_1, \ldots, z_{n-1})\) and \(w(z_1, \ldots, z_{n-1})\), defined near \((z_1, \ldots, z_{n-1})\) such that \(v(z'_1, \ldots, z'_{n-1}) = v(z_1, \ldots, z_{n-1}) = v(z'_1, \ldots, z'_{n-1} = w', \Phi(z_1, \ldots, z_{n-1}, v(z_1, \ldots, z_{n-1})) = 0\) and \(S(w(z_1, \ldots, z_{n-1}), z_1, \ldots, z_{n-1}) = 0\). The latter equation may be replaced, for present purposes, by the equation,
\[ F(w(z_1, \ldots, z_{n-1}), z_1, \ldots, z_{n-1}, v(z_1, \ldots, z_{n-1})) = 0; \]

for, by hypothesis \( R_{S\Phi}^{(n-1, n+1)} \subset R_{F}^{(n, n+1)} \). We now take \( z(z_1, \ldots, z_{n-1}) = v(z_1, \ldots, z_{n-1}) \). The mapping mentioned above therefore establishes the existence of a continuous locally single valued function \( W(z_1, \ldots, z_{n-1}) \) such that

\[ L(W(z_1, \ldots, z_{n-1}), z_1, \ldots, z_{n-1}, v(z_1, \ldots, z_{n-1})) = 0. \]

Let \( P(W, z_1, \ldots, z_{n-1}) \) be the product of all distinct (not necessarily real) factors of \( L(W, z_1, \ldots, z_{n-1}, v(z_1, \ldots, z_{n-1})) \), each factor being irreducible at \((z'_1, \ldots, z'_{n-1})\). Since \( P \) has no repeated factors, its discriminant is not identically 0 near \((z'_1, \ldots, z'_{n-1})\). \( W(z_1, \ldots, z_{n-1}) \), by (8), must then certainly be analytic at points where the discriminant is not 0. Moreover, since \( W \) is bounded, the points where the discriminant vanishes must, at worst, be removable singularities. Since \( W(z_1, \ldots, z_{n-1}) \) is already known to be continuous, it is seen that it is analytic without exception near \((z'_1, \ldots, z'_{n-1})\); hence, in particular, at \((z'_1, \ldots, z'_{n-1})\), as we wished to prove.

**Theorem 7.** If \( W(z_1, \ldots, z_n) \) is a given self conjugate function, single valued on \( R_{F}^{(n, n+1)} \) and analytic at the regular points of \( R_{F}^{(n, n+1)} \), and if it vanishes at the origin but does not vanish identically the points where \( W = 0 \) comprise a finite number of configurations of the form \( R_{S\Phi}^{(n-1, n+1)} \), at least for non-specialized \( z_1, \ldots, z_n \).
In this connection we need

Definition 6. A theorem involving certain variables, \( z_1, \ldots, z_n \) is said

to hold for non specialized \( z_1, \ldots, z_n \) if the theorem holds after the \( z \)'s

have been subjected to a suitably chosen non-singular homogeneous linear

transformation with real coefficients (even if the theorem is not valid for

the original \( z \)'s).

Proof of Theorem 7. By Theorem 2 there exists a real distinguished polynomial

in \( W, L = W^P + A_{p} (z_1', \ldots, z_n')W^{P-1} + \cdots + A_{p} (z_1', \ldots, z_n') \) such that the

function \( W(z_1', \ldots, z_n') \) defined by \( L = 0 \) is identical with the given \( W \). If

\( W = 0 \) at a point \( (w', z_1', \ldots, z_n') \) of \( R_F \), it is clear that \( A_{p} (z_1', \ldots, z_n') \)

= 0. Using non-specialized \( z_1', \ldots, z_n' \), Theorem 1 shows that all the points

where \( A_{p} = 0 \) can be gleaned from a finite number of equations of the form

\[
\Phi(z_1', \ldots, z_{n-1}', z_n') = 0,
\]

where \( \Phi \) is an \( R \)-irreducible real distinguished

polynomial in \( z_n' \). Hence by Theorem 5, the point \( (w', z_1, \ldots, z_n') \) on \( R_F \)

where \( W = 0 \) corresponds to a point on a suitably chosen configuration

\( R_S \Phi(n-l, n+1) \) of which there are but a finite number of possibilities. It

remains to prove that \( W \equiv 0 \) on at least one of such configurations. That is,

we must show that either \( W \equiv 0 \) on \( R_S \Phi(n-l, n+1) \), or else there is another

configuration \( R_S \Phi(n-l, n+1) \), also containing \( (w', z_1', \ldots, z_n') \) on which

\( W \equiv 0 \).

Let us fix attention on a particular \( R_S \Phi(n-l, n) \), consisting of those

points \( (z_1, \ldots, z_{n-l}, z_n) \) near the origin including the given

\( (z_1', \ldots, z_{n-l}', z_n') \) for which \( \Phi(z_1, \ldots, z_{n-l}, z_n) = 0 \). On \( R_S \Phi(n-l,n) \), \( A_{p} \equiv 0 \), so that the equation \( L(W, z_1, \ldots, z_n) = 0 \) on \( R_S \Phi(n-l, n) \) has at least one
vanishing root at every point in a neighborhood of \((z_1', \ldots, z_{n-1}', z_n')\) on \(\mathbb{R}_{\Phi}(n-1, n)\). The points on \(\mathbb{R}_{\Phi}(n, n+1)\) corresponding to points where \(\Phi = 0\), by Theorem 5, lie on a finite number of manifolds \(\mathbb{R}_{\Phi}(n-1, n+1)\). Since \(W\) is given as single valued and continuous on \(\mathbb{R}_{\Phi}(n, n+1)\), at least one of these manifolds \(\mathbb{R}_{\Phi}(n-1, n+1)\) contains a point \((w^*, z_1', \ldots, z_n')\) in a neighborhood of which there is an \((n-1)\)-complex-dimensional continuum in which \(W \equiv 0\). Otherwise the above italicized statement could not be true. Since, by Theorem 6, \(W\) is analytic at every regular point of \(\mathbb{R}_{\Phi}(n-1, n+1)\), analytic continuation shows that \(W \equiv 0\) on this \(\mathbb{R}_{\Phi}(n-1, n+1)\).

This shows that there always exists a positive finite number of configurations of the form \(\mathbb{R}_{\Phi}(n-1, n+1)\) on each of which \(W \equiv 0\). But to show that the given point \((w', z_1', \ldots, z_n')\) lies on one of these amounts to showing that we may take \(w^* = w'\).

If \(w', z_1', \ldots, z_n'\) are all real, we may proceed by transferring the origin to the given point \((w', z_1', \ldots, z_n')\) by means of equations of the form \(\omega = w' - w', \xi_i = z_i - z_i'\). The properties of simultaneous solutions of \(F(w, z_1, \ldots, z_n) = 0\) and \(W(w; z_1, \ldots, z_n) = 0\) near \((w', z_1', \ldots, z_n')\) are then closely linked with the properties of the simultaneous solutions of

\[ f(\omega, \xi_1, \ldots, \xi_n) = 0 \text{ and } V(\omega; \xi_1, \ldots, \xi_n) = 0 \]

where

\[ f(\omega, \xi_1, \ldots, \xi_n) = F(\omega + w', \xi_1 + z_1', \ldots, \xi_n + z_n') \]

and

\[ V(\omega; \xi_1, \ldots, \xi_n) = W(\omega + w, \xi_1 + z_1, \ldots, \xi_n + z_n) \]

both vanish by hypothesis at the new-origin \(\omega = \xi_1 = \cdots = \xi_n = 0\). Moreover \(f\) is analytic near this origin and is real when \(\omega, \xi_1', \ldots, \xi_n'\) are real. \(f\) is
a polynomial in $\omega$ with leading coefficient equal to 1, and with other coefficients analytic in $\zeta_1', \cdots, \zeta_n'$; and the last one, at least, vanishes at the origin. Suppose furthermore that $f$ does not admit a polynomial factor of a similar type but of lower degree. Otherwise we would replace $f$ by one of its suitably chosen factors, thus perhaps losing some of our solutions but retaining enough for present purposes. It is now easy to see, by considerations like those used to establish the Weierstrass Preparation Theorem, that all coefficients, except the leading coefficient, of $f$ must vanish at the origin. In other words $f$ is a real irreducible distinguished polynomial in $\omega$. $V$ is analytic and single valued on $R_f(n, n+1)$; but it does not vanish identically on $R_f(n, n+1)$, because, if it did, $W$ would vanish identically on an $n$-complex dimensional continuum near $(\omega', z_1', \cdots, z_n')$ and would therefore, by analytic continuation, vanish identically throughout $R_f(n, n+1)$.

Hence, applying to $R_f(n, n+1)$ and $V$ the same considerations already applied to $R_f(n, n+1)$ and $W$, we state: There always exists a positive finite number of Riemann configurations of the form $R_{SF}(n-1, n+1)$ on each of which $V \equiv 0$. All such configurations naturally contain the new origin and since they represent $(n-1)$-complex-dimensional continua near $(w', z_1', \cdots, z_n')$ on $R_f(n, n+1)$ on which $W \equiv 0$, it is clear that a suitably chosen $R_{SF}(n-1, n+1)$ may be obtained containing the point $f(w', z_1', \cdots, z_n')$ in a neighborhood of which $W$ vanishes on an $(n-1)$-complex-dimensional continuum. As before, we then have $W \equiv 0$ on this $R_{SF}$.

Finally, if $w', z_1', \cdots, z_n'$ are not all real, we proceed as before, except that, since $f(\omega, \zeta_1, \cdots, \zeta_n)$ is no longer necessarily real, we replace $f$ by $\overline{f}$, where $\overline{f}$ is the conjugate imaginary of $f$, and we similarly replace $V$ by $\overline{V}$. 

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THEOREM 8. Let \( G_i(x_1, \ldots, x_\nu) \) be real and analytic in a neighborhood of the origin \((i = 1, 2, \ldots, \mu)\). Suppose all the \( G_i \)'s vanish at the origin but do not vanish identically and that \( \mu < \nu \). Then, using non-specialized variables, the simultaneous solutions of the equations

\[
G_i(x_1, \ldots, x_\nu) = 0, \quad i = 1, \ldots, \mu.
\]

in the neighborhood of the origin coincide with a positive finite number of \( \mathbb{R} \)-Riemann configurations of dimension \( n \geq \nu - \mu \) imbedded in \( \nu \) dimensions.

That is, there exists at least one real \( \mathbb{R} \)-irreducible distinguished polynomial \( F(w, z_1, \ldots, z_n) \) in \( w \) with coefficients analytic in a suitably chosen number \( n \geq \nu - \mu \) of independent variables \( z_1, \ldots, z_n \), giving rise to an \( \mathbb{R} \)-configuration \( \mathbb{R}_F(n, n+1) \) on which it is possible to define \( \nu - n \) self conjugate complex single valued functions \( w_1, \ldots, w_{\nu-n} \) vanishing at the origin and analytic at every regular point of \( \mathbb{R}_F \), which \( w \)'s, upon interpreting \( z_1, \ldots, z_n \), \( w_1, \ldots, w_{\nu-n} \) suitably as independent real linear combinations of \( x_1, \ldots, x_\nu \), annual identically the left members of (9), when the \( G_i \)'s are thus considered as single valued functions of \( z_1, \ldots, z_n \) on \( \mathbb{R}_F(n, n+1) \). Moreover every solution of (9) can be represented thus, using but a finite number of \( F \)'s (not all necessarily with the same value of \( n \)).

Proof: We proceed by induction on \( \mu \). When \( \mu = 1 \), Theorem 8 is true by Theorem 1. We make the inductive hypothesis that Theorem 8 is true when \( \mu = k \) and prove that it must then hold when \( \mu = k + 1 \). By our inductive hypothesis, all solutions of the first \( k \) of our equations (9) lie on a positive finite number of \( \mathbb{R} \)-configurations of some dimensionality \( n \geq \nu - k \).
Such a configuration is associated, as explained in the statement of the theorem with an $R$-irreducible real distinguished polynomial $F(w, z_1, \ldots, z_n)$ and a set of functions $w_1, \ldots, w_{\nu-n}$ single-valued on $R_F(n, n+1)$ which make the left members of the first $k$ equations (9) vanish identically when they are thought of as single-valued functions on $R_F(n, n+1)$. We now wish to characterize those points of $R_F(n, n+1)$, where the left member of the $(k + 1)$th equation also vanishes. We write this $(k + 1)$th equation in the form

$$W(w_1, \ldots, w_{\nu-n}, z_1, \ldots, z_n) = 0,$$

$W$ is self conjugate, single valued on $R_F(n, n+1)$, and analytic at regular points of $R_F(n, n+1)$, and vanishes at the origin, since the same is true of $w_1, \ldots, w_{\nu-n}$.

**Case 1.** $W \equiv 0$ on $R_F(n, n+1)$. Then the same solution valid for the first $k$ equations is valid also for $k + 1$ equations. The dimensionality of the $R$-configuration is therefore still $n \geq \nu - k > (\nu - k) - 1 = \nu - (k + 1)$, thus completing the induction in Case 1.

**Case 2.** $W \not\equiv 0$ on $R_F(n, n+1)$. Then, by Theorem 7, the points on $R_F(n, n+1)$, for which $W = 0$ comprise a positive finite number of $R$-configurations of the form $R_{\Phi}(n-1, n+1)$, represented, say by

$$S(w, z_1, \ldots, z_{n-1}) = 0, \quad \Phi(w_{\nu-n+1}, z_1, \ldots, z_{n-1}) = 0,$$
where \( w_{\nu - n + 1} = z_n \), with non-specialized \( z_1, \ldots, z_n \). By Theorem 4, we may now find an \( R_{F^*}(n-1, n) \) on which \( w \), as well as \( w_{\nu - n + 1} \), is single valued and self conjugate. Since \( w_1, \ldots, w_{\nu - n} \) are single-valued and self conjugate on \( R_{\Phi^*}(n-1, n+1) \subseteq R_F(n, n+1) \), the same is true for \( R_{F^*}(n-1, n) \). We thus find that the simultaneous solutions of the \( \mu = k + 1 \) equations (9) lie on a positive number of \( R \)-configurations of some dimensionality \( \bar{n} = n - 1 \geq (\nu - k) - 1 = \nu - (k + 1) \), thus completing the induction.

**Theorem 9.** Let \( F(w, z_1, \ldots, z_\mu) \) be an \( R \)-irreducible real distinguished polynomial in \( w \). Suppose also that there are infinitely many real distinct points \( (a_{1h}, \ldots, a_{nh}) \) such that \( \lim_{h \to \infty} (a_{1h}, \ldots, a_{nh}) = (0, \ldots, 0) \) and such that \( F(w, a_{1h}, \ldots, a_{nh}) = 0 \) has at least one real root \( w = r_h \) corresponding to each \( h = 1, 2, 3, \ldots \). Then one of the following two alternatives must hold (with nonspecialized \( z_1, \ldots, z_n \)):

Either (1) there must exist \( n \) continuous real functions \( z_2(z_1), \ldots, z_n(z_1) \), \( w(z_1) \) defined for real values of \( z_1 \) of sufficiently small absolute value, vanishing at the origin and analytic except possibly at the origin, such that \( F(w(z_1), z_1, z_2(z_1), \ldots, z_n(z_1)) = 0 \) near the origin;

Or (2) there must exist two sets of \( n \) analytic real functions \( \bar{z}_2(z_1), \ldots, \bar{z}_n(z_1), \bar{w}(z_1) \), and \( \hat{z}_2(z_1), \ldots, \hat{z}_n(z_1), \hat{w}(z_1) \), all \( 2n \) functions being defined for positive real values of \( z_1 \) of sufficiently small absolute value, with the following properties: They all tend to 0 as \( z_1 \to 0 \).

\[ F(w(z_1), z_1, \bar{z}_2(z_1), \ldots, \bar{z}_n(z_1)) = 0, \quad F(\hat{w}(z_1), z_1, \hat{z}_2(z_1), \ldots, \hat{z}_n(z_1)) = 0; \]

and not all of the following \( n \) identities hold: \( \bar{z}_2(z_1) = \hat{z}_2(z_1), \ldots, \bar{z}_n(z_1) = \hat{z}_n(z_1), \) \( z_n(z_1) = \bar{w}(z_1) = \hat{w}(z_1) \). (That is, the two sets of \( n \) functions are distinct).
Proof: We use induction on $n$.

If $n = 1$, we notice that there is a deleted neighborhood $N$ of the origin where the discriminant $D(z_1)$ of the equation $F(w, z_1) = 0$ does not vanish. Otherwise $D(z_1)$ would vanish infinitely often near the origin and, being analytic, it would have to vanish identically, in which case $F$, contrary to hypothesis, would not be irreducible. Since, then, $D(z_1) \neq 0$ in $N$, the equation $F(w, z_1) = 0$, for each $z_1 \in N$, has a number of distinct roots depending analytically on $z_1$ and tending to 0 as $z_1 \to 0$.

For real $z_1$, these roots are real or occur in conjugate imaginary pairs. As $z_1$ varies continuously on a real interval, a certain root $w(z_1)$ can not exchange its property of being real for the property of being one of a pair of conjugate imaginary roots (or vice versa) unless $z_1$ passes through a point where $D(z_1) = 0$. This can happen only at the origin. Our assumption is to the effect that $F(w, z_1) = 0$ does have at least one real solution in $N$ either for some particular value of $z_1 > 0$ or for $z_1 < 0$. Hence there will be at least one real analytic solution $w(z_1)$ defined for $z_1 > 0$, which solution will tend to 0 as $z_1 \to 0$. If there were only one such real solution for $z_1 > 0$, the degree of the equation would be odd, since complex roots occur in pairs. Since the degree of $F$ is unaltered when $z_1$ passes through 0, there would also be (an odd number of) real solutions in $N$ for $z_1 < 0$, which solutions would also depend analytically on $z_1$. Hence in either case the theorem is established in the case $n = 1$.

Suppose inductively that the theorem is true with $n$ replaced by $n - 1$. We then wish to show that it holds as stated with $n = 2$. 

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Since the points \((a_{1h}, \ldots, a_{nh})\) are distinct and since the \(z_1, \ldots, z_n\) are not specialized, we can, by confining attention to a suitable subsequence, assume that \(a_{1h} \neq 0\), \(h = 1, 2, 3, \ldots\).

Introduce \(z_{1h}(t) = a_{1h}t\). Then \(F(w, z_{1h}(t), \ldots, z_{nh}(t)) = 0\) has at least one real root when \(t = 1\), namely \(r_h\) (cf. the hypothesis of the theorem).

If, for a particular value of \(h\), the discriminant is not 0 for \(0 < t \leq 1\) and for \(-\delta < t < 0\), where \(\delta\) is a sufficiently small positive number, we replace \(t\) by \(a_{1h}^{-1}z_1\), so that \(z_2, \ldots, z_n\) appear as functions of \(z_1\) and the discriminant, which now depends only on \(z_1\), does not vanish in a deleted neighborhood of the origin, containing the point \(z_1 = a_{1h}^{-1} \neq 0\). Hence an argument like that used above (in the case \(n = 1\)) shows at once that the theorem is true. In the contrary case, for each value of \(h\), the discriminant either vanishes at \(t = 1\) or vanishes at \(t = t_h\) for some positive \(t_h < 1\) but does not vanish for \(t_h < t \leq 1\). Notice, in this connection, that if the discriminant vanishes for infinitely many negative values of \(t\) having 0 as a limit point it would vanish identically for \(-\delta \leq t \leq 1\), so that this case is automatically included in the case where the discriminant vanishes at \(t = 1\). Hence there will be infinitely many distinct real points \((b_{1h}, \ldots, b_{nh})\), converging (as \(h \to \infty\)) on the origin, at each of which the discriminant \(D(z_1, \ldots, z_n)\) must vanish and also at each of which the equation \(F(w, z_1, \ldots, z_n) = 0\) has at least one real solution \(w = c_h\).

By Theorem 5, the points on \(R_{\Phi}(n, n+1)\) where \(D(z_1, \ldots, z_n) = 0\) form a finite number of \(R\)-configurations \(R_{\Phi}^{\Phi}(n-1, n+1)\), where \(\Phi\) is always a factor of \(D\) and is also an \(R\)-irreducible polynomial in one of the (non-specialized) \(z\)'s, say \(z_n\), with coefficients analytic in the other \(z\)'s. Since the number
\[ \mathcal{R}_{\Phi}(n-1, n+1) \subseteq \mathcal{R}_{\Phi}(n, n+1), \] we shall have

\[ F(w(z_1), z_1, z_2(z_1), \ldots, z_n(z_1)) \equiv 0 \] as we wished to prove.

Under the second alternative, we infer the existence of two distinct sets of \((n-1)\) analytic real functions, \(\tilde{z}_2(z_1), \ldots, \tilde{z}_{n-1}(z_1), \tilde{w}(z_1)\) and \(\hat{z}_2(z_1), \ldots, \hat{z}_{n-1}(z_1), \hat{w}(z_1)\) all \(2n - 2\) functions being defined, say, for positive real values of \(z_1\) of sufficiently small absolute value, such that

\[ F^*(\tilde{w}(z_1), z_1, \tilde{z}_2(z_1), \ldots, \tilde{z}_{n-1}(z_1)) \equiv 0 \quad \text{and} \quad F^*(\hat{w}(z_1), z_1, \hat{z}_2(z_1), \ldots, \hat{z}_{n-1}(z_1)) \equiv 0. \]

These functions all tend to 0 as \(z_1 \to 0\). We then set

\[ \tilde{w}(z_1) = R(\tilde{w}(z_1); z_1, \tilde{z}_2(z_1), \ldots, \tilde{z}_{n-1}(z_1)) \]

\[ \tilde{z}_n(z_1) = R(\tilde{w}(z_1); z_1, \tilde{z}_2(z_1), \ldots, \tilde{z}_{n-1}(z_1)) \]

\[ \hat{w}(z_1) = R(\hat{w}(z_1); z_1, \hat{z}_2(z_1), \ldots, \hat{z}_{n-1}(z_1)) \]

\[ \hat{z}_n(z_1) = R(\hat{w}(z_1); z_1, \hat{z}_2(z_1), \ldots, \hat{z}_{n-1}(z_1)). \]

Again, since \(\mathcal{R}_{\Phi}(n-1, n+1) \subseteq \mathcal{R}_{\Phi}(n, n+1)\), we know that

\[ F(\tilde{w}(z_1), z_1, \tilde{z}_2(z_1), \ldots, \tilde{z}_n(z_1)) \equiv 0 \] and

\[ F(\hat{w}(z_1), \tilde{z}_2(z_1), \ldots, \hat{z}_n(z_1)) \equiv 0. \]
If the set \( \{ \bar{z}_2(z_1), \ldots, \bar{z}_{n-1}(z_1) \} \) is distinct from the set \( \{ \hat{z}_2(z_1), \ldots, \hat{z}_{n-1}(z_1) \} \), there is nothing further to be proved. Otherwise \( \bar{w}^*(z_1) \neq \hat{w}^*(z_1) \) by our inductive hypothesis. It follows that either \( \bar{w}(z_1) \neq \hat{w}(z_1) \) or \( \bar{z}_n(z_1) \neq \hat{z}_n(z_1) \). For we know that \( \bar{w}(z_1) \) is the same real linear combination of \( \bar{w}(z_1) \) and \( \bar{z}_n(z_1) \) that \( \hat{w}^*(z_1) \) is of \( \hat{w}(z_1) \) and \( \hat{z}_n(z_1) \).

Hence if \( \bar{w}(z_1) = \hat{w}(z_1) \) and \( \bar{z}_n(z_1) = \hat{z}_n(z_1) \), we would have the absurd result that \( \bar{w}^*(z_1) = \hat{w}^*(z_1) \). This completes the proof.
BIBLIOGRAPHICAL NOTES

1. Birkhoff, G. D. Sur le problème restreint des trois corps (second
mémorandum). Annali della reale Scuola Normale Superiore
di Pisa, ser. 2, vol. 5, 1936, pp. 1 - 42. Or his
The quotation of the text is from p. 40 of the Annals',
or p. 707 of the Collected Papers. The sketch here given
of the Wintner-Strömgren termination principle lacks
the enormous detail of Wintner's treatment, but it is
just as lucid and far more economical of the reader's
time.

2. Bliss, G. A. A Generalization of Weierstrass' Preparation Theorem
for a Power Series in Several Variables. Transactions,
133 - 145. Bliss's criticism of Poincaré 1 is that
Poincaré's method may introduce extraneous solutions.

3. Bochner, S. Several Complex Variables (with W. T. Martin), Princeton
1948, 216 pp. The last chapter, following Rückert's
modern algebraic treatment of complex local analytic
varieties, affords an alternative method, which with
suitable modifications could undoubtedly be applied to
our problem in the domain of reals.

Dissertation, Göttingen (1905). This paper has not
been available to me for first hand inspection.
According to Bliss, it employs the elimination method
of Kronecker and is thus possibly similar to the
Lefschetz treatment.

5. Kronecker, L. Grundzüge einer arithmetischen Theorie der algebraischen
Berlin, Reimer 1882. Published in Leopold
Kronecker's Werke, Leipzig 1897, and in Crelle's Journal
für die reine und angewandte Mathematik, Vol. 92,
pp. 1 - 122. For a more convenient reference to Kronecker's

6. Lefschetz, S. Complete families of periodic solutions of differential
equations. Commentarii Mathematici Helvetici, vol. 28
(1954), pp. 341 - 345. This paper, applying the
Kronecker elimination method, arrives in surprisingly
short order at a result substantially equivalent to our
Theorem 8. Some details, however, are omitted; and,
after a serious attempt to supply both the missing details
and the essential additional Theorem 9, it was decided
that, all things considered, it would be simpler to use
the Osgood method.
Martin, T. W. (see Bochner).

Osgood, W. F.
1. Lehrbuch der Funktionentheorie, 2nd volume, first section, Berlin 1924. In the footnote on p. 107 appears the criticism of Poincaré, wherein Osgood comments that it can only be guessed what Poincaré meant by his use of the words "restent distinctes quand on annule tous les x."

Poincaré, H.
1. Les méthodes nouvelles de la mécanique céleste, vol. 1, Paris, 1892. The elimination theory of pp. 71-72 is closely allied with the author's thesis and is subject to criticisms of both Bliss and Osgood.


Puiseux, V.
1. Recherches sur les fonctions algébriques. Journal de Mathématiques pures et appliquées (Liouville), vol. 15 (1850), pp. 365-480. The original source of classical results on the development of analytic functions near an algebraic branch point in fractional power series, referred to by Lefschetz, Poincaré, and Wintner.

Rickert, W.

Weierstrass

Wintner, A.
1. Grundlagen einer Genealogie der periodischen Bahnen im restringierten Dreikörperproblem. Erste Mitteilung. Beweis des E. Strömgren's dynamischen Abschlussprinzips der periodischen Bahngruppen. Mathematische Zeitschrift, vol. 34, (1932), pp. 321-349. The heart of this lengthy paper is on pp. 344-347. At the bottom of p. 344, the author appeals to geometric intuition. He returns on p. 347 to consider the possibility of a rigorous proof by induction on the dimensionality of the space. He apparently appeals to results of Puiseux for the two dimensional case and gives a somewhat vague reference to Poincaré 1 for the elimination process to be used in the induction. As previously mentioned it is this work of Poincaré which has been criticized by Bliss and Osgood.
PART III

THE STRÖMGREN-WINTNER PRINCIPLE ON THE TERMINATION OF FAMILIES OF
PERIODIC SOLUTIONS

by
Daniel C. Lewis, Jr.

The Strömgren-Wintner principle was originally formulated for the
equations of celestial mechanics. These equations always admit a first
integral, namely the energy integral or, when rotating coordinate axes are
used as in the restricted problem of three bodies, the integral of Jacobi.
The principle is concerned with families of periodic solutions that are
not necessarily isoenergetic. That is, the value of the first integral
changes, in general, as we pass from one member of the family to another.

In our present formulation (Theorem 3) we give a theory immediately
applicable to autonomous systems containing a parameter \( \mu \). This formu-
lation would seem to be more suitable for most applications other than
those of celestial mechanics. Nevertheless our theory is a definite
generalization of the Strömgren-Wintner principle, since the latter may
be obtained from it by suitable specialization. In doing this, our
quantity \( \mu \) is to be thought of as the value of the first integral, or
some simple function thereof; and the first integral itself is used to
lower the order of the system from \( 2m \), where \( m \) is the number of degrees of
freedom, to \( 2m - 1 \). The "vector" \( x \) is then to be thought of as having
\( 2m - 1 \) components, while \( \| x - y \| \) is the distance between two "points", \( x \) and
\( y \), in some convenient metric. The words "point" and "vector" are in this
context synonymous. It is, of course, not necessary to employ the same coordinates over the entire manifold of states of motion. An explicit inspection of the equations of the n-body problem will reveal the nature of this manifold $R_\mu$ for any fixed $\mu$ or of the "region" $R = \bigcup_{\mu} R_\mu$. Indeed, from the fact that the kinetic energy is positive definite in the velocities, we see at once that the boundary of $R$ can only be reached by allowing one or more of the bodies to recede to infinity, or by allowing one or more of them to approach collision, or by allowing the energy (or Jacobi) constant to approach infinity. In this way the alternatives $A_1, A_2, A_3$ of our Theorem 3, to be proved in the sequel, yield at once the famous Strömberg-Wintner conditions for the termination of a family of periodic solutions in the n-body problem or in the restricted problem of three bodies.

In the sequel $x$ and $f$ are to represent n-vectors, while $t$ and $\mu$ will represent real variables.

Suppose the system

\[
\frac{dx}{dt} = f(x, \mu),
\]

has a periodic solution $x(t, \mu)$ with period $T(\mu)$ for each value of $\mu$ in a certain open interval, whose right hand end point can, without loss of generality, be taken as 0. It will also be assumed that $f(x, \mu)$ is defined and analytic if $(x, \mu)$ is in an $(n + 1)$ dimensional open region $R$, having a closed bounded $(n + 1)$ dimensional sub region $R^*$ whose intersection $R^*_\mu$ by the hyperplane $\mu = \mu_0$ is not vacuous if $-K < \mu_0 < 0$, where $K$ is a positive number. Let the given periodic solution $x(t, \mu)$ lie in $R^*_\mu$ for $-K < \mu < 0$ and let the period $T(\mu)$ be bounded for $\mu$ in this same interval.

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THEOREM 1. Under the assumptions just specified, the system (1) also has at least one periodic solution, when \( \mu = 0 \), with the following property: Denoting this periodic solution by \( x(t, 0) \) and the period by \( T(0) \), there exists a sequence \( \mu_1, \mu_2, \ldots \), with \( -K < \mu_i < 0 \) and \( \mu_i \to 0 \) as \( i \to \infty \), such that

\[
\lim_{i \to \infty} T(\mu_i) = T(0) \quad \text{and} \quad \lim_{i \to \infty} x(t, \mu_i) = x(t, 0) \quad \text{uniformly.}
\]

Proof. Let \( I^* \) be a bounded closed interval containing all the numbers \( T(\mu) \) for \( -K < \mu < 0 \). From the compactness of the product space \( \mathbb{R}^* \times I^* \), we know that we may select a sequence \( \mu_1, \mu_2, \ldots \) with \( -K < \mu_i < 0 \) and with \( \lim_{i \to \infty} \mu_i = 0 \) such that \( \lim_{i \to \infty} x(0, \mu_i) = x_0 \), for some \( x_0 \in \mathbb{R}^* \), and \( \lim_{i \to \infty} T(\mu_i) = T_0 \), for some \( T_0 \in I^* \). \( ||f|| \) is bounded for \( (x, \mu) \in \mathbb{R}^* \). Hence the derivatives of \( x(t, \mu_i) \) are uniformly bounded for all \( i \) and for \( 0 \leq t \leq T^* \), where \( T^* \) is the right hand end point of \( I^* \). Hence the sequence \( x(t, \mu_1), x(t, \mu_2), \ldots \) either converges uniformly on the interval \( 0 \leq t \leq T^* \) to a vector function \( x(t, 0) \), or else a suitably chosen subsequence has this property (in accordance with Ascoli's theorem). Hence, without loss of generality, we may write

\[
\lim_{i \to \infty} x(t, \mu_i) = x(t, 0) \quad \text{uniformly for } 0 \leq t \leq T^*
\]

with \( x(0, 0) = x_0 \). Evidently also \( x(t, 0) \in \mathbb{R}^* \quad \mathbb{R}^* \quad \mathbb{R} \).

We shall show next that \( x(T_0, 0) = x_0 \). From the periodicity of \( x(t, \mu_i) \) it is known that

\[
(2) \quad x(T(\mu_i), \mu_i) = x(0, \mu_i) \to x_0 \quad \text{as } i \to \infty.
\]
Since $x(t,0)$ is the uniform limit of continuous vector functions, it is continuous itself. Hence, if $\xi > 0$, $N_1$ may be chosen so that

$$||x(T(\mu_1), 0) - x(T_o, 0)|| < \frac{1}{2} \varepsilon \text{ for } i > N_1.$$  

Again using the fact that the convergence is uniform, we select $N_2$ independently of $t$, so that

$$||x(t, \mu_1) - x(t, 0)|| < \frac{1}{2} \varepsilon \text{ for } i > N_2.$$  

Setting $t = T(\mu_1)$ in (4) and using the triangle inequality applied to (3) and (4), we obtain

$$||x(T(\mu_1), \mu_1) - x(T_o, 0)|| < \varepsilon$$  

for $i > \max(N_1, N_2)$. Hence, by (2), we see, on allowing $\mu_1$ to tend to 0, that

$$||x_0 - x(T_o, 0)|| \leq \varepsilon$$  

for any $\xi > 0$. Hence

$$x(T_o, 0) = x_0 = x(0, 0).$$  

Using the uniform continuity for $f$ in the compact space $\mathbb{R}^n$, it is readily proved that $\lim_{i \to \infty} f(x(t, \mu_1), \mu_1) = f(x(t, 0), 0)$ uniformly for
0 ≤ t ≤ T^*, from which fact it is easy to see that x(t, 0) satisfies (1), when μ = 0, on the interval 0 ≤ t ≤ T^*.

Finally from (5), it is clear that this solution may be extended so as to be defined for all t; and that this extended solution x(t, 0) admits T_0 as a period. This completes the proof of the theorem.

Notice that the point x(0, 0) = x_0 set up as a limit point of the set of x(0, μ) need not be unique. Even, if x(t, μ) depends continuously on μ, we can not expect uniqueness in x(0, μ), without introducing further hypotheses. To see this, it is only necessary to remark that, since the system (1) is autonomous, the given family of periodic solutions can always be replaced by an "equivalent" family x(t + T(μ), μ), where T(μ) can be continuous for -K < μ < 0 but might oscillate rather wildly as μ → 0. For example T(μ) might be sin (μ^{-1}). It is, however, possible to prove the following:

**THEOREM 2.** If the dependence of x(t, μ) on μ is continuous, the initial point x_0 = x(0, 0) furnished by the previous theorem is either unique or else the set of such points is perfect (i.e., each such point is a limit point of other such points).

**Proof.** Suppose that we have two distinct periodic solutions x(t, 0) and \( \tilde{x}(t, 0) \) with distinct initial points x_0 and \( \tilde{x}_0 \) such that

\[
\lim_{i \to \infty} x(0, \mu_1) = x_0 \text{ and } \lim_{i \to \infty} x(0, \mu'_{1}) = \tilde{x}_0,
\]

where the \( \mu \)'s and \( \mu' \)'s are all negative and tend to 0 as \( i \to \infty \). Without loss of generality (by considering suitable subsequences, if necessary) we may assume \(- K < \mu_1 < \mu'_{1} < \mu_2 < \mu'_{2} < \mu_3 < \mu'_{3} < \cdots \) and that
\[ |x(0, \mu_1) - x_0| < \frac{2}{3} \delta \] and \[ |x(0, \bar{\mu}_1) - \bar{x}_0| < \frac{2}{3} \delta \] for \( i = 1, 2, 3, \ldots \)

where \( \delta \) is any positive number less than the distance between the two points \( x_0 \) and \( \bar{x}_0 \). It will be sufficient to show the existence of a third point \( \bar{x}_0 \) at a distance \( \frac{1}{3} \delta \) from \( x_0 \) and a sequence of \( \bar{\mu}_1 \)'s tending to 0 through negative values, such that \( \lim_{t \to \infty} x(t, \bar{\mu}_1) \) converges uniformly to a periodic solution of (1) with \( \mu = 0 \) having initial point \( \bar{x}_0 \).

To this end, we notice that the continuity of \( x(0, \mu) \) requires the existence of a number \( \mu_1^* \) such that \( \mu_1 < \mu_1^* < \bar{\mu}_1 \) and such that
\[ |x(0, \mu_1^*) - x_0| = \frac{1}{3} \delta \]
This is because the continuous function \( |x(0, \mu) - x_0| \) is less than \( \frac{1}{3} \delta \) for \( \mu = \mu_1 \) and is certainly greater than \( \frac{2}{3} \delta \) for \( \mu = \bar{\mu}_1 \), so that there is surely an intermediate value of \( \mu \), namely \( \mu_1^* \), where
\[ |x(0, \mu) - x_0| \]

is exactly equal to \( \frac{1}{3} \delta \). As in the proof of Theorem 1 we use the compactness of \( \mathbb{R}^* \times I^* \) so as to choose \( \bar{\mu}_1, \bar{\mu}_2, \ldots \) as a suitable subsequence of \( \mu_1^*, \mu_2^*, \ldots \), such that \( \lim_{i \to \infty} x(0, \bar{\mu}_i) \) exists and such furthermore that \( \lim_{i \to \infty} x(t, \bar{\mu}_i) \) converges uniformly to a periodic solution of (1) with initial point \( \bar{x}_0 = \lim_{i \to \infty} x(0, \bar{\mu}_i) \). Since \[ |x(0, \bar{\mu}_1) - x_0| = \frac{2}{3} \delta \], it is clear also that \[ |\bar{x}_0 - x_0| = \frac{1}{3} \delta \].

In order to discuss the existence of further periodic solutions, say, for \( \mu > 0 \), it is convenient to consider a local analytic "surface of section" through the initial point \( x_0 \) of a periodic solution \( x(t, 0) \) of (1) for \( \mu = 0 \). We hereby assume that \( x(t, 0) \) is not an equilibrium solution. Let this surface of section be denoted by \( S \) and let it consist of those points in a neighborhood \( N \) of \( x_0 \) which satisfy the equation...
\[ S(x) = 0, \]

where \( S(x) \) is any scalar function of the vector \( x \), defined and analytic for \( x \in \mathbb{N} \) and such that

\[ S(x_0) = 0 \]

\[ \frac{\partial S(x_0)}{\partial x} f(x_0, 0) \neq 0. \]

In this connection it should be observed that the neighborhood \( N \) may be chosen so small that \( N \subseteq \mathbb{R} \) for \( |\mu| < K \), at least if the positive number \( K \) is taken small enough. Here \( \mathbb{R} \) denotes the \( n \) dimensional section obtained from the \( n + 1 \) dimensional region \( R \) by taking \( \mu = \bar{\mu} \). The proof of this fact runs as follows: \( R \), being open and containing \((x_0, 0)\), must contain an \((n + 1)\) dimensional neighborhood \( M \) of \((x_0, 0)\). Hence there is a positive number \( K \exists (x, \mu) \in M \) for \( |\mu| < K \) and \( ||x - x_0|| < K \). Hence \( x \in M \subseteq \mathbb{R} \) for \( |\mu| < K \) and \( ||x - x_0|| < K \). It therefore suffices to let \( N \) consist of those points \( x \) for which \( ||x - x_0|| < K \).

Conditions (7) and (8) merely express the requirement that the periodic solution curve \( x = x(t, 0) \) cuts \( S \) at \( x_0 \) at an angle different from 0.

The same will therefore also be true of those sufficiently nearby periodic solutions \( x = x(t, \mu_i) \) of which \( x = x(t, 0) \) is the uniform limit as \( \mu_i \to 0 \) and \( i \to \infty \). Without loss of generality we may choose the notation so that this is true for \( i = 1, 2, \ldots \). We also lose no generality in writing
\begin{align*}
S(x(0, \mu_1)) = 0, \quad i = 1, 2, \ldots
\end{align*}

In fact, if the solution curve \( x = x(t, \mu_1) \) cuts \( S \) when \( t = t_1 \), we have only to modify the equations of the solution curve so that they appear in the form \( x = x(t + t_1, \mu_1) = x^*(t, \mu_1) \). Evidently both (1), for \( \mu = \mu_1 \), and (9) are satisfied with \( x \) replaced by \( x^* \), and then finally the notation is modified by dropping the asterisk.

From the continuity theorems for differential equations of the form (1), we can find a neighborhood \( N_1 \) of \( x_0 \subset N \) and a positive number \( K_1 \leq K \), such that any solution of (1), in case \( |\mu| < K_1 \), which intersects \( S \cap N_1 \) at a point \( x_1 \), for \( t = t_0 \) will intersect \( S = S \cap N \) again for the first time for \( t > t_0 \) at a point \( \bar{x} \).

Let the transformation of \( S \cap N_1 \) into \( S \) which thus sends \( x \) into \( \bar{x} \) be denoted by \( T_{\mu_1} \). This transformation is easily shown to be analytic, at least if \( N_1 \) and \( K_1 \) are sufficiently small.

Any fixed point of \( T_{\mu_1} \) corresponds to a periodic solution of (1) cutting \( S \) at this fixed point. Examples of such fixed points are: \( x_0 = x(0, 0) \), fixed under \( T_0 \), and \( x(0, \mu_1) \), fixed under \( T_{\mu_1} \). We are interested in investigating the existence of still other fixed points for still other values of \( \mu \).

The condition that a point \( x \), sufficiently near to \( x_0 \), should be a fixed point of \( T_{\mu_1} \) may be written in the form

\begin{align*}
F(x, \mu) = 0
\end{align*}
where \( F \) is an \( n \)-vector analytic function of \( x \) and \( \mu \). Evidently (10) is equivalent to \( n \) scalar equations, of which one is simply the equation \( S(x) = 0 \) requiring the point \( x \) to lie on \( S \), while the other \((n-1)\) equations actually express the fact that \( x \) is a fixed point of the transformation \( T_\mu \) of the \((n-1)\)-cell \( S \cap N \) into \( S \cap N \).

The point \( x_0 \), of course, is a fixed point when \( \mu = 0 \), so that

\[
F(x_0, 0) = 0,
\]

and we, of course, have infinitely many other fixed points corresponding to the periodic solutions \( x(t, \mu_i), i = 1, 2, \ldots \), where \( \mu_i \) tends to 0 through negative values as \( i \to \infty \) and where also \( x(t, \mu_i) \to x(t, 0) \).

Upon setting \( x(0, \mu_i) = x_1 \), we thus see that \( x_1 \) is a fixed point under \( T_{\mu_1} \), so that we have

\[
F(x_1, \mu_1) = 0, \text{ where } (x_1, \mu_1) \to (x_0, 0).
\]

Of course, if the jacobian of \( F \) with respect to \( x \) is different from 0 at \((x_0, 0)\), we can, by the implicit function theorem, solve the vector equation (10) for \( x \) as an analytic vector function of \( \mu \) for \(|\mu|\) sufficiently small. Thus, in this case, we always get periodic solutions for all sufficiently small positive value of \( \mu \). If, however, the jacobian is zero, the theory of singular solutions of systems of real analytic equations shows that, under condition (12), there must exist at least two distinct analytic arcs.
or curves $C_1$ and $C_2$ in $R$ having a common end point at $(x_0, 0)$ such that any point on either $C_1$ or $C_2$ satisfies (10). For convenience, we shall say that $C_1$ enters to the right of the origin if it contains points $(x, \mu)$ for which $\mu > 0$. If either $C_1$ or $C_2$ enters to the right, we must have periodic solutions for all sufficiently small positive values of $\mu$. Otherwise, neither $C_1$ nor $C_2$ enter to the right. We then have two distinct families of periodic solutions each having $x(t, 0)$ as a limiting periodic solution.

In other words, it is always possible to find a one parameter family of periodic solutions $x[t, \lambda, \mu(\lambda)]$ depending continuously on a suitably chosen parameter $\lambda$ for $-1 < \lambda < +1$, such that $\mu(0) = 0$ and $x[t, 0, 0]$ is the known periodic solution for $\mu = 0$ through $x_0$, and such that whenever $\mu(\lambda_1) = \mu(\lambda_2)$ for $-1 < \lambda_1 < \lambda_2 < +1$, the solutions $x[t, \lambda_1, \mu(\lambda_1)]$ and $x[t, \lambda_2, \mu(\lambda_2)]$ are essentially distinct. (i.e. there exists no constant $c$ such that $x[0, \lambda_1, \mu(\lambda_1)] = x[c, \lambda_2, \mu(\lambda_2)]$). Moreover the parameter $\lambda$ may be chosen so that both $\mu(\lambda)$ and $x(t, \lambda, \mu(\lambda))$ are analytic in $\lambda$ except possibly at $\lambda = 0$, where, of course, they are nevertheless continuous.

We shall say that under the circumstances described above that the family of periodic solutions $x(t, \mu)$ given originally for $-K < \mu < 0$ does not "terminate" at $\mu = 0$. We have therefore proved, under the hypotheses $H_0$, $H_1$, $H_2$, $H_3$ enumerated below that a family of periodic solutions can not terminate at $\mu = 0$.

$H_0$: $f(x, \mu)$ is analytic for $(x, \mu) \in R$.

$H_1$: The period, $T(\mu)$, is bounded, say $< A$, for $-K < \mu < 0$.

$H_2$: $[x(t, \mu), \mu] \in \mathbb{R}^*$, where $\mathbb{R}^*$ is a closed bounded region $\subset \mathbb{R}$.

$H_3$: $x(t, 0)$ is not an equilibrium solution.
Hence, if a family of periodic solutions satisfying a system of the form (1) does terminate (i.e., does not not terminate) it means that at least one of the above hypotheses fails to hold. We can thus state the following definitive theorem:

**Theorem 3.** If \( f(x, \mu) \) is an analytic vector function defined for \((x, \mu) \in \mathbb{R}\) and if the system (1) admits for \(-K < \mu < 0\) a periodic solution \(x(t, \mu)\) with period \(T(\mu)\), then one, at least, of the following alternatives must hold:

**A_1.** There is a sequence \(\mu_1 < \mu_2 < \mu_3 < \cdots < 0\) such that \(\lim_{i \to \infty} \mu_i = 0\) and \(\lim_{i \to \infty} T(\mu_i) = \infty\).

**A_2.** There exists a sequence \(t_1, t_2, \cdots\) and a sequence \(\mu_1 < \mu_2 < \cdots < 0\) such that \(\lim_{i \to \infty} \mu_i = 0\) and \(\lim_{i \to \infty} |x(t_i, \mu_i)| = \infty\) or else \((x(t_i, \mu_i), \mu_i)\) approaches a boundary point of \(R\).

**A_3.** There exists a sequence \(\mu_1 < \mu_2 < \cdots < 0\) such that \(\lim_{i \to \infty} \mu_i = 0\) and \(\lim_{i \to \infty} x(t, \mu_i)\) exists and is independent of \(t\).

**A_4.** The family does not terminate at \(\mu = 0\).

It is hardly necessary to add that alternative \(A_1\) holds, if hypothesis \(H_1\) fails to hold \((i = 1, 2)\). Alternatives \(A_3\) surely holds if all of the possible limit periodic solutions, \(x(t, 0)\), discussed in Theorem 1 turn out to be equilibrium points, so that \(H_2\) must fail to hold no matter how the possibly non-unique \(x(t, 0)\) is chosen. Finally, alternative \(A_4\) is the residual situation which must hold when \(H_1, H_2,\) and \(H_3\) are known to hold.
Of course, in the foregoing discussion, one may replace \( \mu \) by \( \mu_0 + \mu \)
where \( \mu_0 \) is a constant. It is then clear how non-termination may be defined at an arbitrary value \( \mu_0 \) of the parameter \( \mu \) as we approach \( \mu_0 \) from either the right or the left.

The equation (1), to which this paper is devoted, represents an autonomous system. We focussed attention on such systems partly because, for historical reasons, it seemed desirable to show as directly as possible the connection with the Strömgren-Wintner principle, and partly because some of the most interesting applications are to autonomous systems (cf. Theorem 4 below).

Somewhat similar results, however, can be proved also in the non-autonomous case where the right hand member of (1) depends periodically and analytically on \( t \). The non-autonomous case is even somewhat simpler in some respects, because all periodic solutions have the same period, say \( 2\pi \), which is the same as the period of the right hand member. Thus, alternative \( A_1 \) does not occur in the non-autonomous case. Actually there seems little reason to discuss a second theory for the non-autonomous case. Indeed the non-autonomous case can almost be reduced to the autonomous case of one higher order. For suppose we consider the system

\[
\frac{dz}{d\theta} = Z(z, p, \theta, \mu), \quad \frac{dp}{d\theta} = P(z, p, \theta, \mu),
\]

where \( z \) and \( Z \) are \((n-1)\)-vectors, \( p, \theta, \mu \) are real variables, while \( Z \) and \( P \) are periodic in \( \theta \) with period \( 2\pi \). The system (13) may also be written in the form,
\[
\frac{dz}{dt} = Z(z, p, \theta, \mu), \quad \frac{dp}{dt} = P(z, p, \theta, \mu), \quad \frac{d\theta}{dt} = 1,
\]

where \( \theta \) is no longer the independent variable. Invoke the transformation
\( x = e^p \cos \theta, \ y = e^p \sin \theta \), which is analytic and non-singular in the large, although \( \theta \) is determined only modulo \( 2\pi \) when \( x \) and \( y \) are given. This
indeterminacy of \( \theta \) is not important because of the periodicity of \( Z \) and
\( P \) in \( \theta \). Hence \( \overline{Z}(z, x, y, \mu) = Z(z, p, \theta, \mu) \) and \( \overline{P}(z, x, y, \mu) = P(z, p, \theta, \mu) \)
may readily be defined as long as \( x \) and \( y \) are not both zero and
\( x^2 + y^2 = \log \sqrt{x^2 + y^2}, \ \mu \) are values for which \( Z \) and \( P \) were originally defined.

A simple calculation shows that (14), and hence the non-autonomous system
(13) of order \( n \), is essentially equivalent to the following autonomous
system of order \( (n + 1) \):

\[
\frac{dz}{dt} = \overline{Z}(z, x, y, \mu)
\]

(15)

\[
\frac{dx}{dt} = x\overline{P}(z, x, y, \mu) - y
\]

\[
\frac{dy}{dt} = y\overline{P}(z, x, y, \mu) + x
\]

This system is only defined for \( x^2 + y^2 > 0 \), of course; and from the fact
that \( x \frac{dy}{dt} - y \frac{dx}{dt} = x^2 + y^2 \) for any solution of (15), it is not difficult
to reintroduce the angle \( \theta \) such that \( x = \sqrt{x^2 + y^2} \cos \theta, \ y = \sqrt{x^2 + y^2} \sin \theta, \ \theta = t + \text{const} \). Hence, any period of \( x(t) \) and \( y(t) \) would have to be a period
of \cos (t + c) and of \sin (t + c). Hence all periodic solutions of (15) must have a period which is an integral multiple of $2\pi$. There is thus a thorough going correspondence between the periodic solutions of (13) and (15). Theorem 3 applies to the special case of (15) and hence, indirectly to (13) with the alternative $A_1$ omitted.

This process of reducing the non-autonomous system to the case of an autonomous system of one order higher is the exact opposite of the more usual device (cf. for example the author's article J. Rat. Mech. and Analysis, vol. 4 (1955)), in which the solutions of an autonomous system near a given periodic solution are studied with the help of a non-autonomous system of one order lower. Notice, however, that the transformation for this latter purpose is usually valid only near the given periodic solution, while the transformation employed here is valid in the large. Hence the exclusive study of the autonomous case is perhaps more worthwhile than the exclusive study of the non-autonomous case.

We conclude with the following theorem which is of considerable importance in applications. For convenience in terminology we agree that two solutions $x(t)$ and $\bar{x}(t)$ of the autonomous system (1) are not essentially distinct if there is a constant $c$ such that $x(t + c) \equiv \bar{x}(t)$.

**THEOREM 4.** Suppose the system (1) is defined and analytic for $x \in S$ and for $-K < \mu < L$ where $S$ is an open region, while $K$ and $L$ are positive numbers. Suppose that for each $\mu$ in the interval $(-K, 0)$ there is a periodic solution in a closed region $S^* \subseteq S$. Suppose also that for each $\mu$ in the interval $(-K, L)$ there is essentially at most one periodic
solution in S and that if such a periodic solution exists (for a certain value of μ) it must lie in $S^k$ and its least period $T(\mu)$ can not exceed a positive number $P$. Suppose finally that $S$ contains no equilibrium points.

Then there exists a continuous family of periodic solutions defined for $-K \leq \mu \leq L$ and analytic in $\mu$ except perhaps at a finite set of points on the interval $[-K, L]$. Also $x(t, \mu) \epsilon S^k$ and the period $T(\mu) < P$.

Proof. By Theorem 3, the family, originally given for $-K < \mu < 0$, does not terminate at $\mu = 0$. Since for each $\mu$ there is by hypothesis essentially at most one periodic solution in $S$, the non-termination of the family can only occur by continuation for positive values of $\mu$. We therefore have a continuous family of periodic solutions for $0 \leq \mu < \alpha$ and analytic except possibly at $\mu = 0$, where $\alpha$ is some positive number. Let $\mu_1$ be the least upper bound of the set of $\alpha$'s for which this is true.

If $\mu_1 < L$, it appears that the same hypotheses holding for the point $\mu = 0$ hold also for $\mu = \mu_1$. Hence we find that the family does not terminate at $\mu_1$ and we can repeat the argument and find a number $\mu_2$ greater than $\mu_1$ such that we have a continuous family of solutions on the interval $\mu_1 \leq \mu < \mu_2$ and analytic except at $\mu_1$. The process can evidently be repeated, at least if $\mu_2 < L$. Proceeding in this way, we obtain a monotonic sequence $0 < \mu_1 < \mu_2 < \mu_3 < \cdots$, such that on the interval $\mu_1 \leq \mu < \mu_1 + 1$ we have a continuous family of periodic solutions analytic except at $\mu = \mu_1$ and not terminating at any $\mu_1 < L$.

We next assert that for some integer $k$ this sequence will come to an end because $L \leq \mu_k$. For, if this were not true, we would have
\[ \lim_{i \to \infty} \mu_i^* = \mu^* \leq L. \]

But then Theorem 3 also shows that the family not only does not terminate at $\mu^*$, but that there must be a number $\delta > 0$ such that the family is analytic for $\mu^* - \delta < \mu < \mu^*$. But because of the definition of $\mu^*$ as the limit of the monotonically increasing sequence of $\mu_i^*$'s, we know that there is some value of $i$ for which $\mu^* - \delta < \mu_i < \mu^*$. Since the family is not analytic at $\mu_i$, we have a contradiction. Hence, after repeating the operation a finite number of times we reach the right hand end point $\mu = L$.

Similarly we can reach the left hand end point $\mu = -K$ in a finite number of steps, thus obtaining the additional information, about the family originally given for $-K < \mu < 0$, that the family is analytic, except possibly for a finite number of points on the interval $-K < \mu < 0$, that it exists also at $\mu = -K$ and is continuous for $-K \leq \mu < 0$.

The truth of the last sentence of the theorem is trivially obvious.

This theorem has applications for many systems in which it can proved in an a priori manner that, if a certain type of periodic solution exists for certain values of $\mu$ on an interval $I$, it would be unique and its amplitude and period would have to satisfy certain inequalities. The application of the theorem then consists in concluding the existence of the family over the whole interval $I$ from a knowledge of its existence over any sub interval of $I$.

A well known example of such a system is the van der Pol system

\[ \dot{x} = y, \dot{y} = -x + \mu (x^2 - 1)y, \]

where we take $S$ to be the whole $xy$-plane.
exclusive of the origin and $S^*$ to be a suitably chosen annular region centered at the origin. Using these methods in a straightforward way, one can thus furnish an analytic proof of the existence of a periodic solution over the whole infinite interval $0 < \mu < +\infty$, and, of course, also for $-\infty < \mu < 0$. Moreover, we can assert that this family of solutions must depend analytically on $\mu$ except possibly at a denumerable set of points having no finite limit point.
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