PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS CONTAINING A PARAMETER

by

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Introduction

Concerning a system of differential equations of the type

\[
\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n, \mu, t) = X_i(x, \mu, t), \quad i = 1, \ldots, n,
\]

in which the \(X_i\)'s are sufficiently regular real functions of the indicated arguments and are periodic in \(t\) with period \(2\pi\) (say), Poincaré has proved that the existence of a known periodic solution, \(x_i = x_i(t) = x_i(t + 2\pi)\) when \(|\mu| = 0\) implies the existence of periodic solutions for all sufficiently small \(|\mu| > 0\), at least if the variational equations (appropriate to the given periodic solution) have no non-trivial periodic solutions with period \(2\pi\). This condition we shall refer to as Poincaré's hypothesis H. In the analytic case it is furthermore proved that these solutions admit power series expansions in \(\mu\). Unfortunately the usefulness of these important classical results of Poincaré is severely limited by lack of information on how to calculate a definite interval for \(\mu\) for which these periodic solutions exist. Furthermore, in the analytic case, the fact that convergent series for the periodic solutions exist and that the first few terms may even be written down explicitly is, for some purposes (such as numerical calculation), of limited interest in the absence of information as to how rapidly this convergence takes place.

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The purpose of this paper is to obtain definite estimates for the size of the interval for \( \mu \) and to obtain, in the analytic case, upper bounds for the remainders obtained by cutting the series off at the \( n^{th} \) terms.

An estimate of the size of the \( \mu \)-interval can be done in many ways. In particular, it is possible to review Poincaré's original proof and obtain such an appraisal in terms of bounds for the \( X \)'s and their first and second partial derivatives and in terms of a lower bound for the absolute value of a certain determinant, the non-vanishing of which is essentially Poincaré's hypothesis \( H \), previously mentioned. Poincaré used the implicit function theorem, and therefore a "quantitative" form of this theorem is necessarily involved in order to specify definitely the size of the neighborhood in which the implicit functions are defined. Such a form for the implicit function theorem suitable for this purpose is given, for instance, as a lemma on p. 129 of Reference 1. The final theorem on periodic solutions resulting from such an analysis is very awkward and the estimates do not appear to be very sharp. Hence we shall omit results of this type.

It turns out, however, that the method of integral equations and successive approximations, previously used by L. Lichtenstein and E. Hölder, is admirably suited for our purpose. We obtain, in Part I, our appraisal in terms of certain Lipschitz quantities pertinent to certain functions built out of the \( X \)'s and in terms of certain quantities belonging to the Green's matrix (for periodic boundary conditions) of the variational equations. The existence of this Green's matrix is guaranteed by the hypothesis \( H \). It will be shown that the estimate obtained in this way is, in certain senses, a best possible one. Moreover the result does not depend explicitly on \( n \) and hence can probably be generalized so as to apply to more general spaces. This is not true of the cruder estimates.
Apropos of the reference made above to work of Lichtenstein and Hölder, it should be stated that neither of them was interested in the problem of estimating the $\mu$-interval. They were, instead, interested in establishing the existence of periodic solutions for sufficiently small $|\mu|$ in certain special problems in which the hypothesis $H$ did not hold. This meant that the Green's matrix was available only in a generalized sense, a fact which added greatly to the difficulty of their problems. In contrast to these problems, the present paper is concerned only with the simplest case, where the hypothesis $H$ is assumed. The problem of estimating the $\mu$-interval in the more complicated and, in some respects, more interesting cases will be considered in a future paper.

In Part II, we indicate how the methods of Part I may be modified in the analytic case so as to yield periodic solutions for all complex values of $\mu$ whose absolute values do not exceed an explicitly given positive constant $L$. These periodic solutions, which are, of course, complex in general, though real when $\mu$ is real, are bounded with an explicitly given bound. We can therefore apply the Cauchy appraisal to the developments in powers of $\mu$. This yields results of the desired character on the rapidity of convergence.

All the results in Part II are consequences of bounds for certain functions built out of the $X$'s valid for complex domains; whereas Part I is based upon bounds assumed to be valid only in real domains.

The material following the proof of Theorem 1 in Part I is not necessary for understanding Part II.
Part I

By use of an obvious change of variable we can assume that the given periodic solution of the system described in the introduction is \(x_1 = 0, \ i = 1, 2, \ldots, n\). If we then set

\[ a_{ij}(t) = \left( \frac{\partial x_i}{\partial x_j} \right)_1 = \ldots = x_n = \mu = 0 \]

and \(F_i(x_1, \ldots, x_n, \mu, t) = F(x, \mu, t) = X_i(x, \mu, t) - \sum_{j=1}^{n} a_{ij}(t)x_j\), we see that the given system can be written in the prepared form

\[
\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j + F_i(x, \mu, t) \quad i = 1, \ldots, n,
\]

in which the \(F_i\)'s and their first partial derivatives with respect to the \(x_i\)'s vanish when \(x_1 = \ldots = x_n = \mu = 0\). Our main result, embodied in Theorem 1, is based on rather complicated looking hypotheses applied to the functions \(F_1, \ldots, F_n\) (there denoted as a vector \(F\)). The purpose of Lemma 1 is merely to show that these hypotheses are actually fulfilled if the \(F_i\)'s are of class \(C^1\) in a suitable domain. In other respects, the following sequence of lemmas, corollaries, and theorems should be self-explanatory.

**Lemma 1.** Let \(F_i(x, \mu, t), i = 1, \ldots, n\), and its first partial derivatives with respect to \(x_1, \ldots, x_n, \mu\) be continuous in \(x_1, \ldots, x_n, \mu, t\) for \(\sum_{i=1}^{n} x_i^2 \leq r^2, |\mu| \leq \lambda, \) and for \(0 \leq t \leq 2\pi\). Let \(F_i(0,0,t) = F_{ix_j}(0,0,t) = 0\), \(i, j = 1, \ldots, n\). Then non-negative functions \(B(t), \Phi(\zeta, \mu, t), \) and \(\Psi(\zeta, \mu, t)\) of the indicated arguments \((0 \leq \zeta \leq r)\) exist, such that the following five properties hold:
\[
(I) \lim_{\zeta \to 0, \mu \to 0} \Phi(\zeta, \mu, t) = 0 \text{ uniformly with respect to } t, \quad 0 \leq t \leq 2\pi, \text{ with } \Phi(\zeta, \mu, t)
\]
monotonic increasing in \( \zeta > 0 \).

\[
(II) \int_0^{2\pi} B(t) \, dt, \quad \int_0^{2\pi} \Phi(\zeta, \mu, t) \, dt, \quad \text{and } \int_0^{2\pi} \psi(\zeta, \mu, t) \, dt \text{ exist (for } \zeta = r \text{ and } |\mu| \leq \lambda)\n\]

\[
(III) \sum_{i=1}^{n} |F_i(x_i, \mu, t)|^2 \leq \Phi(\zeta, \mu, t) \zeta^2 + B(t) \mu^2 \quad \text{for } \zeta^2 = \sum_{i=1}^{n} x_i^2 \leq r^2
\]

and for \( |\mu| \leq \lambda \).

\[
(IV) \sum_{i=1}^{n} |F_i(x_i, \mu, t) - F_i(x'_i, \mu, t)|^2 \leq \psi(\zeta, \mu, t) \left( \sum_{i=1}^{n} x_i^2 \right) \left( \sum_{i=1}^{n} x_i'^2 \right),
\]

as long as \( \sum_{i=1}^{n} x_i^2 \) and \( \sum_{i=1}^{n} x_i'^2 \) are both \( \zeta^2 \leq r^2 \) and \( |\mu| \leq \lambda \).

\[
(V) \lim_{\zeta \to 0, \mu \to 0} \psi(\zeta, \mu, t) = 0 \text{ uniformly with respect to } t.
\]

It is evident at once that both \( \Phi \) and \( B \) can be chosen in an infinite number of ways, if they can be chosen at all. For instance, \( \Phi \) and \( B \) both retain their essential properties if they are multiplied by any continuous function (of \( t \)) algebraically \( \geq 1 \). We evidently prove more than required, if we can produce a \( \Phi(\zeta, \mu) \) independent of \( t \) and monotonic in \( \zeta \) such that \( \lim_{\mu \to 0, \zeta \to 0} \Phi(\zeta, \mu) = 0 \) and a \( B \) independent of \( t \), such that (III) holds. The following proof on the other hand makes no claim that it produces a least \( \Phi(\zeta, \mu, t) \) or a least \( B(t) \).
By the mean value theorem

\[ F_i(x, \mu, t) = \sum_{j=1}^{n} F_{ix_j} (x^*, \mu^*, t)x_j + F_{i\mu} (x^*, \mu^*, t)\mu, \]

where \(x_1^*, \ldots, x_n^*\) and \(\mu^*\) are suitably chosen numbers whose absolute values do not exceed those of the corresponding unstarred quantities. From the continuity of the derivatives of the \(F_i\) and the fact that \(F_{ix_j} (0, 0, t) = 0\), we can let

\[ a = \text{max. of } |F_{ix_j} (x, \mu, t)| \text{ for } \zeta \leq \zeta, |\mu| \leq \lambda \text{ and } 0 \leq t \leq 2\pi, \]

and we let

\[ \varphi_{ij} (\zeta, \mu) = \text{max of } |F_{ix_j} (x^*, \mu^*, t)| \text{ for } \zeta = \left( \sum_{k=1}^{n} (x_k^*)^2 \right)^{\frac{1}{2}} \leq \zeta \leq r, \]

for \( |\mu^*| \leq |\mu| \leq \lambda, \) and for \( 0 \leq t \leq 2\pi \). Then \( \lim_{\zeta \to 0, \mu \to 0} \varphi_{ij} (\zeta, \mu) = 0 \). (Otherwise, there would exist a sequence of points in the \(n+2\) dimensional space of the \(x\)'s, \(\mu\), and \(t\), with the \(x\)'s and \(\mu\) approaching zero, such that at these points \(|F_{ix_j}| > \) some positive number \(\epsilon\). From the compactness of our space, we may assume that these points approach a limit point \((0, \ldots, 0, 0, t)\) at which the continuous function \(|F_{ix_j}| \) would therefore have to take on a value different from zero, in fact \(\geq \epsilon\), contrary to hypothesis). Also it is clear that \(\varphi_{ij}\) will be monotonic in \(\zeta\). From Schwarz's inequality, we have

\[ \left( \sum_{j=1}^{n} F_{ix_j} (x^*, \mu^*, t)x_j \right)^2 \leq \left( \sum_{j} \varphi_{ij}^2 \right) \left( \sum_{j} x_j^2 \right) = \left( \sum_{j} \varphi_{ij}^2 \right) \zeta^2. \]

Hence we obtain from (2) the fact that

\[ |F_i (x, \mu, t)| \leq \left( \sum_{j} \varphi_{ij}^2 \right)^{\frac{1}{2}} \zeta + \beta |\mu| \]

\[ |F_i (x, \mu, t)|^2 \leq 2 \left( \sum_{j} \varphi_{ij}^2 \right) \zeta^2 + 2\beta^2 |\mu|^2. \]
Hence, it is evident that, by setting \( \bar{\phi}(\zeta, \mu) = 2 \sum \phi_{ij}^2 \) and \( B = 2n \phi^2 \), we obtain a \( \bar{\phi} \) and \( B \) having the required properties.

To obtain \( \psi(\zeta, \mu, t) = \psi(\zeta, \mu) \) we start again from the mean value theorem, in which \( (x^*) \) is a suitably chosen point in \( n \) dimensional space on the straight line segment between \( (x) \) and \( (x') \):

\[
|F_i(x, \mu, t) - F_i(x', \mu, t)|^2 = |\sum_{j=1}^{n} F_{ixj} (x^*, \mu, t)(x_j - x_j')|^2 \\
\leq \left( \sum_{j=1}^{n} \phi_{ij}(\zeta, \mu)|x_j - x_j'| \right)^2 \leq \left( \sum_{j=1}^{n} \phi_{ij}^2 \right) \left( \sum_{j=1}^{n} |x_j - x_j'|^2 \right).
\]

Summing over \( i \), we thus obtain (IV) with \( \psi(\zeta, \mu) = \sum \phi_{ij}^2 \) and, of course, (V) follows from the known facts about the \( \phi_{ij} \).

**Lemma 2.** Consider the linear differential system,

\[
\frac{dx}{dt} = A(t)x + f(t),
\]

where \( x \) and \( f \) are \( n \) dimensional vectors, \( A \) is an \( n^2 \)-matrix. \( A \) and \( f \) are known continuous functions of \( t \), periodic with period \( 2\pi \). Let \( X(t) \) be the \( n^2 \)-matrix, such that \( \frac{dX}{dt} = A(t)X \) and \( X(0) = I \), the unit matrix. (That such a matrix will always exist and be non-singular follows from existence theorems for systems of differential equations and from the well known fact that \( \det X(t) = \exp \int_0^t \text{trace } A(s)ds \). Suppose that \( \det (X(2\pi) - I) \neq 0 \). Finally, we define the matrix \( G(t,s) \) as follows:

\[
G(t,s) = -X(t)\left( X(2\pi) - I \right)^{-1} \left[ X(s) \right]^{-1} \text{ for } s < t \\
G(t,s) = -X(t) \left( I + \left[ X(2\pi) - I \right]^{-1} \right) \left[ X(s) \right]^{-1} \text{ for } t \leq s.
\]
Then there is a unique periodic solution of (3), given on the interval \( 0 \leq t \leq 2\pi \) by the formula

\[
x(t) = \int_0^{2\pi} G(t,s)f(s)ds.
\]

This lemma is a special case of a well known theorem on boundary value problems. We therefore omit the proof, which is quite obvious anyway. The \( G(t,s) \) is the so-called Green's matrix mentioned in the introduction.

In the sequel we use the notation \( ||v|| \) to represent \( \left( \sum_{i=1}^{n} v_i^2 \right)^{\frac{1}{2}} \), if \( v \) is a vector with components \( v_1, \ldots, v_n \), and we use the same symbol to denote \( \left( \sum_{i,j=1}^{n} v_{ij}^2 \right)^{\frac{1}{2}} \) if \( v \) is an \( n \times n \) matrix with element \( v_{ij} \) in the \( i \)th row and \( j \)th column.

**Lemma 3.** Let \( G(t,s) \) be the matrix defined in Lemma 2. Let \( f(s) \) be an arbitrary continuous vector function defined on \( 0 \leq s \leq 2\pi \). Then there exists an integrable scalar function \( |\gamma(t)| \), independent of \( f(s) \) and defined for \( 0 \leq t \leq 2\pi \), such that

\[
\left\| \int_0^{2\pi} G(t,s)f(s)ds \right\|^2 \leq |\gamma(t)| \cdot \int_0^{2\pi} ||f(s)||^2 ds.
\]

The proof consists in the remark that we may actually take \( \gamma(t) = \int_0^{2\pi} ||G(t,s)||^2 ds \), as it is very easy to prove with the help of the Schwarz inequality. We omit details. Nevertheless, this particular choice of \( \gamma(t) \) is not always the most advantageous. In fact, we find it convenient to record one further lemma about this matter.
Lemma 4. If $G(t,s)$ happens to be diagonal, we may choose $\Gamma(t) = \max_0^{2\pi} \int_0^2 G_{ij}(t,s)^2 ds$, where $G_{ij}(t,s)$ is the element in the $i$th row and $j$th column of $G(t,s)$.

This is also an easy consequence of the Schwarz inequality, the details of which are left to the reader.

Theorem 1. Consider the differential system,

$$\frac{dx}{dt} = A(t)x + F(x,\mu, t),$$

where $x$ and $F$ are vectors and $A(t)$ is a matrix. The latter two are periodic in $t$ with period $2\pi$. $\mu$ is a scalar real parameter. $F(x,\mu, t)$ is defined and continuous for $||x|| \leq r$, $|\mu| \leq \lambda$, and for all values of $t$. $A(t)$ satisfies the conditions of Lemma 2, including the existence of the Green's matrix, $G(t,s)$. Suppose there exist three non-negative scalar functions $B(t)$, $\Phi(\xi, \mu, t)$, and $\psi(\xi, \mu, t)$ defined for $0 \leq t \leq 2\pi$, $0 \leq \xi \leq r$, and $|\mu| = \lambda$, such that:

(I) $\lim_{\xi \to 0, \mu \to 0} \Phi(\xi, \mu, t) = 0$ uniformly with respect to $t$ with $\Phi$ monotonically increasing with increasing $\xi$.

(II) $\int_0^{2\pi} B(s) ds$, $\int_0^{2\pi} \Phi(\xi, \mu, s) ds$, and $\int_0^{2\pi} \psi(\xi, \mu, t) dt$ exist.

(III) $||F(x,\mu, t)||^2 \leq \Phi(||x||, \mu, t) \cdot ||x||^2 + B(t)\mu^2$.

(IV) $||F(x,\mu, t) - F(x', \mu, t)||^2 \leq \psi(\xi, \mu, t) \cdot ||x - x'||^2$, for $||x||, ||x'|| \leq \xi$.

(V) $\lim_{\xi \to 0, \mu \to 0} \psi(\xi, \mu, t) = 0$ uniformly with respect to $t$. 
Let \( L = \max \, |t| \), where \( |t| \) is any function satisfying the specifications of Lemma 3.

Then there exists just one periodic solution \( x = x(t, \mu) = x(t + 2\pi, \mu) \) of (7) for \( |\mu| \leq L \) such that \( |x(t, \mu)| \leq \varphi \), where \( \varphi \leq r \) and \( L \leq \lambda \) are any two positive numbers chosen so that

\[
M \left( \mu^2 + \frac{2\pi}{0} \Phi(\xi, \mu, s) \, ds \right) + \mu^2 \int_0^{2\pi} B(s) \, ds \leq \varphi^2
\]

for \( |\mu| \leq L \), \( \xi \leq r \), and so that

\[
M \int_0^{2\pi} \psi(\rho, \mu, t) \, dt \leq \varphi^2 < 1 \quad \text{for} \quad |\mu| \leq L.
\]

Moreover \( x(t, \mu) \) is continuous simultaneously in \( t \) and \( \mu \).

Before proceeding to the proof of this theorem, it should be remarked that the \( \varphi \) and \( L \) mentioned above can always be chosen (in an infinite number of ways) so as to satisfy the specified conditions. For example, we could, because of (I) and (V) take \( \varphi \) and \( L^* \) so small that

\[
\frac{2\pi}{0} \Phi(\xi, \mu, s) \leq (1 - \varphi) M^{-1}, \quad \text{and} \quad M \int_0^{2\pi} \psi(\xi, \mu, s) \, ds \leq \varphi^2
\]

as long as \( \xi \leq r \) and \( |\mu| \leq L^* \). We then choose

\[
L_{**} = \frac{1}{2} \left( \frac{r M}{2\pi} \int_0^{2\pi} B(s) \, ds \right)^{-\frac{1}{2}} \quad \text{or} \quad \infty \quad \text{if} \quad B(s) \equiv 0,
\]

and take \( L = \min \left( L^*, L_{**} \right) \). Hence for \( |\mu| \leq L \), we have

\[
M \mu^2 \int_0^{2\pi} B(s) \, ds \leq M \left( L_{**} \right)^2 \int_0^{2\pi} B(s) \, ds = \varphi \rho^2
\]
\[ M \int_0^{2\pi} \Phi(C(s), \mu, s) \, ds \leq K \rho^2 (1 - \epsilon) \mu^{-1} = (1 - \epsilon) \rho^2 \]

as long as \( C \leq \rho \). By addition of these last two inequalities we get (8) as required, while (9) is evidently also satisfied.

The construction of a periodic solution of (7) proceeds by the method of successive approximations. According to Lemma 2, any periodic solution of (7) must satisfy the following system of integral equations

\[ x(t) = \int_0^{2\pi} G(t, s) F(x(s), \mu, s) \, ds, \quad \text{for } 0 \leq t \leq 2\pi, \quad (10) \]

and, conversely, any solution of (10) is a solution of (7) for \( 0 \leq t \leq 2\pi \) with \( x(0) = x(2\pi) \) and hence yields a periodic solution of (7) for all \( t \). Let \( x^0(t) \) be an arbitrary continuous vector for \( 0 \leq t \leq 2\pi \) with \( x^0(0) = x^0(2\pi) \) and such that \( ||x^0(t)|| \leq \rho \). We now define inductively, for each value of \( \mu \) numerically less than or equal to \( L \), the vectors, \( x^1(t), x^2(t), \ldots, x^k(t), \ldots \) by means of the following formulas:

\[ x^k(t) = \int_0^{2\pi} G(t, s) F(x^{k-1}(s), \mu, s) \, ds, \quad k = 1, 2, \ldots \quad (11) \]

In order to show that this definition is effective we prove by induction that \( x^k(t) \) is continuous and that \( ||x^k(t)|| \leq \rho \) for \( 0 \leq t \leq 2\pi \), so that the right hand members of (11) exist for all \( k \) (since \( \rho \leq r \)). Moreover we prove simultaneously that \( x^k(0) = x^k(2\pi) \). For by our inductive hypothesis \( x^{k-1}(0) = x^{k-1}(2\pi) \), and \( x^{k-1}(t) \) is continuous, and \( ||x^{k-1}(t)|| \leq \rho \). Hence, from Lemma 2, \( x^k(0) = x^k(2\pi) \) and \( x^k(t) \) is continuous (and even differentiable, since it is a solution of
\[ \frac{dx}{dt} = Ax + F(x^{k-1}(t), \mu, t) \]. Moreover, from Lemma 3 applied to (11) we find that

\[ ||x^k(t)||^2 \leq \gamma(t) \cdot \int_0^{2\pi} ||F(x^{k-1}(s), \mu, s)||^2 ds. \] (12)

Now, from (III), we have

\[ ||F(x^{k-1}(s), \mu, s)||^2 \leq \Phi(||x^{k-1}(s)||, \mu, s) \cdot ||x^{k-1}(s)||^2 + B(s)\mu^2 \]

\[ \leq \Phi(\eta, \mu, s)^\alpha + B(s)\mu^2, \]

where \( \eta = \max ||x^{k-1}(s)|| \leq \rho \). In the last step, we use the monotonicity of \( \Phi \), of course. Hence, from (8), we find that

\[ \int_0^{2\pi} ||F(x^{k-1}(s), \mu, s)||^2 ds \leq \gamma^2 \cdot M^{-1}. \]

From (12) and the fact that \( \gamma(t) \leq K \), it follows at once that \( ||x^k(t)||^2 \leq \rho^2 \) as desired. Thus the successive approximations exist, and moreover, from Lemma 2, it is easily seen that \( x^k(0) = x^k(2\pi), k = 1, 2, \ldots \).

We next prove that \( \lim_{k \to \infty} x^k(t) \) converges uniformly for \( 0 \leq t \leq 2\pi \) and \( |\mu| \leq L \) to a vector function which we denote by \( x(t) \). For this purpose, we find from (11), upon setting \( w^k(t) = x^k(t) - x^{k-1}(t) \), that

\[ ||w^{k+1}(t)||^2 = \int_0^{2\pi} G(t, s) \left\{ F(x^k(s), \mu, s) - F(x^{k-1}(s), \mu, s) \right\} ds \]

which, by Lemma 3, does not exceed

\[ \gamma(t) \cdot \int_0^{2\pi} ||F(x^k(s), \mu, s) - F(x^{k-1}(s), \mu, s)||^2 ds, \]
which, in turn by (IV), does not exceed
\[
\int_0^{2\pi} \psi(\rho, \mu, s) \cdot \left\| w^k(s) \right\|^2 ds \leq M \int_0^{2\pi} \psi(\rho, \mu, s) ds \cdot \max \left\| w^k(t) \right\|^2.
\]
It follows from (9) that \( \max \left\| w^{k+1}(t) \right\| \leq \varphi \max \left\| w^k(t) \right\| \), whence an inductive process shows that
\[
\max \left\| w^{k+1}(t) \right\| \leq \varphi^k \max \left\| w^1(t) \right\|.
\]
In other words \( \left\| x^{k+1}(t) - x^k(t) \right\| \leq k^{\text{th}} \) term of a convergent geometric series of constant terms. Hence \( \lim_{k \to \infty} x^k(t) = x(t) \), say, uniformly for \( 0 \leq t \leq 2\pi \) and for \( \left| \mu \right| \leq L \).

Moreover, from the uniform continuity of \( F(x, \mu, t) \) it is clear that
\[
\lim_{k \to \infty} F(x^k(s), \mu, s) = F(x(s), \mu, s) \text{ uniformly.}
\]
Hence passing to the limit in (11) we get (10) as required. This shows that there is at least one periodic solution \( x(t) \) of (7) for \( \left| \mu \right| \leq L \) such that \( \left| x(t) \right| \leq p \).

If there were a second such solution \( \overline{x}(t) \), we could write (10) with \( x \) replaced by \( \overline{x} \), and then by subtraction we would obtain
\[
\left\| \overline{x}(t) - x(t) \right\|^2 = \left\| \int_0^{2\pi} G(t, s) \left\{ F(\overline{x}(s), \mu, s) - F(x(s), \mu, s) \right\} ds \right\|^2.
\]
Hence, using Lemma 3 and (IV), we find that
\[
\left\| \overline{x}(t) - x(t) \right\|^2 \leq \int_0^{2\pi} \left\| F(\overline{x}(s), \mu, s) - F(x(s), \mu, s) \right\|^2 ds \leq M \int_0^{2\pi} \psi(\rho, \mu, s) ds \cdot \max \left\| \overline{x}(s) - x(s) \right\|^2.
\]
Thus by (9), \( \max ||\bar{x}(t) - x(t)|| \leq C \max ||\bar{x}(t) - x(t)|| \), and since \( C < 1 \),
\( \max ||\bar{x}(t) - x(t)|| = 0 \), or \( \bar{x}(t) \equiv x(t) \). The solution is therefore unique.

Finally, one easily proves by induction that the successive approximations are continuous simultaneously in \( t \) and \( \mu \). Their convergence is uniform with respect to \( \mu \) as well as \( t \), from which the last statement in the theorem is seen also to be correct.

Our next purpose is to investigate the question as to whether there is any sense in which the theorem just proved is a best possible one. Naturally, we must contrive an example in which the inequalities are replaced by equalities or, at least, almost so. From a consideration of Lemmas 3 and 4, it seems desirable to take an example in which \( G(t,s) \) is diagonal. The inequality of Schwarz is seen to be fundamental and it is well known that in certain circumstances this inequality becomes an equality. On account of the fact that \( G(t,s) \) is discontinuous, whereas we want \( F(x,\mu,t) \) to be continuous, it is not hard to see that these circumstances can not be produced. Nevertheless, we can almost produce them, and we therefore will use the following lemma which makes precise the circumstances and sense in which the Schwarz inequality is almost replaced by an equality.

**Lemma 5.** Suppose that \( f(x) \) and \( g(x) \) are integrable and bounded on the interval \( a \leq x \leq b \), with \( |f|, |g| \leq A \). Suppose also that \( k \) is a constant such that
\( |g(x) - kf(x)| \leq \gamma \) except possibly on a point set of measure \( \leq \eta_2 \). Then the following inequality is valid:

\[
\left( \int_a^b f(x)g(x)dx \right)^2 \geq \left( \int_a^b f(x)^2dx \right)\left( \int_a^b g(x)^2dx \right) - R(\gamma_1, \gamma_2, A),
\]

where \( R(\gamma_1, \gamma_2, A) = 2(b - a)^2 A^2 \gamma_1^2 + 4(b - a)A^4 \gamma_2 \).
To prove this, let \( E = \int_a^b f^2 dx \cdot \int_a^b g^2 dx - \left( \int_a^b f g dx \right)^2 \). Set \( \sigma(x) = g(x) - k f(x) \) and then write \( E \) as a double integral:

\[
E = \frac{1}{2} \int_a^b \int_a^b \left[ f(x)g(y) - f(y)g(x) \right]^2 \, dx \, dy = \frac{1}{2} \int_a^b \int_a^b \left[ f(x)\sigma(y) - f(y)\sigma(x) \right]^2 \, dx \, dy.
\]

Remembering that \( |\sigma(x)| \leq \eta \) except possibly on a point set of measure \( \eta_2 \), we see at once that

\[
E = |E| \leq \frac{1}{2}(b - a)^2 (2A \eta_1)^2 + \frac{3}{2}(b - a) \eta_2 4A^4
\]

from which the stated conclusion follows immediately from the definition of \( E \).

We can always render a piecewise continuous function completely continuous by altering its definition on a point set of arbitrarily small measure. Hence with the help of the theorem (of Weierstrass) for the uniform approximation of continuous periodic functions by trigonometric polynomials the following corollary to Lemma 5 (with \( k = 1 \)) becomes obvious:

**Corollary 1.** If a positive number \( \varepsilon \) is given and, if \( g(t) \) is piecewise continuous for \( 0 \leq t = 2\pi \), it is always possible to find a function \( f(t) \), periodic with period \( 2\pi \) and analytic for all \( t \), such that

\[
\left( \int_0^{2\pi} f(t)g(t)dt \right)^2 \geq \left( \int_0^{2\pi} f(t)^2 dt \right) \left( \int_0^{2\pi} g(t)^2 dt \right) - \varepsilon
\]

and such that

\[
\left| \int_0^{2\pi} f(t)^2 dt - \int_0^{2\pi} g(t)^2 dt \right| < \varepsilon.
\]
A slightly different form of this corollary will be used, namely:

**Corollary 2.** If a positive number $\delta$ is given and if $g(t)$ is piecewise continuous for $0 \leq t \leq 2\pi$, it is always possible to find a function $f(t)$, periodic with period $2\pi$ and analytic for all $t$, such that

$$(1 + \delta^2) \left( \int_0^{2\pi} f(t)g(t)\,dt \right)^2 \geq \left( \int_0^{2\pi} f(t)^2\,dt \right) \left( \int_0^{2\pi} g(t)^2\,dt \right).$$

**Proof.** We assume that $\int_0^{2\pi} g(t)^2\,dt \neq 0$, as otherwise the Corollary is trivial. Then we can choose $\varepsilon > 0$ so small that

$$(15) \quad \delta^2 \left( \int_0^{2\pi} g^2\,dt \right) \left( \int_0^{2\pi} g^2\,dt - \varepsilon \right) > \varepsilon (1 + \delta^2).$$

Then, upon choosing $f$ as in Corollary 1, from (14) we may write

$$\int_0^{2\pi} f^2\,dt > \int_0^{2\pi} g^2\,dt - \varepsilon,$$

so that (15) may be modified as follows:

$$(16) \quad \delta^2 \left( \int_0^{2\pi} g^2\,dt \right) \left( \int_0^{2\pi} f^2\,dt \right) > \varepsilon (1 + \delta^2).$$

Multiplying the two sides of (13) by $(1 + \delta^2)$, we obtain

$$(17) \quad (1 + \delta^2) \left( \int_0^{2\pi} fg\,dt \right)^2 \geq \left( \int_0^{2\pi} f^2\,dt \right) \left( \int_0^{2\pi} g^2\,dt \right) + \delta^2 \left( \int_0^{2\pi} f^2\,dt \right) \left( \int_0^{2\pi} g^2\,dt \right) - \varepsilon (1 + \delta^2).$$

The proof is completed by combining (16) with (17).
Theorem 2. Theorem 1 is a "best possible theorem of its type" in the following sense: If \( L, S, \lambda, \) and \( M \) are given positive numbers, with \( L < \lambda \), it is possible to find a system (7) of arbitrary order \( n \), satisfying the hypotheses of Theorem 1 (with \( \max |\Gamma(t)| \) equal to the assigned number \( M \)), such that the inequalities (8) and (9) are satisfied for suitably chosen \( \rho > 0 \) in conjunction with the preassigned \( L \), but such that a periodic solution does not exist when \( \mu = L(1 + \delta^2)^{3/2} \), although of course, by Theorem 1, it must exist for \( |\mu| \leq L \).

It is emphasized that \( L, S, \lambda, \) and \( M \) may be preassigned quite independently of each other, except for \( L < \lambda \). In particular \( L \) may be arbitrarily large and \( L(1 + \delta^2)^{3/2} - L \) may be arbitrarily small.

The theorem is trivial if \( \lambda < L(1 + \delta^2)^{3/2} \), because, in that case, the right hand members of (7) are not even defined when \( \mu = L(1 + \delta^2)^{3/2} \). We therefore assume in the sequel that \( \lambda \geq L(1 + \delta^2)^{3/2} \).

The assignment of \( M = \max |\Gamma(t)| \) amounts roughly to putting a bound on the Green's matrix. Without such a restriction, Theorem 2 becomes easier to prove. The first step in the proof is to choose a positive number \( a \) which satisfies the equation

\[
18 \quad M = \frac{e^{2\pi a} + 1}{2\pi a(e^{2\pi a} - 1)}.
\]

Since the expression on the right becomes infinite as \( a \to 0 \) and since it approaches 0 as \( a \to \infty \), it is clear that such a root exists.

We next consider a system \( S \) of the form

\[
19 \quad \frac{dx_i}{dt} + ax_i = c_i \mu f(t), \quad i = 1, \ldots, n,
\]

where \( c_1, \ldots, c_n \) are any real numbers, such that \( \sum_{i=1}^{n} c_i^2 = 1 \), and where \( f(t) \) is
continuous and periodic with period $2\pi$. This function $f(t)$ will be chosen in a particular way in the sequel. In any event, a system $S_g$ of this form is associated with a diagonal Green's matrix, $G(t,s) = h(t,s)I$, where $I$ is the unit $n^2$ matrix and $h(t,s) = e^{2\pi a(e^{2\pi a} - 1)^{-1}} e^{a(s-t)}$ for $0 \leq s \leq t \leq 2\pi$ and $h(t,s) = (e^{2\pi a} - 1)^{-1} e^{a(s-t)}$ for $0 \leq t < s \leq 2\pi$, so that by Lemma 2, a unique periodic solution of $S_g$ exists for every value of $\mu$, and is given by the formula,

$$
(20) \quad x_i(t) = c_i \mu \int_0^{2\pi} h(t,s)f(s)ds, \quad i = 1, 2, \ldots, n.
$$

In accordance with Lemma 4, we may take $\Gamma(t) = \int_0^{2\pi} h(t,s)^2ds$ which turns out, upon elementary evaluation of the integral, to be the constant $M$ (cf. (18)). Thus for this matrix, we have $\max \Gamma(t) = M$ as desired.

We next choose a number $t^*$ on the interval $0 \leq t \leq 2\pi$, hold it fast, and set $g(s) = h(t^*,s)$. With this $g(s)$ and with the preassigned $\mu$ given in the statement of the theorem, we select a function $f(s)$ according to the specifications of Corollary 2 of Lemma 5. Since

$$
\int_0^{2\pi} g^2ds = \int_0^{2\pi} [h(t^*,s)]^2ds = M,
$$

we thus find from this corollary that

$$
(21) \quad (1 + \delta^2) \left( \int_0^{2\pi} f(s)g(s)ds \right)^2 > M \int_0^{2\pi} f(s)^2ds.
$$

We now introduce a system $S$, of the form (7), which is identical with the system $S_g$ except that it is not regarded as defined for all values of the $x_i$'s and of $\mu$. Namely, we let

$$
(22) \quad F_i(x,\mu, t) = c_i \mu f(t)
$$
for $|\mu| \leq \lambda$, for all $t$, and for

$$
|x| \leq r = \text{Li}^{\frac{1}{2}} \left( \int_0^{2\pi} f(s)^2 \, ds \right)^{\frac{1}{2}}.
$$

(23) The last equality is introduced at this point as the definition of $r$. The $F$'s are not regarded as defined for $|\mu| > \lambda$ or and for $|x| > r$. From (22) we find that $|F(x,\mu,t)|^2 = \mu^2 |f(t)|^2$ while $|F(x,\mu,t) - F(x',\mu,t)| = 0$, as long as $|x| \leq r$ and $|\mu| \leq \lambda$. Hence, in the notation of Theorem 1, we take $F(x,\mu,t) = \psi(x,\mu,t) = 0$ and $B(t) = \left[ f(t) \right]^2$, thus satisfying conditions I – V. It then follows from (23) that the inequalities (8) and (9) are satisfied (the latter trivially) for $|\mu| \leq \lambda$ and $t^* \leq \tau = r$. Hence, the system $S$ does have a unique periodic solution for $|\mu| \leq \lambda$. But, for $\mu = L(1 + \delta^2)^{\frac{1}{2}}$, any periodic solution of $S$ would also have to be a solution of $S_{\mu}$. But the unique periodic solution of $S_{\mu}$ for $\mu = L(1 + \delta^2)^{\frac{1}{2}}$ is outside the region of definition of the $F$'s at $t = t^*$. For, denoting this solution of $S_{\mu}$ simply by $x_i(t)$, $i = 1, \cdots, n$, we find, on setting $\mu = L(1 + \delta^2)^{\frac{1}{2}}$ in (20), that

$$
\left[ x_i(t^*) \right]^2 = c_i^2 L^2 (1 + \delta^2) \left( \int_0^{2\pi} h(t^*,s)f(s)\,ds \right)^2
$$

$$
= c_i^2 L^2 (1 + \delta^2) \left( \int_0^{2\pi} g(s)f(s)\,ds \right)^2
$$

$$
> c_i^2 L^2 M \int_0^{2\pi} f(s)^2 \, ds,
$$

where the inequality follows from (21). Hence, summing over $i$ and remembering that

$$
\sum c_i^2 = 1,
$$

we find from (23) that $|x(t^*)| > r$. Hence, the system $S$ can have no periodic solution for $\mu = L(1 + \delta^2)^{\frac{1}{2}}$, as we wished to prove.
The weakness of Theorem 2 and the example given to establish it is in the trivial satisfaction of (9) and the identical vanishing of the "Lipschitz quantities" $\Phi$ and $\Psi$. We therefore prove the following theorem which in this respect is less trivial, but has the defect of not allowing us to preassign the value of $M$. It ought to be possible to combine the results of Theorems 2 and 3 into a single theorem, but, in spite of considerable effort, I have not been able to do so in a satisfactory manner.

Theorem 3. Theorem 1 is a "best possible theorem of its type" in the following sense: If $L$, $S$, and $\lambda$ are given positive numbers with $\lambda > L$, it is possible to find an essentially non-linear system (7) of arbitrary order $n$, satisfying the hypotheses of Theorem 1, such that the inequalities (8) and (9) are satisfied for a suitably chosen $\varphi > 0$ in conjunction with the preassigned $L$, but such that a periodic solution does not exist when $\mu = L \left( 1 + S^2 \right)^{\frac{3}{2}}$, although, of course, by Theorem 1, it does exist for $|\mu| \leq L$.

We say that the system is "essentially linear", if the definition of $F(x, \mu, t)$ can be extended for $||x|| > r$ so that $F(x, \mu, t)$ is linear (not necessarily homogeneous) in the $x$'s for $||x|| < + \infty$. Otherwise, it is essentially non-linear.

As in the preceding theorem and for the same reason, we confine attention to the case that $\lambda > L(1 + S^2)^{\frac{3}{2}}$.

We first observe that the function $\eta(a) = \pi a (e^{2\pi a} + 1)(e^{2\pi a} - 1)^{-1} - 1$ is monotonically increasing with increasing $a$ and $\lim_{a \to 0} \eta(a) = 0$. Hence we can choose a positive number $a$ so small that

\begin{equation}
0 < \eta(a) < \delta^4 (1 + \delta^2)^{-1}.
\end{equation}
We then consider a system $S_g$ of the form

$$\frac{d}{dt}x_i + ax_i = k\mu(x_i^2 + x_i^2)^{\frac{3}{2}},$$  \hspace{1cm} (25)$$

where $k > 0$ is determined so that

$$\frac{(1 + \Delta^2)^{-1} < \frac{n(\frac{2\pi a}{2\pi a} + 1)}{a(\frac{2\pi a}{2\pi a} - 1)}}{k^2L^2(1 + \Delta^2)} < 1$$  \hspace{1cm} (26)$$

while $c_1, \ldots, c_n$ are arbitrary. Since the right hand members do not contain $t$ explicitly and since each of them involves only one of the $x_i$'s, it is easy to show that the only periodic solutions of $S_g$ are the constant solutions,

$$x_i = c_i \mu(a^2 - k^2\mu^2)^{-\frac{3}{2}}, \hspace{1cm} i = 1, \ldots, n,$$  \hspace{1cm} (27)$$

where, for $|\mu| \leq L(1 + \Delta^2)^{\frac{1}{2}}$, we certainly have $a^2 - k^2\mu^2 \neq 0$ in virtue of (26) and the fact that $\frac{e^{-2\pi a}}{2\pi a} < 1$ for any positive value of $a$ and, hence, in particular for the value of $a$ here to be considered. This system $S_g$, like the one considered in Theorem 2, is also associated with a diagonal Green's matrix, $G(t, x) = h(t, s)I$, given by the same formulas as previously. Again, in accordance with Lemma 4, we take

$$\int_0^{2\pi} h(t, s)^2 ds = \frac{e^{2\pi a} + 1}{2a(e^{2\pi a} - 1)} = \mu.$$  \hspace{1cm} (28)$$

We now introduce a system $S$, of the form (7), which is identical with the system $S_g$, except for the region of definition of its right hand members. Namely, we let

$$F_i(x, \mu, t) = k\mu(x_i^2 + x_i^2)^{\frac{1}{2}}.$$  \hspace{1cm} (29)$$
for $|\mu| \leq \lambda$, for all $t$, and for

$$|x| \leq r = \left[\frac{2nM L^2}{(1 - 2nMk^2L^2)}\right]^\frac{1}{2}$$

where, in this definition of $r$, $B = k^2||c||^2$ and $1 - 2nMk^2L^2 > 0$ because of (28) and (26). The $F$'s are not regarded as defined for $|\mu| > \lambda$ or and for $||x|| > r$. From (29) we find that

$$||F(x, \mu, t)||^2 = k^2\mu^2||x||^2 + k^2||c||^2\mu^2$$

$$||F(x, \mu, t) - F(x', \mu, t)||^2 \leq (\max \frac{\partial F_i}{\partial x_i})^2 \cdot ||x - x'||^2$$

$$\max \frac{\partial F_i}{\partial x_i} < |k\mu|.$$  

Hence, in the notation of Theorem 1, we may take $\Phi(\zeta, \mu, t) = \psi(\zeta, \mu, t) = k^2\mu^2$ and $B(t) = B = k^2||c||^2$, thus satisfying conditions I - V. It then follows from (30), (28), and (26) that the inequalities (3) and (9) are satisfied for $|\mu| \leq L$ and $\zeta \leq \rho = r$. Hence the system $S$ does have a unique periodic solution for $|\mu| \leq L$. But, for $\mu = L(1 + \mu^2)^{\frac{1}{4}}$, any periodic solutions of $S$ would also be a periodic solution of $S_{\mu}$. The unique periodic solution of $S_{\mu}$ for $\mu = L(1 + \mu^2)^{\frac{1}{4}}$ is, however, the constant solution

$$x_i = c_i k L(1 + \mu^2)^{\frac{1}{4}} (a^2 - k^2 L^2 (1 + \mu^2))^{-\frac{1}{2}} \quad (\text{cf. 27})$$

Since $||c||^2 k^2 = B$, we find for this solution,

$$||x||^2 = BL^2(1 + \mu^2)(a^2 - k^2 L^2 (1 + \mu^2))^{-1}.$$
Using (30) to eliminate $B$, we have

$$||x||^2 = \frac{r^2(1 - 2\pi M k^2 L^2)(1 + \delta^2)}{2\pi M(a^2 - k^2 L^2(1 + \delta^2))}.$$ 

From (26) and (28) we have

$$\frac{1}{2\pi M(1 + \delta^2)} < k^2 L^2(1 + \delta^2) < \frac{1}{2\pi M}.$$ 

Hence we find that

$$(31) \quad ||x||^2 > \frac{r^2(1 + \delta^2) - 1}{2\pi M a^2 - (1 + \delta^2)^{-1}} = r^2 \frac{\delta^2 + \delta^4}{\delta^2 + \gamma(1 + \delta^2)}$$

where $\gamma = 2\pi M a^2 - 1$ is evidently by (28) the same $\gamma$ which occurs in (24). Hence, from this last inequality, we have $\gamma(1 + \delta^2) < \delta^4$. Therefore $(\delta^2 + \delta^4)/(\delta^2 + \gamma(1 + \delta^2)) > 1$ so that from (31) we see finally that $||x|| > r$. In other words, the only periodic solution of $S$ for $\mu = L(1 + \delta^2)^{1/3}$ lies outside the region of definition of $S$. Hence $S$ can have no periodic solution for $\mu = L(1 + \delta^2)^{1/3}$, as we wished to prove.

It is interesting to notice the totally different nature of the examples used in the proofs of Theorems 2 and 3. In Theorem 2, the use of Lemma 5 on the reversal of the Schwarz inequality is essential, whereas in Theorem 3 the method of proof depends upon taking $a$ to be very small when $\delta$ is small. Such a system approximates the borderline case when the hypothesis $H$, mentioned in the introduction, is not fulfilled.
Part II

Just as in Part I, we take our given system in the prepared form

\[
\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j + F_i(x, \mu, t),
\]

in which the \( F \)'s and their first derivatives with respect to the \( x \)'s vanish when \( x_1 = \cdots = x_n = \mu = 0 \). But we shall now regard the \( F \)'s as defined for complex values of \( x_1, \cdots, x_n, \mu \), in which variables the \( F \)'s will be assumed to be analytic.

If the \( F \)'s are real when the \( x \)'s and \( \mu \) are real (which is the case of principal interest in both celestial and non-linear mechanics), then they will be, a fortiori, of class \( C^1 \), and it would then always be possible to introduce functions \( B(t), \Phi(\xi, \mu, t), \) and \( \psi(\xi, \mu, t) \) having the properties described in Lemma 1 of Part I.

All subsequent results of Part I continue to hold for this real analytic case.

In Part II, however, the \( B, \Phi, \) and \( \psi \) to be introduced in Lemma 1, will be generally different from the \( B, \Phi, \) and \( \psi \) of Part I, even in the real analytic case. It was deemed preferable to preserve the analogy between the methods and results of Parts I and II rather than to adhere to a strict consistency of notation between the two parts. Moreover any assumption about the reality of the \( F \)'s for real values of the \( x \)'s and \( \mu \) would be irrelevant to future considerations. No further reference will be made to such a hypothesis, even though our results will, of course, apply in particular to the real analytic case, and yield information about it not obtainable elsewhere.

In the following sequence of lemmas and theorems, it will be possible to omit most of the proofs because of the close analogy with the corresponding statements of Part I.
Lemma 1. Let $F_i(x, \mu, t)$, $i = 1, \ldots, n$, be defined for complex values of $x_1, \ldots, x_n$, and $\mu$, and for real values of $t$, such that \( \sum_{i=1}^{n} |x_i|^2 \leq r^2 \), $|\mu| \leq \lambda$, and $0 \leq t \leq 2\pi$. We assume continuity with respect to all $n + 2$ variables, and analyticity with respect to $x_1, \ldots, x_n$, and $\mu$. Let $F_i(0, 0, t) \equiv F_i(x, 0, t) \equiv 0$. Then real non-negative functions $B(t)$, $\Phi(\xi, |\mu|, t)$ and $\psi(\xi, |\mu|, t)$ of the indicated arguments $(0 \leq \xi \leq r)$ exist, such that the following five properties hold.

(I) \[ \lim_{\xi \to 0, \mu \to 0} \Phi(\xi, |\mu|, t) = 0 \text{ uniformly with respect to } t, \quad 0 \leq t \leq 2\pi, \text{ with } \Phi \text{ monotonic increasing in } \xi. \]

(II) $B(t)$, $\Phi(\xi, |\mu|, t)$, and $\psi(\xi, |\mu|, t)$ are all integrable with respect to $t$ from 0 to $2\pi$ (for $\xi \leq r$ and $|\mu| \leq \lambda$).

(III) \[ \sum_{i=1}^{n} |F_i(x, \mu, t)|^2 \leq \Phi(\xi, |\mu|, t) \xi^2 + B(t) |\mu|^2 \text{ for } \xi^2 = \sum_{i=1}^{n} |x_i|^2 \leq r^2 \]

and for $|\mu| \leq \lambda$.

(IV) \[ \sum_{i=1}^{n} |F_i(x', \mu, t) - F(x, \mu, t)|^2 \leq \psi(\xi, |\mu|, t) \left( \sum_{i=1}^{n} |x_i' - x_i|^2 \right) \text{ for } \sum_{i=1}^{n} |x_i|^2 \]

and \[ \sum_{i=1}^{n} |x_i|^2 \leq \xi^2. \]

(V) \[ \lim_{\xi \to 0, \mu \to 0} \psi(\xi, |\mu|, t) = 0 \text{ uniformly with respect to } t. \]

Suppose we set $x_k^*(u) = x_k u$ and $\mu^* = \mu u$. Then

(2) \[ F_i(x, \mu, t) = \int_{0}^{1} \left( \sum_{j=1}^{n} F_{ix_j^*}(x_j^*, \mu, t)x_i + F_{ix_j}(x_j, \mu^*, t)\mu \right) du. \]

We let $\beta = \max$ of $|F_{ix_j}(x_j, \mu, t)|$ for $\xi \leq r$, $|\mu| \leq \lambda$, and $0 \leq t \leq 2\pi$, and we let $\varphi_{ij}(\xi, |\mu|) = \max$ of $|F_{ix_j}(x_j^*, \mu^*, t)|$ for $\xi^* = \left( \sum_{k=1}^{n} |x_k^*|^2 \right)^{\frac{1}{2}} \leq \xi \leq r$, for
\[ |\mu^*| \leq |\mu| \leq \lambda, \text{ and for } 0 \leq t \leq 2\pi. \] Then we can show as in the corresponding situation of Lemma 1 Part I, that \[ \lim_{\zeta \to 0, \mu \to 0} \phi_{ij}(\zeta, |\mu|) = 0 \text{ and that } \phi_{ij}(\zeta, |\mu|) \]
will be monotonic in \( \zeta \). We can also write \[ \left( \sum_{j=1}^{n} |F_{ix_j}(x^*, \mu^*, t)| \cdot |x_j| \right)^2 \leq \left( \sum_{j=1}^{n} \phi_{ij}^2 \right) \left( \sum_{j=1}^{n} |x_j|^2 \right) \]
which by Schwarz's inequality does not exceed \[ \left( \sum_{j=1}^{n} \phi_{ij}^2 \right) \left( \sum_{j=1}^{n} |x_j|^2 \right) = \left( \sum_{j=1}^{n} \phi_{ij}^2 \right) \zeta^2. \]

Hence we obtain from (2) the fact that \[ |F_{i}(x, \mu, t)| \leq \left( \sum_{j=1}^{n} \phi_{ij}^2 \right)^{\frac{1}{2}} \zeta + \beta |\mu| \]
\[ |F_{i}(x, \mu, t)|^2 \leq 2 \left( \sum_{j=1}^{n} \phi_{ij}^2 \right) \zeta^2 + 2\beta^2 |\mu|^2, \]
and we then set \( \Phi(\zeta, |\mu|, t) = \Phi(\zeta, |\mu|) = 2 \sum_{i, j} \phi_{ij}^2 \text{ and } B = 2n\beta^2 \), thus obtaining a \( \Phi \) and \( B \) having the required properties.

To obtain \( \psi(\zeta, |\mu|, t) \) we start with the relation
\[ F_{i}(x, \mu, t) - F_{i}(x', \mu, t) = \int_{0}^{1} \left( \sum_{j=1}^{n} F_{ix_j}(x^*, \mu, t)(x_j - x'_j) \right) \, du \]
where now \( x^*_k = x'_k + u(x_k - x'_k) \). We easily see that \[ \sum_{k} |x^*_k|^2 \leq \zeta^2 \text{ if } \sum_{k} |x'_k|^2 \text{ and } \sum_{k} |x_k|^2 \text{ are both less than or equal to } \zeta^2 \text{ while } 0 \leq u \leq 1. \] Leaving further details to the reader, we find that we may take \( \psi(\zeta, |\mu|, t) = \psi(\zeta, |\mu|) = \sum_{i, j} \phi_{ij}^2 \).

**Lemma 2.** Consider the linear differential system

\[ \frac{dx}{dt} = A(t)x + f(t) \]
where $x$ and $f$ are $n$ dimensional vectors with complex components, and $A$ is an $n^2$-matrix with complex elements. $A$ and $f$ are known continuous functions of the real variable $t$, periodic with period $2\pi$. Let $X(t)$ be the $n^2$-matrix such that $dX(t)/dt = A(t)X(t)$ and $X(0) = I$, and suppose that $\det (X(2\pi) - I) \neq 0$, so that we may set

$$G(t,s) = -X(t) \left\{ I(\frac{1}{2} + \frac{1}{2} \text{ sign } (s-t)) + \left[ X(2\pi) - I \right]^{-1} \right\} \left[ X(s) \right]^{-1}.$$  

Then there is a unique periodic solution of (3), given on the interval $0 \leq t \leq 2\pi$ by the formula

$$(5) \quad x(t) = \int_0^{2\pi} G(t,s)f(s)ds.$$  

Moreover, if $f(t)$, in addition to depending on $t$, also depends analytically on a parameter $\mu$, in such wise that $f$ is continuous simultaneously in $t$ and $\mu$, for $|\mu| \leq L$, then the same may be said of the vector $x(t)$ given by (5).

Except for the addition of the final statement, this lemma is but a slight modification of the corresponding Lemma 2 of Part I. We therefore omit the proof.

We set $||v|| = \left( \sum_{i=1}^{n} |v_i|^2 \right)^{\frac{1}{2}}$, if $v$ is a vector with complex components $v_1, \ldots, v_n$, and we use the same symbol to denote $\left( \sum_{i,j} |v_{ij}|^2 \right)^{\frac{1}{2}}$ if $v$ is an $n^2$-matrix with complex elements $v_{ij}$ in the $i^{th}$ row and $j^{th}$ column.

Lemma 3. Let $G(t,s)$ be the matrix defined in Lemma 2. Let $f(s)$ be an arbitrary continuous vector function with complex components defined on $0 \leq s \leq 2\pi$. Then there exists an integrable bounded real scalar function $\gamma(t)$, independent of $f(s)$ and defined for $0 \leq t \leq 2\pi$, such that
\[ \| \int_0^{2\pi} G(t,s)f(s)ds \|^2 \leq \| \gamma(t) \| \int_0^{2\pi} ||f(s)||^2 ds. \]

As in Part I, the reader may show, with the help of the triangle inequality for complex numbers, and with the help of the Schwarz inequality for both sums and integrals, that we may actually take \( \gamma(t) = \int_0^{2\pi} ||G(t,s)||^2 ds. \)

**Theorem 1.** Consider the differential system

\[ \frac{dx}{dt} = A(t)x + F(x,\mu, t), \]

where \( x \) is the unknown vector function of \( t \) with complex components, where \( A(t) \) is as described in Lemma 2, and where the components of the vector function \( F(x,\mu, t) \) are periodic in the real variable \( t \) with period \( 2\pi \) and satisfy on \( 0 \leq t \leq 2\pi \) all the hypotheses of Lemma 1. Let \( B(t), \Phi(\zeta,\mu, t), \) and \( \psi(\xi,\mu, t) \) be any set of non-negative functions satisfying Conditions I - V of Lemma 1, the existence of which functions is assured by this same lemma. Let \( M = 1.u.b. \gamma(t) \), where \( \gamma(t) \) is any function satisfying the specifications of Lemma 3. Finally let \( \rho \leq r \) and \( L \leq \lambda \)

be any two positive numbers chosen so that

\[ M \int_0^{2\pi} \Phi(\rho,\mu, s) ds + u^2 \int_0^{2\pi} B(s) ds \leq \rho^2 \]

and

\[ M \int_0^{2\pi} \psi(\rho,\mu, t) dt \leq \rho^2 < 1 \]

for \( |\mu| \leq L \).
Then there exists just one periodic solution \( x = x(t, \mu) = x(t + 2\pi, \mu) \) of (7) for \( |\mu| \leq L \) such that \( ||x(t, \mu)|| \leq p \). Moreover \( x(t, \mu) \) is continuous simultaneously in \( t \) and \( \mu \) and is analytic in \( \mu \) for \( |\mu| < L \).

The proof of this theorem is essentially the same as the proof of Theorem 1 of Part II with only the added remark that the uniform convergence of the successive approximations with respect to \( t \) and \( \mu \) is, so far as the latter is concerned, valid in the circular region \( |\mu| \leq L \) of the complex \( \mu \)-plane. Since the successive approximations themselves are analytic in \( \mu \) (as may be proved by induction with the help of the last statement of Lemma 2), it follows from the Weierstrass theorem on uniformly convergent sequences of analytic functions that \( x(t, \mu) \) must be analytic in \( \mu \) for \( |\mu| < L \).

Theorem 2. Let \( x(t, \mu) \) be the solution of (7), the existence and analyticity \( (\text{with respect to } \mu) \) of which was proved in the previous theorem. If the power series expansions of the components of \( x(t, \mu) \) be written as follows:

\[
(10) \quad x_1(t, \mu) = \sum_{k=1}^{\infty} p_{1k}(t)\mu^k, \quad |\mu| < L,
\]

then the coefficients satisfy the inequalities,

\[
(11) \quad |p_{1k}(t)| \leq pL^{-k}
\]

and the remainder obtained upon cutting the series off after the \((m-1)\)th term cannot exceed in absolute value the quantity,

\[
(12) \quad \left( \frac{pL}{L - |\mu|} \right)^m (\frac{|\mu|}{L})^m.
\]
This theorem is an obvious corollary of the preceding theorem, because of the known fact that \( |x_i(t, \mu)| \leq \|x(t, \mu)\| \leq p \) for \( |\mu| \leq L \). The inequalities (11) are the well known Cauchy inequalities for the coefficients of a power series; and (12) follows obviously from (11).

References


