A COMBINATORIAL ANALYSIS OF THE EIGENVALUES OF THE LAPLACIAN MATRICES OF COGRAPHS

by

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Abstract

Cographs, also known as complement reducible graphs or decomposable graphs, are a recursively defined class of graphs built from a single vertex by the operations of disjoint union and join. The eigenvalues of a cograph's Laplacian matrix are nonnegative integers. In this thesis, we explore the combinatorial significance of cograph Laplacian eigenvalues. We show that the second-smallest eigenvalue $\lambda_2(C)$ of a cograph C, also known as the algebraic connectivity of C, is equal to the vertex connectivity of C. We give necessary and sufficient conditions for $\lambda_2(C)$ to be unique, and provide a characterization of the Fiedler vector, the eigenvector of $\lambda_2(C)$, when $\lambda_2(C)$ is unique. Finally, we give a relationship between the nonzero eigenvalues of C and the twin numbers of C, generalizing a result due to Merris stating that the nonzero eigenvalues of a threshold graph T are equal to the Ferrer's conjugate of the degree sequence of T.

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Finally, thank you to my family for twenty-two years of love and support. I hope I didn't drive you too crazy.

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1 Introduction

In this thesis, we explore combinatorial interpretations of the Laplacian eigenvalues of cographs, a subclass of perfect graphs. Cographs have been studied in the past for a number of reasons. We are interested in cographs because cographs have the property that every Laplacian eigenvalue of a cograph C is a nonnegative integer. My advisor likes to say that in combinatorics, if a number is always a nonnegative integer, it must be counting something. To that end, this research aims to understand cograph Laplacian eigenvalues in the context of other combinatorial graph properties. We show that the largest Laplacian eigenvalue of a cograph C is equal to the number of vertices in the largest connected component of C. We also show that the second-smallest Laplacian eigenvalue of a cograph C is equal to the vertex connectivity, or the number of vertices in a minimum vertex cut, of C.

This paper expands on previous related work on graph Laplacians and on cographs; see [12], [3], [10], [8] for a review of the existing literature. In particular, in [10], Russell Merris gives a characterization of the Laplacian eigenvalues of threshold graphs, a proper subclass of cographs: the nonzero Laplacian eigenvalues of a threshold graph T are equal to the Ferrer's conjugate of the degree sequence of T. We generalize this result to a characterization of the Laplacian eigenvalues of cographs.

In Section 2, we discuss relevant background information and existing results. In 2.2, we review basic results and properties of Laplacian matrices of graphs. In 2.3, we discuss threshold graphs, a subclass of cographs, and present the major result due to Merris relating the eigenvalues of threshold graphs to the degree sequence of threshold graphs. In Section 3, we prove several new results on the eigenvalues of cographs. In Section 4, we outline open questions and future research directions relating to the Laplacian eigenvalues of cographs.

The graphs discussed in this thesis are simple: every graph is undirected, with no loops or multiple edges. Familiarity with basic graph theory terms and results is assumed; see [14] for any undefined terms or notation.

2 Overview

2.1 Definitions

We begin by defining two commutative, associative, binary graph operations.

Definition 2.1. Let G_1 and G_2 be graphs. The *disjoint union* of G_1 and G_2, denoted $G = G_1 + G_2$, is the graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The *join* of G_1 and G_2, denoted $G = G_1 \vee G_2$, is the graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{(i, j) \mid i \in V(G_1), j \in V(G_2)\}$.

The operations of disjoint union and join are used to construct cographs, leading to the following definition.

Definition 2.2. A graph G is a *cograph* if it can be generated from a single vertex by the operations of disjoint union and join:

- K_1 is a cograph.
- If G and H are cographs, then $G + H$ is a cograph.
- If G and H are cographs, then $G \vee H$ is a cograph.

It follows that every connected cograph C can be written as $C = C_1 \vee C_2$ for cographs C_1 and C_2, and every disconnected cograph C can be written as $C = C_1 + C_2$ for cographs C_1 and C_2.

Cographs are also commonly defined by the operations of disjoint union and complement, giving the following alternative definition.

Definition 2.3. A graph G is a *cograph* if it can be generated from a single vertex by the operations of disjoint union and complement:

- K_1 is a cograph.
- If G and H are cographs, then $G + H$ is a cograph.
- If G is a cograph, $\overline{G}$ is a cograph.

It follows that cographs form the smallest nonempty class of graphs that is closed under complementation, join, and disjoint union. The operations of join, disjoint union, and complement are related by the following identities. If G and H are graphs, then

$$\overline{G + H} = \overline{G} \vee \overline{H}$$

and

$$\overline{G \vee H} = \overline{G} + \overline{H}.$$

We prove the following observation.

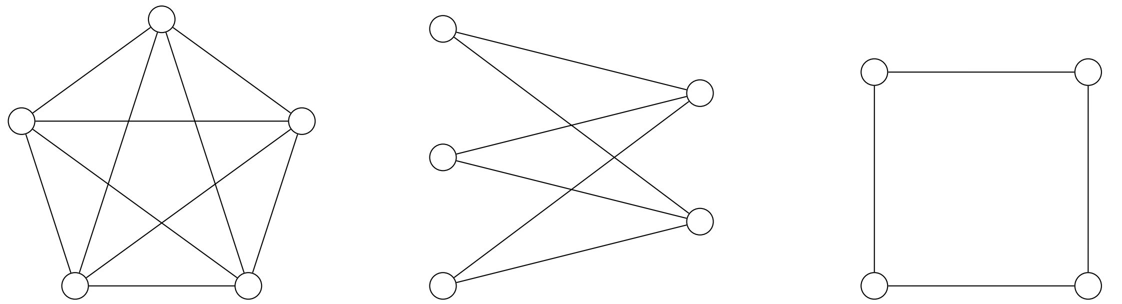
Proposition 2.4. *Suppose C is a connected cograph. Then, $\overline{C}$ is not connected.*

Proof. Let C be a connected cograph. C can be written as the join of two cographs, so $C = C_1 \vee C_2$ for cographs C_1 and C_2. Then,

$$\overline{C} = \overline{C_1 \vee C_2} = \overline{C_1} + \overline{C_2},$$

so $\overline{C}$ is disconnected. $\square$

Cographs can also be characterized by a minimal forbidden subgraph.



(a) K_5 (b) $K_{3,2}$ (c) C_4

Figure 2.1: Several examples of cographs are shown above. K_5 is shown in (a); every complete graph K_n is a cograph. $K_{3,2}$ is shown in (b); every complete bipartite graph $K_{m,n}$ is a cograph. C_4 is shown in (c). No cycle of length greater than four is a cograph, since every C_n with $n \geq 5$ has an induced P_4.

Proposition 2.5 ([3]). *A graph G is a cograph if and only if G does not contain P_4, the path on four vertices, as an induced subgraph.*

It follows from Proposition 2.5 that if C is a cograph, then every induced subgraph of C is a cograph and every component of C is a cograph. We give several examples of cographs in Figure 2.1.

2.2 The Laplacian Matrix

We are interested in cographs because of the properties of their Laplacian eigenvalues. Here, we define the Laplacian matrix and review some basic properties and results.

Definition 2.6. The *Laplacian matrix* L of a graph G is given by $L = D - A$, where D is the degree matrix of G and A is the adjacency matrix of G.

Laplacian matrices are well-studied in the field of spectral graph theory. See [1], [13] for more information on the Laplacian matrix. We review some relevant properties below.

Proposition 2.7. *Let G be a graph and let L be the Laplacian matrix of G. Then,* *(1) L is symmetric and positive semidefinite.*

(2) 0 is an eigenvalue of L with eigenvector equal to the all ones vector.

(3) The eigenvalue 0 has multiplicity one if and only if G is connected.

These are standard results in spectral graph theory. We present the proofs here for convenience.

Proof. Suppose L is the Laplacian matrix of a graph G.

(1) Since $L = D - A$ and D and A are symmetric, L is symmetric. A matrix M is positive semidefinite if $x^T M x \geq 0$ for every nonzero real vector x. Consider $x^T L x$ for nonzero real vector x. We have

$$\begin{aligned} x^T L x &= x^T D x - x^T A x \\ &= \sum_{i=1}^{n} d_i x_i^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \\ &= \sum_{i=1}^{n} d_i x_i^2 - \sum_{(i,j) \in E(G)} 2 x_i x_j \\ &= \sum_{(i,j) \in E(G)} (x_i - x_j)^2. \end{aligned}$$

Since $x^T L x$ can be expressed as the sum of squares, $x^T L x \geq 0$. Therefore, L is positive semidefinite.

Positive semidefinite matrices have nonnegative eigenvalues, so the eigenvalues of L are nonnegative real numbers.

(2) Let the ones vector be denoted $\vec{1}_n$. Consider $L\vec{1}_n = (D - A)\vec{1}_n$. It's easy to verify that

$$D\vec{1}_n = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \quad \text{and} \quad A\vec{1}_n = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix},$$

where d_i is the degree of vertex v_i. Therefore, $L\vec{1}_n = 0$, and $\vec{1}_n$ is an eigenvector for the eigenvalue 0.

(3) Suppose G is connected. Let v be an eigenvector for the eigenvalue 0, so $Lv = 0$. Then,

$$v^T L v = 0 = \sum_{(i,j) \in E(G)} (v_i - v_j)^2,$$

as shown in the proof of (1) above. Therefore, if $(i, j) \in E(G)$, $v_i = v_j$. Since G is connected, it follows that v must be a constant vector. Since the only eigenvector for 0 is the constant vector, the multiplicity of 0 is one.

Now, suppose G is not connected, so G has at least two components G_1 and G_2. Consider the vectors v_1 and v_2 given by

$$v_1(i) = \begin{cases} 1 & \text{if } i \in G_1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad v_2(i) = \begin{cases} 1 & \text{if } i \in G_2 \\ 0 & \text{otherwise} \end{cases}.$$

Because L is positive semidefinite, $Lv = 0$ if and only if $v^T L v = 0$. Since v_1 and v_2 are constant on connected components, it follows from the formula for $v^T L v$ given above that $Lv_1 = Lv_2 = 0$. Moreover, v_1 and v_2 are orthogonal, so the multiplicity of 0 is at least two.

$\square$

We call the multiset of eigenvalues of the Laplacian matrix of G the *spectrum* of G and denote it by $\Lambda(G)$. By convention, $\Lambda(G) = \{0, \lambda_2, \ldots, \lambda_n\}$, where $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. If G and H are graphs with Laplacian matrices L_G and L_H such that $G \cong H$, $L_G = PL_HP^T$ for some permutation matrix P. Therefore, if $G \cong H$, $\Lambda(G) = \Lambda(H)$, so the spectrum of G is a graph invariant.

For any matrix M with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$,

$$\sum_{i=1}^{M} \lambda_i = \mathrm{tr}(M) = \sum_{i=1}^{n} m_{ii}.$$

Therefore, we have the following.

Proposition 2.8. *If L is the Laplacian matrix of a graph G, $tr(L) = \sum_{i=1}^{n} d_i$, so*

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i = 2|E(G)|,$$

where d_i for $i \in \{1, \ldots, n\}$ are the degrees of the vertices of G and $|E(G)|$ is the number of edges of G.

We now show the following relationship between the spectrum of a graph G and the spectrum of $\overline{G}$.

Proposition 2.9. *Let G be a graph with spectrum $\Lambda(G) = \{0, \lambda_2, \ldots, \lambda_n\}$. The spectrum of $\overline{G}$ is given by $\Lambda(\overline{G}) = \{0, n - \lambda_n, \ldots, n - \lambda_2\}$.*

Proof. Let L represent the Laplacian matrix of G. The Laplacian matrix of $\overline{G}$ is given by $nI - J - L$, where I is the $n \times n$ identity matrix, and $J = 1_{n \times n}$ is the $n \times n$ ones matrix. Suppose $\lambda \neq \lambda_1$ is an eigenvalue of L with eigenvector v, so $Lv = \lambda v$. Since $\vec{1}_n$ is the eigenvector for λ_1, v is orthogonal to $\vec{1}_n$. Then,

$$(nI - J - L)v = nv - Jv - \lambda v = (n - \lambda)v.$$

$\square$

Because cographs are defined recursively, their Laplacian matrices can also be defined recursively. In the following two propositions, we give recursive formulas for the spectra of disconnected and connected cographs.

Proposition 2.10. *Let C be a disconnected cograph such that $C = C_1 + C_2$, where C_1 is a cograph on n_1 vertices with Laplacian matrix $L(C_1) = L_1$ and spectrum given by $\Lambda(C_1) = \{0, \lambda_2, \ldots, \lambda_{n_1}\}$, and C_2 is a cograph on n_2 vertices with Laplacian matrix $L(C_2) = L_2$ and spectrum given by $\Lambda(C_2) = \{0, \nu_2, \ldots, \nu_{n_2}\}$. Then, the spectrum of C is given by*

$$\Lambda(C) = \{0, \lambda_2, \ldots, \lambda_{n_1}, 0, \nu_2, \ldots, \nu_{n_2}\}.$$

Proof. Suppose $C = C_1 + C_2$ as defined above. Then, the Laplacian of C is given by

$$L(C) = L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}.$$

Suppose $\lambda \in \Lambda(C_1)$ and let v be the eigenvector of λ. Consider the length-$(n_1 + n_2)$ vector given by $\begin{bmatrix} v \\ 0 \end{bmatrix}$. Then,

$$\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 v \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} v \\ 0 \end{bmatrix},$$

where the last equality holds since v is an eigenvector of L_1 with eigenvalue λ. Therefore, every eigenvalue λ of C_1 is also an eigenvalue of C.

Similarly, suppose $\nu \in \Lambda(C_2)$ and let v be the eigenvector of ν. Consider the length-(n_1+n_2) vector given by $\begin{bmatrix} 0 \\ v \end{bmatrix}$.

$$\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ L_2 v \end{bmatrix} = \nu \begin{bmatrix} 0 \\ v \end{bmatrix},$$

where again, the last equality holds since v is an eigenvector of L_2 with eigenvalue ν. Therefore, every eigenvalue ν of C_2 is also an eigenvalue of C. $\square$

Proposition 2.11. *Let C be a connected cograph such that $C = C_1 \vee C_2$, where C_1 is a cograph on n_1 vertices with Laplacian matrix $L(C_1) = L_1$ and spectrum given by $\Lambda(C_1) = \{0, \lambda_2, \ldots, \lambda_{n_1}\}$, and C_2 is a cograph on n_2 vertices with Laplacian matrix $L(C_2) = L_2$ and spectrum given by $\Lambda(C_2) = \{0, \nu_2, \ldots, \nu_{n_2}\}$. Then, the spectrum of C is given by*

$$\Lambda(C) = \{0, \lambda_2 + n_2, \ldots, \lambda_{n_1} + n_2, \nu_2 + n_1, \ldots, \nu_{n_2} + n_1, n_1 + n_2\}.$$

Proof. We use the fact that

$$C = C_1 \vee C_2 = \overline{\overline{C_1} + \overline{C_2}},$$

and apply the results of Proposition 2.9 and Proposition 2.10. By Proposition 2.9, we know that

$$\Lambda(\overline{C_1}) = \{0, n_1 - \lambda_{n_1}, \ldots, n_1 - \lambda_2\}$$

and

$$\Lambda(\overline{C_1}) = \{0, n_2 - \nu_{n_2}, \ldots, n_2 - \nu_2\}.$$

By Proposition 2.10, we know that

$$\Lambda(\overline{C_1} + \overline{C_2}) = \{0, n_1 - \lambda_{n_1}, \ldots, n_1 - \lambda_2, 0, n_2 - \nu_{n_2}, \ldots, n_2 - \nu_2\}.$$

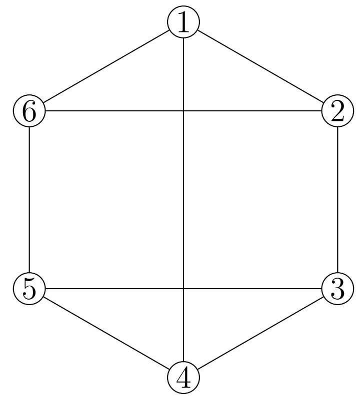


Figure 2.2: A graph G with $\Lambda(G) = \{0, 2, 3, 3, 5, 5\}$. Note that G is not a cograph. G contains an induced P_4; for example, the path on four vertices given by $1 \sim 4 \sim 3 \sim 2$.

Finally, applying Proposition 2.9 once more, we see that

$$\Lambda(\overline{\overline{C_1} + \overline{C_2}}) = \{0, \lambda_2 + n_2, \ldots, \lambda_{n_1} + n_2, \nu_2 + n_1, \ldots, \nu_{n_2} + n_1, n_1 + n_2\}.$$

□

It follows from the recursive formulas that the eigenvalues of cographs are always integers. Graphs whose Laplacian eigenvalues are always nonnegative integers are called Laplacian integral. This property motivates the research in this paper; our results explore the combinatorial significance of cograph integer eigenvalues, relating them to other combinatorial properties of cographs. Though this paper is concerned with the eigenvalues of cographs, cographs are not the only graphs with integer eigenvalues: there exist Laplacian integral graphs that are not cographs. An example is given in Figure 2.2. See [7] for a discussion of Laplacian integral graphs that are not cographs. Although some Laplacian integral graphs are not cographs, we have the following characterization of cographs.

Proposition 2.12. *A graph C is a cograph if and only if every induced subgraph of C is Laplacian integral.*

Proof. If C is a cograph, every induced subgraph of C is also a cograph, so every induced

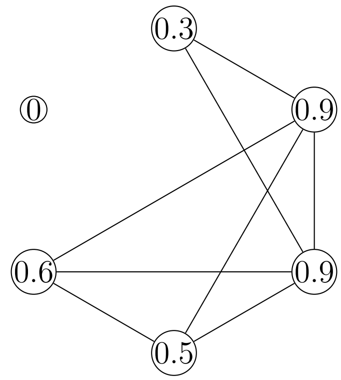


Figure 2.3: An example of a threshold graph. The values of the weight function f are given as the vertex labels.

subgraph of C is Laplacian integral.

If C is not a cograph, then C contains P_4 as an induced subgraph. The eigenvalues of P_4 are $\Lambda(P_4) = \{0, 2-\sqrt{2}, 2, 2+\sqrt{2}\}$, so C has an induced subgraph that is not Laplacian integral. $\square$

2.3 Threshold Graphs

Before we discuss the Laplacian eigenvalues of cographs, we consider the special case of threshold graphs, a subclass of cographs. Threshold graphs are defined as follows.

Definition 2.13. A graph G is a *threshold graph* if there is a weight function of the vertices $f : V(G) \to \mathbb{R}$ such that $u \sim v$ if and only if $f(u) + f(v) \geq 1$.

We give an example of a threshold graph in Figure 2.3.

To further characterize threshold graphs, we give the following definitions.

Definition 2.14. A vertex v of a graph G is an *isolated vertex* if v is not adjacent to any vertex in G.

Definition 2.15. A vertex v of a graph G is a *universal vertex* if v is adjacent to every other vertex in G.

Isolated and universal vertices are naturally related to threshold graphs. In particular, we prove the following.

Proposition 2.16. *A threshold graph T either has a universal vertex or an isolated vertex.*

Proof. Let T be a threshold graph on n vertices given by the weight function f. Let v_1 be a minimum weight vertex and v_n be a maximum weight vertex. There are two cases.

1. $f(v_1) + f(v_n) \geq 1$.

   Since v_1 is a minimum weight vertex, $f(v) \geq f(v_1)$ for all $v \in V(T)$. Therefore, if $f(v_1)+f(v_n) \geq 1$, then $f(v)+f(v_n) \geq 1$ for all $v \in V(T)$, and v_n is a universal vertex.

2. $f(v_1) + f(v_n) < 1$.

   Since v_n is a maximum weight vertex, $f(v) \leq f(v_n)$ for all $v \in V(T)$. Therefore, if $f(v_1)+f(v_n) < 1$, then $f(v_1)+f(v) < 1$ for all $v \in V(T)$, and v_1 is an isolated vertex.

□

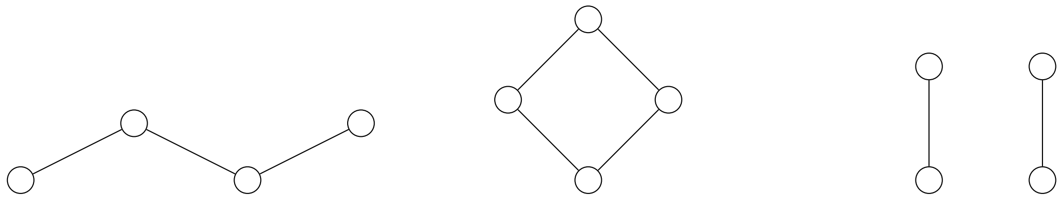
We use this fact to show that threshold graphs are a subclass of cographs.

Proposition 2.17. *Threshold graphs are cographs.*

Proof. It follows from Proposition 2.16 that a threshold graph can be constructed from a single vertex by adding either an isolated vertex or a universal vertex. Adding an isolated vertex corresponds to a disjoint union operation and adding a universal vertex corresponds to a join operation, so threshold graphs are cographs. □

Threshold graphs also have a minimal forbidden subgraph characterization.

Proposition 2.18 ([2])**.** *A graph G is a threshold graph if and only if G does not contain P_4, C_4, or $2K_2$ as induced subgraphs.*



(a) P_4 (b) C_4 (c) $2K_2$

Figure 2.4: The minimal forbidden subgraphs of a threshold graph.

It is obvious from the minimal forbidden subgraph characterizations of threshold graphs and cographs that threshold graphs are a proper subset of cographs.

Threshold graphs arise in many applications and have been studied extensively. See [9] for a survey of threshold graphs. In this paper, we are interested in threshold graphs because the Laplacian eigenvalues of threshold graphs are well understood. We review the relevant results below.

Definition 2.19. The *degree sequence* of a graph G, given by $d(G) = (d_1, \dots, d_n)$, is the nonincreasing sequence of nonzero degrees of vertices of G.

The degree sequence is a graph invariant, so two isomorphic graphs have the same degree sequence. The degree sequence does not uniquely determine a graph; in general, two nonisomorphic graphs can have the same degree sequence. However, for threshold graphs, we have the following result.

Proposition 2.20 ([11])**.** *Let G and H be threshold graphs, and let $d(G)$ and $d(H)$ be the degree sequences of G and H, respectively. If $d(G) = d(H)$, then $G \cong H$.*

In other words, threshold graphs are uniquely determined by their degree sequences.

The degree sequence of a threshold graph T is related to the spectrum of T through Ferrer's diagrams, which are combinatorial tools for analyzing integer partitions.

Definition 2.21. The *Ferrer's diagram* of a nonincreasing sequence of integers $s = (s_1, \dots, s_n)$ consists of n rows of blocks such that row i has s_i blocks.

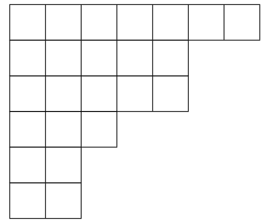


Figure 2.5: The Ferrer's diagram for the sequence $s = (7, 5, 5, 3, 2, 2)$.

(a) $s = (7, 5, 5, 3, 2, 2)$.

(b) $s^* = (6, 6, 4, 3, 3, 1, 1)$.

Figure 2.6: An example of the Ferrer's conjugate.

An example of a Ferrer's diagram is shown in Figure 2.5.

Definition 2.22. The *Ferrer's conjugate* of a nonincreasing sequence of integers $s = (s_1, \ldots, s_n)$ is the nonincreasing sequence s^* whose Ferrer's diagram is the transpose of the Ferrer's diagram of s.

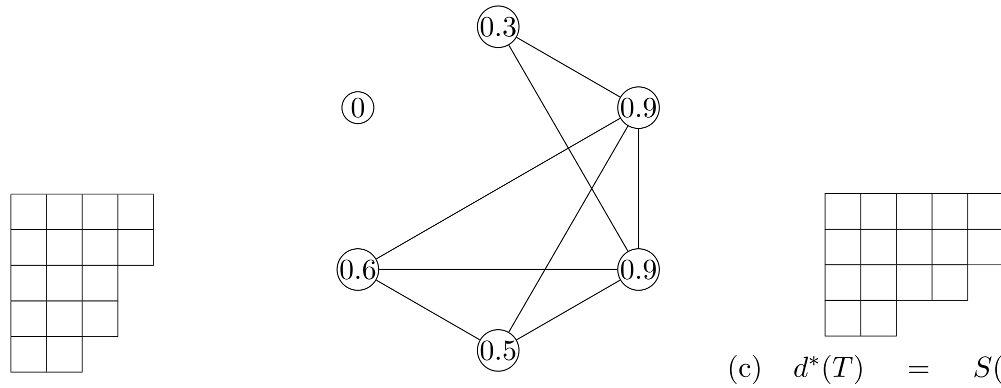
We also give the following equivalent definition.

Definition 2.23. The *Ferrer's conjugate* of a nonincreasing sequence of integers s is the nonincreasing sequence s^*, where

$$s_i^* = |\{x \in s \mid x \geq i\}|.$$

Ferrer's conjugates are used to understand the eigenvalues of threshold graphs in the following theorem.

Theorem 2.24 ([10])**.** *Let T be a threshold graph. The nonzero eigenvalues of T are equivalent to the Ferrer's conjugate of the nonzero degree sequence of T.*



(a) $d(T) = (4, 4, 3, 3, 2)$.

(b) T

(c) $d^*(T) = S(T) = (5, 5, 4, 2)$.

Figure 2.7: The example threshold graph from figure 2.3 is shown in (b). In (a), the Ferrer's diagram of the degree sequence is given. In (c), the Ferrer's conjugate of the degree sequence is given. The spectrum of T is $\Lambda(T) = \{0, 0, 2, 4, 5, 5\}$, and $S(T) = (5, 5, 4, 2)$. We see here that the Ferrer's conjugate of the degree sequence is equivalent to the nonzero eigenvalues of T.

It is useful to consider the spectrum of a threshold graph T as a nonincreasing sequence $S(T) = (\lambda_n, \lambda_{n-1}, \ldots, \lambda_m)$, where m is the index of the smallest nonzero eigenvalue. By Proposition 2.8, $S(T)$ and $d(T)$ are both integer partitions of $2|E|$. Theorem 2.24 gives a relationship between these partitions, stating that $d^*(T) = S(T) = (\lambda_n, \ldots, \lambda_m)$.

We give an example of Theorem 2.24 in Figure 2.7.

The following is an immediate consequence of Theorem 2.24.

Definition 2.25. Two graphs G and H are said to be *cospectral* if $\Lambda(G) = \Lambda(H)$.

Proposition 2.26. *Let T_1 and T_2 be threshold graphs. Then, T_1 and T_2 are cospectral if and only if $T_1 \cong T_2$.*

Proof. If $T_1 \cong T_2$, T_1 and T_2 are cospectral. Suppose T_1 and T_2 are cospectral. By Theorem 2.24, $\Lambda(T_1) = \Lambda(T_2)$ implies that $d(T_1) = d(T_2)$. By Proposition 2.20, $d(T_1) = d(T_2)$ implies that $T_1 \cong T_2$. $\square$

Theorem 2.24 gives a combinatorial interpretation of the eigenvalues of threshold graphs, but it does not hold for all cographs. In Section 3.2, we provide a generalization of this result to all cographs.

3 Cograph Results

We begin our discussion of cograph eigenvalues with a consideration of $\lambda_{\max}(C)$, the largest eigenvalue of a cograph C.

Proposition 3.1. *Let C be a cograph and let $\lambda_{max}(C)$ be the largest eigenvalue of C. Then, $\lambda_{max}(C)$ is equal to the number of vertices in the largest nontrivial connected component of C.*

Proof. Suppose C is a connected cograph on n vertices and $\Lambda(C) = \{0, \lambda_2, \dots, \lambda_n\}$. By Proposition 2.9, $\Lambda(\overline{C}) = \{0, n - \lambda_n, \dots, n - \lambda_2\}$. Since C is a connected cograph, $\overline{C}$ is disconnected, so $\lambda_2(\overline{C}) = n - \lambda_n = 0$, and $\lambda_n = n$. Therefore, if C is connected, then $\lambda_{\max}(C)$ is equal to the number of vertices in C.

Now, suppose C is a disconnected cograph on n vertices. Let the connected components of C be given by $C_1, \dots, C_m$. Then, the spectrum $\Lambda(C)$ is given by the multiset union of the spectra of $\Lambda(C_i)$ for $i \in \{1, \dots, m\}$, and $\lambda_{\max}(C) = \max_i(\lambda_{\max}(C_i))$. Since the components C_i are connected, $\lambda_{\max}(C_i)$ represents the number of vertices in component C_i, and $\max_i(\lambda_{\max}(C_i)) = \lambda_{\max}(C)$ is the largest connected component of C. $\square$

3.1 Algebraic connectivity and the Fiedler vector

In this section, we consider the second-smallest Laplacian eigenvalue of a graph G, $\lambda_2(G)$, also known as the algebraic connectivity. Algebraic connectivity is studied at length in

[5]. For any graph G, the algebraic connectivity $\lambda_2(G)$ is bounded above by the vertex connectivity $\kappa(G)$ [5]. We show that when G is a cograph, $\lambda_2(G) = \kappa(G)$.

Proposition 3.2. *Let C be a cograph. Then, $\lambda_2(C) = \kappa(C)$.*

Proof. Let C be a cograph. If C is not connected, then $\lambda_2(C) = 0$ and $\kappa(C) = 0$, so $\lambda_2(C) = \kappa(C)$. Therefore, suppose C is connected and let $\kappa(C) = k$. Let S be the vertices in a minimum vertex cut set and let $C - S$ be the disconnected cograph given by deleting the vertices in S. Since $\kappa(C) = k$, $|S| = k$. Note that minimizing $|S|$ is equivalent to maximizing $|C - S|$.

Suppose the eigenvalues of C are given by $\Lambda(C) = \{0, \lambda_2, \ldots, \lambda_n\}$. Since C is connected, $\overline{C}$ is not connected. By Proposition 2.9, the eigenvalues of $\overline{C}$ are given by $\Lambda(\overline{C}) = \{0, n - \lambda_n, \ldots, n - \lambda_2\}$, so the largest connected component of $\overline{C}$ has size $n - \lambda_2$.

Let the connected components of $\overline{C}$ be given by $\{\overline{C_1}, \ldots, \overline{C_m}\}$, where

$$|V(\overline{C_1})| \leq |V(\overline{C_2})| \leq \cdots \leq |V(\overline{C_m})| = n - \lambda_2.$$

Since

$$\overline{C} = \overline{C_1} + \overline{C_2} + \cdots + \overline{C_m},$$

we have that

$$C = C_1 \vee C_2 \vee \cdots \vee C_m.$$

It is clear that since $C - S$ is disconnected, $C - S$ can only contain vertices from a single component C_i. Therefore, $|C - S| \leq |C_m|$, since $|C_m|$ is the size of the largest component.

Since $\overline{C_i}$ is a connected cograph, C_i is disconnected for all i. Therefore, to maximize the size of the disconnected set $C - S$, we let $C - S = C_m$, the complement of the largest connected component in $\overline{C}$. Since $|C_m| = n - \lambda_2$,

$$|C - S| = n - k = n - \lambda_2,$$

and,

$$\lambda_2 = k = \kappa(G),$$

as desired. $\square$

We also prove necessary and sufficient conditions for λ_2 to be unique.

Theorem 3.3. *Let C be a cograph and let $\lambda_2(C)$ be the second-smallest eigenvalue of C. Then, $\lambda_2(C)$ is unique if and only if C has a unique minimum cut set S and $C - S$ has exactly two components.*

Proof. Let C be a cograph and suppose C is disconnected with $\Lambda(C) = \{0, \lambda_2, \ldots, \lambda_n\}$. We say $\lambda_2 = 0$ is unique if $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_i > 0$ for $i \geq 3$. Suppose $\lambda_2 = 0$ is unique. Since the multiplicity of 0 is equal to the number of connected components, C has exactly two components. Therefore, λ_2 is equal to the number of vertices in the minimum cut set $S = \{\}$, and $C - S = C$ has exactly two components. Conversely, suppose C has exactly two components. Then, $\lambda_2 = 0$ is unique. So Theorem 3.3 holds when C is disconnected.

Now, suppose C is connected. First, we show that if $\lambda_2(C)$ is unique, then C has a unique minimum cut set S and $C - S$ has exactly two components. We proceed by induction on the number of vertices. The base case, P_3, is trivial to verify.

Let C be a connected cograph such that $\lambda_2(C)$ is unique. Let $C = C_1 \vee C_2$, where C_1 is a cograph on n_1 vertices whose spectrum is given by $\Lambda(C_1) = \{0, \alpha_2, \ldots, \alpha_{n_1}\}$ and C_2 is a cograph on n_2 vertices whose spectrum is given by $\Lambda(C_2) = \{0, \beta_2, \ldots, \beta_{n_2}\}$. Then,

$$\Lambda(C) = \{0, \alpha_2 + n_2, \ldots, \alpha_{n_1} + n_2, \beta_2 + n_1, \ldots, \beta_{n_2} + n_1, n_1 + n_2\}.$$

The second-smallest eigenvalue $\lambda_2(C)$ is either $\alpha_2 + n_2$ or $\beta_2 + n_1$. Without loss of generality, let $\lambda_2(C) = \alpha_2 + n_2$. Then, a minimum cut set of C is of the form $S \cup C_2$, where S is a minimum cut set of C_1.

Since $\lambda_2(C)$ is unique, α_2 is unique. By induction, C_1 has a unique minimum cut set S_1 and $C_1 - S_1$ has exactly two components. Therefore, C has a unique minimum cut set $S_1 \cup C_2$. Further, $C - S_1 \cup C_2 = C_1 - S_1$, so $C - S_1 \cup C_2$ has exactly two components. Thus, we have shown that if $\lambda_2(C)$ is unique, then C has a unique minimum cut set S and $C - S$ has exactly two components.

Now, we show that if C has a unique minimum cut set S and $C - S$ has exactly two components, then $\lambda_2(C)$ is unique. We proceed by induction on the number of vertices. The base case, P_3, is trivial to verify.

Suppose C is a connected cograph such that C has a unique minimum cut set S and $C - S$ has exactly two components. Let $C = C_1 \vee C_2$, where C_1 is a cograph on n_1 vertices whose spectrum is given by $\Lambda(C_1) = \{0, \alpha_2, \ldots, \alpha_{n_1}\}$ and C_2 is a cograph on n_2 vertices whose spectrum is given by $\Lambda(C_2) = \{0, \beta_2, \ldots, \beta_{n_2}\}$. Then,

$$\Lambda(C) = \{0, \alpha_2 + n_2, \ldots, \alpha_{n_1} + n_2, \beta_2 + n_1, \ldots, \beta_{n_2} + n_1, n_1 + n_2\}.$$

A cut set of C must be of the form $S_1 \cup C_2$ or $S_2 \cup C_1$, where S_1 and S_2 represent cut sets of C_1 and C_2, respectively. Without loss of generality, let the minimum cut set of C be given by $S = S_1 \cup C_2$. It follows that $\lambda_2(C) = \alpha_2 + n_2$. Since C has a unique minimum cut set, C_1 must have a unique minimum cut set. Since $C - S$ has exactly two components and $C - S = C - S_1 \cup C_2 = C_1 - S_1$, $C_1 - S_1$ has exactly two components. By induction, $\lambda_2(C_1) = \alpha_2$ is unique. Therefore, $\lambda_2(C) = \alpha_2 + n_2$ is unique. Thus, we have shown that if C has a unique minimum cut set S and $C - S$ has exactly two components, then $\lambda_2(C)$ is unique. $\square$

The eigenvector associated with the second-smallest eigenvalue is known as the Fiedler vector. The Fiedler vector is studied at length in [6]. In particular, Fiedler showed the following result.

Proposition 3.4 ([6])**.** *Let G be a connected graph and let $v_2(G)$ be the eigenvector of the second-smallest eigenvalue $\lambda_2(G)$. Let $V^{\geq}(G) = \{v_i \in V(G) \mid v_2(G)_i \geq 0\}$, and let $V^{\leq}(G) = \{v_i \in V(G) \mid v_2(G)_i \leq 0\}$. Then, the subgraphs induced by $V^{\geq}(G)$ and $V^{\leq}(G)$ are connected.*

We present here a characterization of the Fiedler vector of a connected cograph C when $\lambda_2(C)$ is unique.

Proposition 3.5. *If $\lambda_2(C)$ is unique, then the Fiedler vector of C, $v_2(C)$, defines a partition of $V(C)$ into three sets: the unique minimum cut set S and the two connected components of $C - S$. Specifically, let $V^0(C) = \{v_i \in V(C) \mid v_2(C)_i = 0\}$, $V^-(C) = \{v_i \in V(C) \mid v_2(C)_i < 0\}$, and $V^+(C) = \{v_i \in V(C) \mid v_2(C)_i > 0\}$. Then, $V^0(C) = S$, and $V^+(C)$ and $V^-(C)$ are the two components of $C - S$.*

Proof. We proceed by induction on the number of vertices. The base case, P_3, is trivial to verify.

Let C be a cograph such that $\lambda_2(C)$ is unique. If $\lambda_2(C)$ is unique, C is connected, so let $C = C_1 \vee C_2$, where C_1 is a cograph on n_1 vertices whose spectrum is given by $\Lambda(C_1) =$

$\{0, \lambda_2, \ldots, \lambda_{n_1}\}$ and C_2 is a cograph on n_2 vertices whose spectrum is given by $\Lambda(C_2) = \{0, \nu_2, \ldots, \nu_{n_2}\}$. Then,

$$\Lambda(C) = \{0, \lambda_2 + n_2, \ldots, \lambda_{n_1} + n_2, \nu_2 + n_1, \ldots, \nu_{n_2} + n_1, n_1 + n_2\}.$$

The second-smallest eigenvalue of C, $\lambda_2(C)$, is either $\lambda_2 + n_2$ or $\nu_2 + n_1$. Suppose without loss of generality that $\lambda_2(C) = \lambda_2 + n_2$. Since $\lambda_2(C)$ is unique, λ_2 is unique. By Theorem 3.3, C_1 has a unique minimum cut set S_1 and $C_1 - S_1$ has two components. By induction, the Fiedler vector of C_1 is a vector $v_2(C_1)$ that partitions $V(C_1)$ into S_1 and the two components of $C_1 - S_1$. Let $V^0(C_1)$, $V^+(C_1)$, and $V^-(C_1)$ be as defined above, so $V^0(C_1) = S_1$ and $V^+(C_1)$ and $V^-(C_1)$ are the two components of $C_1 - S_1$.

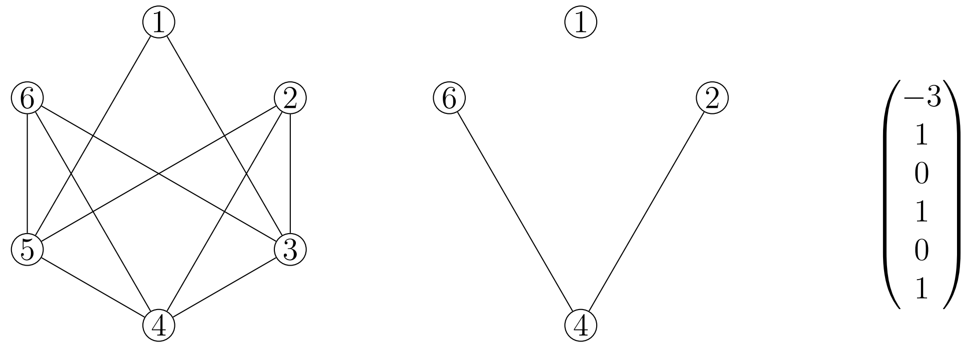
We know that the unique minimum cut set of C is $S = S_1 \cup C_2$, and that the components of $C - S$ are the same as the components of $C_1 - S_1$. Now, consider the partition given by $v_2(C)$. The Fiedler vector of C is given by

$$v_2(C) = \begin{bmatrix} v_2(C_1) \\ 0 \end{bmatrix}$$

Therefore, $V^0(C) = V^0(C_1) \cup C_2$, $V^+(C) = V^+(C_1)$, and $V^-(C) = V^-(C_1)$. So the Fiedler vector of C gives a partition of $V(C)$ into the unique minimum cut set S and the two connected components of $C - S$ as defined above.

Note that since $\lambda_2(C)$ is unique, any eigenvector of $\lambda_2(C)$ must be a nonzero scalar multiple of the eigenvector $v_2(C)$ constructed in this proof. Therefore, this result holds for any choice of eigenvector for $\lambda_2(C)$. $\square$

In Figure 3.1, we give an example demonstrating the results of Theorem 3.3 and Proposition 3.5.



(a) C (b) $C - S$ (c) $v_2(C)$

Figure 3.1: In this example, we consider the cograph C shown in (a) above. The spectrum of C is $\Lambda(C) = \{0, 2, 3, 4, 5, 6\}$. C has a unique minimum cut set $S = \{3, 5\}$, and $C - S$, shown in (b) above, has exactly two components, as proven in Theorem 3.3. The eigenvector for $\lambda_2(C)$ is given in (c) above. Here, $V^0(C) = \{3, 5\}$, $V^-(C) = \{1\}$, and $V^+(C) = \{2, 4, 6\}$. We see that $V^0(C) = S$, and $V^-(C)$ and $V^+(C)$ give the two connected components of $C - S$, as proven in Proposition 3.5.

3.2 Eigenvalues and Twin Numbers

Now that we have characterized the algebraic connectivity of cographs, we broaden our focus to the full spectrum. We present here a relationship between the eigenvalues of a cograph C and the sizes of its twin classes, generalizing the resulting relating eigenvalues and the degree sequence in threshold graphs described in Section 2.3. We begin by defining the notion of twin vertices.

Definition 3.6. Let G be a graph. Two vertices v and w are *twins* if $N(v) - w = N(w) - v$, where

$$N(v) = \{u \in V(G) \mid u \sim v\}$$

denotes the neighborhood of v. We write $u \approx v$ to represent that u and v are twins.

Proposition 3.7. *The relation of being twins is an equivalence relation.*

Proof. The relation is trivially reflexive and symmetric. We show here that it is transitive. Let G be a graph, let $u, v, w \in V(G)$ be vertices of G, and suppose $u \approx v$ and $v \approx w$. Then, $N(u) - v = N(v) - u$ and $N(v) - w = N(w) - v$.

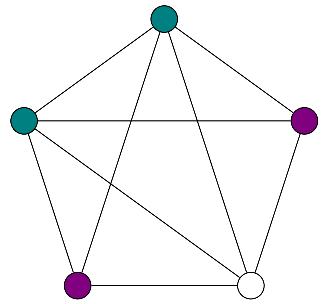


Figure 3.2: Example of twin vertices in a cograph. In this graph, the green vertices are adjacent twins and the purple vertices are nonadjacent twins.

We want to show that $N(u) - w = N(w) - u$. Consider $x \in N(u) - w$, so $x \sim u$ and $x \neq w$. There are two cases: $x = v$ and $x \neq v$. First, suppose $x \neq v$. Since $x \sim u$ and u and v are twins, $x \sim v$. Since $x \sim v$ and v and w are twins, $x \sim w$. Therefore, $x \in N(w) - u$. Now, suppose $x = v$. Since $u \sim v$ and v and w are twins, $u \sim w$. Since $u \sim w$ and u and v are twins, $v \sim w$. Therefore, $v \in N(w) - u$.

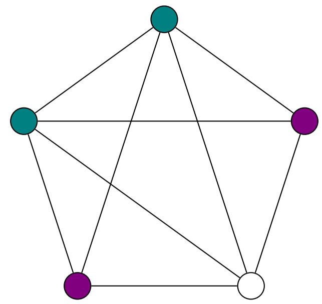
We have shown that for all $x \in N(u) - w$, $x \in N(w) - u$, so $N(u) - w \subseteq N(w) - u$. By an equivalent argument, $N(w) - u \subseteq N(u) - w$. Therefore, $N(u) - w = N(w) - u$, and $u \approx v$. $\square$

Now that we have shown that the relation of being twins is an equivalence relation, we give the following definitions.

Definition 3.8. The *twin partition* of a graph G is the partition of the vertices into their equivalence classes under the relation of being twins. We call the parts of the twin partition the *twin classes* of G and denote them $T_1, \ldots, T_k$.

The vertices in a twin class form a clique if they are pairwise adjacent twins. The vertices in a twin class form an independent set if they are pairwise nonadjacent twins.

Definition 3.9. The *twin numbers* $t_1, \ldots, t_k$ of a graph G are the sizes of the twin classes.



(a)

	T_1	T_2	T_3
twin number t_i	2	2	1
degree d_i	4	3	4

(b)

Figure 3.3: The example cograph C from Figure 3.2 is shown in (a). The twin classes are T_1, T_2, and T_3, where T_1 is the green vertices, T_2 is the purple vertices, and T_3 is the white vertex. The twin numbers and the degrees are shown in (b) above.

Definition 3.10. The *degree* of a twin class T_i, denoted d_i, is the degree of the vertices in the twin class.

It is known that every cograph with at least two vertices has at least one pair of twins [3]. Therefore, the twin partition of a cograph is always nontrivial and at least one twin number is greater than one.

We also define two special types of twin classes.

Definition 3.11. A *universal class* of a cograph C is a twin class containing all universal vertices of C.

Definition 3.12. An *isolated class* of a cograph C is a twin class containing all isolated vertices of C.

We have the following proposition.

Proposition 3.13. *Let C be a cograph with twin classes $T_1, \ldots, T_k$, and suppose T_1 is an isolated class. Then, the nonzero eigenvalues of C are equal to the nonzero eigenvalues of $C - T_1$, the cograph given by deleting the vertices in T_1.*

Proof. Let C be a cograph on n vertices, and let T_1 be an isolated class on n_1 vertices. Then, C can be written as $C = (C - T_1) + G(T_1)$, where $C - T_1$ is the graph given by deleting the vertices in T_1 from C, and $G(T_1)$ is the subgraph induced on the vertices of T_1. Since T_1 is an isolated class, $G(T_1)$ is an edgeless graph and every eigenvalue of $G(T_1)$ is 0. Therefore, by the recursive formula for cograph eigenvalues, the nonzero eigenvalues of C are equal to the nonzero eigenvalues of $C - T_1$. $\square$

We now present a relationship between the twin classes of a cograph C and the twin classes of $\overline{C}$.

Proposition 3.14. *Let C be a cograph with twin classes $T_1, \ldots, T_k$ and twin numbers $t_1, \ldots, t_k$. Then, the twin classes of $\overline{C}$ are $T_1, \ldots, T_k$ and the twin numbers of $\overline{C}$ are $t_1, \ldots, t_k$. Further, if class T_i has degree d_i in C, then T_i has degree $n-1-d_i$ in $\overline{C}$.*

Proof. We show that u and v are twins in C if and only if u and v are twins in $\overline{C}$. Suppose u and v are twins in C, so $N_C(u) - v = N_C(v) - u$, where $N_C(v)$ denotes the neighborhood of v in C. Then, $V - N_C(u) - u - v = V - N_C(v) - u - v$. We know that $V - N_C(u) - u = N_{\overline{C}}(u)$, where $N_{\overline{C}}(u)$ is the neighborhood of u in $\overline{C}$. Similarly, $V - N_C(v) - v = N_{\overline{C}}(v)$. Therefore, $N_{\overline{C}}(u) - v = N_{\overline{C}}(v) - u$, so u and v are twins in $\overline{C}$.

Since $\overline{\overline{C}} = C$, if u and v are twins in $\overline{C}$, u and v are twins in C.

It follows that the twin classes and the twin numbers of $\overline{C}$ are the same as the twin classes and the twin numbers of C. If a vertex v has degree d in C, then v has degree $n-1-d$ in $\overline{C}$, so if class T_i has degree d_i in C, T_i has degree $n-1-d_i$ in $\overline{C}$. $\square$

The twin partition is relevant to our study of cographs because the eigenvalues of a cograph C can be expressed as a sum of twin numbers of C.

Theorem 3.15. *Let C be a cograph. Let $t_1, \ldots, t_k$ be the twin numbers of C, and let $d_1, \ldots, d_k$ be the degrees of the twin classes. Every nonzero eigenvalue λ_j of C can be ex-*

pressed as a sum of twin numbers, so

$$\lambda_j = \sum_{i \in I_j} t_i$$

where $I_j \subseteq \{1, \ldots, k\}$ *represents the indices of the twin numbers that contribute to* λ_j*.*

Further, every twin number t_i *contributes to* d_i *eigenvalues. For a given* i*,*

$$|\{j \in \{1, \ldots, n\} | i \in I_j\}| = d_i.$$

Proof. We proceed by induction on the number of vertices. The base cases K_2 and $\overline{K_2}$ are trivial to verify.

Consider a cograph C. There are two cases: C is connected and C is disconnected.

Suppose C is not connected. We consider two cases. First, suppose C does not have any isolated vertices. Let $C = C_1 + C_2$, where C_1 is a cograph on n_1 vertices whose spectrum is given by $\Lambda(C_1) = \{0, \lambda_2, \ldots, \lambda_{n_1}\}$, and C_2 is a cograph on n_2 vertices whose spectrum is given by $\Lambda(C_2) = \{0, \nu_2, \ldots, \nu_{n_2}\}$. The spectrum of C is given by

$$\Lambda(C) = \{0, 0, \lambda_2, \ldots, \lambda_{n_1}, \nu_2, \ldots, \nu_{n_2}\}.$$

Let the twin classes of C_1 be $S_1, \ldots, S_l$ and let the twin classes of C_2 be $T_1, \ldots, T_k$. We want to determine the twin classes of C. Since C has no isolated vertices, C_1 and C_2 have no isolated vertices.

Consider S_i, a twin class of C_1. Since S_i is not an isolated class, there exists $w \in V(C_1)$ such that the vertices in S_i are adjacent to w. Since no vertex in C_2 is adjacent to w, the vertices in S_i are not twins with any vertices in C_2. Therefore, S_i is a twin class of C. An equivalent argument shows that T_j is a twin class of C when T_j is a twin class of C_2.

Then, the twin classes of C are $\{S_1, \ldots, S_l, T_1, \ldots, T_k\}$ and the twin numbers of C are $\{s_1, \ldots, s_l, t_1, \ldots, t_k\}$. If S_i has degree d_i in C_1, then S_i has degree d_i in C. Similarly, if T_i has degree d_i in C_2, then T_i has degree d_i in C.

Consider a nonzero eigenvalue of C of the form λ_j. Since λ_j is an eigenvalue of C_1, by induction, $\lambda_j = \sum_{i \in I_j} s_i$. Therefore, λ_j can be written as a sum of twin numbers of C. By an equivalent argument, every nonzero eigenvalue ν_j of C can be written as a sum of twin numbers of C. Thus, every nonzero eigenvalue of C can be written as a sum of twin numbers of C.

Now, consider a twin number s_i of C. By induction, s_i contributes to d_i eigenvalues in C_1, where d_i is its degree in C_1 and its degree in C. In C, s_i contributes to the same d_i eigenvalues. Similarly, a twin number t_i of C contributes to d_i eigenvalues, where d_i is its degree in C_2 and its degree in C. Therefore, every twin number of C contributes to d eigenvalues, where d is its degree in C.

Now, suppose C is a cograph with an isolated class J. Then, $C = J + C'$, where C' has no isolated vertices. Let the twin classes of C' be $\{T_1, \ldots, T_k\}$ and the twin numbers of C' be $\{t_1, \ldots, t_k\}$. Then, the twin classes of C are $\{J, T_1, \ldots, T_k\}$ and the twin numbers of C are $\{j = |J|, t_1, \ldots, t_k\}$. The twin class J has degree 0; every other class T_i with degree d_i in C' has degree d_i in C.

By Proposition 3.13, the nonzero eigenvalues of C are the nonzero eigenvalues of C'. By induction, if λ_j is a nonzero eigenvalue of C', $\lambda_j = \sum_{i \in I_j} t_i$. Therefore, every nonzero eigenvalue of C can be written as a sum of twin numbers of C.

Now, consider a twin number t_i of C. Then, t_i is a twin number of C' and t_i contributes to d_i eigenvalues in C'. Therefore, t_i contributes to d_i eigenvalues in C. The twin number j does not contribute to any nonzero eigenvalue, and class J has degree 0. So every twin number contributes to d eigenvalues, where d is the degree of the corresponding twin class.

Thus, we have shown that the theorem is true if C is not connected.

Now, suppose $C = C_1 \vee C_2$, and consider $\overline{C} = \overline{C_1} + \overline{C_2}$. We have shown that if $T_1, \ldots, T_k$ are the twin classes of $\overline{C}$ and $t_1, \ldots, t_k$ are the twin numbers of $\overline{C}$, then every eigenvalue of $\overline{C}$ can be written as a sum of twin numbers. By Proposition 3.14, the twin classes and the twin numbers of C are the same as those of $\overline{C}$. Let $\Lambda(C) = \{0, \lambda_2, \ldots, \lambda_n\}$, so $\Lambda(\overline{C}) = \{0, n - \lambda_n, \ldots, n - \lambda_2\}$. Then, $n - \lambda_j = \sum_{i \in I_j} t_i$, where I_j is the set of indices representing which twin numbers contribute to $n - \lambda_j$ in $\overline{C}$. Because $n = \sum_{i=1}^{k} t_i$, we can write λ as

$$\lambda_j = \sum_{i=1}^{k} t_i - \sum_{i \in I_j} t_i = \sum_{i \notin I_j} t_i.$$

Therefore, λ_j can be written as the sum of twin numbers of C.

Now, consider a twin number t_i of $\overline{C}$. We know that t_i contributes to d_i eigenvalues of $\overline{C}$, where d_i is the degree of t_i in $\overline{C}$. By the above construction, we see that t_i contributes to eigenvalue j of C if and only if it does not contribute to eigenvalue j of $\overline{C}$. Therefore, t_i contributes to $n - 1 - d_i$ eigenvalues in C. By Proposition 3.14, $n - 1 - d_i$ is the degree of T_i in C. Therefore, every twin number of C contributes to d eigenvalues, where d is its degree in C. □

We give an example of Theorem 3.15 here. Consider the cograph C given in Figure 3.4. The spectrum of C is $\Lambda(C) = \{0, 3, 3, 4, 4, 5, 7\}$.

The twin classes of C are $T_1 = \{1, 3, 5\}$, $T_2 = \{6, 7\}$, and $T_3 = \{2, 4\}$. In Table 3.1, we give the twin numbers and degrees of the twin classes. In particular, $t_1 = 3$, $t_2 = 2$, and $t_3 = 2$, and $d_1 = 4$, $d_2 = 4$, and $d_3 = 3$. We represent the result of Theorem 3.15 using a

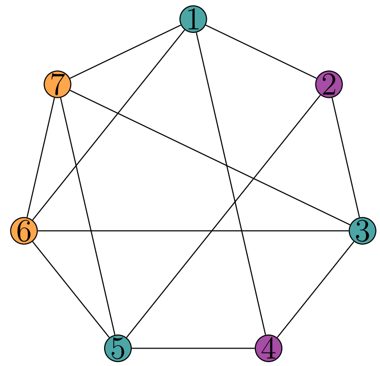


Figure 3.4: Example cograph C. The twin classes $T_1 = \{1, 3, 5\}$, $T_2 = \{6, 7\}$, and $T_3 = \{2, 4\}$.

	T_1	T_2	T_3
twin numbers t_i	3	2	2
degree d_i	4	4	3

Table 3.1: The twin numbers and the degrees of the twin classes of cograph C, shown in Figure 3.4.

	$t_1 = 3$	$t_2 = 2$	$t_3 = 2$
7	$\times$	$\times$	$\times$
5	$\times$	$\times$	
4		$\times$	$\times$
4		$\times$	$\times$
3	$\times$		
3	$\times$		
0			

Table 3.2: This contribution table shows which twin numbers contribute to which eigenvalues of C, shown in Figure 3.4. For example, t_1 and t_2 contribute to $\lambda_6 = 5$. Notice that the total number of squares marked by an $\times$ in column i is equal to d_i. The squares marked by an $\times$ in row j represent I_j, the set of indices of twin numbers that contribute to eigenvalue j.

contribution table.

Definition 3.16. The *contribution table* of a cograph C is a table where each row represents an eigenvalue of C and each column represents a twin number of C. There is an $\times$ in cell $[i, j]$ of the contribution table if and only if twin number t_j contributes to eigenvalue λ_i.

The contribution table of C is given in Table 3.2 and shows which twin numbers contribute to which eigenvalues of C.

Recall that by Theorem 2.24, the nonzero eigenvalues of a threshold graph T are equal to the Ferrer's conjugate of the degree sequence of T. Theorem 3.15 provides a generalization of this theorem to all cographs. We detail here how Theorem 3.15 relates to Theorem 2.24.

Proposition 3.17. *Let T be a threshold graph, and suppose u and v are vertices of T. If $d(u) = d(v)$, then u and v are twins.*

Proof. Let $f : V(T) \to \mathbb{R}$ be the weight function associated with T. Without loss of generality, suppose $f(u) \geq f(v)$. If $d(u) = d(v)$, then $|N(u) - v| = |N(v) - u|$. Suppose $w \in N(v) - u$, so $v \sim w$ and $f(v) + f(w) \geq 1$. Then, $f(u) + f(w) \geq f(v) + f(w) \geq 1$, so $u \sim w$ and $w \in N(u) - v$. Therefore, $N(v) - u \subseteq N(u) - v$. Since $N(v) - u \subseteq N(u) - v$ and $|N(v) - u| = |N(u) - v|$, $N(v) - u = N(u) - v$, and u and v are twins. □

Therefore, two vertices u and v of threshold graph T are twins if and only if $d(u) = d(v)$. This result motivates the following definition.

Definition 3.18. Let G be a graph on n vertices.. The *degree classes* D_i of G for $i = 0, \ldots, n-1$ form a partition of G, where

$$D_i = \{v_j \in V(G) \mid d(v_j) = i\}.$$

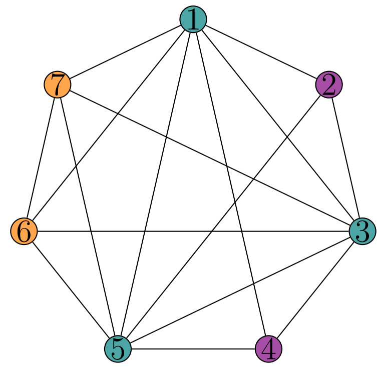


Figure 3.5: Example threshold graph T. The twin classes are $T_1 = \{1, 3, 5\}$, $T_2 = \{6, 7\}$, and $T_3 = \{2, 4\}$. The spectrum of T is $\Lambda(T) = \{0, 3, 3, 5, 7, 7, 7\}$.

It follows from Proposition 3.17 that the twin classes of a threshold graph T are exactly the nonempty degree classes of T.

Now, consider a threshold graph T with degree sequence $d(T) = (d_1, \ldots, d_n)$. Let $D_1, \ldots, D_{n-1}$ be the degree classes of T, let $t_i = |D_i|$ be the twin number of D_i and let $d_i = i$ be the degree of D_i. By Theorem 2.24, the eigenvalues of T can be written as follows. Let $(\lambda_n, \ldots, \lambda_m)$ be the nonincreasing sequence of nonzero eigenvalues of T. Then,

$$\lambda_{n+1-j}(T) = |\{v \in V(T) \mid d(v) \geq j\}|.$$

We can rewrite this equation as

$$\lambda_{n+1-j}(T) = \sum_{i=j}^{n-1} |D_i| = \sum_{i=j}^{n-1} t_i.$$

Therefore, every eigenvalue λ_j can be written as a sum of twin numbers, so every nonzero eigenvalue of T is the sum of twin numbers of T. Further, from the above equation, it is clear that t_i contributes to λ_{n+1-j} when $i \geq j$. Therefore, t_i contributes to $i = d_i$ eigenvalues. This demonstrates that Theorem 2.24 is a special case of Theorem 3.15.

In Figure 3.5, we give an example of Theorem 3.15 applied to a threshold graph T. Table 3.3

	T_1	T_2	T_3
size	3	2	2
degree	6	4	3

Table 3.3: The twin numbers and the degrees of the twin classes of threshold graph T, shown in Figure 3.5.

	$t_1 = 3$	$t_2 = 2$	$t_3 = 2$
7	$\times$	$\times$	$\times$
7	$\times$	$\times$	$\times$
7	$\times$	$\times$	$\times$
5	$\times$	$\times$	
3	$\times$		
3	$\times$		
0			

Table 3.4: This contribution table shows which twin numbers contribute to which eigenvalues in T. For example, t_1 and t_2 contribute to $\lambda_4 = 5$. Notice that the total number of squares marked by an $\times$ in column i is equal to d_i. The squares marked by an $\times$ in row j represent I_j, the set of indices of twin numbers that contribute to eigenvalue j.

(a) $S(T) = (7, 7, 7, 5, 3, 3)$. (b) $d(T) = (6, 6, 6, 4, 4, 3, 3)$.

Figure 3.6: The Ferrer's diagrams of (a) S, the nonzero eigenvalues of T, and (b) $S^*(T) = d(T)$, the degree sequence of T.

	$t_1 = 3$	$t_2 = 2$	$t_3 = 2$
7	□□□	□□	□□
7	□□□	□□	□□
7	□□□	□□	□□
5	□□□	□□	
3	□□□		
3	□□□		
0			

Table 3.5: The expanded contribution table for T. Notice that the Ferrer's diagram present in the expanded contribution table is the same as the Ferrer's diagram of $\Lambda(T)$, shown in Figure 3.6.

shows the twin numbers and degrees of T, and Table 3.4 shows the contribution table for T. Notice that a Ferrer's diagram of the eigenvalues of T is implicit in the contribution table of T. Table 3.5 shows the *expanded contribution table* of T, which is created by replacing every $\times$ in column j by t_j blocks.

4 Open Questions and Future Work

4.1 The Twin Reduction Graph

Consider a cograph C with twin classes $T_1, \ldots, T_k$. Let $v \in T_i$. Suppose $w \in T_i$ is another vertex in the same twin class as v. Then, v and w are equivalent vertices with respect to the edge structure of C. In particular, we only need to know the behavior of v to fully determine the behavior of every vertex $w \in T_i$. This motivates the following definition.

Definition 4.1. Suppose C is a cograph with twin classes $T_1, \ldots, T_k$. The *twin reduction graph* of C, denoted R_C, is the subgraph induced by $\{x_1, \ldots, x_k\}$, where $x_i \in T_i$ is a representative of class T_i.

Every twin class T_i of C is either a clique or an independent set. We write that T_i has type $\vee$ to denote that T_i is a clique, and we write that T_i has type $+$ to denote that T_i is an independent set. Figure 4.1 gives an example of a reduction graph, where every twin class is represented by a single vertex. Let $V(R_C) = \{r_1, \ldots, r_k\}$ be the vertices of the reduction graph, where r_i represents class T_i. Note that the graph structure of R_C, in addition to vertex labels of r_i giving the twin numbers and types of the classes T_i, fully determines the cograph C.

We use the twin reduction graph to formalize the notion that two cographs are "similar." In particular, we give the following definition.

(a) (b)

Figure 4.1: (a): A cograph given by $C = (\mathbf{1} \vee \mathbf{3} \vee \mathbf{5}) \vee (\mathbf{2} + \mathbf{4} + \mathbf{7} \vee \mathbf{6})$. (b) The twin reduction graph R_C. The vertex labels are the twin numbers and the types of the twin classes.

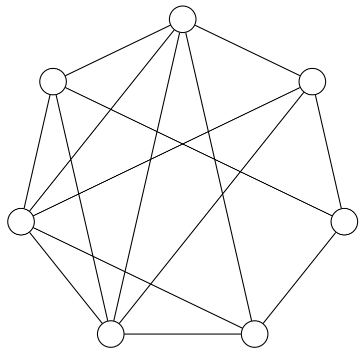
(a) R_{C_1} (b) R_{C_2}

Figure 4.2: (a): Twin reduction graph of a cograph given by $C_1 = (\mathbf{1} \vee \mathbf{3} \vee \mathbf{5}) \vee (\mathbf{2} + \mathbf{4} + \mathbf{7} \vee \mathbf{6})$ with spectrum $\Lambda(C_1) = \{0, 3, 3, 5, 7, 7, 7\}$. (b): Twin reduction graph of a cograph given by $C_2 = (\mathbf{1} + \mathbf{3} + \mathbf{5}) \vee (\mathbf{2} + \mathbf{4} + \mathbf{7} \vee \mathbf{6})$ with spectrum $\Lambda(C_2) = \{0, 3, 3, 4, 4, 5, 7\}$. Note that $\Lambda(C_1)$ and $\Lambda(C_2)$ have five eigenvalues in common, as predicted by Conjecture 4.3.

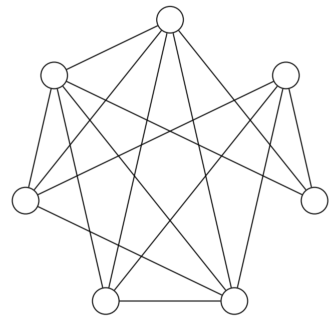
Definition 4.2. Let R_{C_1} and R_{C_2} be twin reduction graphs of cographs C_1 and C_2, and suppose the vertices of R_{C_1} and R_{C_2} are labeled by the twin numbers of the twin classes they represent. In particular, if $V(R_{C_1}) = \{r_1, \ldots, r_k\}$, where r_i represents class T_i, then r_i has label $t_i = |T_i|$. We say that cographs C_1 and C_2 are *equivalent* if $R_{C_1} = R_{C_2}$. In particular, C_1 and C_2 are equivalent if the graphs R_{C_1} and R_{C_2} are isomorphic and if the twin numbers of the vertices $V(R_{C_1})$ and $V(R_{C_2})$ are identical.

We want to use this notion of equivalence to understand the relationship between the spectra of two similar cographs. We attempt to do so in the following conjecture.

Conjecture 4.3. *Let C_1 and C_2 be cographs with equivalent reduction graphs R_{C_1} and R_{C_2}. Let k be the number of twin numbers of C_1 and C_2. Let I be the indices of the twin classes*



(a) C_1



(b) C_2

Figure 4.3: Two nonisomorphic cographs C_1 and C_2 with equal eigenvalues and degree sequences. $\Lambda(C_1) = \Lambda(C_2) = \{0, 3, 4, 4, 6, 6, 7\}$ and $d(C_1) = d(C_2) = (5, 5, 5, 4, 4, 4, 3)$.

whose types are identical in C_1 and C_2. Then, C_1 and C_2 have at least

$$k + \sum_{i \in I} (t_i - 1)$$

eigenvalues in common.

An example of this conjecture is given in Figure 4.2. Proving this conjecture is one direction for future work on cograph eigenvalues.

4.2 Other Properties of Cograph Eigenvalues

Other properties related to cograph Laplacian eigenvalues could be interesting areas of future study. For example, we showed in Proposition 2.26 that two threshold graphs T_1 and T_2 are cospectral if and only if $T_1 \cong T_2$. This does not hold for cographs; there exist nonisomorphic cospectral cographs. Figure 4.3 shows two nonisomorphic cographs with equal spectra and degree sequences. A further exploration into the construction or properties of cospectral cographs is a potential avenue for further research.

Another interesting area of future study involves the distribution of threshold graph and cograph eigenvalues. It is proven in [4] that for a random cograph C on n vertices, for $v \in C$, and for $0 \leq i \leq n-1$, $P(d(v)) = i = \frac{1}{n}$. In other words, the distribution of the vertex degrees of a random cograph is uniform. We know that the Laplacian eigenvalues of a threshold graph T are related to the degree sequence of T. Therefore, it may be possible to prove results on the distribution of the eigenvalues of a threshold graph given the distribution of the vertex degrees. Such research could potentially be extended to results on all cographs.

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Biographical Statement

Tara Abrishami grew up in Northern Virginia and graduated from Johns Hopkins University in 2019 with bachelor's and master's degrees in mathematics. While in college, she was a writer in residence at Crater Lake National Park, studied abroad in Strasbourg, France, and was a mathematics researcher in the U.K. She will be a fellow at the Voting Rights Data Institute in Boston in summer 2019, and will begin her PhD at Princeton University in the fall.