ON BLACK HOLE INFORMATION PARADOX IN AdS$_3$/CFT$_2$

by

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Abstract

The AdS/CFT duality, or the holographic principle, is currently our best tool for studying quantum gravity. The only way we know how to define quantum gravity non-perturbatively is through CFTs, which allow definitive, sharp calculations. Thus many questions about quantum gravity can be answered using CFTs if we know how to do the right calculation, perturbatively or even non-perturbatively.

Due to the infinite dimensional symmetry of two dimensional CFTs, AdS$_3$/CFT$_2$ is one of the few places where we can do explicit calculations. In this thesis, we developed several tools for studying quantum gravity in three-dimensional AdS spacetime using two-dimensional CFTs, aiming at eventually resolving the black hole information paradox. The results obtained in this thesis include the followings.

- In Chapter 2, we developed a method to compute the so called Virasoro vacuum blocks in 2d CFTs using degenerate operators. Several results were obtained using this method, including a result that is of order $1/c^3$, which corresponds to a three-loop calculation in AdS$_3$ gravity.

- In Chapter 3, we charted the breakdown of semiclassical gravity by analyzing the Virasoro conformal blocks to high numerical precision, and found a power law decay for the two-point function of light primary operators in a black hole microstate background.

- In Chapter 4, we obtained an exact formula that defines a bulk scalar proto-field interacting with gravity in terms of CFT operators.
• In Chapter 5, we discovered the breakdown of bulk locality at a new length scale via computing the two-point function of this bulk proto-field.

• In Chapter 6, we studied the horizon of a black hole microstate using this bulk proto-field, and showed the breakdown of the semiclassical gravity near the horizon in the Euclidean signature.

• In Chapter 7, we obtained formulas for proto-fields with general gravitational dressing, and generalized our bulk reconstruction proposal to bulk fields charged under a U(1) Chern-Simons gauge theory. As an application, we computed the one-graviton-loop correction to the bulk-boundary propagator in a BTZ black hole background.
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Acknowledgments

A little bit more than five years ago, I was deciding which graduate school to attend for my Ph.D. study. I finally decided to come to Johns Hopkins University, mainly because Prof. Jared Kaplan just joined Hopkins as a faculty member. Indeed, this was one of the rightest decisions I’ve made so far in my life. It has been a wonderful journey being a student of Jared, from who I learned a lot about cutting-edge physics research, and life. This thesis would not be possible without Jared’s kind guidance, support, encouragement and trust. I’ll leave Hopkins soon, but I believe that what I learned from Jared in the past five years will have a huge impact on my life in the future. I’m really grateful to Jared for being the nicest advisor that I can imagine, and a wonderful person.

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Publications

Chapter 2 – 7 of this thesis are based on the papers [1, 2, 3, 4, 5, 6]. These papers are the result of collaboration between the author of this thesis and the other authors of these papers.


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2.2 Plots comparing the exact behavior from eq. (2.4.14) (black, dashed) for $1 - F(t)$ in the limit $c \to \infty$ with $cz$ fixed, to the heavy-light approximation (2.4.10) (red, solid). Left: $h_L = h_H = \frac{1}{2}$, Right: $h_L = \frac{3}{2}, h_H = \frac{3}{10}$. $F(t)$ and $t$ are as in figure 2.1. Note that both curves only include contributions from the vacuum block, neglecting double-trace operators which could affect an AdS3 calculation.

3.1 This figure suggests the analytic continuations necessary to obtain a heavy-light correlator with increasing (Lorentzian) time separation between the light operators. We take $r \lesssim 1$ to avoid singularities on the lightcones displayed on the left; one can also use $r$ as a proxy for a Euclidean time separation between the light operators.
3.2 This figure shows the Penrose diagram for an energy eigenstate black hole in AdS, suggesting the role of ingoing and outgoing modes behind the horizon and their relationship with local CFT operators. Analytic continuation provides a painfully naive but instrumentally effective method for studying correlators behind the horizon.

3.3 The $q(z)$ map takes the universal cover of the $z$-plane (the sphere with punctures at $0, 1, \infty$) to $|q| < 1$. This figure suggests the relationship between the $z$ plane, the unit $\rho$ disk, and the unit $q$ disk, with branch cuts indicated with colored lines [8]. The relations between these variables are $q = e^{-\pi \frac{K(1-z)}{K(z)}}$ and $z = \frac{4\rho}{(1+\rho)^2}$, and the inverse transformations are $z = \left(\frac{\theta_2(q)}{\theta_3(q)}\right)^4$ and $\rho = \frac{z}{(1+\sqrt{1-z})^2}$. The Virasoro blocks converge throughout $|q| < 1$, with OPE limits occurring on the $q$ unit circle.

3.4 This figure displays contours of constant $|q|$ inside the $\rho$ unit circle, which corresponds to the entire $z$-plane via $z = \frac{4\rho}{(1+\rho)^2}$. Since this is only the first sheet of the $z$-plane, it corresponds to the region in the $q$-disk enclosed by the two blue lines connecting $\pm 1$ in figure 3.3. The correlator can have singularities in the OPE limits $\rho \to -1, 1$ and these correspond to $q \to -1, 1$ as well. Away from these limits $|q| < |\rho|$ and the $q$-expansion converges much more rapidly than the $\rho$ expansion.

3.5 These plots display the maximum $|q|$ where the $q$-expansion converges for various choices of parameters. Convergence improves when $h_L$ and $h_H$ move closer to $c/24$ and when $c$ decreases. The intermediate primary dimension $h$ seems to have little effect on convergence. These plots define ‘convergence’ as $\left|\frac{\mathcal{V}_{2,25N}(q)}{\mathcal{V}_N(q)} - 1\right| < 10^{-5}$, where $\mathcal{V}_M$ includes an expansion up to order $q^M$. . . .
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3.7 In this figure we plot $R = \frac{c' \log V(c,q)}{c \log V(c',q)}$ with $c' \equiv \frac{11}{10} c$ in order to test the semiclassical limit. As we increase $c$, the semiclassical limit becomes a better approximation and $R \to 1$, but even for $c = 2.1$ the blocks are remarkably well approximated by the semiclassical form. For the larger choices of $c$ the functions have similar shapes up to an overall rescaling; this suggests that the first $1/c$ correction is dominating the discrepancy $R - 1$. In the OPE limit $q \to 0$ the semiclassical limit always applies. We find similar results for non-vacuum blocks.

3.8 Heavy-light Virasoro vacuum blocks switch from an initial exponential decay to a slow, universal power law decay at roughly the time scale $t_d = t_D - b$, where the constant offset $b$ depends on the choice of $r$ in $z = 1 - re^{-it}$. The vertical axis is $\log |V|$, while the horizontal axis is the Lorentzian time $t$. The black lines are full Virasoro vacuum blocks computed to order $q^{1200}$. This polynomial truncation stops converging in the shaded region. The yellow dashed lines are the semiclassical vacuum blocks using methods of [9]. The red dashed lines are the time scale (3.3.3). The blue dashed lines are the power law $at^{-\frac{3}{2}}$ with $a$ properly chosen to match the full blocks.
3.9 This figure displays the time $t_d$ at which the semiclassical vacuum blocks drop below the exact vacuum blocks. The dashed line is a fit to the analytic prediction $t_D \equiv \frac{\pi c}{6h_L}$ with an empirical offset $t_d = t_D - 2.6$; the offset depends on the choice of $r$ with $z = 1 - re^{-it}$. Note that the data with smaller values of $c$ is noisy, but the larger values fit the linear behavior extremely well. The plot includes a variety of choices for $\frac{h_L}{c}$. 

3.10 The late time behavior of various non-vacuum Virasoro blocks. The vertical axis is $\log |\mathcal{V}|$ and the horizontal axis is the time $t$. The black lines are full Virasoro blocks computed to order $q^{1200}$, plotted using $z = 1 - re^{-it}$ with $r = 0.3$. The polynomial truncation no longer converges in the shaded region. The blue dashed lines are the power law $at^{-\frac{3}{2}}$ with the constant $a$ fitted to the blocks. We refer to the time and height of the maxima as $t_{\text{max}}$ and $|\mathcal{V}|_{\text{max}} = 16^{h-\frac{c-1}{24}}|\tilde{\mathcal{V}}|_{\text{max}}$. 

3.11 These plots show a host of data demonstrating that $|\tilde{\mathcal{V}}|_{\text{max}}$ and $t_{\text{max}}$ have simple dependence on $\frac{h}{c}$ when $h \gtrsim h_H$ (recall $\alpha_h \equiv \sqrt{1 - \frac{44}{c}}$) for a large variety of different choices of $h_H$. For all of these plots we choose $c = 10$, but we have found that the results are robustly $c$-independent. These plots use $h_L = \frac{c}{30}$, but $h_L$ dependence is mild, as seen in figure 3.12. 

3.12 We have found empirically that the time and height of the maxima of heavy-light Virasoro blocks have a simple dependence on both $h$ and $h_H$. This figure shows linear fits used to obtain the parameters $a_{\text{height}}$ and $A_t$ defined in equations (3.3.6) and (3.3.5). These plots both have $c = 10$. Each point is obtained from the slope of $\log|\tilde{\mathcal{V}}|_{\text{max}}$ and $t_{\text{max}}$ as linear functions of $\log \frac{h}{c}$ and $|\alpha_h|$ respectively (we’ve used points with $\frac{h}{c} = \frac{n}{3}$ for $n = 1, 2, \cdots, 30$). We find that both plots are robustly $c$-independent for $c \gtrsim 5$, as expected in the semiclassical limit. We see explicitly that there is little dependence on $h_L$; in the $a_{\text{height}}$ plot the variation with $h_L$ is almost invisible.
3.13 These plots show a variety of parameter choices where the behavior of Virasoro blocks on the timescale $e^S$ (green vertical line), and even $e^{e^S}$ (blue vertical line), are visible. Yellow lines indicate semiclassical behavior, while the light blue fit corresponds to $t^{-\frac{3}{2}}$. Recall $S = \frac{\pi^2}{4} cT_H$ with $2\pi T_H = \sqrt{\frac{24h_H}{c} - 1}$, so some plots have relatively large $c$ and small $T_H$, while others have order one $T_H$ but small $c$. In all cases we see that the $t^{-\frac{3}{2}}$ late-time decay persists on these exponentially long timescales. These plots all display vacuum blocks, but we have found similar behavior with $h > 0$. 

3.14 The behavior of the coefficients of the $q^{2n}$ term in the polynomial $H$ in (3.2.4) compared to the prediction (3.3.8). The horizontal axis is $\log n$ and the vertical axis is $\log c_n$, where $c_n$ is the coefficient of $q^{2n}$ in $H$. The red lines are power-laws $an^s$ with the constant $a$ determined by the fit.

3.15 In this plot, we compare the exact and semiclassical blocks. One can see that at the positions of the semiclassical forbidden singularities, the exact blocks are smooth. Fixing $h_L$ and $\frac{h_H}{c}$ as we increase $c$, the exact blocks approach the semiclassical block in the region between the origin and the first forbidden singularity. However, beyond the first forbidden singularity the exact blocks deviate greatly as we increase $c$. This indicates that we have passed a Stokes line (emanating from the forbidden singularity) and some other semiclassical saddle dominates the exact blocks in the large $c$ limit. The gray line is the position of $t = i\beta \frac{3}{2}$.

3.16 In this figure we compare the semiclassical and exact blocks associated with $\mathcal{O}(t)$ and $\mathcal{O}(t + i\beta \frac{3}{2})$. The plot suggests that the semiclassical approximation remains valid for correlators of $\mathcal{O}(t + i\beta \frac{3}{2})$. We implement time dependence via $z = 1 - re^{-it}$ and so a shift by $i\beta \frac{3}{2}$ simply corresponds to a different choice of $r$. Corresponding trajectories in the unit $q$ disk are pictured in figure 3.6. Apparently the semiclassical approximation works well at $t + i\beta \frac{3}{2}$. 

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3.17 The figure on the left shows a contour plot of the function $|\Sigma_H \frac{\partial^2}{\partial \phi^2} \phi|$ from equation (3.4.2) in the $\rho$ unit disk with $h_L = 1$ and $h_H = \frac{c}{4}$. The figure on the right is the deviation of the exact and semiclassical Virasoro vacuum blocks with the same parameters and $c = 60$. The positions of the forbidden singularities are indicated with black dots. The plot on the left can be viewed as a kind of analytic prediction for the deviation plotted on the right.

3.18 The figure on the left shows a contour plot of the function $|\Sigma_H \frac{\partial^2}{\partial \phi^2} \phi|$ from equation (3.4.2) in the $\rho$ unit disk with $h_L = 1$ and $h_H = \frac{c}{30}$. In this case $h_H < \frac{c}{24}$, so the heavy-light block does not include a black hole – instead it corresponds to a light probe interacting with a deficit angle in AdS$_3$. Thus there are no forbidden singularities, and the semiclassical approximation is reliable in a much larger region as compared to figure 3.17 (note the difference in scales). The figure on the right is the deviation of the exact and semiclassical Virasoro vacuum blocks with the same parameters and $c = 60$. The plot on the left can be viewed as a kind of analytic prediction for the deviation plotted on the right.

4.1 This figure portrays a bulk-boundary OPE block used to compute the correlator (4.1.2). The red line denotes the gravitational or Chern-Simons Wilson line, while the blue circle suggests radial quantization around the block, so that it creates a definite linear combination of Virasoro descendants of the identity. The explicit computation involves line integrals over stress tensor correlators.
5.1 Left: The trace of the monodromy matrix $M$ computed numerically as a function of $\rho$ and $\rho g'(\rho)$, for three values of $\rho$: $\rho = 0.4$ (black, solid), $\rho = 0.5$, (red, dashed), and $\rho = 0.6$, (blue, dotted). Right: The critical value of $\sigma_c$ (black, solid) where the semiclassical part of the $\langle\phi\phi\rangle$ correlator first develops an imaginary piece, as a function of $h/c$. For comparison, we show (red, dashed) the analytic small $h/c$ behavior, $\sigma_c \approx (9\sqrt{3}h/c)^{1/3}$ from (5.3.23), and (blue, dot-dashed) the large $h$ behavior $\sigma_c \approx \log(2 + \sqrt{3})$, from (5.3.38).

5.2 This plot compares the ratios of successive coefficients in the holomorphic and mixed terms contributing the the full propagator. We see that at large $c$, the coefficients of $\rho^n$ grow at the same rate, meaning that the holomorphic propagator provides a good estimate for the behavior of the full propagator.

5.3 This figure displays the scalar-graviton one-loop diagram that contributes to $\langle\phi(X_1)\phi(X_2)\rangle$ at order $1/c$. There is also a contact interaction, but the associated diagram vanishes. The computation is performed in appendix D.1.1.

5.4 This figure displays fits to logarithms of ratios of successive coefficients in the $\rho$ expansion of equation (5.6.1) up to the 400th order. In all cases we have set $h = 0$ identically, and the value of $c$ increases from the top to the bottom of the plot, ranging from 1.5 to $10^5$. Each line corresponds to one of the points on Fig. 5.5, but for legibility we have only included every fifth point.

5.5 In this plot we used the fits of Fig. 5.4 to extract an approximate asymptotic ratio $\frac{a_{n+1}}{a_n}$, which was then used to identify $\sigma_*$, the scale at which bulk locality appears to break down, for each value of $c$. For very small values of $c$ we find $\sigma_*$ of order the AdS scale, so that at $c \rightarrow 1$ we smoothly match the large $h$ results of section 5.3.3, as indicated by the red line. At large $c$ we enter the flat space regime of small $\sigma_*$, where we extract the fit $\sigma_* \propto c^{-0.27}$. Varying the details of the fitting shifts the exponent, but we consistently find that it lies between 0.25 and 0.28.
5.6 This figure displays the scale at which the propagator breaks down as we approach the semiclassical limit; for each value of $c$, we’ve taken a range of values for the ratio $\frac{h}{c}$. The data was extracted in the same way as in Fig. 5.5. We see that at large $h$ we approach the convergence bound $\sigma_* = R_{AdS} \log(2 + \sqrt{3})$ from the exact result of section 5.3.3. We have also shown (blue, dashed) the result from the numeric semiclassical computation in section 5.3.4, and find that it agrees with the radius of convergence analysis for the large $c$ ($= 20,000$) points shown above.

5.7 This plot shows the same data as figure 5.5 for the radius of convergence $\sigma_*$ as a function of $c$, except here we compare the radii of convergence of the holomorphic and the mixed contributions to the full correlator. Due to numerical limitations we only include the coefficients $a_n$ and $b_n$ up to $n = 300$, which means that these data are not as reliable as those of figure 5.5. Nevertheless, this plot provides clear evidence that the parametric scaling of $\sigma_*$ with $c$ is the same for the full and holomorphic propagators when we restrict to the $z\bar{z}$ plane.

6.1 This figure depicts a Euclidean bulk-boundary correlator in a black hole microstate. Although we have forced the correlator to live on the Euclidean BTZ geometry, due to violations of the KMS condition the correlator will be multivalued on the Euclidean time circle, and so must have a branch cut. Thus semiclassical predictions for bulk correlators must breakdown. In particular, as the Euclidean time circle shrinks to vanishing size at the horizon, it would seem that exact bulk correlators must differ significantly from their semiclassical limits at the Euclidean horizon.

6.2 Caption for LOF
6.3 **Left:** This figure depicts a Euclidean bulk-boundary correlator $|V_{0}^{\text{semi}}|$ on the BTZ ‘cigar’ geometry, focusing on slices at fixed $r$, where we can easily study Euclidean time periodicity. **Right:** These plots display the semiclassical bulk-boundary correlator $V_{0}^{\text{semi}}$ on constant-$r$ slices. The semiclassical correlator is periodic in $t_{E}$, and its range of variation becomes smaller as we approach the horizon $r = r_{+}$, where it is constant in $t_{E}$. The red dashed line is $t_{E} = \beta$ and the parameters are $\frac{h_{L}}{c} = 1, h_{L} = 1$.

6.4 The blue lines are the exact result $|V_{0}^{\text{exact}}|$ and the yellow lines are the semiclassical $|V_{0}^{\text{semi}}|$. From top to bottom the rows of plots correspond to $c = 8.1, 16.1, 32.1, 64.1$, respectively. Other parameters for these plots are $h_{L} = 0.01, \frac{h_{L}}{c} = 100$, and $r_{+} \approx 50$. The first two plots in each row are in the region whose distance from the horizon is much smaller than the AdS radius. The red dashed line is $t_{E} = \beta$ and the gray dashed line is $t_{E} = \pm \beta/2$. The exact results in the visible plot range have converged to better than $10^{-13}$ precision (the precision of convergence is defined as in figure 6.2).

6.5 This is a plot of $|V_{0}^{\text{semi}}|$ zoomed in to the tip of the Euclidean ‘cigar’, with $r_{+} < r < 1.025r_{+}$ and $0 < t_{E} < \beta$. The radial coordinate of the disk is $r - r_{+}$ and the angular direction is $\frac{2\pi}{\beta} t_{E}$; the BTZ angular coordinate $\theta = 0$. The center of the plot is the position of the Euclidean horizon and $r_{+} \approx 49$. Notice that $|V_{0}^{\text{semi}}|$ is smooth everywhere in this plot, in contrast to $|V_{0}^{\text{exact}}|$ in figure 6.6 below.

6.6 These are plots of $|V_{0}^{\text{exact}}|$ for $h_{L} = 0.01, h_{L}/c = 100$ but with different values of $c$. These are plotted in the same region and use the same range as figure 6.5 for ease of comparison. These results have converged to better than $10^{-10}$ accuracy except for a tiny region at the origin of the disk (i.e. the white point at the center).
6.7 These plots show the difference between the exact and semiclassical results: 
\[ \frac{V_0^{\text{exact}} - V_0^{\text{semi}}}{V_0^{\text{semi}}} \] in the same region as figure 6.5 and 6.6. They have the same parameters as figure 6.6: \( h_L = 0.01, h_H/c = 100 \). The difference between exact result and semiclassical result is numerically small because we’ve chosen very small \( h_L = \frac{1}{100} \) for better convergence, and this means that both the exact result and the semiclassical result are very close to 1. Qualitatively, we can see that as we increase \( c \), the agreement between the exact result and the semiclassical result improves. The exact results have converged to better than \( 10^{-10} \) accuracy.

7.1 This figure suggests many aspects of our reconstruction strategy. We reconstruct a bulk operator \( \phi \) connected by a Wilson line or ‘gravitational dressing’ that attaches to the boundary at \( z_0 \). Correlators of \( \phi \) with stress tensors will only have singularities when the stress tensors approach \( z_0 \). When \( \phi \) acts on the vacuum, it creates a state that we expand in radial quantization, and so when we define \( \phi \) we assume it is surrounded by empty space. As our methods are ultimately based on symmetry, we only compute the proto-field \( \phi \) as a linear combination of a CFT2 primary \( \mathcal{O} \) and its Virasoro descendants. All of these statements have analogs for the \( U(1) \) charged \( \phi \) we discuss as a warm-up in section 7.2, with \( T \to J \).

7.2 This figure indicates the relationship between singularities in the current \( J(z) \), the bulk equations of motion, and Wilson lines. Analogous statements hold for gravity and connect singularities in \( T(z) \) to the gravitational dressing.

7.3 This figure suggests the use of a diffeomorphism to generate a natural gravitational dressing for a CFT living on a curved space, such as the cylinder. A curved dressing in the \((u, x, \bar{x})\) coordinates, with the CFT living on a plane, is mapped using equation (7.3.10) to a radial dressing in global AdS and BTZ black hole backgrounds.
B.1 This figure shows a half-completed computation with max order \( q^{12} \); each cell \( H_i^{(k)} \) represents 2 to 4 distinct terms \( H_{m_l n_l}^{(k)} \) with \( m_l n_l = i \). The cyan row, order \( q^8 \), is currently being computed, and the red diagonal contains the terms which are being used in the computation of the cyan row. The purple cells have already been computed and are being stored for future use, and the white cells have not been computed yet or have been deleted to save RAM. The row with \( k = 0 \) (which would be at the bottom) contains the seed terms \( H^{(0)} = H_{m,n}^{(0)} = 1 \) and is not shown.

B.2 This figure corresponds to the top-right plot of figure 3.10, but includes a match to the semiclassical blocks obtained using the methods of [9], which allow for \( h, h_L \propto c \). The poorly fitted dashed line is the approximation of equation (3.2.10), which assumes \( h, h_L \ll c \), and clearly provides a much less reliable fit for these parameter values.

B.3 We have found empirically that the time and height of the maxima of heavy-light Virasoro blocks have a simple dependence on both \( h \) and \( h_H \). This figure shows data on the parameters \( b_{\text{height}} \) and \( b_{\text{time}} \) defined in equations (3.3.6) and (3.3.5). These plots both have \( c = 10 \). Each point is obtained from linear fitting of data points at \( \frac{h}{c} = \frac{2}{3} \) for \( n = 1, 2, \cdots, 30 \). We see explicitly that there is very little dependence on \( h_L \), especially at large values of \( h_H \).

B.4 This is a version of figure 3.12 where we have zoomed out to show the small \( \frac{h_U}{c} \) region. The zoomed-out points with \( \frac{c}{h_L} = (30, 35, 40) \) more closely fit slopes \((0.221, 0.233, 0.242)\), which are shown as solid lines; the \((0.521, 0.515, 0.509)\) fits for large \( \frac{h_U}{c} \) are shown as dotted lines.

B.5 This figure shows the coefficients \( c_n \) of the \( q^{2n} \) expansion of \( H \). We plot \( |c_n| \) as a function of \( n \), with both \( n \) and \( c_n \) on log scales, for increasing \( c \) with \( \frac{h_L}{c} \) and \( \frac{h_U}{c} \) held constant. The sign of the \( c_n \) are illustrated by the color of the points, with blue for positive coefficients and red for negative coefficients.
C.1 Dashed (solid) lines are graviton (scalar) propagators. 328

F.1 Witten diagram for $\langle JO^\dagger \phi \rangle$ 403

F.2 Witten diagram for $\langle \phi^\dagger \phi \rangle$ 405
Chapter 1

Introduction

General relativity (GR) and quantum field theory (QFT) are currently our best understanding of nature. GR describes gravity, while QFT (or the Standard Model) describes the remaining three interactions: electromagnetic, weak and strong interaction. It has been almost one century since theoretical physicists first tried to unify them, to find the theory that we call quantum gravity. We’ve learned a lot during this long journey. We now have perturbative string theory, which has been claimed to be the theory of quantum gravity. Nevertheless, we still don’t have a very good understanding of what non-perturbative quantum gravity is. From the studying of string theory, we now know that there’s the AdS/CFT duality, which has been one of the main tools for studying quantum gravity in the past two decades. This thesis is about using the AdS/CFT duality to study one of the most prominent problems in quantum gravity: the black hole information paradox.

The AdS/CFT duality, or the holographic principle, is the duality between quantum gravitational theories in the bulk of $d+1$-dimensional asymptotically Anti-de Sitter spacetime (AdS$_{d+1}$) and $d$-dimensional Conformal Field Theories (CFT$_d$) on the boundary. Currently, the only way we know how to define quantum gravity non-perturbatively is through CFTs, which allow definitive, sharp calculations. Thus many questions about quantum gravity can be answered using CFTs if we know how to do the right calculation, perturbatively or even
AdS$_3$/CFT$_2$ is one of the few places where we can do explicit calculations, due to the infinite dimensional symmetry of two dimensional CFTs. Therefore, this thesis will focus on using 2d CFTs to understand quantum gravity in 3d AdS. In the rest of this introduction, I’ll explain what the black hole information paradox is and how we attack it in AdS$_3$/CFT$_2$.

Black Hole Information Paradox

On the one hand, black holes are solutions to Einstein’s equations in general relativity. On the other hand, black holes are quantum mechanical, since they have temperature, entropy, and they radiate. These properties make black holes the natural object to study if we want to understand quantum gravity. Furthermore, more than four decades ago, Hawking showed that information thrown into a black hole would be lost eventually, violating the unitarity of quantum mechanics. Resolving this black hole information paradox would be a major breakthrough in our understanding of quantum gravity. This is because any consistent theory of quantum gravity must either give us an answer to this question or explain why this question does not make sense.

Hawking’s calculation was based on local gravitational effective field theory (EFT) in the semiclassical limit. It has become clear that the most striking form of the paradox is the disagreement between unitarity and the local gravitational EFT in AdS. AdS/CFT strongly suggests that information is recovered, since CFTs are unitary theories. Therefore, it seems that the semiclassical bulk EFT must break down in some way. The discrepancy between exact quantum gravity (i.e., unitary CFTs) and the semiclassical bulk EFT can already be studied in terms of CFT correlation functions. But eventually, to fully resolve the information paradox in AdS/CFT, we must understand how to reconstruct AdS physics in terms of boundary CFT data and observables, because it’s crucial to be able to directly compare the semiclassical approximation of bulk EFT with the exact CFT results in the bulk, especially in the vicinity of a black hole horizon. In this thesis, we’ve developed several
tools to study the black hole information paradox along these lines.

**Heavy-light Virasoro Blocks from Degenerate Operators (Chapter 2)**

On the CFT side, the black hole information paradox can be studied via computing the following heavy-light four-point function,

$$\langle \mathcal{O}_H|\mathcal{O}_L(1)\mathcal{O}_L(z)|\mathcal{O}_H \rangle,$$

(1.0.1)

where the heavy operators $\mathcal{O}_H$ with conformal dimension $h_H > c/24$ are dual to BTZ black hole microstates in AdS$_3$ and the light operators $\mathcal{O}_L$ with dimension $h_L \sim \mathcal{O}(1)$ act as probes. Here, $c$ is the central charge of the 2d CFT, and according to AdS/CFT, it’s dual to the gravitational coupling constant $G_N$ in AdS$_3$ via $c \sim \frac{1}{G_N}$. Using the Virasoro symmetry of 2d CFTs, we can decompose this four-point function into Virasoro conformal blocks. In some limits, this four-point function is dominated by the Virasoro vacuum block, which is a universal contribution to (1.0.1) that captures all the gravitational interactions between the light operators and the heavy operators. Therefore, if we are interested in AdS$_3$ gravity, this is the right object to study. In Chapter 2, we developed a computational technique by analytically continuing from degenerate operators to compute this Virasoro vacuum blocks. We used this technique to compute a bunch of results, including a result that is of order $1/c^3$, which corresponds to a three-loop calculation in AdS$_3$ gravity.

**Beyond Perturbation Theory in CFTs (Chapter 3)**

In the semiclassical approximation, a CFT two-point function probing an AdS black hole will decay exponentially at late times. This is a manifestation of information loss. Indeed, the semiclassical limit of (1.0.1) decays exponentially, which confirms that the heavy-light four-point function is the right object to study if we want to understand the black hole information paradox. But to resolve it, we must at least go beyond the semiclassical limit.
In Chapter 3 we computed the Virasoro blocks of (1.0.1) exactly at finite central charge to high precision numerically. It turns out that the initial exponential decay will transition to a universal power law decay. This shows the breakdown of semiclassical gravity explicitly and confirms the expectation that the black hole information paradox must be resolved by non-perturbative effects.

Exact Bulk Reconstruction (Chapter 4)

Studying the heavy-light four-point function (1.0.1) on the boundary can help understand why the black hole information paradox exists (computationally because we took the semiclassical limit), and gives us some hints about how to resolve it (we need to be able to compute things non-perturbatively). But to eventually resolve the information paradox, we must understand how to construct the bulk gravitational theory from the boundary CFTs. The reason is that we really need to understand what’s happening near the black hole horizons and the the black hole interior in order to see if the bulk gravitational EFT breaks down there. And if it does, we want to understand why.

In AdS/CFT, bulk reconstruction is a hard problem. It is potentially ambiguous, partially due to the gauge redundancies of the gravitational theory in AdS. However, in AdS$_3$/CFT$_2$, after fixing the gauge, it turns out that it’s possible to do bulk reconstruction exactly at finite central charge, via using the Virasoro symmetry of 2d CFTs. In Chapter 4 we gave an exact definition of a bulk scalar proto-field $\phi$ in AdS$_3$ in terms of CFT operators. This definition includes all the gravitational interactions, and the correlation functions of this bulk field agree with gravitational perturbation theory when expanded at large central charge. Our definition is valid in all vacuum AdS$_3$ geometries (including BTZ black holes), thus enables us to study interesting bulk physics, e.g., bulk locality and black hole horizons.
Breakdown of Bulk Locality (Chapter 5)

Quantum gravity must be non-local, partially due to gauge redundancies. To preserve unitarity, local bulk EFT that depends on the approximate existence of local bulk observables must also break down in some way. But to show the breakdown of bulk locality explicitly was difficult previously, partially due to lack of computational tools. Now, this is made possible by computing the two-point function of our exact bulk proto-field $\langle \phi \phi \rangle$ non-perturbatively. In Chapter 5, we show explicitly that in vacuum AdS$_3$, bulk locality breaks down at a new length scale, which is between the AdS radius and the Planck scale. This new length scale appears to be the same as the smallest string length scale in AdS$_3$ in known stable string compactifications. It'll be interesting to see whether this is just a coincidence in the future through other approaches.

Black Hole Horizons (Chapter 6)

An important question toward resolving the information paradox is whether there is a "firewall" at the black hole horizon; that is, whether bulk EFTs break down there dramatically. This was under active debate during the past several years, partially due to lack of reliable result of bulk reconstruction near the black hole horizon. Our non-perturbative definition of a bulk scalar proto-field $\phi$ is valid in black hole microstates, and thus provides a way to determine the (non-)existence of firewalls through concrete calculations. We can now study observables like

$$\langle O_H | O_L \phi_L | O_H \rangle,$$

and send $\phi_L$ deep into the bulk to probe the black hole horizon created by $O_H$. I should emphasize that these are purely CFT calculations, and the bulk physics is emergent. Our result about the black hole horizon in Euclidean signature is reported in Chapter 6, where we showed the breakdown of the semiclassical bulk EFT near the horizon. The Lorentzian black hole horizon case (especially the experience of an infalling observer) is more interesting and
more challenging. Unfortunately, it’s still work in progress and not included in this thesis.

**Bulk Proto-fields with General Gravitational Dressing (Chapter 7)**

In Chapter 7, we generalize the definition of the bulk proto-field of Chapter 4, and define proto-fields with general gravitational dressing. We first study bulk fields charged under a $U(1)$ Chern-Simons gauge theory as an illustrative warm-up, and then generalize the results to gravity. As an application, we compute a gravitational loop correction to the bulk-boundary correlator in the background of a black hole microstate, and then verify this calculation using a newly adapted recursion relation. Branch points at the Euclidean horizon are present in the $1/c$ corrections to semiclassical correlators.

The content of this thesis is organized in roughly the following orders: from perturbative (Chapter 2) to non-perturbative (Chapter 3), and from the boundary (Chapter 2-3) into the bulk (Chapter 4-7), with the ultimate goal of understanding the black hole horizon.
Chapter 2

Degenerate Operators and the $1/c$ Expansion: Lorentzian Resummations, High Order Computations, and Super-Virasoro Blocks

This chapter is based on the following paper:


Abstract

One can obtain exact information about Virasoro conformal blocks by analytically continuing the correlators of degenerate operators. This technique can be used to explicitly resolve information loss problems in $\text{AdS}_3/\text{CFT}_2$. In this chapter we use the technique to perform
calculations in the small $1/c \propto G_N$ expansion: (1) we prove the all-orders resummation of logarithmic factors $\propto \frac{1}{c} \log z$ in the Lorentzian regime, demonstrating that $1/c$ corrections directly shift Lyapunov exponents associated with chaos, as claimed in prior work, (2) we perform another all-orders resummation in the limit of large $c$ with fixed $cz$, interpolating between the early onset of chaos and late time behavior, (3) we explicitly compute the Virasoro vacuum block to order $1/c^2$ and $1/c^3$ with external dimensions fixed, corresponding to 2 and 3 loop calculations in AdS$_3$, and (4) we derive the heavy-light vacuum blocks in theories with $\mathcal{N} = 1, 2$ superconformal symmetry.

2.1 Introduction and Summary

The infinite dimensional Virasoro algebra profoundly constrains the dynamics of Conformal Field Theories (CFTs) in two dimensions. Certain “rational” theories have operator algebras that truncate, allowing them to be solved exactly. But despite their phenomenological relevance and beauty, rational theories are small islands in a largely uncharted sea of 2d CFTs. Furthermore, we can only study quantum gravity in AdS$_3$ by analyzing CFTs with large central charge $c$, and relatively little is known about these ‘irrational’ 2d CFTs.

Although it appears that large $c$ CFTs cannot be solved exactly, it is still possible to take some of the methods [10, 11] that make rational CFTs tractable and apply them [12] to irrational theories. The reason is that correlation functions in any CFT can be decomposed into Virasoro conformal blocks $\mathcal{V}_{h,\bar{h},c}(z)$ as

$$\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\mathcal{O}_3(z)\mathcal{O}_4(0) \rangle = \sum_{h,\bar{h}} P_{h,\bar{h}} \mathcal{V}_{h,h,c}(z)\mathcal{V}_{\bar{h},\bar{h},c}(\bar{z}).$$

The Virasoro blocks are the contributions to the Operator Product Expansion (OPE) of

\footnote{We have explicitly indicated the decomposition into a product of holomorphic and anti-holomorphic parts.}
$O_3(z)O_4(0)$ from irreducible representations of the Virasoro algebra

$$[L_n,L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \quad (2.1.2)$$

The $h_i, \bar{h}_i$ are weights of the external operators $O_i$, while $h, \bar{h}$ are intermediate operator weights. When $O_1 = O_2$ and $O_3 = O_4$, a universal contribution in equation (2.1.1) is the Virasoro vacuum block, which encapsulates the exchange of any number of pure AdS3 ‘graviton’ states between the external operators.

The Virasoro blocks have turned out to be extremely useful as a source of information about gravity in AdS3, and in fact BTZ black hole [13] thermodynamics [14] emerges in a universal, theory-independent way from the heavy-light, large central charge limit of the Virasoro blocks [15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Information loss from black hole physics appears to be due to the behavior of the blocks in this limit [16, 20, 12]. The blocks are also the basic components of the conformal bootstrap program [25, 26, 27, 28, 29]. Knowing their explicit forms would greatly assist the study of 2d CFTs and 3d gravity using the bootstrap [30, 14, 31, 32, 33, 34, 35].

Each conformal block depends only on the quantum numbers $(h_i, h, c)$ of the representations involved and not on the specific theory. A strategy for computing the blocks is to work with a theory where operator truncation occurs, and then use the fact that the result is theory-independent. This technique becomes even more useful when augmented by the fact that the conformal blocks are analytic functions of their defining quantum numbers, so that one can compute the blocks for special values of the external dimensions $h_i$ and then analytically continue. In this paper, we will use this technique to develop an efficient method to compute and study the blocks order-by-order in a $1/c$ expansion, and to perform certain all-orders Lorentzian resummations.\footnote{Currently, closed form expressions for Virasoro blocks have been obtained in an expansion about various limits, such as $h \to \infty$ [36], as well as at $c \to \infty$ in the all light and the heavy-light limit [15, 16, 17, 18, 19, 21, 22, 20]. In addition, as a function of $c$ and of intermediate operator dimensions, the Virasoro blocks are meromorphic functions with only simple poles. These properties imply recursion relations [37, 36] that efficiently compute the series expansion [38] of the vacuum blocks near $z = 0$ with generic $h_i, h, c$.}
We will organize the series expansion in terms of the ansatz\(^3\)

\[
\mathcal{V}_{h_H, h_L, 0, c}(z) = \exp \left[ h_L \sum_{n,m=0}^{\infty} \left( \frac{1}{c} \right)^m \left( \frac{h_L}{c} \right)^n f_{mn}(\eta_H, z) \right], \tag{2.1.3}
\]

where \(\eta_H = \frac{h_H}{c}\) is fixed at large \(c\), and we will compute the function \(f_{mn}\). This is a natural expansion to use in the semiclassical limit \([36, 39, 40, 15, 22]\), which keeps only the terms with \(m = 0\), but direct calculations \([17]\) indicate that it is also justified to all orders in the \(1/c\) expansion (see section 2.2 for a more detailed discussion). We explicitly compute up to order \(1/c^3\), i.e. \(m + n \leq 3\), relegating many of the detailed forms of the functions \(f_{mn}\) to appendix A.1.

We have verified that our results match with a direct computation of the blocks to high orders in a series expansion in \(z\), providing further direct evidence for the validity of our methods and for their application to information loss \([12]\). We also apply this method to compute the heavy-light super-Virasoro vacuum block in the case of \(\mathcal{N} = 1\) and \(\mathcal{N} = 2\) superconformal symmetry. With some straightforward but tedious work, the method could certainly be extended to theories with more supersymmetry, which have been studied recently using the bootstrap \([33]\).

One physically interesting regime where our techniques prove to be particularly effective is in the limit where \(z \to 0\) in the Lorentzian region. This limit would be trivial in the Euclidean regime, \(\bar{z} = z^*\), where \(z \to 0\) is the OPE limit of a conformal block and is therefore dominated by the primary state contribution. However, the regime of small \(z\) becomes highly non-trivial after analytically continuing through a branch cut to the Lorentzian sheet. In particular, correlators in the Lorentzian regime depend on the order of the operators, and continuing along different paths before taking \(z \to 0\) can produce different results, which generally include singularities at small \(z\).

\(^3\)In this paper, we denote by \(\mathcal{V}\) the vacuum Virasoro block component of the correlator \(\langle \mathcal{O}_L(0)\mathcal{O}_L(z)\mathcal{O}_H(1)\mathcal{O}_H(\infty) \rangle\), while we use \(\tilde{\mathcal{V}}\) for the normalized vacuum block, i.e. the vacuum block component of \(\langle \mathcal{O}_L(0)\mathcal{O}_L(z)\mathcal{O}_H(1)\mathcal{O}_H(\infty) \rangle \langle \mathcal{O}_L(0)\mathcal{O}_L(z)\mathcal{O}_H(1)\mathcal{O}_H(\infty) \rangle \). \(\tilde{\mathcal{V}}\) begins with \(\tilde{\mathcal{V}} = 1 + \cdots\) in the small \(z\) expansion.
This behavior is related to a variety of fascinating physical phenomena, including bulk singularities [41], black hole scrambling [42], and universal CFT causality constraints [43], to name a few. This regime was studied at subleading order in the large $c$, heavy-light limit of the vacuum block [44], where it was found that certain $\frac{1}{c} \log(z)$ terms appear. These were argued to be $1/c$ corrections to the power-law behavior of singular terms. In particular, at leading non-trivial order the growth of the singular terms is $\sim z^{-1}$, which after mapping to the thermal cylinder $z = e^{2\pi i (t+x)/\beta}$ corresponds to exponential growth with “Lyapunov” exponent $\frac{2\pi}{\beta}$. In [44] a logarithmic correction at the next order in $1/c$ was argued to be the leading term in a correction to this exponent, shifting it to $\frac{2\pi}{\beta} (1 + \frac{12}{c})$. In section 2.4, we will prove that there are indeed an infinite series of terms of the form

$$\frac{1}{cz} \left( \frac{\log(z)}{c} \right)^n$$

(2.1.4)

with exactly the correct coefficients to resum into a correction to the Lyapunov exponent. This might also be viewed as a quantum correction to the Regge trajectory. We will also provide a simple way to understand subleading logarithms, ie terms of the form $\frac{1}{cz} \left( \frac{\log(z)}{c} \right)^n$ with $m > 1$.

As noted in [44], there are also power-law corrections that are larger than the logarithmic corrections. In the limit $c \to \infty$ with $cz$ fixed, there is an infinite sequence of terms of the form $(cz)^{-n}$ that survive. The “Lyapunov” regime, where the onset of scrambling first takes place, is the regime of large $cz$ and is well-described by the first few terms in a $1/c$ expansion. However, eventually the behavior transitions to the “Ruelle” regime [7], related to the decay of quasi-normal mode excitations around a BTZ black hole, and to describe this regime of small $cz$ one must sum all the leading terms. As we will see, this series is asymptotic, so one must Borel resum it. We will show how to do this in subsection 2.4.2, with a remarkably
simple result that interpolates between the “Lyapunov” regime and the “Ruelle” regime:

$$\lim_{c z \to \infty} z^{2h_L} \mathcal{V}_{h_L, h_0}(z) = G(h_H, h_L, \frac{icz}{12\pi}) + G(h_L, h_H, \frac{icz}{12\pi}),$$

$$G(h_1, h_2, x) \equiv (x)^{2h_1} (2h_2 - 2h_1) _1F_1(2h_1, 1 + 2h_1 - 2h_2, x). \quad (2.1.5)$$

It should be remembered, however, that non-vacuum blocks may also significantly affect the behavior of the correlator at intermediate and late times.

The idea that makes these Lorentzian resummations possible is that analytic continuation from the Euclidean to the Lorentzian region simply transforms a degenerate vacuum block into a finite sum of degenerate blocks. In other words, when evaluated on the second (Lorentzian) sheet, the degenerate vacuum block $\mathcal{V}_{(1,s)}(z)$ becomes a linear combination of $s$ degenerate blocks, which are to be evaluated on the first (Euclidean) sheet. Once we understand the behavior of $\mathcal{V}_{(1,s)}(z)$ in the Lorentzian region for all $s$, we can use this to obtain the physical vacuum blocks with general $h_L$. We justify and implement these ideas in section 2.4.

The outline of this paper is as follows. In section 2.2 we review degenerate operators and outline our method of computation. The in section 2.3 we use the method to compute the heavy-light vacuum Virasoro block at order $1/c$, and the all-light Virasoro block up to order $1/c^3$, which would correspond to a 3-loop gravitational calculation in AdS$_3$. We use two methods, one based on solving differential equations, and another based on a $1/c$ expansion of the Coulomb gas formalism. In section 2.4 we state and prove various results on the resummation of logarithms and singularities in the Lorentzian regime, and discuss the application of these results to the study of quantum chaos. Finally, in section 2.5 we derive the super-Virasoro vacuum block for $\mathcal{N} = 1, 2$ superconformal symmetry. Various technical details have been relegated to the appendices.
2.2 Degenerate Operators and Heavy-Light Virasoro Blocks

In this section, we will review the properties of degenerate operators\(^4\) and explain how to use them to extract information concerning the Virasoro vacuum block in the large central charge or \(c \gg 1\) limit. A degenerate operator is a Virasoro primary operator with null descendants, which means that some of its Virasoro descendants have vanishing norm. When discussing degenerate states it is useful to introduce a parameter \(b\) so that

\[
c \equiv 1 + 6 \left(b + \frac{1}{b}\right)^2.
\]  

(2.2.1)

In this work, we take the \(c \to \infty\) limit via \(b \to \infty\). In this notation, the simplest example of a null state is the second level descendant

\[
(L^2_{-1} + b^2 L_{-2}) |h_{1,2}\rangle = 0.
\]  

(2.2.2)

One can check using the Virasoro algebra of equation (2.1.2) that the level 2 Gram matrix

\[
\begin{pmatrix}
\langle hL^2_{-1}|h\rangle & \langle hL^2_{-2}|h\rangle \\
\langle hL^2_{-2}|h\rangle & \langle hL^2_{-1}|h\rangle
\end{pmatrix}
\]  

(2.2.3)

has a vanishing determinant when the holomorphic dimension satisfies \(h_{1,2} = -\frac{1}{2} - \frac{3}{4b^2}\); the level two descendant in equation (2.2.2) is the corresponding null vector. In general, degenerate states can only occur for holomorphic dimensions satisfying the Kac formula

\[
h_{r,s} = \frac{b^2}{4}(1 - r^2) + \frac{1}{4b^2}(1 - s^2) + \frac{1}{2}(1 - rs)
\]  

(2.2.4)

\(^4\)See [45] or [46] for more systematic reviews.
for positive integers $r, s$. This formula determines the values of dimension $h$ when the Kac determinant, of which equation (2.2.3) is an elementary example, vanishes. Notice that $r \leftrightarrow s$ simply corresponds with $b \leftrightarrow 1/b$. For rational models, $b^2 (\equiv - \frac{p}{p'})$ is a rational number, and consequently so are $h_{r,s}$ and $c$. In this work we will mainly be interested in general (irrational) values of $b$ and $h_{r,s}$.

Null conditions such as (2.2.2) translate into differential equations for the correlation functions involving a degenerate state. This follows because within a correlator with operators of dimension $h_i$, the Virasoro generators $L_m$ act as differential operators due to the stress tensor Ward identities. In the simplest case of $\mathcal{O}_{1,2}$, we have:

$$\left( \partial_z^2 + \left( \frac{1 + b^{-2}}{z} + \frac{b^{-2}}{1 - z} \right) \partial_z + \frac{b^{-2} h_H}{(1 - z)^2} \right) \frac{\langle \mathcal{O}_H(\infty) \mathcal{O}_H(1) \mathcal{O}_{1,2}(z) \mathcal{O}_{1,2}(0) \rangle}{\langle \mathcal{O}_H(\infty) \mathcal{O}_H(1) \rangle \langle \mathcal{O}_{1,2}(z) \mathcal{O}_{1,2}(0) \rangle} = 0. \quad (2.2.5)$$

At $b \to \infty$, $\mathcal{O}_{1,2}$ has dimension $h_{1,2} \to -\frac{1}{2}$ and is a light “probe” operator. The other operator, $\mathcal{O}_H$, has arbitrary weight $h_H$. Equation (2.2.5) is a version of the hypergeometric differential equation; it is an exact relation for this correlator and its conformal blocks. One of its solutions, corresponding to the vacuum conformal block, is given by

$$\frac{\langle \mathcal{O}_H(\infty) \mathcal{O}_H(1) \mathcal{O}_{1,2}(z) \mathcal{O}_{1,2}(0) \rangle}{\langle \mathcal{O}_H(\infty) \mathcal{O}_H(1) \rangle \langle \mathcal{O}_{1,2}(z) \mathcal{O}_{1,2}(0) \rangle} = (1 - z)^{\beta_H} F_1 \left( 1 + b^{-2}, 2\beta_H, 2(1 + b^{-2}), z \right), \quad (2.2.6)$$

where $\beta_H$ is a parameterization of the operator dimension $h_H$ and is related to its Coulomb gas charge:

$$\beta_H = \frac{1}{2b} \left( Q - \sqrt{Q^2 - 4h_H} \right),$$

$$h_H = b \beta_H (Q - b \beta_H),$$

$$Q \equiv b + b^{-1}. \quad (2.2.7)$$

We will be interested both in the “light-light” limit, where $h_H$ is $\mathcal{O}(1)$, as well as in the
“heavy-light” limit where $b^{-2}h_H$ is fixed in the large $b$ limit. In the heavy-light limit, $O_H$ represents a heavy operator generating a background probed by $O_{1,2}$. More specifically, in a putative AdS$_3$ dual description, $O_H$ will create either a deficit angle or BTZ black hole [13]. At $c \to \infty$ in the heavy-light limit, (2.2.6) simplifies to

$$e^{-\frac{1}{2}t_E} \sin\left(\frac{\pi T_H t_E}{\pi T_H}\right),$$

(2.2.8)

where $t_E = -\log(1 - z)$ is the Euclidean time and $T_H = \frac{1}{2\pi} \sqrt{\frac{24h_H}{c}} - 1$ is the Hawking temperature of a BTZ black hole created by acting with $O_H$ on the vacuum.

More generally, the vacuum block for the correlator $\langle O_H(\infty)O_H(1)O_{r,s}(z)O_{r,s}(0)\rangle$ satisfies a finite order differential equation for all of the degenerate operators $O_{r,s}$. Since the conformal blocks depend only on the parameters $h_i, h_p, b$, and not on the particular theory, this suggests that one can compute them in general by solving the resulting differential equations. Of course, there is an obvious obstacle: the light weights $h_{r,s}$ are not quite independent free parameters. We can dial their value by changing their indices $r$ and $s$, but within some limitations. First, $r$ and $s$ must be positive integers, and at large $c > 0$ this means $h_{r,s}$ are always in the non-unitary regime. This is not as significant a limitation as it may seem, because the conformal blocks are meromorphic functions of $h_{r,s}$ (for a detailed discussion see [12]). Thus, one can hope to analytically continue the blocks as a function of integer $(r, s)$ to non-integer values. And in fact this method was used in [12] to study contributions to the vacuum block that are non-perturbatively small in the large $c$ limit, which are associated with the resolution of information loss problems.

A second, more serious obstacle is that increasing $r$ and $s$ produces new differential equations of increasingly high orders. Thus, solving for more values of $h_{r,s}$ requires solving increasingly complicated differential equations of increasingly high order. We will see that this translates into increasing complexity in using the method to solve for the vacuum block

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5Continuing a function on the integers to the entire complex plane generally requires some additional knowledge of its behavior at $\infty$; we will see that order-by-order in the large $c$ expansions we employ in this paper, the required information is provided by the OPE.
at increasingly high orders in $1/c$. Nevertheless, comparison with other methods [15, 16, 17] suggests that this may be the most efficient available procedure for determining the large $c$ vacuum blocks, especially if one wishes to go beyond the semi-classical limit.

To be more precise about the method we use, we write a generic vacuum block (which will not involve any degenerate operators) in a double expansion in $1/c$ and $hL/c$:

$$V_{hH,hL,0,c}(z) = \exp \left[ hL \sum_{n,m=0}^{\infty} \left( \frac{1}{c} \right)^m \left( \frac{hL}{c} \right)^n f_{mn}(\eta_H, z) \right], \quad (2.2.9)$$

where $\eta_H = \frac{hH}{c}$. This ansatz can be justified as follows. In the semiclassical limit of $c \to \infty$ with all $h_i/c$ fixed, we have a great deal of evidence [36, 39, 40, 15, 22] that the vacuum block can be written as $\exp \left( c g\left( \frac{hL}{c}, \frac{hH}{c}, z \right) \right)$ for some function $g$ that is analytic in $hL/c$ and $hH/c$ in a neighborhood around the origin. This explains why equation (2.2.9) does not contain terms such as e.g. $hL^2/c^2$, which would behave very differently in the semi-classical limit. Corrections to the semi-classical limit can then be expanded in powers of $1/c$, leading to equation (2.2.9).

The exponential form of the ansatz will be convenient for our purposes, but beyond the semiclassical limit it is not obligatory, and it would be just as natural to write the $1/c$ corrections in a power-series multiplying the exponential semiclassical part. One can also justify the ansatz to any order in $z$ via a direct, brute force computation [17] of the vacuum block using the Virasoro algebra. Finally, note that although we have expanded the ansatz in a heavy-light limit, the conformal blocks will be symmetric under $hL \leftrightarrow hH$.

\footnote{A limitation of this method is that it is much more complicated to get results for general internal dimension $hI$ of the conformal block. The reason for this is that once the external dimensions $hL, hH$ are fixed and $O_L$ is chosen to be a degenerate operator, then the dimensions of the allowed internal operators are also fixed to lie in a finite set. In principle, one could hope to get around this by using the fact that degenerate operators $O_{1,s}$ contain more and more operators in the OPE as $s$ is increased, and in the limit that $s$ becomes large one would have access to a tower of operators with a discretum of dimensions. However, each order in $1/c$ has a complicated dependence on $hI$, in contrast to the simple polynomial dependence on the external dimensions $hL, hH$ (for instance, the $c \to \infty$ piece is the global block, which is independent of $hH$ and $hL$ but a hypergeometric function $2F_1(hI, hI, 2hI, z)$ of $hI$), so extracting this dependence from the discretum of exchanged operators really requires the entire infinite tower, which in turn requires solving the large $c$ degenerate blocks in the $s \to \infty$ limit. Thus we are focusing entirely on the vacuum Virasoro block in this paper.}
As mentioned previously, the vacuum block in equation (2.2.9) is analytic in $h_L$ and $h_H$ [36, 12]. Therefore, when taking $h_L = h_{r,s}$, we must recover the null state vacuum block such as solution (2.2.6). Matching order by order in $\frac{1}{c}$ and $\frac{h_L}{c}$ to these solutions, we can determine $f_{mn}(\eta_H, z)$. Note that our knowledge of the block in the heavy-light semiclassical limit strongly constrains its behavior at large values of the external dimensions, so it seems very unlikely that there are any ambiguities in the analytic continuation from $h_L = h_{r,s}$.

The method can be generalized to study theories with supersymmetry. In particular, we work out heavy-light large $c$ limit of the holomorphic part of super-Virasoro vacuum blocks with $\mathcal{N} = 1, 2$ supersymmetries in section 2.5. It turns out that the super-Virasoro vacuum block of the lowest component fields in these theories do not get contributions from the fermionic supersymmetry generators at leading order of the large $c$ limit, so they largely match with results extrapolated from [16], but it is interesting to understand the supermultiplet structure and the correlators of superconformal descendant fields.

Although the method is straightforward, it becomes quite tedious beyond the first few orders in (2.2.9). When $r$ is large, it becomes a non-trivial task to construct the null state differential equation for $\phi_{1,r}$, which is a complicated $r$-th order differential equation whose exact solutions can be difficult to compute. But in specific limits of physical interest, these equations simplify greatly and become extremely useful in determining key properties of the higher order quantum corrections. One example of these are the leading log terms in the $\frac{1}{c}$ corrections when all four operators are light, which we discuss in section 2.4. Such terms play an important role in the growth of quantum chaos [47, 42, 44] and can be computed efficiently with the $\phi_{1,r}$ null state differential equations.

Another very useful way to get higher order corrections in the large $c$ limit is to use the Coulomb gas formalism, which provides a straight-forward construction of integral representations for the Virasoro blocks involving degenerate operators. We have used it in [12] in order to study the non-perturbative part of the vacuum Virasoro block in the large $c$ asymptotic expansion. In this work, we will show that directly expanding the integrand in
the Coulomb gas formalism provides an efficient way to obtain higher order terms in (2.2.9). This method is discussed in section 2.3.3.

### 2.3 Computing the $1/c$ Expansion of the Vacuum Block

In this section, we will use the computational method explained in last section to calculate the higher order corrections to the Virasoro block. The idea is to assume that the general heavy-light vacuum block $V$ can be written as the ansatz (2.2.9). When $O_L$ is a degenerate operator, $V$ satisfies a null-state differential equation. At order $\frac{1}{c^r}$, there are $p + 1$ functions $\{f_{0,p}, f_{1,p-1}, \ldots, f_{p,0}\}$. Each one appears with a different power of $h_L$ in its coefficient, i.e. $\log V \supset h_L^{p+1} f_{m,n}$. By (2.2.4), the degenerate operators $h_{1,s}$ with $r = 1$ have weights

$$h_{1,s} = \frac{1}{2} (1 - s) + \frac{1}{4b^2} (1 - s^2) \approx \frac{1}{2} (1 - s) + \frac{3}{2c} (1 - s^2) + O\left(\frac{1}{c^2}\right),$$

(2.3.1)

that are $O(1)$ at $c \to \infty$. For any choice of $s$, the operator $O_{1,s}$ produces a differential equation that we can solve for $V$ and expand at large $c$ to obtain the $O(c^{-p})$ term as

$$\log V \supset \frac{1}{c^p} \left(h_{1,s} f_{p,0} + h_{1,s}^2 f_{p-1,1} + \cdots + h_{1,s}^{p+1} f_{0,p}\right),$$

(2.3.2)

Unfortunately, for a single fixed $s$, $h_{1,s}$ is just a number and therefore knowledge of (2.3.2) does not allow one to separate out the different contributions $f_{mn}$. To accomplish this, one needs to take $p + 1$ different degenerate operators, which give $p + 1$ differential equations to be solved for these $p + 1$ $f_{mn}$ functions.

This is the procedure that we will implement in sections 2.3.1 and 2.3.2 in order to

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7More precisely, $h_{1,s}$ is not just a fixed number but rather a fixed function of $c$. However, because of the relation

$$h_{1,s} = h_{1,s}^{(0)} + \frac{1}{b^2} (h_{1,s}^{(0)} - (h_{1,s}^{(0)})^2),$$

(2.3.3)

where $h_{1,s}^{(0)} = \lim_{c \to \infty} h_{1,s}$, we are free to perform an expansion in powers of $h_{1,s}$ or in powers of $h_{1,s}^{(0)}$, since the difference between the two just corresponds to a redefinition of the $f_{m,n}$ functions.
obtain $1/c^p$ corrections up to $p = 3$, corresponding to 3-loop gravitational effects in AdS$_3$. In section 2.3.3 we will study a different method that uses the Coulomb gas formalism to replace differential equations with integrals.

A convenient and efficient formalism for keeping track of the null state differential equations at large $c$ was developed in [48, 45]. Let $D_{1,s}$ be the following matrix:

\[
D_{1,s} = -J_- + \sum_{m=0}^{\infty} \left( \frac{J_+}{b^2} \right)^m L_{-m-1},
\]

(2.3.4)

where $J_\pm$ are matrix generators of the spin $(s - 1)/2$ representation of $SU(2)$:

\[
(J_0)_{ij} = \frac{1}{2}(s - 2i + 1)\delta_{ij},
\]

\[
(J_-)_{ij} = \begin{cases} 
\delta_{i,j+1} & (j = 1, 2, \ldots, s - 1) \\
0 & \text{else} 
\end{cases},
\]

\[
[J_+, J_-] = 2J_0, \quad [J_0, J_\pm] = \pm J_\pm.
\]

(2.3.5)

\[
(J_+)_{ij} = \begin{cases} 
i(s - i)\delta_{i+1,j} & (i = 1, 2, \ldots, s - 1) \\
0 & \text{else} 
\end{cases}.
\]

Then, the null state equation of motion is given by the equation $f_0 = 0$ after eliminating $f_1, \ldots, f_{s-1}$ from the equation

\[
D_{1,s} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_s \end{pmatrix} = \begin{pmatrix} f_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

(2.3.6)

At infinite $c$ with $h_H$ held fixed and $O(1)$, one can manifestly drop all terms in the sum in $D_{1,s}$ except for $m = 0$, so the null state manifestly becomes

\[
L_{-1}^s O_{1,s} = 0,
\]

(2.3.7)
and the infinite $c$ differential equation for the conformal block becomes

$$\partial_s^2 \mathcal{V}(z) = 0, \quad (2.3.8)$$

where the factor of $z^{s-1}$ arises because our convention for $\mathcal{V}(z)$ factors out the $\langle \mathcal{O}_1 s(z) \mathcal{O}_1 s(0) \rangle$ two-point function. More generally, allowing $h_H$ to be $\mathcal{O}(c)$ with $h_H/c$ fixed, at infinite $c$ the differential equation for $\mathcal{V}(z)$ becomes [12, 48, 45]

$$\left[ \prod_{\begin{array}{c} k = - (s-1) + 2j \\

j = 0, \ldots, s-1 \end{array}} \left( \partial_t - \frac{k}{2} \sqrt{1 - 24 \eta_H} \right) \right] e^{\frac{z-1}{2} t} \mathcal{V}(t) = 0, \quad (2.3.9)$$

where $t = - \log(1 - z)$.

Let us see how this works in the simplest case, namely at lowest order in $1/c$ in (2.2.9). At order $c^0$, $\log \mathcal{V} \supset h_L f_{00}(\eta_H, z)$, there is only one unknown function $f_{00}$, which means that we only need the differential equation (2.3.9) with $s = 2$:

$$\left( \frac{d^2}{dt^2} - \frac{1 - 24 \eta_H}{4} \right) e^{\frac{z-1}{2} t} e^{h_{1,2} f_{00}(z(t))}, \quad (2.3.11)$$

with $h_{1,2} \simeq - \frac{1}{2}$. In terms of $z = 1 - e^{-t}$, we obtain

$$f_{00}'' = \frac{12 \eta_H}{(1 - z)^2} + \frac{1}{2} (f_{00}')^2. \quad (2.3.12)$$

And the solution that corresponds to the vacuum block is

$$f_{00}(\eta_H, z) = -(1 - 2\pi i T_H) \log(1 - z) - 2 \log \left( \frac{1 - (1 - z)^{2\pi i T_H}}{2\pi i T_H} \right). \quad (2.3.13)$$

\footnote{For later reference, the exact equation for the vacuum block for $\mathcal{O}_{1,2}$ is

$$\left( \partial_z^2 + \left( \frac{21 + b^2}{z} + \frac{b^2}{1 - z} \right) \partial_z + \frac{b^2 h_H}{(1 - z)^2} \right) \left( z^{2h_{1,2}} \mathcal{V}(h_{1,2}, \eta_H, z) \right) = 0. \quad (2.3.10)$$

with $b^2 = - \frac{3}{2(2h_{1,2} + 1)}$ in this equation.}

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with \( T_H = \frac{1}{2\pi} \sqrt{24\eta_H - 1} \). This reproduces the result for the heavy-light limit of the vacuum block first found in [15].

### 2.3.1 The Heavy-Light Virasoro Vacuum Block at Order 1/c

At order 1/c, there are two functions \( f_{01} \) and \( f_{10} \):

\[
\log V \supset \frac{h_L}{c} (f_{10} + h_Lf_{01})
\]

which means that we need to use both the \( h_{1,2} \) null-state equation (2.3.10) and the \( h_{1,3} \) null-state equation:

\[
0 = \left( \frac{1}{z^2} \frac{d^3}{dz^3} z^2 + \left( \frac{4 \frac{1}{b^2} - \frac{1}{z}}{z} \right) \frac{d^2}{dz^2} + \frac{4(\frac{1}{b^2} + \frac{2}{b^7})h_H(2 - z)}{(1 - z)^3 z} \right. \\
\left. + \left( \frac{2}{b^2}(9 - 13z + (5 + 2h_H)z^2) + \frac{1}{b^2}(3 + (z - 3)z) \right) \frac{d}{dz} \right) \left( z^{2h_{1,3}V(h_{1,3}, \eta, z)} \right)
\]

(2.3.14)

with \( b^2 = -\frac{2}{h_{1,3} + 1} \) in this equation and \( h_{1,3} \simeq -1 - \frac{12}{c} - \frac{156}{c^2} + O(1/c^3) \) in the large \( c \) limit. The \( c^0 \) order of equation (2.3.14) only involves \( f_{00} \) and the solution for it is exactly equation (2.3.12).

At order 1/c, equation (2.3.10) and (2.3.14) give the following two equations for \( f_{00}, f_{01} \) and \( f_{10} \):

\[
0 = F''_1 - f'_{00}F'_1 - 8f''_{00} - 5f''_{00} \frac{12(2z - 1)}{z(z - 1)} f'_{00} - \frac{12(z + 1)}{(z - 1)z^2}
\]

\[
0 = F''_2 - 3f'_{00}F''_2 + 3 \left( f''_{00} - f''_{00} + \frac{8\eta_H}{(z - 1)^2} \right) F'_2 + 12f''_{00} + \left( \frac{24 - 48z}{(z - 1)z} - 72f'_{00} \right) f''_{00} + 36f'''_{00} \frac{24(2z - 1)}{z(z - 1)} f''_{00} \frac{12((50\eta_H + 3)z^2 - z - 1)}{(z - 1)^2z^2} f'_{00} + \frac{24((25\eta_H + 1)z^3 - 2z + 1)}{(z - 1)^3z^3}
\]

(2.3.15)
where we define

\[ F_1 = 2f_{10} - f_{01}, \quad F_2 = f_{10} - f_{01}. \]

Note that the differential equations (2.3.15) involve the ‘zeroth order’ term \( f_{00} \), which also appears at higher orders, since \( \log V \supset h_{1,s}f_{00} = (\frac{1-s}{2} + \frac{1-s^2}{4c^2})f_{00} \). There are a few significant simplifications that occurred in the above equations. First, \( f_{10} \) and \( f_{01} \) show up only as a certain combination \((F_1 \text{ and } F_2)\) in each equation. The reason is that these equations come from the leading term in \( \frac{h_{1,s}f_{10}}{c} + \frac{h^2_{1,s}f_{01}}{c} \). Since \( h_{1,s} = \frac{1-s}{2} + \mathcal{O}(1/c) \), the leading term in \( \frac{h_{1,s}f_{10}}{c} + \frac{h^2_{1,s}f_{01}}{c} \) is

\[ \frac{1-s}{2}f_{10} + \left(\frac{1-s}{2}\right)^2f_{01}. \]

A similar phenomenon continues to be true for higher order calculations. This means that these differential equations can be solved independently for \( F_1 \) and \( F_2 \). Second, only the derivatives of \( F_1 \) and \( F_2 \) show up in these equations. This allows one to solve for the derivatives first, and then integrate. We have found this allows one to obtain a closed form expression for \( F_1(z) \) directly using Mathematica; on the other hand, the differential equation for \( F_2 \) is too complicated to be solved this way. Since the solutions are known from previous work [20] (see also [22, 49] for semi-classical results), one can substitute them into equations (2.3.15) and verify them. For completeness, these solutions are included in appendix A.1.

### 2.3.2 The All-Light Virasoro Vacuum Block at Order \( 1/c^2 \) and \( 1/c^3 \)

At order \( 1/c^2 \), there are three functions \( f_{20}, f_{11} \) and \( f_{02} \):

\[ \log V \supset \frac{h_L}{c^2}(f_{20} + h_Lf_{11} + h^2_Lf_{02}). \]

To fully determine them, one needs to solve the \( h_{1,2} \) and \( h_{1,3} \) null-state equations and also the \( h_{1,4} \) null state equation at order \( 1/c^2 \). These equations are complicated, but at least one can expand them in terms of \( \eta_H \equiv \frac{h c}{c} < 1 \) and obtain the result as an expansion in \( \eta_H \).
Define the expansion of $f_{mn}$ as

$$f_{mn} = \sum_{k=0}^{\infty} \eta_{n}^{k+1} f_{mnk} \quad \text{for } m \text{ or } n > 0 \quad (2.3.16)$$

$$f_{00} = -2 \log(z) + \sum_{k=0}^{\infty} \eta_{n}^{k+1} f_{00k}$$

where the $-2 \log(z)$ in $f_{00}$ is because we include the prefactor $z^{-2h_L}$ in the definition of the vacuum block. Since the vacuum block $V(h_L, h_H, z)$ is symmetric under the exchange $h_L \leftrightarrow h_H$, in our convention, this means that $f_{ijk} = f_{ikj}$.

The linear $\eta_H$ and $\eta_H^2$ terms at order $1/c^2$ are

$$\log V \supset \frac{h_L}{c^2} \left( \eta_H (f_{200} + h_L f_{110} + h_H^2 f_{020}) + \eta_H^2 (f_{201} + h_L f_{111} + h_H^2 f_{021}) \right).$$

At order $\eta_H^1$, using the symmetry under the exchange of $h_L$ and $h_H$, we have $f_{110} = f_{101}, f_{020} = f_{002}$, which can be calculated by expanding $f_{10} (A.1.1)$ and $f_{00} (2.3.13)$ in terms of $\eta_H$. So the only unknown at this order is $f_{200}$, which means that we only need to solve the $h_{1,2}$ null-state equation at this order to get this term. The result is

$$f_{200} = \frac{1728(z^2 - 1) \left( \zeta(3) - \text{Li}_3(1 - z) \right)}{z^2} + \frac{288 \text{Li}_2(z)(7(z - 2)z - 12(z - 1) \log(1 - z))}{z^2}$$

$$- \frac{1728(z - 2) \text{Li}_3(z)}{z} - \frac{144(z - 1) \log^2(1 - z)(6(z + 1) \log(z) - 7z + 7)}{z^2} + 1128$$

$$+ \frac{12(24\pi^2(z^2 - 1) + (z - 2)z) \log(1 - z)}{z^2} + \frac{288(z - 2)(z - 1)^2 \log^3(1 - z)}{z^3}. \quad (2.3.17)$$

We have also checked that these results do satisfy the $h_{1,3}$ and $h_{1,4}$ null-state equations.

At order $\eta_H^2$, only $f_{021} = f_{012}$ can be determined by expanding the result we already have (that is, $f_{01}$), and we need to solve the $h_{1,2}$ and $h_{1,3}$ null-state equations at this order to get $f_{201}$ and $f_{111}$. These results are complicated and given in appendix A.1.

Using the symmetry $f_{ijk} = f_{ikj}$, we can also determine the linear $\eta_H$ terms at order $1/c^3$
by just using the the $h_{1,2}$ null-state equation. At this order,

$$\log V \supset \frac{h_L \eta_H}{c^3} (f_{300} + h_L f_{210} + h_L^2 f_{120} + h_L^3 f_{030}).$$

Since $f_{210} = f_{201}$, $f_{120} = f_{102}$ and $f_{030} = f_{003}$, only $f_{300}$ cannot be obtained by expanding the results we already have, that’s why we only need the $h_{1,2}$ null-state equation. These results are also given in appendix A.1.

### 2.3.3 Integral Formulas from the Coulomb Gas

As we mentioned in the previous sections, computation of $f_{mn}$ at higher orders becomes extremely technically challenging, because upon the substitution $h_L \to h_{r,s}$ one needs to solve a differential constraint equation of order $rs$. However, an integral representation of the solutions to constraint equations such as (2.2.5) are known, thanks to the Coulomb gas formalism [50, 51, 45]. This method makes it possible to write down explicit expressions for all $f_{mn}$ in terms of multiple elementary integrals.

Explicitly, the vacuum block component of $\langle O_{1,s}(0)O_{1,s}(z)O_H(1)O_H(\infty) \rangle$, where $O_{1,s}$ is a light degenerate operator, is given by the following integral representation:

$$\tilde{V}_{1,s}(z) = N_{1,s} \left( \prod_{i=1}^{s-1} \int_0^1 dw_i \right) (1 - z)^{(s-1)\beta_H} e^{\mathcal{I}_{1,s}}, \quad (2.3.18)$$

where the action $\mathcal{I}_{1,s}$ is

$$\mathcal{I}_{1,s} = \sum_{i=1}^{s-1} \left\{ \frac{s-1}{\beta^2} \log \left[ w_i(1 - w_i) \right] - 2\beta_H \log(1 - zw_i) \right\} \frac{2}{\beta^2} \sum_{1 \leq i < j \leq s-1} \log(w_i - w_j). \quad (2.3.19)$$

with $\beta_H$ given by (2.2.7). We have also introduced a normalization factor $N_{1,s}$ such that
\( \tilde{V}_{1,s}(0) = 1 \). Notice that \( N_{1,s} \) is independent of \( h_H \). Perturbatively in \( b \), it is given by

\[
N_{1,s}(b) = 1 + \frac{4(s - 1)^2 - 3(s - 1)(s - 2)}{2b^2} + \mathcal{O}(b^{-4}) .
\] (2.3.20)

In the limit \( b \to \infty \) with fixed \( \beta_H \), we can expand the integrand of (2.3.18) in \( 1/b \):

\[
\tilde{V}_{1,s}(z) = N_{1,s}(b)(1 - z)^{(s-1)\beta_H} \int_0^1 \left( \prod_{i=1}^{s-1} dw_i \right) (1 - zw_i)^{-2\beta_H}
\]

\[
\times \sum_{k=0}^{\infty} \frac{1}{k!b^{2k}} \left( \sum_{i=1}^{s-1} (s - 1)K_i - \sum_{1 \leq i < j \leq s-1} 2U_{ij} \right)^k .
\] (2.3.21)

To lighten the notation, we denote

\[
K_i = \log(w_i(1 - w_i)) , \quad U_{ij} = \log|w_i - w_j| .
\] (2.3.22)

In the rest of this section, we will show how to extract various \( f_{mn} \) from the integral (2.3.21). The general strategy is very simple. Recall that we postulated the ansatz of the vacuum block to be

\[
\tilde{V}_{h_{H},h_{L},0,c}(z) = z^{2h_L} \exp \left[ h_L \sum_{n,m=0}^{\infty} \left( \frac{1}{c} \right)^m \left( \frac{h_L}{c} \right)^n f_{mn}(\eta_H,z) \right] .
\] (2.3.23)

When we set \( h_L = h_{1,s} = \frac{1-s}{2} + \frac{1-s^2}{4b^2} \) in the above ansatz and compare it with (2.3.21), we can read off the \( f_{mn} \) functions.

### 2.3.3.1 Leading Order at Large \( c \)

Let us begin by computing the well-known \( c = \infty \) heavy-light vacuum block as a warm-up. Upon substitution \( h_L \to h_{1,s} \), Eq. (2.3.23) in leading order in \( b \) is simply \( z^{2h_L} \exp(\frac{1-s}{2} f_{00}) \).
Denoting \( X_{1,s} \) the \((s-1)\) dimensional integral in (2.3.21), the comparison implies

\[
z^{2\eta} e^{\frac{1-\xi}{2}} f_{00} = N_{1,s}^{(0)} X_{1,s}^{(0)} = \prod_{i=1}^{s-1} \left[ (1-z)^{\beta_H} \int_{0}^{1} dw_i (1-zw_i)^{-2\beta_H} \right], \quad (2.3.24)
\]

where the superscript denotes the powers in \( \frac{1}{\beta} \), e.g. \( X_{1,s} = X_{1,s}^{(0)} + \frac{1}{\beta^2} X_{1,s}^{(1)} + \ldots \). From the above equation one immediately obtains that

\[
f_{00} = -2 \log \left( \frac{(1-z)^{\beta_H} - (1-z)^{1-\beta_H}}{(1-2\beta_H)} \right). \quad (2.3.25)
\]

Noting that in large \( b \) limit \( \beta_H \to \frac{1-\sqrt{1-24\eta}}{2} + \mathcal{O}(b^{-2}) \), we recognize the above equation in agreement with (2.3.13).

**2.3.3.2 Expansion at Order 1/c**

Now we arrive at the sub-leading order in \( c \). They are two functions, \( f_{10} \) and \( f_{01} \), to be determined at this order. The comparison of (2.2.9) with (2.3.21) yields

\[
\frac{1-s^2}{4} (f_{00} + 2 \log z) + \frac{1-s}{2} \left( \frac{f_{10}}{6} + \frac{f_{01}}{6} \frac{1-s}{2} \right) = \frac{N_{1,s}^{(1)} X_{1,s}^{(0)}}{N_{1,s}^{(0)} X_{1,s}^{(1)}}. \quad (2.3.26)
\]

In the above equation, \( N_{1,s}^{(0)} \) and \( N_{1,s}^{(1)} \) on the RHS are obtained from (2.3.20), while \( X_{1,2}^{(0)} \) and \( X_{1,2}^{(1)} \) are represented by elementary integrals. Staring at (2.3.21), one finds that

\[
X_{1,s}^{(0)} = \left( \int_{w} 1 \right)^{s-1}, \quad X_{1,s}^{(1)} = X_{1,s}^{(0)} \left( (s-1)^2 \frac{f_{w}}{f_{w_1}} \frac{K_1}{1} - (s-1)(s-2) \frac{f_{w_1} f_{w_2} U_{12}}{[f_{w_1}]^2} \right). \quad (2.3.27)
\]

Here we have used the abbreviation

\[
\int_{w_i} f(w_1, \ldots, w_n) \equiv (1-z)^{\beta_H} \int_{0}^{1} dw_i (1-zw_i)^{-2\beta_H} f(w_1, \ldots, w_n). \quad (2.3.28)
\]
Combining all these pieces of information, one can easily solve for \( f_{10} \) and \( f_{01} \):

\[
f_{10} = -18 - 6(f_{00} + 2 \log z) - 12 \frac{\int_{w_1} w_2 U_{12}}{[\int_{w_1} 1]^2},
\]

\[
f_{01} = 12 + 6(f_{00} + 2 \log z) + 24 \frac{\int_{w_1} K_1}{\int_{w_1} 1} - 24 \frac{\int_{w_1} w_2 U_{12}}{[\int_{w_1} 1]^2}, \tag{2.3.29}
\]

where \( f_{00} \) is given by (2.3.25). Now what remains to be computed are the two integrals in the expressions above. After some cumbersome but straightforward algebra, one has

\[
\frac{\int_{w_1} K_1}{\int_{w_1} 1} = \int_0^1 dw (1-zw)^{-2\beta_H} \log |w(1-w)| \int_0^1 dw (1-zw)^{-2\beta_H}
\]

\[
\left( -\alpha H_{-\alpha} + \alpha \log \left(-\frac{1}{z}\right) - 2 - (1-z)^\alpha \left( 2 + \alpha \left( \psi^{(0)}(\alpha) + \log \left( \frac{z}{z-1} \right) + \gamma \right) \right) \right)
\]

\[
+ \frac{2F_1 \left( 1, \alpha + 1; \frac{1}{1-z} \right) \frac{\alpha}{1-z} (1-z)^{\alpha}}{2} \frac{2F_1 \left( 1, 2 - \alpha; \frac{1}{1-z} \right)}{(1 - (1-z)^{\alpha})^{-1}},
\]

\[
\frac{\int_{w_1} w_2 U_{12}}{[\int_{w_1} 1]^2} = \int_0^1 dw_1 \int_0^1 dw_2 [(1-zw_1)(1-zw_2)]^{-2\beta_H} \log |w_1 - w_2| \int_0^1 dw (1-zw)^{-2\beta_H}
\]

\[
\left( \frac{\int_0^1 dw (1-zw)^{-2\beta_H}}{2} \right)^2
\]

\[
= \left[ \frac{i\pi \alpha + 8(1-z)^\alpha - 2\alpha \log(z) + (1-z)^{2\alpha} \left( i\pi \alpha - 2\alpha \log\left( \frac{z}{z-1} \right) - 1 \right) - 1}{2\alpha} \right]
\]

\[
+ (1-z)^{2\alpha} \left( B(1-z, -\alpha, 0) - 3B \left( \frac{1}{1-z}, \alpha, 0 \right) \right) + B \left( \frac{1}{1-z}, -\alpha, 0 \right) - 3B(1-z, \alpha, 0)
\]

\[
+ \frac{\pi \cot(\pi \alpha) - 2H_{\alpha + (1-z)^{2\alpha} \left( \pi \cot(\pi \alpha) - 2H_{\alpha} \right) \right]}{2} (1 - (1-z)^{\alpha})^{-2},
\]

where \( B(x, \beta, 0) = \frac{x^{\beta} F_1(1, \beta, 1+\beta, x)}{\beta} \) is the incomplete Beta function, \( H_{\alpha} \) is the harmonic function, \( \gamma \) is the Euler gamma constant, \( \psi(x) = \Gamma'(x) / \Gamma(x) \) is the digamma function and the parameter \( \alpha \) is related to the Hawking temperature by \( \alpha \equiv \sqrt{1 - 24\eta_H} = 2\pi iT_H \). Having (2.3.30) and (2.3.31) plugged into the expression of \( f_{10} \) and \( f_{01} \) (2.3.29), it is straightforward to show that they match the results obtained in [20], which are also given in appendix A.1.

One can easily continue this procedure to higher orders in the \( 1/c \) expansion, but for
brevity we spare the reader the details, since the lengthy $1/c^2$ and $1/c^3$ results have already been given in section 2.3.2 and in appendix A.1.

### 2.4 All-Orders Resummations in the Lorentzian Regime

Our main focus in this section is to understand how the large $c$ vacuum Virasoro block behaves in the Lorentzian regime. More specifically, we are interested in the behavior of the block after the argument $z$ is analytically continued across the branch cut emanating from $z = 1$ and then taken to small values of $|z|$ on the second sheet. The behavior of CFT correlators in this regime has interesting implications for causality [43, 52, 53, 54] and a fascinating interpretation in terms of chaos [55, 42, 56, 47, 7, 44, 57].

In this section we will show that quantum corrections to the Lyapunov exponent resum to all orders, and that one can also resum the full $1/cz$ expansion in order to obtain an interpolation between the early onset of chaos and late time effects associated with thermalization. These are the Lyapunov and Ruelle regions of figure 2.1. We refer the reader to [44] for a pertinent review of chaotic correlators and Lyapunov exponent bounds in the context of CFT$_2$ at large central charge.

#### 2.4.1 Resummation of $\frac{1}{c} \log z$ Effects

Consider the Virasoro vacuum block in a large $c$ expansion with external dimensions fixed. In a $1/c$ expansion, the leading correction near $z \sim 0$ after analytically continuing around the branch cut emanating from $z = 1$ is of the form

$$F(z) \approx 1 - \frac{48i\pi h_L h_R}{cz} + \ldots \quad (2.4.1)$$

The first term comes from the vacuum itself, while the second term is due entirely to the exchange of a single quasi-primary stress tensor or ‘graviton’ state along with its global conformal descendants. The quantity $F$ is the contribution of the vacuum block to the
Figure 2.1: Plot of the behavior of $1 - F(t)$ as a function of time $t$ in the limit $c \to \infty$ with $cz$ fixed, with $h_L = h_H = \frac{1}{2}$. $F(t)$ is absolute value of the out-of-order correlator $\langle O_L O_H O_L O_H \rangle_\beta / \langle O_L O_L \rangle \langle O_H O_H \rangle$, and $t \equiv -\log(cz/6)$. The initial “Lyapunov” growth and the later “Ruelle” decay are labeled as in [7]. We have plotted only the contribution of an approximation to the vacuum Virasoro block, but the result has the qualitative features expected of the full correlator.

out of time order correlator $\langle O_H O_L O_H O_L \rangle_\beta$ in a thermal background, normalized by the $\langle O_H O_H \rangle \langle O_L O_L \rangle$; it is plotted in figure 2.1.

As $z$ decreases towards 0, the $1/c$ correction grows like $z^{-1}$ and becomes increasingly important. Similarly, higher order terms in $1/c$ can become important at sufficiently small $z$ as well. In this subsection we will show how to resum one set of contributions that grow large at small $z$, namely the terms that are leading logs in the $1/c$ expansion. That is, we will see that terms of the form $(1/zc)(\log(z)/c)^n$ appear exactly in the combination

$$
\frac{A}{cz^{1+\gamma/c}} = \frac{A}{zc} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{-\gamma \log(z)}{c} \right)^n,
$$

with constants $A = -48i\pi h_L h_H$, $\gamma = 12$. We provide another derivation of this resummation in appendix A.2. We also checked the coefficients of these terms by analytically continuing the $f_{m00}$ functions given in appendix A.1 directly to the second sheet up to and including $1/c^3$ corrections. These effects provide a quantum correction [44] to the Lyapunov exponents that characterize the early onset of chaos.

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For degenerate external operators, there is a particularly transparent way of understanding this logarithmic resummation, because only a finite number of Virasoro blocks appear in any channel. The crucial point is that passing through the branch cut in $z$ simply reshuffles one linear combination of blocks into a different linear combination.\(^9\) In other words, on the second (Lorentzian) sheet, the vacuum block is equal to a sum of degenerate blocks evaluated on the first (or Euclidean) sheet.

For the degenerate operator $O_{1,s}$, the operators in the $O_{1,s} \times O_{1,s}$ OPE are degenerate operators $O_{1,p}$ with $p = 1, 3, \ldots, 2s - 1$. These have dimension

$$h_{1,p} = -\frac{1}{2}(p - 1) \left(1 + \frac{1}{2}b^{-2}(p + 1)\right).$$

(2.4.3)

where we recall $c \approx 6b^2 \gg 1$. So for a given value of $s$, analytic continuation of $z$ around 1 transforms the vacuum block into a linear combination of terms of the form

$$\tilde{V}_{(1,s)}(z) \sim \sum_{q=0}^{s-1} c_q(h_{1,s}, h_H) \frac{1}{b^{2q}} \frac{1}{z^{q(1+\frac{1}{2})}} f_q(h_{1,s}, h_H, z),$$

(2.4.4)

where $q \equiv \frac{(p-1)}{2}$ and the $f_q(z) \sim 1 + O(z)$ parts of the blocks have a regular series expansion around $z \sim 0$. In the above, $c_q$ and $f_q$ are functions of $b$ as well but we have factored out explicit powers of $b^{-2}$ so that they have a finite limit at $b \to \infty$. The reason this prefactor of $b^{-2q}$ must be present is that $c_q$ vanishes up to $O(b^{-2q+2})$, by the following argument. If we expand at large $b$, we know that the $b^{-2q+2}$ term is a $(q-1)$-th order polynomial in $h_{1,s}$, and therefore given by $q$ coefficients.\(^{10}\) These coefficient can be fixed by looking at the OPE of the $q$ degenerate operators $\{O_{1,s}\}_{1 \leq s \leq q}$. From the above description of the $O_{1,s} \times O_{1,s}$ OPE, we know that none of the operators $\{O_{1,s}\}_{1 \leq s \leq q}$ contains the $O_{1,2q+1}$ operator, therefore this

\(^9\)One way of understanding this is that crossing symmetry $z \to 1 - z$ acts as a linear operator that changes blocks in one channel into blocks in the other channel. In the other channel, taking $z$ around 1 acts on each block by simply introducing a phase $(1 - z)^{h_I} \to e^{2\pi i h_I} (1 - z)^{h_I}$ given by the weight $h_I$ of the corresponding primary operator. Transforming back to the original channel again acts with the inverse of the first linear operator, producing a linear combination of blocks in the original channel.

\(^{10}\)These coefficients are functions of $z$ and $h_H$. 

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operator does not appear at $O(b^{-2q+2})$ or lower. But, $c_q$ is just the OPE coefficient for the $O_{1,2q+1}$ operator; therefore the lowest order where it appears is $b^{-2q}$.

Now, to see explicitly the behavior of leading logs, sub-leading logs, sub-sub-leading logs, etc, we can simply expand in large $b$ and look for terms of order $(b^{-2}\log(z))^n$, $b^{-2}(b^{-2}\log(z))^n$, etc. Logarithms manifestly arise only from expanding an exponent $n$ in the above expression, so any term of the form

$$(b^{-2})^m(b^{-2}\log(z))^n$$

must come from expanding an exponent $n$ times after expanding the prefactor up to $m$-th order. There are manifestly no terms with $m = 0$. Terms with $m = 1$ must clearly come from the first term, $q = 1$, and are the leading logs. Consequently, we immediately see that all these leading logs arise from the expansion of the term

$$\frac{c_1(h_{1,s},h_H)f_1(h_{1,s},h_H,z)}{b^2 z^{1+2b^{-2}}}$$

and thus manifestly just resum back to this form. This result holds for all values of $s$. Since we expect that the vacuum block $\mathcal{V}$ is analytic in $h_L$, and because this result obtains for $h_L = h_{1,s}$ for all $s$, we expect that it also holds if we analytically continue to general $h_L$. The correction to the power-law in the denominator is

$$1 \rightarrow 1 + 2b^{-2} = 1 + \frac{12}{c} + O(c^{-2}),$$

proving equation (2.4.2) with $\gamma = 12$. This also provides a quick alternative check of the magnitude and sign of the correction to this (Lyapunov) exponent.

The above considerations also make it easy to understand the effect of sub-leading logs, sub-sub-leading logs, etc. For instance terms with $m = 2$ must come from either the first...
term or the second term in (2.4.4), and therefore are of the form

$$z^{2h_1,s} \langle O_H(\infty)O_H(1)O_{1,s}(z)O_{1,s}(0) \rangle \supset$$

$$\sum_{n=0}^{\infty} (b-2)^2 \left( \frac{[c_1 f_1(z)]_{O(b-2)} (-2b^{-2} \log(z))^n}{zn!} + \frac{[c_2 f_2(z)]_{O(b^0)} (-6b^{-2} \log(z))^n}{z^2 n!} \right).$$

(2.4.8)

It is easy to expand in large $b$ to obtain similar higher order results.

### 2.4.2 Resumming Leading Singularities in $\frac{1}{cz}$

Resumming the leading logarithms tells us something about the functional form of the large $c$ expansion, but because of the power-law singularities $\sim (cz)^{-n}$, the leading logs never dominate the behavior of the blocks. In this subsection, we will derive and resum the leading $(cz)^{-n}$ singularities, which do give the dominant behavior at small $z$ in the limit $c \to \infty$ with $cz$ fixed.

![Figure 2.2: Plots comparing the exact behavior from eq. (2.4.14) (black, dashed) for $1-F(t)$ in the limit $c \to \infty$ with $cz$ fixed, to the heavy-light approximation (2.4.10) (red, solid). Left: $h_L = h_H = \frac{1}{2}$, Right: $h_L = \frac{3}{2}, h_H = \frac{3}{10}$. $F(t)$ and $t$ are as in figure 2.1. Note that both curves only include contributions from the vacuum block, neglecting double-trace operators which could affect an AdS$_3$ calculation.](image)

The arguments in the previous subsection already provide a significant amount of information on the coefficients of these singularities in equation (2.4.4): they are polynomials
in $h_L$ and $h_H$ of order $n$, they have to vanish when $h_L$ is a degenerate operator $h_{1,s}$ with $s \leq n$, and they have to be symmetric in $h_L \leftrightarrow h_H$. This in fact completely determines the coefficients $c_q(h_L, h_H)$ of equation (2.4.4) up to an $h_H, h_L$-independent prefactor:

$$c_q(h_L, h_H) = a_q(2h_L)_q(2h_H)_q, \quad (2.4.9)$$

where $a_q$ depends only on $q$ and not on $h_H$ or $h_L$. To obtain its value, we just need to calculate it for some chosen $h_H$, in the limit $c \to \infty$. A convenient choice is $h_H = \eta_H c$ fixed, followed by $\eta_H$ small, since in that case we know from the form of the heavy-light blocks in the $c \to \infty$ limit that, on the second sheet [42, 44], the vacuum block is [15]

$$z^{2h_L}V(z) \approx \left(1 - \frac{1}{1 - \frac{24i\pi h_H}{cz}}\right)^{2h_L}. \quad (2.4.10)$$

Series expanding in $1/c$, we can read off the $c_q$ coefficients in this limit and determine the prefactor $a_q$, with the result\(^\text{11}\)

$$c_q(h_L, h_H) = \frac{(2i\pi)^q(2h_H)_q(2h_L)_q}{q!}. \quad (2.4.11)$$

Substituting these coefficients back into the sum over singular terms

$$\sum_{q=0}^{\infty} \frac{c_q(h_L, h_H)}{b^{2q}z^q}, \quad (2.4.12)$$

we see that the sum on $q$ is an asymptotic series, ie it has zero radius of convergence. One can nevertheless Borel resum it:

$$B(t) = \sum_{q=0}^{\infty} \frac{c_q t^q}{q!} = {}_2F_1(2h_L, 2h_H, 1, 2i\pi t). \quad (2.4.13)$$

\(^{11}\)Note that since the approximation (2.4.10) retains some of the $h_H$-dependence and all of the $h_L$-dependence of the coefficients $c_q$ in its $1/c$ series expansion, this also provides a non-trivial consistency check of equation (2.4.9).
Performing the Borel integral $\int_0^\infty e^{-t}B(\frac{t}{cz})dt$, we obtain a relatively compact expression for the resummation of the leading singular terms:

$$\lim_{cz \to \infty} (z^{2h_L}) V(z) = G\left(h_H, h_L, \frac{icz}{12\pi}\right) + G\left(h_L, h_H, \frac{icz}{12\pi}\right) \tag{2.4.14}$$

where

$$G(h_1, h_2, x) \equiv (x)^{2h_1}(2h_2-2h_1) \, _1F_1(2h_1, 1+2h_1-2h_2, x). \tag{2.4.15}$$

This might be compared with the integral formulas from [56] derived from AdS physics. As one might expect, we see that the singular terms all resum into something that shuts down at $z \sim 0$. The two terms above decay like $z^{2h_H}$ and $z^{2h_L}$, respectively. Suggestively, these exponents would naively correspond to the contributions from a $\mathcal{O}_H\mathcal{O}_H$ double-trace operator and a $\mathcal{O}_L\mathcal{O}_L$ double-trace operator. This is closely related to the fact that if one takes the expression for the vacuum block in the heavy-light limit

$$V \propto \left(\frac{\pi T_H}{\sin^2(\pi T_H(t+i\phi))}\right)^{2h_L} \left(\frac{\pi \tilde{T}_H}{\sin^2(\pi \tilde{T}_H(t-i\phi))}\right)^{2h_L} \tag{2.4.16}$$

and promotes it to a periodic function of $\phi$ (which the full correlator must be) by adding all its images under $\phi \to \phi + 2\pi n$, then this generates additional contributions in the conformal block decomposition that behave like double-trace operators in the $\mathcal{O}_L\mathcal{O}_L$ OPE. It is interesting that, unlike the global conformal blocks, the Virasoro conformal blocks thereby “know” about double-trace operator contributions in the same channel $\mathcal{O}_L\mathcal{O}_L \to \mathcal{O}_H\mathcal{O}_H$ as the vacuum.

Adopting the nomenclature of [7], the above expression interpolates between the “Lyapunov” regime, where $c$ is large with $cz$ fixed and large, and the “Ruelle” regime, where $c$ is
large with $cz$ fixed and small. For $h_H = h_L$, the expression simplifies somewhat:

$$\lim_{h_H \to h_L} G(h_L, h_H, x) + G(h_H, h_L, x) = x^{2h_L} U(2h_L, 1, x). \quad (2.4.17)$$

where $U(a, b, x)$ is a confluent hypergeometric function.\textsuperscript{12} It is particularly simple at $h_L = 1/2$, since $U(1, 1, x) = e^x \Gamma(0, x)$. In figure 2.1, we have plotted the resulting behavior for the correlator (only including the vacuum block contributions) interpolating between the Lyapunov and Ruelle regime for $h_L = h_H = \frac{1}{2}$. In figure 2.2, we compare the behavior of the vacuum block with that of the approximate formula (2.4.10) from the heavy-light limit. Although all of these plots only include vacuum block contributions, they seem to agree with qualitative expectations for the behavior of the full correlator.

We make one final comment on the relation of this result to the heavy-light limit. One open question has been whether or not taking the heavy-light limit, then analytically continuing around $z \sim 1$, and finally taking $c$ large with $h_L, h_H$, and $cz$ fixed is the same as simply analytically continuing the exact Virasoro block and then taking the limit $c$ large with $h_L, h_H$, and $cz$ fixed. So far, all indications are that these different orders of limits do commute for the $O(1/c)$ singular term (2.4.1), which was the main interest of [42], but in the above we see explicitly that they do not commute for most other terms. In particular, taking the heavy-light limit followed by small $h_H$ completely discards the contribution in (2.4.15) that decays like $z^{2h_H}$, since by inspection we see that (2.4.10) contains only the $\sim (cz)^{2h_L}$ piece at small $cz$. This is perhaps not so surprising, since the full result has to be symmetric under $h_L \leftrightarrow h_H$, but taking the heavy-light limit breaks this symmetry and makes the $O(z^{2h_H})$ contributions become formally non-perturbative $\sim e^{2h_H \eta \log(z)}$. By contrast, by working out the exact coefficient of the leading singularities, we have kept the $h_H \leftrightarrow h_L$ symmetry at all stages of the computation.

\textsuperscript{12}For $b \not\in \mathbb{Z}$,

$$U(a, b, x) = \frac{\Gamma(b - 1)}{\Gamma(a)} z^{1-b} _1 F_1 (a-b+1, 2-b, x) + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} _1 F_1 (a, b, x) \quad (2.4.18)$$
2.5 Heavy-Light Super-Virasoro Vacuum Blocks at Large $c$

Similar to the case of non-supersymmetric CFTs that we have been discussing so far, in two-dimensional superconformal theories (SCFTs) there are degenerate operators whose correlators satisfy super null-state differential equations. In this section, we will use these super null-state equations to calculate the large $c$ heavy-light super-Virasoro vacuum block for these degenerate operators, and then analytically continue the result to obtain the super-Virasoro vacuum block for operators with general conformal dimensions. Specifically, we will focus on the holomorphic part of the Neveu-Schwarz (NS) sector of 2d $\mathcal{N} = 1$ [58, 59, 60, 61, 62, 63] and $\mathcal{N} = 2$ [64, 65, 66, 67, 68, 69] SCFTs (see e.g. [70] for a review of these theories). Previously, the $\mathcal{N} = 1$ super-Virasoro blocks in NS sector have been studied using recursion relations [71, 72, 73], while those of $\mathcal{N} = 2$ are less investigated [74, 33].

2.5.1 The $\mathcal{N} = 1$ Super-Virasoro Vacuum Block

2.5.1.1 Brief review of 2d $\mathcal{N} = 1$ SCFTs

In the $\mathcal{N} = 1$ super-space, a point is denoted by $Z \equiv (z, \theta)$, where $\theta$ is a Grassmann variable. A primary superfield $\Phi_h(Z)$ of conformal dimension $h$ can be expanded in terms of $\theta$ as $\Phi_h(Z) = \phi_h(z) + \theta \psi_{h+\frac{1}{2}}(z)$, where $\phi_h(z)$ and $\psi_{h+\frac{1}{2}}(z)$ are two component fields with conformal dimension $h$ and $h + \frac{1}{2}$, respectively. In the NS sector, the energy-momentum superfield $\mathcal{T}(Z)$, which has conformal dimension $3/2$, can be expanded around the origin as

$$\mathcal{T}(Z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{1}{2^{r}} \frac{1}{z^{r+3/2}} G_r + \theta \sum_{n \in \mathbb{Z}} \frac{1}{z^{n+2}} L_n,$$

where the fermionic generators $G_r$ are the supersymmetry generators and the bosonic generators $L_n$ are Virasoro generators. The (anti-)commutation relations between these generators
are:
\[ [L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}, \]
\[ \{G_r, G_s\} = 2L_{r+s} + \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \delta_{r+s,0}, \]
\[ [L_n, G_r] = \left( \frac{n}{2} - r \right) G_{n+r}, \quad m, n \in \mathbb{Z}; \quad r, s \in \mathbb{Z} + \frac{1}{2}. \]

(2.5.2)

The singular terms in the OPE of \( T(Z_1) \) and \( \Phi(Z_2) \) are

\[ \mathcal{T}(Z_1) \Phi(Z_2) \sim \frac{h \theta_{12}}{Z_{12}^2} \Phi(Z_2) + \frac{1}{2Z_{12}} D_2 \Phi(Z_2) + \frac{\theta_{12}}{Z_{12}} \partial_2 \Phi(Z_2), \]

where \( Z_{ij} = z_{ij} - \theta_i \theta_j, z_{ij} = z_i - z_j \) and \( D_i = \partial_{\theta_i} + \theta_i \partial_{z_i} \). Descendant superfields are obtained by acting on a primary with \( L_{-n} \) and \( G_{-r} \) for \( n, r > 0 \). From the above OPE, one can derive that correlation functions with one descendant superfield can be written in terms of a differential operator acting on the correlation functions with only primary superfields [63] via

\[ \langle (L_{-n} \Phi_1)(Z_1)X \rangle = \mathcal{L}_{-n} \langle \langle \Phi_1 \rangle(Z_1)X \rangle, \quad \langle (G_{-r} \Phi_1)(Z_1)X \rangle = \mathcal{G}_{-r} \langle \langle \Phi_1 \rangle(Z_1)X \rangle, \quad (2.5.3) \]

where \( X = \Phi_2(Z_2) \cdots \Phi_N(Z_N) \) is an assembly of primary superfields, and \( \Phi_i \) has conformal dimension \( h_i \). These two super-differential operators are

\[ \mathcal{L}_{-n} = -\sum_{i=2}^{N} Z_{i1}^{-n} [(1 - n)(h_i + \frac{1}{2} \theta_i D_i) + Z_{i1} \partial_{z_i}] \langle \Phi_1(Z_1)X \rangle, \]
\[ \mathcal{G}_{-r} = -\sum_{i=2}^{N} Z_{i1}^{-(r+\frac{1}{2})} [(2r - 1)h_i \theta_i + Z_{i1}(D_i - 2\theta_i \partial_{z_i})] \langle \Phi_1(Z_1)X \rangle. \]

(2.5.4)

\( N \)-point functions of the superfields \( F_N \equiv \langle \Phi_1(Z_1)\Phi_2(Z_2) \cdots \Phi_N(Z_N) \rangle \) should be invariant under the global superconformal transformations generated by \( L_{\pm 1}, L_0, G_{\pm \frac{1}{2}} \), which leads to

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the superconformal Ward identities [62]:
\[
L_{-1} : \sum_{i=1}^{N} \partial_{z_i} F_N = 0, \quad G_{-\frac{1}{2}}, G_{\frac{1}{2}} : \sum_{i=1}^{N} (\partial_{\theta_i} - \theta_i \partial_{z_i}) F_N = \sum_{i=1}^{N} (2h_i \theta_i + z_i (\theta_i \partial_{z_i} - \partial_{\theta_i})) F_N = 0, \\
L_0 : \sum_{i=1}^{N} (2z_i \partial_{z_i} + 2h_i + \theta_i \partial_{\theta_i}) F_N = 0, \quad L_1 : \sum_{i=1}^{N} (z_i^2 \partial_{z_i} + z_i (2h_i + \theta_i \partial_{\theta_i})) F_N = 0.
\]

Due to these constraints, the two-point function is fixed (up to normalization) to be
\[
\langle \Phi_1 (Z_1) \Phi_2 (Z_2) \rangle = \frac{1}{Z_{21}^{2h_1}} \delta_{h_1, h_2} = \left( \frac{1}{z_{21}} + \theta_1 \theta_2 \frac{2h_1}{z_{21} h_1 + 1} \right) \delta_{h_1, h_2}.
\]

where each term on the RHS corresponds to a two-point function of the component fields.

2.5.1.2 \( \mathcal{N} = 1 \) Super-Virasoro Vacuum Blocks at Leading Order

The heavy-light super-Virasoro vacuum block \( \mathcal{V}_{\Phi_L \Phi_L \Phi_H \Phi_H} \) is the contribution to the heavy-light four-point function \( \langle \Phi_L (Z_1) \Phi_L (Z_2) \Phi_H (Z_3) \Phi_H (Z_4) \rangle \) from an irreducible representation of the superconformal algebra whose highest weight state is the vacuum \( |0 \rangle \). In the following calculation, we will take the heavy-light limit, meaning that
\[
\eta_H \equiv \frac{h_H}{c}, h_L \text{ fixed as } c \to \infty.
\]

Our result of this part is \( \mathcal{V}_{\Phi_L \Phi_L \Phi_H \Phi_H} \) given in (2.5.8) with \( f_{h_L} \) and \( g_{h_L} \) given in (2.5.17) and (2.5.19).

As the four-point function, the super-Virasoro vacuum block \( \mathcal{V}_{\Phi_L \Phi_L \Phi_H \Phi_H} \) also satisfies the superconformal Ward identities. There are eight coordinate variables (four Grassmann even and four Grassmann odd) in \( \mathcal{V}_{\Phi_L \Phi_L \Phi_H \Phi_H} \) and it satisfies five global superconformal Ward identities, which means that there are only three independent superconformal invariants, two Grassmann even and one Grassmann odd. The two Grassmann even invariants that we
choose are \[59\]
\[
x_0 \equiv \frac{Z_{12}Z_{34}}{Z_{13}Z_{24}}, \quad x_1 \equiv \frac{Z_{14}Z_{23}}{Z_{13}Z_{24}} - (1 - x_0).
\] (2.5.7)

It is easy to verify that \(x_1^2 = 0\) and superformal Ward identities fix \(\mathcal{V}_{\Phi_L\Phi_L\Phi_H\Phi_H}\) (which is Grassmann even) to be of the following general form:

\[
\mathcal{V}_{\Phi_L\Phi_L\Phi_H\Phi_H} = \frac{1}{Z_{21}Z_{34}^2} [f_{h_l}(x_0) + x_1 g_{h_l}(x_0)].
\] (2.5.8)

The conformal dimensions of the degenerate fields in the NS sector of an \(\mathcal{N} = 1\) SCFTs can be parameterized by

\[
h_{r,s} = \frac{[(m + 2)r - ms]^2 - 4}{8m(m + 2)}, \quad c = \frac{3}{2} - \frac{12}{m(m + 2)}, \quad r, s \in \mathbb{Z}^+; \quad r - s \in 2\mathbb{Z}.
\] (2.5.9)

and the corresponding null-state is at level \(\frac{rs}{2}\). The first non-trivial null state \((r = 1, s = 3)\) is

\[
\left(\frac{2}{2h_{1,3} + 1} L_{-1}G_{-1/2} - G_{-3/2}\right) |\Phi_{1,3}\rangle = 0,
\] (2.5.10)

with \(h_{1,3} = -\frac{1}{2} - \frac{3}{c} + \mathcal{O}(1/c^2)\) in the large \(c\) limit. If \(\Phi_L = \Phi_{1,3}\) in the heavy-light four-point function \(\langle \Phi_L\Phi_L\Phi_H\Phi_H\rangle\), then

\[
\left\langle \left(\frac{2}{2h_{1,3} + 1} L_{-1}G_{-1/2} - G_{-3/2}\right) \Phi_{1,3}(Z_1) \Phi_{1,3}(Z_2) \Phi_H(Z_3) \Phi_H(Z_4) \right\rangle = 0.
\] (2.5.11)

Using (2.5.4), we get a null-state equation satisfied by the four-point function, which also satisfied by the super-Virasoro vacuum block. Simplifying this equation using the superconformal Ward identities (\(\mathcal{L}_{-1}\) becomes just \(\partial_{z_1}\) and \(G_{-1/2}\) becomes just \(D_1 = \partial_{\theta_1} + \theta_1 \partial_{z_1}\) ), we find

\[
\left\{ \frac{2\partial_{z_1}(\partial_{\theta_1} + \theta_1 \partial_{z_1})}{2h_{1,3} + 1} + \sum_{i=2}^4 \left[ Z_{i1}^{-1} (\partial_{\theta_1} - \theta_1 \partial_{z_1} + 2\theta_1 \partial_{z_i}) + 2h_i \theta_1 Z_{i1}^{-1} \right] \right\} \mathcal{V}_{\Phi_{1,3}\Phi_{1,3}\Phi_H\Phi_H} = 0.
\]

This is a super-differential equation with two unknown function \(f_{h_{1,3}}(x_0)\) and \(g_{h_{1,3}}(x_0)\). To
solve it, we can expand it in terms of $\theta_i$s and require that all the coefficients of $\theta_i$s equal to zero. First, we can send the $z_i$s to $(0, z, 1, \infty)$, in which case, $x_0$ and $x_1$ become

$$x_0 \rightarrow z + \theta_1 \theta_2 - z \theta_1 \theta_3,$$

$$x_1 \rightarrow \theta_1 \theta_2 - \theta_1 \theta_3 + \theta_2 \theta_3.$$

Expanding the super-differential equation in terms of $\theta_i$s, we get two differential equations from the coefficients of $\theta_1$ and $\theta_2$ (differential equations from coefficients of other $\theta_i$s are dependent with these two). In the large $c$ limit, with $h_{1,3} = -\frac{1}{2} - \frac{3}{c} + \mathcal{O}(1/c^2)$ and $\eta_H = \frac{h_H}{c}$ fixed, the leading order ($c^0$) of these two equations are

$$ \left( z - 1 \right)^2 \left( z f''_{h_{1,3}}(z) + 2 f'_{h_{1,3}}(z) \right) + z \eta_H f_{h_{1,3}}(z) = 0,$$

$$z \left( f''_{h_{1,3}}(z) + g'_{h_{1,3}}(z) \right) + 2 f'_{h_{1,3}}(z) + g_{h_{1,3}}(z) = 0.$$

Solving these equations and fixing the constants of integration to match the expansion of the vacuum block in terms of small $z$, we find

$$f_{h_{1,3}}(z) = z^{-1} e^{-\frac{1}{2} f_{00}(z)}, \quad \text{(2.5.12)}$$

$$g_{h_{1,3}}(z) = \frac{1}{z} - \frac{f_{h_{1,3}}(z)}{z} - f'_{h_{1,3}}(z). \quad \text{(2.5.13)}$$

where $f_{00}(z)$ is defined in equation (2.3.13). These solutions only apply to $h_L = h_{1,3}$ in the large $c$ limit. But the appearance of $f_{00}(z)$ in $f_{h_{1,3}}(z)$ gives us some hints for how to

\[\text{For later reference, the exact differential equations are}\]

$$f''_{h_{1,3}} + \frac{2(3-z)h_{1,3} + 3z - 1}{2(z-1)} f'_{h_{1,3}} - \frac{(2h_{1,3} + 1) h_H}{(z-1)^2} f_{h_{1,3}} + \frac{2h_{1,3} + 1}{2(z-1)^2} g_{h_{1,3}} = 0$$

$$f''_{h_{1,3}} + \frac{(6-4z)h_{1,3} + 2z - 1}{2(z-1)^2} f'_{h_{1,3}} + g'_{h_{1,3}} + \frac{z - 2(z-2)h_{1,3}}{2(z-1)^2} g_{h_{1,3}} = 0$$
analytically continue to find $f_{h_L}(z)$ for general $h_L$, which is what we are going to do in the following. After getting $f_{h_L}(z)$, we can use it to obtain $g_{h_L}(z)$ without using the null-state equations.

Expanding both sides of the vacuum block of the superfields (2.5.8) in terms of $\theta_i$s and matching the coefficients of $\theta_i$s, we can obtain relations between vacuum blocks of the component fields and the functions $f_{h_L}(z)$ and $g_{h_L}(z)$:

\begin{align}
\mathcal{V}_{\phi_L\phi_L\phi_H\phi_H} &= z^{-2h_L} f_{h_L}(z), \\
\mathcal{V}_{\psi_L\psi_L\phi_H\phi_H} &= -z^{-2h_L} \left( f'_{h_L}(z) - \frac{2h_L f_{h_L}(z)}{z} + g_{h_L}(z) \right). 
\end{align}

where $\mathcal{V}_{\phi_L\phi_L\phi_H\phi_H}$ is from the term without $\theta_i$ in it and $\mathcal{V}_{\psi_L\psi_L\phi_H\phi_H}$ is from the coefficient of $\theta_1 \theta_2$. In (2.5.15), the minus sign in front is due to the fact that $\theta$ anti-commutes with $\psi$. Using equation (2.5.14) for $h_{1,3} = -\frac{1}{2} + \mathcal{O}(1/c)$ in the leading large $c$ limit, we have $\mathcal{V}_{\phi_{1,3}\phi_{1,3}\phi_H\phi_H} = z f_{h_{1,3}} = e^{-\frac{1}{2} f_{00}(z)}$, which suggests that for general $h_L$, we should have

\begin{equation}
\mathcal{V}_{\phi_L\phi_L\phi_H\phi_H} = e^{h_L f_{00}(z)}. 
\end{equation}

Using equation (2.5.14) again, we have

\begin{equation}
f_{h_L}(z) = z^{2h_L} e^{h_L f_{00}(z)}. 
\end{equation}

From equation (2.5.16), one can see that the super-Virasoro vacuum block $\mathcal{V}_{\phi_L\phi_L\phi_H\phi_H}$ in $\mathcal{N} = 1$ SCFTs is the same as the vacuum block in non-susy CFTs at leading order of the large $c$ limit. We explain in detail why this is true in appendix A.3, but the basic point is that in this limit, only the pure Virasoro generators contribute to the sum over intermediate states in this block.

---

These vacuum blocks are normalized such that the first term of the small $z$ expansion of a vacuum block $\mathcal{V}_{\mathcal{O}_L(0)\mathcal{O}_L(z)\mathcal{O}_H(1)\mathcal{O}_H(\infty)}$ is $\langle \mathcal{O}_L(0)\mathcal{O}_L(z) \rangle$. 

---
To get \( g_{h_L}(z) \), we need to know \( \mathcal{V}_{\psi_L\psi_L\phi_H\phi_H} \). At leading order of the large \( c \) limit, the only difference between \( \mathcal{V}_{\psi_L\psi_L\phi_H\phi_H} \) and \( \mathcal{V}_{\phi_L\phi_L\phi_H\phi_H} \) (up to normalization) is that the conformal dimensions of the light operators are different \( (h_{\phi_L} = h_L, h_{\psi_L} = h_L + \frac{1}{2}) \). Since we know \( \mathcal{V}_{\phi_L\phi_L\phi_H\phi_H} = e^{h_L f_{00}(z)} \), we can immediately see that

\[
\mathcal{V}_{\psi_L\psi_L\phi_H\phi_H} = 2h_L e^{(h_L + \frac{1}{2}) f_{00}(z)},
\]

(2.5.18)

where the prefactor \( 2h_L \) is due to our convention of the vacuum block and can be read off from the two-point function of superfields (2.5.6). Equating the above vacuum block to (2.5.15), we find

\[
g_{h_L}(z) = -2h_L z e^{(h_L + \frac{1}{2}) f_{00}(z)} + \frac{2h_L f_{h_L}(z)}{z} - f'_{h_L}(z).
\]

(2.5.19)

One can check that setting \( h_L = -\frac{1}{2} \) gives us back \( g_{h_{1,3}} \) (2.5.13).

Having the expressions for \( f_{h_L} \) and \( g_{h_L} \), we can restore their argument to \( x_0 \), then other super-Virasoro vacuum blocks of the component fields can be read off from the expansion of \( \mathcal{V}_{\Phi_L\Phi_L\Phi_H\Phi_H} \) (2.5.8) in terms of the \( \theta_i \) variables.

### 2.5.2 The \( \mathcal{N} = 2 \) Super-Virasoro Vacuum Block

#### 2.5.2.1 Brief Review of 2d \( \mathcal{N} = 2 \) SCFTs

In the \( \mathcal{N} = 2 \) superspace, a point is denoted by \( Z \equiv (z, \theta, \bar{\theta}) \), where \( z \) is the usual complex coordinate, while \( \theta \) and \( \bar{\theta} \) are two Grassmann coordinates. The energy-momentum superfield can be expanded as

\[
\mathcal{J}(Z) = J(z) + \theta \mathcal{G}(z) - \bar{\theta} G(z) + \theta \bar{\theta} 2 T(z).
\]

(2.5.20)

---

15This point can be seen from the commutation relations of the Virasoro generators with these component fields (A.3.1), and at leading order of large \( c \) limit, only Virasoro generators contribute to these two vacuum blocks.
where $J(z)$ is the $U(1)$ $R$-current. The mode expansions are defined in the usual way

$$J(z) = \sum_{n \in \mathbb{Z}} \frac{J_n}{z^{n+1}}, \ G(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{G_r}{z^{r+\frac{3}{2}}}, \ \overline{G} = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{\overline{G}_r}{z^{r+\frac{3}{2}}}, \ T(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}.$$ 

The full $\mathcal{N} = 2$ superconformal algebra of these generators takes the following form:

\[
\begin{align*}
[L_m, L_n] &= (m - n) L_{m+n} + \frac{c}{12} (m^3 - m) \delta_{m+n,0}, \\
[L_m, G_r] &= \left( \frac{m}{2} - r \right) G_{m+r}, \quad [L_m, \overline{G}_r] = \left( \frac{m}{2} - r \right) \overline{G}_{m+r}, \\
[J_m, J_n] &= \frac{c}{3} m \delta_{m+n,0}, \quad [L_m, J_n] = -n J_{m+n}, \\
[J_m, G_r] &= G_{m+r}, \quad [J_m, \overline{G}_r] = -\overline{G}_{m+r}, \\
\{G_r, G_s\} &= 2 L_{r+s} + (r - s) J_{r+s} + \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \delta_{r+s,0}, \\
\{G_r, \overline{G}_s\} &= 0, \quad m, n \in \mathbb{Z}; \ r, s \in \mathbb{Z} + \frac{1}{2}.
\end{align*}
\] (2.5.21)

A superfield $\Phi(Z)$ can be expanded in terms of $\theta$ and $\overline{\theta}$ as

$$\Phi^q_h(Z) = \phi^q_h(z) + \theta \psi^q_{h+\frac{1}{2}}(z) + \overline{\theta} \overline{\psi}^q_{h+\frac{1}{2}}(z) + \theta \overline{\theta} \lambda^q_{h+1}(z), \tag{2.5.22}$$

where the superscripts and subscripts are the conformal dimensions and $U(1)$ charges of the component fields. The OPE of $\mathcal{J}(Z_1)$ and $\Phi(Z_2)$ is

$$\mathcal{J}(Z_1) \Phi(Z_2) \sim \frac{2h \theta_1 \overline{\theta}_1}{Z_{12}^2} \Phi(Z_2) + \frac{\theta_1 D_2 - \overline{\theta}_1 \overline{D}_2}{Z_{12}} \Phi(Z_2) + \frac{2 \theta_1 \overline{\theta}_1}{Z_{12}} \partial_{z_2} \Phi(Z_2) + \frac{q}{Z_{12}} \Phi(Z_2),$$

where the super derivatives and super-translationally invariant distance are

$$D_i = \partial_{\theta_i} + \overline{\theta}_i \partial_{z_i}, \quad \overline{D}_i = \partial_{\overline{\theta}_i} + \theta_i \partial_{z_i}, \quad Z_{ij} \equiv z_{ij} - \theta_i \overline{\theta}_j - \overline{\theta}_i \theta_j, \tag{2.5.23}$$

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with \( z_{ij} = z_i - z_j \), \( \theta_{ij} = \theta_i - \theta_j \) and \( \overline{\theta}_{ij} = \overline{\theta}_i - \overline{\theta}_j \).

The highest weight states in the NS sector are characterized by their eigenvalues under \( L_0 \) and \( J_0 \):

\[
L_0 |\Phi\rangle = h |\Phi\rangle, \quad J_0 |\Phi\rangle = q |\Phi\rangle,
\]

and they satisfy

\[
L_n |\Phi\rangle = J_n |\Phi\rangle = G_r |\Phi\rangle = \overline{G}_r |\Phi\rangle = 0, \quad \text{for } n, r > 0.
\]

Acting on primary superfield with \( L_{-n}, G_{-r}, \overline{G}_{-r}, J_{-n} (n, r > 0) \), we get the descendent superfields. Using the OPE of \( J \) and \( \Phi \), one can show that the correlation function with one descendant superfield can be written in terms of a super-differential operator acting on a correlation function with only primary fields:

\[
\langle (L_{-n}\Phi_1)(Z_1)X \rangle = \mathcal{L}_{-n} \langle \Phi_1(Z_1)X \rangle, \quad \langle (J_{-n}\Phi_1)(Z_1)X \rangle = \mathcal{J}_{-n} \langle \Phi_1(Z_1)X \rangle,
\]

\[
\langle (G_{-r}\Phi_1)(Z_1)X \rangle = \mathcal{G}_{-r} \langle \Phi_1(Z_1)X \rangle, \quad \langle (\overline{G}_{-r}\Phi_1)(Z_1)X \rangle = \overline{\mathcal{G}}_{-r} \langle \Phi_1(Z_1)X \rangle.
\]

where \( X = \Phi_2(Z_2) \cdots \Phi_N(Z_N) \) is an assembly of primary fields with conformal dimension \( h_i \) and \( U(1) \) charge \( q_i \). These super-differential operators are [69]

\[
\mathcal{L}_{-n} = -\sum_{i=2}^{N} Z_{i1}^{-n} \left[ (1 - n)(h_i + \frac{1}{2} \theta_{i1} D_i + \frac{1}{2} \overline{\theta}_{i1} \overline{D}_i) + Z_{i1} \partial_{z_i} - \frac{q_i}{2} \theta_{i1} \overline{\theta}_{i1} Z_{i1}^{-1} n(1 - n) \right],
\]

\[
\mathcal{J}_{-n} = -\sum_{i=2}^{N} Z_{i1}^{-n} \left( \overline{\theta}_{i1} \overline{D}_i - \theta_{i1} D_i + 2 \theta_{i1} \overline{\theta}_{i1} \partial_{z_i} + q_i - 2 h_i \theta_{i1} \overline{\theta}_{i1} n Z_{i1}^{-1} \right),
\]

\[
\mathcal{G}_{-r} = -\sum_{i=2}^{N} Z_{i1}^{-r-\frac{1}{2}} \left[ (r - \frac{1}{2}) \theta_{i1} (2 h_i + q_i + \overline{\theta}_{i1} \overline{D}_i) + Z_{i1} (D_i - 2 \theta_{i1} \partial_{z_i}) \right],
\]

\[
\overline{\mathcal{G}}_{-r} = -\sum_{i=2}^{N} Z_{i1}^{-r-\frac{1}{2}} \left[ (r - \frac{1}{2}) \overline{\theta}_{i1} (2 h_i - q_i + \theta_{i1} D_i) + Z_{i1} (D_i - 2 \theta_{i1} \partial_{z_i}) \right].
\]

N-point correlation functions of the primary superfields should be invariant under the
global super-conformal transformations generated by $L_{\pm 1}, L_0, J_0, G_{\pm \frac{1}{2}}, \bar{G}_{\pm \frac{1}{2}}$, which leads to the superconformal Ward identities (A.4.1). These Ward identities completely fix the two-point functions to be

\[
\langle \Phi_1(Z_1)\Phi_2(Z_2) \rangle = \frac{1}{Z_{21}^{2h_1}} e^{\theta_{12} \bar{\theta}_{12}} \delta_{q_1 + q_2, 0} \delta_{h_1, h_2}
\]

\[
= \left( \frac{1}{z_{21}^{2h_1}} + \frac{-q_2}{z_{21}^{2h_1+1}} \theta_{1} \bar{\theta}_{1} + \frac{-2h_1 + q_2}{z_{21}^{2h_1+1}} \theta_{1} \bar{\theta}_{2} + \frac{-2h_1 - q_2}{z_{21}^{2h_1+1}} \theta_{2} \bar{\theta}_{2} + \frac{-q_2}{z_{21}^{2h_1+1}} \theta_{2} \bar{\theta}_{2} \right) \delta_{q_1 + q_2, 0} \delta_{h_1, h_2},
\]

up to a normalization constant. Each term in the above equation corresponds to a two-point function of the component fields. Notice that only the two-point function of the lowest component field $\phi$ is normalized as usual.

### 2.5.2.2 Super Null-State Equations

The heavy-light super-Virasoro vacuum block $V_{\Phi^{-q_{HL}}_{L} \Phi^{q_{HL}}_{H}}$ is the contribution to the heavy-light four-point function $\langle \Phi^{-q_{HL}}_{L}(Z_1)\Phi^{q_{HL}}_{L}(Z_2)\Phi^{-q_{H}}_{H}(Z_3)\Phi^{q_{H}}_{H}(Z_4) \rangle$ from an irreducible representation of the superconformal algebra whose highest weight state is the vacuum $|0\rangle$. In this paper, we will take the following heavy-light limit:

\[h_L, q_L, \eta_H \equiv \frac{h_H}{c}, \eta_q \equiv \frac{q_H}{c} \text{ fixed as } c \to \infty.\]

Our main result of this part is $V_{\Phi^{-q_{HL}}_{L} \Phi^{q_{HL}}_{L} \Phi^{-q_{H}}_{H} \Phi^{q_{H}}_{H}}$ given in (2.5.29), with

\[F(x_0, x_1, x_2, x_3, x_4) \text{ given in (2.5.31) and the } g_{i,h_L} \text{ functions given in next subsection 2.5.2.3.}\]

Superconformal Ward identities fix the vacuum block (and the four-point function) to take the following form [67]

\[
V_{\Phi^{-q_{HL}}_{L} \Phi^{q_{HL}}_{L} \Phi^{-q_{H}}_{H} \Phi^{q_{H}}_{H}} = \frac{1}{Z_{21}^{2h_L} Z_{34}^{2h_H}} \exp \left( q_L \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}} + q_H \frac{\theta_{34} \bar{\theta}_{34}}{Z_{34}} \right) F(x_0, x_1, x_2, x_3, x_4),
\]

(2.5.29)
where $F(x_0, x_1, x_2, x_3, x_4)$ is a function of five superconformal invariants

$$
x_0 = \frac{Z_{12} Z_{34}}{Z_{13} Z_{24}}, \quad x_1 = \frac{Z_{14} Z_{23}}{Z_{13} Z_{24}} + x_0 - 1,$$

$$
x_2 = \frac{\theta_{23} \bar{\theta}_{23}}{Z_{23}} + \frac{\theta_{34} \bar{\theta}_{34}}{Z_{34}} - \frac{\theta_{24} \bar{\theta}_{24}}{Z_{24}},$$

$$
x_3 = \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}} + \frac{\theta_{24} \bar{\theta}_{24}}{Z_{24}} - \frac{\theta_{14} \bar{\theta}_{14}}{Z_{14}},$$

$$
x_4 = \frac{\theta_{13} \bar{\theta}_{13}}{Z_{13}} + \frac{\theta_{34} \bar{\theta}_{34}}{Z_{34}} - \frac{\theta_{14} \bar{\theta}_{14}}{Z_{14}}. $$

(2.5.30)

It is easy to verify that these super-conformal invariants satisfy the relations

$$
x_1^3 = 0, \quad x_2^2 = 0, \quad x_3^2 = 0, \quad x_4^2 = 0, \quad x_1 x_2 = x_1 x_3 = x_1 x_4 = 0,$$

$$
x_2 x_3 x_0 = x_2 x_4, \quad x_2 x_3 = x_2 x_4 + x_3 x_4, \quad x_1^2 = 2 x_2 x_3 x_0 (1 - x_0).$$

which means that the most general form of $F(x_0, x_1, x_2, x_3, x_4)$ can be written as

$$F = g_{0,h_L}(x_0) + x_1 g_{1,h_L}(x_0) + x_2 g_{2,h_L}(x_0) + x_3 g_{3,h_L}(x_0) + x_4 g_{4,h_L}(x_0) + x_2 x_3 g_{5,h_L}(x_0).$$

(2.5.31)

The conformal dimensions of the degenerate fields in the NS sector of $\mathcal{N} = 2$ SCFTs can be parameterized by\(^{16}\) [69]

$$h_{r,s} = \frac{r^2 - 1}{8} t - \frac{rs}{4} + \frac{s^2 - 1}{8t} - \frac{4q^2 - 1}{8t}, \quad c = 3 - 3t, \quad r \in \mathbb{Z}^+; s \in 2\mathbb{Z}^+. $$

(2.5.32)

For each degenerate field with dimension $h_{r,s}$, there is a null-field at level $\frac{rs}{2}$. The first non-trivial null-state ($r = 1, s = 2$) is :

$$\left[(q - 1) L_{-1} - (2h_{1,2} + 1) J_{-1} + G_{-\frac{1}{2}} \bar{G}_{-\frac{1}{2}}\right] \Phi_{\frac{1}{2}} = 0,$$

(2.5.33)

\(^{16}\)Besides $h_{r,s}$, there are other degenerate fields whose conformal dimensions can be parameterized by $h_k = kq + \frac{k}{4} t (k^2 - \frac{1}{4}), k \in \mathbb{Z} + \frac{1}{2}$ and having a null field at level $|k|$, but these will not be used in this paper.
with \( h_{1,2} = \frac{e^{-3q_L^2}}{6-2c} = -\frac{1}{2} + \frac{3(q_L^2-1)}{2c} + O(1/c^2) \). Notice that the \( U(1) \) charge \( q \) is a free parameter here. If \( h_L = h_{1,2} \) in the heavy-light four-point function, then

\[
\langle \left[ (\phi_L - 1) \partial_{z_1} - (2h_{1,2} + 1) \mathcal{J}_{-1} + G_{-\frac{1}{2}} \overline{\mathcal{G}}_{-\frac{1}{2}} \right] \Phi_{1,1,2}^{-q_L} (Z_1) \Phi_{1,1,2}^{q_L} (Z_2) \Phi_H^{-q_H} (Z_3) \Phi_H^{q_H} (Z_4) \rangle = 0.
\]

Using equations (2.5.26), we get a super-differential equation satisfied by the four-point function, which is also satisfied by the vacuum block \( \mathcal{V}_{\Phi_L^{-q_L} \Phi_L^{q_L} \Phi_H^{-q_H} \Phi_H^{q_H}} \). Simplifying this super-differential equation using the superconformal Ward identities (A.4.1) \( \mathcal{L}_{-1} \to \partial_{z_1}, \mathcal{G}_{-\frac{1}{2}} \to \mathcal{D}_1 \) and \( \overline{\mathcal{G}}_{-\frac{1}{2}} \to \mathcal{D}_1 \), we find

\[
\left[ (\phi_L - 1) \partial_{z_1} - (2h_L + 1) \mathcal{J}_{-1} + \mathcal{D}_1 \mathcal{D}_1 \right] \mathcal{V}_{\Phi_{1,1,2}^{-q_L} \Phi_{1,1,2}^{q_L} \Phi_H^{-q_H} \Phi_H^{q_H}} = 0, \tag{2.5.34}
\]

with \( \mathcal{J}_{-1} \) given in (2.5.27) and \( \mathcal{D}_1, \mathcal{D}_1 \) given in (2.5.23).

To solve this super-differential equation, we can expand it in terms of \( \theta_i \)'s and \( \overline{\theta}_i \)'s to get six independent differential equations to solve for the six unknown functions\(^{17}\), \( g_{0,h_{1,2}}(z), \cdots, g_{5,h_{1,2}}(z) \). These solutions \( g_{i,h_{1,2}}(z) \) only apply to those vacuum blocks whose light operators are degenerate operators with \( h_L = h_{1,2} \). To get \( g_{i,h_L}(z) \) for general \( h_L \) we need to analytically continue these solutions, as what we did for the non-susy Virasoro blocks. But in the non-susy case, there was only one unknown function and we already knew its anzatz for general \( h_L \) (2.2.9), so things were easier there. Here, we have six \( g_{i,h_{1,2}}(z) \) functions and some of them are complicated and hard to know how to analytically continue them. But it turns out that once we solve the differential equation for \( g_{0,h_{1,2}}(z) \), then analytically continue the solution to get \( g_{0,h_L}(z) \), we can derive the other \( g_{i,h_L}(z) \) functions from it, which will be shown in next subsection 2.5.2.3. The equation that only involves \( g_{0,h_{1,2}}(z) \) is

\[
g_{0,h_{1,2}}^{\prime
d} (z) + \left( \frac{6q_L \eta_q}{z - 1} + \frac{2}{z} \right) g_{0,h_{1,2}}^{\prime} (z) + \frac{6z \eta_H + 3\eta_q (3z(q_L^2 - 1) \eta_q + (z - 2)q_L)}{(z - 1)^2 z} g_{0,h_{1,2}}(z) = 0.
\]

\(^{17}\)Again, we send the coordinates \( z_i \) to \((0, z, 1, \infty)\), in which case, \( x_0 \to z + \cdots \), where \( \cdots \) represents terms proportional to \( \theta_i, \overline{\theta}_i \) or their products.
The solution is

\[ g_{0,h_1,2}(z) = z^{-1} e^{-\frac{1}{2}\tilde{f}(z)} (1 - z)^{-3\eta q_L}, \]  

(2.5.35)

where

\[ \tilde{f}(z) = -(1 - \tilde{a}) \log (1 - z) - 2 \log \left( \frac{1 - (1 - z)^{\tilde{a}}}{\tilde{a}} \right), \]

(2.5.36)

with \( \tilde{a} = \sqrt{1 - 24\eta_H + 36\eta_i^2} \). In the above solution, the constants of integration have been fixed such that the first term in the expansion of \( g_{0,h_1,2}(z) \) in small \( z \) is 1, which corresponds to the vacuum block.

### 2.5.2.3 Solutions for General \( h_L \)

In this subsection, we are going to analytically continue \( g_{0,h_1,2} \) to get \( g_{0,h_L} \), then use it to derive the other \( g_{i,h_L} \) functions. Expanding the ansatz (2.5.29) in terms of \( \theta_i \)s and \( \bar{\theta}_i \)s, we can express the vacuum blocks of the component fields in terms of \( g_{i,h_L}(z) \)\(^{18}\):

\[ \mathcal{V}_{\phi_L^{-q_L}} \phi_L^{q_L} \phi_H^{-q_H} \phi_H^{q_H} = z^{-2h_L} g_{0,h_L}, \]

(2.5.37)

\[ \mathcal{V}_{\psi_L^{-q_L-1}} \psi_L^{q_L+1} \phi_H^{-q_H} \phi_H^{q_H} = -z^{-(2h_L+1)} \left[ (q_L - 2h_L) g_{0,h_L} + z g_{1,h_L} + g_{3,h_L} + z g_{0,h_L}' \right], \]

(2.5.38)

\[ \mathcal{V}_{\psi_L^{-q_L+1}} \psi_L^{q_L-1} \phi_H^{-q_H} \phi_H^{q_H} = z^{-(2h_L+1)} \left[ (q_L + 2h_L) g_{0,h_L} - z g_{1,h_L} + g_{3,h_L} - z g_{0,h_L}' \right], \]

(2.5.39)

\[ \mathcal{V}_{\phi_L^{-q_L} \phi_L^{q_L} \phi_H^{-q_H} \phi_H^{q_H}} = z^{-(2h_L+1)} \left[ -q_L g_{0,h_L} - g_{3,h_L} + \frac{z}{z - 1} g_{2,h_L} \right], \]

(2.5.40)

\[ \mathcal{V}_{\phi_L^{-q_L} \phi_L^{q_L} \phi_H^{-q_H} \phi_H^{q_H}} = -z^{-(2h_L+1)} \left( q_L g_{0,h_L} + g_{3,h_L} + z g_{4,h_L} \right), \]

(2.5.41)

\[ \mathcal{V}_{\phi_L^{-q_L} \phi_L^{q_L} \phi_H^{-q_H} \phi_H^{q_H}} = z^{-2h_L} \left[ g_{0,h_L}' - \frac{4h_L}{z} \left( g_{0,h_L} + g_{1,h_L} \right) + \frac{2h_L(2h_L + 1)}{z^2} g_{0,h_L} \right] + 2g_{1,h_L}' + \frac{1}{1 - z} \left( q_L g_{2,h_L} + g_{5,h_L} + \frac{q_L}{z} g_{4,h_L} \right). \]

(2.5.42)

---

\(^{18}\)These vacuum blocks are normalized such that the first term of the small \( z \) expansion of a vacuum block \( \mathcal{V}_{\phi_L(0)\phi_L(z)\phi_H(1)\phi_H(\infty)} \) is \( \langle \phi_L(0) \phi_L(z) \rangle \).
The basic idea of these derivations is to derive the vacuum blocks on the LHS, then solve the above equations to get the functions $g_{i,h_L}(z)$ on the RHS.

First, the most important function is $g_{0,h_L}(z)$, which is associated with $\mathcal{V}_{\phi_L^{-q_L} \phi_L^{q_L} \phi_H^{-q_H} \phi_H^{q_H}}$. From equation (2.5.37), for $h_L = h_{1,2} = -\frac{1}{2} + O(1/c)$, we have $\mathcal{V}_{\phi_{1,2}^{-q_L} \phi_{1,2}^{q_L} \phi_H^{-q_H} \phi_H^{q_H}} = z g_{0,h_{1,2}} = e^{-\frac{1}{2} \tilde{f}(z)}(1 - z)^{-3\eta q_L}$, which suggests that for general $h_L$, we should have

$$\mathcal{V}_{\phi_L^{-q_L} \phi_L^{q_L} \phi_H^{-q_H} \phi_H^{q_H}} = e^{h_L \tilde{f}(z)}(1 - z)^{-3\eta q_L}. \quad (2.5.43)$$

Indeed, this matches the Virasoro vacuum block for CFT$_2$s with a global $U(1)$ symmetry, which have been computed in [16]. The fact that this super-Virasoro vacuum block only gets contributions from the Virasoro generators and $U(1)$ generators at leading order of the large $c$ limit can be seen from the commutation relations of these generators with the component field $\phi$, as we explain in the appendix A.4.2. Using equation (2.5.37) again, we have

$$g_{0,h_L}(z) = z^{2h_L} e^{h_L \tilde{f}(z)} (1 - z)^{-3\eta q_L}. \quad (2.5.44)$$

Next, to get $g_{1,h_L}(z)$ and $g_{3,h_L}(z)$, we need to know the blocks $\mathcal{V}_{\psi_L^{-q_L} \psi_L^{q_L} \phi_H^{-q_H} \phi_H^{q_H}}$ and $\mathcal{V}_{\overline{\psi}_L^{-q_L} \overline{\psi}_L^{q_L} \phi_H^{-q_H} \phi_H^{q_H}}$. At leading order of the large $c$ limit, the only differences between these two blocks and $\mathcal{V}_{\phi_L^{-q_L} \phi_L^{q_L} \phi_H^{-q_H} \phi_H^{q_H}}$ are that the conformal dimensions and $U(1)$ charges of the light fields are different (note that the conformal dimensions of $\psi_L$ and $\overline{\psi}_L$ are $h_L + \frac{1}{2}$, while that for $\phi_L$ is $h_L$), which means that we can change the parameters accordingly in the expression of $\mathcal{V}_{\phi_L^{-q_L} \phi_L^{q_L} \phi_H^{-q_H} \phi_H^{q_H}}$ to get these two blocks:

$$\mathcal{V}_{\psi_L^{-q_L} \psi_L^{q_L} \phi_H^{-q_H} \phi_H^{q_H}} = (2h_L - q_L)e^{(h_L + \frac{1}{2}) \tilde{f}(z)} (1 - z)^{-3\eta(q_L + 1)} \equiv z^{-2h_L - 1} g_{q_L + 1}(z),$$

$$\mathcal{V}_{\overline{\psi}_L^{-q_L} \overline{\psi}_L^{q_L} \phi_H^{-q_H} \phi_H^{q_H}} = (2h_L + q_L)e^{(h_L + \frac{1}{2}) \tilde{f}(z)} (1 - z)^{-3\eta(q_L - 1)} \equiv z^{-2h_L - 1} g_{q_L - 1}(z).$$

where the prefactor $2h_L \mp q_L$ is due to our convention of the definition of the vacuum blocks
and can be read off from the two-point function (2.5.28). Equating these two blocks to equations (2.5.38) and (2.5.39) respectively, we can solve for \(g_{1,hL}(z)\) and \(g_{3,hL}(z)\)

\[
g_{1,hL} = \frac{1}{2z} \left( 4h_L g_{0,hL} - 2z g'_{0,hL} - g_{qL+1} - g_{qL-1} \right),
\]

\[
g_{3,hL} = \frac{1}{2} \left( -2q_L g_{0,hL} - g_{qL+1} + g_{qL-1} \right).
\]

The remaining functions \(g_{2,hL}(z)\), \(g_{4,hL}(z)\) and \(g_{5,hL}(z)\) are related to the vacuum blocks \(\mathcal{V}_{\phi_L^{qL}\phi_H^{-qH}}\), \(\mathcal{V}_{\phi_L^{qL}\phi_H^{-qH}}\) and \(\mathcal{V}_{\phi_L^{qL}\phi_H^{-qH}}\) (2.5.40-2.5.42), respectively. As is shown in appendix A.4.4, \(\lambda^q_L\) can be written as descendant fields plus a Virasoro and \(U(1)\) primary

\[
\lambda^q_L(z) = \frac{12h^2 - 3q^2}{2c - 3q^2} (J_1\phi^q_L)(z) + \frac{q(c - 6h)}{2c - 3q^2} (L_1\phi^q_L)(z) + \tilde{\lambda}^q_L(z)
\]

The Virasoro and \(U(1)\) primary part \(\tilde{\lambda}^q_L\) has conformal dimension \(h_L + 1\) and \(U(1)\) charge \(q_L\), which are the same as \(\lambda^q_L\). Using this decomposition, we can calculate these three vacuum blocks from \(\mathcal{V}_{\phi_L^{qL}\phi_H^{-qH}}\). Some details for performing these calculations are given in appendix A.4.4. Equating these three vacuum blocks to equations (2.5.40), (2.5.41) and (2.5.42), we can solve for \(g_{2,hL}(z)\), \(g_{4,hL}(z)\) and \(g_{5,hL}(z)\). At leading order of the large \(c\) limit, these functions are

\[
g_{2,hL} = \frac{3zq_L q^2 (q^2_L - 4h^2_L) g_{0,hL} + 2(z - 1)h_L g_{3,hL} + (z - 1)q_L g'_{0,hL}}{2h_L z},
\]

\[
g_{4,hL} = g_{2,hL} - g_{5,hL},
\]

\[
g_{5,hL} = \frac{(z - 1)z}{4} \left( \frac{(z - 1)q^2_L}{h^2_L} + 4 \right) g''_{0,hL} + (z - 1) \left( 2z g'_{2,hL} - 4h_L g_{1,hL} + q_L g_{4,hL} \right)
\]

\[
- q_L g_{2,hL} - \frac{(z - 1) (6zq_L (4h^2_L - q^2_L) + q^2_L (2(z - 2)h_L - z) + 16h^2_L)}{4h^2_L} g'_{0,hL}.
\]
$$+ \left( \frac{(4h_L^2 - q_L^2)}{2h_L z} \right)^2 \left( \frac{(z^2 e^{\bar{f}} - 1)(1 - z)(2h_L + 1) + 3(z - 2)z \eta q_L}{4h_L^2 - q_L^2} + \frac{9\eta^2 z^2}{2h_L} \right) g_{0,h_L}.$$  \hspace{1cm} (2.5.50)

with $\bar{f}$ in $g_{5,h_L}$ given in (2.5.36). One can easily check that setting $h_L = h_{1.2} \simeq \frac{1}{2}$, these $g_{i,h_L}(z)$ functions will become $g_{i,h_{1.2}}(z)$ and they are indeed the solutions to the null-state equation (2.5.34) at leading order of the large $c$ limit. Restoring the argument of $g_{i,h_L}$ to $x_0$, other super-Virasoro vacuum blocks of the component fields can be read off from the expansion of $V_{\Phi_L^{q_L} \Phi_H^{q_H}} \Phi_L^{q_L} \Phi_H^{q_H}$ in terms of $\theta_i$s and $\bar{\theta}_i$s. We’ve checked the first few terms of the expansion of these other blocks in terms of small $z$, and they match the results from the direct calculation of these blocks. \(^{19}\)

\(^{19}\)By direct calculation, we mean to calculate the vacuum block by inserting the vacuum state and its descendants into the four-point function, and then summing over all these contributions.
Chapter 3

A Numerical Approach to Virasoro Blocks and the Information Paradox

This chapter is based on the following paper:


Abstract

We chart the breakdown of semiclassical gravity by analyzing the Virasoro conformal blocks to high numerical precision, focusing on the heavy-light limit corresponding to a light probe propagating in a BTZ black hole background. In the Lorentzian regime, we find empirically that the initial exponential time-dependence of the blocks transitions to a universal $t^{-\frac{3}{2}}$ power-law decay. For the vacuum block the transition occurs at $t \approx \frac{\pi c}{6h_L}$, confirming analytic predictions. In the Euclidean regime, due to Stokes phenomena the naive semiclassical approximation fails completely in a finite region enclosing the ‘forbidden singularities’. We emphasize that limitations on the reconstruction of a local bulk should ultimately stem from distinctions between semiclassical and exact correlators.
3.1 Introduction and Summary

Many of the most challenging conceptual problems in theoretical physics were only resolved after physicists discovered how to ‘shut up and calculate’ a large variety of observables to high precision. For example, our modern understanding of quantum field theory was only developed after the physics community had decades of experience with perturbative calculations. And it is hard to imagine how decoherence could have been understood without the temporary crutch provided by the Copenhagen interpretation and its instrumental approach to the Born rule.

Though we have struggled with the black hole information paradox for decades, major progress has been possible through the development of AdS/CFT. Resolving the information paradox in AdS/CFT will require a precise understanding of bulk reconstruction and its limitations. Although reconstruction presents thorny conceptual problems, the limitations on reconstruction should ultimately stem from discrepancies between the predictions of gravitational effective field theory in AdS and conformal field theory. This means that to make progress, it will be crucial to be able to directly compare the approximate correlation functions of bulk EFT and the exact correlators of the CFT.

AdS$_3$/CFT$_2$ may provide the best opportunity for such comparisons. Many features of quantum gravity in AdS$_3$ can be understood ‘kinematically’ as a consequence of the structure of the Virasoro algebra. To be specific, the Virasoro conformal blocks have a semiclassical large central charge limit that precisely accords with expectations from AdS$_3$ gravity, reproducing the physics of light objects probing BTZ black holes. In the semiclassical approximation, the Virasoro blocks exhibit information loss in the form of ‘forbidden singularities’ and exponential decay at late times [15, 16, 17, 20, 9, 23]. Moreover, these problems can be partially addressed by performing explicit analytic calculations [12]. The blocks can also be computed directly from AdS$_3$ [21, 75, 24, 18, 19, 76, 77, 78].
In this work we will investigate the discrepancies between semiclassical gravity and the exact CFT by computing the Virasoro blocks numerically to very high precision. This is possible via a slightly non-trivial implementation of the Zamolodchikov recursion relations [37, 36, 79]. We discuss the blocks and the algorithm in detail in section 3.2 and appendix B.1. For the remainder of the introduction we will explain the physics questions to be addressed and summarize the results.

**When is the Semiclassical Approximation Valid?**

The Virasoro conformal blocks have a semiclassical limit. CFT\(_2\) correlators can be written in a Virasoro block decomposition as

\[
\langle \mathcal{O}_1(0)\mathcal{O}_2(z)\mathcal{O}_3(1)\mathcal{O}_4(\infty) \rangle = \sum_{h,\tilde{h}} P_{h,\tilde{h}} V_{h_i,\bar{h},c}(z) V_{h_i,\bar{h},c}(\bar{z})
\]  

(3.1.1)

the holomorphic Virasoro blocks \(V_{h_i,\bar{h},c}\) depend on the holomorphic dimensions \(h_i\) of the primary operators \(\mathcal{O}_i\), on an intermediate primary operator dimension \(h\), and on the central charge \(c\). A semiclassical limit emerges when \(c \to \infty\) with all \(h_i/c\) and \(h/c\) fixed; the blocks take the form

\[
V = e^{-\frac{c}{36}f\left(\frac{h_i}{c}, \frac{h}{c}, z\right)}
\]  

(3.1.2)

It is natural to ask about the range of validity of this approximation – how does it depend on the kinematic variable \(z\) and the ratios \(h_i/c\) and \(h/c\)?

One reason to ask is simultaneously speculative and pragmatic – one might like to know if it is possible to explore AdS\(_3\) quantum gravity in the lab by engineering an appropriate CFT\(_2\) (for a concrete idea see [80]). But gravity will only be a good description if the semiclassical limit provides a reasonable approximation at accessible values of \(c\). Unfortunately, even in the semiclassical limit the Virasoro blocks are not known in closed form for general parameters.
Figure 3.1: This figure suggests the analytic continuations necessary to obtain a heavy-light correlator with increasing (Lorentzian) time separation between the light operators. We take $r \lesssim 1$ to avoid singularities on the lightcones displayed on the left; one can also use $r$ as a proxy for a Euclidean time separation between the light operators.

But we can partially test the validity of this limit by computing $\frac{c_2}{c_1} \log \frac{\mathcal{V}(c_1)}{\mathcal{V}(c_2)}$ for $c_2 \approx c_1$, as this ratio will be 1 when the semiclassical limit holds. We plot this ratio in figure 3.7, which shows that the blocks adhere to the semiclassical form of equation (3.1.2) remarkably well (up to an important caveat to be discussed later).

We would also like to understand if the semiclassical limit breaks down in specific kinematic regimes associated with quantum gravitational effects in AdS$_3$. This is what we will explore next.

**Information Loss and OPE Convergence in a New Regime**

Information loss can be probed using the correlators of light operators in a black hole background [81], as illustrated in 3.1. When computed in a BTZ or AdS-Schwarzschild geometry, these correlators decay exponentially as we increase the time separation $t$ between the light operators, even as we take $t \to \infty$. This behavior represents a violation of unitarity for a theory with a finite number of local degrees of freedom on a compact space. Thus it is interesting to see how it is resolved by the exact CFT description.
As a first step, one would like to understand how unitary CFTs are able to mimic bulk gravity, including the appearance of information loss. This arises from the heavy-light large central charge approximation [12, 23, 9] of the Virasoro blocks. Thus it seems to be very universal, as it is largely independent of CFT data. The next step is to understand how finite $c$ physics corrects this approximation, and what CFT data are involved in resolving information loss.

It is useful to think about the Fourier representation of both the correlator and the individual Virasoro blocks. For the full correlator this is

$$
\langle \mathcal{O}_H(\infty)\mathcal{O}_L(t)\mathcal{O}_L(0)\mathcal{O}_H(-\infty) \rangle = \int dE \lambda^2(E)e^{iEt}
$$

where $\lambda(E)$ is the OPE coefficient density and we have taken $z = 1 - e^{-it}$ to study Lorentzian time separations between the light probe operators. At large $t$ we probe the fine structure of $\lambda(E)$, which means that the least analytic features of $\lambda(E)$ dominate the late time limit. Practically speaking, this means that the very late time limit probes the discrete nature of the spectrum, and we become sensitive to the fact that $\lambda^2(E)$ is a sum of delta functions. At early or intermediate times we only discern the coarse features of $\lambda(E)$.

There are at least five different timescales associated with black holes in AdS/CFT. The inverse temperature $\beta = \frac{2\pi}{|\alpha_H|}$ where $\alpha_H \equiv \sqrt{1 - \frac{24\hbar H}{c}}$ sets the shortest relevant scale, where $h_H$ is the holomorphic dimension of $\mathcal{O}_H$. The scale $\beta \log c$ estimates the time it takes for infalling matter to be scrambled [82, 47]. At times of order the entropy $S = \frac{\pi^2 c}{3\beta}$, heavy-light correlators cease their exponential decay; this is also the evaporation timescale for black holes in flat spacetime. We expect that the typical energy splittings among neighboring eigenstates to be of order $e^{-S}$, which means that at times of order $e^S$ we will be sensitive to the discreteness of the spectrum. Finally, on timescales of order $e^{eS}$ the phases of the eigenstates can come back into approximate alignment, leading to recurrences.

As discussed in detail in section 3.3, what we find is that the Virasoro blocks with
$h_L < \frac{c}{24} < h_H$ behave very differently at early and late times, as was presaged by analytic results [12]:

- The blocks with intermediate operator dimension $h \lesssim \frac{c}{24}$ are well-described by their semiclassical limit [15, 16, 9] for

$$t \lesssim t_D \equiv \frac{\pi c}{6h_L} \quad (3.1.4)$$

When $h > \frac{c}{24}$ the blocks are also well-described by the semiclassical limit at early times, but we do not have a precise formula quantifying ‘early’.

- Heavy-light blocks with $h \gtrsim h_H$ initially grow, as was found from a semiclassical analysis [9]. We find that they reach a maximum

$$|V|_{\text{max}} \approx 16^{h - \frac{c}{24}} \left(\frac{h}{c}\right)^{-\frac{5}{2}h_H} \text{ at } t_{\text{max}} \approx A_t \sqrt{\frac{24h}{c} - 1} \quad (3.1.5)$$

and then subsequently decay. The factor $\frac{5}{2}$ comes from empirical fits; the function $A_t(h_H)$ is always order one and is approximately linear in $h_H^{-1}$. Other sub-leading behavior is discussed in section 3.3.

- Numerical evidence indicates that all heavy-light Virasoro blocks decay as

$$|\mathcal{V}(t \gg t_D)| \propto t^{-\frac{3}{2}} \quad (3.1.6)$$

at late times, independent of $h$ and $c$, as long as $h_H > \frac{c}{24} > h_L$. We present evidence that this decay persists beyond the exponentially long timescale $\sim e^{S}$, so we believe that it represents the true asymptotic behavior of the heavy-light blocks.

From the point of view of the $\frac{1}{c} \propto G_N$ expansion, the universal late-time power-law decay comes from non-perturbative effects. If this behavior persists to all times, as our empirical
evidence indicates, then the late time behavior of CFT$_2$ correlators must come from an
infinite sum over Virasoro blocks in the heavy-light channel.$^1$

From a pure CFT perspective, the late Lorentzian time behavior represents a new limit in
which the bootstrap may be analytically tractable. Most analytic bootstrap results, including
the Cardy formula [83], OPE convergence [84], and the lightcone OPE limit [85, 86] arise
in a similar way. In fact, because the expansion of CFT$_2$ correlators in the uniformizing
$q$-variable, defined in (3.2.2), converges everywhere, including in deeply Lorentzian regimes,
it affords the opportunity to explore many new ‘analytic bootstrap’ limits.

Forbidden Singularities and Bulk Reconstruction

It is interesting to understand when exact CFT correlators differ markedly from predictions
obtained from a semiclassical AdS description. The late time regime we discussed above
provides one example of this phenomenon. As we discuss here and in section 3.4, there are
also Euclidean regimes where the semiclassical approximation to the Virasoro blocks fails

---

$^1$In the $O_H O_L \rightarrow O_H O_L$ OPE channel, the late time behavior can be understood from the discreteness
of the spectrum, without including states with energies $E \gg h_H$. It appears that in the $O_L O_L \rightarrow O_H O_H$
channel one needs to include states of arbitrarily high energy.
Correlation functions in CFT\textsubscript{2} must be non-singular away from the OPE limits where local operators collide [8, 12]. The Virasoro conformal blocks must have this same property [9]. But in the semiclassical approximation, the blocks develop additional ‘forbidden singularities’ [12] that represent a violation of unitarity. These singularities are a signature of semiclassical black hole physics in AdS\textsubscript{3}. They arise because thermal correlators exhibit a Euclidean-time periodicity under $t \rightarrow t + i\beta$, and so the OPE singularities have an infinite sequence of periodic images. The exact Virasoro blocks are not periodic, but in the semiclassical approximation they develop a periodicity at the inverse Hawking temperature $\beta = \frac{1}{T_H}$ associated with a BTZ black hole in AdS\textsubscript{3}.

By studying the Virasoro vacuum block in the vicinity of potential forbidden singularities, one can show that at finite $c$ the singularities are resolved in a universal way [12] via an analytic computation. This method predicts the kinematic regimes where non-perturbative effects should become important; it can be extracted from equation 3.4.2 and the results are displayed in figure 3.17. Thus it is interesting to investigate the divergence between the exact blocks and their semiclassical approximation more generally. We study this question numerically in section 3.4.

Discrepancies between exact and semiclassical CFT correlators near the forbidden singularities could have implications for the reconstruction of AdS dynamics. Bulk reconstruction in black hole backgrounds is rather subtle [87, 88, 89, 90], and perhaps requires some understanding of the analytic continuation of CFT correlators. But there is also a very simple and physical reason to expect that the analytic properties of correlators could have something to do with black hole interiors [87].

As emphasized by Raju and Papadodimas [91, 92], a field operator behind the horizon consists of both ingoing and outgoing modes, but only the ingoing modes can be immediately associated with local CFT operators. This issue is portrayed in figure 3.2. The analytic continuation of local operators by $t \rightarrow t + i\frac{\beta}{2}$ provides a naive, instrumental source for the
outgoing modes.\(^2\) Thus it is natural to ask whether the exact and semiclassical correlators differ significantly at \(t + \frac{i\beta}{2}\), which is 'halfway' to the first forbidden singularity.

We will observe in section 3.4 that the exact and semiclassical correlators behave very similarly at these points, though they seem to differ markedly both very near (within \(\frac{1}{\sqrt{\epsilon}}\)) and beyond the first forbidden singularity. The results can be seen in figure 3.15. As previously discussed [12], we expect that Stokes and anti-Stokes lines emanate from the locations of the forbidden singularities, so that different semiclassical saddle points dominate in different regions of the \(q\)-unit disk. It appears that different saddles dominate as we cross the locations of the forbidden singularities, so that the naive semiclassical blocks (the saddles that dominate near \(q = 0\)) are not a good approximation beyond the first singularity. In fact the semiclassical approximation appears to break down in a finite kinematic region, as shown in figure 3.17. Furthermore, the existence of such regions seems to depend in an essential way on the presence of a black hole, i.e., a state with energy above the Planck scale \((h_H > \sqrt{\epsilon})\), as semiclassical/exact agreement is excellent when \(h_H < \sqrt{\epsilon}\), as we see in figure 3.18.

Perhaps future investigations will uncover bulk observables that are sensitive to Stokes phenomena in the large \(c\) expansion of the Virasoro blocks. We hope that the black hole information paradox can be understood with more precision and detail through such calculations. This work takes steps in that direction by identifying new kinematic regimes where the semiclassical limit breaks down badly and by providing results for the correct non-perturbative Virasoro blocks.

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\(^2\)This idea has significant problems. Although it may be applied to single-sided black holes, which are our object of study, it cannot then apply to the case of eternal black holes involving two different entangled CFTs. But even in the single-sided case, there is a problem because the ingoing and outgoing modes must commute, yet this property may fail when we use \(\mathcal{O}(t)\) and \(\mathcal{O}(t + \frac{i\beta}{2})\) for the ingoing and outgoing modes [91, 93]. It can be imposed by fiat if we take an appropriate linear combination of correlators with different analytic continuations. But this seems to require a form of state-dependence. We have discussed this procedure rather than e.g., mirror operators [91, 92] because it is easier to define in an unambiguous way. We thank Suvrat Raju and Daniel Harlow for correspondence on these issues.
3.2 Kinematics, Convergence, and the Semiclassical Limit

A great deal of information about the behavior of CFT\textsubscript{2} correlation functions is encoded in the structure of the Virasoro conformal blocks. We are interested in 4-pt correlators of primary operators, which can be written as

\[
\langle \mathcal{O}_1(0)\mathcal{O}_2(z)\mathcal{O}_3(1)\mathcal{O}_4(\infty) \rangle = \sum_{h \bar{h}} P_{h,\bar{h}} \mathcal{V}_{h,\bar{h},c}(z) \mathcal{V}_{\bar{h},h,c}(\bar{z})
\]

(3.2.1)

where the \( P_{h,\bar{h}} \) are products of OPE coefficients. The \( \mathcal{V}_{h,\bar{h},c}(z) \) are the holomorphic Virasoro blocks, which will be the main object of study in this work. The blocks are uniquely fixed by Virasoro symmetry and depend only on the external dimensions \( h_i \), the exchanged primary operator dimension \( h \), and the central charge \( c \). Often it will convenient to write \( z = \frac{4\rho}{(1+\rho)^2} \) so that the full \( z \)-plane lies inside the \( \rho \) unit circle \[84\]. The Virasoro blocks are not known in closed form, but they can be computed order-by-order in a series expansion using recursion relations. We provide a brief summary here, leaving the details to appendix B.1.

There are two versions of the Zamolodchikov recursion relations (for a nice review see \[38\]). The first \[37\] is based on writing \( \mathcal{V}_{h,\bar{h},c}(z) \) as a sum over poles in the central charge \( c \), plus a remainder term that survives when \( c \to \infty \) with operator dimensions fixed. The second \[36\], which is more powerful, arises from expanding the blocks as a sum of poles in the intermediate dimension \( h \) plus a remainder term that survives as \( h \to \infty \). The remainder term can be computed from the large \( h \) limit of the Virasoro blocks \[79, 36\]. This large \( h \) limit of the blocks takes a simple form when written in terms of the uniformizing variable

\[
q(z) = e^{i\pi \tau(z)} \equiv e^{-\frac{\pi K(1-z)}{K(z)}}
\]

(3.2.2)

where \( K(z) \) is the elliptic function

\[
K(z) = \frac{1}{2} \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-zt)}}
\]

(3.2.3)
The $q$-coordinate can be derived from the accessory parameter/monodromy method in the semiclassical limit [40] or from a quantization of the theory on the pillow metric [8]. It has the remarkable feature that $q(z)$ covers the full multisheeted $z$-plane (the sphere with punctures at $0, 1, \infty$), as depicted in figure 3.3. The Virasoro blocks can then be written in the form

$$\mathcal{V}_{h,h_i,c}(z) = (16q)^h \frac{e^{-c}}{24} z^{\frac{e-1}{24}} e^{-h_1-h_2} (1-z)^{\frac{e-1}{24}} e^{-h_2-h_3} \left[ \theta_3(q) \right]^{\frac{e-1}{2}} - 4 \sum_{i=1}^4 h_i H(c, h, h_i, q) \quad (3.2.4)$$

where $H(c, h, h_i, q)$ can be obtained from the recursion relation:

$$H(c, h, h_i, q) = 1 + \sum_{m,n=1}^{\infty} q^{mn} R_{m,n} H(c, h_{m,n} + mn, h_i, q) \quad (3.2.5)$$

We note that this recursion relation naturally produces a series expansion in the variable $q$. For more details along with the definitions of the quantities appearing in these equations see appendix B.1.

In this work, we will be using the recursion relations to obtain the $q$-expansion of the Virasoro blocks to very high orders. It appears that most prior implementations of the Zamolodchikov recursion relations could not reach the $N \sim 1000$ that we will study.\(^3\) Our improvements are fairly elementary, and are based on computing and storing the specific coefficients of powers of $q$ in $H(c, h_{m,n} + mn, h_i, q)$, as we describe in more detail in appendix B.1. The computational time complexity of our algorithm is roughly $O(N^3(\log N)^2)$, while it seems that some earlier implementations scaled exponentially with $N$. The maximum $N$ is limited by memory consumption, with memory usage scaling roughly as $O(N^3 \log N)$. We have verified our code by comparing to a number of previous results, including prior implementations, the semiclassical blocks, blocks computed by brute force from the Virasoro algebra, and the special case of degenerate external operators.

\(^3\)Prior implementations such as this code [39] and other similar, modestly improved versions we are aware of. Perhaps [94] are using roughly the same algorithm we describe. We have only used laptops; one could perhaps achieve $N \sim 10^4$ with more computing power.
Figure 3.3: The $q(z)$ map takes the universal cover of the $z$-plane (the sphere with punctures at $0, 1, \infty$) to $|q| < 1$. This figure suggests the relationship between the $z$ plane, the unit $\rho$ disk, and the unit $q$ disk, with branch cuts indicated with colored lines [8]. The relations between these variables are $q = e^{-\pi \frac{K(1-z)}{K(z)}}$ and $z = \frac{4\rho}{(1+\rho)^2}$, and the inverse transformations are $z = \left(\frac{\theta_2(q)}{\theta_4(q)}\right)^4$ and $\rho = \frac{z}{(1+\sqrt{1-z})^2}$. The Virasoro blocks converge throughout $|q| < 1$, with OPE limits occurring on the $q$ unit circle.

### 3.2.1 Kinematics and Convergence of the $q$-Expansion

Both the correlator and the Virasoro blocks in equation (3.2.1) can have singularities in the OPE limits, which occur when $z \to 0, 1, \infty$. Generically we expect branch cuts in the $z$-plane running between these three singularities. So for our purposes, the most remarkable feature of the variable $q(z)$ is that the region $|q| < 1$ covers not only the complex $z$-plane, but also every sheet of its cover. The relationship of the $z$ plane and its branch cuts to the region $|q| < 1$ [8] is depicted in figure 3.3. The Zamolodchikov recursion relations provide an expansion for the Virasoro blocks that converges for all $|q| < 1$, which means that they can provide a good approximation to the 4-pt correlator in any kinematic configuration. In particular, we can use the $q$-expansion to study the Lorentzian regime with arbitrary time-orderings for the operators.

The existence of the $q$-variable implies that in CFT$_2$, there are an infinite number of distinct regimes where the bootstrap equation may be analytically tractable. In the case of
\(d \geq 3\), one can study the OPE limit \(z \to 1\) using conformal blocks expanded in the OPE limit of small \(z\), and this implies various exact results about the properties of large spin operators [85, 86, ?]. However, because the Euclidean OPE in \(d \geq 3\) does not converge deep in the Lorentzian region, one cannot study other OPE channels in the same way. This obstruction disappears in \(d = 2\), where one must be able to reproduce all of the distinct OPE limits \(|q| \to 1\) pictured in figure 3.3 using the small \(q\) expansion. The large Lorentzian time limit that we will discuss in section 3.3 provides a physically motivated example of this idea.

We will be studying numerical approximations to the Virasoro blocks based on a large-order expansion in the \(q\) variable. Thus to understand the convergence properties of our expansion, it may be useful to map out the regions of constant \(|q|\). For this purpose we can use the coordinate \(\rho(z)\) defined via \(z = \frac{4\rho}{(1+\rho)^2}\) [84], because the entire \(z\)-plane can be easily visualized as the region \(|\rho| < 1\). The operators at \(z = 1\) and \(\infty\) are mapped to \(\rho = 1\) and \(-1\), respectively. In figure 3.4 we have plotted contours of constant \(|q|\) in the \(\rho\)-coordinate system. In figure 3.5 we present results on the convergence region of the \(q\)-expansion of the Virasoro blocks for various values of the dimensions and central charge.

A kinematic configuration that will be of particular interest represents \(z = 1 - re^{-it}\) (and \(\bar{z} = 1 - re^{-it}\) as well) and is depicted in the AdS/CFT context in figure 3.1. With this setup we can study the correlator of light operators \(\mathcal{O}_L(z)\mathcal{O}_L(0)\) at timelike separation in the background created by a heavy operators \(\mathcal{O}_H\). At large times \(t\), this correlator can be used as a probe of information loss in pure state black hole backgrounds, as we will discuss in section 3.3. On the \(z\) plane, the late time behavior is obtained by analytically continuing the conformal block around the branch-cut starting at \(z = 1\) multiple times. Explicitly, the Lorentzian value of the \(q\) variable is obtained with the following analytic continuation of the elliptic integral:

\[
K|_{z \to 1-e^{-it}} = K(1 - re^{-it}) - 2i \left[ \frac{1}{2} - \frac{t}{2\pi} \right] K(e^{-it}),
\]

where the elliptic functions on the right-hand side are evaluated on the principle sheet with
Figure 3.4: This figure displays contours of constant $|q|$ inside the $\rho$ unit circle, which corresponds to the entire $z$-plane via $z = \frac{4\rho}{(1+\rho^2)}$. Since this is only the first sheet of the $z$-plane, it corresponds to the region in the $q$-disk enclosed by the two blue lines connecting $\pm 1$ in figure 3.3. The correlator can have singularities in the OPE limits $\rho \to -1, 1$ and these correspond to $q \to -1, 1$ as well. Away from these limits $|q| < |\rho|$ and the $q$-expansion converges much more rapidly than the $\rho$ expansion.

the branch-cut chosen to be $z \in [1, \infty)$. At large $t$ we have

$$q(t) \approx 1 + \frac{i\pi^2}{t} - \frac{\pi^4 + 2\pi^3 g(r,t)}{2t^2} + \cdots \quad (3.2.7)$$

where

$$g(r,t) = \frac{K (1 - e^{-it}r)}{K(e^{-it}r)} + 2i \left[ \frac{t + \pi}{2\pi} \right] - \frac{it}{\pi} \quad (3.2.8)$$

with the elliptic function $K(z)$ are taken on their principle sheet, so that $g(r,t)$ is periodic in $t$. This means that $|q|^2 \approx 1 - \frac{\pi^2}{t^2}(g + g^*) + \cdots$ and the real part Re$[g(r,t)] > 0$, so that $|q| < 1$ for all $t$, as expected. In the limit that $r \ll 1$, we have $g(r,t) \approx \frac{1}{\pi} \log \frac{16}{r}$, which leads to the estimate

$$|q|^2 \approx 1 - \frac{2\pi^2 \log \frac{16}{r}}{t^2} \quad (3.2.9)$$
Figure 3.5: These plots display the maximum $|q|$ where the $q$-expansion converges for various choices of parameters. Convergence improves when $h_L$ and $h_H$ move closer to $c/24$ and when $c$ decreases. The intermediate primary dimension $h$ seems to have little effect on convergence. These plots define ‘convergence’ as $\left| \frac{V_{0.95N}(q)}{V_N(q)} - 1 \right| < 10^{-5}$, where $V_M$ includes an expansion up to order $q^M$.

in the limit of $r \ll 1$ and $t \to \infty$. Thus we can translate between convergence in $|q|$ and $t$; very roughly, we expect that working to order $q^N$ will allow us to probe $t \propto \sqrt{N}$ at large $N$. We can visualize the trajectory of $q(r,t)$ for various $r$ in figure 3.6.

### 3.2.2 Review of Blocks and Adherence to the Semiclassical Form

Much is known about the Virasoro blocks in various limits. In the limit $c \to \infty$ with all dimensions held fixed, the Virasoro blocks simply reduce to global conformal blocks, which are hypergeometric functions. Corrections to this result up to order $1/c^3$ are known explicitly [1]. In the heavy-light limit, where we take $c \to \infty$ with two ‘heavy’ operator dimensions $h_H \propto c$, and the two ‘light’ dimensions $h_L$ and the intermediate operator dimension $h$ fixed,
the blocks take the form \[ \mathcal{V} = (1 - w)^{h_L \alpha_H^{-1}} \left( \frac{w}{\alpha_H} \right)^{h - 2 h_L} {}_2F_1(h, h, 2h, w) \] (3.2.10)

where \( w \equiv 1 - (1 - z)^{\alpha_H} \) and \( \alpha_H \equiv \sqrt{1 - \frac{24 h_H}{c}} \). Note that when \( h_H > \frac{c}{24} \), we have \( \alpha_H = 2\pi iT_H \) where \( T_H \) is the Hawking temperature of a corresponding BTZ black hole. In the case of the vacuum block, which is \( h = 0 \), the \( 1/c \) corrections to this limit are also known explicitly [20] for any \( h_H/c \). Finally, in the semiclassical large \( c \) limit, where all dimensions \( h_i, h \propto c \), there is overwhelming evidence that the blocks take the form

\[ \mathcal{V} = e^{-\frac{\pi}{\alpha_H} \left( \frac{h_H}{c} \frac{h}{c} z \right)} \] (3.2.11)

as though they are derived from a semiclassical path integral (and in fact they have an \( \text{sl}(2) \) Chern-Simons path integral representation [77]). The semiclassical saddle points have been
classified [9], and in some kinematic limits we can determine the behavior of \( f \) analytically. In particular, the large Lorentzian time behavior of \( f \) with the kinematics of figure 3.1 and \( h_L < \frac{c}{24} < h_H \) has been determined [9]. The result is that the leading semiclassical contribution always decay exponentially at sufficiently large times\(^4\) at the rate

\[
\mathcal{V}(t) \approx \exp \left[ -\frac{c}{12} |\alpha_H|(1 - \alpha_L) |t| \right]
\]

where \( \alpha_L = \sqrt{1 - \frac{24h_L}{c}} \) and \( \alpha_H = 2\pi i T_H \) with \( T_H \) the corresponding Hawking temperature. As we will review in section 3.3, this demonstrates that information loss due to black hole physics [81] occurs as a consequence of the behavior of the individual Virasoro blocks [12, 9]. Finally, some exact information about the behavior of the Virasoro blocks can be obtained by studying degenerate states [12].

Most of these approximations hold in the large central charge limit when the kinematic configuration is held fixed. But the deviations between the exact and semiclassical Virasoro blocks may depend importantly on the kinematics. As we will discuss in detail below, the semiclassical blocks have ‘forbidden singularities’ that are absent from the exact blocks [12]. We also find that as expected [12, 9], the exact and semiclassical blocks have very different behavior at large Lorentzian times. More generally, we would like to map out the kinematic regimes where non-perturbative corrections to the semiclassical Virasoro blocks become large.

But at a more basic level, it is interesting to ask how large \( c \) must be before the semiclassical limit of the Virasoro blocks provides a reasonable approximation to their behavior. This has immediate implications for the possibility of constructing a 2d CFT and probing quantum gravity in an experimental lab. A natural way to probe the existence of the semiclassical limit is by studying the ratio of logarithms of blocks

\[
R \equiv \frac{c_2 \log \mathcal{V}(c_1, q)}{c_1 \log \mathcal{V}(c_2, q)} \equiv 1
\]

\(^4\)As we increase the intermediate operator dimension this behavior may not set in until later and later times. Here we are studying late times with all other parameters held fixed.
Figure 3.7: In this figure we plot $R = \frac{c'}{c \log V(c,q)}$ with $c' = \frac{11}{10} c$ in order to test the semiclassical limit. As we increase $c$, the semiclassical limit becomes a better approximation and $R \to 1$, but even for $c = 2.1$ the blocks are remarkably well approximated by the semiclassical form. For the larger choices of $c$ the functions have similar shapes up to an overall rescaling; this suggests that the first $1/c$ correction is dominating the discrepancy $R - 1$. In the OPE limit $q \to 0$ the semiclassical limit always applies. We find similar results for non-vacuum blocks.

at somewhat different central charges $c_1$ and $c_2$. If the semiclassical limit of equation (3.2.11) is a good approximation, then this quantity will be 1, but otherwise we expect it to deviate from 1 by effects of order $\frac{1}{c}$. In figure 3.7 we explore this ratio and find that the semiclassical form $V \approx e^{cf}$ provides a remarkably good approximation for very small values of $c$.

There is an important caveat that we will return to in section 3.4. An infinite number of distinct semiclassical saddle points can contribute to the Virasoro blocks in the large $c$ limit [9]. Thus it is possible that $V \approx e^{cf}$ for some $f$, but that due to Stokes phenomena, the dominant saddle $f$ changes as we move in the $q$ unit disk. So although the semiclassical limit may appear to describe the blocks well for all $q$, as indicated by figure 3.7, in fact the saddle that is leading near $q \approx 0$ may be sub-leading at general $q$. Thus the naive semiclassical blocks may differ greatly from the exact blocks; in fact we will find this to be the case in section 3.4.

Nevertheless, figure 3.7 suggests that we should be optimistic about probing semiclassical CFT$_2$ correlators in the lab! It would be very interesting to engineer a CFT$_2$ with $c > 1$ and no conserved currents aside from the stress tensor [80].
3.3 Late Time Behavior and Information Loss

One sharp signature of information loss in AdS/CFT [81] is the exponential decay of correlation functions at large time separations in a black hole background. This can be studied using heavy-light 4-point functions in CFT [15]. As portrayed in figure 3.1, we can interpret this correlator as the creation and subsequent measurement of a small perturbation to an initial high-energy state. In a unitary theory on a compact space with a finite number of local degrees of freedom, this initial perturbation cannot completely disappear. But a computation in the black hole background displays eternal exponential decay, capturing the physical effect of the signal falling into the black hole. At a more technical level, the exponential decay rate can be obtained from the quasinormal mode spectrum of fields propagating in the black hole geometry.

The simplest way to see that heavy-light correlators cannot decay forever is to expand in the $\mathcal{O}_H \mathcal{O}_L \rightarrow \mathcal{O}_H \mathcal{O}_L$ channel, giving

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(t)\mathcal{O}_L(0)\mathcal{O}_H(-\infty) \rangle = \sum_E \lambda^2(E)e^{iEt}$$  \hspace{1cm} (3.3.1)

where $\lambda^2(E)$ is a product of OPE coefficients. Because the sum on the right-hand side is discrete, the correlator must have a finite average absolute value at late times. When $h_H \gtrsim \frac{c}{2\pi} \gg h_L$, we expect the states contributing in (3.3.9) to be a chaotic collection of $e^S$ blackhole microstates with energy near that of $\mathcal{O}_H$, and with $S = \frac{\pi^2}{3}T_Hc$. The amplitude will initially decay due to cancellations between the essentially random phases, but these cancellations cannot become arbitrarily precise. Roughly speaking, the decay should stop when the correlator reaches $\sim e^{-S}$ and begins to oscillate chaotically. At a more detailed level, the time dependence can change qualitatively on timescales of order $S$ and $e^S$ as different features of $\lambda^2(E)$ come into play [95, 96, 97, 98].

In this work, we will not study the $\mathcal{O}_H \mathcal{O}_L \rightarrow \mathcal{O}_H \mathcal{O}_L$ channel directly. Instead we work in the channel where $\mathcal{O}_H \mathcal{O}_H \rightarrow \mathcal{O}_L \mathcal{O}_L$, which is related to the first channel by the bootstrap
equation (or by modular invariance in the case of the partition function [97]). In this channel we are sensitive to the exchange of states between the heavy and light operators. For example, pure ‘graviton’ states in AdS$_3$ correspond to the exchange of the Virasoro descendants of the vacuum, which are encapsulated by the Virasoro vacuum block. Other heavy-light Virasoro blocks include a specific primary state along with its Virasoro descendants, which one can think of as gravitational dressing. We are interested in this channel because heavy-light Virasoro blocks encode many of the most interesting features of semiclassical gravity. We would like to understand to what extent the exact Virasoro blocks know about the resolution of information loss.

It is convenient to think of the time dependence of the Virasoro blocks as coming from a potentially continuous $\lambda_h(E)$ associated with each block, via

$$V_h(t) = \int dE \lambda_h^2(E) e^{iEt} \quad (3.3.2)$$

where $h$ labels the dimension of an intermediate Virasoro primary operator $\mathcal{O}_h$ in both the $\mathcal{O}_H(x)\mathcal{O}_H(0)$ and $\mathcal{O}_L(x)\mathcal{O}_L(0)$ OPEs. Roughly speaking, the late time dependence of $V_h(t)$ will come from the least analytic features of $\lambda_h^2(E)$.

For example, in the leading semiclassical limit, heavy-light Virasoro blocks decay exponentially at late times at a universal rate given in equation (3.2.12). This semiclassical behavior comes from a function $\lambda_h^2(E)$ that is smooth on the real axis, but has poles in the complex $E$-plane. In AdS$_3$ these poles can be interpreted as the quasinormal modes of a BTZ black hole background (at least for small $h$). A straightforward contour deformation of equation (3.3.2) connects these poles to the exponential decay.

At sufficiently late times, the physics of the quasinormal modes will be subdominant to less analytic features in $\lambda_h^2(E)$. For example, if $\lambda_h^2(E)$ exhibits thresholds of the form $(E - E_*)^{p-1}$ with $E_*$ real, then $V(t)$ will inherit a power-law behavior $t^{-p}$ at late times. And if $\lambda_h^2(E)$ receives delta function type contributions, then $V(t)$ will have a finite average
absolute value at late times. If such features are present in \( V_h(t) \), then it is natural to investigate the timescale where \( V_h(t) \) transitions from exponential decay to some other late-time behavior.

The full CFT\textsubscript{2} correlator should not become much smaller than \( \sim e^{-S} \). Since Virasoro blocks associated with light operators initially decay exponentially, one might naively expect that \( V_h(t) \) should change qualitatively after a time of order \( S \). More specifically, for heavy-light correlators dominated by the vacuum block, we would expect a departure from exponential decay by a time

\[
t_D = \frac{\pi c}{6h_L}
\]  

up to an unknown order one factor. This argument is rather weak, since the full correlator might not behave like the light-operator Virasoro blocks. However, the same prediction for \( t_D \) was derived from an analysis of non-perturbative effects \cite{12} in the vacuum block. We discuss the equation that led to that prediction in section 3.4.3.

We will see empirically that Virasoro blocks with small \( h \) do undergo a transition at a timescale remarkably close to \( t_D \). Furthermore, at late times the behavior of the heavy-light Virasoro blocks appears to be a universal power-law:

\[
|V_{h_L,h_H,h,c}(t \gg t_D)| \propto t^{-\frac{3}{2}},
\]  

where we require \( h_H \geq \frac{1}{24} \), so that at least one external operator is heavy enough to create a blackhole. When the intermediate dimension \( h \gtrsim h_H \) the late time power-law behavior remains the same, although the transition time then also depends on \( h \) (and we do not have an analytic prediction to compare to). This universal behavior suggests a threshold \( \sqrt{E - E_\ast} \) in \( \lambda^2(E) \), which seems to correspond with random matrix behavior \cite{99, 96, ?}. Our results indicate that the \( t^{-\frac{3}{2}} \) power-law persists to timescales \( \sim e^S \), so individual heavy-light Virasoro blocks are not sensitive to the discreteness of the spectrum.
These results show that the time-dependence of the heavy-light Virasoro blocks has some qualitative similarities with that of the Virasoro vacuum character after an $S$ transformation and the analytic continuation $\beta \rightarrow \beta + it$ [97]. Both the heavy-light blocks with small $h$ and the vacuum character have an initial exponential-type decay, though the precise time-dependence is rather different. The heavy-light blocks and the vacuum character have the same power-law decay at late times, though non-vacuum characters decay with a different late-time power-law [97].

In what follows we will study the heavy-light blocks $V_h(t)$ empirically to establish the robust features of their time-dependence. We also translate the late-time $t^{-3/2}$ behavior into a statement about the coefficients of $q^N$ in $V_h(q)$ at large orders in the $q$-expansion, as one might hope to derive this asymptotic behavior for the coefficients using the Zamolodchikov recursion relations. One might also compute $\lambda^2_h(E)$ directly using the crossing relation [100, 101]. Finally we discuss the implication of our results for the late time behavior of the correlator.

### 3.3.1 Numerical Results and Empirical Findings

#### 3.3.1.1 Vacuum Virasoro Blocks

Using the methods discussed in section 3.2, we compute the vacuum Virasoro blocks at late times. Figure 3.8 shows the result along with a comparison to the semiclassical blocks computed using semi-analytic methods [9]. For numerical convenience we avoid certain rational values of $c$ to prevent singularities in intermediate steps of the computation.

Using the numerical result of the full Virasoro blocks, we can measure the departure time $t_d$ when the semiclassical block drops below the exact block. We compare this measured value to the prediction of (3.3.3) in figure 3.9. The logic leading up to (3.3.3) is only valid parametrically, so it is remarkable that it agrees with the measured $t_d$ up to a small constant shift. Note that we parameterize the time dependence via $z = 1 - re^{-it}$, and this constant shift depends on $r$. We have also checked that $t_d$ is primarily controlled by the ratio $\frac{h_c}{c}$ and
Figure 3.8: Heavy-light Virasoro vacuum blocks switch from an initial exponential decay to a slow, universal power law decay at roughly the time scale $t_d = t_D - b$, where the constant offset $b$ depends on the choice of $r$ in $z = 1 - re^{-it}$. The vertical axis is $\log |\mathcal{V}|$, while the horizontal axis is the Lorentzian time $t$. The black lines are full Virasoro vacuum blocks computed to order $q^{1200}$. This polynomial truncation stops converging in the shaded region. The yellow dashed lines are the semiclassical vacuum blocks using methods of [9]. The red dashed lines are the time scale (3.3.3). The blue dashed lines are the power law at $-\frac{3}{2}$ with $a$ properly chosen to match the full blocks.

has a very weak dependence on $h_H$ and $c$.

Around the time $t_D$, all vacuum blocks show an obvious change of behavior from an initial exponential decay to a much slower power law decay. To very good accuracy, the power of this decay seems to be $t^{-\frac{3}{2}}$ universally in all of the parameter space we were able to explore with an external operator with dimension $h_H > \frac{1}{24}$. A few examples are provided in figure 3.8, but we tested this behavior with hundreds of different parameter choices.
Figure 3.9: This figure displays the time $t_d$ at which the semiclassical vacuum blocks drop below the exact vacuum blocks. The dashed line is a fit to the analytic prediction $t_D \equiv \frac{\pi c}{6h_L}$ with an empirical offset $t_d = t_D - 2.6$; the offset depends on the choice of $r$ with $z = 1 - re^{-it}$. Note that the data with smaller values of $c$ is noisy, but the larger values fit the linear behavior extremely well. The plot includes a variety of choices for $\frac{h_H}{c}$.

### 3.3.1.2 General Virasoro Blocks

The non-vacuum blocks also exhibit universal $t^{-\frac{3}{2}}$ late-time decay. The difference from the vacuum case is that we no longer have a simple estimate for the time scale of the transition. In particular, we find that generically the non-vacuum blocks grow at early times, reach a maximum at time $t_{\text{max}}$, and then start to decay, finally settling down to the $t^{-\frac{3}{2}}$ power law behavior. These features are illustrated by examples in figure 3.10.

From the data plotted in figure 3.11, we see that beyond the blackhole threshold $h > \frac{c}{24}$, the timescale $t_{\text{max}}$ has a simple dependence on parameters. We can fit it to the ansatz

$$t_{\text{max}} = A_t |\alpha_h| + b_{\text{time}}$$

with $\alpha_h = \sqrt{1 - \frac{24h}{c}}$ and obtain $A_t$ and $b_{\text{time}}$ empirically. The parameter $A_t$ is almost a linear function of $\frac{c}{h_H}$, as can be seen in figure 3.12, with virtually no dependence on other parameters such as $h_L$. It approaches $A_t \approx \frac{c}{2h_H} + \text{constant}$ when $h_H \gtrsim \frac{c}{2}$. For smaller values of $h_H$ we find $\frac{1}{2} \geq \frac{dA_t}{d(c/h_H)} \geq \frac{1}{5}$. We cover a larger range of $h_H$ in figure B.4 in the appendix, which displays the variation in $A_t$. 

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Figure 3.10: The late time behavior of various non-vacuum Virasoro blocks. The vertical axis is $\log |\mathcal{V}|$ and the horizontal axis is the time $t$. The black lines are full Virasoro blocks computed to order $q^{1200}$, plotted using $z = 1 - re^{-it}$ with $r = 0.3$. The polynomial truncation no longer converges in the shaded region. The blue dashed lines are the power law $a t^{-\frac{1}{2}}$ with the constant $a$ fitted to the blocks. We refer to the time and height of the maxima as $t_{\text{max}}$ and $|\mathcal{V}|_{\text{max}} = 16^{h - \frac{c}{c} - 1} |\tilde{\mathcal{V}}|_{\text{max}}$.

On the left of figure 3.11 we plot $|\tilde{\mathcal{V}}|_{\text{max}}$, which is the maximum of the absolute value of the block after extracting a universal prefactor via $|\mathcal{V}|_{\text{max}} = 16^{h - \frac{c}{c} - 1} |\tilde{\mathcal{V}}|_{\text{max}}$. We see that $|\tilde{\mathcal{V}}|_{\text{max}}$ also has a simple dependence on $\frac{h}{c}$. We can perform a similar fit for $|\tilde{\mathcal{V}}|_{\text{max}}$, and we find that

$$
\frac{\log |\tilde{\mathcal{V}}|_{\text{max}}}{c} = a_{\text{height}} \frac{h_H}{c} \log \frac{h}{c} + b_{\text{height}}
$$

(3.3.6)

Empirically we obtain $a_{\text{height}} \approx -2.5$ from the fit in figure 3.12. The $b_{\text{time}}$ and $b_{\text{height}}$ parameters do not fit a simple pattern; we provide some data on these parameters in figure B.3 in the appendix. These fits led to the result summarized by equation (3.1.5) in the introduction, which neglects the small offsets from the $b$-parameters. We expect that $|\mathcal{V}|_{\text{max}}$ and $t_{\text{max}}$ are controlled by semiclassical physics (for example, see figure B.2), so it would be
interesting to try to prove these empirical relations using analytic results [9] on the semi-classical time-dependence. In principle these results could also be obtained from an AdS calculation involving black holes and deficit angles.

### 3.3.1.3 Probing Exponentially Large Timescales

Formally, we are interested in high-energy pure states corresponding to BTZ black holes, which have a large entropy $S = \frac{\pi^2}{3} c T_H$ in the large central charge limit where AdS gravity provides a reliable description. This suggests that timescales of order $e^S$ will be unreachably large. Nevertheless, by considering either small $c$ or small $T_H$, we can probe order one $S$, and thus reach $t \sim e^S$ within the range of convergence of our numerics.

In fact, the plot on the bottom-right of figure 3.8 is already in this regime. Due to its low temperature of $T_H \approx 0.03$ in AdS units, we have $S \approx 3.3$ so that times of order $e^S \approx 27$ are within the range of convergence. Thus this plot already suggests that the $t^{-\frac{3}{2}}$ power-law decay persists to exponentially large timescales. In figure 3.13 we have displayed four other choices of parameters where timescales of order $e^S$, and even $e^{es}$, are visible within the range of convergence. Two examples have order one $T_H$ and small $c$, while two others have very
Figure 3.12: We have found empirically that the time and height of the maxima of heavy-light Virasoro blocks have a simple dependence on both $h$ and $h_H$. This figure shows linear fits used to obtain the parameters $a_{\text{height}}$ and $A_t$ defined in equations (3.3.6) and (3.3.5). These plots both have $c = 10$. Each point is obtained from the slope of $\log |\tilde{V}_{\text{max}}|$ and $t_{\text{max}}$ as linear functions of $\log h/c$ and $|\alpha_h|$ respectively (we've used points with $h/c = \frac{n}{3}$ for $n = 1, 2, \ldots, 30$). We find that both plots are robustly $c$-independent for $c \gtrsim 5$, as expected in the semiclassical limit. We see explicitly that there is little dependence on $h_L$; in the $a_{\text{height}}$ plot the variation with $h_L$ is almost invisible.

small $T_H$ and relatively large $c$. In all cases we see that the $t^{-\frac{3}{2}}$ late-time decay persists on these exponentially large timescales. This provides good evidence that the heavy-light Virasoro blocks really do decay in this way at very late times. This means that these blocks are not sensitive to the discreteness of the spectrum in other channels.

3.3.2 Power Law Behavior of $q$-Expansion Coefficients

We have observed an apparently universal late-time power-law behavior in the heavy-light Virasoro blocks $V_h(t)$. One might try to derive this behavior by studying its implications for the $q$-expansion. In fact, for a large region of parameter space, the $t^{-\frac{3}{2}}$ decay translates to a power law growth of the coefficients in the $q$ expansion.

To see this, we note that at late times $q$ approaches 1 with a rate given by (3.2.7). This implies that $\theta_3(q) \sim \sqrt{t}$, which means that the prefactor in (3.2.4) behaves like $t^{\frac{1}{2}}(\frac{c+1}{2} - 8(h_H + h_L))$ at late times. In order to have the entire block decay as $t^{-\frac{3}{2}}$, the polynomial part $H(c, h, h_L, h_H, q)$ must cancel all $c$ and $h_i$ dependence in the prefactor. This
Figure 3.13: These plots show a variety of parameter choices where the behavior of Virasoro blocks on the timescale $e^S$ (green vertical line), and even $e^{e^S}$ (blue vertical line), are visible. Yellow lines indicate semiclassical behavior, while the light blue fit corresponds to $t^{-\frac{3}{2}}$. Recall $S = \frac{c}{3} \pi t$ with $2\pi T_H = \sqrt{\frac{2h_H c}{c} - 1}$, so some plots have relatively large $c$ and small $T_H$, while others have order one $T_H$ but small $c$. In all cases we see that the $t^{-\frac{3}{2}}$ late-time decay persists on these exponentially long timescales. These plots all display vacuum blocks, but we have found similar behavior with $h > 0$.

means:

$$H(t) = \sum_{n=0}^{\infty} c_n g(t)^{2n} \sim t^{4\left(h_H + h_L - \frac{c}{16} - \frac{9}{16}\right)} \quad (3.3.7)$$

A power law in the late time behavior of the $H$ can be directly related to the large order behavior of the $q$-expansion coefficients $c_n$. We find that $c_n \sim n^s$ with

$$s = 4 \left(h_H + h_L - \frac{c}{16} - \frac{9}{16}\right) \quad (3.3.8)$$

where $s$ is the dominant power of the coefficient growth, and we are assuming that $H(t)$ does grow at large $t$, which roughly requires $h_H > \frac{c}{16}$. Examples of this behavior are shown in
figure 3.14. If $H(t)$ decays at late times, then there must be cancellations in the sum over $q^n$, and we cannot predict such a simple power-law.

![Graphs showing the behavior of the coefficients of the $q^{2n}$ term in the polynomial $H$ in (3.2.4) compared to the prediction (3.3.8). The horizontal axis is log $n$ and the vertical axis is log $c_n$, where $c_n$ is the coefficient of $q^{2n}$ in $H$. The red lines are power-laws $an^s$ with the constant $a$ determined by the fit.](image)

So in addition to directly computing the late time values of the Virasoro blocks, we can test whether the blocks follow the $t^{-\frac{3}{2}}$ decay simply by comparing the coefficients of the $q$-expansion of $H$ to the prediction (3.3.8). This is actually a more efficient method that allows us to access certain regimes, such as larger $c$ and $h_H$ of the parameter space where the direct Virasoro block calculation converges poorly.

However, the prediction (3.3.8) is less universal than the $t^{-\frac{3}{2}}$ behavior. For example, outside the regime where $H(t)$ grows, the coefficients $c_n$ can have alternating signs, so that there are large cancellations between different terms in the $q$-expansion. Then the magnitude of the coefficients will no-longer follow the simple pattern depicted in figure 3.3.8. Empirically,
another example is when $\frac{\hbar}{c}$ is small. In this case the coefficients are pretty small and show complicated irregular behaviors. Examples can be see in figure B.5 in the appendix. Yet in all cases the overall late time behavior of the heavy-light Virasoro blocks is still the $t^{-\frac{3}{2}}$ power law.

One would hope to derive the power-law behavior $c_n \sim n^s$ using the Zamolodchikov recursion relations. Unfortunately, it appears that this behavior arises from a large number of cancellations between much larger terms. Thus we leave this problem to future work.

### 3.3.3 Implications for Information Loss and the Bootstrap

In the semiclassical limit, heavy-light Virasoro blocks decay exponentially at late times. We do not expect that perturbative corrections in $G_N = \frac{3}{2c}$ will alter this conclusion, and to first order this has been demonstrated explicitly [20]. Thus the late time power-law behavior of the exact blocks represents a non-perturbative correction that ameliorates information loss (insofar as information loss is tantamount to late-time decay). However, since the Virasoro blocks continue to decay, albeit much more slowly, this effect does not solve the information loss problem. For this we need an infinite sum over Virasoro blocks in the $O_L O_L$ OPE channel.\(^5\)

Let us examine the correlator as a sum over blocks from the point of view of the bootstrap equation [25, 26, 27]. This equation dictates that\(^6\)

\[
\langle O_H(\infty) O_L(t) O_L(0) O_H(-\infty) \rangle = \sum_E \lambda^2_{LH}(E) e^{iEt} = \sum_{h,\bar{h}} P_{h,\bar{h}} V_h(t) \bar{V}_{\bar{h}}(t) \tag{3.3.9}
\]

Here we have equated a sum over energies in the $O_H O_L$ OPE channel with a sum over heavy-light Virasoro blocks in the $O_L O_L$ OPE channel. In $d > 2$ dimensions this equation would be

\(^5\)Of course we are assuming that we are dealing with a chaotic large $c$ theory, rather than e.g. a rational CFT. For special values of the external dimensions and $c$, such as those corresponding to degenerate external operators, the individual Virasoro blocks may not decay at late times.

\(^6\)We are being schematic to emphasize the time dependence. One should define $z = 1 - e^{-t+i\phi}$ and $\bar{z} = 1 - e^{-t-i\phi}$ in the Euclidean region, and then analytically continue $t \to it$, so that both channels depend on the coordinates $t$ and $\phi$ pictured in figure 3.1. We are suppressing these details.
meaningless at large $t$, because we would be well outside the regime of convergence of the OPE expansion on the right-hand side. Remarkably, as discussed in section 3.2.1, the Virasoro block decomposition converges for all values of $t$, so it is possible to try to ‘solve’ for the coefficients $P_{h, \bar{h}}$ by equating the large $t$ behavior of both sides. More generally, one could take the limit $|q| \to 1$ with various phases for $q$ and derive new, potentially analytic regimes for the bootstrap (this is non-trivial because it could enable a partial analytic treatment without requiring a complete solution to the bootstrap equation). The only obvious obstruction to this procedure is that we do not have simple analytic formulas for the Virasoro blocks in such limits.

As we have already noted, equation (3.3.9) can only be satisfied at late times if we have an infinite number of Virasoro blocks contributing on the right-hand side. Such infinite sums are compulsory in order to reproduce conventional OPE limits [83, 84, 85, 86]. But it is easy to see that the Cardy formula and the asymptotic expectations on $P_{h, \bar{h}}$ from Euclidean crossing or the light-cone OPE limit are insufficient to account for the late-time behavior. The reason is that conventional arguments require the large $h, \bar{h}$ terms in equation (3.3.9) to reproduce either the identity (vacuum) or perhaps the contribution of low dimension or low twist operators in the crossed channel. These would correspond to the very small $E$ region of $\lambda_{LH}(E)$. But the late time behavior arises from the collective contributions of $\sim e^S$ states with large $E \sim h_H + \bar{h}_H$, not from the small $E$ states.\footnote{Here we are imagining subtracting off the contributions from the expectation values $\langle \mathcal{O}_H | \mathcal{O}_L | \mathcal{O}_H \rangle$. These are generically expected to be exponentially suppressed [102] in holographic CFT$_2$.}

In this regard there is an amusing connection with Maldacena’s original discussion [81] of the large time behavior. He suggested that in a black hole background, contributions from the vacuum, corresponding to the $E = 0$ term in equation (3.3.9), might resolve the information loss problem. But the vacuum in the $\mathcal{O}_H \mathcal{O}_L$ OPE channel just corresponds with the Cardy-type growth (or more precisely OPE convergence [84] type growth) of $P_{h, \bar{h}}$. So this simple OPE convergence growth fails to account for the late time behavior for the same reason that Maldacena’s suggestion did not resolve the information loss problem.
In summary, the late-time bootstrap equation (3.3.9) cannot be solved without providing a more refined asymptotic formula for $P_{h,\bar{h}}$ at large $h, \bar{h}$. However, it does not appear that a discrete spectrum in the $\mathcal{O}_L\mathcal{O}_L$ channel is required to obtain the correct late-time behavior. We will not pursue this in detail since we only have some rough empirical information about the behavior of $\mathcal{V}_h(t)$, but it might be interesting to study this bootstrap equation for the case of the partition function [97] where the Virasoro characters are known in closed form.

3.4 Euclidean Breakdown of the Semiclassical Approximation

3.4.1 Some Philosophy

Eventually, we hope to learn about bulk reconstruction – and its limitations – by comparing exact CFT correlators to their semiclassical approximations. It is not clear whether this is possible, even in principle, due to ambiguities in the reconstruction process associated with bulk gauge redundancies (see e.g. [103] for a recent discussion). For now we will take a very instrumental approach, or in other words, we will try to ‘shut up and calculate’ some potentially interesting observables.

The information paradox pits local bulk effective field theory in the vicinity of a horizon against quantum mechanical unitarity. But in the strict semiclassical limit, information is lost and the (approximate) CFT correlators agree precisely with perturbative AdS field theory or string theory. Thus one would expect that bulk reconstruction should be possible in this approximation, since we have allowed the local bulk theory to ‘win’ the fight, at the expense of unitarity.\footnote{This suggests that solving the reconstruction problem in the strict semiclassical limit should not have much to do with the information paradox or the existence of firewalls [104], except insofar as it is a first step towards the problem of bulk reconstruction from the data and observables of the exact CFT. As an alternative perspective, one might claim that even in the semiclassical limit reconstructing black hole interiors is impossible because firewalls are completely generic.}

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But even in the semiclassical limit, bulk reconstruction has been controversial [87, 88, 89, 90]. On an intuitive level, this is because correlators at infinity must have exponential sensitivity to ‘observe’ physics near or behind a black hole horizon. At an instrumental level, this means that there may be obstructions to the existence of smearing functions mapping boundary to bulk observables. These issues can be avoided by going to momentum space [91, 92], or perhaps via an appropriate analytic continuation [87] or cutoff procedure [90].

Another elementary issue with semiclassical bulk reconstruction is pictured in figure 3.2. The problem is that only the ingoing modes behind the horizon can be reconstructed in an obvious way from the degrees of freedom of a single CFT [87]. This can be understood by considering the extended AdS-Schwarzschild spacetime, or simply by studying Rindler space. Field theory degrees of freedom behind the horizon appear as a linear combination of modes from the left and right ‘wedges’, but in a single-sided black hole, only one asymptotic region is present.

If the goal is simply to compute correlators behind the horizon of a single-sided black hole, then there is a naive, instrumental way to obtain outgoing modes. One can obtain correlators that behave like those of the other asymptotic region by analytically continuing [87] CFT operators $O(t, x)$ in Euclidean time to $\tilde{O}(t, x) \equiv O\left(t + \frac{i\beta}{2}, x\right)$. This procedure has an important flaw – operators on opposite sides of the black hole should commute, but $O$ and $\tilde{O}$ may not. Nevertheless, we can force $O$ and $\tilde{O}$ to commute (by definition) if we choose an appropriate but ad hoc analytic continuation procedure for correlators involving $O$ and $\tilde{O}$. Conceptually, this does not seem to be an improvement on state-dependent mirror operators [91, 92], which represent a modification of quantum mechanics. In fact, our procedure implements its own form of state-dependence, since the analytic continuations will depend on all of the other local operators inserted into the correlator. However, the prescription does have the simple advantage of being relatively precise and unambiguous.

In any case, we are led to a very simple question – do the correlators of operators like $O\left(t + \frac{i\beta}{2}, x\right)$ receive large non-perturbative corrections? Do the semiclassical Virasoro blocks
provide a good approximation to the exact blocks with these kinematics?

3.4.2 Forbidden Singularities and Thermofield Doubles

The questions raised in the previous section can be explored using the methods of this paper. They are also closely related to observations about information loss [12]. Finite-temperature correlation functions must satisfy the KMS condition, which for identical operators just means that $\langle \mathcal{O}(t, x)\mathcal{O}(0) \rangle_\beta$ must be periodic in Euclidean time with period $\beta$. It has been shown that in the large central charge limit with $h_H > \frac{c}{24}$, heavy-light Virasoro blocks appear thermal.\(^9\) Since the 4-point correlator has an OPE singularity

$$\langle \mathcal{O}_H(0)\mathcal{O}_L(z)\mathcal{O}_L(1)\mathcal{O}_H(\infty) \rangle = \frac{1}{(1 - z)^2h_L} + \cdots \quad (3.4.1)$$

as $z \to 1$, in the heavy-light semiclassical limit, it will also have singularities at $z_n = 1 - e^{n\beta}$ for all integers $n$.

While such singularities are permissible for correlators in the canonical ensemble, they are forbidden [8, 12] from 4-point correlators of local operators in unitary CFTs. They are also forbidden from individual Virasoro blocks at finite central charge [12, 9]. Thus exact Virasoro blocks completely disagree with their semiclassical counterparts at $z_n = 1 - e^{n\beta}$, the locations of the singularities. So to summarize, we know that the exact and semiclassical blocks match at $z = 0$, and completely disagree at $z = 1 - e^{n\beta}$ for $n \neq 0$. Thus it is natural to wonder whether the semiclassical blocks are a good approximation at $z = 1 - e^{-\frac{\beta}{2} - it}$, which corresponds to the location of $\mathcal{O}(t + i\frac{\beta}{2})$. More generally we would like to understand the kinematical regimes where the (leading) semiclassical approximation breaks down.

We observe from figure 3.15 that as expected, the exact Virasoro blocks do not have forbidden singularities. Nevertheless one might have expected to see bumps or local maxima

---

\(^9\)The vacuum block is exactly periodic. The general case in equation (3.2.10) would be periodic except for the branch cuts of the hypergeometric function, but these do not obstruct the KMS condition for the full correlator, and are compatible with the Virasoro block decomposition of correlators obtained from BTZ black hole backgrounds [16].
Figure 3.15: In this plot, we compare the exact and semiclassical blocks. One can see that at the positions of the semiclassical forbidden singularities, the exact blocks are smooth. Fixing $h_L$ and $h_H$ as we increase $c$, the exact blocks approach the semiclassical block in the region between the origin and the first forbidden singularity. However, beyond the first forbidden singularity the exact blocks deviate greatly as we increase $c$. This indicates that we have passed a Stokes line (emanating from the forbidden singularity) and some other semiclassical saddle dominates the exact blocks in the large $c$ limit. The gray line is the position of $t = \frac{i\beta}{T}$.

at $z_n = 1 - e^{n\beta}$, whereas the exact correlator simply grows as a function of $z \in [0, 1)$. In fact local maxima are prohibited because the exact blocks are analytic functions of $q$ and $z$ away from the true OPE singularities.\(^\text{10}\) Thus the semiclassical approximation breaks down badly beyond the first forbidden singularity.

We compare the exact and semiclassical blocks at finite time in figure 3.16. We see that the semiclassical blocks remain a good approximation to correlators of $\mathcal{O}(t + \frac{i\beta}{T})$ as long as we avoid the long-time region of $t \propto S$ that was discussed in section 3.3. In particular, there is not a significant difference between the quality of the semiclassical approximation to correlators of $\mathcal{O}(t + \frac{i\beta}{T})$ and $\mathcal{O}(t)$. The most naive interpretation of this fact is that non-perturbative quantum gravitational effects do not obstruct local physics across the horizon of

\(^\text{10}\)Moreover it is not too surprising that a finite series expansion of the exact blocks simply grows in the region where the semiclassical blocks have forbidden singularities. For example, the finite-order series expansion of a function like $\frac{1}{(1-x)(2-x)^2}$ will grow monotonically on the positive real $x$-axis; one can only see the correct behavior on $x \in (1, 2)$ by summing the full series and analytically continuing around $x = 1$.\)
Figure 3.16: In this figure we compare the semiclassical and exact blocks associated with \( \mathcal{O}(t) \) and \( \mathcal{O}(t + \frac{i\beta}{2}) \). The plot suggests that the semiclassical approximation remains valid for correlators of \( \mathcal{O}(t + \frac{i\beta}{2}) \). We implement time dependence via \( z = 1 - re^{-it} \) and so a shift by \( \frac{i\beta}{2} \) simply corresponds to a different choice of \( r \). Corresponding trajectories in the unit \( q \) disk are pictured in figure 3.6. Apparently the semiclassical approximation works well at \( t + \frac{i\beta}{2} \).

Figure 3.16: In this figure we compare the semiclassical and exact blocks associated with \( \mathcal{O}(t) \) and \( \mathcal{O}(t + \frac{i\beta}{2}) \). The plot suggests that the semiclassical approximation remains valid for correlators of \( \mathcal{O}(t + \frac{i\beta}{2}) \). We implement time dependence via \( z = 1 - re^{-it} \) and so a shift by \( \frac{i\beta}{2} \) simply corresponds to a different choice of \( r \). Corresponding trajectories in the unit \( q \) disk are pictured in figure 3.6. Apparently the semiclassical approximation works well at \( t + \frac{i\beta}{2} \).

Figure 3.16: In this figure we compare the semiclassical and exact blocks associated with \( \mathcal{O}(t) \) and \( \mathcal{O}(t + \frac{i\beta}{2}) \). The plot suggests that the semiclassical approximation remains valid for correlators of \( \mathcal{O}(t + \frac{i\beta}{2}) \). We implement time dependence via \( z = 1 - re^{-it} \) and so a shift by \( \frac{i\beta}{2} \) simply corresponds to a different choice of \( r \). Corresponding trajectories in the unit \( q \) disk are pictured in figure 3.6. Apparently the semiclassical approximation works well at \( t + \frac{i\beta}{2} \).

3.4.3 Fate of the Semiclassical Approximation from Analytics and Numerics

We do not have to rely entirely on numerics to explore the regime of validity of the semiclassical limit. It has been shown that the vacuum block’s forbidden singularities have a universal resolution due to non-perturbative effects in central charge. Specifically, the heavy-light vacuum block (with \( h_L \) and \( \frac{h_u}{c} \) held fixed at large \( c \)) should obey an approximate differential
Figure 3.17: The figure on the left shows a contour plot of the function $|\Sigma_H \frac{\mathcal{V}''}{\mathcal{V}'}|$ from equation (3.4.2) in the $\rho$ unit disk with $h_L = 1$ and $h_H = \frac{c}{4}$. The figure on the right is the deviation of the exact and semiclassical Virasoro vacuum blocks with the same parameters and $c = 60$. The positions of the forbidden singularities are indicated with black dots. The plot on the left can be viewed as a kind of analytic prediction for the deviation plotted on the right.

The figure on the left shows a contour plot of the function $|\Sigma_H \frac{\mathcal{V}''}{\mathcal{V}'}|$ from equation (3.4.2) in the $\rho$ unit disk with $h_L = 1$ and $h_H = \frac{c}{4}$. The figure on the right is the deviation of the exact and semiclassical Virasoro vacuum blocks with the same parameters and $c = 60$. The positions of the forbidden singularities are indicated with black dots. The plot on the left can be viewed as a kind of analytic prediction for the deviation plotted on the right.

The left plot shows contours of $|\Sigma_H \frac{\mathcal{V}''}{\mathcal{V}'}|$ for $h_L = 1$. We see that this function becomes large and makes important contributions in the immediate vicinity of the forbidden singularities, though at sufficiently large $c$ the right-hand side of equation (3.4.2) will remain

$$h_L g_H(\tau) \frac{\mathcal{V}(\tau)}{\mathcal{V}'(\tau)} - 1 = \frac{6}{c} \Sigma_H(\tau) \frac{\mathcal{V}''(\tau)}{\mathcal{V}'(\tau)}$$

where $\tau = -\log(1 - z)$ is a Euclidean time variable, and this equation neglects terms of order $1/c^2$ and higher as well as effects that are less singular near the forbidden singularities. We provide the functions $g_H$ and $\Sigma_H$ in appendix B.2.1. This differential equation also predicts [12] that the semiclassical vacuum block will receive large non-perturbative corrections after a Lorentzian time of order $\frac{S_{BH}}{h_L T_H} \propto \frac{c}{h_L}$. That prediction was corroborated in section 3.3.

Neglecting the term proportional to $\frac{1}{c}$ on the right-hand side, equation (3.4.2) is solved by the semiclassical heavy-light vacuum block. But when the right-hand side of this equation becomes large, non-perturbative effects come into play, resolving the forbidden singularities. We plot contours of the function $|\Sigma_H \frac{\mathcal{V}''}{\mathcal{V}'}|$ for $h_L = 1$ in figure 3.17. We see that this function becomes large and makes important contributions in the immediate vicinity of the forbidden singularities, though at sufficiently large $c$ the right-hand side of equation (3.4.2) will remain...
Figure 3.18: The figure on the left shows a contour plot of the function $|\Sigma_H \frac{V'}{V}|$ from equation (3.4.2) in the $\rho$ unit disk with $h_L = 1$ and $h_H = \frac{c}{30}$. In this case $h_H < \frac{c}{24}$, so the heavy-light block does not include a black hole – instead it corresponds to a light probe interacting with a deficit angle in AdS$_3$. Thus there are no forbidden singularities, and the semiclassical approximation is reliable in a much larger region as compared to figure 3.17 (note the difference in scales). The figure on the right is the deviation of the exact and semiclassical Virasoro vacuum blocks with the same parameters and $c = 60$. The plot on the left can be viewed as a kind of analytic prediction for the deviation plotted on the right.

small a finite distance away from these singularities. At a more detailed level, the function $|\Sigma_H \frac{V'}{V}|$ can be compared directly to the deviation of the numerical and semiclassical vacuum block. We plot contours of the ratio of the exact and semiclassical blocks in the $\rho$ unit disk, corresponding to the entire Euclidean $z$-plane in figure 3.17 (recall that we compared various kinematic variables in figures 3.3 and 3.4).

Our numerical results demonstrate that the semiclassical approximation breaks down in a finite region enclosing the forbidden singularities. We believe this phenomenon occurs because Stokes and anti-Stokes lines (for review see e.g. [106]) emanate from the forbidden singularities, as has been demonstrated for the correlators of degenerate operators [12]. As we cross Stokes lines, the coefficients of semiclassical saddles change by discrete jumps. Across anti-Stokes lines saddles exchange dominance.

Near the OPE configuration $z \propto \rho \propto q \approx 0$ where the light operators collide, a special ‘original’ semiclassical saddle dominates the large $c$ limit [12] of the Virasoro blocks. But in
a finite region near the forbidden singularities, different semiclassical saddles [9] can come to dominate, and the original saddle may become sub-leading. In other words, analytic continuation in the kinematic variables does not commute with the large $c$ limit. Non-perturbative effects can dramatically alter the behavior of CFT$_2$ correlation functions with these kinematics, supplanting the naive semiclassical limit and the perturbation expansion around it.

It would be fascinating if the black hole interior depends in some way on the behavior of CFT correlation functions in these regimes. Note that when $h_H < \frac{c}{24}$, so that the heavy background state does not correspond to a black hole, the original semiclassical approximation remains good throughout the Euclidean region. We demonstrate this explicitly in figure 3.18. So the breakdown of the semiclassical limit exhibited in figure 3.17 really does depend on the presence of a black hole, and is not a general feature of all Virasoro blocks at large central charge.

### 3.5 Discussion

We would eventually like to resolve the black hole information paradox by doing the right calculation. In the context of AdS/CFT, this means discerning under what circumstances, if any, bulk reconstruction is possible near and behind black hole horizons.

If firewalls [104] are completely generic, or if bulk reconstruction is sufficiently ambiguous, then this could be a fools errand. But even in this case, one can still hope for a more constructive argument rather than various reductio ad absurdums [93]. For example, one would like to reconstruct the ‘experience’ of a collapsing spherical shell, and explicitly compute the timescale beyond which subsequent infallers will not see a smooth (or well-defined) geometry.

But let us imagine that the strict semiclassical limit is not misleading and black holes often have smooth interiors. In this case, violations of bulk locality should arise from the
difference between computations in the semiclassical limit and the exact CFT observables (or perhaps meta-observables). This sort of approach has already been successfully pursued in the context of local bulk scattering [8]. We have identified gross differences between exact and semiclassical CFT correlators in both the late Lorentzian time and the Euclidean regime. These do not seem to affect a certain naive bulk reconstruction algorithm, but perhaps they do afflict more sophisticated methods yet to be developed. Hopefully we have done some of the right calculations but do not yet know how to give them the right interpretation. In the case of quantum mechanics and QFT, we were in that sort of boat for decades.
Chapter 4

An Exact Operator That Knows Its Location

This chapter is based on the following paper:


Abstract

We use conformal symmetry to define an AdS$_3$ proto-field $\phi$ as an exact linear combination of Virasoro descendants of a CFT$_2$ primary operator $\mathcal{O}$. We find that both symmetry considerations and a gravitational Wilson line formalism lead to the same results. The operator $\phi$ has many desirable properties; in particular it has correlators that agree with gravitational perturbation theory when expanded at large $c$, and that automatically take the correct form in all vacuum AdS$_3$ geometries, including BTZ black hole backgrounds. In the future it should be possible to use $\phi$ to probe bulk locality and black hole horizons at a non-perturbative level.
4.1 Introduction

To resolve the black hole information paradox in AdS/CFT, we must understand how to describe local AdS dynamics in terms of CFT data and observables. Unfortunately, bulk gauge redundancies could render AdS reconstruction ambiguous, and the existence of black holes at high-energies suggests that local physics may not be well-defined. We will argue that the Virasoro symmetry of CFT\(_2\) provides a sort of beachhead into AdS\(_3\), making it possible to exactly define a bulk ‘proto-field’ \(\phi\) as a specific linear combination of Virasoro descendants of a given local primary operator \(\mathcal{O}\).

The simplest AdS/CFT observable is the vacuum bulk-boundary correlator

\[
\langle \phi(X)\mathcal{O}(P) \rangle = \frac{1}{(P \cdot X)^{\Delta}},
\]

which is determined by conformal symmetry up to an overall constant. From this correlator alone one can derive a formula for a proto-field \(\phi(X)\) as a linear combination of global conformal descendants of the primary operator \(\mathcal{O}\) [107, 87, ?, 108]. At this level, bulk reconstruction is purely kinematical, following entirely from the assumption that conformal transformations act on \(\phi\) as AdS isometries.

In the case of AdS\(_3\)/CFT\(_2\), Virasoro conformal transformations act as asymptotic symmetries. So it is natural to expect that the bulk-boundary correlator should be uniquely determined in any geometry that can be related to the vacuum by a Virasoro symmetry. In rather different words, we expect that all correlators of the form

\[
\langle \phi(X)\mathcal{O}(z, \bar{z})T(z_1)\cdots T(z_n)\bar{T}(\bar{w}_1)\cdots \bar{T}(\bar{w}_m) \rangle
\]

(4.1.2)

can be determined by symmetry once we fix a gauge for the bulk gravitational field. This leads to a unique expression for a Virasoro proto-field operator \(\phi(X)\) as a linear combination
of Virasoro descendants of the CFT$_2$ primary $\mathcal{O}$. These proto-field operators will automatically ‘know’ about the bulk geometry associated with heavy distant sources, meaning that they perform bulk reconstruction at an operator level. In this paper we will explain how to identify and explicitly compute $\phi(X)$ as a CFT$_2$ operator. We will be led to the potentially surprising conclusion that an exact (non-perturbative in $c$) condition uniquely determines $\phi$ in our Fefferman-Graham type gauge.

We will determine $\phi(X)$ in two distinct but ultimately equivalent ways. The first is based on an extension of gravitational Wilson lines [109, 21, 110, 111, 24, 77, 78] as OPE blocks [112]. We will introduce a ‘bulk-boundary OPE block’ that encapsulates the projection of the (non-local) operator $\phi(X)\mathcal{O}(x)$ onto the vacuum sector. This provides an explicit prescription for computing all correlators of the form of equation (4.1.2). Our second method is based purely on imposing Virasoro symmetry, resulting in a very simple, non-perturbative definition for $\phi(X)$. This also makes it possible to determine the correlators of equation (4.1.2) via a simple recursion relation. The proto-field operator that we will obtain has a number of desirable properties:

- Virasoro transformations act on the scalar field $\phi(X)$ as infinitesimal bulk diffeomorphisms preserving the gauge. At the semiclassical level, $\phi(X)$ obeys the Klein-Gordon equation in any vacuum geometry.

- Correlators of $\phi$ with stress tensors are causal and have only those singularities dictated by the gravitational constraints [113, 114, 115], matching bulk perturbation theory. Correlators of $\phi(X)$ reduce to those of $\mathcal{O}(x)$ when we extrapolate $\phi(X)$ to the boundary. Equation (4.1.2) reduces to $\langle \mathcal{O} \cdot \mathcal{T} \cdots \rangle$; in fact there is a simple recursion relation that computes vacuum correlators, generalizing well-known relations [46] for correlators of CFT$_2$ primaries with stress tensors.

With our exact definition for $\phi(X)$, it is possible to study the impact of non-perturbative gravitational effects on bulk observables. This means that one could study $\phi(X)\phi(Y)$ at short
distances, and directly probe near black hole horizons without relying on bulk perturbation theory.

There is a large literature on bulk reconstruction in AdS/CFT employing a variety of philosophies and methods, for example [107, 116, 87, 117, 113, 114, 115, 118, 91, 119, 108, 120, 121, 112, 122, 123, 124, 125, 126]. The most common approach expresses bulk fields in terms of local CFT operators integrated against a kernel [107, 87, 117]. We will take a somewhat different approach [127, 128, 129]; our scalar operator \( \phi(y, 0, 0) \) will be expressed in a boundary operator expansion\(^2\) (BOE) [130]

\[
\phi(y, 0, 0) = \sum_{N=0}^{\infty} \lambda_N y^{2h+2N} L_{-N} \tilde{L}_{-N} O(0)
\]  

(4.1.3)

where \( L_{-N} \) and \( \tilde{L}_{-N} \) are linear combinations of products of Virasoro generators at level \( N \), and \( \lambda_N = \frac{(-1)^N}{N!(2h)^N} \). In the global limit \( (c \to \infty) \), we have \( \lim_{c \to \infty} L_{-N} = L_{-1}^N \). At finite \( c \), we will show that \( L_{-N} O \) satisfies the bulk primary conditions

\[
L_m L_{-N} O = 0, \quad \text{for } m \geq 2.
\]  

(4.1.4)

and similarly for \( \tilde{L}_{-N} O \). Roughly speaking, these conditions say that \( \phi \) is as primary as it can be and still move around under AdS bulk isometries. In the smearing function language, we are computing \( \phi \) as an infinite sum of operators\(^3\) of the schematic form \( O, [T \bar{T}^2 O], \cdots, [T \bar{T}^2 T \bar{T} \bar{O}^4 O], \cdots \), though we will not express our results in this way.

The outline of this paper is as follows. In section 4.2 we explain the bulk-boundary

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1We believe the proposal in [125, 126] is different from ours.

2The idea of performing bulk reconstruction using a boundary operator expansion was briefly discussed in [127]. The global AdS results have been worked out by M. Paulos in unpublished work. Note that the boundary operator expansion appears local on the boundary, but due to the infinite sum it should really be viewed as a non-local CFT operator, for the same reason that \( e^{x \phi}(0) = \phi(x) \) should not be viewed as a local operator at the origin.

3As was shown by Kabat and Lifschytz [114, 115], because of the gravitational gauge constraints \( \phi \) must include contributions from the scalar descendants of quasi-primaries with non-zero spin, such as \( \partial^\mu \partial^\nu[T_{\mu\nu} O] \), even though \( \phi \) itself is a bulk scalar field. Thus it’s not entirely clear how smearing functions can be used to describe our results.
OPE block idea, and then show how the vacuum $\phi(X)\mathcal{O}(z)$ OPE block can be derived using gravitational or Chern-Simons Wilson lines. We begin section 4.3 by providing an exact algebraic definition for $\phi$ compatible with the results of section 4.2. Then we show that this simple definition follows from considerations of symmetry. We solve for $\phi$ explicitly in various cases, and then show how our definition leads to new recursion relations for correlators of $\phi$ with boundary stress tensors. We collect various technical results and background material in the appendices. Appendix C.1.3 may be useful for readers who are most familiar with the HKLL [87] smearing procedure, and want to understand how our approach, in the simple global conformal case, can be reduced to theirs. All formulas in this paper are written in Euclidean signature.

### 4.2 Bulk Reconstruction from Gravitational Wilson Lines

The operator product expansion (OPE) expresses a product of separated local operators $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$ as an infinite sum of local operators at a single point. It is very natural to gather the contributions to the OPE that come from a single conformal primary and its descendants. This has been dubbed [112] an ‘OPE block’. In the case of CFT$_2$, the Virasoro OPE blocks can be computed using Chern-Simons Wilson lines [77].

In this work we will be studying an AdS$_3$ proto-field $\phi(X)$ as a CFT$_2$ operator, and we focus on Euclidean signature. Although $\phi(X)$ may be somewhat non-local, on the border of a sufficiently large region in the CFT containing $\phi(X)$, we expect that it should still be possible to perform a radial quantization, as shown in figure 4.1. This suggests that we can study OPE blocks involving $\phi(X)$ and other operators. We will be focusing on the simplest such object, the scalar Virasoro vacuum OPE block

$$
\phi(y, 0, 0)\mathcal{O}(z, \bar{z}) = \frac{y^{2h}}{(y^2 + z\bar{z})^{2h}} + \cdots
$$

(4.2.1)
Figure 4.1: This figure portrays a bulk-boundary OPE block used to compute the correlator (4.1.2). The red line denotes the gravitational or Chern-Simons Wilson line, while the blue circle suggests radial quantization around the block, so that it creates a definite linear combination of Virasoro descendants of the identity. The explicit computation involves line integrals over stress tensor correlators.

where the ellipsis denotes non-identity Virasoro generators (e.g. \( L_{-6} \bar{L}_{-4} \)) with coordinate-dependent coefficients, and we have labeled \( \phi \) using the coordinates of the AdS_3 vacuum metric

\[
ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2} \tag{4.2.2}
\]

Note that we have already identified the contribution of the identity operator in equation (4.2.1) as the vacuum correlator \( \langle \phi \mathcal{O} \rangle \), which is fixed by conformal symmetry. All of the remaining terms in equation (4.2.1) would be fixed if we knew all correlators of the form (4.1.2), because the Virasoro generators are just the modes in an expansion of the stress tensors \( T(z) \) and \( \bar{T}(\bar{z}) \).

Building on prior work [77], we will make the following proposal for the \( \phi \mathcal{O} \) OPE block.
The general asymptotically AdS$_3$ vacuum metric can be written\textsuperscript{4} as \cite{131, 132}

\[ ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2} - \frac{6T(z)}{c}dz^2 - \frac{6\bar{T}(\bar{z})}{c}d\bar{z}^2 + y^2\frac{36T(z)\bar{T}(\bar{z})}{c^2}d\bar{z}d\bar{z} \]  

(4.2.3)

This amounts to a choice of gauge for the bulk gravitational field. Normally the objects $T(z)$ and $\bar{T}(\bar{z})$ appearing in this equation are viewed as classical functions, but let us instead view them as CFT$_2$ stress tensor operators. We define the bulk-boundary OPE block as the operator defined by the propagation of a (quantum, first-quantized) particle from the location of $O$ on the boundary to that of $\phi$ in the bulk. Formally, this means that the bulk-boundary OPE block can be thought of as a world-line path integral

\[ \phi(X)O(0)|_{\text{vac}} = \int DY(\tau) e^{-m \int_0^X d\tau \sqrt{g_{\mu\nu}Y^\mu Y^\nu}}, \]  

(4.2.4)

where on the LHS we are restricting to the vacuum sector of the operator product. On the RHS we interpret $g_{\mu\nu}$ as a quantum operator dependent on $T, \bar{T}$ as defined in equation (5.2.1), and $Y^\mu(\tau)$ is world-line connecting $\phi$ and $O$. The mass $m$ of the particle will be related to the dimension of $O$ by $m^2 = 2h(2h - 2)$. Equation (4.2.4) defines the OPE block as an infinite sum of products of line integrals of the CFT$_2$ stress tensor. We have sketched the OPE block in figure 4.1.

In a certain sense, we will use equation (4.2.4) more for conceptual purposes than for computation ones. To use (4.2.4) directly would require defining the path integral measure precisely; in practice, we will circumvent this kind of issue by starting with the exact CFT result for (4.2.4) in the AdS vacuum and uplifting to nonzero $T, \bar{T}$ by performing diffeomorphisms, as we describe below. Nevertheless, it is useful to bear equation (4.2.4) in mind as it intuitively captures what we are trying to achieve in defining $\phi O$, and furthermore it

\textsuperscript{4}Our construction requires an ability to specify this gauge. This will be possible sufficiently close to the boundary of AdS, but may fail in the presence of sources, or in the context of non-trivial boundary topologies. We expect that the construction may break down in regimes where the dynamics are inconsistent with our gauge choice. We leave a detailed discussion of the regime of validity to future work.
should agree with our practical definition in a semiclassical limit where ambiguities in the path integral measure do not arise. So when we compute the bulk-boundary OPE block in the presence of operators with dimensions $h_H \propto c$ at large $c$, then we can approximate $\phi \mathcal{O}$ by including only the semiclassical expectation value $\langle T(z) \rangle \propto \frac{h_H}{c}$. This immediately leads to the correct $\phi \mathcal{O}$ correlators in a semiclassical background, such as that of a BTZ black hole. Relatedly, our prescription will also lead to a $\phi(\mathcal{X})$ that satisfies the Klein-Gordon equation in the semiclassical metric of equation (5.2.1). We review this elementary fact in appendix C.1.1. We also provide a more detailed discussion of (4.2.4) and its regulation in appendix C.2.

In the remainder of this section we will use equation (4.2.4) to explicitly compute various correlation functions, and demonstrate that the results reduce to those of [77] when we take $\phi$ to the boundary. In fact we will find that we can reformulate equation (4.2.4) in terms of sl$(2)$ Wilson lines as

$$
\phi(y, z_2, \bar{z}_2) \mathcal{O}(z_1, \bar{z}_1)|_{\text{vac}} = P \left\{ e^{\int_{z_1}^{z_2} dz A_z + \int_{\bar{z}_1}^{\bar{z}_2} d\bar{z} \bar{A}_z} \left( \frac{y}{y^2 + x \bar{x}} \right)^{\Delta} \right\}_{x=\bar{x}=0}. \tag{4.2.5}
$$

As we will explain in section 4.2.2 (where we also define the notation), this is the most natural generalization of prior Chern-Simons Wilson line results [77] to the case of the bulk-boundary OPE block. It also makes manifest the fact that as we take $\phi$ to the boundary, we recover the structure of the more conventional $\mathcal{O}(z_2)\mathcal{O}(z_1)$ Virasoro OPE block.

### 4.2.1 Computing $\phi(\mathcal{X})\mathcal{O}(0)$ from a Diffeomorphism

We will use two facts to formulate an operational definition of equation (4.2.4) that can be used for practical computations. The first is that in pure AdS$_3$, the first-quantized path integral reduces to $e^{-2\sigma}$ where $\sigma$ is the (renormalized) length of a geodesic connecting $\mathcal{O}$ and $\phi$. The second fact is an explicit diffeomorphism [132] that relates metrics of the form (5.2.1) to the pure AdS$_3$ metric. We will elevate this diffeomorphism to an operator equation,
defining a change of coordinates parameterized by a function $f_T(z)$ that maps the pure AdS$_3$
metric to the operator-valued vacuum metric of equation (5.2.1). Then we can use the first
fact to evaluate the bulk-boundary OPE block as a functional of $f_T(z)$, which itself depends
on the operator $T(z)$. These ideas were inspired by very similar methods that have been
used to evaluate Chern-Simons Wilson lines [77] in order to compute Virasoro OPE blocks;
we will see in section 4.2.2 that this is not an accident.

The first fact means that in a vacuum metric

$$ds^2 = \frac{du^2 + dwd\bar{w}}{u^2}, \quad (4.2.6)$$

we can write the bulk-boundary correlator as

$$\phi(u, 0, 0)O(w, \bar{w}) = \left(\frac{u}{u^2 + w\bar{w}}\right)^{2h}. \quad (4.2.7)$$

In the CFT vacuum, this is an exact CFT result, just the standard scalar bulk-to-boundary
propagator that can be derived purely from symmetries of the CFT. But now we will general-
ize it by viewing the coordinates $(u, w, \bar{w})$ as the result of an operator valued diffeomorphism
from a general vacuum metric of the form of equation (5.2.1). The diffeomorphism takes the
form [132]

$$\begin{align*}
    w &\rightarrow f(z) - \frac{2y^2(f'(z))^2f''(\bar{z})}{4f'(z)f''(\bar{z}) + y^2f''(z)f''(\bar{z})} \\
    \bar{w} &\rightarrow \bar{f}(\bar{z}) - \frac{2y^2(\bar{f}'(\bar{z}))^2f''(z)}{4f'(z)f''(\bar{z}) + y^2f''(z)f''(\bar{z})} \\
    u &\rightarrow y\frac{4(f'(z)\bar{f}'(\bar{z}))^{3/2}}{4f'(z)f''(\bar{z}) + y^2f''(z)f''(\bar{z})} \quad (4.2.8)
\end{align*}$$

and is parameterized by the independent holomorphic and anti-holomorphic functions $f(z)$
and $\tilde{f}(\tilde{z})$. This diffeomorphism has the property that the transformed metric is precisely

$$
\begin{aligned}
ds^2 &= \frac{dy^2 + dzd\bar{z}}{y^2} - \frac{1}{2} S(f, z) dz^2 - \frac{1}{2} S(\tilde{f}, \tilde{z}) d\bar{z}^2 + y^2 \frac{S(f, z) S(\tilde{f}, \tilde{z})}{4} dz d\bar{z} \\
&= \frac{S(f, z)}{2} dz^2 - \frac{S(\tilde{f}, \tilde{z})}{2} d\bar{z}^2 + y^2 S(f, z) S(\tilde{f}, \tilde{z}) dz d\bar{z} 
\end{aligned}
$$

(4.2.9)

where

$$
S(f, z) \equiv \frac{f'''(z)f'(z) - \frac{3}{2}(f''(z))^2}{(f'(z))^2} = \frac{12}{c} T(z) 
$$

(4.2.10)

is the Schwarzian derivative. Thus the diffeomorphism maps pure AdS$_3$ to a general vacuum-sector metric with a non-vanishing stress tensor. Applying this operator valued diffeomorphism to (4.2.7), we obtain the vacuum sector bulk-boundary OPE block$^5$

$$
\phi(y, z_2, \bar{z}_2) \mathcal{O}(z_1, \bar{z}_1) |_{\text{vac}} = (f'(z_1) \tilde{f}'(\tilde{z}_1))^h \left( \frac{u_2}{u_2^2 + (w_2 - w_1)(\bar{w}_2 - \bar{w}_1)} \right)^{2h},
$$

(4.2.11)

where $u_2, w_2, \bar{w}_2$ are $u, w, \bar{w}$ in (4.2.8) evaluated at $(y, z_2, \bar{z}_2)$, and $w_1, \bar{w}_1$ are evaluated at $(0, z_1, \bar{z}_1)$. This is the key formulation of the bulk-boundary OPE block that will be used in this paper.

To evaluate (4.2.11), we need to solve equation (4.2.10) and its anti-holomorphic equivalent for the functions $f(z)$ and $\tilde{f}(\tilde{z})$, determining them as functionals of the stress tensor operators $T(z), \tilde{T}(\tilde{z})$. Then we can evaluate equation (4.2.7) by expanding the coordinates $u, w, \bar{w}$ in terms of $f, \tilde{f}$. To carry out this procedure explicitly in $1/c$ perturbation theory, we write

$$
f(z) = z + \frac{1}{c} f_1(z) + \frac{1}{c^2} f_2(z) + \cdots
$$

(4.2.12)

$^5$Note that in deriving this equation, we cut off the divergent near boundary integral at a constant $y$ plane as oppose to the constant $y_\infty$ plane used in (4.2.7). This shift results in the $(w'(z_1)\bar{w}'(\bar{z}_1))^h$ factor that is essential to reproduce the transformation property of a boundary Virasoro primary.
and then solve for the $f_n$ in terms of $T$ using equation (4.2.10). The first two $f_n$ are determined by the differential equations

$$f''_1(z) - 12T(z) = 0$$

$$2f^{(3)}_1(z)f'_1(z) + 3f''_1(z)^2 - 2f^{(3)}_2(z) = 0$$ (4.2.13)

so for example, the first equation simply leads to $f_1(z) = -6f_0^z dz' (z - z')^2 T(z')$. Once we solve for the $f_n$, we can expand (4.2.11) to find the bulk-boundary OPE block\(^6\)

$$\log \phi(y, 0, 0) \mathcal{O}(z, \bar{z}) = 2h \log \left( \frac{y}{z\bar{z} + y^2} \right) + \left[ \frac{h(z\bar{z} + y^2) f'_1(z) - 2\bar{z}f_1(z)}{c(z\bar{z} + y^2)} \right]_{\mathcal{K}_T} + \ldots$$ (4.2.14)

where the ellipsis denotes both the conjugate anti-holomorphic $\mathcal{K}_{\bar{T}}$ terms as well as the perturbation series at order $1/c^2$ and above. The order $1/c$ terms $\mathcal{K}_T$ and $\mathcal{K}_{\bar{T}}$ are line-integrals of the stress tensors $T$ and $\bar{T}$ against specific kernels. For example, by combining terms above we find that

$$\mathcal{K}_T = \frac{12h}{c} \int_0^z dz' \int_0^{z'} d\bar{z}' \frac{(y^2 + z'\bar{z})(z - z')}{y^2 + z\bar{z}} T(z')$$ (4.2.15)

and similarly for the anti-holomorphic $\mathcal{K}_{\bar{T}}$. In the limit $y \to 0$ we recover the kernels [77] for the standard ‘boundary-boundary’ $\mathcal{O}(z)\mathcal{O}(0)$ OPE block.

At the next order we would obtain the new kernels $\mathcal{K}_{TT}, \mathcal{K}_{\bar{T}\bar{T}}$, and also the mixed kernel $\mathcal{K}_{T\bar{T}}$ which are computed explicitly in appendix C.4.1. The results are

$$\mathcal{K}_{TT} = \frac{72h}{c^2} \int_0^z dz' \int_0^{z'} d\bar{z}' \int_0^{z'} d\bar{z}'' \frac{(z - z')^2 (y^2 + z\bar{z}'')^2}{(z\bar{z} + y^2)^2} T(z') T(z'')$$

\(^6\)We took the logarithm because it renders computations simpler and more transparent [77], but one could easily deal with the full OPE block directly instead. Taking the logarithm of an operator is not at all innocuous in general, but due to our choice of regulator it will not present any problems.
\[ K_{T\bar{T}} = -\frac{72\hbar y^2}{c^2 (z\bar{z} + y^2)^2} \int_0^z dz' (z - z')^2 \int_0^\bar{z} d\bar{z}' (\bar{z} - \bar{z}')^2 T(z') \bar{T}(\bar{z}') \]  

(4.2.16)

for the bulk-boundary OPE blocks. Note that the first reduces to the expected \( \mathcal{O}(z)\mathcal{O}(0) \) kernel (compare to equation 4.40 of [77]) at this order, while the \( K_{T\bar{T}} \) kernel vanishes as \( y \to 0 \), again matching with the expectations for the boundary (where OPE blocks factorize into holomorphic \( \times \) anti-holomorphic parts). In the next subsection we will present an alternative derivation that makes this matching explicit to all orders in \( 1/c \).

### 4.2.2 Connection with Chern-Simons Wilson Lines

The \( \text{sl}(2) \) Wilson line formulation in [77] (based on the earlier work [109]) of the standard OPE block takes the form

\[ \mathcal{O}(z_2)\mathcal{O}(z_1) \supset W(z_2, z_1) = P \left\{ e^{\int_{z_1}^{z_2} dx^\mu A^a_\mu(z)L^a_x} \right\} \left. \frac{1}{x^2\hbar} \right|_{x=0}. \]  

(4.2.17)

First, we will review the notation and some of the results from [77], and then we will see how to generalize (4.2.17) to the expression (4.2.5) above.

In the Wilson line expression (4.2.17), \( P \) indicates ‘path-ordering’, the \( A_\mu \)s are the \( \text{sl}(2) \) gauge fields, and the \( L^a_x \) are the corresponding generators. The variable \( x \) is an auxiliary coordinate introduced so that \( L^a_x \) can be written in an infinite dimensional representation,

\[ L^1 \cong L_{-1} = \partial_x, \quad L^0 \cong L_0 = x\partial_x + \hbar, \quad L^{-1} \cong L_1 = \frac{1}{2}x^2\partial_x + \hbar x. \]  

(4.2.18)

Equation (4.2.17) is the holomorphic part of the OPE block, and a similar anti-holomorphic piece is present in the full block. The boundary condition on \( A_\mu \) that leads to Virasoro symmetry is

\[ A_z|_{y=0} = L^1 + \frac{12}{c} T(z)L^{-1}. \]  

(4.2.19)
For boundary operators $O$, we can push the Wilson line connecting $O(z_2)$ and $O(z_1)$ onto the boundary so that only the above behavior at $y = 0$ is necessary. When we move one of the $O$s into the bulk to position $(y, z_2, \bar{z}_2)$, we will first take the Wilson line to be along the boundary from $(0, z_1, \bar{z}_1)$ to $(0, z_2, \bar{z}_2)$, and then to go directly to the bulk point $(y, z_2, \bar{z}_2)$ along constant $(z_2, \bar{z}_2)$. Making the gauge choice $A_y = 0$, the second part of the Wilson line is trivial.

In [77], it was shown that the path-ordered term $P\left\{ e^{ \int_{z_1}^{z_2} dz^{\mu} A^{\mu}_{\mu}(z) L^a_{z} } \right\}$ could equivalently be written as

$$ e^{ \frac{12h}{c} \int_{z_1}^{z_2} ds T(z) x_T(z) } $$

after promoting $x$ everywhere to an operator $x_T(z_1)$ that is defined as the (operator valued) solution to the differential equation

$$ -x_T'(z) = 1 + \frac{6T(z)}{c} x_T^2(z), \quad x_T(z_2) = 0. \quad (4.2.21) $$

In other words,

$$ W(z_2, z_1) = \left( e^{ \int_{z_1}^{z_2} dz \frac{12T(z) x_T(z)}{c x_T(z_1)^2} } \frac{1}{x_T(z_1)^2} \right)^h. \quad (4.2.22) $$

A key point was that $x_T$ is closely related to the uniformizing coordinates $f_T$ defined through the Schwarzian in 4.2.10. In particular,

$$ \frac{1}{x_T(z)} = \frac{f_T'(z)}{2 f_T(z)^2} - \frac{f_T''(z)}{f_T'(z) - f_T(z)}. \quad (4.2.23) $$

automatically satisfies the constraint (4.2.21).

Now we are ready to derive (4.2.5). The starting point will be our general philosophy that
φ in a general background follows from φ in the AdS vacuum combined with the operator-valued transformation (4.2.8). This results in the bulk-boundary OPE block for φO given by (4.2.11). Our goal will be to write (4.2.11) in terms of the Wilson line building blocks. For concision, let us define the exponential

\[ E_T \equiv e^{\int_{z_1}^{z_2} dz' T(z') x_T(z')} \]  

(4.2.24)

From the constraint equation (4.2.21), we have

\[ \log E_T = - \int_{z_1}^{z_2} dz' \frac{1 + x_T'(z)}{x_T(z)} = \log \left( \frac{2(f_T'(z_2))^{\frac{3}{2}}(f_T'(z_1))^{\frac{3}{2}}}{2(f_T'(z_i))^2 + (f_T(z_2) - f_T(z_1))f_T''(z_1)} \right). \]  

(4.2.25)

Furthermore, we see that the OPE block to has the correct semiclassical limit [77]

\[ W(z_2, z_1) \approx E_T^2 \frac{1}{x_T^2(z_1)} = \frac{f_T'(z_2)f_T'(z_1)}{(f_T(z_2) - f_T(z_1))^2}. \]  

(4.2.26)

It is now a straightforward matter to compare (4.2.11) to the RHS of

\[ P \left\{ e^{\int_{z_1}^{z_2} dz A_\phi + \int_{z_1}^{z_2} dz \tilde{A}_\phi} \right\} \left( \frac{y}{y^2 + x_T x} \right)^\Delta \bigg|_{x=\tilde{x}=0} \approx E_T^\Delta E_T^\Delta \left( \frac{y}{y^2 + x_T x} \right)^\Delta \]  

(4.2.27)

expanded out in terms of their dependence on \( f_T, \tilde{f}_T \) and confirm that they agree. Thus the conclusion is that the methods of 4.2.1 are entirely consistent with those from [77], and all of the techniques from that paper apply equally well to the bulk-boundary OPE. In particular, one can compute the integration kernels \( K_{T\ldots\tilde{T}\ldots} \) very efficiently to high orders using the \( x_T \) variables [77]; this is a significant technical improvement compared to solving equations like (4.2.13) directly.

We can go further and obtain a simple form for the generalization of (4.2.5) to the case of spinning fields and operators as well. We relegate the details of the derivation to appendix

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7To be systematic, one can just solve for \( x_T \) and \( E_T \) in terms of \( f'(z_2), f''(z_2) \) and substitute.
C.4.3 and simply quote the result here:

\[
\langle A_{\mu_1...\mu_\ell}(y, z_2, \bar{z}_2) \mathcal{O}_{h,\bar{h}}(z_1, \bar{z}_1) \rangle = P \left\{ e^{\int_{z_1}^{z_2} dz_2 A_z + \int_{\bar{z}_1}^{\bar{z}_2} d\bar{z_2} \bar{A}_{\bar{z}}} \right\} t_{\mu_1}^{\mu_1} \ldots t_{\mu_\ell}^{\mu_\ell} K_{\mu_1',...\mu_\ell'}(y, x, \bar{x}).
\]  

(4.2.28)

Here, \( \mathcal{O}_{h,\bar{h}} \) is a boundary field of weight \((h, \bar{h})\) and \( A_{\mu_1,...,\mu_\ell} \) is a bulk field with \( \ell = h - \bar{h} \geq 0 \) (a similar expression holds for \( \ell \leq 0 \)). The factor \( K_{\mu_1,...,\mu_\ell} \) is the vacuum AdS bulk-boundary propagator that we describe in detail in appendix C.4.3, and the tensor \( t_{\mu}^{\nu} \) is a diagonal matrix of the form

\[
t_{y}^{y} = 1, \quad t_{z}^{z} = 1 + \frac{6}{c} \frac{T(z_2) y^4}{x^2}, \quad t_{\bar{z}}^{\bar{z}} = 1 + \frac{6}{c} \frac{\bar{T}(\bar{z}_2) \bar{x}^2}{x^2}.
\]  

(4.2.29)

Although we have not pursued it directly in this paper, these results can be used to study the reconstruction of massive spinning fields in the bulk.

### 4.2.3 Evaluating Vacuum Sector Correlators

In this section we will use the bulk-boundary OPE block to compute correlators of \( \phi \mathcal{O} \) with products of local stress tensors. These correlators repackage all of the information about the overlap of \( \phi \mathcal{O} \) with the Virasoro vacuum sector.

Since \( \langle \phi \mathcal{O} \rangle \) is simply given by the first term in equation (4.2.14), ie

\[
\langle \phi(y, 0, 0) \mathcal{O}(z, \bar{z}) \rangle = \left( \frac{y}{y^2 + z^2} \right)^{2h},
\]

the simplest non-trivial correlator is \( \langle \phi \mathcal{O} T \rangle \). It can be computed using (4.2.15), giving

\[
\frac{\langle \phi(y, 0, 0) \mathcal{O}(z, \bar{z}) T(z_1) \rangle}{\langle \phi(y, 0, 0) \mathcal{O}(z, \bar{z}) \rangle} = \langle K_{T} T(z_1) \rangle = \frac{12h}{c} \int_{0}^{z} \frac{dz'}{2(z-z')(y^2+z'^2)} \frac{c}{2(z'-z_1)^4}
\]

\[
= \frac{h z^2}{z_1^3 (z_1 - z)^2} \left( z_1 + \frac{2y^2(z_1 - z)}{y^2 + z \bar{z}} \right)
\]  

(4.2.30)

The computation is suggested pictorially in figure 4.1. This result matches bulk gravitational perturbation theory using AdS$_3$ Feynman diagrams in our chosen gauge, as we show explicitly
in appendix C.4.4. This is no surprise, as the definition in equation (4.2.4) essentially reproduces gravitational perturbation theory in a first quantized language.

Note that the singularity structure of equation (4.2.30) as \( z_1 \) approaches 0 encodes the (expectation value of) the commutator\(^{8}\) of the stress tensor with \( \phi \). The fact that the most singular term is of order \( z_1^{-3} \) provides a first hint of a powerful symmetry structure that we will discuss in section 4.3.

Naively, one might expect that equation (4.2.30) is only the first term in an infinite perturbation series for this correlation function. However, the higher order contributions need to be regulated in a way that is consistent with Virasoro symmetry and with the fixed dimension \( 2h \) for the scalar CFT operator \( O \). In the context of Chern-Simons Wilson lines, we proposed a prescription for regulating multi-\( T \) correlators in Appendix C.2 of [77] that produces the correct Virasoro OPE blocks. In appendix C.2, we argue that this regulator can be derived from the generating function of multi-\( T \) correlators. Applying this same regulator for the bulk-boundary OPE block, we find that all higher order contributions to \( \langle \phi O T \rangle \) vanish. Thus we claim that equation (4.2.30) is the exact result for this correlation function. We will provide another argument that equation (4.2.30) is exact in section 4.3.

We can also compute the correlators \( \langle \phi O T \rangle \) and \( \langle \phi O \bar{T} \rangle \). We provide details of the computations in appendix C.4.2. The results are that

\[
\frac{\langle \phi (y, 0, 0) O(z, \bar{z}) T(z_1) T(z_2) \rangle}{\langle \phi (y, 0, 0) O(z, \bar{z}) \rangle} = \frac{c}{2(z_1 - z_2)^4} + \frac{h^2 z^4 \left( z_1 \bar{z} z + y^2 (3z_1 - 2z) \right)}{z_1^3 z_2^3 (z - z_1)^2 (z - z_2)^2 (z + y^2)^2} \tag{4.2.31}
\]

\[
+ \frac{2h z^2 \left( y^2 z \bar{z} z_1 z_2 (z_1 + z_2) - 4z_1 z_2 \right)}{(z - z_1) (z - z_2) z_1^3 z_2^3 (z_2 - z_1)^2 (z + y^2)^2} z^2 (z_1 - z_2)^2
\]

\[
+ \frac{y^4 \left( z_1 z_2 (z_1 + z_2) - 3z_1^2 z_2^2 - 2z_1 - 2z_2 \right)}{(z + y^2)^2 (z_1 - z_2)^2 (z_1 - z_2)^2}
\]

\[\text{To any order in a small } y \text{ expansion, the operator-valued commutator } [\phi(y, 0, 0), T(z_1)] \text{ will be a sum of Virasoro descendants of } O(0). \text{ However, at finite } y \text{ this commutator cannot be interpreted as a local CFT operator at } z, \text{ for the same reason that } \phi \text{ itself does not have this interpretation – it is an infinite sum of local operators, and so it is not local.}\]
These reduce to the expected $\mathcal{OO}$ correlators as $y \rightarrow 0$. We should also emphasize that in the semiclassical limit, where we include sources with dimensions $h_H \propto c$ as $c \rightarrow \infty$, the correlators of $\phi$ will take the correct form. This follows automatically from the definition of the OPE block in equation (4.2.4) and the form of the vacuum metric in equation (5.2.1). We can compute correlators in a BTZ black hole background when we include a heavy operators $\mathcal{O}_H(\infty)\mathcal{O}_H(0)$, which lead to $\frac{1}{z}(T(z)) = \frac{h_H}{c} \frac{1}{z^2}$ in the semiclassical limit. We hope to study these correlators at a non-perturbative level in the future.

### 4.3 An Exact Algebraic Definition for the Proto-Field $\phi(X)$

Our regulated bulk-boundary OPE block computes vacuum sector correlators exactly, and this suggests that we can obtain an exact definition for the proto-field $\phi$ built from the Virasoro primary $\mathcal{O}$. Now we provide this definition in a simple algebraic form, which originates from symmetry considerations. Our $\phi(y,0,0)$ will satisfy

$$L_m \phi(y,0,0) |0\rangle = 0, \quad \bar{L}_m \phi(y,0,0) |0\rangle = 0, \quad m \geq 2. \quad (4.3.1)$$

This follows from the fact that $\phi$ is a scalar and the bulk points $(y,0,0)$ are invariant under bulk Virasoro transformations generated by $L_m$ with $m \geq 2$. We explain this in detail in section 4.3.1 and appendix C.3.
In the following discussion, we will write $\phi(y,0,0)$ as an expansion in small $y$ or the boundary OPE expansion (BOE)$^9$

$$\phi(y,0,0) |0\rangle = \sum_{N=0}^{\infty} y^{2h+2N} |\phi\rangle_N$$  (4.3.2)

where $|\phi\rangle_N$ is a level $N$ Virasoro descendant of $O$ in both holomorphic and anti-holomorphic sectors, since we are defining the proto-field $\phi$ to be made of $O$ and its descendants.$^{10}$

Then the conditions (4.3.1) for $\phi(y,0,0)$ will be equivalent to saying that $|\phi\rangle_N$ satisfies the following ‘bulk primary’ conditions:

$$L_m |\phi\rangle_N = 0, \quad T_m |\phi\rangle_N = 0, \quad \text{for } m \geq 2.$$  (4.3.3)

That is, $\phi(y,0,0)$ will be a sum over these operators $\phi_N$ of different levels. The $|\phi\rangle_N$ is, in a sense, as close as possible to being a primary itself while still living in the bulk (ie it is a primary that is not quasi-primary). It is an eigenstate of $L_0$ and is annihilated by all higher generators except $L_1$. We will say more about the non-trivial action of $L_1$ in appendix C.3.

In particular, the conditions (4.3.3) imply that at each level, $|\phi\rangle_N$ factorizes, and can be written in the following form

$$|\phi\rangle_N = \lambda_N \mathcal{L}_{-N} \mathcal{L}_{-N} |O\rangle,$$

$$\lambda_N = \frac{(-1)^N}{N! (2h)^N}.$$  (4.3.4)

where $\mathcal{L}_{-N}$ (and $\mathcal{L}_{-N}$) are linear combinations of products of holomorphic (and anti-holomorphic) Virasoro generators at level $N$. Note that, the holomorphic and anti-holomorphic conditions $^9$In the conventional BCFT case, the bulk theory is a CFT (see [133] for a nice discussion). An identical expansion also applies when studying non-gravitational QFTs in AdS [127], because boundary dilatations correspond to a bulk isometry. When the bulk theory is gravitational, one cannot use pure symmetry or OPE type arguments to prove that this expansion converges, but our results suggest that it can be determined exactly to all orders in $y$ after bulk gauge fixing. It seems reasonable to expect that the small $y$ expansion of $\phi$ would have a finite radius of convergence, since no terms like $\sim e^{-1/y}$ are allowed by scaling symmetry. We also explain in appendix C.1.3 that symmetry arguments dictate this global conformal BOE result [127, 108]

$^{10}$More generally, a full bulk field would have terms like $y^{\tilde{h}'}+k |O_{h',\tilde{h}}\rangle$, where $O_{h',\tilde{h}}$ is not a descendant of $O$. 

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above are independent, which means that $\mathcal{L}_{-N}$ will just be $\mathcal{L}_{-N}$ with $L$ replaced by $\mathcal{L}$.

The conditions (4.3.3) will uniquely determine $|\phi\rangle_N$ (or $\mathcal{L}_{-N}$) up to an overall normalization (will be explained below). The overall normalization of $|\phi\rangle_N$ is fixed by

$$L_1^N \mathcal{L}_1^N |\phi\rangle_N = (-1)^N N! (2h)_N |\mathcal{O}\rangle.$$  \hspace{1cm} (4.3.5)

This normalization condition is based on the requirement that we correctly reproduce the vacuum correlator $\langle \phi \mathcal{O} \rangle$, that is, $\langle \phi(y, 0, 0) \mathcal{O}(z, \bar{z}) \rangle = \langle \phi^{\text{global}} \mathcal{O} \rangle = \left( \frac{y}{y^2 + z \bar{z}} \right)^{2h}$. $\phi^{\text{global}}$ here is the global bulk field in the HKLL reconstruction [87], which we explain in C.1.3 is equivalent to

$$\phi^{\text{global}}(y, 0, 0) |0\rangle = \sum_{N=0}^{\infty} y^{2h+2N} \lambda_N L_1^N \mathcal{L}_1^N |\mathcal{O}\rangle.$$ \hspace{1cm} (4.3.6)

So the requirement that $\langle \phi \mathcal{O} \rangle = \langle \phi^{\text{global}} \mathcal{O} \rangle$ implies that

$$\mathcal{L}_{-N} |\mathcal{O}\rangle = L_1^N |\mathcal{O}\rangle + (\text{other quasi-primaries and their descendants})$$ \hspace{1cm} (4.3.7)

where the terms in the parenthesis are all orthogonal to $\mathcal{O}$ and its global descendants, and will not contribute when computing $\langle \phi \mathcal{O} \rangle$. They are then fixed by solving (4.3.3). When acting on $|\phi\rangle_N$ with $L_1^N \mathcal{L}_1^N$, the terms in the parenthesis will be killed, that’s why we have the normalization condition (4.3.5).\footnote{Specifically, $L_1^N \mathcal{L}_1^N |\phi\rangle_N = L_1^N \mathcal{L}_1^N |\phi^{\text{global}}\rangle = \lambda_N L_1^N \mathcal{L}_1^N L_1^N \mathcal{L}_1^N |\mathcal{O}\rangle = (-1)^N N! (2h)_N |\mathcal{O}\rangle$.} It’s also true that in the large $c$ limit, our $\phi$ will reduce to $\phi^{\text{global}}$, as will be shown in 4.3.2.2 that the terms in the parenthesis are suppressed at large $c$. 

Now let us explain why the conditions (4.3.3) uniquely determine $\mathcal{L}_{-N}$. It is easy to see that they are equivalent to the equations

$$L_{m_1} \cdots L_{m_i} |\phi\rangle_N = 0, \quad \sum_i m_i = N$$ \hspace{1cm} (4.3.8)

(and similarly for the anti-holomorphic part) where $L_{m_1} \cdots L_{m_i}$ represents the set of all level
$N$ products of Virasoro generators with at least one $L_{m_i}$ with $m_i \geq 2$. That is, $L_{m_1} \cdots L_{m_i}$ does not include $L_1^N$. These conditions say that when $L_{m_1} \cdots L_{m_i}$ decreases the level of $|\phi\rangle_N$ back to level zero, the result vanishes. There are $p(N) - 1$ independent ways (because we exclude $L_1^N$) to lower $|\phi\rangle_N$ to level zero, and thus $|\phi\rangle_N$ must satisfy $p(N) - 1$ constraint equations. Since all the level $N$ descendants of $|\mathcal{O}\rangle$ form a $p(N)$ dimensional space, the above condition will fix the bulk field up to an overall constant. So $\phi(y, 0, 0)$ will be uniquely fixed\textsuperscript{12} by the constraints (4.3.3) and the normalization condition (4.3.5).

In section 4.3.1 we motivate the definition of $\phi$ using Virasoro symmetry and the fact that $\phi$ is a bulk scalar field. We then solve these conditions in various cases in section 4.3.2. In section 4.3.3, we show that our definition of $\phi(y, 0, 0)$ leads to a powerful recursive algorithm to compute correlators of the form of equation (4.1.2), extending standard recursion relations for correlators of stress tensors with local CFT\textsubscript{2} primary operators. The results exactly agree with those obtained from the bulk-boundary OPE block in section 4.2.

### 4.3.1 Virasoro Transformations of $\phi(X)$

In this section we will derive (4.3.1) using the fact that $\phi$ must transform as a bulk scalar. This means that under a coordinate transformation, $\phi(z, \bar{z}, y) \rightarrow \phi(z', \bar{z}', y')$.

We would like to understand the transformation of $\phi$ under the action of Virasoro, which is defined on the boundary by $(z, \bar{z}) \rightarrow (g(z), \bar{g}(\bar{z}))$. We will constructively demonstrate that there is a unique extension of an infinitesimal boundary Virasoro transformation preserving the Fefferman-Graham gauge. Infinitesimally, we have

$$\epsilon L_m(y, z, \bar{z}, S, \bar{S}) \equiv \epsilon(\delta_m y, \delta_m z, \delta_m \bar{z}, \delta_m S, \delta_m \bar{S}).$$

(4.3.9)

where $S, \bar{S}$ parameterizes the metric and are defined in (4.2.10). Then the transformation of

\textsuperscript{12}This means that $\phi$ has been fixed exactly (ie non-perturbatively in $c$, and not just to all orders in a $1/c$ expansion) to all-orders in powers of $y$. It’s less clear if we have determined $\phi$ exactly in both $c$ and $y$ simultaneously, though it would appear that we have for cases where $\phi$ is inserted into a correlator where the sum over $y^{2N}$ powers has a finite radius of convergence.
φ under an infinitesimal Virasoro generator $L_m$ is determined by its scalar property:

$$L_m \phi(z, \bar{z}, y) = (\delta_m y \partial_y + \delta_m z \partial + \delta_m \bar{z} \bar{\partial}) \phi(z, \bar{z}, y)$$  \hspace{1cm} (4.3.10)

This transformation rule is expected to hold within correlation functions.

We work out the gauge preserving extension of $L_m$ in Appendix C.3, with the result

$$\delta_m y = \frac{1}{2}(m + 1)y z^m$$  \hspace{1cm} (4.3.11)

$$\delta_m \bar{z} = \frac{zm^{-1} \left( (m^2 + m + z^2 S(z)) \bar{S}(\bar{z}) y^4 - 4z^2 \right)}{y^4 S(z) \bar{S}(\bar{z}) - 4}$$  \hspace{1cm} (4.3.12)

$$\delta_m \bar{z} = \frac{2m(m + 1)y^2 z^{m-1}}{y^4 S(z) \bar{S}(\bar{z}) - 4}$$  \hspace{1cm} (4.3.13)

We have verified that these results agree with the action of $L_m$ computed using contour integrals [46] of the stress tensor correlators from section 4.2.3. These results have several notable features. First, they reduce to the expected form of a Virasoro transformation on the boundary:

$$\lim_{y \to 0} (\delta_m y, \delta_m z, \delta_m \bar{z}) = (0, z^{m+1}, 0).$$  \hspace{1cm} (4.3.14)

Secondly, the transformation on the coordinates depends on the starting metric through $(S, \bar{S})$. This fact is easy to understand because if no such dependency existed, then we would not be able preserve the Fefferman-Graham form of the metric in general.

The central feature of these transformations is that for $m \geq 2$, points on the line $(y, 0, 0)$ are left invariant:

$$\delta_m (y, 0, 0) = 0 \quad \text{for } m \geq 2.$$  \hspace{1cm} (4.3.15)
Using the scalar property (4.3.10), we find that

\[ L_m \phi(y, 0, 0) |0\rangle = 0, \quad \text{for } m \geq 2. \]

(4.3.16)

Including the constraints from \( \bar{L}_m \), we arrive at conditions (4.3.1) satisfied by \( \phi(y, 0, 0) \).

One can also motivate the conditions (4.3.1) satisfied by \( \phi(y, 0, 0) \) by consideration of causality [113, 114, 115, 121]. Correlators of \( \phi(y, 0, 0) \) with boundary stress tensors \( T(z) \) necessarily have singularities of the form \( \frac{1}{z^2} \), as the stress tensor must be sensitive to the energy-momentum ‘charge’ of the bulk field, as well as \( \frac{1}{z^3} \) singularities, since special conformal transformations move \( \phi \) around in the bulk.\(^{13}\) However, one may wish to forbid branch cuts and higher order singularities such as \( \frac{1}{z^n} \) with \( n \geq 4 \). Our \( \phi(y, 0, 0) \) is constructed to satisfy these requirements. The conditions on \( \phi \) are equivalent to stipulating that the singular terms in the OPE of the stress energy tensor \( T(z) \) with \( \phi(y, 0, 0) \) are

\[ T(z) \phi(y, 0, 0) \sim \frac{L_{-1} \phi(y, 0, 0)}{z} + \frac{L_0 \phi(y, 0, 0)}{z^2} + \frac{L_1 \phi(y, 0, 0)}{z^3}. \]

(4.3.17)

So there will be no higher order singularities in correlators of \( \phi \) with any number of \( T \).

This property also holds for the individual components \( \phi_N \). One can also see this explicitly in the correlators \( \langle \phi OT \rangle, \langle \phi OTT \rangle, \) and \( \langle \phi OT\bar{T} \rangle \) that we computed using bulk-boundary OPE blocks in section 4.2.3, where there are no singularities beyond \( \frac{1}{z^3} \), including in the expansions of these expressions in \( y \).

### 4.3.2 Solving for \( \phi(X) \) Explicitly

In this section, we will solve the conditions (4.3.3) and the normalization condition (4.3.5) for \( \phi(y, 0, 0) \) explicitly. We will focus on the holomorphic part of \( |\phi\rangle_N = \lambda_N \mathcal{L}_{-N} \mathcal{L}_{-N} |O\rangle \) and solve for \( \mathcal{L}_{-N} \), since \( \bar{\mathcal{L}}_{-N} \) is just the anti-holomorphic conjugate. In terms of \( \mathcal{L}_{-N} \), the

\(^{13}\) These singularities could move to a different location in a different gauge, but they cannot be eliminated entirely [114].
conditions are

\[ L_m \mathcal{L}_N |\mathcal{O}\rangle = 0, \quad \text{for } 2 \leq m \leq N \]  

\[ L_1^N \mathcal{L}_N |\mathcal{O}\rangle = N!(2h)_N |\mathcal{O}\rangle \]  

We first provide an example at low orders in section 4.3.2.1, and then we obtain an exact, all orders solution in terms of orthogonal quasi-primaries in 4.3.2.2. We also solve these conditions in the large \( c \) limit up to order \( \mathcal{O}(c^{-2}) \) in appendix C.4.5.3.

4.3.2.1 Explicit Solutions at Low Orders

It is obvious that \( |\phi\rangle_0 = |\mathcal{O}\rangle \) and \( |\phi\rangle_1 = -\frac{1}{2h}L_{-1} \tilde{L}_{-1} |\mathcal{O}\rangle \), and so the first non-trivial case arises at the next level. At level 2, an arbitrary \( \mathcal{L}_{-2} \) is given by \( \mathcal{L}_{-2} = b_1L_{-1}^2 + b_2L_{-2} \) and the conditions are

\[ L_2 \left( b_1L_{-1}^2 + b_2L_{-2} \right) |\mathcal{O}\rangle = 0, \]  

\[ L_1^2 \left( b_1L_{-1}^2 + b_2L_{-2} \right) |\mathcal{O}\rangle = 2!(2h)_2 |\mathcal{O}\rangle. \]  

Solving these two equations for \( b_1 \) and \( b_2 \), we find

\[ \mathcal{L}_{-2} = \frac{(2h+1)(c+8h)}{(2h+1)c+2h(8h-5)} \left( L_{-1}^2 - \frac{12h}{c+8h}L_{-2} \right) \]  

and \( |\phi\rangle_2 \) is given by \( |\phi\rangle_2 = \lambda_2 \mathcal{L}_{-2} \tilde{L}_{-2} |\mathcal{O}\rangle \). One can continue this process at higher orders (we also computed \( |\phi\rangle_3 \) and \( |\phi\rangle_4 \) in Appendix C.4.5.1.), although the explicit expressions become rather complicated. Instead we will see how to solve these equations in general in terms of quasi-primaries.
4.3.2.2 Solution in Terms of Quasi-Primaries

We know that $|\phi\rangle_N$ can be written as the sum of the level $N$ descendants of $\mathcal{O}$. These descendants can be decomposed into quasi-primaries (global primaries) and their global conformal descendants. In this subsection, we will show that the coefficients in this decomposition are determined by the norms of the quasi-primaries. We already saw an obvious example in the global case, as the global descendant $L_{-1}^N \bar{L}_{-1}^N |\mathcal{O}\rangle$ appears as

$$|\phi\rangle_N \supset (-1)^N \frac{1}{N! (2h)_N} L_{-1}^N \bar{L}_{-1}^N |\mathcal{O}\rangle = (-1)^N \frac{L_{-1}^N \bar{L}_{-1}^N |\mathcal{O}\rangle}{|L_{-1}^N |\mathcal{O}\rangle|^2}$$

(4.3.23)

where $|L_{-1}^N |\mathcal{O}\rangle|^2 \equiv \langle \mathcal{O} | L_{-1}^N L_{-1}^N |\mathcal{O}\rangle = N!(2h)_N$. We will show that phenomenon is a general feature of the quasi-primary decomposition.

Suppose $\mathcal{L}_{-N}^{\text{quasi}}$ is a linear combination of Virasoro generators that acts on $|\mathcal{O}\rangle$ to create a quasi-primary at level $N$, with the coefficient of $L_{-1}^N$ in $\mathcal{L}_{-N}^{\text{quasi}}$ normalized to 1. For example, at level two there is a unique $\mathcal{L}_{-2}^{\text{quasi}} = L_{-1}^2 - \frac{2(2h+1)}{3} L_{-2}$. Since there are many quasi-primaries at level $N$, we will take the quasi-primary created by our chosen generator $\mathcal{L}_{-N}^{\text{quasi}}$ to be orthogonal to all of the other level $N$ quasi-primaries, and normalized to contain exactly $L_{-1}^N$.

In what follows we will treat the holomorphic and anti-holomorphic descendants of $\mathcal{O}$ separately, since at each level $\phi_N$ factorizes. Then we will combine the holomorphic and anti-holomorphic pieces and correctly normalize them. Let us define the coefficient of $\mathcal{L}_{-N}^{\text{quasi}} |\mathcal{O}\rangle$ in $|\phi\rangle_N = \lambda_N \mathcal{L}_{-N} |\mathcal{O}\rangle$ to be $b_N$, that is

$$|\phi\rangle_N \supset b_N \mathcal{L}_{-N}^{\text{quasi}} |\mathcal{O}\rangle$$

(4.3.24)

\footnote{The number of quasi-primaries at level $N$ is $p(N) - p(N-1)$, where $p(N)$ is the number of partitions of $N$.}

\footnote{Via an abuse of notation, here $|\phi\rangle_N = \lambda_N \mathcal{L}_{-N} |\mathcal{O}\rangle$, but it should be clear from the context whether $\mathcal{L}_{-N}$ is included in the definition of $|\phi\rangle_N$ or not.}
When we take the inner product of $|\phi\rangle_N$ with $L_{-N}^{\text{quasi}} |\mathcal{O}\rangle$, we obtain

$$
\langle \mathcal{O} | (L_{-N}^{\text{quasi}})^\dagger |\phi\rangle_N = b_N \langle \mathcal{O} | (L_{-N}^{\text{quasi}})^\dagger L_{-N}^{\text{quasi}} |\mathcal{O}\rangle \equiv b_N |L_{-N}^{\text{quasi}} \mathcal{O}\rangle^2,
$$

(4.3.25)

because $L_{-N}^{\text{quasi}} |\mathcal{O}\rangle$ is orthogonal to all other states in $|\phi\rangle_N$.

Now, using the conditions defining $\phi_N$, we have

$$
\langle \mathcal{O} | \left( (L_{-N}^{\text{quasi}})^\dagger - L_1^N \right) |\phi\rangle_N = 0
$$

(4.3.26)

because all of the terms in $(L_{-N}^{\text{quasi}})^\dagger - L_1^N$ will include at least one $L_m$, with $m \geq 2$, and according to the conditions (4.3.8), these terms will all annihilate $|\phi\rangle_N$. Using the normalization condition

$$
L_1^N |\phi\rangle_N = \frac{(-1)^N}{N!(2h)_N} L_1^N L_{-1}^N |\mathcal{O}\rangle = (-1)^N |\mathcal{O}\rangle,
$$

(4.3.27)

equation (4.3.26) leads to

$$
b_N = \frac{(-1)^N}{|L_{-N}^{\text{quasi}} \mathcal{O}\rangle^2}.
$$

(4.3.28)

So we have shown that the coefficient of the level $N$ quasi-primary $L_{-N}^{\text{quasi}}$ will be given by the inverse of its norm. Actually, one can show that this is also true even for the global descendants of the quasi-primaries. The holomorphic part of $|\phi\rangle_N$ will be given in the following form:16

$$
|\phi\rangle_N \propto (-1)^N \left( \frac{L_1^N}{|L_{-1}^N |\mathcal{O}\rangle^2} + \frac{L_{-1}^{\text{quasi}}}{|L_{-N}^{\text{quasi}} |\mathcal{O}\rangle^2} + \frac{L_{-1} L_{-1}^{\text{quasi}}}{|L_{-1}^{\text{quasi}} L_{-1}^N |\mathcal{O}\rangle^2} + \cdots + \frac{L_m L_{-1}^{\text{quasi}}}{|L_m L_{-N}^{\text{quasi}} |\mathcal{O}\rangle^2} + \cdots \right) |\mathcal{O}\rangle.
$$

Including the anti-holomorphic part and accounting for the overall coefficient (ie requiring

16It is easy to see $|L_{-1}^m L_{-(N-m)}^{\text{quasi}} |\mathcal{O}\rangle^2 = m!(2(h + N - m))_m |L_{-(N-m)}^{\text{quasi}} |\mathcal{O}\rangle^2$. 

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the coefficient of $L_{-1}^N L_{-1}^N$ to be $\lambda_N = \frac{(-1)^N}{|L_{-1}^N \mathcal{O}|^2}$, we find

$$|\phi\rangle_N = (-1)^N |L_{-1}^N \mathcal{O}|^2 \left( \frac{L_{-1}^N}{|L_{-1}^N \mathcal{O}|^2} + \frac{L_{N}^{\text{quasi}}}{|L_{-N}^{\text{quasi}} \mathcal{O}|^2} + \frac{L_{-1}^N L_{(N-1)}^{\text{quasi}}}{|L_{-1}^N L_{(N-1)}^{\text{quasi}} \mathcal{O}|^2} + \cdots \right)$$

$$\times \left( \frac{L_{-1}^N}{|L_{-1}^N \mathcal{O}|^2} + \frac{L_{N}^{\text{quasi}}}{|L_{-N}^{\text{quasi}} \mathcal{O}|^2} + \frac{L_{-1}^N L_{(N-1)}^{\text{quasi}}}{|L_{-1}^N L_{(N-1)}^{\text{quasi}} \mathcal{O}|^2} + \cdots \right) |\mathcal{O}\rangle$$

as the exact solution for $|\phi\rangle_N$ in terms of orthogonal quasi-primaries with our chosen normalization. Note that in a large $c$ expansion, the norms of the non-trivial quasi-primaries (and their descendants) will be proportional to positive powers of $c$, so that their contributions will be suppressed. But at finite $c$, or for $h \gtrsim c$, their contributions will be on equal footing with the global conformal descendants $\phi$.

As an illustration of the above result, $\lambda_2 L_{-2}$ in $|\phi\rangle_2$ derived in equation (6.2.9) of last section can be written in the following form:

$$\lambda_2 L_{-2} = \frac{L_{-1}^2}{2! (2h)_2} + \frac{L_{-1}^2 - \frac{2(2h+1)}{3} L_{-2}}{2(2h+1) ((2h+1)c + 2h(8h-5))} = \frac{L_{-1}^2}{|L_{-1}^2 \mathcal{O}|^2} + \frac{L_{-2}^{\text{quasi}}}{|L_{-2}^{\text{quasi}} \mathcal{O}|^2}$$

with $|L_{-2}^{\text{quasi}} \mathcal{O}|^2 = \frac{2}{9} ((2h+1)c + 2h(8h-5))$. We also explicitly compute $|\phi\rangle_3$ and $|\phi\rangle_4$ in Appendix C.4.5.1.

### 4.3.3 Recursion Relation for Stress-Tensor Correlators

In section (4.2.3) we computed correlators of the form $\langle \phi \mathcal{O} T \cdots T \cdots \rangle$ using the bulk-boundary OPE block. In this section, we will derive a recursion relation that can be used to calculate these correlators. Specifically, we will express correlators with $n+1$ stress tensors in terms of a differential operator acting on correlators with fewer stress tensors. This relation generalizes the well-known case of $\langle \mathcal{O} \mathcal{O} T \cdots T \cdots \rangle$ correlators [45], which can be derived recursively from the two point function $\langle \mathcal{O} \mathcal{O} \rangle$ using the Virasoro Ward identity.
Suppose we know the correlator with \( n \) insertions of \( T \) and \( m \) insertions of \( \bar{T} \),

\[
G_{n,m} \equiv \left\langle T(z_1) \cdots T(z_n) \bar{T}(\bar{w}_1) \cdots \bar{T}(\bar{w}_m) \mathcal{O}(z, \bar{z}) \phi(y, 0, 0) \right\rangle, \tag{4.3.31}
\]

and now we consider the case of one more \( T \) insertion,

\[
G_{n+1,m} \equiv \left\langle T(z_1) \cdots T(z_n) T(z_{n+1}) \bar{T}(\bar{w}_1) \cdots \bar{T}(\bar{w}_m) \mathcal{O}(z, \bar{z}) \phi(y, 0, 0) \right\rangle. \tag{4.3.32}
\]

A key feature of stress tensor correlators such as \( G_{n+1,m} \) is that as \( z_{n+1} \to \infty \), the correlator vanishes. This means that \( G_{n+1,m} \) is completely determined by its poles in the \( z_{n+1} \) variable. Thus \( G_{n+1,m} \) can be computed by taking the OPE of \( T(z_{n+1}) \) with all the other operators in \( G_{n+1,m} \) and only keeping the singular terms. We know the singular terms in the OPE of \( T(z_{n+1}) \) with \( \mathcal{O}(z, \bar{z}) \) and \( T(z_i) \), which are

\[
T(z_{n+1}) \mathcal{O}(z, \bar{z}) \sim \frac{h \mathcal{O}(z, \bar{z})}{(z_{n+1} - z)^2} + \frac{\partial \mathcal{O}(z, \bar{z})}{z_{n+1} - z},
\]

\[
T(z_{n+1}) T(z_i) \sim \frac{c}{2 (z_{n+1} - z_i)^4} + \frac{2 T(z_i)}{(z_{n+1} - z_i)^2} + \frac{\partial T(z_i)}{z_{n+1} - z_i}.
\]

The conditions of equation (4.3.3) tell us that the singular terms in the OPE of \( T(z_{n+1}) \) with \( \phi(y, 0, 0) \) are given by

\[
T(z_{n+1}) \phi(y, 0, 0) \sim \frac{L_1 \phi(y, 0, 0)}{z_{n+1}^3} + \frac{L_0 \phi(y, 0, 0)}{z_{n+1}^2} + \frac{L_{-1} \phi(y, 0, 0)}{z_{n+1}}. \tag{4.3.33}
\]

Writing \( |\phi\rangle \) as a sum over \( |\phi\rangle_N \), that is \( |\phi\rangle = \sum_{N=0}^{\infty} y^{2h+2N} |\phi\rangle_N \), we know that the effect of \( L_0 \) on \( |\phi\rangle \) is to pull down a factor of \( h + N \) for each \( |\phi\rangle_N \). This is equivalent to taking the derivative with respect to \( y \), so we have

\[
L_0 \phi(y, 0, 0) = \frac{1}{2} y \partial_y \phi(y, 0, 0). \tag{4.3.34}
\]
And it’s easy to see that

\[ L_{-1} \phi (y, 0, 0) = \partial_y \phi (y, x, \bar{x}) \big|_{x, \bar{x} = 0}. \]  

(4.3.35)

Because of translation invariance, the action of \( L_{-1} \) on \( \phi (y, 0, 0) \) is equal to a holomorphic partial derivative of all of the other operators, namely \( \mathcal{O} (z, \bar{z}) \) and other \( T \)s in the correlator \( G_{n,m} \).

The term \( L_1 \phi (y, 0, 0) \) is more subtle. In general, at finite \( c \) we cannot write it as a simple differential operator acting on \( \phi (y, 0, 0) \) itself (see appendix C.3 for more details). But since \( L_{-1} \) annihilates the vacuum, i.e., \( \langle 0 | L_1 = (L_{-1} | 0) \rangle^\dagger = 0 \), we can commute \( L_1 \) with all the other operators on the left. Since we know the action of \( L_1 \) on \( \mathcal{O} \) and the stress tensor,\(^\text{17} \) we can evaluate its action on \( \phi \) within the vacuum sector correlator \( G_{n,m} \).

Combining all the above facts, we obtain a recursion relation for computing \( G_{n+1,m} \) from \( G_{n,m} \) and \( G_{n-1,m} \):

\[
G_{n+1,m} = \left( -\frac{\partial_z + \sum_{i=1}^n \partial_{z_i}}{z_{n+1}} + \frac{\partial_y}{z_{n+1}^2} - \frac{z (2h + z\partial_z)}{z_{n+1}^3} + \sum_{i=1}^n \frac{-z_i (4 + z_i\partial_{z_i})}{z_{n+1}^3} \right) G_{n,m}
+ \left( \frac{h}{(z_{n+1} - z)^2} + \frac{\partial_z}{(z_{n+1} - z)} + \sum_{i=1}^n \left( \frac{2}{(z_{n+1} - z_i)^2} + \frac{\partial_{z_i}}{z_{n+1} - z_i} \right) \right) G_{n,m}
+ \sum_{i=1}^n \frac{\langle T (z_1) T (z_2) \cdots T (z_{i-1}) T (z_{i+1}) \cdots T (z_n) \bar{T} (w_1) \cdots \bar{T} (w_m) \mathcal{O} (z, \bar{z}) \phi (y, 0, 0) \rangle}{2 (z_{n+1} - z_i)^4}
\]

(4.3.36)

We display the origin of all of these terms in appendix C.4.6. In appendix C.4.6, we also use this recursion relation to easily reproduce the correlators \( \langle \phi OT \rangle, \langle \phi OTT \rangle \) and \( \langle \phi OT \bar{T} \rangle \) computed in section 4.2.3 using the bulk-boundary OPE block.

One can derive an identical recursion relation with \( T \leftrightarrow \bar{T} \) for adding insertions of the

\(^{17}\)The commutators of \( L_1 \) with \( \mathcal{O} \) and \( T \) are simply

\[
[L_1, \mathcal{O} (z, \bar{z})] = z (2h + z\partial_z) \mathcal{O} (z, \bar{z}),
\]

\[
[L_1, T (z_i)] = z_i (4 + z_i\partial_{z_i}) T (z_i).
\]
anti-holomorphic stress tensor. Together, these relations precisely determine all vacuum sector correlators of \( \phi \mathcal{O} \). In other words, one can view these recursion relations as an alternative definition for the proto-field \( \phi \), which is entirely equivalent to the definition (4.3.16) and the bulk-boundary OPE block prescription and accompanying regulator from section 4.2.

4.4 Discussion

It is natural to conjecture [113] that complete, interacting scalar fields \( \Phi(X) \) in AdS\(_3\) should be written as

\[
\Phi(X) = \sum_\mathcal{O} \lambda_\mathcal{O} \phi_\mathcal{O}(X) \tag{4.4.1}
\]

where the sum runs over all scalar CFT\(_2\) primaries, and the coefficients \( \lambda_\mathcal{O} \) are constrained by consistency and causality [113, 114, 115, 121]. Our work does not shed much light on the questions of existence, (non-)uniqueness, and efficient determination of the \( \lambda_\mathcal{O} \).

However, we have proposed a formula for the local AdS\(_3\) proto-field operator \( \phi_\mathcal{O} \) built from a specific CFT\(_2\) primary \( \mathcal{O} \) and its Virasoro descendants.\(^{18}\) We argued that our choice of \( \phi_\mathcal{O} \) has a number of desirable properties, including healthy vacuum-sector correlators that match bulk Witten diagrams, a natural interpretation in any semiclassical vacuum geometry, and Virasoro symmetry transformations implemented as bulk diffeomorphisms. But perhaps the most surprising and intriguing aspect of our analysis is that we have determined \( \phi_\mathcal{O} \) exactly, based on the extremely simple condition of equation (4.3.1).

Profound lore based on diffeomorphism gauge redundancy and black hole physics suggests that local observables in gravitational theories may be ambiguous\(^{19}\) or ill-defined. Hopefully our formalism will provide a context where these ideas can be made more precise. It may be

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\(^{18}\)This may be enough to reconstruct the (toy?) case of a CFT with a low-energy spectrum that is dual to AdS\(_3\) gravity coupled to a free bulk scalar field.

\(^{19}\)For an example of an interesting recent discussion see [103].
that AdS$_3$ differs significantly from the case of higher dimensions (or AdS$_3 \times X$ spacetimes), where most aspects of bulk gravitational physics cannot be fixed by symmetry, and the gravitational dynamics can depend on many parameters. In CFT$_{\geq 3}$ this difference arises because the OPE of the stress tensor is largely unconstrained, in marked contrast with the CFT$_2$ case.

**How Non-Local is $\phi$?**

Our construction of $\phi$ was based on a series expansion in the radial coordinate $y$, which may be viewed as a gravitational version of the boundary operator expansion of boundary CFT. The non-locality of $\phi$ (as a CFT operator) arises from the fact that it has been expressed as an infinite sum of local operators. In the global conformal case, one can precisely relate the standard HKLL smearing function to the boundary operator expansion (see appendix C.1.3), making the non-locality of $\phi$ manifest. The extent of the non-locality displayed by the exact Virasoro $\phi$ remains less clear. It should be possible to evaluate this region by computing correlators of $\phi$ with local CFT operators and investigating the convergence properties of the infinite sum.$^{20}$ There may be a more direct method involving a non-perturbative generalization of the smearing procedure.

These questions will be of particular interest when we move from Euclidean to Lorentzian signature. Lorentzian CFT correlators can be obtained from their Euclidean counterparts by analytic continuation, but we do not know to what extent this holds for bulk dynamics. At the very least we will need to have a better understanding of bulk diffeomorphisms, including large transformations to new gauges. From the bulk or Wheeler-DeWitt perspective, the formation and evaporation of a black hole can be pure gauge!

Many recent works have focused on the relationship between bulk and boundary domains of dependence [134, 88, 90, 135, 123] in Lorentzian signature. Some of this work [136] was

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$^{20}$This suggests an amusing exercise – one might Borel resum the boundary operator expansion for $\phi$. It seems plausible that the summation defining the Borel series (operator) would appear local, in the sense that its series expansion would converge in correlators with all other local operators.
motivated by putative ambiguities in bulk reconstruction associated with the fact that a bulk operator $\phi(X)$ can be expressed using smearing functions supported on different boundary domains [87]. These ambiguities do not exist for non-gravitational AdS field theory and its non-local boundary dual, as in this case $\phi(X)$ is precisely well-defined. Thus it appears that these ambiguities can only arise from non-perturbative gravitational effects. It would be interesting to exhibit such effects explicitly and to characterize their physical significance in the bulk; perhaps this is possible in AdS$_3$/CFT$_2$.

**Bulk Locality and Horizons**

The primary motivation for studying $\phi$ is to investigate bulk locality and physics near and beyond black hole horizons [137, 138, 104, 93].

The breakdown of bulk locality can be analyzed using scattering in AdS/CFT [41, 8]. However, one can attack the problem much more directly by studying the operator product $\phi(X)\phi(Y)$ and its expectation value. The correlator $\langle \phi(X)\phi(Y) \rangle$ can differ greatly from that of a free bulk scalar field because it includes the exchange of arbitrary Virasoro descendants of $\mathcal{O}$, or in the language of multi-trace operators, states such as $\langle [T\partial^2 T\bar{T}\bar{\partial}^4 \mathcal{O}] \rangle$ built from the OPE of $\mathcal{O}$ with any number of stress tensors. Since the contribution of these states has been fixed exactly, one can compute $\langle \phi(X)\phi(Y) \rangle$ at finite operator dimension $h$ and central charge $c$ and as a function of the geodesic separation between the bulk operators. When $h \ll c$ one might hope to see the breakdown of bulk locality at the Planck scale, and for heavy operators with $h \gg c$ one might see indications of the horizon radius (or some other pathology associated with bulk fields dual to very heavy CFT states). More generally, we would expect that the bulk OPE expansion of $\phi(X)\phi(Y)$ does not exist.

We can also use correlators like $\langle \phi \mathcal{O}_H \mathcal{O}_H \rangle$ and $\langle \phi \phi \mathcal{O}_H \mathcal{O}_H \rangle$ to probe the vicinity of black hole horizons. In these and other high-energy states, we may find that $\phi$ breaks down deep$^{21}$ in the bulk, and it will be interesting to understand when and how. Previously it was unclear

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$^{21}$Due to quantum gravitational effects, or from a failure of our Fefferman-Graham gauge choice.
how to study such observables in a non-trivial way, since it seemed that one would need to rely on bulk perturbation theory to define them. It appears that our construction surmounts this particular obstacle.

Many aspects of black hole thermodynamics are encoded in the Virasoro algebra at large central charge [15, 16, 23, 139, 140], including various non-perturbative effects that resolve or ameliorate information loss problems [12, 9, 2]. This means that it should be possible to learn about bulk physics in the presence of black holes using Virasoro technology.

Furthermore, general considerations [141] borne out by non-perturbative investigations of Virasoro blocks [2] show that in Euclidean space, pure high-energy quantum states look very different from the BTZ black hole solution in the vicinity of the horizon. This follows from the fact that thermal and BTZ correlators are periodic in Euclidean time, while pure state correlators display completely unsuppressed violations of this periodicity [2]. Thus we have reason to believe that correlators like $\langle \phi \phi O_H O_H \rangle$ will tell us about interesting structures near the Euclidean horizon. By decomposing correlators into Virasoro blocks, we can learn which of these effects are universal, and which depend on the details of the CFT data.

Of course the real question is whether black hole horizons appear innocuous to infalling Lorentzian observers. We hope to address some of these questions soon.
Chapter 5

The AdS$_3$ Propagator and the Fate of Locality

This chapter is based on the following paper:


Abstract

In the last chapter, we used Virasoro symmetry considerations to propose an exact formula for a bulk proto-field $\phi$ in AdS$_3$. In this chapter we study the propagator $\langle \phi \phi \rangle$. We show that many techniques from the study of conformal blocks can be generalized to compute it, including the semiclassical monodromy method and both forms of the Zamolodchikov recursion relations. When the results from recursion are expanded at large central charge, they match gravitational perturbation theory for a free scalar field coupled to gravity in our chosen gauge.

We find that although the propagator is finite and well-defined at long distances, its perturbative expansion in $G_N = \frac{2}{2c}$ exhibits UV/IR mixing effects. If we nevertheless interpret $\langle \phi \phi \rangle$ as a probe of bulk locality, then when $G_N m_\phi \ll 1$ locality breaks down at the new
short-distance scale $\sigma_s \sim \sqrt[3]{G_N R_{AdS}^3}$. For $\phi$ with very large bulk mass, or at small central charge, bulk locality fails at the AdS length scale. In all cases, locality ‘breakdown’ manifests as singularities or branch cuts at spacelike separation arising from non-perturbative quantum gravitational effects.

5.1 Introduction and Summary

General Relativity does not seem, at first glance, so very different from other effective quantum field theories. When we study GR in a perturbative expansion about a semiclassical background, it is tempting to interpret localized gravitational and matter fluctuations as the degrees of freedom that define the space of states. But the area-law entropy of black hole thermodynamics [142] starkly conflicts with this picture, which was already suspect due to considerations of diffeomorphism gauge redundancy [143]. So we must ask, to what extent can the conflicting viewpoints of local bulk effective field theory and holography be reconciled?

Our goal is to understand the limitations of bulk locality in a concrete, quantitative way in the context of AdS$_3$/CFT$_2$. This is a necessary step towards the resolution of the black hole information paradox in AdS/CFT [144, 145, 146], because the most striking form of the paradox is a disagreement between unitarity and effective field theory in the bulk that depends on the approximate existence of local bulk observables.

We recently proposed an exact definition [3] of a local bulk proto-field $\phi$ associated with a specific CFT$_2$ primary operator $O$. Physically, one can think of $\phi$ as the nearest one can get to defining a free scalar field coupled to AdS$_3$ gravity in a specific coordinate system, neglecting loops of $\phi$ itself. This bulk field operator automatically ‘knows’ about the dynamical gravitational background, or in other words it is ‘gravitationally dressed’. A simple algebraic definition [3] for $\phi$ exists because, roughly speaking, quantum gravity matrix elements in AdS$_3$ are determined by Virasoro symmetry.
In this work we will study the simplest local bulk observable, the vacuum propagator \( K = \langle \phi \phi \rangle \). We will compare perturbation theory in \( G_N = \frac{3}{2c} \), semiclassical methods, and exact numerical results. The computations we will present are possible because \( \phi \) has a very natural definition in CFT\(_2\), which means that many techniques for the efficient calculation of conformal blocks can be generalized to the study of \( \phi \) correlators. In particular, both the semiclassical ‘monodromy method’ [147, 39, 15, 16, 23] and the Zamolodchikov recursion relations [37, 36, 79] can be adapted and recruited to our cause.

In the remainder of this section we will separately summarize the technical machinery we have developed and the physical results we have obtained.

**Notation**

Throughout this paper we use \( h \) to refer to the conformal dimension of the primary operator \( O \) dual to the bulk proto-field \( \phi \) with mass \( m^2 R_{AdS}^2 = 4h(h-1) \). The CFT\(_2\) central charge is \( c = \frac{3R_{AdS}^2}{2G_N} \). When discussing the propagator \( K \equiv \langle \phi(X_1)\phi(X_2) \rangle \) (we use \( K \) and \( \langle \phi \phi \rangle \) to denote the propagator interchangeably) we often use the kinematic variable \( \rho = e^{-2\sigma} \), where \( \sigma(X_1, X_2) \) is the geodesic distance between the bulk points in the vacuum. In our coordinate system, the metric of empty AdS\(_3\) is

\[
ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2}.
\]

(5.1.1)

We compute the propagator in the AdS\(_3\) vacuum throughout. Explicitly, we have \( X_1 = (y_1, z_1, \bar{z}_1) \), \( X_2 = (y_2, z_2, \bar{z}_2) \) and \( \rho \equiv \frac{\xi^2}{(1+\sqrt{1-\xi^2})^2} \) with \( \xi \equiv \frac{2y_1y_2}{y_1^2+y_2^2+z_1\bar{z}_2} \), where \( \rho = \xi = 1 \) corresponds to vanishing separation in the bulk. In this coordinate system, the free field propagator, which we’ll denote as \( K_{\text{global}} \equiv \langle \phi \phi \rangle_{\text{global}} \) is given by

\[
K_{\text{global}} = \langle \phi \phi \rangle_{\text{global}} = \frac{\rho^h}{1-\rho}.
\]

(5.1.2)

\(^1\)The subscript 'global' here means that \( K_{\text{global}} \) is the 2-pt function of \( \phi_{\text{global}} \), which is the reconstruction of a bulk field using only global conformal symmetry.
$K_{\text{global}}$ is the large $c$ limit of $K$, i.e. $K_{\text{global}} = \lim_{c \to \infty} K$. We also study the ‘holomorphic part’ of $K$ due to purely holomorphic gravitons, which we denote as $K_{\text{h}0} = \langle \phi \phi \rangle_{\text{h}0}$ and define in Section 5.2.2.2.

### 5.1.1 Summary of Technical Developments

In section 5.2 we briefly review $\phi$ [3], and then discuss the properties of its correlators. We introduce the technically useful notion of the ‘holomorphic part’ $\phi_{\text{h}0}$, which corresponds in perturbation theory to computing $\phi$ correlators while only incorporating holomorphic gravitons. We show that knowledge of the $\phi_{\text{h}0}$ propagator $K_{\text{h}0}$ can be combined with purely global-conformal information to determine the complete $\phi$ propagator. We also emphasize that at two-loops and beyond the full propagator is not spherically symmetric as a consequence of our gauge choice. The full propagator depends on both $\rho$ and an angle of inclination with respect to the $z$–$\bar{z}$ plane.

Conformal blocks in CFT$_2$ exponentiate in the semiclassical approximation of large central charge where ratios of conformal dimensions to the central charge, $h/c$, are held fixed. This is dual to the semiclassical limit $G_N \to 0$ with $G_N m$ fixed in AdS$_3$. In section 5.3 we show that the propagator also has a semiclassical limit, and we derive a generalization of the monodromy method [147, 39, 15, 16, 23] that computes the semiclassical $K_{\text{h}0}$. We then use this method to obtain the semiclassical propagator to order $\frac{h^2}{c}$ in equation (5.3.18) and at large$^2 h$ in equation (5.3.35). At infinite $h$, we are able to go beyond the semiclassical limit and derive an exact expression for the block in (5.3.37). Finally, while we cannot obtain an analytic expression for the semiclassical part of the correlator at general $h/c$, it is straightforward to use the monodromy method to determine it numerically. We apply this technique to determine the critical value of the geodesic distance where the semiclassical part first develops an imaginary piece; the result is summarized in figure 5.1.

$^2$At large $h$ it is natural to define a variable $q$, as in equation (5.3.36); the variables $\xi, \rho, q$ play a similar role here as the variables $z, \rho, q$ used in the study of Virasoro blocks, though here we gain no advantage in convergence by using $q$ in place of $\rho$.  

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In section 5.4 we derive recursion relations that compute the propagator exactly as an expansion in small $\rho$ and $q$ (these are long-distance expansions). We find generalizations of both the $c$ and $h$ Zamolodchikov recursion relations [37, 36, 79], though we mostly use $h$-recursion as it is more efficient algorithmically. It is summarized by equation (5.4.10). The large $h$ limit of equation (5.3.37) is a crucial ingredient needed for $h$-recursion.

We discuss the perturbative expansion of the propagator in section 5.5. The full one-loop result is equation (5.5.2). We show in section 5.5.1 and appendix D.1 that our result agrees with the bulk one-loop Witten diagram. We also provide an explicit unitarity-based argument in the appendix, which ultimately relates the one-loop correction to the tree-level correlator $\langle \phi \mathcal{O} T \rangle$ computed previously [3].

5.1.2 UV/IR Mixing in Perturbation Theory

Before we analyze the interesting features of non-perturbative gravity, we must discuss a surprising result that is already visible at one-loop in gravitational perturbation theory! As discussed in detail in section 5.5.1, we find that in the short-distance limit $\sigma \ll R_{AdS}$, the one-loop corrected bulk propagator takes the form

\[
\langle \phi \phi \rangle \approx \frac{1}{\sigma} \left( 1 + \frac{3G_NR_{AdS}^3}{2\sigma^4} - \frac{G_NR_{AdS}(10 + m^2 R_{AdS}^2)}{4\sigma^2} + \cdots \right). \tag{5.1.3}
\]

Notice that the one-loop correction is very singular at short-distances, so that it competes with the free field propagator at $\sigma_* \sim \sqrt{\frac{G_NR_{AdS}^3}{\sigma}}$. In contrast, we might have expected a one-loop correction that scaled like $\frac{G_N}{\sigma}$, so that it only became important for separations of order the Planck length. Instead we have discovered an intermediate scale that mixes the UV Planck scale with the IR scale of $R_{AdS}$. This UV/IR mixing is not what one would expect for a local observable$^3$ in a local theory, and it would lead to power-law IR divergences if we were to take the flat space limit of AdS.

$^3$Of course our $\phi$ must be accompanied by ‘gravitational Wilson lines’, in the same way that the physical electron field must be attached to a Wilson line. So $\phi$ correlators are not truly local.

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One could interpret this result as an indication that we should modify the definition of $\phi$ or $K$ to eliminate this UV/IR mixing. For a variety of reasons discussed in section 5.5.1, it would seem that the required modifications would have to be rather consequential. In particular, since our results agree with bulk Witten diagrams at large $c$, the same modifications also apply to these Witten diagram calculations in AdS$_3$. Nevertheless, we believe this is an interesting avenue for future exploration. For the rest of this summary (and most of the paper) we will just study the naive vacuum $\phi$ propagator and assume that its correlators provide a meaningful probe of bulk locality, but one should keep in mind the caveat that the results could be different if we were to identify an observable free from UV/IR mixing.

### 5.1.3 Physics of the Exact Propagator and the Breakdown of Locality

By construction, $\phi$ is a real scalar field and its propagator should be a real-valued function. Both the propagator $K$ and the holomorphic part $K_{\text{holo}}$ should not develop imaginary parts, because there are no states for $\phi$ quanta to decay into.$^4$ So if we find that the exact $K$ or $K_{\text{holo}}$ develop imaginary parts at spacelike separation, then we may interpret this as a violation of bulk unitarity, even though the CFT itself remains perfectly healthy; potentially, the proto-field $\phi$ may be indicating the presence of an instability that arises when two $\phi$s are brought close together in the full bulk gravitational theory, where a complete bulk field would include not only $\mathcal{O}$ and its descendants but other states as well. In order to develop an imaginary part at a distance $\sigma_*$, $K$ must exhibit a singularity at spacelike separation, which also represents a direct violation of bulk locality.

In the global or $c = \infty$ limit we have $K_{\text{global}} = \lim_{c \to \infty} K = \frac{\rho^h}{1-\rho}$, which is real and finite for all spacelike geodesic separations $\sigma = -\frac{1}{2} \log \rho$. Furthermore, to all orders in

---

$^4$One can formalize these expectations for a local $\phi$ using the Kallen-Lehmann representation. Readers may wonder if $\phi$ can decay into gravitons, but this is forbidden for the proto-field. Specifically, all correlators $\langle \phi T \cdots \tilde{T} \cdots \tilde{T} \cdots \rangle$ vanish because $\phi$ is a linear combination of descendants of the Virasoro primary $\mathcal{O}$. In order for these correlators to be turned on, one would have to also include dressing of $\phi$ by operators that are not Virasoro descendants of $\mathcal{O}$. 

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perturbation theory, the propagator remains real and finite. However, we find that in various limits the exact propagator develops new singularities (branch cuts) indicating violations of bulk locality. Specifically:

- When studying light bulk fields with $h \ll c$, we can resum the the full $1/c$ expansion in the short-distance limit (section 5.5.2). The result is ambiguous, but generically includes an imaginary piece associated with the length scale $\sigma_\ast \propto c^{-1/4}$. We obtain substantial numeric evidence (section 5.6.1) that the light-field propagator develops a singularity at a finite separation that scales as $\sigma_\ast \propto c^{-1/4}$. In figure 5.7 we display evidence that the full and holomorphic propagators show the same scaling of $\sigma_\ast$ and $c$.

- Semiclassical results at $c \to \infty$ with $\frac{h}{c} \ll 1$ fixed indicate an apparent breakdown of locality at $\sigma_\ast \propto \left(\frac{h}{c}\right)^{\frac{1}{3}}$ (section 5.3.2). This is corroborated by semiclassical numerics (section 5.3.4) and by exact numerics (figure 5.6), which also demonstrate that our semiclassical results are reliable at large $c$ and spacelike separation.

- In the heavy bulk field limit $h \gg c$, we find the exact propagator analytically (section 5.3.3) and demonstrate that it develops a branch cut at $\sigma_\ast = R_{AdS} \log(2 + \sqrt{3}) \approx 1.32 R_{AdS}$. Thus for heavy bulk fields, locality breaks down at the AdS scale. We find numerically that in the limit of large $c$ and fixed $h/c$, our results smoothly interpolate (figure 5.6) between large $h$ and the fixed $h \ll c$ scaling $c^{-1/4}$. Moreover, we show that the behavior at large $h$ and at very small $c$ appear to be identical (figure 5.5), with locality breaking down at the same numerical multiple of the AdS scale in both cases.

Aside from the surprising UV/IR mixing effect and associated emergent scale $\sigma_\ast$ discussed above, this is roughly what one might have expected. Bulk locality makes approximate sense in gravitational perturbation theory, but breaks down due to non-perturbative gravitational effects in an explicitly quantifiable way. Light fields in theories with a large separation of scales between $G_N$ and $R_{AdS}$ can be local to a high degree of precision, but outside this regime bulk locality breaks down at the AdS length.
5.2 Structure of $\phi$ Correlators

In recent work [3] we provided an exact definition for the bulk scalar proto-field $\phi(y, z, \bar{z})$ as a linear combination of a primary CFT$_2$ scalar $\mathcal{O}$ and its Virasoro descendants.\(^5\) We refer to $\phi$ as (merely) a proto-field because its existence follows entirely from symmetry considerations in AdS$_3$/CFT$_2$. One might expect that full scalar fields\(^6\) can be represented as infinite sums of proto-fields [113, 114, 115, 121]. We can also think of the proto-field as a free scalar field in AdS$_3$ coupled to pure quantum gravity, where loops of $\phi$ itself have been neglected.

The proto-field is interesting because it encodes an infinite sum of quantum gravitational effects, which involve Virasoro (CFT stress tensor) matrix elements. For example, we will see that the propagator $\langle \phi(X_1)\phi(X_2) \rangle$ includes graviton loops to all-orders. The proto-field is labeled by a bulk point $(y, z, \bar{z})$ associated with a specific coordinate system (or gauge choice) where AdS$_3$ vacuum metrics take the form [131, 132]

\[
ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2} - \frac{6T(z)}{c}dz^2 - \frac{6\bar{T}(\bar{z})}{c}d\bar{z}^2 + y^2\frac{36T(z)\bar{T}(\bar{z})}{c^2}dzd\bar{z} \tag{5.2.1}
\]

for holomorphic functions $T(z), \bar{T}(\bar{z})$. We emphasize that the proto-field operator depends in an essential way on this gauge choice; were we to choose a different gauge, we would obtain a different bulk operator. The dependence on the gauge will appear explicitly later on, where we will see that in our gauge, the full propagator $\langle \phi(X)\phi(Y) \rangle$ is not spherically symmetric.

We will briefly review the definition of $\phi$ in section 5.2.1; for detailed explanations and derivations we refer the reader to [3]. The operator $\phi$ and its correlators do not factorize into a product of holomorphic and anti-holomorphic parts. However, it is possible to define a ‘holomorphic’ part $\phi_{\text{holo}}$, by which we mean that we only include the effects of holomorphic

\(^5\)There have been many approaches to reconstruction, for an incomplete sample see [107, 116, 87, 117, 113, 114, 115, 118, 91, 119, 108, 120, 129, 121, 112, 122, 123, 124, 125, 126]. Our prescription matches perturbation theory when expanded in $G_N \propto 1/c$ (see appendix D.4 of [3] and section 5.5.1 of this work), but this property does not seem to hold for some other approaches [126] that attempt to leverage Virasoro symmetry.

\(^6\)Full bulk scalar fields may not exist, and to the extent that they do exist, their definition may be ambiguous. These are interesting issues but we will not be addressing them here, as we will only be studying proto-fields and their correlators.
gravitons on $\phi$. We explain these facts and define $\phi_{\text{holo}}$ in section 5.2.2. This notion is useful because full $\phi$ correlators can be determined from $\phi_{\text{holo}}$ correlators using additional data that only depends on global conformal information. Throughout this paper we will primarily be studying $\phi_{\text{holo}}$ correlators.

5.2.1 Brief Review of the AdS$_3$ Proto-field $\phi$

We define the operator $\phi(y, z, \bar{z})$ using a Boundary Operator Expansion (BOE)

$$
\phi(y, z, \bar{z}) = \sum_{N=0}^{\infty} y^{2h+2N} \phi_N(z, \bar{z}).
$$

(5.2.2)

Each operator $\phi_N(z, \bar{z})$ can be defined by first translating $z \rightarrow 0$ and then applying the operator/state correspondence to study the state $|\phi\rangle_N = \phi_N(0, 0)|0\rangle$. These states are then defined by the bulk primary conditions

$$
L_m |\phi\rangle_N = 0, \quad \bar{L}_m |\phi\rangle_N = 0, \quad \text{for } m \geq 2.
$$

(5.2.3)

along with a normalization condition

$$
L_1^N \bar{L}_1^N |\phi\rangle_N = (-1)^N N! (2h)_N |O\rangle.
$$

(5.2.4)

The bulk primary condition can be given the simple, physical interpretation that the line $(y, 0, 0)$ is fixed by $L_{m \geq 2}$ in our gauge. The normalization condition simply guarantees that we recover the global conformal bulk reconstruction when $c = \infty$. These conditions have a unique solution [3], which can be conveniently written

$$
\phi(y, 0, 0) = \sum_{N=0}^{\infty} y^{2h+2N} \lambda_N L_N \bar{L}_N O(0),
$$

(5.2.5)
where \( \lambda_N \equiv \frac{(-1)^N}{(2h)_N N!} \) and the \( \mathcal{L}_N \) are a certain linear combination of holomorphic Virasoro generators\(^7\) at level \( N \), and similarly for the anti-holomorphic \( \bar{\mathcal{L}}_N \). When \( c \to \infty \) with other parameters held fixed, our prescription reduces to the global conformal bulk reconstruction of \( \phi \) that can be obtained from the ‘HKLL kernel’ [107, 87, 108], and we have the simplification \( \mathcal{L}_N \to L^N_{-1} \).

This prescription for \( \phi \) can be motivated in a number of ways; for details see [3]. When Virasoro transformations are realized as bulk diffeomorphisms preserving the gauge choice of equation (5.2.1), our definition emerges by demanding that \( \phi(y, z, \bar{z}) \) transforms as a bulk scalar field. Alternatively, one can arrive at our prescription by studying correlators of \( \phi \) with \( \mathcal{O}(x) \) and any number of stress tensors \( T(z_i) \) and \( \bar{T}(\bar{z}_i) \). After gauge fixing, Virasoro symmetry appears to determine these correlators exactly [77, 3], and their specification is equivalent to our definition of \( \phi \). In more conventional terms, our definition of \( \phi \) should agree with bulk gravitational perturbation theory to all orders in \( G_N = \frac{3}{2c} \), and this has been verified explicitly to order \( 1/c^3 \) for some observables. In section 5.5.1 we will verify the agreement between one-loop gravitational perturbation theory and our prescription for the propagator \( \langle \phi \phi \rangle \).

**Solution for \( \phi \) Using Quasi-Primaries**

For various purposes it is useful to solve for \( \phi_N \) explicitly in terms of quasi-primary states, which are annihilated by \( L_1 \) but not \( L_m \) with \( m \geq 2 \). Importantly, we will take the quasi-primaries to be orthogonal, and we fix their overall normalization by demanding that a level

\(^7\)For example, the explicit solution at level 2 is

\[
\mathcal{L}_{-2} = \frac{(2h+1)(c+8h)}{(2h+1)(c+2h(8h-5))} \left( L_{-1}^2 - \frac{12h}{c+8h} L_{-2} \right) 
\]

with \( \bar{\mathcal{L}}_{-2} \) only differing by \( L_{-n} \to \bar{L}_{-n} \).
$M$ quasi-primary includes the term $L_{-1}^M$ with overall coefficient $1$. In this basis, we showed\textsuperscript{8} that \([3]\)

$$\phi_N = (-1)^N |L_{-1}^N O| \left( \frac{L_{-1}^N}{|L_{-1}^N O|^2} + \frac{L_{-1}^{\text{quasi}}}{|L_{-1}^{\text{quasi}} O|^2} + \frac{L_{-1}^{\text{quasi}} (N-1)}{|L_{-1}^{\text{quasi}} (N-1) O|^2} + \cdots \right)$$

$$\times \left( \frac{\tilde{L}_{-1}^N}{|\tilde{L}_{-1}^N O|^2} + \frac{\tilde{L}_{-1}^{\text{quasi}}}{|\tilde{L}_{-1}^{\text{quasi}} O|^2} + \frac{\tilde{L}_{-1}^{\text{quasi}} (N-1)}{|\tilde{L}_{-1}^{\text{quasi}} (N-1) O|^2} + \cdots \right) O, \quad (5.2.8)$$

where the notation is slightly schematic, as each term represents a sum over all quasi-primaries at the indicated level.

Once we establish the overall coefficient of the quasi-primary contributions at level $(N, \bar{N})$, the contributions of all global conformal descendants of these quasi-primaries are fixed. Thus much of the non-trivial information required to define correlators of $\phi(X)$ is encoded in sums over inverse normalization factors

$$C_N \equiv \sum_{i=1}^{p(N)-p(N-1)} \frac{1}{|L_{-1}^{\text{quasi},i} O|^2}, \quad (5.2.9)$$

where the sum includes all quasi-primaries at level $N$ ($p(N)$ denotes the number of integer partitions, and the super-script $i$ denotes the $i$-th quasi-primary at level $N$). We can take advantage of this fact by finding efficient methods for isolating and determining the $C_N$ \cite{148}, and then recombining them to compute $\phi$ correlators.

\textsuperscript{8}For clarity, by $L_{-1}^{\text{quasi}_N}$ we mean $L_{-1}^{\text{quasi}_N}$ acting on $O$ creates a level $N$ quasi-primary, while $L_N$ defined in equation (5.2.5) is the sum of all level $N$ contributions to $\phi$ and it’s given by

$$L_N = L_{-1}^N + \frac{|L_{-1}^N O|^2}{|L_{-1}^{\text{quasi}} O|^2} L_{-2}^N + \frac{|L_{-1}^N O|^2}{|L_{-1}^{\text{quasi}} O|^2} L_{-2}^{\text{quasi}} + \cdots \quad (5.2.7)$$

and $|L_{-1}^N O|^2 = |\tilde{L}_{-1}^N O|^2 = (2h)_N N! = \frac{1}{|\lambda_N|}$.
5.2.2 ‘Holomorphic’ Parts Determine Full Correlators

In CFT, many observables can be decomposed into holomorphic and anti-holomorphic parts. For example, the conformal partial waves or conformal blocks involve sums over all states related by conformal symmetry. Since the symmetry algebra is a product of holomorphic and anti-holomorphic Virasoro algebras, conformal blocks can thus be written as products $\mathcal{V} \times \bar{\mathcal{V}}$. This feature leads to many convenient simplifications. Due to the $y$-dependence of $\phi(y, z, \bar{z})$, this property does not hold for $\phi$ correlators, but we can still take advantage of something almost as useful, which can be summarized by equations (5.2.13), (5.2.14), and (5.2.21).

5.2.2.1 The Full Correlator $\langle \phi \phi \rangle$

Computing $\langle \phi \phi \rangle$ using the quasi-primary decomposition in equation (5.2.8) is useful because distinct quasi-primaries (and their global descendants) have vanishing two-point correlators. So we can write $\langle \phi \phi \rangle$ as a sum over contributions from different quasi-primaries, that is,

$$\langle \phi(y_1, z_1, \bar{z}_1) \phi(y_2, z_2, \bar{z}_2) \rangle = \sum_{n, \bar{n}, i, j} \langle \phi_{n, \bar{n}}^{i, j}(y_1, z_1, \bar{z}_1) \phi_{n, \bar{n}}^{i, j}(y_2, z_2, \bar{z}_2) \rangle,$$

where the sum $(n, \bar{n})$ is over different levels for the quasi-primaries, and the sum $(i, j)$ is over all of the different quasi-primaries at level $(n, \bar{n})$. By $\phi_{n, \bar{n}}^{i, j}$ we denote the contribution to $\phi$ from the quasi-primary $L_{-n}^{\text{quasi}, i}L_{-\bar{n}}^{\text{quasi}, j}\mathcal{O}$ and all its global descendants

$$\phi_{n, \bar{n}}^{i, j}(y, z) \equiv y^{2h+2n} \sum_{m=0}^{\infty} (-1)^{n+m} y^{2m} |L_{-1}^{n+m}\mathcal{O}|^2 \frac{L_{-1}^{m}L_{-\bar{n}}^{\text{quasi}, i}\mathcal{O}}{|L_{-1}^{m}L_{-\bar{n}}^{\text{quasi}, i}\mathcal{O}|^2} \frac{L_{-1}^{m+n-\bar{n}}L_{-\bar{n}}^{\text{quasi}, j}\mathcal{O}}{|L_{-1}^{m+n-\bar{n}}L_{-\bar{n}}^{\text{quasi}, j}\mathcal{O}|^2},$$

where without loss of generality, we assume $n \geq \bar{n}$. The above equation can be read off from equation (5.2.5) and equation (5.2.8). As we’ll show in Appendix D.2, $\langle \phi_{n, \bar{n}}^{i, j} \phi_{n, \bar{n}}^{i, j} \rangle$ is given by

$$\langle \phi_{n, \bar{n}}^{i, j} \phi_{n, \bar{n}}^{i, j} \rangle = \frac{1}{|L_{-n}^{\text{quasi}, i}L_{-\bar{n}}^{\text{quasi}, j}\mathcal{O}|^2} \mathcal{F}_{n, \bar{n}}(h),$$

(5.2.12)
where $F_{n, \pi}(h)$ only depends on the level of the quasi-primary $(n, \pi)$ and it’s symmetric under exchange of $n$ and $\bar{n}$. So we can write $\langle \phi \phi \rangle$ as

$$\langle \phi \phi \rangle = \sum_{n, \pi} \left( \sum_{i,j} \frac{1}{|L_{-n}^{\text{quasi}, i} L_{-\pi}^{\text{quasi}, j}|^2} \right) F_{n, \pi}(h) \equiv \sum_{n, \pi=0}^{\infty} C_{n, \pi} F_{n, \pi}, \quad (5.2.13)$$

where we define $C_{n, \pi}$ to be the sum over the inverse of all quasi-primaries at level $(n, \pi)$, and it factorizes as

$$C_{n, \pi} = C_n C_{\pi} = \left( \sum_{i=1}^{p(n) - p(n-1)} \frac{1}{|L_{-n}^{\text{quasi}, i} O|^2} \right) \left( \sum_{j=1}^{p(\pi) - p(\pi-1)} \frac{1}{|L_{-\pi}^{\text{quasi}, j} O|^2} \right). \quad (5.2.14)$$

So we only need to compute $C_n$ to determine $C_{n, \pi}$. In section 5.4, we’ll show that $C_n$ can be obtained by modifying Zamolochikov’s recursion relations for Virasoro blocks. In the semiclassical limit, the $C_n$ can also be determined by the monodromy method of section 5.3.

To get $\langle \phi \phi \rangle$, we also need to compute $F_{n, \pi}$. We will show in Appendix D.2 that $F_{n, \pi}$ is given by the Kampe de Feriet (KdF)

$$F_{0,3}^{2,2}.$$ 

$$F_{n, \pi} \equiv \left( \frac{y_1 y_2}{z_{12} z_{12}} \right)^{2n} (2h \pi)_{2l} \left[ n! (2h)_n \right] \left[ l! (2h \pi)_l \right]^{2} \quad (5.2.16)$$

$$\times F_{0,3}^{2,2} \left( 2h_n, 2h_{\pi} : \begin{array}{c} 2h_n - n, 2h_{\pi} - n; n + 1, n + 1; \\ 2h_n, 2h_{\pi} - l, 2h_{\pi} - l; l + 1, l + 1; \end{array} ; \frac{y_1^2}{z_{12} z_{12}}, -\frac{y_2^2}{z_{12} z_{12}} \right),$$

with

$$h_n \equiv h + n, \quad h_{\pi} \equiv h + \pi, \quad l \equiv n - \pi. \quad (5.2.17)$$

The general form of a KdF series is given by

$$F_{r,s}^{p,q} \left( a_1, \ldots, a_p : b_1, b_1' ; \ldots ; b_q, b_q' ; c_1, \ldots, c_r : d_1, d_1' ; \ldots ; d_s, d_s' , x, y \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1)_{m+n} \cdots (a_p)_{m+n} (b_1)_m (b_1')_n \cdots (b_q)_m (b_q')_n x^m y^n}{(c_1)_{m+n} \cdots (c_r)_{m+n} (d_1)_m (d_1')_n \cdots (d_s)_m (d_s')_n m! n!} \quad (5.2.15)$$

so these can be viewed as a generalization of hypergeometric functions. We only need some of the simplest examples of these functions.
As far as we know, there is no closed form expression\(^{10}\) for the general KdF series \(F_{2,2}^{2,2,0,3}\). But in the case that \(n = 0\) (or \(n = 0\)), the above KdF series is given by an Appell \(F_4\) function, which in our case greatly simplifies to

\[
F_{n,0} = (2h)^{2n} \frac{\rho^{h+n}}{1-\rho}.
\]  

(5.2.18)

where \(\rho = e^{-2\sigma}\) with \(\sigma = \log \frac{1+\sqrt{1-\xi^2}}{\xi}\) and \(\xi = \frac{2y_1y_2}{y_1+y_2+\xi_1\xi_2}\). Here, \(\sigma\) is the geodesic separation between the two bulk operators in pure \(\text{AdS}_3\). Note that the global (or free field) bulk-bulk propagator \(\langle \phi\phi \rangle_{\text{global}}\) is given by \(F_{0,0}\):

\[
\langle \phi\phi \rangle_{\text{global}} = F_{0,0} = \frac{\rho^h}{1-\rho}.
\]

This means that the general \(F_{n,0}\) is just proportional to the global bulk-bulk propagator with a shifted bulk mass \(h \rightarrow h + n\). We’ll see that these \(F_{n,0}\) can be summed to give the holomorphic correlator \(\langle \phi\phi \rangle_{\text{holo}}\), which we define next.

### 5.2.2.2 The Holomorphic Correlator \(\langle \phi\phi \rangle_{\text{holo}}\)

The definition of \(\phi\) involves a sum over products of Virasoro generators \(L_N\) and \(\overline{L}_N\), which are related by \(L_n \leftrightarrow \overline{L}_n\). We can define the non-trivial holomorphic part of \(\phi\) as

\[
\phi_{\text{holo}} (y, z, \overline{z}) = \sum_{N=0}^{\infty} y^{2h+2N} \lambda_N L_N \left( \overline{L}_1 \right)^N O(z, \overline{z}).
\]  

(5.2.19)

by replacing \(\overline{L}_N\) with its \(c \rightarrow \infty\) limit \(\overline{L}_1^N\). This simplified operator \(\phi_{\text{holo}}\) is useful because, roughly speaking, it encodes all of the non-trivial quantum gravity information in \(\phi\). As we will show, its two-point function

\[
\langle \phi\phi \rangle_{\text{holo}} \equiv \langle \phi_{\text{holo}} (y_1, z_1, \overline{z}_1) \phi_{\text{holo}} (y_2, z_2, \overline{z}_2) \rangle
\]  

(5.2.20)

\(^{10}\)In Appendix D.2 we present an integral expression for \(F_{2,2}^{2,2,0,3}\) in terms of hypergeometric functions.
involves all the $C_n$ coefficients. In large $c$ perturbation theory, the holomorphic part $\phi_{\text{holo}}$ can be understood as the result of including only holomorphic gravitons $h_{zz}$ while neglecting $h_{\bar{z}z}$. Thus $\phi_{\text{holo}}$ will have valid correlators of the form $\langle \phi_{\text{holo}} O T \cdots T \rangle$, but it will not have valid correlators with the anti-holomorphic stress tensor $\mathcal{T}$. This means that the holomorphic propagator $\langle \phi \phi \rangle_{\text{holo}}$ includes holomorphic graviton loops, but not mixed or anti-holomorphic loops.

As one can easily see, $\langle \phi \phi \rangle_{\text{holo}}$ defined in equation (5.2.20) can be written as

$$\langle \phi \phi \rangle_{\text{holo}} = \sum_{n=0}^{\infty} C_n \mathcal{F}_{n,0},$$

(5.2.21)

since we defined $\phi_{\text{holo}}$ in equation (5.2.19) such that it contains no information about anti-holomorphic Virasoro generators (thus $\bar{n} = 0$ and $C_0 = 1$). So just as with $\langle \phi \phi \rangle_{\text{global}}$, the holomorphic propagator $\langle \phi \phi \rangle_{\text{holo}}$ will only depend on $\rho$, which means that in our Fefferman-Graham gauge, $\langle \phi \phi \rangle_{\text{holo}}$ is spherically symmetric. This is not true for $\langle \phi \phi \rangle$, which will depend on another variable besides $\rho$, specifically an angle with respect to the $z$-$\bar{z}$ plane, captured for example by the ratio $y_1/y_2$.

Since the contribution to $\langle \phi \phi \rangle$ from $\mathcal{F}_{0,\pi}$ is the same as $\mathcal{F}_{n,0}$, we can write $\langle \phi \phi \rangle$ as

$$\langle \phi \phi \rangle = 2 \langle \phi \phi \rangle_{\text{holo}} - \langle \phi \phi \rangle_{\text{global}} + \langle \phi \phi \rangle_{\text{mixed}},$$

(5.2.22)

where $\langle \phi \phi \rangle_{\text{mixed}}$ is the contribution from $\mathcal{F}_{n,\pi}$ with $n, \pi > 0$ and the subtraction of $\langle \phi \phi \rangle_{\text{global}} = \mathcal{F}_{0,0}$ is necessary because we count it twice in the first term.

In this paper, we will focus mostly on $\langle \phi \phi \rangle_{\text{holo}}$. In section 5.3, we will use monodromy method to obtain the semiclassical limit of $\langle \phi \phi \rangle_{\text{holo}}$, and in section 5.4, we will provide two recursion relations for computing $\langle \phi \phi \rangle_{\text{holo}}$ exactly. We provide some discussion of the mixed and holomorphic terms in section 5.4.2, and we provide one important and physically relevant comparison, restricted to the $z$-$\bar{z}$ plane, in figure 5.7.
5.3 Semiclassical Limit

When studying quantum gravity, it is always important to make contact with the semiclassical limit of general relativity, where $G_N \to 0$ with products like $G_N M$ fixed. This limit of GR appears directly at the kinematical level in CFT$_2$ [147, 39, 15, 16, 23]. Conformal blocks in CFT$_2$ (which are determined by Virasoro symmetry) have a semi-classical limit of the form $e^{c f}$ as we take $c \to \infty$ with the ratios of scaling dimensions to the central charge, $h/c$, held fixed. This has a beautiful connection with AdS$_3$ gravity via $G_N = \frac{3}{2c}$ in AdS units, with scaling dimensions playing the role of AdS$_3$ masses.

Correlators of $\phi$ also behave nicely in this semiclassical limit. The bulk propagator can be approximated by

$$\langle \phi(y_1, z_1, \bar{z}_1) \phi(y_2, z_2, \bar{z}_2) \rangle \approx e^{c g(h/c, \xi, r)} \quad (5.3.1)$$

for some function $g$ at large $c$. In this section we will show how to compute the semiclassical $g_{\text{holo}}$ using a generalization of the ‘monodromy method’ [147, 40, 39, 15] that has been used to compute conformal blocks. Then we will apply our method to calculate $g_{\text{holo}}$ perturbatively in small $\frac{h}{c}$, and more importantly, to obtain $g_{\text{holo}}$ in the limit $h \to \infty$ in section 5.3.3. In fact, we will be able to determine the large $h$ limit of $\langle \phi \phi \rangle_{\text{holo}}$ exactly, and this will be an important seed for very efficient recursion relations discussed in section 5.4. As one might expect for the trans-Planckian $h \gg c$ regime, the large $h$ limit of the propagator exhibits a breakdown of bulk locality. In section 5.3.2 we also obtain some explicit analytic results to all-orders in $h/c$ in the short-distance limit.

5.3.1 Generalizing the Monodromy Method to $\langle \phi \phi \rangle$

The monodromy method for Virasoro conformal blocks was developed by Zamolodchikov in [37, 36]; for some recent reviews see [40, 15]. The basic idea is that the $\mathcal{O}(c)$ piece “$g$” in the exponent of 5.3.1 is unaffected by adding extra ‘light’ operators with $\mathcal{O}(1)$ conformal
weights inside the correlator. Therefore, one can add a degenerate operator

\[ \hat{\psi}_{2,1}(z) \] (5.3.2)

that has a null Virasoro descendant at level 2. Correlators of this degenerate operator must obey a second order differential equation. In the case of \( \phi \), let us define the “wavefunction” \( \psi \) to be the three-point function

\[ \psi \equiv \langle \hat{\psi}_{2,1}(z)\phi(X_1)\phi(X_2) \rangle. \] (5.3.3)

Because of \( \hat{\psi}_{2,1} \)'s null descendant, \( \psi \) obeys the following differential equation

\[ \partial_z^2 \psi(z, X_1, X_2) + \frac{6}{c} T(z, X_1, X_2) \psi(z, X_1, X_2) = 0, \] (5.3.4)

where \( T(z, X_1, X_2) \) is the stress tensor evaluated in the presence of the two \( \phi \)s:

\[ T(z, X_1, X_2) = \frac{\langle T(z)\phi(X_1)\phi(X_2) \rangle}{\langle \phi(X_1)\phi(X_2) \rangle}. \] (5.3.5)

In (5.3.4), \( T(z, X_1, X_2) \) acts like a Schrodinger potential for \( \psi \). It is fixed in terms of the \( \langle \phi \phi \rangle \) correlator by recursion relations that follow from the \( T \phi \) OPE. Unlike boundary primary operators, \( \phi \) has a third-order pole in its OPE with \( T \), due to the fact that it transforms non-trivially under special conformal transformations \( L_1 \) at the origin. When both holomorphic and anti-holomorphic stress tensors contribute, the action of \( L_1 \) is somewhat complicated:

\[ L_1 \phi(y, 0, 0) = -y^2 \frac{\bar{\partial} + y^2 \bar{\partial} \bar{T}(0)\bar{\partial}}{1 - y^{3/2} \bar{T}(0)T(0)} \phi(y, 0, 0). \] (5.3.6)

We will just develop the monodromy method for the “holomorphic” \( \langle \phi \phi \rangle \) correlator, where \( \bar{T} \) contributions are absent (it would be interesting to study the full case, which is more complicated, in the future). Then, \( L_1 \) acts much more simply, and the singular terms in the
OPE of $T$ and $\phi$ are the following:

$$T(z)\phi(y, w, \bar{w}) \sim \frac{y^2 \partial_w \phi(y, w, \bar{w})}{(z - w)^3} + \frac{1}{2} \frac{y \partial_y \phi(y, w, \bar{w})}{(z - w)^2} + \frac{\partial_w \phi(y, w, \bar{w})}{z - w}.$$  \hspace{1cm} (5.3.7)

Another significant simplification of the holomorphic correlator is that it depends only on $\rho$ or equivalently $\xi = \frac{2y_1 y_2}{y_1^2 + y_2^2 + z_1 z_2}$; in other words, the holomorphic correlators are still invariant under the AdS isometries, despite the gauge fixing.

We can evaluate $T(z, X_1, X_2)$ by summing over its poles at $z_1$ and $z_2$, and the residues are given by derivatives of the exponent $g$ \textsuperscript{11} in (5.3.1):

$$\frac{T}{c} = -\frac{y_1^2 \partial_{z_1} g}{(z - z_1)^3} + \frac{1}{2} \frac{y_1 \partial_y g}{z - z_1} + \frac{\partial_{z_1} g}{(z - z_1)^3} + \frac{1}{2} \frac{y_2 \partial_{y_2} g}{(z - z_2)^3} + \frac{\partial_{z_2} g}{z - z_2}.$$  \hspace{1cm} (5.3.8)

Finally, without loss of generality we can take $z_1 = 0, y_1 = 1, z_2, y_2 \to \infty$ with $z_2/y_2 = 1$ fixed. \textsuperscript{12} In this limit, using the fact that $g$ depends on the coordinates $X_i$ only through the invariant combination $\xi$, $T(z, X_1, X_2)$ simplifies to

$$\frac{T}{c} = \xi g'(\xi) \left( \frac{z - \xi(z^2 + 1)}{2z^3} \right).$$  \hspace{1cm} (5.3.10)

The solutions for $\psi$ are given by the differential equation (5.3.4) with this potential.

The final input into the monodromy method is that the solutions for $\psi$ have fixed monodromy when $z$ is taken along closed paths that encircle other operators in the correlator. This follows from the fact that when $\hat{\psi}_{2,1}$ fuses with an operator, only two possible operator

\textsuperscript{11}Since we will be focusing on $g_{\text{holo}}$ from this point on, we will denote it using $g$ to reduce clutter.

\textsuperscript{12}Any non-zero value of $z_2/y_2$ is allowed without loss of generality. Taking different positions for the two $\phi$s in the correlator leads to different forms of the potential $T$, and consequently different solutions for the wavefunction $\psi$. However, as long as the geodesic distance between the positions is the same, the monodromy of the solutions does not depend on the specific values of the coordinates. Another, slightly more complicated limit we could take is $z_1 = \bar{z}_1 = 0, z_2 = 1$ and $y_1 = y_2 = 1$, in which case the potential takes the form

$$T(z) = \frac{\xi \xi \mathcal{O}((\mathcal{O} + 1)(z - 1)z)g'(\xi)}{2(z - 1)^3z^3}.$$  \hspace{1cm} (5.3.9)

And in Appendix D.3, we also show that we can use the bulk-bulk OPE to obtain the leading term of the above equation.

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dimensions are allowed in its OPE. In our case, when \( \hat{\psi}_{2,1} \) fuses with one of the \( \phi \)s, it can only produce operators \( O_\beta \) that have weight \( h_\beta \) satisfying\(^\text{13}\)

\[
h_\beta - h_\phi - h_\psi = \frac{1}{2} \left( 1 \pm \sqrt{1 - \frac{24h_\phi}{c}} \right). \tag{5.3.11}
\]

The LHS above is the power of the leading singularity of \( O_\beta \) in the \( \psi \times \phi \) OPE, so when \( z \) circles one of the \( \phi \)s, the monodromy matrix of the two solutions to (5.3.4) must have eigenvalues

\[
M_\pm = -e^{\pm i\pi \Lambda_h}, \quad \Lambda_h \equiv \sqrt{1 - \frac{24h_\phi}{c}}. \tag{5.3.12}
\]

In summary, the monodromy method for \( \langle \phi \phi \rangle_{\text{holo}} \) is that one solves (5.3.4) for \( \psi(z, \xi) \) with \( T \) given by (5.3.10), and then fixes \( g(\xi) \) by demanding that the monodromy matrix for the two solutions have eigenvalues given by (5.3.12) as \( z \) encircles the origin.

### 5.3.2 Perturbation Theory in \( \frac{h}{c} \) and an All-Orders Resummation

Let us see how to apply the monodromy method in the limit of small \( h/c \). To first order, \( g \sim O(\frac{h}{c}) \), and since \( cg \sim h \) is independent of \( c \) at this order we should just rederive the \( h \)-dependence of the free scalar propagator in AdS\(_3\).

At zero-th order in \( h/c \), \( T \) vanishes, so the solutions for \( \psi \) are just

\[
\psi^{(1)} = 1, \quad \psi^{(2)} = z. \tag{5.3.13}
\]

Both of these have trivial monodromy, consistent with \( -e^{i\pi \Lambda_h} = 1 + O(h/c) \) at leading order.

At next order, we demand monodromies of \( -e^{i\pi \Lambda_h} = 1 \pm \frac{12i\pi h}{c} \). We will use the method of

\(^{13}\)One may ask how is it possible for the three-point function \( \langle \hat{\psi}_{2,1} \phi \phi \rangle \) to be non-zero at all if \( \hat{\psi}_{2,1} \) can only fuse with \( \phi \) to produce operators of weight \( h_\beta \). To make sense of this puzzle, one should remember that we are just computing the semiclassical piece of \( \langle \phi \phi \rangle \), which is insensitive to additional light operators in the correlator. So, one can think of the correlator as really being \( \langle \hat{\psi}_{2,1} \phi \phi O' \rangle \), where \( O' \) is another light operator whose OPE with \( \phi \) contains \( O_\beta \).
separation of variables. The Wronskian of the zero-th order solutions is trivial

\[ W = \psi^{(1)} \psi^{(2)}' - \psi^{(1)'} \psi^{(2)} = 1. \]  

(5.3.14)

The monodromy matrix \( M_{ij} \) of the first-order solutions as \( z \) goes around the origin are given by the following residue formula:

\[ M_{ij} = \delta_{ij} - 2\pi i \text{res}_{z=0} \left[ \frac{\frac{2}{c} T(z)}{W(z)} \tilde{\psi}^{(i)} \psi^{(j)} \right], \quad \tilde{\psi} \equiv \begin{pmatrix} -\psi^{(2)} \\ \psi^{(1)} \end{pmatrix}. \]  

(5.3.15)

The eigenvalues of \( M \) are

\[ \text{evals}(M) = 1 \pm 6i\pi \xi \sqrt{1 - \xi^2} g'(\xi). \]  

(5.3.16)

Equating \( \text{evals}(M) = 1 \pm 12i\pi \frac{h}{\xi} \), we obtain

\[ e^{cg(\xi)} = \left( \frac{\xi}{1 + \sqrt{1 - \xi^2}} \right)^{2h} = \rho^h. \]  

(5.3.17)

which is indeed the right answer.

We can continue to higher orders in \( h/c \) as well. It becomes somewhat nicer to write expressions in terms of the variable \( \rho \) rather than \( \xi \). At higher orders, rather than writing \( M_{ij} \) in terms of residues of a matrix, one must solve order-by-order for the solutions \( \psi^{(1)} \) and \( \psi^{(2)} \); the non-trivial monodromies arise from logarithms in \( \psi^{(i)} \), and these are fairly easy to deal with by hand. At \( \mathcal{O}(h^2/c) \), we find

\[ cg = h \log \rho + \frac{12h^2}{c} \left( \frac{\rho}{(1 - \rho)^2} + \log(1 - \rho) \right) + \mathcal{O} \left( \frac{h^3}{c^2} \right). \]  

(5.3.18)

in the semiclassical limit. This agrees with bulk gravitational perturbation theory (see section 5.5.1 and appendix D.1) and the methods of section 5.4.

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After working to sufficiently high order using the recursion relation of section 5.4, a pattern emerges and one can conjecture the following ansatz for the all-orders result:

\[
\log \frac{\langle \phi \phi \rangle_{\text{holo}}}{\langle \phi \phi \rangle_{\text{global}}} = \sum_{n=1}^{\infty} \frac{h^{n+1}}{c^n} \left( \frac{2 \times 12^n (2n - 1)!!}{(n+1)!} \log (1 - \rho) + \frac{24^n (3n - 3)!!}{(n+1)!(n-1)!!} \left( \frac{g_n(\rho)}{(1 - \rho)^{3n-1}} \right) \right),
\]

with the first three \( g_n(\rho) \) taking the form

\[
g_1(\rho) = \rho, \quad g_2(\rho) = \frac{1}{12} \rho \left( 7\rho^4 - 41\rho^3 + 73\rho^2 - 33\rho + 6 \right), \quad g_3(\rho) = \frac{1}{192} \rho \left( -42\rho^7 + 366\rho^6 - 1205\rho^5 + 1758\rho^4 - 1112\rho^3 + 606\rho^2 - 209\rho + 30 \right),
\]

and where we have normalized \( g_n(\rho) \) so that \( g_n(1) = 1 \). The leading term involving \( g_1 \) matches equation (5.3.18).

The second term in equation (5.3.19) can be summed if we work to leading order in the short-distance limit \( \rho \to 1 \), i.e. by setting \( g_n(\rho) = 1 \). \(^{14}\) We find

\[
\sum_{n=1}^{\infty} \frac{24^n (3n - 3)!!}{(n+1)!(n-1)!!} \frac{h^{n+1}}{c^n (1 - \rho)^{3n-1}} = \frac{c(1 - \rho)^4}{576} \left( 2F_1 \left( \begin{array}{c} -\frac{2}{3}, \frac{1}{3}; \frac{1}{2} \end{array} \left\{ \frac{15552h^2}{c^2(1 - \rho)^6} \right\} - 1 \right) + h(1 - \rho) \left( 1 - 2F_1 \left( \begin{array}{c} -\frac{1}{6}, \frac{1}{6}; \frac{3}{2} \end{array} \left\{ \frac{15552h^2}{c^2(1 - \rho)^6} \right\} \right) \right)
\]

\(^{14}\)The logarithmic terms in equation (5.3.19) can also be summed to give

\[
\sum_{n=1}^{\infty} \frac{2^{h^{n+1}} 12^n (2n - 1)!!}{c^n (n+1)!} \log (1 - \rho) = \frac{c}{6} \left( 1 - \frac{12h}{c} - \sqrt{1 - \frac{24h}{c}} \right) \log (1 - \rho).
\]

Note that for \( h > \frac{c}{24} \), i.e. above the BTZ black hole threshold, equation (5.3.21) develops an imaginary piece. Unfortunately, we cannot conclude anything from this fact alone, since that result is subdominant to the other terms in equation (5.3.19).
for the function that appears in the exponent of the semiclassical propagator in the short-distance limit.

Both hypergeometric functions in equation (5.3.22) can be expanded as $\rho \to 1$ with fixed $h/c$, and both develop complex parts in this limit. More generally, the hypergeometric functions both have branch cuts running from the point where $(1 - \rho)^6 = \frac{15552h^2}{c^2}$ to $\rho = 1$. This indicates an apparent breakdown of bulk locality in the semiclassical part of the propagator. Note that in terms of the geodesic length $\sigma$, this breakdown occurs at the critical value

$$\sigma_* \approx \left(9\sqrt{3} \frac{h}{c}\right)^{1/3} R_{AdS}$$

(5.3.23)

at large $c$ and small $h/c$. This formula only applies in the regime where $\sigma_* \ll R_{AdS}$, because we were only able to compute the semiclassical result analytically to leading order as $\rho \to 1$.

### 5.3.3 Exact Large $h$ Limit and the Breakdown of Locality

We can also solve the monodromy method in an expansion about large $h$. This limit is interesting for two very different reasons. The first is that large $h$ corresponds with a large bulk mass for $\phi$. So in this case we expect $\phi$ to have a very large effect on the local geometry, potentially leading to the breakdown of bulk locality at macroscopic distances.\(^{15}\) Our second motivation is more technical: as we will demonstrate in section 5.4, the infinite $h$ limit of the correlator is the necessary “seed” for a very efficient recursion relation that can be used to numerically compute the correlator exactly and at any $h$.

At large $h$, the potential $T$ should become large and therefore one can solve the Schrodinger equation for $\psi$ using a WKB approximation. This approach (used for the blocks in [36]), is

\(^{15}\)One might have expected to see indications of black hole physics, since we are studying the limit $h \gg c$ where $\phi$ would have to be interpreted as a sort of ‘black hole field’. We do not see any direct indications of the Hawking temperature or Schwarzschild radius in the $\phi$ propagator, though these parameters must appear in higher-point semiclassical correlators.
easiest to implement if we change variables according to

\[ \psi = (y'(z))^{-1/2}\Psi(y), \]  

(5.3.24)

bringing the Schrodinger equation into the form

\[ \Psi''(y) + U(y)\Psi(y) = \xi g'(\xi)\Psi(y). \]  

(5.3.25)

The coordinate \( y(z) \) that achieves this is

\[
y(z) = \sqrt{3} \int z dt \frac{\sqrt{-t + \xi(1 + t^2)}}{t^{3/2}} \\
= -\frac{2\sqrt{6}}{\sqrt{2 - s}} \left( \frac{(1-z)\sqrt{s(z+1)^2 - 4z}}{2\sqrt{s(z+1)}} + E(\varphi|s) \right),
\]

(5.3.26)

where \( E \) is an elliptic integral, \( \sin \varphi \equiv \frac{2\sqrt{1+z}}{(1+z)^{1/2}}, \) and we have introduced the new coordinate \( s: \)

\[ s \equiv \frac{4\xi}{1 + 2\xi}. \]

(5.3.27)

The new potential \( U(y) \) is

\[ U(y) = \frac{3y''(z)^2 - 2y^{(3)}(z)y'(z)}{4y'(z)^4}. \]

(5.3.28)

Now, the advantage is that \( y \) is a periodic variable - under a monodromy cycle in \( z \), \( y \) shifts because of a corresponding shift in the elliptic integral:

\[ E(\varphi + n\pi, s) = E(\varphi, s) + 2nE(s). \]  

(5.3.29)
Consequently, the $y$ variable lives in a box of size $R$ given by
\[ R = 4\sqrt{6}\frac{E(s)}{\sqrt{2} - s}, \] (5.3.30)

The new form (5.3.25) of the Schrödinger equation is for a particle in a box, having energy $\xi g'(\xi)$; the monodromy condition is that the particle should have quasimomentum $\Lambda_h$. Therefore, in the limit $\Lambda_h \to \infty$, we have
\[ c\xi g'(\xi) = -c\frac{(\pi \Lambda_h)^2}{R^2} = \pi^2 \frac{(2 - s)(\hbar - \frac{c}{24})}{4E^2(s)}. \] (5.3.31)

Equivalently,
\[ cg'(s) = \frac{\pi^2(h - \frac{c}{24})}{2sE^2(s)} + O(c). \] (5.3.32)

The subleading in $1/\hbar$ correction is an $O(c)$ correction, as indicated above. We can obtain this correction as follows. Because $\xi g'(\xi)$ is the eigenvalue in the Schrödinger equation (5.3.25), in the WKB approximation its subleading correction enters as
\[ \log \Psi(y) \approx \int dy \sqrt{\xi(g'(\xi) + \delta g'(\xi)) - U(y)} \approx \sqrt{\xi g'(\xi)} y + \frac{1}{2\sqrt{\xi g'(\xi)}} \int dy (\xi \delta g'(\xi) - U(y)). \]

The monodromy of our leading order solution above is already the correct value, $\Lambda_h$. Demanding that the correction to the monodromy vanish, one therefore obtains\(^{16}\)
\[ \xi \delta g'(\xi) = \frac{1}{R} \int_0^R U(y) dy = \frac{2 - s}{144} \left( \frac{7K(s)}{E(s)} - \frac{2 - s}{2(1 - s)} \right). \] (5.3.33)

Putting this correction together with the leading piece, we obtain the full semiclassical part.

\(^{16}\)We performed the $dy$ integral in (5.3.33) by changing variables to $dz$ and doing the indefinite integral to get a result involving elliptic integrals. The definite integral $\int_0^R dy$ then corresponds to the shift in the $\int dz$ integral under a $z$ monodromy cycle, which is easy to read off using (5.3.29) together with $F(\varphi + n\pi, s) = F(\varphi, s) + 2nK(s)$. 

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of the exponent $c_g$ at $h = \infty$:

$$c_g = \left( h - \frac{c}{24} \right) \int \frac{ds}{2s} \frac{\pi^2}{E^2(s)} + \frac{c}{144} \left[ \log \left( \frac{16(1-s)}{s^2} \right) + 14 \int \frac{ds}{s} \frac{K(s)}{E(s)} \right]. \quad (5.3.34)$$

The integrals over $s$ can all be done in closed form.\(^{17}\) We fix the integration constants by matching to the known, small $\xi$ behavior. The result is that the semiclassical part of the correlator at large $h$ is given by

$$e^{c_g \ h} \geq \ q^{h - \frac{c}{24}} \left( \frac{s}{8} \right)^{\frac{c}{24}} (1 - s)^{\frac{c}{12}} \left( \frac{2E(s)}{\pi} \right)^{-\frac{7c}{36}}. \quad (5.3.35)$$

where we have defined the new variable $q$:

$$q \equiv 4e^{2\pi \frac{E(1-s) - K(1-s)}{E(s)}}. \quad (5.3.36)$$

In addition to the semiclassical part above, the full holomorphic correlator $\langle \phi \phi \rangle_{h_{\text{olo}}}$ at infinite $h$ has a residual piece that is independent of $h$ and $c$. We have not been able to derive this residual piece from first principles, but we believe we were able to obtain the correct formula as follows. The corresponding residual piece in the conformal blocks simply results in a shift $c \to c - 1$ in the formula as compared to the semiclassical result. We tried an ansatz of the form of (5.3.35) where in each of the four places $c$ appears, we allow a separate shift in $c$. Comparing to an exact calculation of the leading small $\xi$ expansion, we fixed these four new parameters and checked that the Ansatz reproduced the correct result to high order in $\xi$.

The final result is

$$\lim_{h \to \infty} \langle \phi \phi \rangle_{h_{\text{olo}}} = q^{h - \frac{c-1}{24}} \left( \frac{s}{8} \right)^{\frac{c-1}{12}} (1 - s)^{\frac{c-13}{12}} \left( \frac{2E(s)}{\pi} \right)^{\frac{19-7c}{36}}. \quad (5.3.37)$$

We emphasize that this is not merely a semiclassical result, but the exact answer at large $h$.

\(^{17}\)For reference, $\int \frac{\pi ds}{\pi E^2(s)} = 4 \frac{E(1-s) - K(1-s)}{E(s)}$ and $\int \frac{ds K(s)}{s E(s)} = \log \left( \frac{s}{E^2(s)} \right)$. 

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The large $h$ limit of the holomorphic correlator has an important feature. As is evident in equation (5.3.36), the $q$ variable becomes complex when $s > 1$, which corresponds to $\rho_c = 7 - 4\sqrt{3} \approx 0.072$, or a physical geodesic separation

$$\frac{\sigma_c}{R_{AdS}} \approx \log(2 + \sqrt{3}) = 1.32$$

in the bulk (for emphasis, we have written the AdS radius explicitly). This represents a breakdown of bulk locality at the AdS scale. Note that this is not merely a relic of the semiclassical approximation, since it applies to the exact holomorphic propagator in the large $h$ limit.

Physically, this failure of bulk locality is not very surprising. We certainly would not have expected to have healthy bulk correlators for a field $\phi$ with extremely large (trans-Planckian!) bulk mass. But it is nevertheless reassuring that we can identify the breakdown of bulk locality in a precise, quantitative way.

![Figure 5.1: Left: The trace of the monodromy matrix $M$ computed numerically as a function of $\rho$ and $\rho g'(\rho)$, for three values of $\rho$: $\rho = 0.4$ (black, solid), $\rho = 0.5$, (red, dashed), and $\rho = 0.6$, (blue, dotted). Right: The critical value of $\sigma_c$ (black, solid) where the semiclassical part of the $\langle \phi \phi \rangle$ correlator first develops an imaginary piece, as a function of $h/c$. For comparison, we show (red, dashed) the analytic small $h/c$ behavior, $\sigma_c \approx (9\sqrt{3} h/c)^{1/3}$ from (5.3.23), and (blue, dot-dashed) the large $h$ behavior $\sigma_c \approx \log(2 + \sqrt{3})$, from (5.3.38).]
5.3.4 Numeric Monodromy Results

Away from the limits of large and small $\hbar/c$, we have not been able to solve the monodromy method for the semiclassical piece $g(\rho)$ of the $\langle \phi \phi \rangle_{\text{holo}}$ correlator in closed form. However, it is straightforward to compute $g$ numerically. Converting to $\rho$ coordinates,

$$
\frac{T}{c} = C_\rho \left( \frac{(\rho + 1)z - 2\sqrt{\rho}(z^2 + 1)}{(1 - \rho)z^3} \right),
$$

where numerically we fix the parameters $\rho$ and $C_\rho \equiv \rho g'(\rho)$, and numerically integrate the wavefunction $\psi$ along a cycle around the origin and match in order to compute the monodromy matrix $M$. Once $C_\rho$ is known for all $\rho$, it can be integrated to obtain $g(\rho)$.

In the left plot of Fig. 5.1, we show $\text{Tr}(M)$ computed in this way as a function of $C_\rho$ for a few values of $\rho$, and for real $C_\rho$. One can invert the relation to find $C_\rho$ as a function of $\hbar/c$ and $\rho$ by looking at the point where

$$
\text{Tr}(M_h) \equiv -2 \cos(\pi \Lambda_h) \quad \left( \Lambda_h \equiv \sqrt{1 - \frac{24\hbar}{c}} \right)
$$

intersects the curve for any specific $\rho$. For small enough $\rho$, the curve will cross $\text{Tr}(M_h)$ at multiple values of $C_\rho$, but by continuity with the small $\rho$ limit, one should take the smallest value for $C_\rho$ in these plots.

The most physically important feature of these numeric results is that each curve has a minimum at some value of $C_\rho$:

$$
\left[ \text{Tr}(M) \right]_{\text{min}}(\rho) \equiv \min_{C_\rho} \text{Tr}(M).
$$

Therefore, if $\text{Tr}(M_h)$ is below this minimum value $[\text{Tr}(M)]_{\text{min}}(\rho)$, then $C_\rho$ must become complex. Note also that the minimum value is an increasing function of $\rho$, so for fixed $\text{Tr}(M_h)$, there is a critical value of $\rho$ where $[\text{Tr}(M)]_{\text{min}}(\rho) = \text{Tr}(M_h)$; for larger $\rho$, $C_\rho$ develops an imaginary piece. The right plot of Fig. 5.1 shows the resulting critical value for $\sigma_c = -\frac{1}{2} \log \rho_c$. 

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in Fig. 5.1. We obtain a satisfying agreement with the results of section 5.3.2 in the limit of small \( h/c \). We compare these results with the exact methods of section 5.4 in figure 5.6.

## 5.4 Computing the Propagator Exactly

In this section we will develop a generalization of the Zamolodchikov recursion relations that make it possible to compute the bulk propagator exactly. The relations produce the exact coefficients for a series expansion of \( K_{\text{holo}} \) in the variable \( \rho \) (and \( q \)). This means that we obtain a long-distance expansion for the propagator, since \( \rho = 0 \) corresponds to geodesic separation \( \sigma \to \infty \).

### 5.4.1 Generalizing the Zamolodchikov Recursion Relations to \( \langle \phi\phi \rangle \)

The recursion relations that we will develop for computing \( \langle \phi\phi \rangle_{\text{holo}} \) are very similar to Zamolodchikov’s recursion relations for computing Virasoro blocks \([37, 36]\). They are based on the large \( c \) limit and large \( h \) limit of the \( \langle \phi\phi \rangle_{\text{holo}} \), as well as the pole structure of \( \langle \phi\phi \rangle_{\text{holo}} \) as a function of \( c \) or \( h \), respectively. In the following, we’ll denote them as the \( c \)-recursion and \( h \)-recursion.

Let us first write the central charge \( c \) in terms of a variable \( b \) as \( c = 13 + 6 (b^2 + b^{-2}) \) and define an function \( A_{m,n}^c \) given by

\[
A_{m,n}^c = \frac{1}{2} \prod_{k=1-m}^{m} \prod_{l=1-n}^{n} \frac{1}{kb + \ell}, \quad (k, l) \neq (0, 0), (m, n) \tag{5.4.1}
\]

which will be an ingredient of both \( c \)-recursion and \( h \)-recursion. It was determined in \([149]\) that \( A_{m,n}^c \) is equal to\(^{18}\)

\[
A_{m,n}^c = \lim_{h \to h_{m,n}(c)} \left( \frac{\langle \mathcal{L}_{mn}^{\text{quasi}} \mathcal{O}^h | \mathcal{L}_{mn}^{\text{quasi}} \mathcal{O}^h \rangle}{h - h_{m,n}(c)} \right)^{-1}, \tag{5.4.2}
\]

\(^{18}\)The coefficient of \((L_{-1})^{mn}\) in \( \mathcal{L}_{mn}^{\text{quasi}} \mathcal{O}^h \) in \([149]\) is also normalized to 1, which is the same as the convention of this paper.
where $h_{m,n}$ is the degenerate-state dimensions (which will be given below) and we put a superscript $h$ on $O$ to emphasize that if we send $h \rightarrow h_{m,n}(c)$ then $L^\text{quasi}_{-mn} O^h$ becomes a level $mn$ null-state.

5.4.1.1 $c$-recursion relation

To obtain the $c$-recursion, we need to know the large $c$ limit of $\langle \phi \phi \rangle_\text{holo}$, which is simply

$$\lim_{c \rightarrow \infty} \langle \phi \phi \rangle_\text{holo} = \langle \phi \phi \rangle_\text{global} = \frac{\rho^h}{1 - \rho}.$$  

As in the Virasoro block case, as a function of $c$ the correlator $\langle \phi \phi \rangle_\text{holo}$ has simple poles at $c = c_{m,n}(h)$. The residue of the pole at $c = c_{m,n}(h)$ must be proportional to the two-point function of $\phi_\text{holo}$ with dimension $h + mn$ and central charge $c_{m,n}(h)$. As shown in Section 5.2.2, $\langle \phi \phi \rangle_\text{holo}$ can be written as

$$\langle \phi \phi \rangle_\text{holo} = \sum_{N=0}^{\infty} C_N (2h)_{2N} \frac{\rho^h + N}{1 - \rho} = \frac{\rho^h}{1 - \rho} \sum_{N=0}^{\infty} \left( \frac{p(N) - p(N-1)}{p(N)} \right) (2h)_{2N} \rho^N, \quad (5.4.3)$$

where we have written $\langle \phi \phi \rangle_\text{holo}$ explicitly as a sum over contributions from different quasi-primaries and their global descendants. So if we write $\langle \phi \phi \rangle_\text{holo}$ as

$$\langle \phi \phi \rangle_\text{holo} = \frac{\rho^h}{1 - \rho} F (h,c), \quad (5.4.4)$$

then in $F (h,c)$, the residues at $c = c_{m,n}(h)$ will include a factor $\rho^{mn} F (h + mn, c_{m,n}(h))$. At the poles, the residues should also includes a factor that will give $C_N$. But this is precisely given by $-\frac{\partial c_{m,n}(h)}{\partial h} A_{m,n}^{c_{m,n}}$, where $\frac{\partial c_{m,n}(h)}{\partial h}$ is the Jacobian factor, because we are considering the poles at $c = c_{m,n}(h)$ while equation (6.4.19) is at the poles of $h = h_{m,n}(c)$. So combining all these facts, we find that $F (h,c)$ is given by the following $c$-recursion relation:

$$F (h,c) = 1 + \sum_{m \geq 2, n \geq 1} -\frac{\partial c_{m,n}(h)}{\partial h} A_{m,n}^{c_{m,n}} (2h)_{2mn} \rho^{mn} F (h + mn, c_{m,n}(h)), \quad (5.4.5)$$
where poles $c_{m,n}(h)$ are given by

$$c_{m,n}(h) = 13 + 6 \left[ (b_{m,n}(h))^2 + (b_{m,n}(h))^{-2} \right], \quad (5.4.6)$$

with

$$(b_{m,n}(h))^2 = \frac{2h + mn - 1 + \sqrt{(m-n)^2 + 4(mn-1)h + 4h^2}}{1 - m^2}, m = 2, 3, \cdots, n = 1, 2 \cdots. \quad (5.4.7)$$

The super-script in $A_{m,n}^c$ means that the $b$ in $A_{m,n}^c$ (equation (6.4.19)) should be substituted by $b_{m,n}(h)$. The factor $(2h)^N$ in equation (5.4.3) is accounted for by the $(2h)^{2mn}$ in the residues of this $c$-recursion (5.4.5).

Compared to the $c$-recursion relation for Virasoro block, we see that besides adding the factor $(2h)^{2mn}$, we simply get rid of the factor in the residues that encodes the information about the three point function between the intermediate state and the external operators.

The existence of this $c$-recursion relation can be traced back to the similarity of our definition of $\phi$ and a projection operator to project the four-point function into Virasoro blocks. We’ve checked this recursion relation by directly computing the $\langle \phi \phi \rangle_{\text{holo}}$ up to level\footnote{For example, at level 2 the $c$-recursion gives}

$$C_2 = - \frac{\partial c_{1,2}(h)}{\partial h} \frac{A_{1,2}^{c_{1,2}}}{c - c_{1,2}(h)} = \frac{9}{2(2h+1)(2ch+c+2h(8h-5))} \quad (5.4.8)$$

which is exactly equal to $\frac{1}{\ell_{-2}^{\text{quasi}}}$ with $\ell_{-2}^{\text{quasi}} = L_{-1}^2 - \frac{2(2h+1)}{3} L_{-2}$. We also used this recursion relation to obtain the semiclassical limit of $\langle \phi \phi \rangle_{\text{holo}}$, and the results agree with those obtained from monodromy method of Section 5.3.2.

### 5.4.1.2 $h$-recursion relation

The $h$-recursion relation is obtained by considering $\langle \phi \phi \rangle_{\text{holo}}$ as a function of $h$ with simple poles at $h = h_{m,n}(c)$. In Section 5.3.3, we already obtained the large $h$ limit of $\langle \phi \phi \rangle_{\text{holo}}$ by...
the monodromy method and a bit of guesswork; we found

$$\lim_{h \to \infty} \langle \phi \phi \rangle_{\text{holo}} = q^{h - \frac{c+1}{24}} \left( \frac{s}{8} \right)^{\frac{c-1}{12}} (1 - s)^{\frac{c-13}{144}} \left( \frac{2E(s)}{\pi} \right)^{\frac{19-7c}{36}} \cdot$$ (5.4.9)

So if we write $\langle \phi \phi \rangle_{\text{holo}}$ as

$$\langle \phi \phi \rangle_{\text{holo}} = q^{h - \frac{c+1}{24}} \left( \frac{s}{8} \right)^{\frac{c-1}{12}} (1 - s)^{\frac{c-13}{144}} \left( \frac{2E(s)}{\pi} \right)^{\frac{19-7c}{36}} H(h, c),$$

then $H(h, c)$ is given by the following recursion relation:

$$H(h, c) = 1 + \sum_{m,n \geq 1} q^{mn} \frac{(2h_{m,n})_{2mn}}{h - h_{m,n}(c)} A_{m,n}^{c} H(h_{m,n} + mn, c),$$ (5.4.10)

where the poles are given by $h_{m,n} = \frac{1}{4} \left( b + \frac{1}{b} \right)^2 - \frac{1}{4} \left( mb + \frac{n}{b} \right)^2$, with $c = 13 + 6 \left( b^2 + b^{-2} \right)$.

The residues of the $h$-recursion are just those of the $c$-recursion but now evaluated at the poles $h_{m,n}(c)$.

The $c$-recursion and $h$-recursion can be solved numerically, in the sense that we can obtain higher order coefficients from lower order coefficients, analogously to the blocks [2, 38]. We discuss the algorithm for implementing these recursions in Appendix D.4 and we have also attached our Mathematica code.

In each iteration of the $c$-recursion, we need to change both $h \to h + mn$ and $c \to c_{m,n}(h)$, whereas in the $h$-recursion, we only need to change $h \to h_{m,n} + mn$. Thus, the implementation of the $h$-recursion is faster than the $c$-recursion by roughly a factor of $N$. Although obtaining $C_N$ from the $c$-recursion is straightforward, one can also use the $h$-recursion to obtain $\langle \phi \phi \rangle_{\text{holo}}$ and then expand the result in terms of $\rho$ to obtain $C_N$, which is faster for higher order coefficients.
5.4.2 Comparison of Full and Holomorphic Propagators

In this section, we exhibit a numerical result comparing the full and holomorphic propagators. First we recall a convenient definition from equation (5.2.22)

\[ \langle \phi \phi \rangle = 2 \langle \phi \phi \rangle_{\text{holo}} - \langle \phi \phi \rangle_{\text{global}} + \langle \phi \phi \rangle_{\text{mixed}}, \]

(5.4.11)

where \( \langle \phi \phi \rangle_{\text{global}} \) is given in (5.1.2). Most of the analytic tools we have developed in this paper apply directly to \( \langle \phi \phi \rangle_{\text{holo}} \). But we can use the recursion relations to numerically compute both \( \langle \phi \phi \rangle_{\text{holo}} \) and \( \langle \phi \phi \rangle_{\text{mixed}} \) to high order, as explained in section 5.2.2.1.

The coordinate system we specified in equation (5.2.1) is not invariant under the isometries of vacuum AdS3. Therefore \( \langle \phi(y_1, z_1, \bar{z}_1)\phi(y_2, z_2, \bar{z}_2) \rangle \) can depend on both the geodesic separation between two points and an angular variable with respect to the \( z-\bar{z} \) plane, such as the ratio \( y_1/y_2 \). However, the holomorphic propagator \( \langle \phi \phi \rangle_{\text{holo}} \) is invariant under the isometries of vacuum AdS3. Specifically, we found20

\[ \langle \phi(y_1, z_1, \bar{z}_1)\phi(y_2, z_2, \bar{z}_2) \rangle_{\text{holo}} = \rho^h \sum_{n=0}^{\infty} a_n \rho^n. \]

(5.4.12)

where \( \rho = e^{-2\sigma} \) with \( \sigma \) the geodesic separation. This nice property does not hold for \( \langle \phi \phi \rangle_{\text{mixed}} \).

We will leave detailed discussion of the dependence of \( \langle \phi \phi \rangle \) on \( y_1/y_2 \) to the future.21 In this section, we focus on computing \( \langle \phi \phi \rangle_{\text{mixed}} \) when the two points are in the same \( z-\bar{z} \) plane, ie when \( y_1 = y_2 \), so that

\[ \langle \phi(y, z_1, \bar{z}_1)\phi(y, z_2, \bar{z}_2) \rangle_{\text{mixed}} = \rho^h \sum_{n=0}^{\infty} b_n \rho^{2n}. \]

(5.4.13)

The coefficients \( b_n \) can be computed exactly using the method outlined in section 5.2.2.

20Compared to equation (5.4.3), we see that \( \sum_{n=0}^{\infty} a_n \rho^n = \frac{1}{1-\rho} \sum_{n=0}^{\infty} C_n (2h) z_n \rho^n \), but the effect of the factor \( \frac{1}{1-\rho} \) is negligible in the following discussion.

21In Appendix D.2 we discuss the properties of the KdF series in general configurations, giving some further information on the angular dependence of the propagator.
Notice that the coefficients $a_n$ in equation (5.4.12) (which are related to $C_n$ in equation (5.4.3)) are always positive, but the coefficients $b_n$ in equation (5.4.13) can be negative. We have displayed the ratios of the growth rates of the coefficients $a_n$ and $b_n$ in figure 5.2. These coefficients grow exponentially at large $n$, indicating that $\langle \phi \phi \rangle$ has a finite radius of convergence in $\rho$. In other words, there is a singularity in $\langle \phi \phi \rangle$ when the two points are separated by a finite distance, signaling a breakdown of locality. We will discuss this phenomenon in great detail in section 5.6.

![Figure 5.2: This plot compares the ratios of successive coefficients in the holomorphic and mixed terms contributing to the full propagator. We see that at large $c$, the coefficients of $\rho^n$ grow at the same rate, meaning that the holomorphic propagator provides a good estimate for the behavior of the full propagator.](image)

Comparing these coefficients, we see numerically that for sufficiently large $c$ and $n$, the two types of coefficients seem to satisfy a rough empirical relation $b_n^2 \sim a_n$. Since $a_n$ and $b_n$ are approximated by exponentials at large $n$, this relation would indicate that, roughly, $b_{2n} \sim a_n$. This is the condition for the holomorphic and full correlators to have similar radii of convergence. Therefore we believe that although many of our analytical results are
explicitly obtained by studying $\langle \phi \phi \rangle_{\text{holo}}$, our conclusions about the physics should also hold approximately for $\langle \phi \phi \rangle$. In figure 5.7 we compare the convergence rates of the holomorphic and full propagators (in the $z$-$\bar{z}$ plane) explicitly.

5.5 Perturbation Theory in $\frac{1}{c}$

We are using CFT$_2$ to learn about AdS$_3$ quantum gravity, so it is very natural to study the expansion of observables in $G_N = \frac{3}{2c}$. In this section we will present the first $1/c$ correction to the propagator, and then a conjectured all-orders formula for light bulk proto-fields in the short-distance limit.

However, we find a surprising and potentially disturbing result, which appears already at one-loop: there are ‘UV/IR mixing’ effects, by which we mean that singular, short-distance terms in $\langle \phi \phi \rangle$ are enhanced by powers of the AdS scale $R_{AdS}$. Specifically, at one-loop and at short distances $\sigma \ll R_{AdS}$, we find

$$\langle \phi \phi \rangle \approx \frac{1}{\sigma} \left( \frac{3G_N R_{AdS}^3}{4\sigma^4} - \frac{G_N R_{AdS}(10 + m^2 R_{AdS}^2)}{8\sigma^2} + \cdots \right).$$  \hspace{1cm} (5.5.1)

Although this is a finite result in AdS$_3$, it does not have a good flat space limit as $R_{AdS} \to \infty$. We believe there are two plausible responses to this state of affairs:

1. One can interpret this UV/IR mixing effect as a signal that $\langle \phi \phi \rangle$ is too non-local in perturbation theory, and thus requires modification in order to obtain an IR safe quantity. Likely this would involve summing over external graviton states in place of the vacuum. We will not pursue this avenue of investigation here, but we believe it is interesting and important to consider, and we plan to return to it in the future.

2. One can ‘bite the bullet’ and simply study $\langle \phi(X_1)\phi(X_2) \rangle$, the exact vacuum propagator. In AdS$_3$ this observable is finite and well-defined in perturbation theory, since AdS$_3$ acts as an IR regulator, and our results for it accord with naive gravitational perturbation
Figure 5.3: This figure displays the scalar-graviton one-loop diagram that contributes to $\langle \phi(X_1)\phi(X_2) \rangle$ at order $1/c$. There is also a contact interaction, but the associated diagram vanishes. The computation is performed in appendix D.1.1.

In section 5.5.1 we will discuss the results of a one-loop gravity calculation, with the technical details relegated to appendix D.1. Then in section 5.5.2 we will present analytic results for the holomorphic propagator with fixed $h \ll c$, to all orders in $1/c$, but in the leading short-distance limit. Our one-loop results exactly match those of the recursion relation of section 5.4, and our all-orders results were obtained by extrapolating from the recursion relations. If one takes the vacuum propagator $\langle \phi\phi \rangle$ seriously as an observable, then our all-orders results suggest that bulk locality breaks down due to non-perturbative effects at a length scale $\sigma_* \sim c^{-1/4}$. We will obtain corroborating evidence for this conclusion numerically in section 5.6.
5.5.1 One-Loop Bulk Gravity and UV/IR Mixing

We find that bulk perturbation theory matches the recursion relations developed in section 5.4, and to leading non-trivial order in $1/c$, both give\(^{22}\)

$$\langle \phi \phi \rangle = \frac{\rho^h}{1 - \rho} \left[ 1 + \frac{12}{c} \left( \frac{\rho (2h^2(\rho - 1)^2 + h(\rho(3\rho - 11) + 2)(\rho - 1) + \rho^2((\rho - 5)\rho + 10))}{(1 - \rho)^4} \right. \right.$$

$$+ 2h\rho^2 \Phi(\rho, 1, 2h + 1) + h\rho^{1-2h} B_\rho(2h + 1, -1) + 2(h - 1)h \log(1 - \rho)) \right]\right], \quad (5.5.2)$$

where $B_\rho$ is the incomplete beta function and $\Phi$ is the Hurwitz Lurch function. In appendix D.1 we explicitly perform the bulk loop calculation, and we also show how a part of this result can be obtained directly from unitarity.

The formula above is complicated, but it simplifies in the short distance limit of $\sigma \ll 1$ with $\rho = e^{-2\sigma}$. The most singular terms are

$$\langle \phi \phi \rangle \approx \frac{1}{\sigma} \left( \frac{3G_N R_{AdS}^3}{2\sigma^4} - \frac{G_N R_{AdS}(10 + m^2 R_{AdS}^2)}{4\sigma^2} \right. \right.$$ 

$$\left. + \cdots \right). \quad (5.5.3)$$

where we have used $G_N = \frac{3}{2c}$ and $m^2 = 2h(2h - 2)$, and we have also included factors of the AdS scale. This result suggests a new length scale

$$\sigma_* \sim \sqrt[4]{G_N R_{AdS}^3}. \quad (5.5.4)$$

Although this follows straightforwardly from perturbative gravitational field theory, the emergence of this new scale is quite surprising. It is indicative of UV/IR mixing and the presence of IR divergences in the flat space limit $R_{AdS} \to \infty$. We do not expect a result like equation (5.5.3) from a well-defined local observable in a local theory. At a computational level, the scale $\sigma_*$ arises from the $\sigma^{-5}$ short distance singularity in equation (5.5.3), which

\(^{22}\) Note that the $1/c$ correction to $\langle \phi \phi \rangle$ is just twice the $1/c$ correction to $\langle \phi \phi \rangle_{\text{holo}}$, because the full propagator gets corrections from both holomorphic and anti-holomorphic gravitons, but no mixed terms, at this order.
can itself be traced to the fact that the bulk ‘graviton’ propagator \([150]\) is proportional to 
\[
\frac{1}{(z_1 - z_2)^4}
\]
and independent of anti-holomorphic coordinate \(\bar{z}\) and the radial direction \(y\). This AdS\(_3\) graviton propagator has been used successfully in other calculations; for example the results of \([77]\) can be re-interpreted \([3]\) as geodesic Witten diagrams \([151]\) that use this graviton propagator to compute conformal blocks. But we would expect a quite different graviton propagator in higher dimensions \([152]\).

Note that the less singular terms in equation (5.5.3) also display UV/IR mixing. There is both a semiclassical effect \(\sim \frac{G_N m^2 R_{AdS}^3}{\sigma^2}\) and a quantum effect \(\sim \frac{G_N R_{AdS}^2}{\sigma^2}\) which are enhanced by \(R_{AdS}\). The former has also been obtained from the monodromy method of section 5.3. This suggests that it may be quite non-trivial to define a fully IR safe modification of \(\langle \phi \phi \rangle\). We should also emphasize that because equation 5.5.3 has been obtained directly from unitarity in appendix D.1, modifying it may require a different choice for the \(\langle \phi OT \rangle\) correlator, which itself follows \([3]\) from a simple tree-level calculation. Modifying \(\langle \phi OT \rangle\) might also jeopardize the ability of \(\phi\) to ‘know its location’ \([3]\) in general semiclassical geometries.

Finally, it is natural to ask whether the one-loop corrected propagator can be used as an ingredient in a complete and gauge-invariant calculation of a CFT correlator. In this way one might approach the short-distance behavior of the propagator indirectly. For example, we could attach \(\langle \phi(X)\phi(Y)\rangle\) to a pair of bulk-boundary scalar propagators at \(X\) and another pair at \(Y\), and then integrate over \(X\) and \(Y\) to obtain a complete Witten diagram for a CFT 4-pt correlator, though it is not clear exactly what CFT quantity such a diagram should correspond to when the fully dressed \(\phi\) propagator is used. When computing Virasoro conformal blocks using Wilsons lines \([77]\), bulk diagrams like figure 5.3 were not included. The connection between the recursion relations of section 5.4 and the Zamolodchikov relations for conformal blocks might also provide further clues. It would be interesting to study these issues further.
5.5.2 All-Orders in $\frac{1}{c}$ in the Short Distance Limit

Now let us study $\frac{1}{c}$ perturbation theory to all orders. We are interested in light fields with $h \ll c$. In fact, the correlator $\langle \phi \phi \rangle$ remains very non-trivial even when $h \to 0$, so for definiteness and simplicity we will focus$^{23}$ on this case, which we have found (numerically) to be representative of the light field regime. Using the recursion relations of section 5.4, we find that the holomorphic part $K_{\text{holo}}(\rho) = \langle \phi \phi \rangle_{\text{holo}}$ of the bulk propagator takes the form$^{24}$

$$K_{\text{holo}} = \frac{1}{1-\rho} \left( 1 + \sum_{n=1}^{\infty} \rho^3 f_n(\rho) \frac{(4n-1)!!}{n!} \left( \frac{12}{c(1-\rho)^4} \right)^n \right), \quad (5.5.5)$$

where $f_n(\rho)$ are polynomials of order $4n - 2$ in $\rho$. For definiteness, the first three are

$$f_1(\rho) = \frac{1}{6} \left( \rho^2 - 5\rho + 10 \right), \quad (5.5.6)$$

$$f_2(\rho) = \frac{1}{1260} \left( 13\rho^6 - 117\rho^5 + 468\rho^4 - 1112\rho^3 + 1833\rho^2 + 195\rho - 20 \right),$$

$$f_3(\rho) = \frac{1}{99786} \left( 41\rho^{10} - 533\rho^9 + 3198\rho^8 - 11718\rho^7 + 29226\rho^6 - 56454\rho^5 + 105078\rho^4 
+ 34722\rho^3 - 3687\rho^2 - 89\rho + 8 \right),$$

and we have computed $f_1(\rho)$ perturbatively in appendix D.1. We have chosen the normalizations so that $f_n(\rho \to 1) = 1$ in order to ensure that the $f_n$ become trivial in the short-distance limit. This means that to leading order in that limit, $K_{\text{holo}}$ takes the very simple form

$$K_{\text{holo}}(\rho \to 1) \approx \frac{1}{1-\rho} \left( 1 + \sum_{n=1}^{\infty} \frac{(4n-1)!!}{n!} \left( \frac{12}{c(1-\rho)^4} \right)^n \right). \quad (5.5.7)$$

$^{23}$This does not imply that the identity operator/vacuum has a bulk dual, as infinitesimal $h$ differs from $h = 0$ identically. Even in the $c = \infty$ limit the propagator is the non-trivial $\frac{1}{1-\rho}$ as $h \to 0$.

$^{24}$These results are really conjectural, as they were discovered by computing the $\rho$ expansion to high orders using the recursion relations of section 5.4 and then identifying a pattern in the result.
So we see that the quantity $c(1 - \rho)^4 \propto c\sigma^4$ indicative of the new bulk length scale $\sigma_* \sim c^{-1/4}$ appears in every term.

The series expansion in $1/c$ has zero radius of convergence because the coefficients grow factorially. But this series is not very exotic, and in fact it can be obtained from the $1/c$ expansion of the well-studied quartic integral

$$Z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \, e^{-\frac{1}{2}z^2 + \frac{12}{c(1-\rho)}z^4}. \quad (5.5.8)$$

This integral can be re-summed either via a Borel transform or by noting that it obeys a second order differential equation. The correspondence between the quartic integral and the combinatorics of the series is easy to explain by considering the computation of $\langle \phi \phi \rangle$ in gravitational perturbation theory. The leading terms at short-distances come from summing all diagrams generated by a bulk cubic coupling of schematic form $\frac{1}{\sqrt{c}}(\partial \bar{z} \phi)^2 h_{zz}$ and ignoring graviton self-interactions. We can count the diagrams in this theory by integrating out the graviton, which leads to a pure quartic interaction for $\phi$ and explains the combinatorics of our result.

One might expect that one could take this leading order diagrammatic argument further, working to all orders in the effective action after integrating out the graviton. Since we are dealing with $\langle \phi \phi \rangle_{\text{horo}}$, one should include only the holomorphic modes of the graviton. We can obtain additional evidence that such an effective action for $\phi$ is possible by computing finite-distance corrections to the correlator. More precisely, we look at small $\sigma$ corrections to the limit with $c\sigma^4$ fixed at large $c$ (equivalently, these are $1/c$ corrections to the large $c$, fixed $c\sigma^4$ limit). In terms of the representation (5.5.5), these corrections are the subleading series coefficients in the $f_n(\rho)s$ in an expansion around $\rho = 1$. We find empirically that these subleading terms are correctly reproduced up to the fourth derivative $f_n^{(4)}(1)$ by the

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25If we define $x = \frac{c^{1/2}}{c(1-\rho)^4}$ then the integral obeys the differential equation

$$16x^2 Z'' + (32x - 6)Z' + 3Z = 0 \quad (5.5.9)$$

which can be solved in terms of incomplete Bessel functions.
following integral expression:

\[(1 - \rho)K_{\text{holographic}} \sim e^{-(\sigma - \frac{\sigma^2}{2} + \frac{\sigma^4}{4})} \left[ -1 + \sqrt{\frac{c\sigma^4}{\pi}} \int_{-\infty}^{\infty} e^{-c\sigma^4(z^2 + a_4(\sigma)z^4 + a_6(\sigma)z^6 + a_8(\sigma)z^8)} \, dz \right], \tag{5.5.10}\]

where we have determined the first few \(a_n\) coefficients to be

\[
a_4(\sigma) = -3 + 6\sigma^2 - \frac{151}{15}\sigma^4, \\
a_6(\sigma) = -27\sigma^2 + \frac{617}{5}\sigma^4, \\
a_8(\sigma) = \frac{3519}{10}\sigma^4, \tag{5.5.11}\]

up to higher order corrections in \(\sigma\). What is notable about this expression is that, by fitting only a few numbers in the \(a_n(\sigma)\) coefficients, we correctly reproduce the first several terms in the \(f_n(\rho)\) expansion around \(\rho = 1\) for all \(n\). It would be very interesting if these \(a_n\) coefficients could be determined directly by integrating out the graviton modes in AdS\(_3\).\(^{26}\)

Coming back to the leading order expression (5.5.7), we can attempt to transform the asymptotic series into an exact function. Either by solving the differential equation (5.5.9) or by Borel resumming, we obtain a one-parameter family of possible results, which are linear combinations of modified Bessel\(^{27}\) functions,

\[
\lim \left[ (1 - \rho)K_{\text{holographic}}(\rho) \right] = e^{-\lambda X} \sqrt{2\pi X} \left( (1 + \kappa) I_{\frac{3}{4}}(X) - \kappa I_{-\frac{3}{4}}(X) \right), \tag{5.5.12}\]

where \(X \equiv \frac{c(1-\rho)^4}{384}\) and the limit is \(c \to \infty\) and \(\rho \to 1\) with \(X\) fixed. One can verify that this function reproduces equation (5.5.7) when expanded in large \(c\).

The parameter \(\kappa\) in equation (5.5.12) is arbitrary, as \(K_{\text{holographic}}\) has the correct perturbative expansion around \(c = \infty\) for any value of this parameter. Thus \(\kappa\) represents a non-perturbative

\(^{26}\)The effective actions in [153] may be a useful tool for such a derivation.

\(^{27}\)They have series expansions \(I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\nu+1)} x^{2k+\nu} \) and are real for \(\nu = \pm \frac{1}{4}\) and \(x > 0\).
ambiguity in the definition of the correlator; it arises because there is a branch cut on the positive real axis in the Borel plane.

If $\kappa$ is real, then equation (5.5.12) will be real for positive $X$, i.e. for positive $c$ and real $\rho$. On the other hand, if the correct choice is not $\kappa \in \mathbb{R}$, then $K_{\text{holo}}$ will be complex. If the propagator has a Kallen-Lehmann representation, then it would seem that its spectral function must develop an imaginary part in this case. A complex value for a scalar propagator usually signals the presence of an instability where $\phi$ quanta decay into other states. However, it is less clear what the precise interpretation is in our case since our $\phi$ is a linear combination of descendants of the scalar primary $\mathcal{O}$. In CFT$_2$ such operators cannot mix with the vacuum sector (i.e. with ‘gravitons’), as correlators like $\langle \phi T \cdots T \rangle$ vanish, and the only interactions we have included are those of $\phi$ with gravity. Thus any $\text{Im}[\kappa] \neq 0$ suggests a non-perturbative violation of unitarity at short distances.

We cannot determine the value of $\kappa$ with the methods of this section. However, in section 5.6 we will take a numeric approach, and argue that the $\langle \phi \phi \rangle$ correlator develops a singularity and likely an imaginary piece at short distances.

## 5.6 Numerics and Locality

Arguments based on black hole thermodynamics and the gauge redundancies of general relativity suggest that local observables do not exist in quantum gravity. However, we have introduced an exact bulk proto-field operator $\phi(X)$ and provided various techniques for computing its correlation functions. While $\phi(X)$ is in some sense a non-local operator, one may nevertheless wonder if its correlation functions exhibit pathologies associated with the failure of bulk locality in quantum gravity.

The propagator depends on the central charge $c$, on the kinematic configuration, and on

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28In part $\phi$ is non-local simply because it includes gravitational dressing; this is analogous to the way that the electron operator is non-local because it must be attached to a Wilson line. But we should also recall the caveat (discussed in section 5.5) that even in bulk gravitational perturbation theory $\langle \phi \phi \rangle$ exhibits a surprising UV/IR mixing, so perhaps $\sigma_*$ can be modified or eliminated by identifying a different observable with better IR behavior.
the conformal dimension $h$ of the CFT$_2$ scalar primary $O$ dual to $\phi$. As we have explained in sections 5.2.2, the full $K$ depends on two independent kinematic variables, although most of the non-trivial information in the full $K$ can be obtained from the holomorphic part $K_{\text{holo}}$. This part only depends on the variable $\rho = e^{-2\sigma}$, where $\sigma(X,Y)$ is the geodesic separation between $X$ and $Y$ in the vacuum. Throughout this section we will mostly focus on $K_{\text{holo}}$, as it is easier to obtain high-orders numerical results for this object, though in figure 5.7 we provide evidence that our conclusions concerning $K_{\text{holo}}$ should also apply to the full $K$.

Recall that in section 5.3.3 we already observed a sharp conflict between bulk locality when in the limit of large $h$. The conflict arose because $K_{\text{holo}}$ developed an imaginary part at the geodesic separation $\sigma_* = R_{AdS} \log(2 + \sqrt{3}) \approx 1.32 R_{AdS}$. Of course we would not have expected correlators of fields with trans-Planckian masses to be local, so this result was not too surprising. In this section we mostly focus on light bulk fields, though the form of the pathologies we uncover will be very similar.

We have studied short-distance locality in three ways. First, in section 5.5 we discussed the $1/c$ expansion of $K_{\text{holo}}$, and observed that the $1/c^n$ corrections can be determined exactly in the short distance limit $\rho \rightarrow 1$. However, the $1/c$ expansion was asymptotic (it has zero radius of convergence), and Borel resumming the series led to a non-perturbative ambiguity. Generically, this means that $\langle \phi \phi \rangle$ develops an instability or unitarity-violating imaginary piece, though there does exist a reality-preserving resolution of the ambiguity.

Second, in section 5.3, we developed methods that allow us to compute the semiclassical part of the correlator numerically and, in some cases, analytically. In particular, we numerically computed the critical value $\sigma_c$ where the semiclassical part develops an imaginary piece. At infinite $h$, where we know the exact (not just semiclassical) correlator, this critical value for the semiclassical part matches that of the exact result. At smaller $h$, the exact correlator could in principle develop additional singularities or imaginary pieces at even larger values of $\sigma$, but it seems very unlikely that quantum effects could cancel the imaginary part of the
Figure 5.4: This figure displays fits to logarithms of ratios of successive coefficients in the $\rho$ expansion of equation (5.6.1) up to the 400th order. In all cases we have set $h = 0$ identically, and the value of $c$ increases from the top to the bottom of the plot, ranging from 1.5 to $10^5$. Each line corresponds to one of the points on Fig. 5.5, but for legibility we have only included every fifth point.

semiclassical propagator.$^{29}$

Third, in the next section, 5.6.1, we will obtain additional numeric evidence that is complementary to the first and second methods, by evaluating $K_{\text{holo}}$ to high orders in the $\rho$ expansion. This numeric high-order behavior provides abundant evidence that the $\rho$-series has a finite radius of convergence, breaking down when $\sigma \propto -\log \rho \propto c^{-1/4}$ when $h \sim \mathcal{O}(c^0)$, and at $\sigma \propto (h/c)^{1/3}$ when $h/c$ is fixed but small in the large $c$ limit. Assuming our numerical extrapolations are correct, this implies that $\langle \phi \bar{\phi} \rangle$ becomes singular at a finite separation, which is a harbinger of the failure of bulk locality. It may be possible to analytically continue the propagator to shorter distances, but one would expect it to develop an imaginary part. In section 5.6.2 we discuss the interpretation of these results.

### 5.6.1 Numerical Results for the Exact $\rho$ Expansion

In this section we will study the AdS$_3$ proto-field propagator numerically to high orders in the $\rho$ expansion. Since $\rho = e^{-2\sigma}$ and $\sigma$ is the geodesic separation between the points, this

$^{29}$We cannot prove this does not happen. However, the semiclassical piece scales differently ($\mathcal{O}(c)$ in the exponent) from the residual piece ($\mathcal{O}(1)$), so a potential cancellation cannot be as simple as the residual piece contributing an exactly opposite phase.
Figure 5.5: In this plot we used the fits of Fig. 5.4 to extract an approximate asymptotic ratio $\frac{a_{n+1}}{a_n}$, which was then used to identify $\sigma_*$, the scale at which bulk locality appears to break down, for each value of $c$. For very small values of $c$ we find $\sigma_*$ of order the AdS scale, so that at $c \to 1$ we smoothly match the large $h$ results of section 5.3.3, as indicated by the red line. At large $c$ we enter the flat space regime of small $\sigma_*$, where we extract the fit $\sigma_* \propto c^{-0.27}$. Varying the details of the fitting shifts the exponent, but we consistently find that it lies between 0.25 and 0.28.

is an expansion around the long-distance limit. So on physical grounds, we should expect the propagator to be well-behaved as $\rho \to 0$. If bulk locality did not break down, then we would expect the radius of convergence of the $\rho$-series to be 1, as is the case for the free field propagator $\frac{\rho^h}{1-\rho^h}$. Instead we will present evidence that:

- The radius of convergence in $\rho$ is strictly less than 1 at finite $c$, which means that $K_{\text{holo}}$ develops a singularity at some finite critical distance $\sigma_*(c) > 0$.

- The failure of convergence occurs at a physical separation in AdS$_3$ that scales as $\sigma_*(c) \propto c^{-p}$ at large $c$. We find $0.25 < p < 0.28$, which approximates the expected $p \approx \frac{1}{4}$ from section 5.5 but appears slightly larger, as shown in figure 5.5. This behavior holds throughout the $h \ll c$ regime.

$\text{Padé approximants to the } \rho \text{ series expansion display a ‘condensation’ of poles that suggest that at distances shorter than } \sigma_c, \text{ the correlator will develop a branch cut.}$
• When \( h \sim c \gg 1 \), convergence fails at a physical separation of order the AdS\(_3\) length. The behavior as \( h \gg c \) connects smoothly with the results of section 5.3.3, as shown in figure 5.6. We also find that for any \( h \), when \( c \approx 1 \) the propagator breaks down at roughly the same distance scale as in the large \( h \); this is indicated with the red line in Fig. 5.6.

Since these results only follow from a numerical analysis, they are not theorems. Readers are encouraged to conduct their own investigations with the attached code implementing the recursion relations of section 5.4.

![Figure 5.6](image-url)

Figure 5.6: This figure displays the scale at which the propagator breaks down as we approach the semiclassical limit; for each value of \( c \), we’ve taken a range of values for the ratio \( \frac{h}{c} \). The data was extracted in the same way as in Fig. 5.5. We see that at large \( h \) we approach the convergence bound \( \sigma_\ast = R_{AdS} \log(2 + \sqrt{3}) \) from the exact result of section 5.3.3. We have also shown (blue, dashed) the result from the numeric semiclassical computation in section 5.3.4, and find that it agrees with the radius of convergence analysis for the large \( c \) (= 20,000) points shown above.

The radius of convergence in \( \rho \) can be analyzed by studying the growth of the coefficients \( a_n \) in the expansion

\[
K_{\text{holo}} = \rho^h \sum_{n=0}^{\infty} a_n \rho^n, \tag{5.6.1}
\]
where the $a_n$ depend implicitly on $h$ and $c$. If the radius of convergence in $\rho$ is less than 1, then the $a_n$ must grow exponentially, which means that as $n \to \infty$ we must have $\frac{a_{n+1}}{a_n} \to r$ for some $r > 1$. However, there will likely be a subleading power-law behavior as well, so that $a_n \approx n^v r^n$ for some $v$. We display a fit to this behavior for 30 values of $c$, ranging from 1.5 to $10^5$ in figure 5.4.

The convergence radius in $\rho$ and thus the value of $r$ will correspond with a physical geodesic distance scale in the bulk $\sigma = -\frac{R_{\text{AdS}}}{2} \log r$. Since $r$ depends implicitly on $c$, if the physical separation is proportional to $c^{-1/4}$, then we should find $\log[r(c)] \propto c^{-1/4}$ at large $c$. We can test this hypothesis by identifying $r(c)$ for a large range of values of $c$, and then fitting a line to $\log[\log r(c)]$ vs $\log c$, as the slope of this line measures the exponent $-\frac{1}{4}$. We have provided such a fit in figure 5.5. Varying the details of the fit changes the exponent $p$ of $c^{-p}$, but in all cases we find that $p$ ranges between about 0.25 and 0.28. Thus the exponent appears systematically slightly larger than would be expected from the analysis of section 5.5. This may be due to the fact that for large values of $c$, we simply do not have enough coefficients $a_n$ to get to the asymptotic regime of very large $n$ necessary to correctly identify the exponent. But this discrepancy may be worthy of further consideration.

We can also study the radius of convergence in the case of general $h$. At large enough $c$, we expect to enter the semiclassical regime where the radius of convergence should depend only on the ratio $h/c$, and indeed this is what we see in figure 5.6. In fact, we find the radius of convergence in this limit exactly matches the results of section 5.3.3, for the critical value for $\sigma$ where an imaginary piece develops. One might have expected a distinctive feature at the BTZ black hole threshold, $h = \frac{c}{24}$, i.e. the value where the corresponding primary state develops a horizon in AdS, but we do not see any such feature, and it is not until $h/c \sim 1$ that the curve starts to flatten out towards its asymptotic large $h$ value.

Interestingly, at small $c$, regardless of $h/c$, the radius of convergence also approaches the large $h$ value $R_{\text{AdS}} \log(2 + \sqrt{3})$. So it appears that this is a fairly generic “strong coupling” result, valid either at small $c$ or at large $h/c$. It would be very interesting to understand a
Figure 5.7: This plot shows the same data as figure 5.5 for the radius of convergence $\sigma_*$ as a function of $c$, except here we compare the radii of convergence of the holomorphic and the mixed contributions to the full correlator. Due to numerical limitations we only include the coefficients $a_n$ and $b_n$ up to $n = 300$, which means that these data are not as reliable as those of figure 5.5. Nevertheless, this plot provides clear evidence that the parametric scaling of $\sigma_*$ with $c$ is the same for the full and holomorphic propagators when we restrict to the $z$-$\bar{z}$ plane.

more physical origin of this scale.

Finally, in figure 5.7 we compare the radius of convergence of the holomorphic propagator to that of the mixed terms (recall that the full propagator $K = 2K_{\text{holo}} + K_{\text{mixed}} - K_{\text{global}}$) in the $z$-$\bar{z}$ plane, following up on the preliminary analysis in section 5.4.2. We only used the coefficients $a_n$ and $b_n$ up to $n = 300$, and so the precise $\sigma_*(c)$ from this plot is not as reliable as that of figure 5.5. However, we can see from the scaling of $\sigma_*$ with $c$ at large $c$ that the radius of convergence of the full correlator seems to scale in the same way as that of $K_{\text{holo}}$. At small $c$ the behavior also appears similar insofar as both correlators break down at the AdS scale, though the precise radius of convergence differs by an order one factor. This result largely justifies our focus in this paper on the simpler $K_{\text{holo}}$, but it would still be interesting to study the full propagator in more detail in future work. We have not studied the full propagator away from the $z$-$\bar{z}$ plane in detail, so it would be very interesting to explore that regime.
5.6.2 Interpreting the $c^{-1/4}$ Length Scale

In the previous section, we found evidence that the $\langle \phi \phi \rangle$ correlator develops a singularity or branch cut at a scale $\sim \mathcal{O}(c^{-1/4})$ when $h$ is fixed as we take the large $c$ limit.\footnote{We showed results for vanishing $h$, but have found the same behavior for fixed small $h$.} This scale is truly quantum, invisible in the semiclassical limit, and in some sense represents an irreducible distance below which locality breaks down. It is natural to ask if this length appears as a fundamental scale in the AdS$_3$ gravity theory itself, and not just in the $\langle \phi \phi \rangle$ correlator.

From an effective theory viewpoint, such a fundamental scale would be extremely surprising,\footnote{For instance, an analogous result in de Sitter in $d = 4$ would imply that quantum gravity effects become relevant at some geometric average length scale between the Planck length and the Hubble radius, long before they would be expected to be important. However, we do not expect UV/IR mixing in higher dimensional theories, so we do not believe this phenomenon can occur.} since it involves both the Planck length $\ell_{pl}$ and the AdS radius $R_{AdS}$. Restoring dimensionful quantities,

$$\sigma_* \sim c^{-1/4} \sim R_{AdS}^{3/4} \ell_{pl}^{1/4}. \quad (5.6.2)$$

One possibility is certainly that this length scale is not fundamental, but rather is an artifact of our definition of the proto-field $\phi$. As we have mentioned, perhaps the correct lesson is that one should attempt to define an improved bulk field that has a good flat space limit.

However, it may be that this scale is truly indicative of the underlying physics of quantum gravity in AdS$_3$. From this point of view, it is interesting to note that the scale $c^{-1/4}$ also arises as the smallest string length $\ell_s$ in known stable, controlled string compactifications in perturbative string theory. The basic reason that $c^{-1/4}$ appears in this context is straightforward to understand. In compactifications of the form $\text{AdS}_3 \times S^3 \times M_4$ where $M_4$ is a 4d compact manifold, the radius of the $S^3$ is the same as the AdS$_3$ length scale $R_{AdS}$, and the size of the $M_4$ cannot be taken smaller than $\ell_s$. So, the 3d Planck length $\ell_{pl}$ is related to the
10d Planck length $\ell_{10}$ by

$$\ell_{\text{pl}} R_{\text{AdS}}^3 \ell_s^4 \lesssim \ell_{10}^8 \lesssim \ell_s^8,$$

(5.6.3)

where we have used the fact that the $M_4$ volume is greater than $\ell_s^4$, and the 10d Planck scale must be smaller than the string scale. Therefore,

$$\ell_s \gtrsim R_{\text{AdS}}^{3/4} \ell_{\text{pl}}^{1/4} \sim \sigma_s.$$

(5.6.4)

As far as we know, no stable AdS$_3$ string compactifications violate this inequality. This may just be a “lamp-post” effect, i.e. stable string compactifications with smaller compact dimensions may exist but simply be much more difficult to find. And the inequality relating $\ell_s$ and $\sigma_s$ may be coincidental. On the other hand, a tantalizing explanation is that spacetime itself breaks down at the scale $\sigma_s$, creating an obstacle to the existence of weakly coupled strings with a smaller string length.\textsuperscript{33}

5.7 Discussion

If a theory’s dynamics are fully non-local, then the underlying spacetime picture loses its meaning, becoming a mere book-keeping device – and the values of fields at different spacetime points become arbitrary independent variables. What makes spacetime more than just a label is some notion of locality, which can be diagnosed using correlations. In a physical spacetime, nearby observables should be highly correlated, whereas correlators of distant fields should be small. In a theory of quantum gravity, however, spacetime may play a role intermediate between these two extremes, with observables becoming more highly correlated as they approach each other, up to a point, beyond which local spacetime is revealed as a

\textsuperscript{33}It would be interesting to study the interplay in compactified theories between the AdS$_3$ gravitational contributions we have included and the contributions from Kaluza Klein modes of the large compact directions. Since the KK modes encode the fact that the theory really involves higher-dimensional gravity, they may soften or remove the UV/IR mixing, similarly to what occurs in computations of the free energy [154].
mere approximation.

While the two-point function of fields in a complete theory of quantum gravity is beyond the scope of presently available techniques, in this paper we have settled for something simpler that contains much of the same physics. We have computed the two-point function \( \langle \phi(X_1)\phi(X_2) \rangle \) of an exact ‘proto-field’ \( \phi \) that is reconstructed in the bulk of AdS\(_3\) in terms of a boundary primary operator \( \mathcal{O} \) and all of its Virasoro descendants. Equivalently, in perturbation theory \( \phi \) correlators are fully dressed by all graviton loops, but without any quantum corrections from matter fields. The proto-field is a quantity defined in the spirit of the conformal bootstrap, in that it leverages the non-perturbative power of the conformal symmetry in the CFT\(_2\) by resumming all contributions in an irreducible representation of the Virasoro algebra.

In this paper we have developed techniques to compute the correlator \( \langle \phi(X_1)\phi(X_2) \rangle \) and characterized some of its most striking features. We have analyzed the distance scale where it develops singularities and imaginary pieces, and we have interpreted these phenomena as an indication of the breakdown of bulk locality, as summarized in section 5.1.3. For light bulk fields at large \( c \), quantum effects produce the most important non-perturbative pathologies. But in the semiclassical limit of fixed \( h/c \) (or \( G_N m_\phi \)) and large central charge, branch cuts and imaginary pieces were already visible. We have found that these semiclassical pathologies are not misleading, as they appear to persist in the exact quantum propagator.

It should be possible to derive our semiclassical results, including the imaginary parts, from a bulk gravity calculation. Such a derivation may shed light on the nature of any physical instabilities associated with these imaginary pieces. Moreover, while the full generalization of our approach to higher dimensions is probably impossible (since in higher dimensions graviton interactions are not fixed by symmetry), semiclassical gravity computations are likely to be tractable. It would also be interesting to connect our results with other work \([136, 155, 103, 156]\) on the breakdown of locality in quantum gravity.

Our results tentatively suggest a more general lesson – when we attempt to define an exact
bulk observable, we may induce small violations of unitarity, even if the underlying CFT is healthy. To test this idea we will need to better understand more general bulk correlators, their dependence on CFT data, and their implications for physical bulk measurements.

We have studied the φ propagator at spacelike separations, as is most natural when we take a Euclidean CFT as our starting point. The Lorentzian correlators of any number of local CFT operators can be precisely determined via the analytic continuation of Euclidean correlators [157, 43]. It is much less clear whether Lorentzian φ correlators can be determined in the same way, because φ arises from an infinite sum of local operators in the CFT and carries an emergent bulk coordinate label. This question may be connected with the gauge-dependence of φ, since any analytic continuation in a bulk coordinate will clearly depend on our choice of the coordinate system! At a pragmatic level, the most obvious next step would be to analytically continue (z, ̄z) to Lorentzian signature, as these coordinates have a natural correspondence with the locations of operators in the boundary CFT. This simply leads to the continuation from σ > 0 to σ < 0. There are several formulas that hint at a simple analytic relation between the correlator at σ and −σ. For example, the semiclassical potential $T$ (5.3.10) is $g'(σ)$ times an anti-symmetric function of σ.34 In any case, it will be very interesting to understand how the non-perturbative non-localities that we have discovered manifest in Lorentzian signature.

We have focused on the propagator of φ because it is the most tractable non-trivial φ correlator. But another quantity that would be extremely interesting to compute is the heavy-light correlator

$$\langle O_H O_H φ_L O_L \rangle,$$  \hspace{1cm} (5.7.1)

where $O_H$ is a heavy operator and $φ_L$ is the bulk proto-field made from $O_L$. This correlator computes the bulk-to-boundary propagator for $φ_L$ in the background of a heavy state such as

34Interestingly, the infinite $h$ result for $⟨φφ⟩$ is formally invariant under $σ → −σ$, though the path from positive to negative $σ$ passes around a branch cut that can break the symmetry and introduce dependence on the operator ordering.
a BTZ black hole, and therefore could be used to probe what happens when \( \phi_L \) approaches a black hole horizon. Many of the methods used in this paper to study \( \langle \phi \phi \rangle \) should be applicable to this heavy-light correlator as well, though the calculations will be more complicated because of the extra operator insertions. Roughly speaking, each invariant contribution to the heavy-light bulk-boundary correlator will have the complexity of a 5-pt conformal block, since \( \phi \) involves a sum over an infinite set of Virasoro descendants.

By computing equation (5.7.1), we may begin to study the properties of black hole microstates without relying on bulk perturbation theory. Major technical and conceptual challenges remain, but it appears that a direct investigation of the horizon may be possible. As a first step, it will be interesting to explore features that emerge at the Euclidean horizon (the tip of the ‘cigar’) as a consequence of the failure of the KMS condition [141, 12, 2, 158, 159] in black hole microstate backgrounds.

A distinct line of inquiry will be to search for an improved observable that is free of the UV/IR mixing we observe in \( \langle \phi \phi \rangle \). Along these lines, it would be rewarding to obtain a definition of bulk fields in other gauges. Another possibility is that rather than evaluating the propagator in the vacuum, one ought to introduce a sum over boundary graviton configurations, along the lines of the way soft photons resolve IR divergences in 4d QED. That is, it may be that the non-IR-safety of \( \phi \) is similar to the physics of Sudakov factors, and when one attempts to produce and detect \( \phi \) particles, one unavoidably produces some gravitons in the process. It would be interesting to try to construct an IR-safe observable, directly related to local measurements in the bulk, and to see to what extent the behavior of \( \langle \phi \phi \rangle \) is modified.

The UV/IR mixing behavior of the propagator might have a gauge invariant footprint in CFT observables. For example, at the level of diagrammatics one would expect \( \langle \phi \phi \rangle \) to contribute as an intermediate propagator in CFT correlators computed using Witten diagrams. Thus it would be interesting to examine the one-loop gravitational corrections [20, 1] to a 4-pt CFT\(_2\) correlator including a scalar exchange. At a deeper level, there is a
simple relationship between the Zamolodchikov recursion relations that compute the exact bulk propagator and the relations that compute Virasoro blocks for 4-pt correlators. Along with the idea of geodesic Witten diagrams [151] for conformal blocks, this may provide a direct avenue for further exploration.
Chapter 6

The Bulk-to-Boundary Propagator in Black Hole Microstate Backgrounds

This chapter is based on the following paper:


Abstract

First-quantized propagation in quantum gravitational AdS$_3$ backgrounds can be exactly reconstructed using CFT$_2$ data and Virasoro symmetry. In this chapter, we develop methods to compute the bulk-to-boundary propagator in a black hole microstate, $\langle \phi_L \mathcal{O}_L \mathcal{O}_H \mathcal{O}_H \rangle$, at finite central charge. As a first application, we show that the semiclassical theory on the Euclidean BTZ solution sharply disagrees with the exact description, as expected based on the resolution of forbidden thermal singularities, though this effect may appear exponentially small for physical observers.
6.1 Introduction

Perturbative gravitational physics in AdS$_3$ is largely determined by the Virasoro algebra of CFT$_2$ [39, 160, 15, 42, 16, 17, 23, 161, 139, 140, 22, 21, 75, 18, 19, 162, 163, 164, 165, 166, 167, 168]. But one can go further, and explicitly compute many nonperturbative quantum gravitational effects [12, 2, 169, 9, 170, 171] as well. These include a prescription for bulk reconstruction that incorporates the exchange of all multi-graviton states [3], and has led to a quantitative prediction for the breakdown of bulk locality at the non-perturbative level in $G_N$ [4]. In this work we will study the heavy-light bulk-boundary correlator

$$ \mathcal{A}(y, z, \bar{z}) = \langle O_{H}(\infty)O_{H}(1)O_{L}(z, \bar{z})\phi_{L}(y, 0, 0) \rangle $$

which can be used to explore the limits of gravitational effective field theory, including in the near horizon region of the black hole microstate created by $O_{H}$. We will primarily focus on the pure graviton contributions to this observable.

In the remainder of this introduction we will discuss an aspect of the information paradox associated with Euclidean correlators. Then we will provide a physical interpretation for the bulk field $\phi$ and a summary of the technology developed to compute universal contributions to $\mathcal{A}$. In this paper we will largely focus on technical machinery, while in future work we hope to use these methods to study infalling observers.

6.1.1 A Problem at the Euclidean Horizon

Black hole microstates can be sharply differentiated from the canonical ensemble using Euclidean correlators [141, 12, 2]. In the canonical ensemble, correlators are subject to the KMS condition, which means that they must be periodic in Euclidean time. Black hole solutions such as BTZ reflect this periodicity directly in their Euclidean geometry.
Figure 6.1: This figure depicts a Euclidean bulk-boundary correlator in a black hole microstate. Although we have forced the correlator to live on the Euclidean BTZ geometry, due to violations of the KMS condition the correlator will be multivalued on the Euclidean time circle, and so must have a branch cut. Thus semiclassical predictions for bulk correlators must breakdown. In particular, as the Euclidean time circle shrinks to vanishing size at the horizon, it would seem that exact bulk correlators must differ significantly from their semiclassical limits at the Euclidean horizon.

In contrast, microstate correlators cannot exhibit this periodicity [12]. If we attempt to parameterize them using BTZ Schwarzschild coordinates, then they must be multivalued on the Euclidean time circle, as pictured in figure 6.1. This suggests that bulk-boundary correlators will be singular at the horizon, where the size of the Euclidean time circle shrinks to zero. One of our goals will be to study bulk-boundary correlators near the Euclidean horizon.

6.1.2 Quantum Gravitational Propagation

We recently derived a prescription [3] for an exact AdS$_3$ proto-field $\phi$ in Fefferman-Graham gauge. Instead of recapitulating the formal definition of $\phi$ (see section 6.2 for those details), let us consider some physical scenarios where $\phi$ plays a natural role. These include first quantized propagation in a quantum gravitational background, and a universe including only a free-field coupled to gravity at low energies.

We can view $\phi$ as a short-hand for an operator sourcing first-quantized propagation in a quantum gravitational background. That is, to all orders in gravitational perturbation theory about a background created by distant sources, in the vacuum sector we have an
operator relation \[3, 164]\]

\[
\phi(X_1)\phi(X_2) = \exp \left[ -m \int_{X_1}^{X_2} ds \sqrt{g_{\mu\nu} \dot{Y}^\mu \dot{Y}^\nu} \right]
\]

(6.1.2)

This formula includes both quantum gravitational interactions with external sources, such as CFT operators, as well as gravitational self-interactions.

However, \(\phi\) does not include loops of matter fields, including itself. To clarify this, consider a complete AdS\(_3\) theory whose sub-Planckian spectrum consists of a single species of scalar particles with purely gravitational interactions. That is, a theory with a low-energy effective action

\[
S_{\text{universe}} = \int d^3x \sqrt{-g} \left( \frac{1}{2} (\nabla \varphi)^2 - \frac{m^2}{2} \varphi^2 + \frac{1}{16\pi G_N} R - 2\Lambda \right)
\]

(6.1.3)

Above the Planck scale, we do not have any particular requirements for the universe other than those imposed upon us by symmetry, unitarity, and crossing.

This universe will be a large \(c\) CFT\(_2\) whose spectrum between the vacuum and the Planck scale\(^1\) consists entirely of a Fock space of states generated by the single-trace operator \(\mathcal{O}\) dual to \(\varphi\), with generalized free theory OPE coefficients \([172, 173, 174]\) modified only by gravitational effects. In this universe, the single-particle component\(^2\) of the effective field \(\varphi\) will correspond with the proto-field operator \(\phi\) constructed from \(\mathcal{O}\). This follows because \(\varphi\) has only gravitational interactions, which are encoded in the Virasoro algebra and were incorporated into the definition of \(\phi\). This universe must contain a Cardy spectrum of black holes at energies \(E > \frac{c}{6}\), so it provides a very convenient laboratory to explore the interactions of particles with black holes, including near horizons. But the reconstructed

\(^1\)By the Planck scale we mean an energy scale \(\lesssim \frac{c}{24}\); the details won’t be important for this informal discussion. We do not know if a CFT like this actually exists, nor do we know of any bottom-up constraints that make the existence of such a CFT appear problematic.

\(^2\)Graviton exchanges in \(n + 1\)-pt correlators induce mixing between \(\varphi\) and \(n\)-particle states, so that \(\langle \varphi \mathcal{O}^n \rangle \neq 0\), whereas \(\phi\) has a vanishing 2-pt function with multi-trace operators. We describe this in more detail in appendix E.4.
proto-field ϕ still differs from the field ϕ, as ϕ only incorporates gravitational loops, and not
loops of itself.

Although we have used the language of perturbation theory to describe ϕ, as we review
in section 6.2.2, ϕ is defined using symmetry considerations at finite c.

Universal Contributions to A

In this work we will mostly focus on the pure graviton contributions to A. But our techniques
can be used to compute more general ‘bulk-boundary Virasoro blocks’, where full Virasoro
representations are exchanged between a pair of boundary operators and a bulk-boundary
pair. So it is natural to ask to what extent the behavior of the full A correlator will differ
in more general holographic CFT\(_2\)s.

One way to partially address this question is by adapting OPE convergence analyses
and large c asymptotics [84, 14, 175, 167] to estimate the effect of new interactions and
high-energy states on A. That is, the correlator can be expanded as

\[
A(y, z, \bar{z}) = \sum_{h, \bar{h}} C_{HH;h,h} C_{LL;h,h} V_{h,\bar{h}} (y, z, \bar{z})
\]  

(6.1.4)

where C are conventional OPE coefficients and V are new bulk-boundary conformal blocks
involving primaries O\(_{h,\bar{h}}\) exchanged between the heavy and light operators.

The convergence rate of this expansion will depend on the kinematic configuration defined
by O\((z, \bar{z})\)φ\((y, 0, 0)\), providing information about the sensitivity of A to high-energy (or spin)
states and OPE coefficients. Near the breakdown of convergence, the correlator A will be
UV sensitive, but in regions where the convergence is rapid, the correlator will be dominated
by the exchange of low-dimension primaries,\(^3\) leading to a universal gravitational prediction.

\(^3\)In the free field + gravity universe at infinite c, the vacuum Virasoro block and its images under crossing
will dominate, as discussed in section 6.2.1. In a more general holographic CFT\(_2\) the correlator will be
dominated by the exchange of low-dimension primaries associated with light bulk fields. [176, 177]
Thus the vacuum or pure gravity contribution\(^4\)

\[
\mathcal{V}_0(y, z, \bar{z}) = \left\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1) \left( \sum_{\{m_i\},\{n_j\}} \frac{L_{-m_1} \cdots L_{-m_i} |0\rangle \langle 0| L_{n_1} \cdots L_{n_i}}{\mathcal{N}_{\{m_i\},\{n_j\}}} \right) \mathcal{O}_L(z, \bar{z}) \phi_L(y, 0, 0) \right\rangle \tag{6.1.5}
\]

will be a major focus of study in this work, though the techniques we develop are also applicable to the calculation of \(\mathcal{V}_{h, \bar{h}}\) associated with the exchange of any state.

The full bulk operator \(\varphi\) will receive other important corrections, as full bulk fields involve sums of proto-fields. In perturbation theory, this means that \(\varphi\) will contain small admixtures of multi-trace operators [117, 113, 121]. Instead of the sum in equation (6.1.4), these effects will appear as sums over the external operators \(\mathcal{O}\) contained in \(\varphi\). We will not explore these effects here, but understanding or constraining their contributions in detail is an important problem as it would shed light on the difference between correlators of proto-fields and full bulk fields.

### 6.1.3 Summary

This work largely consists of technical developments to compute the bulk-boundary Virasoro blocks \(\mathcal{V}_{h, \bar{h}}\) contributing to \(\left\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \phi_L \right\rangle\), with \(\phi_L\) the Fefferman-Graham gauge proto-field [3] defined by the bulk primary condition. We mostly focus on the vacuum block contribution \(\mathcal{V}_0(y, z, \bar{z})\) of equation (6.1.5), though all our methods can be applied to general blocks.

We review the fact that \(\mathcal{V}_0\) determines the physics of propagation in a semiclassical gravitational background in section 6.2. We also briefly review the bulk primary condition and the definition of \(\phi\). Then, in the remaining sections, technical developments include:

- We compute the semiclassical limit \(\mathcal{V}^{\text{semi}}_0\) (section 6.3) and show explicitly that it agrees

\(^4\)For simplicity, we only wrote down the holomorphic descendant states in equation (6.1.5), but since \(\mathcal{V}_0(y, z, \bar{z})\) does not factorize, we also need to include the anti-holomorphic descendant states. We will denote a projection operator like that in equation (6.1.5) as \(\mathcal{P}^{\text{holo}}_h\) and a full projection operator that also includes the anti-holomorphic contributions as \(\mathcal{P}_{h, \bar{h}}\). We mostly consider scalar exchanged states (\(h = \bar{h}\)) and in particular the vacuum (\(h = \bar{h} = 0\)) in this paper so we will often omit \(\bar{h}\) in the subscript.
with BTZ correlators. We develop a monodromy method [37, 36] for computing bulk-boundary blocks. We also define their symmetry transformations precisely, and show that these greatly constrain their form.

- We develop three methods (section 6.4) to compute the bulk-boundary blocks in either a $y$ (radial direction) or $z, \bar{z}$ expansion, but exactly in $h_H, h_L, c$, and attach Mathematica implementations. These methods match the semiclassical BTZ correlators in appropriate limits, as shown in figure 6.2. In Appendix E.2, we used the OPE block method [112, 77, 3, 178] to compute $\mathcal{V}_0$ perturbatively at order $1/c^2$.

On a more conceptual level, in section 6.5 we demonstrate that the semiclassical approximation fails if we interpret $\mathcal{V}_0$ as a correlator on the Euclidean BTZ solution. For explicit results, see figures 6.4 and 6.7. In this regard the Euclidean horizon is a special place where derivatives of the correlator become singular. But in the most conservative interpretation, these singularities may have a non-perturbatively small coefficient.

### 6.2 Brief Technical Review

In this section we provide a very brief review. In section 6.2.1 we discuss BTZ correlators, emphasizing that in the semiclassical limit, they are entirely determined by summing the vacuum Virasoro block over all possible OPE channels [179]. Then in section 6.2.2 we review our bulk reconstruction prescription, and the relation between BTZ Schwarzschild coordinates and other coordinate systems.
6.2.1 Semiclassical Probe Correlators in a BTZ Black Hole Background

The spherically symmetric BTZ black hole background has a Euclidean metric

$$ds^2 = (r^2 - r_+^2)dt_E^2 + \frac{dr^2}{r^2 - r_+^2} + r^2d\theta^2$$

(6.2.1)

with the Lorentzian metric related by $t_E \rightarrow it$. Note that the horizon radius

$$r_+ = 2\pi T_H = \sqrt{\frac{24h_H}{c}} - 1$$

(6.2.2)

where $T_H$ is the Hawking temperature, $h_H$ is the (holomorphic) heavy operator dimension, and $c = \frac{3}{2G_N}$ is the central charge of the CFT$_2$. The full semiclassical bulk-boundary correlator for a free field in this geometry$^5$ is given by the image sum [179]

$$A_{\text{semi}} = \langle \phi O \rangle_{\text{BTZ}} = \left(\frac{r_+}{2}\right)^{2h_L} \sum_{n=-\infty}^{\infty} \frac{1}{r_+ \cosh(r_+(\delta\theta + 2\pi n)) - \sqrt{r_+^2 - r_+^2} \cos(r_+\delta t_E)}$$

(6.2.3)

where $\delta\theta$ and $\delta t_E$ are differences between the bulk and boundary values of the cylindrical coordinates $t_E$ and $\theta$, and $r$ is the location of $\phi$ in the radial direction. The sum guarantees periodicity under $\theta \rightarrow \theta + 2\pi$ for the angular coordinate. The geometry and the correlator are periodic under $t_E \rightarrow t_E + \beta$, enforcing the KMS condition geometrically, and avoiding a conical singularity at the horizon $r = r_+$.

If we take the limit $r \rightarrow \infty$ and rescale the bulk-boundary correlator by $r^{2h_L}$, we obtain a probe CFT 2-pt correlator in the BTZ geometry. This is a semiclassical approximation to a heavy-light CFT 4-pt correlator. In the OPE limit where the light probe operators collide, this 4-pt function has a Virasoro block decomposition. The only Virasoro primary states that propagate in this light-light OPE channel are the vacuum and double-trace operators.

$^5$By this we mean the limit $c \rightarrow \infty$ with $h_L$ and $\frac{h_H}{c}$ fixed, so that the light free field acts as a probe.
The semiclassical vacuum Virasoro block contribution is simply the \( n = 0 \) term of the sum in equation (6.2.3). In other words, in the semiclassical limit

\[
\gamma_0^{\text{semi}} = \left( \frac{r_+}{2} \right)^{2h_L} \frac{1}{\left[ \frac{r}{r_+} \cosh(r_+\delta\theta) - \frac{\sqrt{r^2 - r_+^2}}{r_+} \cos(r_+\delta E) \right]^{2h_L}}
\]

is the bulk-boundary vacuum block, generalizing the semiclassical heavy-light vacuum block [15]. We will show how to obtain this semiclassical result in Section 6.3.

Clearly the \( n \neq 0 \) terms in equation (6.2.3) must also be intimately connected to the Virasoro vacuum block, since all of the terms in the summation have its functional form. From the point of view of the bootstrap, the image sum simply satisfies crossing symmetry in the simplest possible way, as it sums the inherently crossing asymmetric Virasoro vacuum block over all possible OPE channels. This means that in the semiclassical limit, bulk-boundary correlators in a black hole background are fully determined by the vacuum block, suggesting that universal features of AdS\(_3\) quantum gravity can be understood by computing \( \mathcal{V}_0 \) of equation (6.1.5) exactly.

### 6.2.2 CFT Definition of the Bulk Proto-Field

For completeness we will now summarize the definition of the bulk proto-field operator \( \phi \); for derivations and explanations see [3]. In Fefferman-Graham gauge, where the vacuum AdS\(_3\) metric takes the form

\[
ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2} - \frac{1}{2} S(z)dz^2 - \frac{1}{2} \bar{S}(\bar{z})d\bar{z}^2 + y^2 \frac{S(z)\bar{S}(\bar{z})}{4} dzd\bar{z}
\]

(6.2.5)

for general holomorphic and anti-holomorphic functions \( S, \bar{S} \), a bulk scalar proto-field must satisfy the bulk primary conditions [3]

\[
L_n \geq 2 \phi(y, 0, 0)|0\rangle = 0, \quad \bar{L}_n \geq 2 \phi(y, 0, 0)|0\rangle = 0
\]

(6.2.6)
along with the condition that in the vacuum, the bulk-boundary propagator is

$$\langle \mathcal{O}(z, \bar{z}) \phi(y, 0, 0) \rangle = \frac{y^{2h_L}}{(y^2 + z\bar{z})^{2h_L}}$$ \hspace{1cm} (6.2.7)

These conditions uniquely and exactly determine $\phi(y, 0, 0)$ as a CFT operator defined by its series expansion in the radial $y$ coordinate:

$$\phi(y, 0, 0) = y^{2h_L} \sum_{N=0}^{\infty} \frac{(-1)^N y^{2N}}{N!(2h_L)_N} \mathcal{L}_{-N} \bar{\mathcal{L}}_{-N} \mathcal{O}(0)$$ \hspace{1cm} (6.2.8)

The $\mathcal{L}_{-N}$ are polynomials in the Virasoro generators at level $n$, with coefficients that are rational functions of the dimension $h_L$ of the scalar operator $\mathcal{O}$ and of the central charge $c$. For example

$$\mathcal{L}_{-2} = \frac{(2h + 1)(c + 8h)}{(2h + 1)(c + 2h)(8h - 5)} \left( L_{-1}^2 - \frac{12h}{c + 8h} L_{-2} \right)$$ \hspace{1cm} (6.2.9)

Note that in the limit $c \to \infty$, we have $\mathcal{L}_{-N} \to L_{-1}^N$ and $\bar{\mathcal{L}}_{-N} \to \bar{L}_{-1}^N$, and our $\phi$ matches known results [107, 116, 117] for bulk reconstruction in the absence of gravity. In some situations it is convenient to compute the properties of a simpler object, which we refer to as the ‘holomorphic part of $\phi$ [4]; it is defined by replacing the anti-holomorphic $\bar{\mathcal{L}}_{-N} \to \bar{L}_{-1}^N$, so that anti-holomorphic gravitons are neglected.

This CFT operator $\phi$, inserted in correlation functions such as $\langle \phi \mathcal{O} T \rangle$ and $\langle \phi \phi \rangle$ correctly reproduces the result of Witten diagram calculations\(^6\) in the bulk [3, 4]. We will show explicitly in this paper that $\phi$ inserted in states generated by heavy operators correctly reproduces the correlator of a scalar field on the corresponding non-trivial background geometry.

The function $S(z), \bar{S}(\bar{z})$ in the metric (6.2.5) are related to expectation values of the

\(^6\)These Witten diagram calculations were performed in the Fefferman-Grahm gauge to facilitate the comparison. In [125, 126] another construction for $\phi$ was proposed, which differs perturbatively from the bulk reconstruction adopted in this paper.
boundary stress-energy tensor $T(z), \bar{T}(\bar{z})$ by

$$S(z) = \frac{12}{c} T(z), \quad \bar{S}(\bar{z}) = \frac{12}{c} \bar{T}(\bar{z}). \quad (6.2.10)$$

Throughout this paper we will work with $\phi$ defined in Fefferman-Graham gauge, which is natural in the coordinates $(y, z, \bar{z})$, and in virtually all cases of interest we will have

$$T(z) = \frac{h_H}{z^2}, \quad \bar{T}(\bar{z}) = \frac{h_H}{\bar{z}^2}. \quad (6.2.11)$$

due to the presence of heavy operators. The semiclassical metric (6.2.5) is then describing a BTZ black hole in the coordinate system $(y, z, \bar{z})$. However, for clarity, we will almost always express correlators of $\phi$ using the BTZ coordinates $(r, t_E, \theta)$. This is simply a re-labeling of spacetime points, and not a gauge transformation. The relations between the $(y, z, \bar{z})$ coordinates in equation (6.2.5) and BTZ coordinates are a bit subtle, and are worked out in appendix E.1. The result for spherically symmetric black holes is

$$y = \frac{2}{\bar{r}} \left( \frac{r - \sqrt{r^2 - r_+^2} - 1}{r_+^2 + 1} \right) e^{t_E}$$

$$z = \frac{1}{\bar{r}} e^{t_E + i\theta} \quad (6.2.12)$$

$$\bar{z} = \frac{1}{\bar{r}} e^{t_E - i\theta}$$

where

$$\bar{r} \equiv \left( \frac{r + i r_+ \sqrt{r^2 - r_+^2} - 1}{(1 + i r_+)\sqrt{r^2 - r_+^2}} \right)^{1/2} \quad (6.2.13)$$

and $r_+ = \sqrt{\frac{24h_H}{c} - 1}$ is the horizon radius. Notice that for $r^2 < r_+^2 + 1$ the $y$ coordinate must be analytically continued into the complex plane, and that in this range the magnitude of $\frac{y^2}{z\bar{z}}$ remains constant, with only its phase changing with $r$.

For the configuration $\langle \mathcal{O}_H(\infty) \mathcal{O}_H(0) \mathcal{O}_L(1, 1) \phi_L(y, z, \bar{z}) \rangle$ that will be used in Section
6.3.1, we can map to the BTZ coordinates \((r, t_E, \theta)\) via the transformation (6.2.12), since the operator \(O_L\) at \(z = \bar{z} = 1\) has \(t_E = \theta = 0\). This configuration is intuitive and has the nice interpretation of the correlator as a function of the location of \(\phi_L\) with fixed \(O_L\). However, in Section 6.4 (and also parts of Section 6.3), in order to take advantage of the bulk primary condition for computation, we will compute \(V_0\) in the kinematic configuration \(\langle O_H(\infty)O_H(1)O_L(z, \bar{z})\phi_L(y, 0, 0)\rangle\). To map this configuration to the BTZ coordinates \((r, t_E, \theta)\), we first perform a conformal transformation to the new configuration \(\langle O_H(\infty)O_H(0)O_L(1, 1)\phi_L(y', z', \bar{z}')\rangle\) with

\[
y' = \frac{y}{\sqrt{(1 - z)(1 - \bar{z})}}, \quad z' = \frac{1}{1 - z}, \quad \bar{z}' = \frac{1}{1 - \bar{z}}
\]

and then relate the coordinates \((y', z', \bar{z}')\) to \((r, t_E, \theta)\). We obtain the transformation from \(\langle O_H(\infty)O_H(1)O_L(z, \bar{z})\phi_L(y, 0, 0)\rangle\) to the BTZ coordinates \((r, t_E, \theta)\)

\[
y = 2 \left( r - \frac{\sqrt{r^2 - r_+^2 - 1}}{r_+^2 + 1} \right)
\]

\[
z = 1 - \bar{r}e^{-t_E-i\theta}
\]

\[
\bar{z} = 1 - \bar{r}e^{-t_E+i\theta}
\]

We explain more details of this relation in appendix E.1.2. We will be using these relations implicitly when we probe the Euclidean horizon in section 6.5.

Ultimately, all of these coordinates and their relations are merely labels for the non-local CFT operator \(\phi\), which was precisely defined by the bulk primary conditions and equation (6.2.8). From these algebraic conditions, it might not be obvious that \(\phi\) can be interpreted as a field in a dynamical spacetime, nor do these conditions explicitly encode any information about the black hole geometries we will study. The bulk dynamics are entirely emergent.
6.3 Semiclassical Analyses and Symmetry

The purpose of this section is to connect the bulk primary condition reviewed in section 6.2.2 to semiclassical correlation functions. It was implicit in [3] that correlators of the bulk proto-field $\phi$ automatically reconstruct the leading semiclassical free-field correlators in any vacuum AdS backgrounds, including BTZ black holes; in section 6.3.1 we will make this explicit. In section 6.3.2 we explain how the monodromy method can be used to compute semiclassical $\phi$ (bulk) conformal blocks. Finally in section 6.3.3 we will use the symmetry transformation properties of $\phi$ to constrain the coordinate dependence of bulk-boundary Virasoro blocks. We address both $\langle \phi O_O H \rangle$ and a previously unexplained simplification [4] in $\langle \phi \phi \rangle$.

6.3.1 Semiclassical Bulk Correlators from Uniformizing Coordinates

In this section, we will show that in the background of a heavy state $|B\rangle$, vacuum block exchange for the correlator $\langle B|\phi_L O_L |B\rangle$ automatically reconstructs the leading semiclassical bulk-to-boundary propagator in the bulk vacuum geometry corresponding to $|B\rangle$.\(^7\) This treatment generalizes an argument from [16] to bulk conformal blocks.

We restrict to states $|B\rangle$ created by the product of a finite number of local operators $O_i$, so that the sources $O_i$ can be separated by a ball from the boundary points of the probes,\(^8\) and the boundary stress tensor $T(z)$ in the state $|B\rangle$ is holomorphic outside this ball, where we can define the local operator $B(x)$ that corresponds to the state $|B\rangle$. The bulk conformal block is the contribution to $\langle B B \phi_L O_L \rangle$ from the exchange of the vacuum and its Virasoro

\(^7\)By ‘bulk vacuum geometry’, we mean that the bulk stress tensor vanishes, aside from localized sources. For CFT states $|B\rangle$ created by a product of local operators $O_i$ with large scaling dimensions $\Delta_i$, their corresponding bulk stress tensor will be localized to geodesics in the large $\Delta_i$ limit and therefore produce a bulk vacuum geometry. More generally, the bulk vacuum geometry can be viewed as an approximation where bulk sources are treated as localized.

\(^8\)For instance, map to the cylinder, with the light boundary operator $O_L$ at $\infty$ and the boundary point corresponding to the proto-field at $-\infty$, so they are separated from the finite region containing the sources.
descendants between $\phi_L \mathcal{O}_L$ and $B B$:

$$\mathcal{V}_0 \equiv \langle B(\infty) B(0) \mathcal{P}_0 \phi_L(y, z, \bar{z}) \mathcal{O}_L(1) \rangle,$$

(6.3.1)

where $\mathcal{P}_0$ is the projection operator onto the vacuum irrep. The background stress tensor is its expectation value in the state $|B\rangle$:

$$T_B(z) \equiv \langle B|T(z)|B\rangle.$$

(6.3.2)

We are interested in the limit of infinite $c$ with $\frac{1}{c} T_B(z)$ fixed. In this case, one can define uniformizing coordinates $f(z)$, such that they satisfy

$$\frac{12T_B(z)}{c} = S(f, z),$$

(6.3.3)

where $S(f, z)$ is the Schwarzian derivative\textsuperscript{9}, so that $\langle B|T(f(z))|B\rangle = 0$ in the uniformizing coordinates. In other words, the OPE coefficient vanishes for $T(f(z))$ in the operator product $B \times B$, and straightforward power-counting of factors of $c$ shows that at infinite $c$, the OPE coefficients for all powers of $T(f(z))$ (normalized by their two-point functions) vanish as well. This is equivalent to the statement that if $\phi_L$ and $\mathcal{O}_L$ are conformally mapped to the uniformizing coordinates, then at infinite $c$ the only state that contributes in the projection onto the vacuum irrep in (6.3.1) is the vacuum state itself. Therefore in these coordinates, $\langle B B \mathcal{P}_0 \phi_L \mathcal{O}_L \rangle$ is just the usual $\langle \phi_L \mathcal{O}_L \rangle$ bulk-to-boundary propagator in pure AdS.

The transformation of $\mathcal{O}_L$ under $z \rightarrow f(z)$ is simply the usual local scalar operator transformation $\mathcal{O}_L(f(z)) = (f'(z) \bar{f}'(\bar{z}))^{-h_L} \mathcal{O}_L(z)$. For $\phi_L$, the transformation must be extended into the bulk; by definition, $\phi_L$ transforms by extending $z \rightarrow f(z)$ into the bulk such that

\textsuperscript{9}The Schwarzian derivative is defined to be

$$S(f, z) = \{f(z), z\} \equiv \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.$$  

(6.3.4)
Fefferman-Graham gauge is preserved. This extension is given by \((y, z, \bar{z}) \rightarrow (u, x, \bar{x})\) with
\[
\begin{align*}
u &= y - \frac{4(f'(z) \bar{f}'(\bar{z}))^2}{4f'(z)f'(\bar{z}) + y^2 f''(z)\bar{f}''(\bar{z})} \\
x &= f(z) - \frac{2y^2(f'(z))^2 \bar{f}''(\bar{z})}{4f'(z)f'(\bar{z}) + y^2 f''(z)\bar{f}''(\bar{z})} \\
\bar{x} &= \bar{f}'(\bar{z}) - \frac{2y^2(\bar{f}'(\bar{z}))^2 f''(z)}{4f'(z)f'(\bar{z}) + y^2 f''(z)\bar{f}''(\bar{z})}
\end{align*}
\] (6.3.5)

Under this transformation, \(\phi_L\) transforms like a bulk scalar, \(\phi_L(y, z, \bar{z}) \rightarrow \phi_L(y, z, \bar{z}) = \phi_L(u, x, \bar{x})\). So, we have
\[
\langle B(\infty)B(0)P_0\phi_L(y, z, \bar{z})O_L(1) \rangle = (f'(1)\bar{f}'(1))^{h_L} \langle \phi_L(u, x, \bar{x})O_L(f(1), \bar{f}(1)) \rangle \\
= (f'(1)\bar{f}'(1))^{h_L} \left( \frac{u}{u^2 + (x - f(1))(\bar{x} - \bar{f}(1))} \right)^{2h_L}, \quad (6.3.6)
\]
where \(u, x, \bar{x}\) should be understood to be the functions of \((y, z, \bar{z})\) in (6.3.5). This result reproduces the leading semiclassical contribution to the bulk-to-boundary propagator in a general vacuum metric, which we can write in Fefferman-Graham gauge (6.2.5). This follows first of all from the fact that the coordinate transformation (6.3.5) is also the transformation that takes the Fefferman-Graham gauge metric (6.2.5) to be the pure AdS metric
\[
ds^2 = \frac{du^2 + dx d\bar{x}}{u^2}. \quad (6.3.7)
\]
The semiclassical bulk-to-boundary propagator is therefore given by the pure AdS bulk-to-boundary propagator in the new coordinates, which is just (6.3.6), plus a sum over images arising from the fact that the coordinate transformation is typically not single-valued. The result (6.3.6) is just one of these images, but each image can be thought of as just the vacuum block in a particular channel [23]. Moreover, if \(h_L \gg 1\), then there is a sharp transition
between regions where one image dominates and the others are subleading. In this case, one can cleanly think of one image as being the dominant semiclassical contribution, which is reproduced by the bulk vacuum block in the corresponding channel.

In the specific case where the heavy state \( |B\rangle \) is created by a single primary operator \( \mathcal{O}_H \) of weight \( h_H \), we can be more explicit. Using the coordinate transformation (6.3.5) with \( f(z) = z^\alpha, \bar{f}(\bar{z}) = \bar{z}^{\bar{\alpha}} \), we find that the bulk-to-boundary propagator transformed to the Fefferman-Graham coordinates is

\[
\alpha^{h_L} \bar{\alpha}^{h_L} \langle \phi_L (y, z, \bar{z}) \mathcal{O}_L (1, 1) \rangle_{FG} = \left[ \frac{4y\alpha \bar{\alpha} z^{\alpha+1} \bar{z}^{\bar{\alpha}+1}}{4z\bar{z} (z^\alpha - 1) (\bar{z}^{\bar{\alpha}} - 1) + y^2 ((\alpha + 1)z^\alpha + \alpha - 1)(\bar{z}^{\bar{\alpha}} (\bar{\alpha} + 1) + \bar{\alpha} - 1)} \right]^{2h_L}
\]

By the above argument, this is also the semiclassical limit \( V_0^{\text{semi}} \) of the bulk-boundary vacuum block \( \langle \mathcal{O}_H(\infty) \mathcal{O}_H(0) \mathcal{P}_0 \mathcal{O}_L (y, z, \bar{z}) \mathcal{O}_L(1) \rangle \), i.e.

\[
V_0^{\text{semi}} = \alpha^{h_L} \bar{\alpha}^{h_L} \langle \phi_L (y, z, \bar{z}) \mathcal{O}_L (1, 1) \rangle_{FG}.
\]

To obtain the result in the usual BTZ coordinates \((r, t_E, \theta)\), we can use the coordinate transformations (6.2.12), and the result is exactly the same as equation (6.2.4). We have also checked this semi-classical result with the result of \( V_0 \) from the recursion relation (to be introduced in next section) analytically at low orders and numerically up to order \( z^{10} \bar{z}^{10} \) in the limit where \( \frac{h_L}{c} \) is fixed, and \( h_L \ll c \).

### 6.3.2 Monodromy Method

Our goal in this subsection is to extend Zamolodchikov’s monodromy method\(^{10}\) [37, 36] for Virasoro conformal blocks to bulk-boundary blocks with three boundary and one bulk proto-field operator. Although boundary blocks factorize into holomorphic and anti-holomorphic

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\(^{10}\)For a nice pedagogical introduction to the monodromy method, see appendix D of [40].
pieces, once a bulk field enters the correlator this does not occur. In [4], we developed the monodromy method for the two-point function $\langle \phi \phi \rangle$ of two bulk proto-fields in a “holomorphic” version where only the holomorphic stress tensors are included (all global descendants, under either $L_{-1}$ or $\bar{L}_{-1}$, are also included). \footnote{This holomorphic bulk block can be obtained by taking a chiral limit where $c_R \gg c_L$ and in particular $c_R$ is infinitely larger than all the other parameters that determine the correlator, so that the right-moving stress tensors decouple; it can therefore be thought of as a chiral gravity limit.} In this subsection, we will continue to work in this limit for the sake of simplicity, and will relegate some discussion of how to apply the monodromy method to the full block to appendix E.5.

As usual, the monodromy method begins by considering the wavefunction $\psi$ for a degenerate light operator $\hat{\psi}$ acting on the correlator in the large $c$ limit, where it exponentiates to the form

$$\langle O^H(z_1)O^H(z_2)\phi_L(y_3,z_3,\bar{z}_3)O_L(z_4,\bar{z}_4) \rangle = e^{\xi g},$$  

(6.3.10)

with $g \sim O(c^0)$ at large $c$. The wavefunction $\psi$ satisfies the degenerate equation of motion

$$\psi''(z) + \frac{6}{c} T(z) \psi(z) = 0,$$

(6.3.11)

where the potential $T(z)$ is the stress tensor acting on the bulk correlator. Because the bulk field necessarily involves both $z$ and $\bar{z}$ dependence, we will also need to consider the analogous anti-holomorphic degenerate wavefunction $\bar{\psi}$, which satisfies the conjugate of (6.3.11).

The action of the stress tensors $T(z)$, $\bar{T}(\bar{z})$ on the correlator are determined by the singular parts of their OPE with the bulk and boundary operators. For the boundary operators $O_L, O_H$, these singular terms are the standard ones for primary operators and simply depend on the primary operator weights as well as their derivatives, which bring down derivatives of the exponent $g$. For the bulk operator $\phi$, however, the OPE is more complicated:

$$T(z) \phi(y, w, \bar{w}) \sim -y^2 \frac{1 - y^4}{(z - w)^3} \frac{\partial_w + y^2 \bar{T}(\bar{w}) \partial_w}{\bar{T}(\bar{w})} \phi(y, w, \bar{w}) + \frac{1}{2} \frac{y \partial_y \phi(y, w, \bar{w})}{(z - w)^2} + \frac{\partial_w \phi(y, w, \bar{w})}{z - w}. \quad (6.3.12)$$
The origin of the complicated cubic term is the fact that $\phi$ transforms under special conformal transformation $L_1$ by moving around in the bulk in a way that depends on the background geometry. A similar formula holds for the $\bar{T}(\bar{z})\phi(y, w, \bar{w})$ OPE, related to the above one by conjugation. These expressions require some care because, as we will discuss in more detail, the $T, \bar{T}$s that appear on the RHS have singularities that must be regulated appropriately.

We will begin by considering the limit where $h_L/c$ is small, so to leading order $T$ and $\bar{T}$ are just given by their behavior in the heavy state background. For holomorphic backgrounds, i.e. $\bar{h}_H = 0$, we therefore have at leading order in $h_L/c$ that

\[
T(z)\phi(y, w, \bar{w}) \sim -y^2 \frac{\partial_{w}\phi(y, w, \bar{w})}{(z - w)^3} + \frac{1}{2} \frac{y \partial_{y}\phi(y, w, \bar{w})}{(z - w)^2} + \frac{\partial_{\bar{w}}\phi(y, w, \bar{w})}{z - w},
\]

\[
\bar{T}(\bar{z})\phi(y, w, \bar{w}) \sim -y^2 \frac{\partial_{w} + y^2 c T_H(w) \partial_{\bar{w}}}{(\bar{z} - \bar{w})^3} \phi(y, w, \bar{w}) + \frac{1}{2} \frac{y \partial_{y}\phi(y, w, \bar{w})}{(\bar{z} - \bar{w})^2} + \frac{\partial_{\bar{w}}\phi(y, w, \bar{w})}{\bar{z} - \bar{w}},
\]

where $T_H$ includes only the contribution from the heavy boundary operators $O_H$, 

\[
T(z) = T_H(z) + T_L(z), \quad T_H(z) \equiv \frac{\langle T(z)O_H(z_1)O_H(z_2) \rangle}{\langle O_H(z_1)O_H(z_2) \rangle},
\]

and therefore $T_H(w)$ is regular when $\phi$ is separated from the $z$ positions of the heavy operators.

Using the bulk OPE (6.3.12) and the standard boundary OPEs, the potentials for the correlator $\langle O_H(z_1)O_H(z_2)\phi(y_3, z_3, \bar{z}_3)O_L(z_4) \rangle$ are easily seen to be

\[
\frac{6}{c} T(z) = -\frac{y^2 c z_3}{(z - z_3)^3} + \frac{y_3 c y_3}{2(z - z_3)^2} + \frac{c_{z_1}}{z - z_1} + \frac{c_{z_2}}{z - z_2} + \frac{c_{z_3}}{z - z_3} + \frac{c_{z_4}}{z - z_4}
\] 

\[+ \frac{h_H}{(z - z_1)^2} + \frac{h_H}{(z - z_2)^2} + \frac{h_L}{(z - z_4)^2}.
\]

(6.3.15)
for the holomorphic potential and

\[
\frac{6}{c} \bar{T}(\bar{z}) = -\frac{y_3^2}{2} \left( c_{z_3} + y_3^2 \frac{6}{c} T_H (z_3) c_{\bar{z}_3} \right) + \frac{y_3 c_{y_3}}{2 (\bar{z} - \bar{z}_3)^2} + \frac{c_{\bar{z}_1}}{\bar{z} - \bar{z}_1} + \frac{c_{\bar{z}_2}}{\bar{z} - \bar{z}_2} + \frac{c_{\bar{z}_3}}{\bar{z} - \bar{z}_3} + \frac{c_{\bar{z}_4}}{\bar{z} - \bar{z}_4} + \frac{\bar{h}_L}{(\bar{z} - \bar{z}_4)^2}
\]

(6.3.16)

for the anti-holomorphic one, where the \( c_i \)'s are the derivatives of the semiclassical function \( g \):

\[
c_X \equiv \frac{\partial}{\partial X} g.
\]

(6.3.17)

The dependence of the function \( g \) on the positions of the operators must be invariant under global coordinate transformations. An efficient way to impose this constraint is that the potentials \( T(z) \) and \( \bar{T}(\bar{z}) \) must decay at large \( z, \bar{z} \) like \( z^{-4}, \bar{z}^{-4} \), respectively. This constraint imposes six conditions (the first three inverse powers of \( z \) and \( \bar{z} \)), so we are able to eliminate the derivatives with respect to all coordinates except for three, which we will choose to be \( y_3, z_4, \bar{z}_4 \). We set the other six coordinates to

\[
z_1 = \bar{z}_1 = \infty, z_2 = \bar{z}_2 = 1, z_3 = \bar{z}_3 = 0.
\]

(6.3.18)

The remaining derivatives \( c_X \) fixed indirectly by the two Schrodinger equations for \( \psi \) and \( \bar{\psi} \), by demanding that the monodromy of the solutions to these Schrodinger equations along cycles in the complex \( z \) and \( \bar{z} \) plane correspond to the weights of the operators contained within those cycles. Setting the heavy operator to be purely holomorphic, i.e. \( \bar{h}_H = 0 \), makes the anti-holomorphic condition particularly useful, since it means that the monodromy of the \( \bar{\psi} \) solutions around a cycle containing only the points \( \bar{z}_1 \) and/or \( \bar{z}_2 \) must vanish. First of all, this condition immediately implies that the coefficient \( c_{\bar{z}_2} \) of the \( \bar{z} = \bar{z}_2 \) pole in \( \bar{T} \) must vanish; we then obtain the following condition when we eliminate \( c_{\bar{z}_2} \) in terms of the \( y_3, z_4, \bar{z}_4 \).
derivatives:
\[ 0 = -\tilde{h}_L - \frac{1}{2}y_3c_{y_3} - \tilde{z}_4c_{\tilde{z}_4}. \] (6.3.19)

This condition is equivalent to the statement that the correlator depends on \( \tilde{z}_4 \) and \( y_3 \) only in the combination
\[ x \equiv \frac{y_3^2}{z_4\tilde{z}_4} \] (6.3.20)
after we factor out an overall \( y^{-2\tilde{h}_L} \) from the correlator. In other words, the function \( g \) must be of the form
\[ g(y_3, z_4, \tilde{z}_4) = g(z_4, x) - 2\tilde{h}_L \log y_3. \] (6.3.21)

Next, we consider the monodromy of the \( \bar{\psi} \) solutions around the point \( \bar{z}_1 \). This monodromy must also be trivial. In the limit \( \bar{z}_1 \to \infty \) that we have taken, this condition implies that \( \lim_{\bar{z}_1 \to \infty} c_{\bar{z}_1} = 0 \).\(^{12}\) We can then use our solution for \( c_{\bar{z}_1} \) in terms of \( c_{y_3}, c_{z_4}, c_{\tilde{z}_4} \) together with the constraint (6.3.21) on \( g \) to write this condition on \( c_{\bar{z}_1} \) in terms of derivatives of \( g(z_4, x) \):
\[ 0 = - (xz_4 + 2)\tilde{h}_L + xg^{(0,1)}(z_4, x) \left( x^2z_4^6cT_H(0) + x + 1 \right) + x(z_4 - 1)z_4g^{(1,0)}(z_4, x) + xz_4\tilde{h}_L. \] (6.3.22)

The general solution to this equation is of the form
\[ g(z_4, x) = g(z_{\text{eff}}(z_4, x)) - \tilde{h}_L \log(1 - z_4) - \tilde{h}_L \log \left( 1 - \frac{(1 + \frac{2}{x})^2}{\frac{\sigma^2}{4}} \right), \] (6.3.23)

\(^{12}\)This is probably most explicitly seen by changing variables of the Schrodinger equation from \( \bar{z} \) to \( t = \frac{1}{\bar{z}} \), in which case the condition \( \lim_{\bar{z}_1 \to \infty} c_{\bar{z}_1} = 0 \) is simply that the coefficient of the pole of \( \bar{T} \) at \( t = 0 \) must vanish. Since the map \( \bar{z} = 1/t \) maps the point \( \bar{z}_1 = \infty \) to 0, a small cycle around \( t = 0 \) contains only the heavy operator \( O_H(z_1) \).
where we have defined the combination

\[ z_{\text{eff}}(z_4, x) \equiv 1 + (z_4 - 1) \left( \frac{2 - xz_4(\alpha_H - 1)}{2 + xz_4(\alpha_H + 1)} \right)^{\frac{1}{\alpha_H}} \]  

so that it reduces to \( z_4 \) at the boundary \( y_3 = 0 \) \((x = 0)\). This parameterization also depends on the stress tensor in the heavy operator background, through the parameter \( \alpha_H \equiv \sqrt{1 - \frac{24T_H(0)}{c}} \). Remarkably, the dependence on all bulk coordinates has been reduced to the dependence on a single coordinate!

In the limit that \( \phi \) approaches the boundary, the bulk block reduces to the boundary block, so the problem is reduced to the previously solved problem of the boundary block behavior. Note that we did not need to use the holomorphic Schrodinger equation monodromy condition to accomplish this reduction. So far, this result holds only to leading order in the small \( h_L/c \) limit, where we can neglect the subleading pieces of \( T \) that depend on the light operator. It would be interesting to extend this analysis to higher orders, where additional conceptual issues arise due to the necessity of regulating the singularities in \( T(z) \) at \( z = 0 \).

### 6.3.3 Constraining Bulk Correlators Using Symmetries

In this section we will discuss the semiclassical and quantum symmetries of various correlators involving the bulk proto-field \( \phi \). Our main focus is on the heavy-light bulk-boundary propagator, discussed in section 6.3.3.1, but we also discuss the bulk-to-bulk propagator in section 6.3.3.2, and the discrete inversion symmetry in section 6.3.3.3.

#### 6.3.3.1 Heavy-Light Bulk-Boundary Correlator

Because the result (6.3.23) at the end of section 6.3.2 followed essentially from demanding certain residues of \( \tilde{T}(\tilde{z}) \) vanished, it should be equivalent to demanding that the corresponding conformal symmetries are satisfied. In this subsection, we will go through this explicitly, though here we will specialize to the case \( h_L = \bar{h}_L \) for simplicity.
We will apply the method to holomorphic heavy operators with $\bar{\hbar} = 0$ and that therefore
\[ \langle O_L (z, \bar{z}) \phi_L (y, z_3, \bar{z}_3) O_H (z_1) O_H (z_2) \rangle \] have no dependence on $\bar{z}_1, \bar{z}_2$. Now this four-point function depends on seven coordinates and we can fix five of them using the symmetry transformations $L_{-1,0,1}$ and $\bar{L}_{-1,0}$, and we get
\[ \mathcal{A} = \langle O_L (z, 1) \phi_L (y, 0, 0) O_H (1) O_H (\infty) \rangle \] (6.3.25)

The remaining generator $\bar{L}_1$ acts on a bulk point as the vector field [3]
\[ \bar{L}_1 (y', z', \bar{z}') = \left( y' \bar{z}', \frac{4y'^2}{-4 + y'^4 S}, \frac{2y'^4 S}{-4 + S S y'^4} + \bar{z}'^2 \right) \] (6.3.26)

interpreted as a differential operator $\bar{L}_1^A \partial_A$ in the bulk (with $A$ running over $(y', z', \bar{z}')$). Here $S$ is defined as
\[ S (z') = \frac{12}{c} \frac{\langle [O_L (z, 1) \phi_L (y, 0, 0) T (z')] [O_H (1) O_H (\infty)] \rangle}{\langle [O_L (z, 1) \phi_L (y, 0, 0)] [O_H (1) O_H (\infty)] \rangle} \] (6.3.27)

where the brackets represent the normal ordering defined in [77]. In the semiclassical limit subtleties concerning normal ordering are irrelevant. $\bar{S}$ would be defined in a similar way, but it vanishes since we are considering the case that $\bar{\hbar} = 0$.

We can identify a certain linear combination of $\bar{L}_1$ with other global conformal generators that will move $z$ and $y$ while keeping the other coordinate fixed. We will denote this linear combination by $\tilde{L}$. We find that $\tilde{L}$ acts on a bulk point as the vector field:
\[ \tilde{L} (y', z', \bar{z}') = \left( \frac{1}{4} y' (4z' - S(0)y^4 - 2y^2 - 2), -y'^2 - y^2 (z' - 1), \frac{1}{2} ((\bar{z}' - 1) (2\bar{z}' - y^4 S(0)) - y'^4 S(z')) \right) \] (6.3.28)

This transformation is a global conformal symmetry which leaves the vacuum invariant,
\[ \langle [\tilde{L}, O_L (z, 1) \phi_L (y, 0, 0) O_H (1) O_H (\infty)] \rangle = 0 \] (6.3.29)
Therefore, the correlator \( e^I \equiv A \) must be a solution to the differential equation

\[
-h_L y^2 + h_L - y^2 (z - 1) \partial_z I - \frac{1}{2} (1 + y^2) y \partial_y I - \frac{1}{2} y^4 \left( h_L S(0) + S(0) \frac{1}{2} y \partial_y I + \frac{1}{2} y \partial_y S(0) \right) = 0
\]  

(6.3.30)

In the semiclassical limit of \( c \to \infty \) with \( \frac{h_L}{c} \) fixed, we simply have

\[
S(0) = \frac{12h_H}{c} + O \left( \frac{1}{c} \right)
\]  

(6.3.31)

Solving this equation while requiring the \( y \to 0 \) limit to match the boundary heavy-light Virasoro vacuum block, we find

\[
V_{\text{semi}}^0 = y^{-2h_L} \left( \frac{\alpha (1-z)^{\frac{\alpha-1}{2}}}{\alpha + ((1-z)^\alpha - 1) \left( \frac{(\alpha-1)}{2} - \frac{1}{y^2} \right)} \right)^{2h_L}
\]  

(6.3.32)

which agrees with the bulk-boundary vacuum block obtained using the uniformizing coordinates (with \( f(z) = z^\alpha \) and \( \tilde{f}(\tilde{z}) = \tilde{z} \), since we are setting \( \tilde{h}_H=0 \)) and the semiclassical monodromy method in previous subsections.

In the large \( c \) limit with \( h_L, h_H \) fixed, using the OPE block method developed in [77, 3], we can compute the next to leading order correction to \( S(0) \), which is given by

\[
S(0) = \frac{12h_H}{c} + \frac{24h_H h_L}{c^2} \frac{1}{(y^2 + z \tilde{z})} \left[ z \left( 2z((z - 12)z + 12)\tilde{z} - y^2(z(z + 2) + 6) - 12 \right) \right. \\
-12(z - 1) \left. \left( y^2 - (z - 2)z \tilde{z} \right) \log(1 - z) \right] + O(1/c^3)
\]  

(6.3.33)

with \( \tilde{z} = 1 \) for \( S(0) \) defined in (6.3.27). Inserted into (6.3.30), this gives a differential equation satisfied by the vacuum block \( V_0 \) up to order \( O(1/c^2) \). In Appendix E.2, we used the OPE block method to compute \( V_0 \) up to order \( O(1/c^2) \) and checked that the result (with \( \tilde{h}_H = 0 \)) does satisfy this differential equation.
6.3.3.2 Symmetry Analysis of the Propagator $\langle \phi \phi \rangle$

We can perform a similar analysis of the bulk-bulk propagator in the vacuum. In recent work [4] we found that when $\langle \phi(X)\phi(Y) \rangle$ is computed while incorporating only holomorphic gravitons (we denote this as $\langle \phi \phi \rangle_{\text{holo}}$), it depends only the geodesic separation between $X$ and $Y$. We will now explain this fact using symmetry.

We can immediately use the translations $L_{-1}$ and $\bar{L}_{-1}$ to write the propagator as

$$G(y_1, y_2, z, \bar{z}) = \langle \phi(y_1, z, \bar{z})\phi(y_2, 0, 0) \rangle_{\text{holo}}$$  \hspace{1cm} (6.3.34)

The transformations $L_0$ and $\bar{L}_0$ also do not depend on $S$ or $\bar{S}$, and so they act simply, giving the differential equations

$$0 = (y_1 \partial_{y_1} + y_2 \partial_{y_2} + 2z \partial_z) G$$

$$0 = (y_1 \partial_{y_1} + y_2 \partial_{y_2} + 2\bar{z} \partial_{\bar{z}}) G$$  \hspace{1cm} (6.3.35)

These require $G$ to depend on only the quantities $\frac{y_1^2}{z\bar{z}}$ and $\frac{y_2^2}{z\bar{z}}$. This is as far as we can go in general, as the action of $L_1$ and $\bar{L}_1$ depend on $S$ and $\bar{S}$, which themselves will depend on the bulk fields $\phi$.

However, if we are only computing the holomorphic propagator [4], then we can ignore anti-holomorphic gravitons, and so $\bar{S} = 0$. In that case $L_1$ acts simply, so that $G$ must satisfy the addition differential equation

$$\left(y_1z \partial_{y_1} + z^2 \partial_z - y_1^2 \partial_{\bar{z}} + y_2^2 \partial_{\bar{z}}\right) G = 0$$  \hspace{1cm} (6.3.36)

This then implies that

$$\langle \phi \phi \rangle_{\text{holo}} = G \left(\frac{2y_1y_2}{y_1^2 + y_2^2 + z\bar{z}}\right)$$  \hspace{1cm} (6.3.37)
or in words, that the holomorphic propagator can only depend on the geodesic separation (in the AdS\(_3\) vacuum) between the bulk points. It would be interesting to study this method at higher orders in \(1/c\) using the additional \(\tilde{L}_1\) generator and the \(S\) determined by gravitational back-reaction.

### 6.3.3.3 A Note on Inversion Symmetry

CFTs may have a discrete symmetry under inversions in the plane, which take

\[(z, \bar{z}) \rightarrow \left(\frac{1}{\bar{z}}, \frac{1}{\bar{z}}\right)\]  

(6.3.38)

After transforming to the cylinder, inversions correspond to the \(t \rightarrow -t\) time reversal symmetry. The vacuum conformal block of CFT\(_2\) possesses these symmetries in both the \(1/c\) expansion and also at finite central charge. Correlation functions in vacuum AdS and probe correlators in classical BTZ black hole backgrounds also inherit this inversion symmetry. For example, the semiclassical bulk-boundary conformal block in equation (6.2.4) is manifestly symmetric under \(\delta t_E \rightarrow -\delta t_E\).

However, complications arise when extending this symmetry to bulk proto-fields at the quantum level. First, we must extend inversions into the bulk in the \((y, z, \bar{z})\) coordinate system in the chosen Fefferman-Graham gauge. Formally, this is fairly simple. If we obtain the vacuum AdS metric of equation 6.2.5 via maps \(f(z), \bar{f}(\bar{z})\) from the pure AdS metric

\[ds^2 = \frac{du^2 + dx d\bar{x}}{u^2}\]  

(6.3.39)

by the coordinate transformation (6.3.5) [3, 132], then inversions correspond to the identification between unprimed and primed coordinates through the relations

\[u\left(y, f(z), \bar{f}(\bar{z})\right) = u\left(y', f\left(\frac{1}{\bar{z}}\right), \bar{f}\left(\frac{1}{\bar{z}}\right)\right)\]
\[ x \left( y, f(z), \bar{f}(\bar{z}) \right) = x \left( y', f \left( \frac{1}{y} \right), \bar{f} \left( \frac{1}{z} \right) \right) \]  \hspace{1cm} (6.3.40)

\[ \bar{x} \left( y, f(z), \bar{f}(\bar{z}) \right) = \bar{x} \left( y', f \left( \frac{1}{y} \right), \bar{f} \left( \frac{1}{z} \right) \right) \]

Note that because \( S(z) \) in equation (6.2.5) is determined by the Schwarzian derivative of \( f(z) \), it is automatic that equation (6.3.40) is a discrete symmetry of the spacetime. We provide a few examples and details in appendix E.1.3, but although equation (6.3.40) is simple, the relation between the original and primed coordinates may be rather involved.

Beyond heavy-light semiclassical limit, to determine the inversion symmetry transformations explicitly we must incorporate the backreaction on the geometry from \( \phi \) itself. This echoes complications encountered when extending Virasoro transformations, such as equation (6.3.26), to the quantum level in the bulk. To extend the inversion symmetry into the bulk, the coordinates \((y, z, \bar{z})\) must transform in a way that depends on \( S(z) \) and \( \bar{S}(\bar{z}) \).

A further issue arises when interpreting inversion symmetry in F-G coordinates as time reversal in the BTZ coordinate system. The connection between F-G coordinates \((r, t_E, \theta)\) and the BTZ Schwarzschild coordinates \((y, z, \bar{z})\) obtained in section 6.2.2 was semiclassical, and did not account for the backreaction of \( \phi \) or quantum corrections. In other words, the Schwarzschild coordinates were introduced as a re-labeling of the F-G coordinates, and it’s challenging to extend this re-labeling beyond the semiclassical probe limit.

We demonstrate some of these points in appendix E.1.3, where we show explicitly how bulk-boundary correlators transform under the inversion symmetry, including quantum effects in \( 1/c \) perturbation theory. As a consequence of such effects, when the exact correlators are plotted using the semiclassical BTZ coordinates \((r, t_E, \theta)\), they are not manifestly symmetric under a \( t_E \rightarrow -t_E \) reflection. Violations of this symmetry are very small, but become noticeable for BTZ \( r \) coordinates very near the horizon. We emphasize that this apparent asymmetry comes from the application of the (merely) semiclassical coordinate transformations from section 6.2.2.
6.4  Exact Correlators

In this section we discuss two different methods that can be used to automate the calculation
of the bulk-boundary conformal blocks $V_{h,\bar{h}}(y, z, \bar{z})$, where its most convenient to use the
kinematic configuration

$$\langle O_H(\infty)O_H(1)O_L(z, \bar{z})\phi_L(y, 0, 0) \rangle .$$

(6.4.1)

The two direct methods of section 6.4.1 are based on a brute force sum over Virasoro de-
sendants. These methods have the advantage of providing either exact $y$-dependence to
some order in $z$, or (nearly) exact $z$-dependence to fixed order in $y$. Then in section 7.4 we
discuss a generalization of the Zamolodchikov recursion relations; this enables a higher order
numerical evaluations of $V_{h,\bar{h}}(z, y)$. The direct methods are most useful for computing cor-
relators in the Lorentzian regime, as they permit extremely high accuracy in the boundary
coordinate and Lorentzian time. The recursion relation is more efficacious in the Euclidean
regime, where it’s possible to obtain $V_0$ as an expansion in $z, \bar{z}$ with coefficients exact in $y$.
The plots in this paper are made with results from the recursion relation up to order $z^{60}\bar{z}^{60}$.

We have attached Mathematica code implementing these three methods. Figure 6.2
provides visual confirmation that the bulk primary reproduces semiclassical physics in black
hole backgrounds at large $c$.

6.4.1  Direct Calculations

The bulk-boundary blocks can be directly evaluated in two ways. The first leverages the
simplicity of the bulk primary condition, while the second attempts to exploit the availability
of high-precision information [2] on the boundary blocks. Thus the first method computes
$V_h(y, z)$ exactly in $y$ but only to low-order in $z$ (practically up to order $\sim z^{14}$), while the
second method computes the blocks only to low order in $y$, but to extremely high precision
in the boundary coordinates (so the result can be written in terms of the $q$ coordinate [36, 8],

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Figure 6.2: These plots compare the exact (blue, \( \log(|V_{exact}^0|) \)) and semiclassical (pink, \( \log(|V_{semi}^0|) \)) correlators for different values of \( r \). The parameters for these plots are \( c = 30.1, h_L = 0.505, \frac{h_H}{c} = 4 \), so that \( r_+ \approx 9.7 \). The semiclassical approximation is excellent for these values of \( t_E \) and \( r \). The gray dashed lines are \( \pm \beta/2 \). We used the exact result from recursion up to order \( z^{60} \bar{z}^{60} \), with convergence \( \frac{|V_{exact}^0(60 \text{ orders}) - V_{exact}^0(59 \text{ orders})|}{V_{exact}^0(60 \text{ orders})} < 10^{-12} \).

which provides far better convergence, along with the ability to analytically continue deep into the Lorentzian regime).

### 6.4.1.1 Using the Bulk Primary Condition

Consider the direct evaluation of the general bulk-boundary conformal block

\[
V_h(y, z, \bar{z}) = \left\langle \mathcal{O}_H(\infty) \mathcal{O}_H(1) \left( \sum_{\{m_i\}, \{n_j\}} \frac{L_{-m_1} \cdots L_{-m_i} |h\rangle \langle h| L_{n_j} \cdots L_{n_1}}{\mathcal{N}(m_i, n_j)} \right) \mathcal{O}_L(z, \bar{z}) \phi_L(y, 0, 0) \right\rangle
\]

(6.4.2)

For simplicity we have only explicitly included a holomorphic intermediate primary \( |h\rangle \) along with a sum over holomorphic Virasoro descendants, but in general we would also simultaneously include an anti-holomorphic intermediate state and a sum over anti-holomorphic Virasoro descendants. Due to the presence of \( \phi_L(y, 0, 0) \) this block will not factor into a product of holomorphic and anti-holomorphic contributions, although the coefficients of any given power \( y^{2h_L + 2N} \) do factorize in this way.

We can compute using equation (6.4.2) almost as efficiently as in the pure boundary case.
of $\langle O_H O_H O_L O_L \rangle$. This follows because the bulk primary condition

$$L_{m \geq 2}\phi(y, 0, 0)|0\rangle = 0 \quad (6.4.3)$$

implies that almost all Virasoro generators act trivially on $\phi$, meaning that

$$\langle h|(L_{n_k} \cdots L_{n_2})_N\rho_L(z, \bar{z})\rho_L(y, 0, 0)\rangle = \langle h|(L_{n_k} \cdots L_{n_2})|L_{n_1}, \rho_L(z)|\rho_L(y, 0, 0)\rangle \quad (6.4.4)$$

whenever $n_1 \geq 2$. Thus we can simply extract any string of Virasoro generators. When computing the vacuum block, we have $\langle O_L(z, \bar{z})\rho_L(y, 0, 0)\rangle = \left(\frac{y}{y^2 + \bar{z}}\right)^{2h_L}$ and we can choose a basis where all $n_i \geq 2$, so that all calculations can be performed in this way.

The calculation of the other factors in equation (6.4.2) are just a standard application of the Virasoro algebra, and are easily automated. This makes it possible to compute $V_0(y, z, \bar{z})$ to reasonably high order order (e.g. at least $z^{14}$ for the holomorphic $\phi$) with exact, algebraic coefficients, including the exact $y$ dependence. For example, up to order $z^4$ we find that the contributions from the exchanged vacuum state and its holomorphic descendants are

$$\frac{V_0(y, z, \bar{z})}{\langle O_L(z, \bar{z})\rho_L(y, 0, 0)\rangle} = 1 + \frac{2h_Lh_H(1 + 3x)z^2}{c(1 + x)} + \frac{2h_Lh_H(1 + 2x)z^3}{c(1 + x)} \quad (6.4.5)$$

$$+ \frac{h_Lh_Hz^4}{c(5c + 22)(x + 1)^2} \left(12x(9 + 2c) + (2 + 12x)(h_L + h_H + 5h_Lh_H)\right) + 3x^2 \left(24 + 5c + 6h_L + 10h_H + 30h_Lh_H\right) + 9c + 40 + \cdots$$

where we define $x \equiv \frac{v^2}{\bar{z}}$ and note that when $x \to 0$ this reduces to the usual boundary Virasoro block. We have also verified that these results agree with those of section 7.4, which are based on an adaptation of the Zamolodchikov recursion relations [37]. At large $c$ with

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13 We apologize for the usage of $x$ in several different places in this paper (e.g. $x$ is also used in equation (6.3.7) as the coordinate in the pure Poincare metric). But its meaning should be clear from the context.
$h_H/c$ and $h_L$ fixed, these results match the semiclassical correlators reviewed in section 6.2.1.

These methods imply that terms of order $z^{2n}$ or $z^{2n+1}$ are always given by polynomials of degree $n$ in $x$ times a factor of $\frac{1}{(1+x)^n}$. This follows because each $L_m$ includes only a single $\partial_z$ derivative acting on $\langle O_L(z, z) \phi_L(y, 0, 0) \rangle$, and since $m \geq 2$ we have at most $n$ such derivatives producing the $z^{2n}$ or $z^{2n+1}$ terms. This insight makes it possible to extract the exact $x$ dependence from the methods of section 7.4, which formally only produce a series expansion in the variables $x, z, \bar{z}$. In practice, this is how we study bulk-boundary correlators in the Euclidean region.

### 6.4.1.2 Using Knowledge of the Boundary Correlators

As our starting point, we can instead use the expression

$$V_h(y, z, \bar{z}) = \left\langle O_H(\infty) O_H(1) P_h O_L(z, \bar{z}) \sum_{n=0}^{\infty} \frac{y^{2h_L+2n}}{n!(2h_L)_n} L_{-n} L_{-n} O_L(0) \right\rangle$$

for the bulk-boundary block. The $L_{-n}$ are linear combinations of products of Virasoro generators at level $n$, determined by the bulk primary condition from section 6.2.2, and $P_h$ is the Virasoro projector onto the block with primary dimension $h$. All Virasoro generators $L_m$ commute with $P_h$, so we can compute $V_h$ by commuting the individual Virasoro generators in $L_{-n}$ to the left, where they act on $O_L(z, \bar{z})$ and $O_H(1)$ before annihilating the $\langle 0 | O_H(\infty) \rangle$ state.

This method outputs the coefficient of $y^{2h_L+2n}$ in $V_h$ as a differential operator acting on the boundary Virasoro block

$$V_h(y, z, \bar{z}) = \langle O_H(\infty) O_H(1) P_h O_L(z, \bar{z}) O_L(0) \rangle$$

As a concrete example, in the kinematic configuration $z = \bar{z}$, the first three terms are

$$V_h = \frac{y^{2h_L}(1-z)^{2h_L}}{z^{4h_L}} \left( V_h(z)^2 - y^2 \left( 2h_L V_h(z) - (1-z)z V'_h(z) \right)^2 \right)$$
The boundary blocks $V_h(z)$ can be computed to extremely high precision [2] using the Zamolodchikov recursion relations. In particular, $V_h$ can be computed in the $q$-expansion, which remains convergent after arbitrary analytic continuation into the Lorentzian regime. This last property will make this method very useful for studying Lorentzian bulk-boundary correlators. We have attached Mathematica code implementing this computation.

We can also use this method to compute $A$ directly from the boundary correlator $\langle O_H O_H O_L O_L \rangle$. In particular, in regimes where the boundary correlator is extremely well-approximated by its semiclassical limit, we can simply feed the semiclassical $V_h(z)$ into this algorithm. When our goal is to uncover new effects from bulk reconstruction (rather than from deviations between the exact and semiclassical boundary correlators), this is a useful trick: any deviations between the result and the semiclassical bulk correlator will be due to the difference between extrapolating boundary operators into the bulk via classical bulk wave equations vs via the protofield construction.\footnote{To be more precise, for any heavy-heavy-light-light boundary correlator we can compare a ‘semiclassical’ and an ‘exact’ extrapolation of one of the boundary operators into the bulk. The ‘semiclassical’ extrapolation is defined as using the bulk wave equation for the classical geometry corresponding to the heavy state, whereas the ‘exact’ extrapolation is defined as using the protofield, as in (6.4.8).}

### 6.4.2 Recursion Relations

The Zamolodchikov recursion relations [37, 36, 38] can be adapted to compute the bulk-boundary block $V_h$. This requires a sum over holomorphic and anti-holomorphic Virasoro descendants from both the Virasoro projector $P_h$ and from the definition of $\phi$. Thus the bulk-boundary correlator $V_h$ has the complexity of two coupled 5-pt Virasoro blocks [148]. In this section we will present the $c$-recursion relations for computing $V_h$. 

\[ +y'^{(1+2h_L)}\left(\frac{2h_L(-6z^2h_H+2h_L(-3+cz+8h_L))V_h(z)-2(1-z)(z(c+2h_L(-5+cz+8h_L)))V^n_h(z)+(-1+z)^2z^2(c+8h_L)V''_h(z)}{4h_Lz^4(c+2h_L(-5+cz+8h_L))^2} + \cdots \right) \]
6.4.2.1 Order by Order Factorization of the Bulk-boundary Blocks

At each order of $y$, the proto-field

$$\phi = y^{2h} \sum_{n=0}^{\infty} (-1)^n y^{2n} \lambda_n \mathcal{L}_{-n} \bar{\mathcal{L}}_{-n} \mathcal{O}(z, \bar{z}), \quad \lambda_n = \frac{1}{n!(2h_L)_n}$$

(6.4.9)

decompose into the product of holomorphic and antiholomorphic parts. This will lead to the factorization of the bulk-boundary blocks at each order of $y$. Thus we can compute the “holomorphic” part of the bulk-boundary block first and recover the full block at the end. We define the holomorphic part of the proto-field to be

$$\bar{\varphi}_h^{\text{holo}}(y, z, \bar{z}) \equiv y^{2h} \sum_{n=0}^{\infty} \lambda_n y^{2n} \mathcal{L}_{-n} \mathcal{O}_{h,h}(z, \bar{z}) .$$

(6.4.10)

Then the holomorphic bulk-boundary block is given by

$$\mathcal{V}_{\text{holo}}(h_1, h_2, c) \equiv \langle \mathcal{O}_H(\infty) \mathcal{O}_{H}(1) \mathcal{P}_{h_1}^{\text{holo}} \mathcal{O}_L(z, \bar{z}) \bar{\varphi}_{h_2}^{\text{holo}}(y, 0, 0) \rangle ,$$

(6.4.11)

where the holomorphic projection operator $\mathcal{P}_{h_1}^{\text{holo}}$ only includes the holomorphic descendants of the $\mathcal{O}_{h_1}$. We will introduce a recursion relation to compute $\mathcal{V}_{\text{holo}}(h_1, h_2, c)$ in next sub-section. Eventually, we are interested in $\mathcal{V}_{\text{holo}}(0, h_L, c)$, which will be given as an expansion in terms of $y^2$, that is

$$\mathcal{V}_{\text{holo}}(0, h_L, c) = \left(\frac{y}{z}\right)^{2h_L} \sum_{n=0}^{\infty} \left(\frac{y^2}{z}\right)^n F_n(z) .$$

(6.4.12)

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15 Note that the definition of the holomorphic part of the proto-field $\phi$ is different the definition of that in [4]. The definition here is simply for computational convenience.

16 For the convenience of discussing the recursion relation later on, here we are being more general by setting the dimensions of the intermediate state and the proto-field to be arbitrary $h_1$ and $h_2$. Eventually, we are interested in the case that $h_1 = 0$ and $h_2 = h_L$. 

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where $F_n(z)$ is an expansion in terms of $z$ (starting from $z^0$). And we can obtain the full bulk-boundary vacuum block via

$$V_0 \equiv V(0, h_L, c) = \left( \frac{y}{z} \right)^{2h_L} \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda_n} x^n F_n(z) F_n(\bar{z})$$ (6.4.13)

where $F_n(\bar{z})$ is defined to be $F_n(z)$ with $z$ replaced by $\bar{z}$ and $x \equiv \frac{y^2}{z\bar{z}}$.

The above result is an expansion of $V_0$ in terms of $x, z, \bar{z}$. On the other hand, as explained at the end of section 6.4.1.1, we know that the vacuum block is of the form

$$V_0 = \left( \frac{y}{y^2 + z\bar{z}} \right)^{2h_L} \tilde{V}_0.$$ (6.4.14)

Here $\tilde{V}_0 = 1 + \cdots$ is an expansion of $z, \bar{z}$ with the coefficient of $z^n\bar{z}^m$ being a product of $\frac{1}{(1+x)[m/2]+[n/2]}$ and a polynomial of degree $[m/2] + [n/2]$ in $x$, where $[k]$ means the maximum integer that’s small or equal to $k$. So we can use the coefficients of $z^n\bar{z}^m$ in $\tilde{V}_0$ up to $x^{[m/2]+[n/2]}$ and extract its exact dependence on $x$. Eventually, the result we obtain for the vacuum block $V_0$ is an expansion in terms of $z$ and $\bar{z}$, with coefficients exact in $x$.

### 6.4.2.2 Recursion relation

Now our task is to compute $V_{\text{holo}}(h_1, h_2, c)$. We will show that $V_{\text{holo}}(h_1, h_2, c)$ can be computed via the following recursion relation

$$V_{\text{holo}}(h_1, h_2, c) = V_{\text{holo}}(h_1, h_2, c \to \infty)$$ (6.4.15)

$$+ \sum_{m \geq 2, n \geq 1} \frac{R_{m,n}(h_1, h_2)}{c - e_{m,n}(h_1)} V_{\text{holo}}(h_1 \to h_1 + mn, h_2, c \to e_{mn}(h_1))$$

$$+ \sum_{m \geq 2, n \geq 1} \frac{S_{m,n}(h_1, h_2)}{c - e_{m,n}(h_2)} V_{\text{holo}}(h_1, h_2 \to h_2 + mn, c \to e_{mn}(h_2)), $$
with

\[
R_{m,n} (h_1, h_2) = - \frac{\partial c_{m,n} (h_1)}{\partial h_1} A^{c_{m,n}(h_1)}_{m,n} P^{c_{m,n}(h_1)}_{m,n} \begin{bmatrix} h_H \\ h_H \end{bmatrix} P^{c_{m,n}(h_1)}_{m,n} \begin{bmatrix} h_L \\ h_2 \end{bmatrix}.
\]

\[
S_{m,n} (h_1, h_2) = - \frac{\partial c_{m,n} (h_2)}{\partial h_2} A^{c_{m,n}(h_2)}_{m,n} P^{c_{m,n}(h_2)}_{m,n} \begin{bmatrix} h_1 \\ h_L \end{bmatrix}.
\]

(6.4.16)

We will parametrize the central charge \( c \) in terms of \( b \) as \( c = 13 + 6 (b^2 + b^{-2}) \). The poles \( c_{m,n} (h) \) are given by

\[
c_{m,n} (h) = 13 + 6 \left[ \left( b_{m,n} (h) \right)^2 + \left( b_{m,n} (h) \right)^{-2} \right]
\]

(6.4.17)

with

\[
(b_{m,n} (h))^2 = \frac{2h + mn - 1 + \sqrt{(m - n)^2 + 4 (mn - 1) h + 4h^2}}{1 - m^2}, m = 2, 3, \ldots, n = 1, 2, \ldots.
\]

(6.4.18)

The functions \( A^{c}_{m,n} \) and \( P^{c}_{m,n} \) are given by

\[
A^{c}_{m,n} = \frac{1}{2} \prod_{k=1-m}^{m} \prod_{l=1-n}^{n} \frac{1}{kb + \frac{l}{b}}, \quad (k,l) \neq (0,0),(m,n),
\]

(6.4.19)

and

\[
P^{c}_{m,n} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \prod_{p,q} \frac{\lambda_1 + \lambda_2 + pb + qb^{-1}}{2} \frac{\lambda_1 - \lambda_2 + pb + qb^{-1}}{2}
\]

(6.4.20)

with \( \lambda_i^2 = b^2 + b^{-2} + 2 - 4h_i \). The ranges of \( p \) and \( q \) in the above product are

\[
p = -m + 1, -m + 3, \ldots, m - 3, m - 1,
\]

\[
q = -n + 1, -n + 3, \ldots, n - 3, n - 1.
\]
Note that in $R_{m,n}(h_1, h_2)$, $A_{m,n}^{c,h_1}$ means that the $b$ in $A_{m,n}^c$ should be replaced by $b_{m,n}(h_1)$, and similarly for other terms in $R_{m,n}(h_1, h_2)$ and $S_{m,n}(h_1, h_2)$.

The last piece of information we need for the recursion (6.4.15) is the bulk-boundary global blocks

$$G(h_1, h_2) \equiv \mathcal{V}_{\text{holo}}(h_1, h_2, c \to \infty).$$

(6.4.21)

In the limit that $c \to \infty$, all the Virasoro generators will be suppressed, therefore in the projection operator $P_{h_1}$ and the holomorphic proto-field $\phi_{h_2}^{\text{holo}}$, all that left are the global descendants. Thus we have

$$G(h_1, h_2) = \sum_{m_1, m_2 = 0}^{\infty} y^{2h_2 + 2m_2} \frac{\langle O_H L_{-1}^{m_1} | h_1 \rangle \langle h_1 | L_{-1}^{m_1} O_L(z) L_{-1}^{m_2} | h_2 \rangle}{|L_{-1}^{m_1} | h_1 \rangle|L_{-1}^{m_2} | h_2 \rangle|^2}. $$

(6.4.22)

The details for computing $G(h_1, h_2)$ is provided in Appendix E.3, and the result is given by

$$G(h_1, h_2) = z^h_1 \left( \frac{y^2}{z} \right)^{h_2} \sum_{m_1, m_2 = 0}^{\infty} \frac{(h_1)_{m_1} s_{m_1, m_2} (h_1, h_L, h_2)}{(2h_1)_{m_1} m_1! (2h_2)_{m_2} m_2!} \left( \frac{y^2}{z} \right)^{m_2} \right)^{m_2}$$

(6.4.23)

with [180]

$$s_{k,m}(h_1, h_2, h_3) \equiv \langle h_1 | L_1^{m_1} O_{h_2} (1) L_{-1}^{m_2} | h_3 \rangle$$

(6.4.24)

$$= \sum_{p=0}^{\min(k,m)} \frac{k!}{p! (k-p)!} (2h_3 + m - p)_p (m - p + 1)_p$$

$$\times (h_3 + h_2 - h_1)_{m-p} (h_1 + h_2 - h_3 + p - m)_{k-p}.$$ 

Solving the recursion (6.4.15) will give $\mathcal{V}_{\text{holo}}(h_1, h_2, c)$ as a sum over global blocks

$$\mathcal{V}_{\text{holo}}(h_1, h_2, c) = \sum_{m,n=0}^{\infty} C_{m,n} G(h_1 + m, h_2 + n).$$

(6.4.25)

The global block $G(h_1 + m, h_2 + n)$ is the contribution to $\mathcal{V}_{\text{holo}}$ from a level-$m$ quasi-primary
in $\mathcal{P}_h$ and a level-$n$ quasi-primary in $\phi$. The coefficients $C_{m,n}$ are functions of the operators dimensions and the central charge $c$. As shown in equation (E.3.12), they are related to three point functions of primaries with one or two quasi-primaries and the norms of the quasi-primaries. Specifically, $C_{m,n}G(h_1 + m, h_2 + n)$ computes the total contribution to $V_{h_1}$ from all the level-$m$ quasi-primaries in $\mathcal{P}_h$ and level-$n$ quasi-primaries in $\phi$. One way of understanding the recursion (6.4.15) is that it provides an efficient way of computing these coefficients. More details about the recursion relation and the algorithm for implementing it in Mathematica can be found in Appendix E.3.

After obtaining $V_{h_1}(0, h, c)$, we can use the method discussed in last subsection to compute $\tilde{V}_0$. Concretely, the first several terms of $\tilde{V}_0$ are given by

$$\tilde{V}_0 = 1 + \frac{2(3x + 1)h_L}{c(x + 1)} (z^2 + \bar{z}^2)$$

$$+ \frac{4h_L^2 (5x - 2) + (1 + 2x - 3x^2) h_L + (17x^2 + 12x + 2) h_L^2 + 12x^2 h_L^3 - 4x^2 h_L^4)}{c^2(x + 1)^2 (2h_L + 1)} z^2 \bar{z}^2 + \ldots$$

We’ve checked that all the three methods discussed in this section for computing $V_0$ give the same result, which also agrees with the large $c$ expansion of $V_0$ (Appendix E.2) and the semiclassical result $V_0^{\text{semi}}$ (Section 6.3.1) in the appropriate limits.

In next section, we will compare the result from the recursion with the semiclassical result. For clarity, we will convert all results to the usual BTZ coordinates $(r, t_E, \theta)$, where the semiclassical result is given by

$$V_0^{\text{semi}}(r, t_E, \theta) = \left(\frac{r_+}{2}\right)^{2h_L} \frac{1}{\left[\frac{r}{r_+} \cosh (r_+ \theta) - \sqrt{\frac{r^2}{r_+^2} - 1 \cos (r_+ t_E)}\right]^{2h_L}}.$$ (6.4.27)

As discussed in section 6.2.2 and appendix E.1.2, the right object to compare with $V_0^{\text{semi}}$ is the following

$$V_0^{\text{exact}}(r, t_E, \theta) \equiv (1 - z)^{h_L} (1 - \bar{z})^{h_L} \left(\frac{y}{y^2 + z\bar{z}}\right)^{2h_L} \tilde{V}_0.$$ (6.4.28)
with the coordinate transformation from \((y, z, \bar{z})\) to \((r, t_E, \theta)\) via (6.2.15) and \(\tilde{\mathcal{V}}_0\) as given in (6.4.26). For better visibility of the plots, we will actually divide both \(\mathcal{V}_0^{\text{semi}}\) and \(\mathcal{V}_0^{\text{exact}}\) by \(y^{2h_L}\) (which is not singular in the region we are interested in).

### 6.5 Exploring the Euclidean Horizon

Now we will explore the behavior of the correlator when the bulk operator \(\phi\) approaches the Euclidean horizon\(^{17}\) of a black hole microstate. For simplicity we study spherically symmetric black holes with \(h_H = \bar{h}_H\), and since \(\phi\) is a scalar we have \(h_L = \bar{h}_L\). Our plots always indicate bulk-boundary correlators with no angular separation, so that the correlators depend only on \((r, t_E)\).

The Euclidean horizon is the region where \(r \gtrsim r_+\) with purely Euclidean BTZ time coordinate \(t_E\). We have reason to expect a sharp, order-one deviation between the semiclassical and exact correlators in this region. As one can see from figure 6.3, the classical BTZ geometry and the semiclassical correlators are periodic in Euclidean time. But exact CFT correlators in a pure state (or even in the microcanonical ensemble) cannot be periodic [141, 12]. As illustrated in figure 6.1, the exact CFT correlators must lift to multivalued functions on the ‘cigar’ geometry. This suggests that the correlators will be badly behaved at the Euclidean horizon where the \(t_E\) circle shrinks to zero size. We will confirm this expectation with an explicit numerical computation using the exact correlators. We will also see that the region where the exact and semiclassical correlators differ shrinks as we increase \(c\).

Near the Euclidean horizon, the corresponding Fefferman-Graham coordinates \(z, \bar{z}\) remain in the Euclidean region with \(\bar{z} = z^*\), and thus the correlator can be best approximated using the algorithm of section 7.4. With it we can compute the correlator to order \(z^{60}\bar{z}^{60}\) with coefficients that capture the exact dependence on \(h_H, h_L, c\) and the kinematic \(y\)-coordinate.

\(^{17}\)The bulk field operator \(\phi(y, z, \bar{z})\) was defined in terms of a local CFT\(_2\) primary and its descendants via the bulk primary conditions of section 6.2.2. So when we discuss the ‘horizon’, we are referring to certain values of the \((y, z, \bar{z})\) coordinate labels determined mathematically in terms of the BTZ black hole coordinates \((t, r, \theta)\) through equation (6.2.12). Bulk interpretations of these labels are emergent.
Figure 6.3: **Left:** This figure depicts a Euclidean bulk-boundary correlator $|\mathcal{V}_0^{\text{semi}}|$ on the BTZ `cigar' geometry, focusing on slices at fixed $r$, where we can easily study Euclidean time periodicity. **Right:** These plots display the semiclassical bulk-boundary correlator $\mathcal{V}_0^{\text{semi}}$ on constant-$r$ slices. The semiclassical correlator is periodic in $t_E$, and its range of variation becomes smaller as we approach the horizon $r = r_+$, where it is constant in $t_E$. The red dashed line is $t_E = \beta$ and the parameters are $\frac{h_H}{c} = 1, h_L = 1$.

For clarity, we will convert all results into the usual BTZ coordinates $(r, t_E, \theta)$ as discussed at the end of section 7.4.

To any finite order in $y$, these results should converge for all $|z| < 1$. However, since we are only computing to finite order in the $z$ expansion, the radius of convergence will be smaller, and must be estimated empirically based on the growth of terms in the series expansion. We find that the recursion relations of section 7.4 converge best when $24h_H/c \gg 1$, $h_L \ll 1$, and $c > 1$ is relatively small. For the most part we will focus on this regime, as our goal is to compare the exact and semiclassical correlators as precisely as possible. Note that in this regime there are two relevant length scales in the bulk, the AdS scale $R_{\text{AdS}} = 1$ in our conventions, and the larger horizon scale $r_+ = \sqrt{\frac{24h_H}{c}} - 1 \gg 1$. Typically with our chosen parameters $r_+ \sim 10 - 100$. As explained in section 6.3.3.3, the exact results are not exactly symmetric under $t_E \rightarrow -t_E$ in the BTZ coordinates.

We compared the exact and semiclassical results for small $t_E$ and large $r$ in Figure 6.2 and we found excellent agreement. Now let us investigate $r \approx r_+$, larger $t_E$, and small $c$. In figure 6.4 we have compared the exact and semiclassical correlators as functions of
the Euclidean time $t_E$ for various fixed values of the radius $r$. We see that the exact and semiclassical correlators are very similar for $t_E < \beta$ when $r \gg r_+$, though the correlators deviate significantly for $t_E \approx \beta$, as expected based on the boundary behavior [2]. But as we approach the horizon, the correlators disagree for a greater and greater range of $t_E$ values, such that for $r \approx r_+$ the exact and semiclassical correlators are significantly different for all $t_E$.

We compare the exact and semiclassical correlators on the full Euclidean ‘cigar’ geometry in figures 6.5, 6.6, and 6.7. These plots indicate the full dependence on $r$ and $t_E$, and give some idea of the way the results change with $c$. However the ‘migration’ of the discrepancy from $t_E \approx \beta$ to the full range of $t_E$ is easier to see in figure 6.4. Notice that the semiclassical correlator $V_0^\text{semi}$ in figure 6.5 is smooth everywhere, even at the Euclidean black hole horizon at the center of the disk. This is not the case for the exact correlator $V_0^\text{exact}$ in figure 6.6. The exact correlator also has a discontinuity between $t_E = 0$ and $t_E = \beta$ (the wiggling lines in the plots), which indicates that $V_0^\text{exact}$ is not a periodic function in $t_E$, in contrast to $V_0^\text{semi}$.

**Minimizing Violations of Bulk Effective Field Theory**

Since these results are somewhat preliminary, we would like to interpret them as conservatively as possible. So its natural to ask how to minimize the discrepancy between a naive bulk effective field theory description – i.e. the semiclassical correlator – and the exact correlator.

The discrepancy between the exact and semiclassical correlators becomes unavoidable once we approach $t_E - \beta \sim O\left(\frac{1}{\sqrt{c}}\right)$. And for $|t_E|$ larger than $\beta$ the semiclassical description completely fails. We have now seen that this applies both on the boundary and in the bulk. This unsuppressed effect is due to non-perturbative corrections in the large $c$ limit, though surprisingly, there are already hints of this phenomenon in $1/c$ perturbation theory [20, 12].

However, one can brush this problem under the rug by defining the correlator on the Euclidean cigar using the exact correlator evaluated in the range $t_E \in \left[-\frac{\beta}{2}, \frac{\beta}{2}\right]$. On the boundary, the disagreement between the exact and perturbative correlators will be extremely

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Figure 6.4: The blue lines are the exact result $|\psi_0^{\text{exact}}|$ and the yellow lines are the semiclassical $|\psi_0^{\text{semi}}|$. From top to bottom the rows of plots correspond to $c = 8.1, 16.1, 32.1, 64.1$, respectively. Other parameters for these plots are $\hbar_L = 0.01, \hbar c = 100$, and $r_+ \approx 50$. The first two plots in each row are in the region whose distance from the horizon is much smaller than the AdS radius. The red dashed line is $t_E = \beta$ and the gray dashed line is $t_E = \pm \beta/2$. The exact results in the visible plot range have converged to better than $10^{-13}$ precision (the precision of convergence is defined as in figure 6.2).
Figure 6.5: This is a plot of $|\mathcal{V}_{0,\text{semi}}|$ zoomed in to the tip of the Euclidean ‘cigar’, with $r_+ < r < 1.025r_+$ and $0 < t_E < \beta$. The radial coordinate of the disk is $r - r_+$ and the angular direction is $\frac{2\pi}{\beta}t_E$; the BTZ angular coordinate $\theta = 0$. The center of the plot is the position of the Euclidean horizon and $r_+ \approx 49$. Notice that $|\mathcal{V}_{0,\text{semi}}|$ is smooth everywhere in this plot, in contrast to $|\mathcal{V}_{0,\text{exact}}|$ in figure 6.6 below.

Figure 6.6: These are plots of $|\mathcal{V}_{0,\text{exact}}|$ for $h_L = 0.01$, $h_H/c = 100$ but with different values of $c$. These are plotted in the same region and use the same range as figure 6.5 for ease of comparison. These results have converged to better than $10^{-10}$ accuracy except for a tiny region at the origin of the disk (i.e. the white point at the center).
Figure 6.7: These plots show the difference between the exact and semiclassical results: 
\[ \left| \frac{V_{\text{exact}}^0 - V_{\text{semi}}^0}{V_{\text{semi}}^0} \right| \] in the same region as figure 6.5 and 6.6. They have the same parameters as figure 6.6: \( h_L = 0.01, h_H/c = 100 \). The difference between exact result and semiclassical result is numerically small because we’ve chosen very small \( h_L = \frac{1}{100} \) for better convergence, and this means that both the exact result and the semiclassical result are very close to 1. Qualitatively, we can see that as we increase \( c \), the agreement between the exact result and the semiclassical result improves. The exact results have converged to better than \( 10^{-10} \) accuracy.

(nonde-perturbatively) small for this range of \( t_E \). This fact has been discussed previously [2], as analytic continuation in \( t_E \) to \( \frac{\beta}{2} \) can also be used to mimic the correlators in the double-sided eternal black hole geometry.

Even if the bulk correlators can be smoothly connected between \( t_E = \pm \frac{\beta}{2} \) up to \( \sim e^{-c} \) corrections, then at exponentially small values of \( |r - r_+| \) we might nevertheless see a large deviation from naive effective field theory predictions. This follows because the derivative of the correlator will grow as \( \frac{1}{r - r_+} \), and so eventually even a tiny effect may become significant. But this will only occur at a distance exponentially close to the horizon, and so it’s unclear if it would affect observers.

We also see indications in figure 6.4 that the exact and semiclassical correlators disagree for a greater range of \( t_E \) as \( r \rightarrow r_+ \). We have confirmed this phenomena for some other choices of parameters. Unfortunately, due to the limitations of numerical convergence we do not have the dynamic range to determine if this effect is perturbative or non-perturbative in nature, or to work out its empirical dependence on \( h_L, r_+, \) and \( c \). Hopefully some of
these issues can be clarified through a more detailed analysis, or by directly studying the Lorentzian regime in future work.

6.6 Discussion

The primary purpose of this paper was to develop methods for computing the gravitational contributions to the bulk-boundary propagator in a black hole microstate at finite $G_N$. In CFT\textsubscript{2} terminology, we studied the conformal block decomposition of a 4-pt correlator involving three CFT primaries and a single bulk proto-field $\phi$, which has been defined as a specific infinite sum of Virasoro descendants [3] parameterized by the bulk coordinate $y$. We explored the semiclassical limit of these correlators, and demonstrated that they reduce to known results in the probe or heavy-light limit.

It would be interesting to better understand the convergence of the bulk reconstruction algorithm and of the associated conformal blocks. It would be especially useful to develop an analog of the $q$ variable [36, 8] that can achieve a maximum radius of convergence for these objects. To reach the interior of a microstate black hole, it seems that one must analytically continue through a bulk-boundary light-cone OPE [85, 86], as the bulk field must cross the past lightcone of $O_H$ in the bulk. The $q$ variable allows analytic continuation through infinitely many boundary light-cone limits, so an analog in the bulk might clarify the definition of correlators in the black hole interior.

We performed a preliminary comparison of exact vs semiclassical Euclidean bulk-boundary correlators. Our goal was to understand the bulk implications of the fact that correlators in black hole microstate backgrounds violate the Euclidean periodicity manifest in the classical black hole geometry. The result was that Euclidean bulk correlators deviate from their semiclassical limit in a way that appears to be unsuppressed at $t_E \gtrsim \beta$. The effect appears increasingly impactful as $r \to r_+$, since the naive Euclidean-time circle contracts to zero size. We also found evidence that the effect spreads to a much greater range of $t_E$ as one
approaches very near to the horizon.

The most conservative interpretation still allows for an exponentially suppressed deviation for physical observables. Furthermore, violations of Euclidean-time periodicity in perturbative CFT$_2$ computations [20] are an important case where even for boundary correlators, the distinction between effects that are and are not visible in bulk effective field theory remains to be understood. Note that even if our results have implications for ‘drama’ at the horizon [137, 104], they would not immediately apply to eternal black holes or the canonical ensemble, which satisfy the KMS condition exactly.

Do physical observers see violations of bulk effective field theory outside the horizon, and are there relatively unambiguous predictions for what observers might see inside a black hole? To address these questions, we must investigate the behavior of the Lorentzian correlators pertaining to physical observers. It will also be important to differentiate between corrections to CFT correlators and qualitatively new effects due to the bulk reconstruction process itself. Non-perturbative corrections to reconstruction can dramatically alter the bulk equations of motion and invalidate bulk locality [4]; it is the investigation of such effects in black hole backgrounds that necessitates exact bulk reconstruction.
Chapter 7

AdS$_3$ Reconstruction with General Gravitational Dressings

This chapter is based on the following paper:


Abstract

The gauge redundancy of quantum gravity makes the definition of local operators ambiguous, as they depend on the choice of gauge or on a ‘gravitational dressing’ analogous to a choice of Wilson line attachments. Recent work identified exact AdS$_3$ proto-fields by fixing to a Fefferman-Graham gauge. Here we extend that work and define proto-fields with general gravitational dressing. We first study bulk fields charged under a $U(1)$ Chern-Simons gauge theory as an illustrative warm-up, and then generalize the results to gravity. As an application, we compute a gravitational loop correction to the bulk-boundary correlator in the background of a black hole microstate, and then verify this calculation using a newly adapted recursion relation. Branch points at the Euclidean horizon are present in the $1/c$
corrections to semiclassical correlators.

7.1 Introduction and Summary

A complete description of AdS/CFT requires an exact prescription for bulk reconstruction, which would ideally provide a quantitative guide to its own limitations. This problem may decomposed into two (overlapping) sub-problems:

- Reconstruction of interacting bulk fields from dual boundary ‘CFT’ operators in the absence of AdS gravity. It’s easy to solve this problem for free bulk fields and generalized free theory (GFT) duals, and it can also be addressed order-by-order in bulk perturbation theory [113, 114, 115, 121]. This problem is very similar [127] to the question of how to relate the operators in a CFT which ends at a boundary to the BCFT operators living on that boundary.

- Bulk reconstruction in the presence of gravity. This problem is qualitatively different, because we do not expect local bulk operators to be uniquely defined – they must be associated with a ‘gravitational Wilson line’ or ‘gravitational dressing’. These complications arise because of the gauge redundancy of bulk diffeomorphisms and the universality of the gravitational force.

Both gravitational and non-gravitational interactions seem to require bulk field operators $\Phi(X)$ to include mixtures of infinitely many CFT operators.

In AdS$_3$ the purely gravitational component of bulk reconstruction can be more precisely specified by taking advantage of the relation between bulk gravity and Virasoro symmetry. This makes it possible to solve one aspect of reconstruction exactly. Prior work [3] defined bulk operators by first fixing to Fefferman-Graham gauge, thereby assuming a specific and arbitrarily chosen gravitational dressing. The purpose of this paper is to define bulk proto-fields with much more general gravitational dressings, or equivalently, by defining the bulk
field in a more general gauge.

**Bulk Operators from Symmetry**

The AdS/CFT dictionary specifies that

$$\lim_{y \to 0} y^{-2h} \Phi(y, z, \bar{z}) = O(z, \bar{z})$$

(7.1.1)

for a bulk scalar field \(\Phi(Y)\) and a dual boundary CFT primary \(O\). However, at finite \(y\) the bulk field operator \(\Phi(Y)\) will include an infinite sum of contributions from other primaries. Ultra-schematically, we may write \([113]\)

$$\Phi = O + \sqrt{G_N}[T O] + G_N[T T O] + \cdots$$

$$+ g[O_i O_j] + g^2[O_i O_j O_k] + \cdots$$

(7.1.2)

to indicate that \(\Phi\) includes a mixture of multi-trace operators made from the stress-tensor \(T\) and \(O\), as well as multi-trace operators made from other primaries \(O_i\), with perturbative coefficients that can be computed \([113, 114, 115, 121]\) when such a description applies.

We will be studying the terms in \(\Phi\) involving \(O\) and any number of stress tensors, as these are determined by the Virasoro symmetry\(^1\) in AdS\(_3\). Just as conformal symmetry dictates that CFT correlators must be decomposable as a sum of conformal blocks, bulk scalar fields \(\Phi\) can be written as a sum of bulk proto-field operators \(\phi\) that are fixed by symmetry.

The proto-fields \(\phi\) also have another interpretation, as sources or sinks for one-particle states in a first-quantized worldline action description. Correlators of protofields \(\langle \phi(X_1) \phi(X_2) \cdots \rangle\) with other CFT operators will match to all orders in perturbation theory with the propagation of a particle from \(X_1\) to \(X_2\) in the gravitational background created by these other CFT

\(^1\)There has been much recent work on AdS\(_3\) reconstruction \([3, 164, 122, 4, 128, 120, 129, 181, 182, 5]\). Our results \([3]\) differ from the proposal \([126]\), which produces a field that does not seem to satisfy the interacting bulk equation of motion in a known gauge when expanded perturbatively in \(1/c\) (e.g., compare \(\langle \phi OT \rangle\) correlators to the results of appendix D.4 of \([3]\)).
operators, including the effect of gravitational loops on the propagation. But the proto-field correlators do not include non-gravitational interactions, or mixings with multi-trace operators induced by gravity.

‘Dressings’ and Correlators with Symmetry Currents

Charged operators in gauge theories and local bulk operators in quantum gravity are not gauge-invariant. This means that their definition is ambiguous, and we need to supply more information to fully specify them. This additional information may be a Wilson line, a specific choice of gauge, or a ‘gravitational dressing’ (by this term we roughly mean ‘gravitational Wilson line’). We discuss the relation between these ideas in section 7.2.1.

The necessity and ambiguity of these dressings has a simple interpretation in the CFT.
If we are to write a bulk proto-field $\phi(X)$ as a CFT operator, then the charge and energy in $\phi$ must be visible to the charge $Q$ and spacetime symmetries $D, P_\mu, K_\nu, M_{\mu\nu}$ in the CFT. These quantities can be computed by integrals of $J_\mu(x)$ or $T_{\mu\nu}(x)$ over Cauchy surfaces on the boundary [46], but the specific spacetime distribution of current and energy-momentum associated with $\phi(X)$ is somewhat arbitrary. This explains the ambiguity in $\phi(X)$, and also suggests how it can be fixed – the gauge and gravitational dressings are specified by the form of $\phi(X)$ correlators with $J_\mu(x)$ or $T_{\mu\nu}(x)$.

To make sense of this logic, it must be possible to distinguish the energy-momentum in $\phi(X)$ from that of other sources in any state or correlator. We accomplish this by assuming that $\phi(X)$ is surrounded by vacuum, so that we can define $\phi(X)$ in a series expansion\(^2\) in the bulk coordinate [127], with local CFT operators as coefficients. In this way we can use radial quantization to define $\phi(X)$.

We will specify general gravitational dressings in two equivalent ways. In section 7.3.2 we use a trick: starting with the proto-field defined through Fefferman-Graham [3], we use a diffeomorphism to bend the gravitational dressing. Whereas in section 7.3.4 we take a more abstract route, and simply construct an operator $\phi(X; x_0)$ at a point $X$ in the bulk, but where $T(x)$ on the boundary detects $\phi$’s associated stress-energy at a general point $x_0$.

**Summary of Results**

Our main result is a simple formula for a proto-field with general dressing

$$
\phi(u, x, \bar{x}; x_0, \bar{x}_0) = \sum_{n=0}^{\infty} \sum_{m, \bar{m}=0}^{\infty} \frac{(-1)^n u^{2h+2n}}{n!(2h)_n} \frac{(x-x_0)^m (\bar{x}-\bar{x}_0)^{\bar{m}}}{m!\bar{m}!} L_{-n-m} \bar{L}_{-n-m} O(x_0, \bar{x}_0)
$$

(7.1.3)

The interpretation of this operator is discussed in section 7.3, but roughly speaking, the proto-field is located $(u, x, \bar{x})$ in $\text{AdS}_3$, with its associated energy-momentum localized at

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\(^2\)Another common approach [117] defines bulk fields by integrating local CFT operators over a region. This procedure may have equivalent issues when other local CFT operators are present in the region of integration and OPE singularities are encountered.
On the boundary. The $L_{-N}$ are polynomials in the Virasoro generators determined by the bulk primary condition [3], with coefficients that are rational functions in the central charge $c$ and holomorphic dimension $h$ of $\mathcal{O}$. We verify that this result has the expected correlators with stress tensors $T(x)$. Our formula can be integrated against a positive, normalized distribution $\rho$ via

$$
\phi[\rho](X) \equiv \int d^2x_0 \rho(x_0, \bar{x}_0)\phi(X; x_0, \bar{x}_0)
$$

(7.1.4)

to obtain a very general gravitational dressing for the proto-field.

We also show in section 7.4 that correlators of $\phi(X; x_0, \bar{x}_0)$ can be computed by a further adaptation [4, 5] of Zamolodchikov’s recursion relations [37, 36]. Then in section 7.5 we analytically calculate the $1/c$ correction to the heavy-light, bulk-boundary propagator on the cylinder using a recent quantization [168] of AdS$_3$ gravity. We demonstrate that our analytic result matches that of the recursion relation. We also observe that as expected [20, 12], the analytic $1/c$ correction to the correlator is not periodic in Euclidean time [5], and so it has a branch cut singularity at the Euclidean horizon. This is surprising from the point of view of perturbation theory in a fixed black hole background.

The outline of the paper is as follows. In section 7.2 we provide a detailed discussion of bulk reconstruction for fields charged under a $U(1)$ Chern-Simons field. This serves as a warm-up where many of the ideas can be more straightforwardly illustrated. Then in section 7.3 we turn to gravity, where many of our results are analogous to the simpler $U(1)$ setting. In section 7.4 we adapt a recursion relation to compute correlators of $\phi$ with general dressing. In section 7.5 we explain some rather technical calculations, including the recursion relation in a specific configuration and an analytic computation of the one-loop gravitational correction (i.e., order $1/c$) to a $\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \phi_L \rangle$ correlator. We provide a brief discussion in

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3It’s not entirely clear what operators the full space of gravitational dressings should include, but by letting $\rho$ depend on $X$ we can parameterize a large space of possibilities. An average over $x_0$ is not equivalent to averaging over different exponents in a Wilson line [183, 155], since the average of an exponential is not the exponential of an average. Wilson lines should be path-ordered, so averaging over complete Wilson lines should be the more generally valid approach.
section 7.6. Many technical results are relegated to the appendices.

7.2 Bulk Proto-Fields with $U(1)$ Chern-Simons Charge

This section will serve as a warm-up in preparation for our eventual discussion of bulk gravity, where most of these ingredients will have a direct analog.

7.2.1 Charged Fields, Wilson Lines, and Gauge Fixing

Consider a bulk field $\varphi(X)$ charged under a $U(1)$ gauge symmetry. It transforms as

$$\varphi(X) \rightarrow e^{iq\Lambda(X)} \varphi(X)$$

under the gauge redundancy, so it cannot be regarded as a physical observable. We can remedy this problem in two equivalent ways – by fixing the gauge, or by attaching $\varphi(X)$ to a Wilson line.

The latter approach has the clear advantage that it makes the gauge-invariant nature of our observable manifest. Given a Wilson line

$$W_C(\infty, X) = e^{iq \int_C dx^\mu A_\mu}$$

running from $X$ to infinity, we can form a non-local operator

$$\phi(X) = W_C(\infty, X) \varphi(X)$$

Since gauge transformations do not act at infinity, $\phi$ will be a gauge-invariant observable. However, this means that $\varphi(X)$ itself was highly ambiguous, since $\phi$ now depends on the path of the Wilson line. Note that once we define a gauge-invariant $\phi$ in this way, we can compute observables involving it in any convenient gauge, and we will obtain the same results.
The other (fixing the gauge) approach will be easier to discuss when we generalize to quantum gravity. However, it’s less flexible and can lead to confusing terminology. In this approach we simply fix a gauge, for example by setting some component of the gauge field \( A_y = 0 \), and then compute observables involving \( \phi(X) \) in this gauge. The results will then be well-defined observables. Note that if \( A_y = 0 \) then the Wilson line in the \( \hat{y} \) direction \( W_{\hat{y}} = 1 \) identically, so in this case the underlying gauge invariant observable will be \( \phi = W_{\hat{y}} \phi(X) \). But in general it may not be clear how to compute with our observable in other gauges. And it may seem confusing to refer to an observable defined in a specific gauge as gauge-invariant (though this is in fact true).

Let us develop these ideas in the context of a scalar field \( \phi \) in AdS\(_3\) charged under a \( U(1) \) Chern-Simons theory with level \( k \). The scalar will be dual to a CFT\(_2\) primary operator \( \mathcal{O} \) with conformal dimension \( h \) and charge \( q \), and the gauge field to a holomorphic conserved current \( J(z) \). We will work in Euclidean space with a fixed metric

\[
ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2} \tag{7.2.4}
\]

and in this section we will not include dynamical gravity.

We will be viewing \( \phi(y, z, \bar{z}) \) through the lens of radial quantization, as discussed previously in [3] and pictured in figure 7.1 (the figure denotes the gravitational case, but with \( T \to J \) it also applies to the present discussion). In the CFT \( \phi \) will be a non-local operator, but only because it will be written as an infinite sum of local operators, each a coefficient in the near-boundary or small \( y \) expansion of \( \phi \). If we turn off both gravity and the Chern-Simons interaction, then \( \phi \) is determined by symmetry to be [108]

\[
\phi_0(y, z, \bar{z}) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2h+2n}}{n!(2h)_n} L^n_{-1} \bar{L}^n_{-1} \mathcal{O}(z, \bar{z}) \tag{7.2.5}
\]
This follows from the form of the vacuum bulk-boundary propagator

$$\langle \phi_0(y,z,\bar{z})O^\dagger(w,\bar{w})\rangle = \left(\frac{y}{y^2 + (z-w)(\bar{z}-\bar{w})}\right)^{2h}$$  \hspace{1cm} (7.2.6)

if we expand in $y$ and identify the coefficients with global descendants of $O$.

Now let us define a gauge-invariant charged scalar field. As discussed above, we can do this by simply attaching $\varphi$ to a Wilson line that ends on the boundary at $y = 0$. A very simple choice takes the Wilson line to run in the $\hat{y}$ direction, so that

$$\phi(y,z,\bar{z}) = e^{iq \int_0^y dy' A_y(y',z,\bar{z})} \varphi(y,z,\bar{z})$$  \hspace{1cm} (7.2.7)

This operator also has a very simple definition via gauge fixing – it is simply $\phi$ defined in the gauge $A_y = 0$. This makes it clear that the correlator $\langle \phi(y,0,0)O^\dagger(w,\bar{w})\rangle$ should be equal to the expression (7.2.6).

Correlators involving the boundary current will be non-trivial. Computing in perturbation theory gives

$$\langle J(z_1)O^\dagger(w,\bar{w})\phi(y,0,0)\rangle = \frac{qw}{z_1(z_1-w)} \left(\frac{y}{y^2 + w} \right)^{2h}$$  \hspace{1cm} (7.2.8)

where $O^\dagger$ has charge $-q$. This reduces to $y^{2h}\langle JO^\dagger O\rangle$ in the limit of small $y$, as expected based on the AdS/CFT dictionary.

If we attach $\phi$ to the boundary with a more general Wilson line, then we will have

$$\phi(y,z,\bar{z};z_0,\bar{z}) = e^{iq \int_{z_0}^{X} dY^\mu A_\mu(Y)} \phi(y,z,\bar{z})$$  \hspace{1cm} (7.2.9)

The Wilson line takes some general path from $z_0$ on the boundary to the location of $\phi$ in the bulk, yet our notation does not include information about the path. Since Chern-Simons theory is topological, our $\phi$ will only depend on this path if other charges or Wilson lines
entangle with it. However, we are invoking radial quantization to define $\phi$, meaning that we will be assuming that there aren’t any matter fields or Wilson lines near $\phi$, or between it and the boundary. Thus our results for the operator $\phi$ will be independent of the choice of path, except through the location of $z_0$.

Correlators of this more general bulk field can still be computed in $A_y = 0$ gauge. Since the vacuum equations of motion set $F_{y\mu} = 0$, in this gauge $A_z$ is independent of $y$. Since on the boundary $A_z(0, z, \bar{z}) = \frac{1}{k} J(z)$, this identification must hold for all $y$, so

$$\phi(y, z, \bar{z}; z_0, \bar{z}) = e^{\frac{i q}{k} \int_{z_0}^{z} J(z') dz'} \phi(y, z, \bar{z})\Big|_{A_y = 0}$$

(7.2.10)

This formula requires some regularization to remain consistent with $\mathcal{O}$ correlators and the dictionary $\mathcal{O} = \lim_{y \to 0} \left[ y^{-2h} \phi \right]$. If we expand to first order in $q$ we find

$$\langle J(z_1) \mathcal{O}^\dagger(w, \bar{w}) \phi(y, 0, 0; z_0, 0) \rangle \approx \langle J(z_1) \mathcal{O}^\dagger(w, \bar{w}) \phi(y, 0, 0) \rangle + i \frac{q}{k} \int_{z_0}^0 \langle J(z_1) \mathcal{O}^\dagger(w, \bar{w}) J(z') \phi(y, 0, 0) \rangle$$

$$\approx \frac{q(z_0 - z)}{(z_1 - z_0)(z_1 - z)} \langle \mathcal{O}^\dagger \phi \rangle$$

(7.2.11)

This formula has a nice interpretation, as the singularities in $z_1$ indicate the presence of charge $\pm q$ at $z_0$ and $z$. But higher order corrections involving many $J$ will produce divergent integrals. And even the simpler correlator $\langle \mathcal{O}^\dagger \phi \rangle$ also requires regularization.

In the next sections we will see how to avoid regularization by defining $\phi$ using symmetry when its Wilson line attachments are simple. Then we will extend our results to include general Wilson lines by leveraging the singularity structure of $\phi$ correlators.

### 7.2.2 A Bulk Primary Condition from Symmetry

We can take another approach, and constrain the bulk field $\phi$ using symmetry. If we can determine how to extend CFT symmetries into the bulk, then we can use their action on a charged bulk field to determine how to write it as a sum of CFT operators. This approach

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4We could also just choose to have the Wilson line run along the boundary from $z_0$ to $z$. 230
will provide an exact definition, without needing to regulate Wilson lines, and it will also generalize more directly to gravity.

The CFT current $J(z)$ can be expanded in modes

$$ J(z) = \sum_{n=-\infty}^{\infty} \frac{J_n}{z^{n+1}}. \quad (7.2.12) $$

The global conformal generators have an algebra with $J_n$ with commutation relations

$$ [L_m, L_n] = (m - n) L_{m+n}, $$
$$ [L_m, J_n] = -n J_{n+m}, \quad (7.2.13) $$
$$ [J_m, J_n] = mk\delta_{n+m,0}, $$

where the subscripts of the $L$ generators runs from $-1$ to $1$, and the subscript of the $J$ generators run from $-\infty$ to $\infty$. The current acts on local primary operators via

$$ [J_n, \mathcal{O}(z)] = qz^n \mathcal{O}(z) \quad (7.2.14) $$

which can be derived from the $J(x)\mathcal{O}(z)$ OPE. This means that a finite transformation $e^{i\delta J_n}$ will rephase $\mathcal{O}(z) \rightarrow e^{iq\delta z^n} \mathcal{O}(z)$.

Now we would like to understand how to extend these symmetries so that they act on bulk fields. This requires either a careful specification of the gauge invariant operators, or a choice of gauge. We will take the latter route and choose $A_y = 0$. We can still transform $A_z \rightarrow A_z + \partial_z \lambda(z)$ while preserving this gauge fixing condition. But this is a global (rather than gauge) symmetry transformation, since it acts non-trivially on fields at the boundary $y = 0$.

In the bulk, we expect that a charged field should transform as $\phi \rightarrow e^{i\lambda} \phi$. Since $\lambda$ cannot depend on $y$, this transforms $\phi(X) \rightarrow e^{i\lambda(z)} \phi(X)$. So in $A_y = 0$ gauge, the $J_n$ act on $\phi$ in
the same way that they act on $O$, giving

$$[J_n, \phi(y, z, \bar{z})] = q^n \phi(y, z, \bar{z})|_{A_y=0} \quad (7.2.15)$$

where we have indicated explicitly that this only holds in $A_y = 0$ gauge. This further implies a bulk primary condition

$$[J_n, \phi(y, 0, 0)] = 0 \quad \text{for} \quad n \geq 1 \quad (7.2.16)$$

for the bulk field $\phi$. This condition is the $U(1)$ Chern-Simons version of the gravitational bulk primary condition originally derived in [3]. Along with the requirement that $\phi$ has the correct bulk-boundary propagator in vacuum (7.2.6), this bulk primary condition uniquely determines $\phi(y, z, \bar{z})$ as an expansion in $y$. Furthermore, it is an exact result, and does not require a small coupling expansion.

Notice that a gauge-invariant bulk operator $\phi(y, z, \bar{z})$ attached to the boundary by a Wilson line in the $\hat{y}$-direction must transform in the same way. This simply follows from the fact that $\phi = \varphi$ identically if the latter is defined in the gauge $A_y = 0$.

We can write a formal solution to the bulk primary conditions as

$$\phi(y, z, \bar{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2h)_n} y^{2h+2n} J_{-n} \bar{L}_{-1}^n O(z, \bar{z}) \quad (7.2.17)$$

where $J_{-n} O$ is defined as an $n$th level descendant of $O$ satisfying the bulk primary condition (7.2.16). In appendix F.2.1, we solve the the bulk primary condition exactly for the first several $J_{-n}$s. As in section 3.2.2 of [3], it can be shown that $J_{-n}$ can be written formally in terms of quasi-primaries as

$$J_{-n} O = L_{-1}^n O + n! (2h)_n \sum_{j=1}^{n} \sum_{i} \frac{L_{-1}^{n-j} O_h^{(i)}}{[L_{-1}^{n-j} O_h^{(i)}]^2} \quad (7.2.18)$$
where the $O_{h+j}^{(i)}$ represent the $i$th quasi-primary at level $j$ (so they satisfy $L_1 O_{h+j}^{(i)} = 0$) and the denominator is the norm of the corresponding operator. In writing down this equation, we’ve used the fact that the quasi-primaries can be chosen to be orthogonal to each other. This will be a very useful property for some of the discussions in the following sections. As a concrete example, $J_{-1}$ is given by

$$J_{-1} O = L_{-1} O + \frac{q^2}{2h k - q^2} \left( L_{-1} - \frac{2h}{q} J_{-1} \right) O. \quad (7.2.19)$$

where the second term is a quasi-primary satisfying $L_1 \left( L_{-1} - \frac{2h}{q} J_{-1} \right) O = 0.$

In appendix (F.2.2), we solve the bulk primary condition in the large $k$ limit for the all-order $\frac{1}{k}$ terms in $J_{-n}.$ As shown in both appendix (F.2.1) and (F.2.2), in the $k \to \infty$, we have $J_{-n} = L^n_{-1} + O(1/k),$ so our expansion in $y$ reduces to that of equation (7.2.5). Note that as in [3], $\phi$ is a non-local operator in the CFT due to the infinite sum in its definition.

Since the bulk proto-field has been defined as an expansion in descendants of a local CFT primary, we will often informally discuss the ‘OPE’ of the current $J$ with $\phi.$ Because of the bulk primary condition above, the singular term in the OPE of $J(z_1)$ and $\phi(y, z, \bar{z})$ is very similar to the $J O$ OPE (which is simply $J(z_1) O(z, \bar{z}) \sim \frac{q O(z, \bar{z})}{z_1 - \bar{z}} + \cdots$):

$$J(z_1) \phi(y, z, \bar{z}) \sim \frac{q \phi(y, z, \bar{z})}{z_1 - \bar{z}} + \cdots \quad (7.2.20)$$

where we have used $[J_0, \phi] = q \phi,$ since the descendant operators in $\phi$ all have the same charge $q.$

Using these CFT definitions of the bulk charged scalar $\phi,$ we can compute various correlation functions, such as $\left< \phi O^{\dagger} J \cdots J \right>$ and $\left< \phi^{\dagger} \phi \right>.$ We can first verify that $\phi$ given in (7.2.17) indeed gives the correct bulk-boundary propagator. Using (7.2.18), one can see $\left< \phi O^{\dagger} \right>$ is given by

$$\left< \phi(y, z, \bar{z}) O^{\dagger}(w, \bar{w}) \right> = \left( \frac{y}{y^2 + (z - w)(\bar{z} - \bar{w})} \right)^{2h}. \quad (7.2.21)$$
simply because the quasi-primary terms in $J_n \mathcal{O}$ do not contribute to this two-point function and the calculation reduces to that of $\langle \phi_0 \mathcal{O}^\dagger \rangle$ with $\phi_0$ given by (7.2.5). The bulk-boundary three-point function $\langle JO^\dagger \phi \rangle$ can be computed simply using the OPE of $J \mathcal{O}^\dagger$ and $J \phi$ (equation (7.2.20)), and the result is given by

$$\langle J (z_1) \mathcal{O}^\dagger (w, \bar{w}) \phi (y, z, z) \rangle = q \left( \frac{1}{z_1 - z} - \frac{1}{z_1 - w} \right) \langle \phi (y, z, z) \mathcal{O}^\dagger (w, \bar{w}) \rangle.$$  \hspace{1cm} (7.2.22)

Correlation functions of the form $\langle \phi \mathcal{O}^\dagger \cdots J \rangle$ can then be computed recursively using OPEs used above and the $JJ$ OPE.

For the bulk two-point function $\langle \phi^\dagger \phi \rangle$, we compute up to order $1/k$ using the perturbative approximation to $\phi$ that we derived in appendix (F.2.2), which corresponds to the one photon-loop correction to the bulk propagator. The details of the calculation are given in appendix F.3.1. The result for $\langle \phi^\dagger (y_1, z_1, \bar{z}_1) \phi (y_2, \bar{z}_2, \bar{z}_2) \rangle$ is given by

$$\langle \phi^\dagger \phi \rangle = \frac{\rho^h}{1 - \rho} \left[ 1 - \frac{q^2}{k} \left( \frac{\rho \frac{\rho^2 F_1(1,2h+1;2(h+1); \rho)}{2h+1} + \frac{\rho}{2h} - \log(1 - \rho)} \right) \right] + O \left( \frac{1}{k^2} \right),$$  \hspace{1cm} (7.2.23)

where $\rho = \left( \frac{\xi}{1 + \sqrt{1 - \xi^2}} \right)^2$ with $\xi = \frac{2y_1 y_2}{y_1^2 + y_2^2 + z_1 z_2}$.

In appendix F.3 we also use the bulk Witten diagrams to compute $\langle JO^\dagger \phi \rangle$ and $\langle \phi^\dagger \phi \rangle$, and the results exactly match equations (7.2.22) and (7.2.23). This provides a non-trivial check of our definition of a charged bulk scalar field.

### 7.2.3 Singularities in $J(z)$ and the AdS Equations of Motion

The singularity structure of the $J(z)$ correlators in equation (7.2.22) follow from the bulk equations of motion. So these singularities indicate the placement of Wilson lines attaching bulk charges to the boundary, and vice versa. Let us briefly explain these statements, which are illustrated in figure 7.2.
In the $U(1)$ Chern-Simons theory, the equations of motion are

$$\epsilon^{abc} F_{bc}(X) = \frac{2\pi}{k} j^a(X) \quad (7.2.24)$$

for the bulk matter charge $j^a(X)$ and bulk field strength $F_{ab}$. In the presence of a Wilson line with components in the $\hat{y}$ direction, $j^y$ will receive a delta function contribution localized to the Wilson line. This means that $F_{z\bar{z}}$ must include a delta function. In $A_y = 0$ gauge, we can identify $A_z(y, z, \bar{z}) = \frac{1}{k} J(z)$, so

$$\partial_z J(z) = 2\pi q \delta^2(z, \bar{z}) \quad (7.2.25)$$

or equivalently

$$\oint dz J(z) = 2\pi i q \quad (7.2.26)$$

To satisfy this constraint, the current must have a simple pole $J(z) = \frac{q}{z} + \cdots$, where the ellipsis denotes less singular terms as $z \to 0$. So the singularity structure of correlators with $J(z)$ follows directly from the bulk equations of motion. Conversely, a singularity in $J(z)$ in

Figure 7.2: This figure indicates the relationship between singularities in the current $J(z)$, the bulk equations of motion, and Wilson lines. Analogous statements hold for gravity and connect singularities in $T(z)$ to the gravitational dressing.
correlators with other operators indicates the presence of charge.

This means that we can use the singularity structure of correlators with \( J(z) \) or \( T(z) \) in the case of gravity to help to define a bulk field \( \phi \) with a more general Wilson line attachment or ‘gravitational dressing’. Similar observations also hold in higher dimensions, and may be useful for bulk reconstruction more generally.

7.2.4 Charged Bulk Operators and General Wilson Lines

In section 7.2.2 we constructed a charged bulk scalar field by fixing to the gauge \( A_y = 0 \). In so doing we defined a gauge-invariant bulk field \( \phi(y, z, \bar{z}) \) connected to the boundary by a Wilson line in the \( \hat{y} \) direction. In this section, we are going to construct an exact gauge-invariant bulk field whose associated Wilson line attaches to an arbitrary point \( z_0 \) on the boundary.

There are several ways to approach the construction of this general \( \phi(y, z, \bar{z}; z_0, \bar{z}) \). The most immediate one was already discussed in section 7.2.1, namely including an explicit Wilson line. An issue with this approach is that it requires a regulator for divergences in intermediate calculations, and this makes it difficult to define \( \phi \) non-perturbatively. Another important limitation is that it’s challenging to work with Wilson lines for bulk diffeomorphisms, as would be necessary when we turn to gravity.

Instead of inserting an explicit Wilson line, we can define \( \phi \) using the singularity structure of current correlators. Correlators involving \( J(z_1) \) and the field \( \phi(y, z, \bar{z}) \) defined in \( A_y = 0 \) gauge have singularities as \( z_1 \to z \). These singularities represent the charge of the bulk \( \phi \) on the boundary, as emphasized in section 7.2.3. If a Wilson line connects \( \phi \) to the boundary at \( z_0 \), then instead we expect that correlators involving \( J(z_1)\phi \) will have singularities at \( z_0 \).

Thus we need a way to move the singularities in \( z_1 \) for all correlators involving \( J(z_1) \) and \( \phi \). In fact, we already have many of the technical tools that we need. The level \( n \) descendants \( J_{-n}\mathcal{O} \) defined in section 7.2.2 were constructed so that they would not have any additional singularities in the \( J(z_1)J_{-n}\mathcal{O} \) OPE beyond those already present in \( J(z_1)\mathcal{O} \).
We can use these $J_{-n}$ to move an operator without moving the singularities associated with its charge (we develop this idea in more detail in appendix F.1). The point is that since $J_{-n} = L_{-1}^n + \cdots$ as shown in equation (7.2.18), we have that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} J_{-n} = e^{xL_{-1}} + \cdots \tag{7.2.27}$$

where the ellipsis denotes terms proportional to powers of the charge $q$. So this expression acts like a combination of a translation and a $U(1)$ symmetry transformation. This means that the non-local operator

$$\tilde{O}(z, \bar{z}; z_0, \bar{z}) = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!} J_{-n} O(z_0, \bar{z}) \tag{7.2.28}$$

behaves like a kind of mirage. It is defined so that $J_{z_1} \tilde{O}(z, \bar{z}; z_0, \bar{z})$ only has singularities in $z_1 - z_0$, while correlators of $\tilde{O}(z, \bar{z}; z_0, \bar{z})$ with $O^\dagger(w, \bar{w})$ will instead have singularities when $w - z$ vanishes\textsuperscript{5}. Explicitly, by using the OPEs of $J O^\dagger$ and $J \tilde{O}$ and properties of $J_{-n}$, we have

$$\langle J(z_1) O^\dagger(w, \bar{w}) \tilde{O}(z, \bar{z}; z_0, \bar{z}) \rangle = \frac{q(z_0 - w)}{(z_1 - z_0)(z_1 - w)(z - w)^{2h}(\bar{z} - \bar{w})^{2h}} \tag{7.2.29}$$

so $\tilde{O}(z, \bar{z}; z_0, \bar{z})$ behaves as though it is in two places at once.

We can generalize the above idea to obtain a bulk proto-field $\phi(y_1, z, \bar{z}; z_0, \bar{z})$ at $(y, z, \bar{z})$ with a Wilson line landing at $(z_0, \bar{z})$ on the boundary. This leads to a proposal for a charged bulk field with a more general Wilson line attachment

$$\phi(y, z, \bar{z}; z_0, \bar{z}) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n \frac{y^{2h+2n}}{n!(2h)_n} \frac{(z - z_0)^m}{m!} J_{-n-m} L_{-1}^n O(z_0, \bar{z}) \tag{7.2.30}$$

In fact, this form for $\phi$ is uniquely fixed by demanding that:

\textsuperscript{5}Since only the $L_{-1}^n O$ terms in $J_{-n}$ contribute in $\langle \tilde{O}(z, \bar{z}; z_0, \bar{z}) O^\dagger(w, \bar{w}) \rangle$, the $J_{-n}$ in (7.2.28) become a translation operator, and one can see that we have $\langle \tilde{O}(z, \bar{z}; z_0, \bar{z}) O^\dagger(w, \bar{w}) \rangle = \langle O(z, \bar{z}) O^\dagger(w, \bar{w}) \rangle$. 

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1. $\langle \phi \mathcal{O} \rangle$ takes the vacuum form given in equation (7.2.6)

2. Correlators $\langle \mathcal{O}^\dagger (w, \bar{w}) \mathcal{J}(z_1) \cdots \mathcal{J}(z_n) \phi (y, z, \bar{z}; z_0, \bar{z}) \rangle$ only have simple poles in the $z_i$, which can only occur when $z_i \to z_0$ or $z_i \to w$.

Note that if $z = z_0$, then only the $m = 0$ terms contribute to equation (7.2.30), and $\phi$ reduces to the bulk field of equation (7.2.17). To obtain $\mathcal{O}(z, \bar{z})$ as we take $y \to 0$, we need to simultaneously send $z_0 \to z$; otherwise we obtain the non-local operator $\tilde{\mathcal{O}}(z, \bar{z}; z_0, \bar{z})$ of equation (7.2.28).

We can verify that $\phi (y, z, \bar{z}; z_0, \bar{z})$ is really the desired operator by computing $\langle \mathcal{O}^\dagger \phi \rangle$ and $\langle J \mathcal{O} \phi \rangle$ using the properties of $\mathcal{J}_{-n}$s. For the bulk-boundary propagator $\langle \mathcal{O}^\dagger (w, \bar{w}) \phi (y, z, \bar{z}; z_0, \bar{z}) \rangle$, since the quasi-primaries terms in $\mathcal{J}_{-n} \mathcal{O}$ (equation (7.2.18)) will not contribute, we can simply replace $\mathcal{J}_{-n} \mathcal{O}$ with $L_{n-1} \mathcal{O}$. But then the sum over $m$ becomes exactly a translation operator, and the $\phi$ in (7.2.30) becomes the free-field $\phi_0$ of (7.2.5), and we have

$$\langle \mathcal{O}^\dagger (w, \bar{w}) \phi (y, z, \bar{z}; z_0, \bar{z}) \rangle = \langle \mathcal{O}^\dagger (w, \bar{w}) \phi_0 (y, z, \bar{z}) \rangle = \left( \frac{y}{y^2 + (z - w)(\bar{z} - \bar{w})} \right)^{2h}, \quad (7.2.31)$$

as expected. For $\langle J \mathcal{O}^\dagger \phi \rangle$, we can use the OPEs $J(z) \mathcal{O}^\dagger (w, \bar{w}) \sim \frac{-q\mathcal{O}^\dagger (w, \bar{w})}{z_1 - w} + \cdots$ and $J(z) \phi (y, z, \bar{z}; z_0, \bar{z}) \sim \frac{y}{z_1 - z_0} \phi (y, z, \bar{z}; z_0, \bar{z}) + \cdots$, where the ellipses denotes non-singular terms. $\langle J \mathcal{O}^\dagger \phi \rangle$ can then be computed by only including the singular terms in both OPEs, and we get

$$\langle J(z) \mathcal{O}^\dagger (w, \bar{w}) \phi (y, z, \bar{z}; z_0, \bar{z}) \rangle = q \frac{z_0 - w}{(z_1 - z_0)(z_1 - w)} \left( \frac{y}{y^2 + (z - w)(\bar{z} - \bar{w})} \right)^{2h}. \quad (7.2.32)$$

Note that here the singularities are at $z_1 = w$ and $z_1 = z_0$, while in equation (7.2.22), the singularities were at $z_1 = w$ and $z_1 = z$.

Before concluding this section, we should emphasize that our methods are insensitive to the trajectory that the Wilson line takes from $(y, z, \bar{z})$ in the bulk to the boundary point $z_0$. This is possible because the bulk theory is topological, and we have assumed that $\phi$ is
surrounded by a region of vacuum, so that it’s possible to define \( \phi \) in radial quantization. Our gravitational \( \phi \) will be defined in the same way.

### 7.2.5 More General Bulk Operators from Sums of Wilson Lines

In prior sections we have developed a formalism for exactly defining and evaluating an operator \( \phi(X; z_0) \), where a Wilson line connects \( X \equiv (y, z, \bar{z}) \) to a point \( z_0 \) on the boundary. But there are a host of other choices for gauge-invariant bulk operators. We can explore this space by defining a new bulk operator

\[
\phi[\rho](X) \equiv \int d^2 z_0 \rho(z_0, \bar{z}_0) \phi(X; z_0, \bar{z})
\]

where \( \rho(z_0, \bar{z}_0) > 0 \) and \( \int d^2 z_0 \rho(z_0, \bar{z}_0) = 1 \). The properties of \( \phi[\rho] \) are inherited from \( \phi(X; z_0) \), which is simply the special case where \( \rho(x) = \delta(x - z_0) \). Note that if we like, we can let \( \rho \) depends on \( X \) so that \( \rho \) varies as we move \( \phi \) to different locations \( X \) in the bulk.

Operators like \( \phi[\rho] \) make it possible to study bulk fields with far more general ‘dressings’, which include superpositions of Wilson lines. For example, we might define a bulk field with spherically symmetric dressing. Let the bulk field be located at the point \( (y, z, \bar{z}) \) in Poincare patch and \( (t_E, r, \theta) \) in global coordinates\(^6\), and let the point of attachment of the Wilson line to the boundary be \( z_0 \). We will average with the measure \( \frac{d\theta}{2\pi} \) over the spatial circle on the boundary at fixed \( t_E \). In Poincare patch this means integrating with the measure \( \frac{dz_0}{2\pi i z_0} \) along the contour \( |z_0| = |z|\sqrt{1 + \frac{y^2}{z^2}} \). The integration contour ensures that \( z_0 \) is on the same time slice as the bulk field. In summary, the integration measure is

\[
\rho(z_0, \bar{z}_0) = \frac{1}{2\pi i z_0} \delta\left(|z_0| - |z|\sqrt{1 + \frac{y^2}{z^2}}\right)
\]

\(^6\)The Poincare patch metric \( ds^2 = \frac{dy^2 + dz d\bar{z}}{y^2} \) and the global AdS\(_3\) metric \( ds^2 = (r^2 + 1) dt_E^2 + \frac{dr^2}{r^2 + 1} + r^2 d\theta^2 \) are related by

\[
y = \frac{e^{t_E}}{\sqrt{r^2 + 1}}, \quad z = \frac{re^{t_E + i\theta}}{\sqrt{r^2 + 1}}, \quad \bar{z} = \frac{re^{t_E - i\theta}}{\sqrt{r^2 + 1}}.
\]
Note that $\int d^2 z_0 \rho(z_0, \bar{z}_0) = 1$. The integration of $\langle J^\dagger \phi \rangle$ (equation 7.2.32) then gives

$$
\frac{\langle J(z_1)O^\dagger(w, \bar{w})\phi[\rho](y, z, \bar{z})\rangle}{\langle \phi(y, z, \bar{z})O(w, \bar{w})\rangle} = q \left( \frac{\Theta(|z_1| - |z_0|)}{z_1} - \frac{1}{z_1 - w} \right)
$$

(7.2.36)

where $|z_0|$ denotes $|z|\sqrt{1 + \frac{y^2}{z^2}}$ and $\Theta$ is the Heaviside step function. Since $J_0 = \frac{1}{2\pi i} \oint dz_1 J(z_1)$, we can further contour integrate the above expression with respect to $z_1$ to obtain the total charge of the state. The result is given by

$$
q \left( \Theta(|z_1| - |z_0|) - \Theta(|z_1| - |w|) \right)
$$

(7.2.37)

The interpretaton of this is clear: A state created by $\phi$ or $O^\dagger$ carries charges $q$ and $-q$ respectively, and the charge content of the state at the time of insertion of $J$ depends upon the insertions that occur before that time. This result should be equivalent to Coulomb gauge [183] to lowest order in perturbation theory, but will not agree more generally, because the exponential of an average is not equal to the average of an exponential.

### 7.3 Gravitational Proto-Fields with General Dressing

There are no local gauge invariant operators in gravity. To understand this, observe that diffeomorphism gauge redundancies act on local scalar fields via

$$
\varphi(x) \rightarrow \varphi(x) + \xi^\mu(x) \nabla_\mu \varphi(x)
$$

(7.3.1)

This is similar to $U(1)$ gauge theory, where a charged field transforms as $\varphi \rightarrow \varphi + iq\Lambda(x)\varphi$, insofar as in both cases, local fields by themselves are not gauge-invariant. As we discussed in detail in section 7.2, we can form gauge invariant quasi-local charged operators in a $U(1)$ theory using Wilson line attachments.

Matters are not so simple in the case of gravity, because it is the dependence of $\varphi$ on
the bulk coordinates themselves that renders $\varphi$ gauge non-invariant. Lacking a simple and general notion of a gravitational Wilson line, we will discuss diffeomorphism gauge-invariant operators as ‘gravitationally dressed’ local operators. In this section, we consider gravitational dressings\(^7\) that associate local bulk operators with specific points on the boundary. Such dressings are natural analogs of Wilson lines joining a field in the bulk to a point on the boundary.

We will use the notation $\phi(y, z, \bar{z}; z_0, \bar{z}_0)$ to denote a diffeomorphism invariant bulk proto-field at $(y, z, \bar{z})$ in AdS$_3$ that has a gravitational line dressing landing on the boundary at the point $(z_0, \bar{z}_0)$. The simplified notation $\phi(y, z, \bar{z})$ is used when the boundary point of the gravitational dressing is at $(z, \bar{z})$. We sometimes refer to the path that the associated gravitational dressing takes, but strictly speaking our results are not associated with any particular path. In our formalism all paths are equivalent because we assume that the bulk field can be connected to the boundary through the bulk vacuum.

Our typical setup is pictured in figure 7.1. Some of our analysis will be analogous to the simpler and conceptually clearer $U(1)$ Chern-Simons case discussed in section 7.2. However, we will provide more details about the relationship between diffeomorphisms and gravitational dressing, and explain how some simple correlation functions can be computed.

### 7.3.1 Review of Protofields in Fefferman-Graham Gauge

In recent work [3, 4, 5], an exact gravitational proto-field was defined with a very specific gravitational dressing. The dressing was determined implicitly, by fixing to a Fefferman-Graham or Banados gauge [131] where the metric takes the form

\[
d\hat{s}^2 = \frac{dy^2 + dzd\bar{z}}{y^2} - \frac{6T(z)}{c}dz^2 - \frac{6\bar{T}(\bar{z})}{c}d\bar{z}^2 + y^2\frac{36T(z)\bar{T}(\bar{z})}{c^2}dzd\bar{z}.
\]  
\[\text{(7.3.2)}\]

\(^7\)As discussed in section 7.2.1, one way to define dressed, gauge-invariant operators is to define the operators after fixing to a specific gauge. We will use this method and others in this section.
Away from sources of bulk energy, $ds^2$ may be viewed as an operator whose VEV corresponds to the semiclassical metric. That is, for a CFT state $|\psi\rangle$, the semiclassical metric will be given by $ds^2_{|\psi\rangle} = \langle\psi|d\hat{s}|\psi\rangle$, which is the RHS of the above equation with $T(z) \rightarrow \langle\psi|T(z)|\psi\rangle$ and $\bar{T}(z) \rightarrow \langle\psi|\bar{T}(z)|\psi\rangle$. In this subsection, we are considering the case where the CFT is living on a flat Euclidean plane with coordinates $(z, \bar{z})$. So for the CFT vacuum $|0\rangle$, we have $\langle0|T(z)|0\rangle = 0$ and $\langle0|\bar{T}(z)|0\rangle = 0$, and the bulk metric is the Poincare metric $ds^2_{|0\rangle} = \frac{dy^2 + dzd\bar{z}}{y^2}$.

Once we gauge fix, Virasoro symmetry transformations extend to a unique set of vector fields in the bulk. Demanding that $\phi(y, z, \bar{z})$ transforms as a scalar field under the corresponding infinitesimal diffeomorphisms then implies that it must satisfy the bulk primary conditions \[ [L_n, \phi(y, 0, 0)] = 0, \quad [\bar{L}_n, \phi(y, 0, 0)] = 0, \quad n \geq 2, \] along with the condition that in the vacuum $ds^2_{|0\rangle} = \frac{dy^2 + dzd\bar{z}}{y^2}$, the bulk-boundary propagator is \[ \langle \mathcal{O}(z, \bar{z})\phi(y, 0, 0) \rangle = \frac{y^{2h_L}}{(y^2 + z\bar{z})^{2h_L}}. \] These conditions uniquely determine $\phi(y, 0, 0)$ as a CFT operator defined by its series expansion in the radial $y$ coordinate:

\[ \phi(y, 0, 0) = y^{2h_L} \sum_{N=0}^{\infty} (-1)^N \frac{y^{2N}}{N! (2h_L)_N} \mathcal{L}_{-N} \bar{L}_{-N} \mathcal{O}(0). \]

This is a bulk proto-field in the metric of equation (7.3.2). The $\mathcal{L}_{-N}$ are polynomials in the Virasoro generators at level $N$, with coefficients that are rational functions of the dimension $h$ of the scalar operator $\mathcal{O}$ and of the central charge $c$. They are obtained by solving the bulk-primary conditions (7.3.3) exactly (no large $c$ expansion is required). And as shown

\[ \mathcal{L}_{-2} = \frac{(2h + 1)(c + 8h)}{(2h + 1)c + 2h(8h - 5)} \left( L_{-1}^2 - \frac{12h}{c + 8h} L_{-2} \right). \]
in section 3.2.2 of [3], they can be written formally in terms of quasi-primaries as

$$L_{-N} \mathcal{O} = L_{-1}^N \mathcal{O} + N! (2h)_N \sum_{n=2}^N \frac{L_{-1}^{N-n} \mathcal{O}^{(i)}_{n+n}}{\left| L_{-1}^{N-n} \mathcal{O}^{(i)}_{n+n} \right|^2} \tag{7.3.7}$$

where $\mathcal{O}^{(i)}_{n+n}$ is the $i$th quasi-primary at level $n$ and the denominator is the norm of the corresponding operator. In writing this equation, we have used the fact that the quasi-primaries can be chosen so that they are orthogonal to each other. This will be a very useful property of $L_{-N} \mathcal{O}$ in later discussions.

The correlation function $\langle \phi \mathcal{O} T \rangle$ was computed and it was found that [3]

$$\frac{\langle \phi(y,0,0) \mathcal{O}(z,\bar{z}) T(z_1) \rangle}{\langle \phi(y,0,0) \mathcal{O}(z,\bar{z}) \rangle} = \frac{hz^2}{z_1^2 (z_1 - z)^2} \left( z_1 + \frac{2y^2 (z_1 - z)}{y^2 + z \bar{z}} \right). \tag{7.3.8}$$

Notice that this correlator has a cubic singularity when the position $z_1$ of the energy-momentum tensor $T$ approaches the point $(0,0)$. Roughly speaking, this singularity is the boundary imprint of the energy-momentum of the bulk field. It demonstrates the presence of a gravitational dressing connecting $\phi$ to the boundary point $(0,0)$. The boundary energy-momentum tensor $T(z_1)$ can detect this dressing via the cubic singularity, just as the current $J(z_1)$ registered the charge of a bulk field by a similar, but lower-order, singularity, as discussed in section 7.2.3. On a more formal level, contour integrals $\oint z_1^{2+n} T(z_1)$ surrounding the singularity pick out the action of conformal generators $L_n$ on the bulk field $\phi$.

Using our more general notation $\phi(y, z, \bar{z}; z_0, \bar{z}_0)$, the proto-field $\phi(y, 0, 0)$ given in equation (7.3.5) is the gauge-invariant operator $\phi(y, 0, 0; 0, 0)$ evaluated in the Fefferman-Graham gauge. The gravitational dressing is in the $\hat{y}$ direction. In the next subsection, we are going to construct bulk proto-fields that have more general dressings.
7.3.2 Natural Gravitational Dressings from Diffeomorphisms

In many situations it is convenient to place the CFT on a two-dimensional surface other than the plane. For example, in the presence of a BTZ black hole it’s more natural to place the CFT on a cylinder, as will be discussed in section 7.5. In this section we will demonstrate how to define scalar proto-fields with a correspondingly natural dressing, greatly generalizing the results of section 7.3.1. The ideas are illustrated in figure 7.3.

General vacuum AdS$_3$ metrics with different boundary geometries can be obtained from the CFT on a plane

$$ds^2 = \frac{du^2 + dx d\bar{x}}{u^2} \quad (7.3.9)$$

by the following coordinate transformation [132]

$$u = y \frac{4 \left(f'(z)\bar{f}'(\bar{z})\right)^{\frac{3}{2}}}{4f'(z)f'(\bar{z}) + y^2 f''(z)f''(\bar{z})} \quad (7.3.10)$$

$$x = f(z) - \frac{2y^2 \left(f'(z)\right)^2 \bar{f}''(\bar{z})}{4f'(z)f'(\bar{z}) + y^2 f''(z)f''(\bar{z})}$$

---

In prior work [3, 4, 5] the notation $\mathcal{L}_{\text{quasi}}$ has been used, which is $\mathcal{O}_{n+n}^{(i)}$ here.
\[ \ddot{x} = \ddot{f}(\bar{z}) - \frac{2y^2 \left( \dot{f}'(\bar{z}) \right)^2 f''(z)}{4f'(z)f'(\bar{z}) + y^2f''(z)f''(\bar{z})}. \]

The resulting vacuum AdS\(_3\) metric is then given by

\[ ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2} - \frac{1}{2}S_f(z)dz^2 - \frac{1}{2}\bar{S}_f(\bar{z})d\bar{z}^2 + \frac{y^2S_f(z)\bar{S}_f(\bar{z})}{4}dzd\bar{z}. \] (7.3.11)

where \(S_f(z) \equiv \frac{f''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2\) is the Schwarzian derivative.

In this section, we would like to obtain a formula for a bulk proto-field \(\phi(y, z, \bar{z})\) with a gravitational dressing in the \(\hat{y}\) direction. The difference between this section and section 7.3.1 is that in this section, the CFT is living on a different 2d surface. This may seem confusing, since the boundary here is also described by \((z, \bar{z})\), but here the energy-momentum tensor has a non-vanishing VEV given by \(\langle 0|T(z)|0 \rangle = \frac{c}{12}S_f\). For example, if \(f(z) = e^z\) and \(\bar{f}(\bar{z}) = e^{\bar{z}}\) (see Section 7.3.3.2 for more details about this example), then \((z, \bar{z})\) are naturally coordinates on a cylinder and \(S_f = -\frac{1}{2}\) (i.e. \(\langle 0|T(z)|0 \rangle = -\frac{c}{24}\), which is the expectation value of the energy-momentum tensor on the cylinder). In other words, the metric dual to the CFT vacuum \(|0\rangle\) here is given by (7.3.11) with a \(S_f \neq 0\), whereas in section 7.3.1 the metric dual to the CFT vacuum is given by the Poincare metric \(ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2}\) with planar boundary.

To obtain a bulk proto-field operator \(\phi(y, z, \bar{z})\) with a gravitational dressing in the \(\hat{y}\) direction, we can consider what this operator corresponds to in the \((u, x, \bar{x})\) coordinates of equation (7.3.9). The position \((y, z, \bar{z})\) of this operator will be mapped to \((u(y, z, \bar{z}), x(y, z, \bar{z}), \bar{x}(y, z, \bar{z}))\), and the boundary point of the gravitational dressing \((y = 0, z, \bar{z})\) will be mapped to the boundary point \((u = 0, x_0 = f(z), \bar{x}_0 = \bar{f}(\bar{z}))\). Therefore we can also denote the operator \(\phi(y, z, \bar{z})\) as follows

\[ \phi(y, z, \bar{z}) \equiv \phi(u, x, \bar{x}; x_0, \bar{x}_0) \] (7.3.12)

with the understanding that \((u, x, \bar{x})\) are functions of \((y, z, \bar{z})\) as given by (7.3.11) and \(x_0 = f(z)\) and \(\bar{x}_0 = \bar{f}(\bar{z})\). Although in the \((y, z, \bar{z})\) coordinates, the gravitational dressing is in the \(\hat{y}\) direction, in the \((u, x, \bar{x})\) coordinates the dressing is not in the \(\hat{u}\) direction, since typically
\[ x_0 \neq x \text{ and } \bar{x}_0 \neq \bar{x}. \]

These observations imply that in the \((u, x, \bar{x})\) coordinates, the bulk-boundary propagator should be given by

\[
\langle \phi(u, x, \bar{x}; x_0, \bar{x}_0) \rangle \mathcal{O}(x_1, \bar{x}_1) = \left( \frac{u}{u^2 + (x - x_1)(\bar{x} - \bar{x}_1)} \right)^{2h}. \tag{7.3.13}
\]

The new operator must also satisfy the bulk-primary conditions

\[
[L_n^{(x_0)}, \phi(u, x, \bar{x}; x_0, \bar{x}_0)] = 0, \quad [\bar{L}_n^{(\bar{x}_0)}, \phi(u, x, \bar{x}; x_0, \bar{x}_0)] = 0, \quad n \geq 2 \tag{7.3.14}
\]

The above two conditions uniquely fix \(\phi(u, x, \bar{x}; x_0, \bar{x}_0)\) to be\(^{10}\)

\[
\phi(u, x, \bar{x}; x_0, \bar{x}_0) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n u^{2h+2n} (x - x_0)^m (\bar{x} - \bar{x}_0)^m}{n!(2h)_n m!m!} \mathcal{L}_{-n-m} \bar{\mathcal{L}}_{-n-m} \mathcal{O}(x_0, \bar{x}_0).
\]

\(\tag{7.3.16}\)

Note that the \(\mathcal{L}_{-n-m}\) and \(\bar{\mathcal{L}}_{-n-m}\) here are defined on the boundary complex plane \((x, \bar{x})\), where \(T(x), \bar{T}(\bar{x})\) are quantized by expanding around the point \((x_0, \bar{x}_0)\). It’s obvious that the above equation satisfies the bulk-primary conditions \((7.3.14)\), since the \(\mathcal{L}_{-N}s\) and \(\bar{\mathcal{L}}_{-\bar{N}}s\) are constructed as solutions to such conditions. Using the property \((7.3.7)\) of the \(\mathcal{L}_{-N}s\), it’s easy to see why \(\phi(u, x, \bar{x}; x_0, \bar{x}_0)\) has the correct bulk-boundary propagator \((7.3.13)\) with \(\mathcal{O}(x_1, \bar{x}_1)\), since the quasi-primary terms in \(\mathcal{L}_{-n-m} \bar{\mathcal{L}}_{-n-m} \mathcal{O}\) will not contribute in this two-point function. So when computing \((7.3.13)\), we can simply replace the \(\mathcal{L}_{-n-m} \bar{\mathcal{L}}_{-n-m} \mathcal{O}\) with \(L_{n-1}^{n+m} \bar{L}_{-1}^{n+m} \mathcal{O}\), and the sums over \(m\) and \(\bar{m}\) becomes translation operators. We then have \(\phi(u, x, \bar{x}; x_0, \bar{x}_0) \to \sum_{n=0}^{\infty} \frac{(-1)^n u^{2h+2n}}{n!(2h)_n} L_{n-1}^{n} \bar{L}_{-1}^{n} \mathcal{O}(x, \bar{x})\), which will give us the desired bulk-boundary propagator \((7.3.13)\).

\(^{10}\)Performing one of the summations allows us to simplify and write \(\phi\) as

\[
\phi(u, x, \bar{x}; x_0, \bar{x}_0) = u^{2h} \sum_{N, \bar{N}=0}^{\infty} \frac{(\Delta x)^N (\Delta \bar{x})^{\bar{N}}}{N! \bar{N}!} 2F1 \left( -N, -\bar{N}, 2h, -\frac{u^2}{\Delta x \Delta \bar{x}} \right) \mathcal{L}_{-N} \bar{\mathcal{L}}_{-\bar{N}} \mathcal{O}(x_0, \bar{x}_0). \tag{7.3.15}
\]

where we used \(\Delta x = x - x_0\) and \(\Delta \bar{x} = \bar{x} - \bar{x}_0\) for concision.
In prior work [3] the special case \( f(z) = z, \bar{f}(\bar{z}) = z \) was studied, so that \( y = u, x = z, \bar{x} = \bar{z} \), and only the terms with \( m = \bar{m} = 0 \) contribute in the above equation. In that case the result reduces to

\[
\phi(y, z, \bar{z}; z, \bar{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2h+2n}}{n!(2h)_n} \mathcal{L}_{-n} \bar{\mathcal{L}}_{-n} \mathcal{O}(z, \bar{z}) = \phi(y, z, \bar{z})
\] (7.3.17)

where \( \phi(y, z, \bar{z}) \) here is exactly the bulk proto-field defined in section 7.3.1. And in this case, the gravitational dressing points in the \( \hat{\mu} \) direction, as it coincides with the \( \hat{y} \) direction.

If we take the limit that \( y \to 0 \), then this also implies \( u \to 0 \), and also \( (x - x_0), (\bar{x} - \bar{x}_0) \to 0 \), as can be seen from equation (7.3.10), so we find

\[
\lim_{y \to 0} y^{-2h} \phi(y, z, \bar{z}) = \left( f'(z) \bar{f}'(\bar{z}) \right)^h \mathcal{O}(f(z), \bar{f}(\bar{z})) = \mathcal{O}(z, \bar{z})
\] (7.3.18)

where the Jacobian factor \( \left( f'(z) \bar{f}'(\bar{z}) \right)^h \) comes from \( \lim_{y \to 0} u \) using equation (7.3.10). This is exactly what we expect for a bulk operator in the \((y, z, \bar{z})\) coordinates with a gravitational dressing in the \( \hat{y} \) direction.

Up to now we have been discussing the operator \( \phi(y, z, \bar{z}) \) in the CFT vacuum. But our definition also applies to general CFT states \( |\psi\rangle \). The semiclassical bulk metric associated with \( |\psi\rangle \) is

\[
ds_{\psi}^2 = \frac{dy^2 + dzd\bar{z}}{y^2} - \frac{6T_\psi(z)}{c}dz^2 - \frac{6\bar{T}_\psi(\bar{z})}{c}d\bar{z}^2 + y^2 \frac{36T_\psi(z)\bar{T}_\psi(\bar{z})}{c^2}dzd\bar{z}
\] (7.3.19)

where \( T_\psi(z) \equiv \langle \psi|T(z)|\psi\rangle \) and similarly for \( \bar{T}_\psi(\bar{z}) \). Note that \( T_\psi(z) \) here is the expectation value of the energy-momentum tensor in \( |\psi\rangle \), where the CFT lives on a general 2d surface defined via \( f, \bar{f} \). \( T_\psi(z) \) is related to the \( \langle \psi|T(x)|\psi\rangle \) (since we are focusing on the boundary here, we have \( x = f(z) \)) via the usual transformation rule for the energy-momentum tensor

\[
T_\psi(z) = f'(z)^2 \langle \psi|T(x)|\psi\rangle + \frac{c}{12} S_f,
\] (7.3.20)
where again, \( S_f \) is the Schwarzian derivative. Since on the boundary, \( x, \bar{x} \) are the coordinates on a complex plane, we have \( \langle 0 | T(x) | 0 \rangle = 0 \). So in the vacuum, the metric reduces to that of equation (7.3.11).

An interesting example that we will consider later is a map to the cylinder via \( f(z) = e^z \) and \( \bar{f}(\bar{z}) = e^{\bar{z}} \), where the bulk metric dual to the CFT vacuum is global AdS\(_3\). We will study correlators in a heavy state \( |\psi\rangle = |O_H\rangle \), so that the semiclassical bulk geometry is a BTZ black hole. Specifically, we will study the bulk-boundary propagator \( \langle O_H | \phi O | O_H \rangle \) in such a heavy state in section 7.5.

**Semiclassical Correlators in General Backgrounds**

We would like the bulk proto-field to have the property that in *any* semiclassical background arising from heavy sources in a CFT state \( |\psi\rangle \), the bulk-boundary propagator \( \langle \psi | \phi O | \psi \rangle \) takes the correct form. Formally, this means that as \( c \to \infty \) with heavy source dimensions \( h_H \propto c \), but with the dimension of \( O \) fixed, this correlator must obey the free bulk wave equation in the associated metric of equation (7.3.19). Let us explain why this must be the case.

Given any classical fields \( T_\psi, \bar{T}_\psi \) forming the metric of equation (7.3.19), we will assume that we can identify new diffeomorphisms \( g(z), \bar{g}(\bar{z}) \) so that the metric (7.3.19) arises from a diffeomorphism (7.3.10) with \( g, \bar{g} \) replacing \( f, \bar{f} \). Note that while \( f, \bar{f} \) connect the empty Poincaré patch to a corresponding CFT vacuum (7.3.11), the new functions \( g, \bar{g} \) relate the empty Poincaré patch to a Fefferman-Graham gauge metric (7.3.19) where there are heavy sources\(^{11}\).

The bulk primary conditions of equation (7.3.14) were chosen to guarantee that \( \phi \) transforms as a scalar field in Fefferman-Graham gauge, and so it must transform as a scalar under the map from \( (u, x, \bar{x}) \leftrightarrow (y, z, \bar{z}) \) induced by \( g, \bar{g} \). Then the condition (7.3.13) guarantees that the bulk boundary propagator in \( (u, x, \bar{x}) \) coordinates takes the correct form.

\(^{11}\)We’ll give an example of this statement in section 7.5.
and so we can conclude that it must take the correct form in the semiclassical limit in the very non-trivial metric of equation (7.3.19). Thus the conditions we have used to define \( \phi \) ensure that its semiclassical correlators must take the expected form in the background of any sources, as long as \( \phi \) does not intersect with these sources directly.

### 7.3.3 \( \langle \phi \mathcal{O} T \rangle \) on General Vacuum AdS\(_3\) Metrics and Examples

We can compute \( \langle \phi \mathcal{O} T \rangle \) with \( \phi \) given by (7.3.16) on general vacuum AdS\(_3\) metrics given by (7.3.11). Specifically, we study

\[
\langle T(x_1) \mathcal{O}(x_2, \bar{x}_2) \phi(u, x, \bar{x}; x_0, \bar{x}_0) \rangle \tag{7.3.21}
\]

with

\[
x_0 = f(z), \bar{x}_0 = \bar{f}(\bar{z}), x_1 = f(z_1), x_2 = f(z_2), \text{ and } \bar{x}_2 = \bar{f}(\bar{z}_2) \tag{7.3.22}
\]

Although (7.3.21) is written in terms of the \((u, x, \bar{x})\) coordinates, it should really be understood as a correlation function in the coordinates \((y, z, \bar{z})\) with the metric given by (7.3.11) (and \(S_f\) given by the Schwarzian derivative of \(f(z)\)). More precisely, to obtain the correct \( \langle \phi \mathcal{O} T \rangle \) in the \((y, z, \bar{z})\) coordinates, we would simply transform \(T(x_1)\) and \(\mathcal{O}(x_2, \bar{x}_2)\) to \(T(z_1)\) and \(\mathcal{O}(z_2, \bar{z}_2)\) using the usual transformation rules for the energy-momentum tensor and primary operators, and leave \(\phi\) as it is, since it’s a bulk scalar field.

We can use the OPE of \(T \mathcal{O}\) and \(T \phi\) to evaluate this correlator. Note that when using the OPE of \(T \phi\), we need to expand \(T\) around \(x_0\) instead of \(x\). Explicitly, the singular terms in the OPEs are

\[
T(x_1) \mathcal{O}(x_2, \bar{x}_2) \sim \frac{L_{-1} \mathcal{O}(x_2, \bar{x}_2)}{x_1 - x_2} + \frac{h \mathcal{O}(x_2, \bar{x}_2)}{(x_1 - x_2)^2} + \ldots \tag{7.3.23}
\]

\[
T(x_1) \phi(u, x, \bar{x}; x_0, \bar{x}_0) \sim \frac{L_{-1} \phi(u, x, \bar{x}; x_0, \bar{x}_0)}{x_1 - x_0} + \frac{L_{0} \phi(u, x, \bar{x}; x_0, \bar{x}_0)}{(x_1 - x_0)^2}
\]
\[
\phi(u, x, \bar{x}; x_0, \bar{x}_0) = L_1 \phi(u, x, \bar{x}; x_0, \bar{x}_0) + \frac{L_1 \phi(u, x, \bar{x}; x_0, \bar{x}_0)}{(x_1 - x_0)^3} + \cdots
\]

(7.3.24)

where we’ve used bulk-primary conditions (7.3.14) when writing down the OPE of \( T \phi \). So when computing \( \langle \phi O T \rangle \), we simply include all the singular terms in these two OPEs. The \( L_{-1} \)s will becomes just the differential operators \( \partial_{x_2} \) and \( \partial_{x_0} \), while the terms with \( L_0 \phi \) and \( L_1 \phi \) can be computed by commuting \( L_0 \) and \( L_1 \) with \( O(x_2, \bar{x}_2) \) or by writing them in terms of contour integrals. Eventually, we get

\[
\langle T(x_1) O(x_2, \bar{x}_2) \phi(u, x, \bar{x}; x_0, \bar{x}_0) \rangle = h \left[ \frac{1}{(x_1 - x_2)^2} - \frac{1}{(x_1 - x_0)^2} - 2 \frac{(x_2 - x_0)}{(x_1 - x_0)^3} \right] \langle \phi O \rangle
\]

(7.3.25)

\[
- 2h \left[ - \frac{1}{x_1 - x_2} + \frac{x_2 - x_0}{(x_1 - x_0)^2} + \frac{1}{x_1 - x_0} + \frac{(x_2 - x_0)^2}{(x_1 - x_0)^3} \right] \frac{(\bar{x} - \bar{x}_2)}{u^2 + (x - x_2)(\bar{x} - \bar{x}_2)} \langle \phi O \rangle,
\]

where \( \langle \phi O \rangle = \left( \frac{u}{u^2 + (x - x_2)(\bar{x} - \bar{x}_2)} \right)^{2h} \). Correlation functions of the form \( \langle \phi OT \cdots T \rangle \) can then be computed recursively using the above OPEs and also the OPE of \( TT \).

Here, we give two examples of the above result.

### 7.3.3.1 Example 1: \( \langle \phi OT \rangle \) on Poincare AdS

To obtain the \( \langle \phi OT \rangle \) on the Poincare AdS metric, we use \( f(z) = z, \bar{f}(\bar{z}) = \bar{z} \). In this case, we have \( u = y, x = z, \) and \( \bar{x} = \bar{z} \), and the metric (7.3.11) is simply \( ds^2 = \frac{dy^2 + dz d\bar{z}}{y^2} \). After simplification, \( \langle \phi OT \rangle \) is given by

\[
\langle T(z_1) O(z_2, \bar{z}_2) \phi(y, z, \bar{z}) \rangle = \frac{h(z_2 - z)^2}{(z_1 - z)^3 (z_1 - z_2)^2} \left( z_1 - z + \frac{2y^2 (z_1 - z_2)}{y^2 + (z - z_2)(\bar{z} - \bar{z}_2)} \right) \langle \phi O \rangle
\]

(7.3.26)

This is exactly the result (7.3.8) obtained in [3].
7.3.3.2 Example 2: $\langle \phi O T \rangle$ on Global AdS$_3$

To obtain $\langle \phi O T \rangle$ on global AdS$_3$, whose boundary is a cylinder, we use $f(z) = e^z$ and $\bar{f}(\bar{z}) = e^{\bar{z}}$. From equation (7.3.10), we have

$$u = \frac{4y\sqrt{\xi\bar{\xi}}}{4 + y^2}, \quad x = \frac{4 - y^2}{4 + y^2}\xi, \quad \bar{x} = \frac{4 - y^2}{4 + y^2}\bar{\xi}. \quad (7.3.27)$$

where we’ve defined $\xi \equiv e^z$ and $\bar{\xi} \equiv e^{\bar{z}}$ for notational convenience. The resulting metric is

$$ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2} + \frac{dz^2}{4} + \frac{d\bar{z}^2}{4} + \frac{y^2}{16}dzd\bar{z}, \quad (7.3.28)$$

which is related to the usual global AdS$_3$ metric

$$ds^2 = (r^2 + 1)dt_E^2 + \frac{dr^2}{r^2 + 1} + r^2d\theta^2 \quad (7.3.29)$$

by a simple coordinate transformation

$$z = t_E + i\theta, \quad \bar{z} = t_E - i\theta, \quad y = 2\left( \sqrt{r^2 + 1} - r \right). \quad (7.3.30)$$

The bulk-boundary three-point function $\langle \phi O T \rangle$ on this metric is given by

$$\langle T(\xi_1)O(\xi_2, \bar{\xi}_2)\phi (u, x, \bar{x}; \xi, \bar{\xi}) \rangle$$

$$= \frac{h(\xi - \xi_2)^2}{(\xi_1 - \xi)^3(\xi_1 - \xi_2)^2} \left[ \xi_1 - \xi + \frac{4y^2\xi(\xi_1 - \xi_2)(\bar{\xi} + \bar{\xi}_2)}{(\xi\bar{\xi} + \xi_2\bar{\xi}_2)(y^2 + 4) + (\xi_2 + \xi\bar{\xi})y^2 - 4} \right]$$

$$\times \langle O(\xi_2, \bar{\xi}_2)\phi (u, x, \bar{x}; \xi, \bar{\xi}) \rangle. \quad (7.3.31)$$

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Here, $\langle O \xi_2, \bar{\xi}_2 \rangle \phi (u, x, \bar{x}; \xi, \bar{\xi}) \rangle$ is given by

$$
\langle O \phi \rangle = \left( \frac{u}{u^2 + (x - \xi_2)(\bar{x} - \bar{\xi}_2)} \right)^{2h}
$$

(7.3.32)

$$
= \left( \frac{4y\sqrt{\xi\bar{\xi}}}{(\xi \xi + \bar{\xi}_2\xi_2)(y^2 + 4) + (\bar{\xi}_2\xi + \bar{\xi}\xi_2)(y^2 - 4)} \right)^{2h}.
$$

We have also obtained the same result using the effective field theory of gravitons developed in [168] (see appendix F.4 for details of that calculation).

### 7.3.4 General Gravitational Dressings from Singularity Structure

In this section we will define very general gravitational dressings through a procedure analogous to that of section 7.2.4, where we studied the $U(1)$ Chern-Simons case. To define these bulk proto-fields we will leverage the singularity structure of correlators between $\phi, O$, and any number of stress tensors. We review how the singularity structure of $T(z)$ is determined by Einstein’s equations in appendix F.1.2, generalizing the $U(1)$ case of section 7.2.3. Here the CFT will be living on a flat 2d plane, with the CFT vacuum state corresponding to the pure Poincare metric $ds^2_{0} = \frac{du^2 + dx^2 + d\bar{x}^2}{u^2}$.

The proto-field operator we will identify takes a similar form to that derived in section 7.3.2, but our interpretation here will be different, and more abstract. Another way of motivating this section would be to ask to what extent equation (7.3.16) can be given a general meaning, independent of the diffeomorphism of equation (7.3.10).

#### A General Bulk Proto-Field

The energy associated with the bulk operator $\phi$ must be reflected in the CFT by singularities in $T(x)$ correlators. As in the $U(1)$ Chern-Simons case of section 7.2.4, we can construct a bulk proto-field $\phi(u, x, \bar{x}; x_0, \bar{x}_0)$ by demanding:
1. \( \langle \phi \mathcal{O} \rangle \) must be given by \( \langle \phi(u, x, \bar{x}; x_0, \bar{x}_0) \mathcal{O}(w, \bar{w}) \rangle = \left( \frac{u}{u^2 + (x-w)(\bar{x}-\bar{w})} \right)^{2h} \) in the vacuum.

2. Correlators \( \langle \mathcal{O}(w, \bar{w}) T(x_1) \cdots T(x_n) \bar{T}(x_1) \cdots \bar{T}(\bar{x}_n) \phi(u, x, \bar{x}; x_0, \bar{x}_0) \rangle \) only have poles of up to third order in the \( x_i \), which can only occur when \( x_i \to x_0 \) (along with up to second order poles as \( x_i \to w \)), and equivalently for the antiholomorphic variables.

Note that we allow poles \( \propto \frac{1}{(x_i-x_0)^3} \) in \( T(x_i)\phi(u, x, \bar{x}; x_0, \bar{x}_0) \), whereas only second order poles occur in the OPE of \( T(x_i)\mathcal{O}(w) \). This is because \( \phi \) contains descendants of \( \mathcal{O} \), including the first descendant \( L_{-1}\mathcal{O} \propto \partial \mathcal{O} \), and such operators necessarily induce third-order poles. But higher order singularities are excluded by our assumptions.

The unique operator built from Virasoro descendants of a primary \( \mathcal{O} \) and satisfying these conditions is

\[
\phi(u, x, \bar{x}; x_0, \bar{x}_0) = \sum_{n=0}^{\infty} \sum_{m, \bar{m}} u^{2h+2n} \frac{(-1)^n}{n!(2h)_n} \frac{(x-x_0)^m(\bar{x}-\bar{x}_0)^{\bar{m}}}{m!\bar{m}!} L_{-n-m} \bar{L}_{-n-m} \mathcal{O}(x_0, \bar{x}_0)
\]

(7.3.33)

As a formal power series expansion in CFT operators, this is identical to equation (7.3.16), except that \( x_0 \) and \( \bar{x}_0 \) are now arbitrary, instead of given by \( f(z) \) and \( \bar{f}(\bar{z}) \). As we will explain below, this object can be interpreted as a bulk proto-field with a dressing that follows the geodesic path from \( (u, x, \bar{x}) \) to \( (x_0, \bar{x}_0) \) in any geometry.

One way to interpret this expression is as a modification of equation (7.3.5) (with the notation \( (y, z, \bar{z}) \) replaced by \( (u, x, \bar{x}) \) here), where the boundary imprint of the bulk energy has been translated to \( x_0, \bar{x}_0 \). In appendix F.1 we develop a theory of non-local ‘mirage translation’ operators that move the local energy of CFT operators (singularities in \( T(x) \) correlators) without moving the apparent location of an operator \( \mathcal{O} \) itself (so mirage translations leave OPE singularities between local primaries fixed). Mirage translations also provide an independent motivation for the bulk primary conditions.

To better understand equation (7.3.33), let’s consider a few simplifying limits. If \( x_0 = x \) and \( \bar{x}_0 = \bar{x} \), then only the \( m = 0 \) terms contribute to equation (7.3.33), and \( \phi \) reduces to the
bulk field of equation (7.3.5). To obtain $\mathcal{O}(x_0, \bar{x}_0)$ as we take $u \to 0$, we need to simultaneously send $x \to x_0$; otherwise we obtain a non-local operator $\tilde{\mathcal{O}}(x, \bar{x}; x_0, \bar{x}_0)$, corresponding to $\mathcal{O}(x, \bar{x})$ multiplied with an additional gravitational dressing on the boundary. The non-local operator $\tilde{\mathcal{O}}(x, \bar{x}; x_0, \bar{x}_0)$ can be interpreted is the mirage translation of $\mathcal{O}(x, \bar{x})$, as discussed in appendix F.1.

The non-gravitational limit is also easy to understand. When $c \to \infty$ with $\hbar$ fixed, we have $\mathcal{L}_- \to L^N_{-1}$. In this limit the sum over $m, \bar{m}$ simplifies into a pure translation, and the sums on $m$ simply convert $\mathcal{O}(x_0, \bar{x}_0) \to \mathcal{O}(x, \bar{x})$. Only the sum on $n$ remains, and this reconstructs the non-interacting field defined in equation (7.2.5).

More generally, all of the terms $\mathcal{L}_- \tilde{\mathcal{L}}_- \mathcal{O}$ were constructed so that they only have OPE singularities of fixed order $\leq 3$ with $T(x_1)$, and these singularities only occur when $x_1 \to x_0$. The field $\phi(u, x, \bar{x}; x_0, \bar{x}_0)$ inherits this property. However, as the $\mathcal{L}_-$ act as a kind of modified translation, other local CFT operators constructed from other primaries will behave as though $\phi(u, x, \bar{x}; x_0, \bar{x}_0)$ is located at $(u, x, \bar{x})$ in the bulk.

**The Gravitational Dressing Naturally Follows a Geodesic**

We glossed an important issue when defining equation (7.3.33): for this expression to be meaningful, we must have some way of determining where in the bulk this field lives. In the CFT vacuum the operator is at $(u, x, \bar{x})$ in the Poincaré patch metric. This observation is sufficient to determine the location of $\phi$ in perturbation theory around the Poincaré patch metric. But if we turn on heavy sources, the bulk metric will change by a finite amount. We have not fixed to Fefferman-Graham gauge in $(u, x, \bar{x})$ coordinates, so these coordinates are just labels, which only have unambiguous meaning in perturbation theory or in the limit $u \to 0$.

We can partially resolve this issue by comparing with section 7.3.2. It is easy to see that there exist an infinite family of $f(z), \bar{f}(\bar{z})$ functions so that $x_0, \bar{x}_0 = f(z), \bar{f}(\bar{z})$ and equation (7.3.10) relates $(u, x, \bar{x})$ to $(y, z, \bar{z})$. For each such $f, \bar{f}$ we implicitly define a gauge choice
in \((u, x, \bar{x})\) by pulling back the Fefferman-Graham gauge of the \((y, z, \bar{z})\) coordinate system to \((u, x, \bar{x})\) via the diffeomorphism (7.3.10). So the bulk location of the proto-field could be interpreted as a bulk proto-field in any of these gauges.

However, all interpretations of the proto-field share a common feature. The gravitational dressing in \((y, z, \bar{z})\) can be chosen to be a curve with constant \((z, \bar{z})\), so that it points in the \(\hat{y}\)-direction. In Fefferman-Graham gauge, these curves are all geodesics. This means that the gravitational dressing of \(\phi(u, x, \bar{x}; x_0, \bar{x}_0)\) will follow a geodesic from \((u, x, \bar{x})\) in the bulk to \((x_0, \bar{x}_0)\) on the boundary, in any dynamical metric.

Thus we conclude that \(\phi(u, x, \bar{x}; x_0, \bar{x}_0)\) will behave like a scalar field in the bulk defined by fixing to any gauge where the gravitational field does not fluctuate along the geodesic connecting \((u, x, \bar{x})\) and \((x_0, \bar{x}_0)\). That is, if \(X^u(\lambda)\) are coordinates on this geodesic, then \(h_{\mu\nu}(X(\lambda))\dot{X}^\nu(\lambda) = 0\), where \(h_{\mu\nu}\) is the deviation of the bulk metric from the Poincaré patch form. This leaves the gauge choice elsewhere in spacetime almost entirely undetermined.

**More General Dressings**

To obtain a more general class of dressed bulk \(\phi\) we can smear \(\phi(X; x_0, \bar{x}_0)\) over \((x_0, \bar{x}_0)\) via

\[
\phi[\rho](X) \equiv \int dx_0 d\bar{x}_0 \rho(x_0, \bar{x}_0) \phi(X; x_0, \bar{x}_0)
\]

(7.3.34)

with any positive function \(\rho\) that integrates to one \(\int d^2x_0 \rho(x_0, \bar{x}_0) = 1\). If we work in perturbation theory around the vacuum (or any fixed semiclassical metric), then the location of the protofield in \((u, x, \bar{x})\) coordinates will be unambiguous. Then we can obtain results similar to the \(U(1)\) case in section 7.2.5.

### 7.4 Recursion Relation for Bulk-Boundary Vacuum Blocks

Gravitational dynamics can be probed with correlation functions of the bulk proto-field (7.3.16). For example, bulk locality was studied in [4] by computing the bulk two point
function $\langle \phi \phi \rangle$ and Euclidean black hole horizons were investigated in [5] by computing the vacuum blocks of heavy-light bulk-boundary correlator $\langle O_H \phi_L O_L O_H \rangle$. In those works, recursions relations were derived for computing correlators involving the proto-field of section 7.3.1 (i.e., the special case of (7.3.16) with $f(z) = z$ and $\bar{f}(\bar{z}) = \bar{z}$). Now that we have the more general bulk proto-field (7.3.16), we can also derive recursion relations for computing its correlators.

In this section, we are going to derive a recursion relation for computing the Virasoro vacuum block contribution to

$$\langle O_H(\infty) O_H(1) \phi_L(u, x, \bar{x}; x_0, \bar{x}_0) O_L(0) \rangle,$$

where $\phi_L(u, x, \bar{x}; x_0, x_0)$ (with $h = \bar{h} = h_L$) is given by equation (7.3.16) and the coordinates $\infty, 0, 1$ are on the complex plane with $(x, \bar{x})$ coordinates. In order to be more general and include of case of section 7.3.4, we are going to assume that $x_0, \bar{x}_0$ are arbitrary (i.e., we are not assuming that they are given by $f(z)$ and $\bar{f}(\bar{z})$), although this will not affect the discussion in this section. Although we use the subscripts $H$ and $L$ which usually means “heavy” and “light”, the conformal dimension $h_H$ and $h_L$ in this section are arbitrary. We will study a special case of this result in section 7.5, and compute the bulk-boundary propagator in a black hole microstate background.

### 7.4.1 General Structure of the Vacuum Blocks

As usual, the vacuum block $\mathcal{V}_0$ is obtained via the projection

$$\mathcal{V}_0 = \langle O_H(\infty) O_H(1) P_0\phi_L(u, x, \bar{x}; x_0, \bar{x}_0) O_L(0) \rangle$$


\footnote{Although we don’t develop it in this paper, the recursion relation for computing the bulk two-point function $\langle \phi \phi \rangle$ for $\phi$ given by (7.3.16) should be very similar to the one derived in [4].}
where $P_0$ is the projection operator into the vacuum module\textsuperscript{13} (including holomorphic and anti-holomorphic parts). $\phi_L (u, x, \bar{x}; x_0, \bar{x}_0)$ of equation (7.3.16) can be simplified to the following form:

$$\phi_L (u, x, \bar{x}; x_0, \bar{x}_0) = u^{2h} \sum_{N, \bar{N}=0}^{\infty} A_{N, \bar{N}} (u, \Delta x, \Delta \bar{x}) \frac{L_{-N} L_{-\bar{N}} O_0 (x_0, \bar{x}_0)}{(2h_L)_N (2h_{L\bar{L}})_{\bar{N}} N! \bar{N}!}$$

(7.4.4)

where $\Delta x = x - x_0$ and $\Delta \bar{x} = \bar{x} - x_0$, and

$$A_{N, \bar{N}} (u, \Delta x, \Delta \bar{x}) \equiv (2h_L)_N (2h_{L\bar{L}})_{\bar{N}} (\Delta x)^N (\Delta \bar{x})^{\bar{N}} \binom{2}{2h_L}_N N! \binom{2}{2h_{L\bar{L}}}_{\bar{N}} \binom{2}{2h_L}_N \binom{2}{2h_{L\bar{L}}}_{\bar{N}}$$

(7.4.5)

The extra factors of $(2h_L)_N (2h_{L\bar{L}})_{\bar{N}}$ in the expressions above are inserted for later convenience.

Although the Virasoro vacuum block of (7.4.2) doesn’t factorize into holomorphic and anti-holomorphic parts, we can make use of the fact that it does factorizes for a specific $N$ and $\bar{N}$, since the projection operator $P_0$ factorizes. Similar to the case in [5], we can define the holomorphic part of $\phi$ to be

$$\tilde{\phi}_h^{\text{holo}} (u, x; x_0) \equiv \sum_{N=0}^{\infty} \frac{L_{-N} O_0 (x_0, \bar{x}_0)}{(2h_L)_N N!},$$

(7.4.6)

and we’ll obtain a recursion relation for computing the more general holomorphic block:

$$V_{\text{holo}} (h_1, h_2, c) \equiv \langle O_H (\infty) | O_H (1) | \overset{\text{holo}}{P}_{h_1} \tilde{\phi}_h^{\text{holo}} (u, x; x_0) | O_L (0) \rangle.$$

(7.4.7)

Here the holomorphic projection operator $\overset{\text{holo}}{P}_{h_1}$ only includes the holomorphic descendants of $|O_{h_1}\rangle$. We are considering this more general block for the convenience of discussing the

\textsuperscript{13}To be clear, the projection operator for a representation of the Virasoro algebra with lowest weight state $|O_{h_1}\rangle$ factorizes, that is, $P_{h_1} = \overset{\text{holo}}{P}_{h_1} \overset{\text{anti-holo}}{P}_{h_1}$ and the holomorphic part is given symbolically by

$$\overset{\text{holo}}{P}_{h_1} = \sum_{\{m_i\}, \{n_j\}} \frac{L_{-m_1} \cdots L_{-m_l} |O_{h_1}\rangle \langle O_{h_1} | L_{n_j} \cdots L_{n_l}}{N_{\{m_i\}, \{n_j\}}},$$

(7.4.3)

where $N_{\{m_i\}, \{n_j\}}$ is the inverse of the inner-product matrix between the states.
recursion relation in the next subsection. Eventually, we are interested in the vacuum block \( \mathcal{V}_{\text{holo}}(0, h_L, c) \), and it will be given in the following form

\[
\mathcal{V}_{\text{holo}}(0, h_L, c) = \frac{1}{x_0^{2h_L}} \sum_{N=0}^{\infty} \frac{1}{x_0^N} F_N(x_0)
\]

(7.4.8)

\( F_N(x_0) \) is an infinite series of \( x_0 \), starting with \( F_N(x_0) = 1 + \cdots \) (we’ll explain how to obtain \( F_N(x_0) \) in next subsection). The full vacuum block \( \mathcal{V}_0 \) is then obtained via the following equation

\[
\mathcal{V}_0 = \left( \frac{u}{x_0 \bar{x}_0} \right)^{2h_L} \sum_{N, \bar{N}=0}^{\infty} \frac{A_{N, \bar{N}}(u, \Delta x, \Delta \bar{x})}{(\Delta x_0)^N (\Delta \bar{x}_0)^{\bar{N}}} F_N(x_0) F_{\bar{N}}(\bar{x}_0),
\]

(7.4.9)

where \( F_{\bar{N}}(\bar{x}_0) \) is simply \( F_N(x_0) \) with \( x_0 \) replaced by \( \bar{x}_0 \).

### 7.4.2 Recursion Relation

Our task now is to obtain the recursion relation for computing \( \mathcal{V}_{\text{holo}}(h_1, h_2, c) \) based on the singularity structure of \( \mathcal{V}_{\text{holo}}(h_1, h_2, c) \) as a function of the central charge \( c \). Actually, the recursion relation for computing \( \mathcal{V}_{\text{holo}} \) here is almost the same as the recursion in [5] for computing the \( \mathcal{V}_{\text{holo}} \) in that case\(^{14}\), except that the seed of the recursion is different. We reproduce the recursion relation here for convenience

\[
\mathcal{V}_{\text{holo}}(h_1, h_2, c) = \mathcal{V}_{\text{holo}}(h_1, h_2, c \to \infty) + \sum_{m \geq 2, n \geq 1} \frac{R_{m,n}(h_1, h_2)}{c - c_{m,n}(h_1)} \mathcal{V}_{\text{holo}}(h_1 \to h_1 + mn, h_2, c \to c_{mn}(h_1)) \]

(7.4.10)

\[
+ \sum_{m \geq 2, n \geq 1} \frac{S_{m,n}(h_1, h_2)}{c - c_{m,n}(h_2)} \mathcal{V}_{\text{holo}}(h_1, h_2 \to h_2 + mn, c \to c_{mn}(h_2)).
\]

\(^{14}\)The recursion relation in [5] is an special case of the recursion here, with \( f(z) = z \) and \( \bar{f}(\bar{z}) = \bar{z} \), but the general structures are almost the same.
For more details about the meaning of the symbols $R_{m,n}$, $S_{m,n}$, $c_{m,n}$ and how to solve this recursion relation, please see section 4 and appendix C of [5].

As mentioned above, the seed of the recursion $V_{\text{holo}} (h_1, h_2, c \to \infty)$ is different from that of [5]. In the $c \to \infty$ limit, only global descendants of the intermediate state $|\mathcal{O}_{h_1}\rangle$ and global descendants of $\mathcal{O}_{h_2}$ contribute, so $V_{\text{holo}} (h_1, h_2, c \to \infty)$ is actually the global block, i.e.

$$G (h_1, h_2) \equiv V_{\text{holo}} (h_1, h_2, c \to \infty).$$  \hfill (7.4.11)

$G (h_1, h_2)$ can be obtain by direct computation, as follows

$$G (h_1, h_2) = \sum_{m_1, m_2=0}^{\infty} \frac{\langle \mathcal{O}_H | \mathcal{O}_H | L_{m_1}^0 \mathcal{O}_1 \rangle \langle L_{m_1}^1 \mathcal{O}_1 | L_{m_2}^0 \mathcal{O}_{h_2} (x_0) | \mathcal{O}_L \rangle}{|L_{m_1}^1 |^2 | L_{m_2}^0 |^2}$$

$$= \frac{1}{x_0^{2h_L}} \sum_{m_1, m_2=0}^{\infty} \frac{(h_1 + h_2, h_L)}{(2h_1 + m_1)! (2h_2 + m_2)!} x_0^{h_1 + m_1} x_0^{m_2 + h_2 - h_L},$$  \hfill (7.4.12)

where $\tilde{x}_0 \equiv \frac{1}{x_0}$. We are using $\tilde{x}_0$ here for keeping tracking the origin of each term (so that we can obtain the $F_N$ in (7.4.8) after we compute $V_{\text{holo}}$) and for the convenience of implementation in Mathematica. Here, $\rho_{i,j,k} (h_1, h_2, h_3)$ is the three point functions of global descendants, and it’s given by [180]

$$\rho_{i,j,k} (h_1, h_2, h_3) \equiv \langle L_{-1}^i \mathcal{O}_{h_1} | L_{-1}^j \mathcal{O}_{h_2} (1) | L_{-1}^k \mathcal{O}_{h_3} \rangle$$

$$= (h_1 + i - h_2 - j + 1 - h_3 - k) \cdot s_{ik} (h_1, h_2, h_3)$$  \hfill (7.4.13)

with

$$s_{ik} (h_1, h_2, h_3) = \sum_{p=0}^{\min(i,k)} \frac{i!}{p!(i-p)!} (2h_3 + k - p)_p (i - p + 1)_p$$

$$\times (h_3 + h_2 - h_1)_{k-p} (h_1 + h_2 - h_3 + p - k)_{i-p}.$$  \hfill (7.4.14)

And we only need the $\rho_{i,j,k}$ with $k = 0$.  

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As in [5], solving the recursion produces \( \mathcal{V}_{\text{holo}}(h_1, h_2, c) \) as the following sum\(^{15} \)

\[
\mathcal{V}_{\text{holo}}(h_1, h_2, c) = \sum_{m,n=0}^{\infty} C_{m,n} G(h_1 + m, h_2 + n).
\]

(7.4.16)

Here, \( G(h_1 + m, h_2 + n) \) is the global block (7.4.12) with \( h_1 \to h_1 + m \) and \( h_2 \to h_2 + n \). The summand \( C_{m,n} G(h_1 + m, h_2 + n) \) is the contribution to \( \mathcal{V}_{\text{holo}} \) from all the level \( m \) quasi-primaries \( \mathcal{O}_{h_1+m}^{(i)} \) and level \( n \) quasi-primaries \( \mathcal{O}_{h_2+n}^{(j)} \) (denoted as \( \mathcal{L}^{\text{quasi},i}_{-m} \mathcal{O}_{h_1} \) and \( \mathcal{L}^{\text{quasi},j}_{-n} \mathcal{O}_{h_2} \) in previous papers [4, 5]), and their global descendants. The coefficients \( C_{m,n} \) are exactly the same as the coefficients in [5]. Basically, they encode the three point functions of the quasi-primaries with primaries as follows

\[
C_{m,n} = \sum_{i,j} \left\langle \mathcal{O}_H | \mathcal{O}_H(1) | \mathcal{O}_{h_1+m}^{(i)} \right\rangle \left\langle \mathcal{O}_{h_1+m}^{(i)} | \mathcal{O}_{h_2+n}^{(j)}(1) | \mathcal{O}_L \right\rangle, \quad (7.4.17)
\]

where we've assumed that the quasi-primaries are orthogonalized and the sum \( \sum_{i,j} \) is over all level \( m \) and level \( n \) quasi-primaries. In section 4 and appendix C of [5], we discussed in detail how to obtain the above result and how to compute them using the recursion.

After obtaining \( \mathcal{V}_{\text{holo}}(0, h_L, c) \) as a polynomial of \( \tilde{x}_0 \), the coefficient of \( \tilde{x}_0^N \) is the \( F_N(x_0) \) in (7.4.8). This is why we keep \( \tilde{x}_0 \) explicitly in the global blocks (7.4.12), instead of using the fact that \( \tilde{x}_0 \equiv \frac{1}{x_0} \) to simplify the calculation of the global blocks, because that will mix \( \frac{1}{x_0} \) with the \( x_0 \) in \( F_N(x_0) \), and we will not be able to extract \( F_N(x_0) \). After obtaining \( F_N(x_0) \), we can simply use equation (7.4.9) to compute the full vacuum block \( \mathcal{V}_0 \). The Mathematica code for implementing this recursion relation is attached with this paper.

\(^{15}\)As shown in section 3.2.2 of [3], solving the bulk primary conditions will give us \( \mathcal{L}_{-n} \mathcal{O} \) in terms of quasi-primaries and their global descendants (see the paragraph below (7.3.7) for explanations of the notations here)

\[
\mathcal{L}_{-N} \mathcal{O} = N! (2\hbar)^N \sum_{n=0}^{N} \sum_{i} \frac{L_{-1}^{N-n} \mathcal{O}_{h+n}^{(i)}}{L_{-1}^{N-n} \mathcal{O}_{h+n}^{(i)}}. \quad (7.4.15)
\]

Similarly, \( \mathcal{P}^{\text{holo}}_{h_1} \) can be written in terms of quasi-primaries and their global descendants. So \( \mathcal{V}_{\text{holo}}(h_1, h_2, c) \) defined in (7.4.7) can also be written as a sum over quasi-primaries and their global descendants. In this way, it’s easier to see why \( \mathcal{V}_{\text{holo}}(h_1, h_2, c) \) can be decomposed into a sum over global blocks as in (7.4.16).
Generally, the recursion relations for computing the boundary Virasoro blocks [37, 36, 148] and the bulk-boundary Virasoro blocks consist of two parts. One is the computation of the coefficients $C_{m,n}$. And the other part is the computation of the global blocks. The computation of $C_{m,n}$ is the most complicated part of the recursion relations, but luckily, for most observables of interest, it’s universal. The difference between observables is in the global blocks, which are the seed for the recursion relations.

### 7.5 Heavy-Light Bulk-Boundary Correlator on the Cylinder

Our study of bulk reconstruction was motivated by the desire to understand near horizon dynamics and the black hole information paradox. This program may be advanced by computing correlation functions of bulk proto-fields in a black hole microstate background. One object of interest is the heavy-light bulk-boundary vacuum block $\mathcal{V}_0$ of $\langle O_H | O_L \phi_L | O_H \rangle$ for $\phi_L$ defined in global AdS. When $|O_H\rangle$ is dual to a BTZ black hole microstate ($h_H > \frac{c}{2}\pi$), $\mathcal{V}_0$ will be dual to the bulk-boundary propagator of the light operators in such a background.

In this section, we’ll compute this vacuum block using two different methods. Our first method utilizes the recursion relations introduced in the last section. This method will give us an exact result for the vacuum block as an expansion in the kinematic variables, with coefficients evaluated exactly at finite $c$, including all the gravitational interactions between the light probe operator and the heavy state. Our second method is based on the idea of bulk-boundary OPE blocks [3] (or bulk-boundary bi-local operators) and effective theory for boundary gravitons in AdS$_3$/CFT$_2$ [168]. This second method will give us the vacuum block in a large $c$ expansion with $\frac{h_H}{c}$ fixed. We’ll carry out the calculation up to order $\frac{1}{c}$, which corresponds to the gravitational one-loop correction to the bulk-boundary propagator in a microstate BTZ black hole background. We have verified that the results from these methods agree.
We will also show analytically that the one-loop corrections are singular at the Euclidean horizon. This effect only arises because the $1/c$ corrections to the correlators are not periodic in Euclidean time [20, 12]. When they are interpreted as correlators in the BTZ geometry, they must have a branch cut at the horizon.

Throughout this section, we’ll assume the following limit

$$c \to \infty, \quad h_H \sim O(c), \quad h_L \sim O(1),$$

although our computation using the recursion relation is valid at finite $c$. We’ll study the bulk protofield

$$\phi_L (y, z, \bar{z}) \equiv \phi_L (u, x, \bar{x}; f(z), \bar{f}(\bar{z}))$$

(with $h = \bar{h} = h_L$) to be given by equation (7.3.16) with $f(z) = e^z$ and $\bar{f}(\bar{z}) = e^{\bar{z}}$. In this case, we have

$$u = \frac{4y\sqrt{\xi \bar{\xi}}}{4 + y^2}, \quad x = \frac{4 - y^2}{4 + y^2} \xi, \quad \bar{x} = \frac{4 - y^2}{4 + y^2} \bar{\xi}$$

with $\xi \equiv e^z$ and $\bar{\xi} \equiv e^{\bar{z}}$. As discussed in section (7.3.2), in this case, the CFT is living on the boundary cylinder with coordinates $(z, \bar{z})$ and the bulk metric (7.3.11) that’s dual the CFT vacuum $|0\rangle$ is given by

$$ds^2_{|0\rangle} = \frac{dy^2 + dzd\bar{z}}{y^2} + \frac{dz^2}{4} + \frac{d\bar{z}^2}{4} + \frac{y^2}{16} dzd\bar{z},$$

which becomes the usual global AdS$_3$ metric$^{16}$ (7.3.29) via the coordinate transformation (7.3.30).

In this section, we are interested in studying bulk-boundary propagator in a heavy state background $|O_H\rangle$. The semiclassical bulk metric that’s dual to this heavy state is given by

$^{16}$The reason that we consider this specific $\phi_L$ is because it’s easier to relate the global AdS$_3$ metric to the BTZ black hole metric (since their boundaries are both cylindrical), and it enables us to circumvent some technical (numerical) issues that we encounter in [5].
equation (7.3.19), i.e.

\[ ds^2_{O_H} = \frac{dy^2 + dzd\bar{z}}{y^2} + \frac{1}{4}\alpha^2 dz^2 + \frac{1}{4}\bar{\alpha}^2 d\bar{z}^2 + \frac{\alpha^2\bar{\alpha}^2 y^2}{16} dzd\bar{z}, \]  

(7.5.4)

with \( \alpha = \sqrt{1 - \frac{24h_H}{c}} \) and \( \bar{\alpha} \) its complex conjugate\(^{17}\). This metric is related the usual BTZ black hole metric

\[ ds^2 = (r^2 - r_+^2) dt_E^2 + \frac{dr^2}{r^2 - r_+^2} + r^2 d\theta^2 \]  

(7.5.5)

via a simple coordinate transformation,

\[ z = t_E + i\theta, \quad \bar{z} = t_E - i\theta, \quad r^2 = \frac{(\alpha^2 y^2 - 4)(\bar{\alpha}^2 y^2 - 4)}{16y^2}, \]  

(7.5.6)

where \( \alpha = ir_+ \) and \( \alpha = -ir_+ \) with \( r_+ = \sqrt{\frac{24h_H}{c}} - 1 \). This is why the vacuum block \( V_0 \) of \( \langle O_H|O_L\phi_L|O_H \rangle \) has the interpretation of the bulk-boundary propagator in a BTZ black hole microstate background.

### 7.5.1 Recursion Relation on the Cylinder

In last section, we obtained the recursion relation for computing the bulk-boundary vacuum block in the configuration

\[ \langle O_H(\infty)O_H(1)\phi_L \left(u, x, \bar{x}; f(z), \bar{f}(\bar{z})\right) O_L(0) \rangle. \]  

(7.5.7)

We emphasize again that the coordinates \((0,1,\infty)\) here are on the boundary \((x, \bar{x})\) complex plane (rather than the \((z, \bar{z})\) coordinates). In order to study the bulk-boundary propagator in a heavy background in this section, we’ll consider the following configuration

\[ \langle O_H(\infty)O_H(1)\phi_L \left(u, 1-x, 1-\bar{x}; 1 - f(z), 1 - \bar{f}(\bar{z})\right) O_L(0) \rangle \]  

(7.5.8)

\(^{17}\)In this section, we mostly consider the non-rotating BTZ black holes, but most of our formulas (especially those of section 7.5.2, since we’ve kept \( \alpha \) and \( \bar{\alpha} \) independent) are easily generalized to the rotating case. In the rotating black hole case, the relations between \( \alpha, \bar{\alpha}, \bar{h}_H, \bar{\bar{h}}_H \) and \( r_+, r_- \) are a little bit more complicated.
which is equivalent to $\langle O_H|\phi_L(u,x,\bar{x}; f(z), \bar{f}(\bar{z})) O_L(1)|O_H \rangle$ due to translational symmetry on the complex plane\footnote{Another way of obtaining $\phi_L$ in (7.5.8) is to substitute $f(z)$ and $\bar{f}(\bar{z})$ with $1 - f(z)$ and $1 - \bar{f}(\bar{z})$ in equation (7.3.16), which will not change the metrics (7.5.3).}.

To obtain the bulk-boundary vacuum block $V_0$ of (7.5.8), we just need to adopt the result of last section to the special case considered here. For the configuration in (7.5.8), we have

$$u = \frac{4y\sqrt{\xi \bar{\xi}}}{4 + y^2}, \quad \Delta x = \frac{2y^2}{y^2 + 4} \xi, \quad \Delta \bar{x} = \frac{2y^2}{y^2 + 4} \bar{\xi}, \quad x_0 = 1 - \xi, \quad \bar{x}_0 = 1 - \bar{\xi}. \quad (7.5.9)$$

So we just need to plug the above expressions for $u, \Delta x, \Delta \bar{x}, x_0, \bar{x}_0$ into (7.4.8), (7.4.9) and (7.4.12) to obtain $V_0$. The Mathematica code for computing $V_0$ using the recursion relation is attached with this paper. The first several terms of $V_0$ from the recursion relation are given by

$$\begin{align*}
\frac{V_0}{\left(\frac{u}{x_0 \bar{x}_0}\right)^{2h_L}} &= 1 + \frac{2(z^2 + \bar{z}^2) h_H h_L}{c} - \frac{h_L}{72} \left(z^2 (\bar{z}^2 + 12) - 36 z\bar{z} + 12 \left(\bar{z}^2 + 12\right)\right) s \quad (7.5.10) \\
&+ \frac{h_H h_L (3z^3 \bar{z} h_L - 2z^2 (z^2 + 6) (h_L - 1) + 3z\bar{z}^3 h_L - 12z^2 (h_L - 1))}{3c} s + \cdots
\end{align*}$$

where we’ve defined $s \equiv \frac{y^2}{z \bar{z}}$ and expanded the LHS in terms of small $s, z$ and $\bar{z}$ to get the RHS. We’ve checked that this result is consistent with the semiclassical result and $1/c$ corrections to be computed in next subsection. Since the recursion relation is valid at finite $c$, we can use it to study non-perturbative physics near the black hole horizon. Due to numerical difficulties of obtaining convergent and reliable result near the horizon, we postpone it to future work.

### 7.5.2 Quantum Corrections to $\langle O_H|\phi_L O_L|O_H \rangle$ on the Cylinder

We can use bulk-boundary bi-local operators (as a generalization of the boundary bi-local operators in [168]) and the effective theory for boundary gravitons developed in [168] to
compute the semiclassical limit of $V_0$ and its $1/c$ corrections. We will briefly discuss the physical interpretation of these results at the end of this section.

**Semiclassical Result**

First, we notice that the semiclassical metric (7.5.4) can be obtained from the Poincare patch

$$ds^2 = \frac{du^2 + dx d\bar{x}}{u^2}$$

(7.5.11)

via the coordinate transformation (7.3.10) with

$$f(z) = e^{\alpha z}, \quad \bar{f}(\bar{z}) = e^{\bar{\alpha} \bar{z}},$$

(7.5.12)

where $\alpha = \sqrt{1 - \frac{24h_y}{c}}$ and $\bar{\alpha}$ is the complex conjugate of $\alpha$. This means that semiclassically, the effect of the heavy operators is trivialized via this map back to the $(u, x, \bar{x})$ coordinates\(^\text{19}\).

So the semiclassical result of $V_0$ of $\langle O_H | \phi_L (y, z_1, \bar{z}_1) O_L (z_2, \bar{z}_2) | O_H \rangle$ must be given by the bulk-boundary propagator $\langle \phi_L (u_1, x_1, \bar{x}_1) O_L (x_2, \bar{x}_2) \rangle$ in the $(u, x, \bar{x})$ coordinates, that is

$$V_0 = \left( f' (z_2) \bar{f}' (z_2) \right)_{hl} \left( \frac{u_1}{u_1^2 + (x_1 - x_2)(\bar{x}_1 - \bar{x}_2)} \right)^{2h_L} + O \left( \frac{1}{c} \right)$$

(7.5.13)

where the factor $\left( f' (z_1) \bar{f}' (z_1) \right)_{hl}$ comes from the transformation rule for the primary operator $O_L$ and

$$u_1 = \frac{4y \sqrt{\alpha \bar{\alpha} e^{\alpha z_1 + \bar{\alpha} \bar{z}_1}}}{\alpha \bar{\alpha} y^2 + 4}, \quad x_1 = \frac{e^{\alpha z_1} (4 - \alpha \bar{\alpha} y^2)}{\alpha \bar{\alpha} y^2 + 4}, \quad \bar{x}_1 = \frac{e^{\bar{\alpha} \bar{z}_1} (4 - \alpha \bar{\alpha} y^2)}{\alpha \bar{\alpha} y^2 + 4}$$

and $x_2 = e^{\alpha z_2}, \bar{x}_2 = e^{\bar{\alpha} \bar{z}_2}$. For later convenience, we’ll denote the semi-classical result (the first term in (7.5.13)) as $V_0^{sc}$, and in terms of $(y, \xi, \bar{\xi})$ with $\xi \equiv e^{z_1 - z_2}$ and $\bar{\xi} \equiv e^{\bar{z}_1 - \bar{z}_2}$, it’s

\(^{19}\)Note that the $(u, x, \bar{x})$ in this subsection are different from those of last subsection since $f(z), \bar{f}(z)$ are different now.
given by [179, 5]

\[
\mathcal{V}_0^{sc} = \left( \frac{16 g^2 a^2 \bar{a}^2 \xi \bar{\xi}}{(4 (1 - \xi \bar{\xi}) (1 - \bar{\xi}^2) + y^2 a \bar{a} (1 + \xi) (1 + \bar{\xi})^2)^2} \right)^{h_L} \tag{7.5.14}
\]

The configuration of last subsection corresponds to \( z_1 = z \) and \( z_2 = 0 \), so the \( \xi \) and \( \bar{\xi} \) defined here are the same as those of last subsection.

1/c Corrections

In order to compute the 1/c corrections (gravitational one-loop corrections) to the semiclassical result of \( \mathcal{V}_0 \), we must include perturbations in \( f(z) \) and \( \bar{f}(\bar{z}) \), and be more precise about the central charge \( c \). It turns out that, as in [168], we should use the following \( f(z) \) and \( \bar{f}(\bar{z}) \)

\[
f(z) = e^{\alpha_0 z + \frac{ix(z)}{\sqrt{C}}} \quad \text{and} \quad \bar{f}(\bar{z}) = e^{\bar{\alpha}_0 \bar{z} + \frac{i\bar{x}(\bar{z})}{\sqrt{C}}} \tag{7.5.15}
\]

where \( C = c - 1 \), \( \alpha_0 = \sqrt{1 - 24 \frac{h \mu}{C}} \) and \( \bar{\alpha}_0 \) is the complex conjugate of \( \alpha_0 \). Here \( \epsilon \) and \( \bar{\epsilon} \) are to be understood as operators. We then obtain the large \( c \) expansion (with \( \frac{h \mu}{c} \) fixed) of \( \mathcal{V}_0 \) via

\[
\mathcal{V}_0 = \left\langle \left( f'(z_2) \bar{f}'(\bar{z}_2) \right)^{h_L} \left( \frac{u_1}{u_1^2 + (x_1 - x_2)(\bar{x}_1 - \bar{x}_2)} \right)^{2h_L} \right\rangle \tag{7.5.16}
\]

upon plugging in the expressions of various terms using (7.3.10) with \( f \) and \( \bar{f} \) of (7.5.15), and expanding in large \( c \) or small \( \epsilon, \bar{\epsilon} \). The idea here is roughly the same as the idea of the bulk-boundary OPE blocks used in [3] to compute \( \langle \phi OT \rangle \) and in [5] to compute the large \( c \) expansion of vacuum block of \( \langle O_H \bar{O}_H \phi_L \bar{O}_L \rangle \) (with \( h_H \sim O(1) \)) in Poincare AdS3. It’s also a generalization of the boundary bi-local operators in [168] to the bulk-boundary case.
At leading order of the large $C$ limit (with $h_H \sim \mathcal{O}(C)$), we have

$$
\mathcal{V}_0 = \frac{16y^2\alpha_0^2\tilde{\alpha}_0^2\xi\bar{\xi}}{(4(1 - \xi\bar{\xi})(1 - \bar{\xi}^2) + y^2\alpha_0\bar{\alpha}_0(1 + \xi\bar{\xi})(1 + \bar{\xi}^2))^2} h_L \left(1 + \mathcal{O}\left(\frac{1}{C}\right)\right) \tag{7.5.17}
$$

where $\xi \equiv e^{z_1 - z_2}$ and $\bar{\xi} \equiv e^{\bar{z}_1 - \bar{z}_2}$. At order $1/c$, we’ll get two different kinds of contributions, one from the $\epsilon$ and $\bar{\epsilon}$ terms, and another from the large $c$ expansion of the leading order result (recalling that $C = c - 1$). Since the terms linear in $\epsilon$ or $\bar{\epsilon}$ will have zero expectation value, we’ll only keep the terms quadratic in $\epsilon$ or $\bar{\epsilon}$ in the large $c$ expansion. We can then package the order $1/c$ contributions to $\mathcal{V}_0$ in the following form:

$$
\mathcal{V}_0 = \mathcal{V}_0^{sc} \left(1 + \frac{h_L}{c} \mathcal{V}_{h_L/c} + \frac{h_L^2}{c} \mathcal{V}_{h_L^2/c} + \mathcal{O}\left(\frac{1}{c^2}\right)\right) \tag{7.5.18}
$$

where $\mathcal{V}_0^{sc}$ is the semiclassical result given in (7.5.14). The order $1/c$ terms of the large $c$ expansion of (7.5.17) will be linear in $h_L$ and will be included in $\mathcal{V}_{h_L/c}$.

The order $h_L/c$ term $\mathcal{V}_{h_L/c}$ of (7.5.18) is given by

$$
\mathcal{V}_{h_L/c} = \frac{96}{D^2} (1 - \bar{\xi}^2) \left(\mathcal{V}_{h_L/c}^{(0)} + 4\alpha\bar{\alpha}(1 - \bar{\xi}^2) \mathcal{V}_{h_L/c}^{(2)} \right) y^2 + \bar{\alpha}^2 (1 + \bar{\xi}^2)^2 \mathcal{V}_{h_L/c}^{(4)} y^4 + \frac{\mathcal{V}_{h_L/c}^{sc} + \mathcal{O}(1/c)}{D} + c.c. \tag{7.5.19}
$$

where we’ve defined

$$
D \equiv 4(1 - \xi^2)(1 - \bar{\xi}^2) + y^2\alpha\bar{\alpha}(1 + \xi^2)(1 + \bar{\xi}^2) \tag{7.5.20}
$$

for notational convenience and the various terms in the numerator are given by

$$
\mathcal{V}_{h_L/c}^{(0)} = \frac{1}{12\alpha^2} (1 - \xi^2)^2 ((\epsilon')^2 + (\epsilon''')^2) - \frac{1}{6}\xi^2(\epsilon_1 - \epsilon_2)^2 \tag{7.5.21}
$$

$$
\mathcal{V}_{h_L/c}^{(2)} = \frac{1}{\alpha^3} (4\alpha^2\xi^2(\epsilon_1 - \epsilon_2)\epsilon' + (1 - \xi^2)(2(\xi^2 - 1)\epsilon'\epsilon'' + \alpha(\xi^2 + 1)((\epsilon'))^2 + (\epsilon'')^2))
$$

$$
\mathcal{V}_{h_L/c}^{(4)} = \frac{1}{2} (\xi^2 + 1)^2 ((\epsilon'')^2 - (\epsilon')^2) + \xi^2(\alpha^2(\epsilon_1 - \epsilon_2) - 4\epsilon''')(\epsilon_1 - \epsilon_2) + \frac{4\alpha(2\xi^2 - 1)\epsilon'_{\epsilon'''}(2\xi^2 - 1)\epsilon'''}{\alpha^2}
$$

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with $\epsilon_1 \equiv \epsilon(z_1)$ and $\epsilon_2 \equiv \epsilon(z_2)$ and the primes means the derivatives with respect to their arguments, respectively. The complex conjugate c.c. in Equation (7.5.19) means replacing $\xi, \bar{\xi}, \alpha, \bar{\alpha}$ with $\bar{\xi}, \xi, \bar{\alpha}, \alpha$, respectively, and also changing $\epsilon$ to $\bar{\epsilon}$. The $V_{h_L/c}^{s-c}$ term plus its complex conjugate is from the $1/c$ expansion of (7.5.17), and it’s given by

$$V_{h_L/c}^{s-c} = \left(1 - \alpha^2\right) \left(\frac{2 \left(1 - \xi \bar{\alpha}\right) \left(2\xi\alpha - \alpha \left(1 + \xi\alpha\right) \log(\xi) - 2\right)}{\alpha'^2} - \frac{y^2}{2} \bar{\alpha} \left(1 - \xi\alpha\right) \left(1 + \xi\bar{\alpha}\right) \log(\xi)\right).$$  \hspace{1cm} (7.5.22)

The order $h_L^2/c$ term $V_{h_L/c}^{s-c}$ of (7.5.18) is given by

$$V_{h_L/c} = \frac{\langle V_{h_L/c}^{(0)} \rangle + \langle V_{h_L/c}^{(2)} \rangle y^2 + \langle V_{h_L/c}^{(4)} \rangle y^4}{D^2} + \text{c.c.,} \hspace{1cm} (7.5.23)$$

where $D$ is given by (7.5.20). The numerator is given by

$$\langle V_{h_L/c}^{(0)} \rangle + \langle V_{h_L/c}^{(2)} \rangle y^2 + \langle V_{h_L/c}^{(4)} \rangle y^4 = \frac{1}{-2\alpha^2} \left\langle \left(V_1 + y^2V_2\right)^2\right\rangle \hspace{1cm} (7.5.24)$$

with

$$V_1 = 4 \left(\bar{\xi}\bar{\alpha} - 1\right) \left(\alpha (\epsilon_1 - \epsilon_2) (\xi\alpha + 1) + (1 - \xi\alpha) (\epsilon_1' + \epsilon_2')\right), \hspace{1cm} (7.5.25)$$

$$V_2 = \bar{\alpha} \left(\xi\bar{\alpha} + 1\right) \left(\xi^2 (\xi\alpha - 1) (\epsilon_1 - \epsilon_2) + \alpha (\xi\alpha + 1) (\epsilon_1' - \epsilon_2') + 2 \left(1 - \xi\alpha\right) \epsilon_1''\right),$$

Now to compute $V_{h_L/c}$ and $V_{h_L^2/c}$, we need the $\epsilon$ propagator. This is worked out in [168] using the effective theory for boundary gravitons developed in that paper, and it’s given by\(^{20}\)

$$\langle \epsilon(z_1)\epsilon(z_2) \rangle = \frac{6}{C} \left(2 \ln(1 - \xi) + \Phi(\xi, 1, \alpha) + \Phi(\xi, 1, -\alpha)\right), \hspace{1cm} \xi = e^{z_1 - z_2} \hspace{1cm} (7.5.26)$$

\(^{20}\)There is a subtlety about the ordering of the $\epsilon$ operators in the $\epsilon$ two-point function. When computing $V_0$, we substitute $\langle \epsilon(z_1)\epsilon(z_2) \rangle$ with the symmetric average of the two different ordering $\frac{1}{2} (\langle \epsilon(z_1)\epsilon(z_2) \rangle + \langle \epsilon(z_2)\epsilon(z_1) \rangle)$. This procedure gives a result matching the recursion relation.
where $\Phi(z, s, a)$ is the Lerch transcendant \(^{21}\)

$$
\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n + a)^s}.
$$

(7.5.28)

Using the $\epsilon$ propagator (7.5.26), we can evaluate $\mathcal{V}_{hL/c}$ and $\mathcal{V}_{hL^2/c}$, but they are logarithmically divergent and need to be renormalized. We follow the procedure in [168], and define the renormalized expectation value of the $\langle \mathcal{V}^{(i)}_{hL/c} \rangle$ as follows

$$
\langle \mathcal{V}^{(i)}_{hL/c} \rangle_R = \langle \mathcal{V}^{(i)}_{hL/c} \rangle - \left( \langle \mathcal{V}^{(i)}_{hL/c} \rangle_{\alpha \to 1, \bar{\alpha} \to 1} \right)_{w \to \alpha w, \bar{w} \to \bar{\alpha} \bar{w}, y \to y \sqrt{\alpha \bar{\alpha}}}
$$

(7.5.29)

and similarly for $\langle \mathcal{V}^{(i)}_{hL^2/c} \rangle_R$.

Since the result of $\mathcal{V}_0$ only depends on $z_1 - z_2$, we’ll set $z_1 = z$ and $z_2 = 0$. Eventually, the $\langle \mathcal{V}^{(i)}_{hL/c} \rangle_R$ terms are given by

$$
\langle \mathcal{V}^{(0)}_{hL/c} \rangle_R = e^{\alpha z} \left( \mathcal{F}_1 + 4 \log \alpha + 4 \log \left( 2 \sinh \left( \frac{z}{2} \right) \right) + 2 \right) + (1 - e^{2\alpha z}) \alpha z
$$

(7.5.30)

$$
+ \left( 1 + e^{2\alpha z} \right) \left( H_{-\alpha} + H_{\alpha} - 1 + i\pi - 2 \log \alpha \right) + 2 \left( 1 - e^{\alpha z} \right)^2 \log \left( 2 \sinh \left( \frac{\alpha z}{2} \right) \right),
$$

$$
\langle \mathcal{V}^{(2)}_{hL/c} \rangle_R = \left( 1 - e^{2\alpha z} \right) \left( H_{-\alpha} + H_{\alpha} + i\pi - 1 - 2 \log \alpha + 2 \log \left( 2 \sinh \left( \frac{\alpha z}{2} \right) \right) \right)
$$

$$
+ \left( 1 + e^{2\alpha z} \right) \alpha z + e^{\alpha z} \mathcal{F}_2,
$$

(7.5.31)

$$
\langle \mathcal{V}^{(4)}_{hL/c} \rangle_R = (1 - \xi^\alpha)^2 + 6 \alpha^3 \left( 1 - e^{2\alpha z} \right) z
$$

(7.5.32)

$$
+ 6 \alpha^2 \left( 1 + e^{2\alpha z} \right) \left( H_{-\alpha} + H_{\alpha} + 2 \log \left( 2 \sinh \left( \frac{\alpha z}{2} \right) \right) - 2 \log \alpha - \frac{7}{6} + i\pi \right)
$$

$$
+ 6 \alpha^2 e^{\alpha z} \left( \mathcal{F}_1 - 4 \log \alpha - 4 \log \left( 2 \sinh \left( \frac{z}{2} \right) \right) + 4 \log \left( 2 \sinh \left( \frac{\alpha z}{2} \right) \right) + \frac{7}{3} \right),
$$

\(^{21}\)For $s = 1$ it is related to a certain incomplete Beta function as

$$
B(z, a, 0) = z^a \Phi(z, 1, a).
$$

(7.5.27)
and the $\langle \mathcal{V}_{h_L^2/c}^{(i)} \rangle_R$ terms are given by

$$
\langle \mathcal{V}_{h_L^2/c}^{(0)} \rangle_R = 192 \left( 1 - e^{\bar{\alpha} \bar{z}} \right)^2 \left( \left( \log \sinh \left( \frac{z}{2} \right) - \log \sinh \left( \frac{\alpha z}{2} \right) \right) + \log \alpha + 1 \right) \left( 1 - e^{\alpha z} \right)^2 \\
+ \mathcal{F}_3 + 2e^{\alpha z} \left( H_{-\alpha} + H_{\alpha} + 2 \log \left( 2 \sinh \left( \frac{z}{2} \right) \right) + i\pi \right), 
$$

(7.5.33)

$$
\langle \mathcal{V}_{h_L^2/c}^{(2)} \rangle_R = 48\tilde{\alpha} \left( 1 - e^{\alpha z} \right) \left( 1 - e^{2\tilde{\alpha} \bar{z}} \right) \left( \coth \left( \frac{z}{2} \right) \left( 1 - e^{\alpha z} \right) + 2\alpha \left( e^{\alpha z} + 1 \right) \left( \log \alpha - \log \left( \sinh \left( \frac{\alpha z}{2} \right) \right) + \log \left( \sinh \left( \frac{z}{2} \right) \right) + \frac{1}{2} \right) \right), 
$$

(7.5.34)

$$
\langle \mathcal{V}_{h_L^2/c}^{(4)} \rangle_R = 2\alpha^2 \tilde{\alpha}^2 \left( 1 + e^{\tilde{\alpha} \bar{z}} \right)^2 \left( \frac{6}{\alpha} \coth \left( \frac{\alpha z}{2} \right) \left( 1 - e^{2\alpha z} \right) + 6 \log \alpha \left( 1 + e^{\alpha z} \right)^2 - e^{2\alpha z} + 6 \left( 1 - e^{\alpha z} \right)^2 \left( \log \left( \sinh \left( \frac{z}{2} \right) \right) - \log \left( \sinh \left( \frac{\alpha z}{2} \right) \right) + \frac{1}{6\alpha^2} \right) - 6\mathcal{F}_3 \\
- 1 - 12e^{\alpha z} \left( H_{-\alpha} + H_{\alpha} + 2 \log \left( 2 \sinh \left( \frac{\alpha z}{2} \right) \right) + i\pi - \frac{13}{6} \right) \right). 
$$

(7.5.35)

where we’ve define

$$
\mathcal{F}_1 \equiv \Phi \left( e^z, 1, \alpha \right) + \Phi \left( e^z, 1, -\alpha \right) + \Phi \left( e^{-z}, 1, \alpha \right) + \Phi \left( e^{-z}, 1, -\alpha \right) \\
\mathcal{F}_2 \equiv \Phi \left( e^z, 1, \alpha \right) - \Phi \left( e^z, 1, -\alpha \right) - \Phi \left( e^{-z}, 1, \alpha \right) + \Phi \left( e^{-z}, 1, -\alpha \right) \\
\mathcal{F}_3 \equiv e^{2\alpha z} \Phi \left( e^z, 1, \alpha \right) + \Phi \left( e^z, 1, -\alpha \right) + \Phi \left( e^{-z}, 1, \alpha \right) + e^{2\alpha z} \Phi \left( e^{-z}, 1, -\alpha \right)
$$

for notational convenience.

We checked that these results agree with the recursion relation of last subsection by expanding in small $s \equiv \frac{y^2}{z^2}$, $z$ and $\bar{z}$. We have also verified that when we take $\phi_L$ to the boundary, we recover the $1/c$ correction [20] to the heavy-light vacuum block [15].

---

22The branch cuts in the various functions in the equations (7.5.30)-(7.5.35) were chosen such that when they are expanded in small $z$ assuming $z > 0$ in Mathematica, the result matches that of the recursion relation. One should be careful when evaluating (7.5.30)-(7.5.35) for $z < 0$. 

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One-Loop Corrections Near the Black Hole Horizon

We would like to see what the semiclassical result and the $1/c$ corrections computed here tell us about physics near the Euclidean black hole horizon. For the non-rotating case considered here, the horizon is at $r = r_+ = \sqrt{\frac{24h_H}{c} - 1}$ (which corresponds to $y = \frac{2}{r_+}$). In this case, the semi-classical result $V_0^{s-c}$ of equation (7.5.14) written in terms of the $(r, t_E, \theta)$ (using the coordinate relations (7.5.6)) is given by

$$V_0^{s-c} = \left(\frac{r_+}{2}\right)^{2h_L} \frac{1}{\left(\frac{r}{r_+} \cosh (r_+ \theta) - \sqrt{r^2 - r_+^2} \cos (r_+ t_E)\right)^{2h_L}}. \quad (7.5.36)$$

which is the $n = 0$ terms in the full semiclassical bulk-boundary correlator for a free field in a BTZ black hole given by the image sum in [179]. So one can see that the semiclassical result is periodic in $t_E$, and it’s smooth at the horizon $r = r_+$ (and its dependence on $t_E$ drops out there).

In terms of $(y, z, \bar{z})$, the horizon is at $y = \frac{2}{r_+}$, and at this value of $y$, the $\frac{1}{c}$ corrections $V_{h_1/c}$ and $V_{h_2/c}$ to the vacuum block of $\langle \mathcal{O}_H | \phi_L \mathcal{O}_L | \mathcal{O}_H \rangle$ are finite, since their numerators truncate at order $y^4$, and their denominators are just the same as the denominator of the semiclassical result $V_0^{s-c}$, which is also non-singular at this value of $y$. However, unlike the semiclassical vacuum block, the functions $V_{h_1/c}$ and $V_{h_2/c}$ are not periodic in Euclidean time [20, 12]. This means that the $1/c$ correction to the bulk-boundary heavy-light correlator will have a branch point at the Euclidean horizon. So the singularity of these correlators at the Euclidean horizon arises already in perturbation theory, and does not require non-perturbative effects [5].

7.6 Discussion

Our primary goal has been to develop an exact bulk reconstruction procedure with very general gravitational dressings. The motivation was to enable future investigations into
the dressing-dependence of bulk observables, as these ambiguities present a major caveat when drawing physical conclusions. For example, using our results it should be possible to determine if the breakdown of bulk locality at short-distances in AdS$_3$ [4] persists with a general class of gravitational dressings. We can also investigate BTZ black hole horizons [5], though the necessary numerics may be rather formidable. We took the first steps in this direction in section 7.5. By exploiting the connection between the singularity structure of CFT stress-tensor correlators and gravitational dressings, it may be possible to generalize some of our results to higher dimensions.

Our reconstruction procedure only incorporates effects arising as a mandatory consequence of Virasoro symmetry. With hard work one could add other perturbative interactions, but such methods would likely just reproduce bulk perturbation theory, without providing a deeper understanding of quantum gravity. So our methods are limited, as they are only able to address certain universal features of quantum gravity. Unfortunately, in the case of quantum gravity it would seem that we must either solve toy models completely, and then try to argue that they are representative, or solve a universal sector of a general class of models, and try to argue that the effects from this sector determine the relevant physics. Given the universal nature of the gravitational force, we believe that the latter route is a more compelling way to address locality and near-horizon dynamics.

Most work on bulk reconstruction suffers from a nagging conceptual problem. As physical observers, we do not setup experiments by making reference to the boundary of spacetime. And defining the bulk by reference to the boundary seems even more perverse in a cosmological setting. Furthermore, it has been shown that using the boundary as a reference point leads to fundamental problems, such as bulk fields that do not commute outside the light-cone [155], even at low-orders in gravitational perturbation theory. Perhaps a more sensible approach defines observables relative to other objects in spacetime, just as we define our local reference frame with respect to the earth, solar system, galaxy, and galactic neighborhood. This also seems more in keeping with interpretations of the Wheeler-DeWitt equation [143].
We hope to formalize such a definition of local observables in future work.
Appendix A

Appendix to Chapter 2

A.1 Summary of Corrections to the Vacuum Block

In this appendix, we will list results concerning the large $c$ expansion of the vacuum block.

At order $c^0$ and $1/c$, $f_{00}$ (2.3.13) and $f_{10}, f_{01}$ are known in closed form. The functions $f_{10}$ and $f_{01}$ are known from the work of [20] to be

$$f_{10} = \frac{\csch^2 \left( \frac{\alpha t}{2} \right)}{2} \left[ 3 \left( e^{-\alpha t} B(e^{-t}, -\alpha, 0) + e^{\alpha t} B(e^{-t}, \alpha, 0) + e^{\alpha t} B(e^{t}, -\alpha, 0) + e^{-\alpha t} B(e^{t}, \alpha, 0) \right) ight]$$

$$+ \frac{1}{\alpha^2} + \cosh(\alpha t) \left( -\frac{1}{\alpha^2} + 6H_{-\alpha} + 6H_\alpha + 6i\pi - 5 \right) + 12 \log \left( 2 \sinh \left( \frac{t}{2} \right) \right) + 5$$

$$- t \frac{(13\alpha^2 - 1) \coth \left( \frac{\alpha t}{2} \right)}{2\alpha} + 12 \log \left( \frac{2 \sinh \left( \frac{\alpha t}{2} \right)}{\alpha} \right), \quad (A.1.1)$$

$$f_{01} = 6 \left( \frac{\alpha}{2} \right)^2 \left[ B(e^{-t}, -\alpha, 0) + B(e^{t}, -\alpha, 0) + B(e^{-t}, \alpha, 0) + B(e^{t}, \alpha, 0) \right]$$

$$+ H_{-\alpha} + H_\alpha + 2 \log \left( 2 \sinh \left( \frac{t}{2} \right) \right) + i\pi \right] + 2 \left( \log \left( \alpha \sinh \left( \frac{t}{2} \right) \right) \right) \right) + 1 \right).$$

where $B(x, \beta, 0) = y^{\beta} \frac{\bf{1}_{\beta}(1, \beta, 1 + \beta, x)}{\beta}$ is the incomplete Beta function, $z \equiv 1 - e^{-t}$, $\alpha = \sqrt{1 - 24\eta_H}$, $H_n$ is the harmonic function.

At order $1/c^2$, we calculated the order $\eta_H$ and $\eta_H^2$ terms in the expansion of the vacuum block in the parameter $\eta_H = \frac{hu}{c}$, and at order $1/c^3$, we calculated the linear $\eta_H$ terms.

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At order $1/c^2$,

$$\log \mathcal{V} \supset \frac{h_L}{c^2} \sum_{k=0}^{\infty} \eta_H^{k+1} \left( f_{20k} + h_L f_{11k} + h_L^2 f_{02k} \right).$$

The linear $\eta_H$ terms are $f_{200}, f_{110} = f_{101}$ and $f_{020} = f_{002}$. This first term is given in equation (2.3.17), while the last two terms can be obtained from the expansion of $f_{10}$ and $f_{00}$. The $\eta_H^2$ terms are

$$f_{201} = \frac{432 \left( (z - 1)z(15z - 46) + 4\pi^2(z((z - 2)z - 10) + 12) \right) \log^2(1 - z)}{z^3} + \frac{864(z(z(5z - 44) + 103) - 96) + 33 \log^4(1 - z)}{z^4} + \frac{10368(z - 2)^2 \text{Li}_2(z)^2}{z^2} + \frac{864((9z - 46)(z - 1)^2 + (4z(15 - 2(z - 2)z) - 72) \log(z)) \log^3(1 - z)}{z^3} + \frac{5184\text{Li}_2(z) \log(1 - z)(z(z(7z - 32) + 32) + 2(z - 9)(z - 2)(z - 1) \log(1 - z))}{z^3} + \frac{5184\text{Li}_3(1 - z)(z(z(5z - 14) + 16) - 4(z - 3)(z - 1)(z + 2) \log(1 - z))}{z^3} + \frac{10368\text{Li}_3(z)((z - 2)z + 2((z - 8)z + 8) \log(1 - z))}{z^2} + \frac{20736(z - 2) \text{Li}_4(1 - z)}{z} + \frac{20736((z - 6)z + 6) \left( \text{Li}_4 \left( \frac{z}{z-1} \right) + \text{Li}_4(z) \right)}{z^2} + \frac{12960(z - 2) \text{Li}_2(z)}{z} + \frac{216((z - 2)z^2 + 96(6 - 5z)\zeta(3) - 4\pi^2(z(5z - 14) + 16)z) \log(1 - z)}{z^3} - \frac{144(-525z^2 + 180(z(5z - 14) + 16)\zeta(3) + 8\pi^4(z - 2)z)}{5z^2} + \frac{2592(z(5z - 14) + 16) \log(z) \log^2(1 - z)}{z^2}.

(A.1.2)
\begin{align*}
f_{111} = & \frac{864((3z(z^2 - 8z + 7) + 8\pi^2(2z^2 - 9z + 8)))\log^2(1 - z)}{z^3} + \frac{5184(z - 2)\text{Li}_2(z)}{z} \\
+ & \frac{1728(z(z(3z - 44) + 127) - 136) + 51)\log^4(1 - z)}{z^4} - \frac{41472(z - 1)\text{Li}_2(z)^2}{z^2} \\
+ & \frac{3456((z - 1)((z - 17)z + 21) - 6(z(2z - 9) + 8)\log(z))\log^3(1 - z)}{z^3} \\
+ & \frac{10368((z - 7)z + 7)\log(z)\log^2(1 - z)}{z^2} - \frac{41472((z - 6)z + 6)\text{Li}_3(z)\log(1 - z)}{z^2} \\
+ & \frac{20736\text{Li}_2(z)\log(1 - z)(z((z - 9)z + 9) + (z((z - 14)z + 34) - 22)\log(1 - z))}{z^3} \\
+ & \frac{20736\text{Li}_3(1 - z)(z((z - 7)z + 7) - 2(z(2z - 9) + 8)\log(1 - z))}{z^3} \\
+ & \frac{41472((z - 6)z + 6)\left(\text{Li}_4(z) + \text{Li}_4\left(\frac{z}{z - 1}\right)\right)}{z^2} + 20736\left(2 - \frac{((z - 7)z + 7)\zeta(3)}{z^2}\right) \\
+ & \frac{432(3(z - 2)z^2 + 96(z(2z - 9) + 8)\zeta(3) - 8\pi^2((z - 7)z + 7)z)\log(1 - z)}{z^3},
\end{align*}

(A.1.3)

and \( f_{021} = f_{012} \) can be obtained from the expansion of \( f_{01} \).

At order \( 1/c^3 \),

\[
\log \mathcal{V} \supset \frac{h_L}{c^3} \sum_{k=0}^{\infty} \eta_H^{k+1} \left( f_{30k} + h_L f_{21k} + h_L^2 f_{12k} + h_L^3 f_{03k} \right),
\]

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The linear $\eta_H$ terms are

\[
\begin{align*}
f_{300} &= \frac{864(2z(z^2 - 8z + 17) - 14) + 9 \log(1 - z)}{z^4} - \frac{20736(z - 1)\text{Li}_2(z)^2}{z^2} \\
&\quad + \frac{216 \log^2(1 - z) (8\pi^2 (z^2 - 2) (z^2 - 2) - 108z (z^2 - 1) \log(z) + 73z(z - 1)^2)}{z^3} \\
&\quad - \frac{864(z - 2) \log^3(1 - z) (4 (2z^2 - 3) \log(z) - 9(z - 1)^2)}{z^3} \\
&\quad + \frac{432 \text{Li}_2(z) (73(z - 2)z^2 + 24(z - 1) \log(1 - z)((6 - 4z) \log(1 - z) - 9z))}{z^3} \\
&\quad + \frac{5184(z^2 - 1) \text{Li}_3(z)(9z + 4(z - 2) \log(1 - z))}{z^3} - \frac{20736(2z - 3) \text{Li}_4(z)}{z^2} \\
&\quad - \frac{5184(z - 2) \text{Li}_3(z)(9z + 4(z - 2) \log(1 - z))}{z^2} + \frac{20736(z - 3)(z - 1) \text{Li}_4(z)}{z^2} \\
&\quad + \frac{20736(z - 2) \text{Li}_4(1 - z)}{z} + \frac{192 (1215 (z^2 - 1) \zeta(3) + 320z^2 - 6\pi^4(z - 2)z)}{5z^2} \\
&\quad + \frac{12 ((z - 2) (z^2 - 1728\zeta(3)) + 648\pi^2z(z^2 - 1)) \log(1 - z)}{z^3}, \\
\end{align*}
\]

(A.1.4)

$f_{210} = f_{201}$, and $f_{120} = f_{102}$, $f_{030} = f_{003}$ can be obtained from the expansions of $f_{10}$ and $f_{00}$, respectively.

The terms $f_{200}, f_{201}, f_{111}$ and $f_{300}$ were derived for the first time in this work. We’ve checked these expressions against a direct small $z$ expansion up to $O(z^9)$ using the methods of [11]. We’ve also analytically continue these results to the second sheet and checked that the they do contain the first few terms of (2.4.2) and (2.4.12). Under this analytic continuation,
the various logarithms and polylogarithms have monodromies

\[ \log(1 - z) \rightarrow \log(1 - z) - 2\pi i, \]

\[ \text{Li}_n(z) \rightarrow \text{Li}_n(z) + \frac{2\pi i}{(n - 1)!} \log^{n-1}(z), \]

\[ \text{Li}_n(1 - z) \rightarrow \text{Li}_n(1 - z), \]

\[ \text{Li}_n \left( \frac{z}{z - 1} \right) \rightarrow \text{Li}_n \left( \frac{z}{z - 1} \right) - \frac{2\pi i}{(n - 1)!} \log^{n-1} \left( \frac{z}{z - 1} \right), \]

which can be derived from \( \text{Li}_n(z) = \int_0^z \frac{\text{Li}_{n-1}(t)}{t} \, dt \) and \( \text{Li}_1(z) = -\log(1 - z) \).

### A.2 Direct Derivation of Leading Logs in the Lorentzian Regime

In subsection 2.4.1, we presented a proof that the “leading logs” in the Lorentzian regime resum to form a correction to the leading singularity \((cz)^{-1}\) that appears at \( \mathcal{O}(1/c) \) in a large \( c \) expansion. The proof given was somewhat indirect, however, and in this appendix we will give another proof that is more cumbersome, but more directly connected to the structure of the differential equations for the degenerate operators that are used order-by-order in \( 1/c \) in the rest of the paper. In this appendix, for convenience we define

\[ \bar{\mathcal{V}}(z) = z^{2h_L} \mathcal{V}(z), \]  

(A.2.1)

so that for the vacuum block, \( \bar{\mathcal{V}}(z) \xrightarrow{z \rightarrow 0} 1 \) on the first sheet.

From equation (2.3.7), at large \( c \) the null equation of motion for the degenerate operator \( \mathcal{O}_{1,s} \) takes the form

\[ (L^s_{-1} + \mathcal{O}(1/c))\mathcal{O}_{1,s} = 0. \]  

(A.2.2)
In terms of $\tilde{V}$, (A.2.2) translates into the differential equation

$$(\partial_z^s + O(1/c))\left(z^{s-1}\tilde{V}(z)\right) = 0.$$ (A.2.3)

We will organize the solution to (2.3.8) in a series expansion of $\tilde{V}$:

$$\tilde{V}(z) \equiv \tilde{V}_0(z) + \frac{1}{c}\tilde{V}_1(z) + \frac{1}{c^2}\tilde{V}_2(z) + \ldots.$$ (A.2.4)

The lowest-order term then obey the following differential equation

$$\partial_z^s(z^{s-1}\tilde{V}_0(z)) = 0,$$ (A.2.5)

whose general solution takes the form

$$\tilde{V}_0(z) = \sum_{i=0}^{s-1} \frac{c_i}{z^i}.$$ (A.2.6)

with $s$ free coefficients $c_i$. Of course, the relevant solution for the vacuum at $c \to \infty$ is $c_0 = 1, c_i \neq 0 = 0$. But equally importantly, when we work to higher orders, all the solutions above will continue to be homogeneous solutions, and there will also be one particular solution at each order that arises because of the “source” from the lower order terms.

There is a drastic simplification that occurs if we are interested only in the leading log terms. First, notice that none of the homogeneous solutions (A.2.6) have logarithms in them. As a result, logarithms can be produced only by the “particular” solutions, which are integrals of the lower-order solutions. More precisely, the leading logarithms arise from integrating the lowest order solution and never introducing any “homogeneous” terms, since doing so would reduce the power of the logarithm. Therefore, we can simply perform our analysis directly on the second sheet (where the differential equation must still be satisfied), and the unknown integration constants that enter at each step will not contaminate the
leading logs.

Using the expression (2.3.9), it is straightforward to extend this argument to leading logs in the heavy-light limit as well. At infinite $c$, the general solution to (2.3.9) is

$$\tilde{V}(t) = e^{\frac{c}{2} \sum_{j=0}^{s-1} c_j \exp \left[ t \left( 2j - \frac{s-1}{2} \right) \sqrt{1-24\eta_H} \right]}. \quad (A.2.7)$$

These are all exponentials in $t$, i.e. powers in $(1-z)$. Therefore, logarithms of $z$ can arise only from integrating source terms that are generated from the solution at lower orders.

Let us see how this works in practice, and along the way we will illustrate some points. For simplicity, we will begin by solving for the leading logarithms in the conformal block $\langle \mathcal{O}_H \mathcal{O}_{1,2} \mathcal{O}_{1,2} \rangle$. Once we have gone through this case, it will be easy to see how to generalize to arbitrary degenerate operators.

The exact equation of motion for $\tilde{V}(z)$ is

$$0 = (z - 1) \left( (-4(z - 2)h_{1,2} + 4z - 2) \tilde{V}'(z) + 3(z - 1)z\tilde{V}''(z) \right) - 2z(2h_{1,2} + 1) \tilde{V}(z) h_H \quad (A.2.8)$$

This can be solved in closed form by a hypergeometric function, but to illustrate our points we will solve it in a $1/c$ expansion. At leading order it is just (A.2.5) with $s = 2$. At next order, it is

$$\partial_z^2 (z \tilde{V}_1(z)) = -\frac{6h_H z}{(1-z)^2}, \quad (A.2.9)$$

which is easily solved:

$$\tilde{V}_1(z) = c_0 + \frac{c_1}{z} - \frac{6(z-2)h_H \log(1-z)}{z} \quad (A.2.10)$$

We fix $c_0$ and $c_1$ on the first sheet by demanding that $\tilde{V}_1$ have the correct behavior (i.e., have leading term $\propto z$ in a small $z$ expansion), and then analytically continuing to the second
sheet and taking small \( z \) to find the small \( z \) behavior on the second sheet. Doing this, we find \( c_0 = 12h_H, c_1 = 0 \) on the first sheet. Analytically continuing, this means that on the second sheet,

\[
c_1 = 24i\pi h_H, \quad c_0 = -12i\pi h_H. \tag{A.2.11}
\]

Note that at this order, there are no logarithms \( \log(z) \) in a small \( z \) expansion, even on the second sheet:

\[
\tilde{V}_1(z) = \frac{c_1}{z} + (c_0 - 12h_H) + \mathcal{O}(z). \tag{A.2.12}
\]

To see the emergence of logarithms, we have to work to the next order in \( 1/c \). The equation of motion for \( \tilde{V}_2 \) is

\[
\partial_z^2(z\tilde{V}_2(z)) = -\frac{6h_H z}{(1-z)^2} - \frac{6 \left(z\tilde{V}_1(z)h_H + (z-2)(z-1)\tilde{V}_1'(z)\right)}{(z-1)^2} \tag{A.2.13}
\]

This can also be solved in closed form. It again has two free parameters corresponding to the two homogeneous solutions, which we can fix the same way we fixed them for the free parameters in \( \tilde{V}_1 \). However, we can instead apply an argument that will easily generalize to all higher orders, which is to expand the above equation of motion at small \( z \) directly on the second sheet:

\[
\partial_z^2(z\tilde{V}_2(z)) = 6c_1 \left( (1 - h_H) + \frac{1}{z} + \frac{2}{z^2} + \mathcal{O}(z) \right). \tag{A.2.14}
\]

The solution to the above equation of motion is again easily determined:

\[
\tilde{V}_2(z) = 6c_1 \left( -2\frac{\log(z)}{z} + \log(z) + \mathcal{O}(z) \right) + \frac{d_1}{z} + d_0. \tag{A.2.15}
\]
We do not need to determine the integration constants $d_0, d_1$, because they do not contaminate the leading logs! Since the integration constants are always coefficients of the homogeneous solutions, this feature manifestly continues to all higher orders as well.

The above explicit demonstration was specific to the $O_{1,2}$ block, but it is straightforward to generalize to general degenerate operators. For all degenerate operators $O_{1,s}$, the $1/c$ piece $\tilde{V}_1$ is the same universal function (A.2.10) (in fact, it is just the global conformal block for the stress tensor), with a coefficient that is linear in $h_{s,1}$:

$$\tilde{V}_{1,s} = \frac{2h_H h_{1,s}}{c} z^2 F_1(2, 2, 4, z). \quad (A.2.16)$$

We do not have to appeal to our knowledge that this is the stress tensor conformal block; (A.2.10) is a derivation of $\tilde{V}_{s,1}$ since we know $\tilde{V}_{1,s} = \left( \lim_{c \to \infty} \frac{h_{1,s}}{h_{1,2}} \right) \tilde{V}_{1,2}$. This means that generally, on the second sheet we have $\tilde{V}_1$ is given by

$$c_1^{(s)} = \left( \lim_{c \to \infty} \frac{h_{1,s}}{h_{1,2}} \right) 24i\pi h_H = 24i\pi h_H(s - 1),$$

$$c_0^{(s)} = \left( \lim_{c \to \infty} \frac{h_{1,s}}{h_{1,2}} \right) (-12i\pi h_H) = 12i\pi h_H(s - 1). \quad (A.2.17)$$

Thus, $\tilde{V}_2$ is generally given on the second sheet at small $z$ by

$$\partial_z \left(z^{s-1} \tilde{V}_2 \right) = 12A_s \frac{z \tilde{V}_1(z)}{z^2} + O(1/z)$$

$$= \frac{12c_1^{(s)} A_s}{z^2} + O(1/z), \quad (A.2.18)$$

(where $A_s$ depends on $s$ but will be determined momentarily). We have taken advantage of the fact that $z \tilde{V}_1(z)$ is regular at $z \to 0$ since $c_1/z$ was the most singular term generated at this order, and so by scaling $z \tilde{V}_1(z)/z^2$ is the most singular term generated in the null
equation of motion above. The solution to (A.2.18) is clearly

\[ \tilde{V}_2(z) = 12c_1^{(s)} \frac{\log(z)}{z} \frac{A_s}{(s-2)!} + \mathcal{O}(\log(z)). \] (A.2.19)

We can easily fix \( A_s \) since \( \tilde{V}_2 \) is completely determined for any \( h_{1,s} \) by just the two function \( \tilde{V}_{1,2} \) and \( \tilde{V}_{2,2} \); therefore once we calculate \( \tilde{V}_2 \) for two values of \( s \), we know it for all \( s \). A simple computation shows that \( A_2 = A_3 = 1 \). Demanding consistency of the above equation with all \( r \) immediately fixes

\[ \frac{A_s}{(s-2)!} = 1. \] (A.2.20)

Finally, to get the leading logs, we can just iterate at higher orders, since the only way to get double logs is to integrate single logs (which first appear in \( \tilde{V}_2 \)), and the only way to get triple logs is to integrate double logs (which first appear in \( \tilde{V}_3 \)), etc. So for instance, in the equation of motion for \( \tilde{V}_3 \), we can just look at \( \tilde{V}_2 \) in the source terms, since this is the only contribution that has a single log. But the relation between \( \tilde{V}_3 \) and \( \tilde{V}_2 \) at leading order in \( 1/c \) is the same as the relation between \( \tilde{V}_2 \) and \( \tilde{V}_1 \) at leading order in \( 1/c \):

\[ \partial_z^s \left( z^{s-1} \tilde{V}_3 \right) = 12(s-2)! \frac{z \tilde{V}_2(z)}{z^2} + \mathcal{O}(1/z), \] (A.2.21)

and so on. Keeping track of just the most singular leading log terms, we see that at each order

\[ \tilde{V}_n(z) \supset 12c_1 \frac{\log^{n-1}(z)}{z(n-1)!} + \mathcal{O}(z^0, \log(z)) \rightarrow \tilde{V}_{n+1}(z) \supset -12c_1 \frac{\log^n(z)}{zn!} + \mathcal{O}(z^0, \log(z)), \] (A.2.22)

which proves that the leading singularity in the leading logs exactly exponentiates to all orders.
A.3 Leading Contribution to the Vacuum blocks in $\mathcal{N} = 1$ SCFTs

In this section, we are going to prove that the heavy-light vacuum block $\mathcal{V}_{\phi_L \phi_L \phi_H \phi_H}$ in $\mathcal{N} = 1$ SCFTs is the same as the vacuum block in non-supersymmetric CFTs at leading order of the large $c$ limit, meaning that it only gets contributions from the pure Virasoro generators at this order.

The commutators of the symmetry generators with the component fields of a superfield $\Phi(Z) = \phi_h(z) + \theta \psi_{h+\frac{1}{2}}(z)$ are

$$[L_n, \phi(z)] = z^n [h (n + 1) + z \partial_z] \phi,$$

$$[L_n, \psi(z)] = z^n [(h + \frac{1}{2}) (n + 1) + z \partial_z] \psi,$$

$$[G_r, \phi(z)] = z^{r+\frac{1}{2}} \phi,$$

$$\{G_r, \psi(z)\} = z^{r-\frac{1}{2}} [h (2r + 1) + z \partial_z] \phi, \quad n \in \mathbb{Z}; \quad r \in \mathbb{Z} + \frac{1}{2}. \quad (A.3.1)$$

The vacuum block $\mathcal{V}_{\phi_L \phi_L \phi_H \phi_H}$ is the contribution to $\langle \phi_H(\infty) \phi_H(1) \phi_L(z) \phi_L(0) \rangle$ from an irreducible representation of the super-Virasoro algebra whose highest weight state is the vacuum $|0\rangle$. The vacuum state is annihilated by $L_n$ and $G_r$ for $n \geq -1$ and $r \geq -\frac{1}{2}$. Besides the vacuum state, other states in this representation are the descendants of the vacuum, which can be obtained by acting on the vacuum with $L_{-n}$ and $G_{-r}$ for $n \geq 2$ and $r \geq \frac{3}{2}$. To get the vacuum block, we can insert a projection operator into the four-point function:

$$\mathcal{V}_{\phi_L \phi_L \phi_H \phi_H} = \frac{\langle \phi_H(\infty) \phi_H(1) \mathcal{P}_0 \phi_L(z) \phi_L(0) \rangle}{\langle \phi_H(\infty) \phi_H(1) \rangle}. \quad (A.3.2)$$
At leading order of the large $c$ limit, we can use the approximate projection operator

$$
P_0 \approx \sum_{\{n_i, r_j\}} \frac{G_{-r_j} \cdots G_{-r_1} L_{-n_i} \cdots L_{-n_1} |0\rangle \langle 0| L_{n_1} \cdots L_{n_i} G_{r_1} \cdots G_{r_j}}{L_{n_1} \cdots L_{n_i} G_{r_1} \cdots G_{r_j} G_{-r_j} \cdots G_{-r_1} L_{-n_i} \cdots L_{-n_1}}.
\tag{A.3.3}
$$

with $n_i \in \mathbb{Z}$ and $r_j \in \mathbb{Z} + \frac{1}{2}$, because the states $G_{-r_j} \cdots G_{-r_1} L_{-n_i} \cdots L_{-n_1} |0\rangle$ are orthogonal with each other at this order \(^1\). We can arrange the order of the generators such that $n_i \geq \cdots \geq n_1 \geq 2$ and $r_j \geq \cdots \geq r_1 \geq \frac{3}{2}$. Denote the level of each state as $N + R$, where $N = \sum_{l=1}^{i} n_l$ and $R = \sum_{l=1}^{j} r_l$. Notice that in the above equation (A.3.3), at each level $N + R$, we should only sum over independent states. For example, at level 3, we only have $L_{-3}$, because $G_{-\frac{3}{2}} G_{-\frac{3}{2}} = L_{-3}$ and shouldn’t be included.

Consider a state $G_{-r_j} \cdots G_{-r_1} L_{-n_i} \cdots L_{-n_1} |0\rangle$, its contribution to $V_{\phi_L \phi_L \phi_H \phi_H}$ is

$$
\frac{\langle \phi_H(\infty) \phi_H (1) G_{-r_j} \cdots G_{-r_1} L_{-n_i} \cdots L_{-n_1} |0\rangle \langle 0| L_{n_1} \cdots L_{n_i} G_{r_1} \cdots G_{r_j} \phi_L(z) \phi_L(0) \rangle}{\langle \phi_H(\infty) \phi_H (1) \rangle \langle L_{n_1} \cdots L_{n_i} G_{r_1} \cdots G_{r_j} G_{-r_j} \cdots G_{-r_1} L_{-n_i} \cdots L_{-n_1} \rangle}.
\tag{A.3.4}
$$

In the large $c$ limit, the normalization factor in the denominator scales as

$$
\langle L_{n_1} \cdots L_{n_i} G_{r_1} \cdots G_{r_j} G_{-r_j} \cdots G_{-r_1} L_{-n_i} \cdots L_{-n_1} \rangle \sim c^{N+R}
$$

because the commutation of each pair of generators $G_r$ with $G_{-r}$ or $L_n$ with $L_{-n}$ will give us one power of $c$ (2.5.2). In the numerator, $\langle 0| L_{n_1} \cdots L_{n_i} G_{r_1} \cdots G_{r_j} \phi_L(z) \phi_L(0) \rangle$ is order $O(1)$, because the commutation of these generators with $\phi_L$ will not give us $c$ or $h_H$. And the remaining part in (A.3.4) scales as

$$
\frac{\langle \phi_H(\infty) \phi_H (1) G_{-r_j} \cdots G_{-r_1} L_{-n_i} \cdots L_{-n_1} |0\rangle}{\langle \phi_H(\infty) \phi_H (1) \rangle} \sim h_H^{N+R/2}.
\tag{A.3.5}
$$

The reason is that when we commute one $L_{-n}$ with $\phi_H$ we’ll get one power of $h_H$, but we need to commute two $G_{-r}$s with $\phi_H$ to get one power of $h_H$ as can be seen from the commutation

\(^1\)The proof is similar to that for non-susy CFTs with only Virasoro generators, see appendix B of [15].
relations (A.3.1). So in the heavy-light limit, with \( \eta_H = \frac{\hbar}{c} \) fixed, the contribution of (A.3.4) will be order \( O(c^{-R/2}) \). This means that at order \( c^0 \) (that is, \( R = 0 \)), the heavy-light vacuum block \( \mathcal{V}_{\phi_L \phi_L \phi_H \phi_H} \) in \( \mathcal{N} = 1 \) SCFTs will only get contributions from the pure Virasoro generators, which make it the same as that in non-susy CFTs at leading order. This is also true for the vacuum blocks \( \mathcal{V}_{\psi_L \psi_L \phi_H \phi_H} \).

### A.4 Details of the \( \mathcal{N} = 2 \) SCFT Calculations

#### A.4.1 Superconformal Ward Identities

\( N \)-point functions \( F_N \equiv \langle \Phi_1(Z_1) \Phi_2(Z_2) \cdots \Phi_N(Z_N) \rangle \) in \( \mathcal{N} = 2 \) SCFTs satisfy the following eight superconformal Ward identities

\[
L_{-1} : \sum_{i=1}^{N} \partial_{z_i} F_N = 0, \\
L_{0} : \sum_{i=1}^{N} (2z_i \partial_{z_i} + 2h_i + \theta_i \partial_{\theta_i} + \overline{\theta}_i \partial_{\overline{\theta}_i}) F_N = 0, \\
L_{1} : \sum_{i=1}^{N} (z_i^2 \partial_{z_i} + z_i(2h_i + \theta_i \partial_{\theta_i} + \overline{\theta}_i \partial_{\overline{\theta}_i}) + q_i \theta_i \overline{\theta}_i) F_N = 0, \\
J_{0} : \sum_{i=1}^{N} (\overline{\theta}_i \partial_{\overline{\theta}_i} - \theta_i \partial_{\theta_i} + q_i) F_N = 0 \tag{A.4.1} \\
G_{-\frac{1}{2}}, \overline{G}_{-\frac{1}{2}} : \sum_{i=1}^{N} (\partial_{\overline{\theta}_i} - \theta_i \partial_{z_i}) F_N = \sum_{i=1}^{N} (\partial_{\theta_i} - \overline{\theta}_i \partial_{z_i}) F_N = 0, \\
G_{\frac{1}{2}} : \sum_{i=1}^{N} [z_i (\partial_{\overline{\theta}_i} - \theta_i \partial_{z_i}) - \theta_i (2h_i + q_i + \overline{\theta}_i \partial_{\overline{\theta}_i})] F_N = 0, \\
\overline{G}_{\frac{1}{2}} : \sum_{i=1}^{N} [z_i (\partial_{\theta_i} - \overline{\theta}_i \partial_{z_i}) - \overline{\theta}_i (2h_i - q_i + \theta_i \partial_{\theta_i})] F_N = 0.
\]
Specifically, the three identities corresponding to $L_{-1}$, $G_{-\frac{1}{2}}$ and $\overline{G}_{-\frac{1}{2}}$ were used in the simplification of the super null-state equation (2.5.34).

A.4.2 Leading Contributions to the Vacuum Blocks

Similar to the reasoning of $\mathcal{N} = 1$ (appendix A.3), the vacuum block $\mathcal{V}_{\phi_L^{-q} \phi_H^{-q} \phi_H^{-q}}$ in $\mathcal{N} = 2$ SCFTs will not get contribution from the generators $G_r$ and $\overline{G}_r$ at leading order of large $c$ limit, which makes it the same as the vacuum block of a theory with only Virasoro and $U(1)$ symmetry at this order. This can be seen from the commutation relations of these symmetry generators with the lowest component field $\phi(z)$ of a superfield $\Phi^q (Z) = \phi^q (z) + \theta \overline{\psi}_{h+\frac{1}{2}} (z) + \overline{\theta} \psi_{h+\frac{1}{2}} (z) + \theta \overline{\theta} \lambda^q (z)$. For simplicity, in the following subsections, we only keep the superscripts and subscripts when necessary.

Commutation relations of the generators with the component field $\phi(z)$ are

$$[L_n, \phi(z)] = z^n [(n+1) h + z \partial_z] \phi, \quad [J_n, \phi(z)] = q z^n \phi, \quad [G_r, \phi(z)] = z^{r+\frac{1}{2}} \overline{\psi}.$$  \hspace{1cm} (A.4.2)

The last two commutators are exactly the same as that of the fermionic generator $G_r$ with $\phi$ in $\mathcal{N} = 1$ SCFTs (A.3.1), which upon the same reasoning means that when summing over descendant states of the vacuum to get $\mathcal{V}_{\phi_L^{-q} \phi_H^{-q} \phi_H^{-q}}$, those states having $G_r$ or $\overline{G}_r$ in them will not contribute at leading order of the large $c$ limit. We can also easily see that some other vacuum blocks, such as $\mathcal{V}_{\lambda_L^{-q} \phi_L^{-q} \phi_H^{-q}}$, $\mathcal{V}_{\phi_L^{-q} \lambda_L^{-q} \phi_H^{-q}}$ and $\mathcal{V}_{\phi_L^{-q} \lambda_L^{-q} \phi_H^{-q}}$, also only get contributions from Virasoro and $U(1)$ generators. This point will be used in the calculation of subsection A.4.4. Note that to construct the projection operator for $\mathcal{N} = 2$, the Hermiticity conditions among these generators are $L_n^* = L_{-n}$, $J_n^* = J_{-n}$, $G_r^* = \overline{G}_{-r}$, $\overline{G}_r^* = G_{-r}$. And the vacuum $|0\rangle$ in $\mathcal{N} = 2$ is annihilated by $L_n, J_m, G_r, \overline{G}_r$ for $n \geq -1, m \geq 0, r \geq -\frac{1}{2}$. 

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For completeness, the commutation relations of other component fields are

\[ [L_n, \psi(z)] = z^n [(h + \frac{1}{2})(n + 1) + z\partial_z] \psi, \]

\[ [L_n, \overline{\psi}(z)] = z^n [(h + \frac{1}{2})(n + 1) + z\partial_z] \overline{\psi}, \]

\[ [L_n, \lambda(z)] = z^n [(h + 1)(n + 1) + z\partial_z] \lambda + \frac{1}{2} n(n + 1) q z^{n-1} \phi, \]

\[ \{G_r, \psi(z)\} = \{\overline{G}_r, \overline{\psi}(z)\} = 0 \]

\[ \{G_r, \overline{\psi}(z)\} = z^{r-\frac{1}{2}} [(r + \frac{1}{2})(2h + q) + z\partial_z] \phi + z^{r+\frac{1}{2}} \lambda, \]

\[ \{\overline{G}_r, \psi(z)\} = z^{r-\frac{1}{2}} [(r + \frac{1}{2})(2h - q) + z\partial_z] \phi - z^{r+\frac{1}{2}} \lambda, \]

\[ [G_r, \lambda(z)] = -z^{r-\frac{1}{2}} [(r + \frac{1}{2})(2h + q + 1) + z\partial_z] \psi, \quad \text{(A.4.3)} \]

\[ \overline{[G_r, \lambda(z)]} = z^{r-\frac{1}{2}} [(r + \frac{1}{2})(2h - q + 1) + z\partial_z] \overline{\psi}, \]

\[ [J_n, \psi(z)] = (q + 1) z^n \psi, \]

\[ [J_n, \overline{\psi}(z)] = (q - 1) z^n \overline{\psi} \]

\[ [J_n, \lambda(z)] = q z^n \lambda + 2hn z^{n-1} \phi. \]

A.4.3 Correlation Functions with Descendant Component Fields

In this subsection, we are going to derive the relationships between correlation functions with descendant fields and correlation functions with only primary fields. These relationships are also true for the corresponding vacuum blocks. Specifically, we only consider the lowest component primary field \( \phi_h^q \) and its descendants that are relevant to our calculation.
For correlation functions involving \((L^{-1}\phi)(z)\), since \((L^{-1}\phi)(z) = \partial_z \phi(z)\), we have

\[
\langle (L^{-1}\phi)(z)X \rangle = \partial_z \langle \phi(z)X \rangle \tag{A.4.4}
\]

where \(X\) is an assembly of primary or descendant component fields. If there are more than one \((L^{-1}\phi)\), we just need to take the derivatives in succession with respect to the coordinate of each \((L^{-1}\phi)\).

For correlation functions involving only one descendant \((J^{-n}\phi)\), we have

\[
\langle (J^{-n}\phi)(z_1)Y \rangle = \frac{1}{2\pi i} \oint_{z_1} dz (z - z_1)^{-n} \langle J(z) \phi(z_1)Y \rangle
\]

\[
= -\frac{1}{2\pi i} \sum_{i=2}^{N} \oint_{z_i} dz (z - z_1)^{-n} \frac{q_i \langle \phi(z_1)Y \rangle}{z - z_i} \tag{A.4.5}
\]

\[
= -\sum_{i=2}^{N} \frac{q_i \langle \phi(z_1)Y \rangle}{(z_i - z_1)^{n}}
\]

where \(Y = \phi_2(z_2) \cdots \phi_N(z_N)\) is an assembly of primary fields with conformal dimensions \(h_i\) and \(U(1)\) charge \(q_i\), and we have used the OPE \(J(z)\phi_i(z_i) \sim \frac{q_i \phi_i(z_i)}{z - z_i}\) in the second line.

For correlation functions involving two \((J^{-n}\phi)\)s, we need to know the OPE \(J(z)(J^{-n}\phi)(w)\), which can be written as

\[
J(z)(J^{-n}\phi)(w) = \sum_{k>0} \frac{(J_{k,-n}\phi)(w)}{(z - w)^{k+1}} + \sum_{k>0} \frac{(J_{k,-n}\phi)(w)}{(z - w)^{1-k}} \tag{A.4.6}
\]

In the first sum, since \([J_k,J_{-n}] = \frac{2}{3} k \delta_{k,-n,0}\) and \((J_k\phi)(w) = 0\) for \(k > 0\), only the term with \(k = n\) is non-zero. In the second sum, only the term with \(k = 0\) is singular. So we have

\[
J(z)(J^{-n}\phi)(w) \sim \frac{(J_{n,-n}\phi)(w)}{(z - w)^{n+1}} + \frac{(J_{0,-n}\phi)(w)}{z - w}
\]

\[
\sim \frac{nc \phi(w)}{3 (z - w)^{n+1}} + \frac{q(J_{n}\phi)(w)}{z - w} \tag{A.4.7}
\]

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where \( q \) is the \( U(1) \) charge of \( \phi \) (and \( J_n \) will not change the \( U(1) \) charge) and \( \sim \) means that in the RHS we omit terms that are regular. For \( n = 1 \), the OPE of \( J(z) \) with \((J_{-1}\phi)(w)\) is

\[
J(z) (J_{-1}\phi)(w) \sim \frac{c}{3} \frac{\phi(w)}{(z-w)^2} + \frac{q (J_{-1}\phi)(w)}{(z-w)}
\]  

(A.4.8)

In the calculation of this paper, we only need \( \langle (J_{-1}\phi_1)(z_1)(J_{-1}\phi_2)(z_2)Y \rangle \) with \( Y = \phi_3(z_3) \cdots \phi_N(z_N) \) an assembly of primary fields. Using the above OPE, we have

\[
\langle (J_{-1}\phi_1)(z_1)(J_{-1}\phi_2)(z_2)Y \rangle = \frac{1}{2\pi i} \oint_{z_1} dz \langle J(z) \phi_1(z_1)(J_{-1}\phi_2)(z_2)Y \rangle \frac{1}{z-z_1}
\]

\[
= -\frac{1}{2\pi i} \oint_{z_2} dz \langle \phi_1 \left[ \frac{c}{3} \frac{\phi_2}{(z-z_2)^2} + \frac{q_2 (J_{-1}\phi_2)}{(z-z_2)} \right] Y \rangle
\]

\[
- \frac{1}{2\pi i} \sum_{i=3}^{N} \oint_{z_i} dz \langle \phi_1 (J_{-1}\phi_2)Y \rangle \frac{q_i}{z-z_i}
\]

\[
= \frac{c}{3} \frac{\langle \phi_1\phi_2 Y \rangle}{(z_2-z_1)^2} - \sum_{i=3}^{N} \frac{q_i \langle \phi_1 (J_{-1}\phi_2)Y \rangle}{z_i-z_1}
\]

\[
= \frac{c}{3} \frac{\langle \phi_1\phi_2 Y \rangle}{(z_2-z_1)^2} + \sum_{i=2}^{N} \sum_{j=1,j\neq 2}^{N} \frac{q_i q_j \langle \phi_1\phi_2 Y \rangle}{(z_i-z_1)(z_j-z_2)}
\]

(A.4.9)

where in the second line, we used the OPE of \( J(z)(J_{-1}\phi_2)(z_2) \) and \( J(z)Y \) (or equation (A.4.5)), and in the last line, we used equation equation (A.4.5).

A.4.4 Decomposition of \( \lambda_{h+1}^q \)

In this subsection, we’ll show that \( \lambda_{h+1}^q \) with conformal dimension \( h+1 \) and \( U(1) \) charge \( q \) can be written as

\[
\lambda_{h+1}^q(z) = \frac{12h^2 - 3q^2}{2ch - 3q^2} (J_{-1}\phi_h^q)(z) + \frac{q(c-6h)}{2ch - 3q^2} (L_{-1}\phi_h^q)(z) + \tilde{\lambda}_{h+1}^q(z),
\]

(A.4.10)

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where \( \tilde{\lambda}^q_{h+1} \) is a Virasoro and \( U(1) \) primary with conformal dimension \( h + 1 \) and \( U(1) \) charge \( q \), in the sense that \( L_0 \tilde{\lambda}^q_{h+1} = (h + 1) \tilde{\lambda}^q_{h+1}, \) \( J_0 \tilde{\lambda}^q_{h+1} = q\tilde{\lambda}^q_{h+1} \) and \( L_n \tilde{\lambda}^q = J_n \tilde{\lambda}^q = 0, n \geq 1 \). In the following calculation, for simplicity, we only keep the superscripts and subscripts when necessary.

\( \lambda^q \) can be obtained by acting on the lowest component field \( \phi^q \) with \( G_{-\frac{1}{2}} \) and \( \overline{G}_{-\frac{1}{2}} \):

\[
\lambda^q = \frac{1}{2} \left( G_{-\frac{1}{2}} \overline{G}_{-\frac{1}{2}} - \overline{G}_{-\frac{1}{2}} G_{-\frac{1}{2}} \right) \phi^q = \left( L_{-1} - \overline{G}_{-\frac{1}{2}} G_{-\frac{1}{2}} \right) \phi^q. \tag{A.4.11}
\]

Suppose \( \lambda^q \) can be written as

\[
\lambda^q = AJ_{-1} \phi^q + BL_{-1} \phi^q + \tilde{\lambda}^q, \tag{A.4.12}
\]

where \( A \) and \( B \) are two constants depending on \( h \) and \( q \), and \( \tilde{\lambda}^q \) is a Virasoro and \( U(1) \) primary. Acting on \( (A.4.11) \) and \( (A.4.12) \) with \( L_1 \) and \( J_1 \), we get two equations

\[
L_1 \left( L_{-1} - \overline{G}_{-\frac{1}{2}} G_{-\frac{1}{2}} \right) \phi^q = L_1 \left( AJ_{-1} \phi^q + BL_{-1} \phi^q + \tilde{\lambda}^q \right),
\]

\[
J_1 \left( L_{-1} - \overline{G}_{-\frac{1}{2}} G_{-\frac{1}{2}} \right) \phi^q = J_1 \left( AJ_{-1} \phi^q + BL_{-1} \phi^q + \tilde{\lambda}^q \right).
\]

Using the commutation relation of these generators (2.5.21), we have

\[
q \phi^q = (Aq + 2hB) \phi^q,
\]

\[
2h \phi^q = \left( \frac{Ac}{3} + Bq \right) \phi^q.
\]

Solving these equations, we get

\[
A = \frac{12h^2 - 3q^2}{2ch - 3q^2}, \quad B = \frac{q(c - 6h)}{2ch - 3q^2}. \tag{A.4.13}
\]

which give us the decomposition as equation \( (A.4.10) \). Note the \( A \) is invariant but \( B \) changes
sign when $q$ is changed to $-q$, so the decomposition of $\lambda^{-q}$ is

$$\lambda^{-q} = AJ_{-1}\phi^{-q} - BL_{-1}\phi^{-q} + \tilde{\lambda}^{-q}.$$  \hfill (A.4.14)

The commutation of $\tilde{\lambda}^q$ with the Virasoro and $U(1)$ generators can be derived from those of $\lambda^q$ (A.4.3):

$$[L_n, \tilde{\lambda}^q(z)] = [L_n, \lambda^q(z)] - A[L_n, (J_{-1}\phi^q)(z)] - B[L_n, (L_{-1}\phi^q)(z)],$$  \hfill (A.4.15)

$$[J_n, \tilde{\lambda}^q(z)] = [J_n, \lambda^q(z)] - A[J_n, (J_{-1}\phi^q)(z)] - B[J_n, (L_{-1}\phi^q)(z)].$$

The commutation relations of $J_{-1}\phi^q(z)$ on the RHS can be derived from the OPE of $T(w)$ and $J(w)$ with $J_{-1}\phi^q(z)$:

$$T(w) (J_{-1}\phi^q)(z) \sim \frac{q\phi(z)}{(w-z)^3} + \frac{(h+1)(J_{-1}\phi)(z)}{(w-z)^2} + \frac{\partial_z(J_{-1}\phi)(z)}{w-z},$$  \hfill (A.4.16)

$$J(w) (J_{-1}\phi^q)(z) \sim \frac{c}{3} \frac{\phi(z)}{(w-z)^2} + \frac{q(J_{-1}\phi)(z)}{(w-z)}.$$

and the results are

$$[L_n, (J_{-1}\phi^q)(z)] = z^{n-1}\left[\frac{1}{2}(n+1)nq\phi^q + (h+1)(n+1)z(J_{-1}\phi^q) + z^2\partial_z(J_{-1}\phi^q)\right],$$  \hfill (A.4.17)

$$[J_n, (J_{-1}\phi^q)(z)] = z^{n-1}\left[\frac{c}{3}n\phi^q + zq(J_{-1}\phi^q)\right].$$

The commutation relations of $L_n$ and $J_n$ with $L_{-1}\phi^q(z) = \partial_z\phi^q(z)$ are just the derivative of the commutation relations of $L_n$ and $J_n$ with $\phi^q(z)$ given in (A.4.2):

$$[L_n, (L_{-1}\phi^q)(z)] = z^{n-1}[(n+1)h + (n+1)z\partial_z + z^2\partial_z^2]\phi^q,$$  \hfill (A.4.18)

$$[J_n, (L_{-1}\phi^q)(z)] = qz^{n-1}(n + z\partial_z)\phi^q.$$
Putting everything together, we finally get

\[
[L_n, \tilde{\lambda}^q(z)] = z^n[(h + 1)(n + 1) + z\partial_z]\tilde{\lambda}^q, \quad [J_n, \tilde{\lambda}^q(z)] = qz^n\tilde{\lambda}^q.
\] (A.4.19)

Comparing these two commutations with those for \(\phi^q_h\) (A.4.2), we can see that under the action of Virasoro and \(U(1)\) generators, \(\lambda^q_{h+1}\) acts like \(\phi^q_h\) but with conformal dimension \(h+1\).

To derive the normalization of two-point function \(\langle \tilde{\lambda}^{-q}(z_1)\tilde{\lambda}^q(z_2) \rangle\), we need to use the two-point function of \(\lambda^q\) and \(\lambda^{-q}\), which can be read off from the two-point function of two superfields (2.5.28):

\[
\langle \lambda^{-q}_{h+1}(z_1)\lambda^q_{h+1}(z_2) \rangle = \frac{2h(2h + 1)}{z_{21}^{2h+2}}.
\] (A.4.20)

Substituting the decompositions of \(\lambda^q_{h+1}\) and \(\lambda^{-q}_{h+1}\) in the above two-point function, we can express \(\langle \tilde{\lambda}^{-q}(z_1)\tilde{\lambda}^q(z_2) \rangle\) as

\[
\langle \tilde{\lambda}^{-q}(z_1)\tilde{\lambda}^q(z_2) \rangle = \langle \lambda^{-q}\lambda^q \rangle - A^2 \left\langle (J_{-1}\phi^{-q})(J_{-1}\phi^q) \right\rangle + B^2 \left\langle (L_{-1}\phi^{-q})(L_{-1}\phi^q) \right\rangle
\]

\[- AB \left\langle (J_{-1}\phi^{-q})(L_{-1}\phi^q) \right\rangle + AB \left\langle (L_{-1}\phi^{-q})(J_{-1}\phi^q) \right\rangle.
\] (A.4.21)

The terms on the RHS are easy to calculate using the equations derived in last subsection A.4.3 and the two-point function \(\langle \phi^{-q}(z_1)\phi^q(z_2) \rangle\) = \(\frac{1}{z_{21}^{2\beta}}\). The results are as follow

\[
\left\langle (J_{-1}\phi^{-q})(J_{-1}\phi^q) \right\rangle = q^2 + \frac{\xi}{z_{21}^{2h+2}},
\]

\[
\left\langle (L_{-1}\phi^{-q})(L_{-1}\phi^q) \right\rangle = \partial_{z_1}\partial_{z_2} \frac{1}{z_{21}^{2h}} = \frac{-2h(2h + 1)}{z_{21}^{2h+2}},
\]

\[
\left\langle (J_{-1}\phi^{-q})(L_{-1}\phi^q) \right\rangle = \partial_{z_2} - q \left\langle \phi^{-q}\phi^q \right\rangle = \frac{(2h + 1)q}{z_{21}^{2h+2}},
\]

\[
\left\langle (L_{-1}\phi^{-q})(J_{-1}\phi^q) \right\rangle = \partial_{z_1} q \left\langle \phi^{-q}\phi^q \right\rangle = \frac{-2h(2h + 1)q}{z_{21}^{2h+2}}.
\] (A.4.22)
Plugging these equations and equation (A.4.20) back in equation (A.4.21), we get

\[
\langle \tilde{\lambda}_{h+1}^q (z_1) \tilde{\lambda}_{h+1}^q (z_2) \rangle = \frac{(4h^2 - q^2) (2ch + c - 3 (2h + q^2))}{2ch - 3q^2} \frac{1}{z_2^{2h+2}}. \tag{A.4.23}
\]

Using the decomposition (A.4.10), we can calculate \( V_{\tilde{\lambda} - qL} \phi L \phi H H \phi H \) and \( V_{\tilde{\lambda} - qL} \phi L \phi H H \) from \( V_{\phi L} \phi L \phi H H \phi H \), and then equate these blocks to equations (2.5.40), (2.5.41) and (2.5.42), to solve for \( g_{2,hL}(z) \), \( g_{4,hL}(z) \) and \( g_{5,hL}(z) \), respectively. As we said in subsection A.4.2, at leading order of the large \( c \) limit, these blocks only get contributions from Virasoro and \( U(1) \) generators. Some details for calculating these blocks are as follow:

1. In these calculations, we need to use the relationship between vacuum blocks with descendant fields and vacuum blocks with only primaries. These relationships are the same as those for the corresponding correlation functions, which are derived in subsection A.4.3.

2. The heavy-light vacuum blocks with one light operator being \( \tilde{\lambda} \) and the other light operator being \( \phi \) vanish, \( V_{\tilde{\lambda} - qL} \phi L \phi H H \phi H \phi H = V_{\phi L} \phi L \phi H H \phi H \phi H = 0 \). The reason is just because \( \tilde{\lambda}_L \) and \( \phi_L \) have different conformal dimensions \( (h_{\tilde{\lambda}_L} = h_{\phi_L} + 1) \), and the two-point functions of them vanishes, \( \langle \tilde{\lambda}_L^q \phi_L^q \rangle = \langle \phi_L^q \tilde{\lambda}_L^q \rangle = 0 \).

3. At leading order of the large \( c \) limit, the only difference (up to normalization) between the vacuum blocks \( V_{\tilde{\lambda} - qL} \phi L \phi H H \phi H \phi H \) and \( V_{\phi L} \phi L \phi H H \phi H \phi H \phi H \phi H \phi H \) is that the conformal dimension of the light operators in the former is \( h_L + 1 \) while that of the latter is \( h_L \). So we can just change \( h_L \) to \( h_L + 1 \) in the expression of \( V_{\phi L} \phi L \phi H H \phi H \phi H \phi H \) to get

\[
V_{\tilde{\lambda} - qL} \phi L \phi H H \phi H \phi H \phi H = \frac{(2h_L + 1) (4h_L^2 - q_L^2)}{2h_L} e^{(h_L+1)\hat{f}(z)} (1 - z)^{-3q_L q_L}, \tag{A.4.24}
\]

where the prefactor here is just the prefactor in (A.4.23) in the large \( c \) limit.
Appendix B

Appendix to Chapter 3

B.1 Details of Recursion Relations and Our Algorithm

In this appendix we will present more details about Zamolodchikov’s recursion relations and the algorithm we used to compute with them.

B.1.1 Zamolodchikov’s Recursion Relations

There are actually two Zamolodchikov recursion relations, based on viewing the Virasoro blocks as either a sum over poles in the central charge $c$ or the intermediate state dimension $h$. The latter is more powerful and will be our focus.

The Virasoro block of the four-point function $\langle \mathcal{O}_1(0)\mathcal{O}_2(z)\mathcal{O}_3(1)\mathcal{O}_4(\infty) \rangle$ with central charge $c$, external dimensions $h_i$ and intermediate dimension $h$ takes the following form

$$V_{h,h_i,c}(z) = (16q)^{\frac{h-c-1}{24}} z^{\frac{c-1}{24}-h_1-h_2} (1-z)^{\frac{c-1}{24}-h_2-h_3} \left[ \theta_3(q) \right]^{\frac{c-1}{24}} \prod_{i=1}^{4} h_i H(c,h_i,h,q), \quad (B.1.1)$$

where

$$q = e^{i\pi \tau}, \quad \tau = i \frac{K(1-z)}{K(z)}, \quad (B.1.2)$$
and the inverse transformations is

\[ z = \left( \frac{\theta_2(q)}{\theta_3(q)} \right)^4. \]  \hspace{1cm} (B.1.3)

If we parametrize the central charge \( c \), the external operator dimensions \( h_i \) and the degenerate operator dimensions \( h_{mn} \) as follows

\[ c = 13 + 6 \left( b^2 + \frac{1}{b^2} \right), \quad h_i = \frac{1}{4} \left( b + \frac{1}{b} \right)^2 - \lambda_i^2, \quad h_{m,n} = \frac{1}{4} \left( b + \frac{1}{b} \right)^2 - \lambda_{m,n}^2, \]  \hspace{1cm} (B.1.4)

with

\[ \lambda_{m,n} = \frac{1}{2} \left( \frac{m}{b} + nb \right), \]  \hspace{1cm} (B.1.5)

then the function \( H(b, h_i, h, q) \) can be calculated using the following recursion relation

\[ H(b, h_i, h, q) = 1 + \sum_{m,n \geq 1} \frac{q^{mn} R_{m,n}}{h - h_{m,n}} H(b, h_i, h_{m,n} + mn, q), \]  \hspace{1cm} (B.1.6)

where \( R_{m,n} \) is given by

\[ R_{m,n} = 2 \frac{\prod_{p,q} (\lambda_1 + \lambda_2 - \lambda_{p,q}) (\lambda_1 - \lambda_2 - \lambda_{p,q}) (\lambda_3 + \lambda_4 - \lambda_{p,q}) (\lambda_3 - \lambda_4 - \lambda_{p,q})}{\prod_{k,l} \lambda_{k,l}}, \]  \hspace{1cm} (B.1.7)

and the ranges of \( p, q, k, \) and \( l \) are:

\[ p = -m + 1, -m + 3, \cdots, m - 3, m - 1, \]

\[ q = -n + 1, -n + 3, \cdots, n - 3, n - 1, \]

\[ k = -m + 1, -m + 2, \cdots, m, \]

\[ l = -n + 1, -n + 2, \cdots, n. \]

The prime on the product in the denominator means that \( (k, l) = (0,0) \) and \( (k, l) = (m, n) \).
are excluded. Note that our definition of $\lambda_{p,q}$ differs by a factor of $-\frac{i}{2}$ from the original paper.

In each iteration of the recursion relation B.1.6, the only thing that changes is the value of the intermediate state dimension $h \rightarrow h_{m,n} + mn$, which only depends on the values of $m$ and $n$. For simplicity we’ll omit the arguments and denote $H(b, h_i, h, q)$ as $H$ and $H(b, h_{m,n} + mn, h_i, q)$ as $H_{m,n}$ in the following discussion.

This recursion relation was derived by viewing the Virasoro block $V_h$ as a function of the intermediate dimension $h$, so it can be written as a remainder term that survives when $h \rightarrow \infty$ plus a sum over poles at $h = h_{m,n}$, where $h_{m,n}$ are the dimensions of the degenerate operators. The prefactor in front of $H$ in B.1.1 is the $h \rightarrow \infty$ limit of $V_h$, as can be derived from [37, 36, 79]. The reason that $V_h$ has poles at $h = h_{m,n}$ is because of the existence of the null-operator (whose norm is zero) at level $mn$ of the descendants of $O_{h_{m,n}}$, which usually will make $V_h$ diverge when $h \rightarrow h_{m,n}$. The residue of the pole at $h_{m,n}$ will be proportional to the block $V_{h_{m,n} + mn}$ with intermediate operator being the null-operator with dimension $h_{m,n} + mn$. Thus, these residues will have high powers of $q$, which accounts for the $q^{mn}$ factor in front of $H_{m,n}$ and naturally makes the Virasoro block $V_h$ a series expansion in $q$.

The numerator of the factor $R_{m,n}$ is constructed such that it vanishes when $O_1$ (or $O_3$) belongs to the set of operators allowed by the fusion rule of $O_2O_{h_{m,n}}$ (or $O_4O_{h_{m,n}}$). The denominator of $R_{m,n}$ comes from the norm of the null-state when $h \rightarrow h_{m,n}$ (factoring out $h - h_{m,n}$); as far as we know, although it has passed numerous checks, it’s never been derived from first principles.

### B.1.2 Algorithm

In this paper, we only consider the case that $h_1 = h_2 = h_L$ and $h_3 = h_4 = h_H$. Under this circumstance, $R_{m,n}$ becomes directly proportional to $\lambda^2_{p,q}$, so $R_{m,n} = 0$ whenever $(m, n)$ are

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1. This is easy to see by writing $V_h$ as a sum over contributions from the states in the Verma Module of $O_h$. In this sum, we need to orthogonalize the states, but the zero norm of the null-state will appear as a denominator in this process, which causes the divergence.
both odd, because \((p,q)\) can then be \((0,0)\). This means that every \(H_{m,n}\) with odd \(mn\) is also zero, as every term contributing to it contains at least one \(R_{m_l,n_l}\) with odd \(m_l n_l\). As a consequence of this, only even powers of \(q\) ever appear, and there's no need to compute anything with odd \(mn\). This provides some simplification for the calculation, but it's easy to generalize the following discussion to the case that all \(h_i\)'s are different.

Now we turn to the algorithm we used to compute the recursion relation. The main idea is to sort every contribution to the functions \(H\) and \(H_{m,n}\) by its order in \(q\). By doing this from the beginning of the computation, we are able to use lower-level terms as partial sums for the higher-level terms, saving a great deal of computation.

Denote the coefficient of \(q^k\) in any function \(f\) as \(f^{(k)}\). Then the recursion relation for the coefficients of \(q^k\) in the function \(H\) is

\[
H^{(k)} = \sum_{i=2}^{k} \sum_{m_l n_l = i} \frac{R_{m_l,n_l}}{h - h_{m_l,n_l}} H^{(k-i)}_{m_l,n_l},
\]

where in the first sum \(i\) runs over even integers (odd terms will always be zero, as explained at the beginning of this section) and the second sum counts the ways to write \(i\) as the product of two integers \(m_l\) and \(n_l\), so \(l\) runs from 1 to the number of divisors of \(i\), which we denote as \(\text{div}(i)\). For large \(i\), \(\text{div}(i)\) is roughly of order \(\sim \log i\). Similarly, for the coefficients \(H^{(i)}_{m,n}\) of \(q^i\) in \(H_{m,n}\), we have

\[
H^{(k)}_{m,n} = \sum_{i=2}^{k} \sum_{m_l n_l = i} \frac{R_{m_l,n_l}}{h_{m,n} + mn - h_{m_l,n_l}} H^{(k-i)}_{m_l,n_l},
\]

Notice that in the above two equations, \(H^{(k)}\) and \(H^{(k)}_{m,n}\) only depend on lower order terms \(H^{(k-i)}_{m_l,n_l}\) for which \((k - i) + m_l n_l = k\). As illustrated in Figure B.1, we can perform the calculation from lower rows (small \(k\)) to upper rows (large \(k\)). In this way, when calculating \(H^{(k)}_{m,n}\), all the \(H^{(k-i)}_{m_l,n_l}\)'s are known already (and they are in the diagonal positions, which
Figure B.1: This figure shows a half-completed computation with max order \(q^{12}\); each cell \(H^{(k)}_i\) represents 2 to 4 distinct terms \(H^{(k)}_{m,n}\) with \(m,n = i\). The cyan row, order \(q^8\), is currently being computed, and the red diagonal contains the terms which are being used in the computation of the cyan row. The purple cells have already been computed and are being stored for future use, and the white cells have not been computed yet or have been deleted to save RAM. The row with \(k = 0\) (which would be at the bottom) contains the seed terms \(H^{(0)} = H^{(0)}_{m,n} = 1\) and is not shown.

suggests to store them in diagonals), and there are only \(\sim k \log k\) such terms, so the time complexity is only roughly \(O(N^3(\log N)^2)\). This is better than the literal implementation of the recursion relation (getting the coefficients \(H^{(k)}\) by directly recursing down to \(H^{(0)}_{m,n}\)), which seems to have a complexity of \(O(e^N)\).

There are several other tricks that one can do to even speed up the calculation. For example, one can precompute all of the residue prefactors \(\frac{R_{p,q}}{h_{m,n} + mn - h_{p,q}} \equiv C_{m,n,p,q}\) in B.1.9. There are only \(O(N(\log N)^2)\) of these, so we can save time by computing them in advance and reusing them. Although precomputation dramatically improves performance, it also doubles memory consumption; but since we store the \(H^{(i)}_{m,n}\) in diagonals, this can be ameliorated by deleting them after they’re used, as shown in B.1.

Precomputing \(C_{m,n,p,q}\) can only improve overall speed if each of its terms can be computed in constant time. This is potentially problematic, since \(R_{p,q}\) contains two products of \(O(pq)\) complexity, but it can be solved by filling \(R_{p,q}\) recursively – \(R_{p,q}\) can be computed in \(O(p)\) time from \(R_{p,q-2}\), and there are only \(O(N \log N)\) of them, so the computational complexity
of filling all $R_{p,q}$ is just $O(N^2 \log N)$. These can be further sped up by pairing up terms
to rewrite all of the defining equations in terms of $b^2$ and $\lambda^2_{m,n}$ instead of $b$ and $\lambda_{m,n}$. In
addition to the reduced number of multiplications, this also allows the entire computation
to be done using real numbers when $c > 25$, which is generally an order of magnitude faster.
When $c < 25$, $b^2$ becomes complex, and even though the final coefficients must be real by
unitarity, this only occurs at the very last step in the form of a $b^2 \leftrightarrow \frac{1}{b^2} = (b^2)^*$ symmetry.

We have implemented this algorithm in both Mathematica and C++(with Mathematica integration). The Mathematica notebook is included as a companion to this paper, while the C++ implementation is maintained at https://github.com/chussong/virasoro. The C++ implementation is about one order of magnitude faster, and the coefficients used in this paper were obtained using it. The C++ implementation has used the GMP [184],
MPFR [185], MPC [186], and MPFR C++ [187] numerical libraries. On standard personal
computers we were able to compute the $H^{(k)}$ to $k = 1000$ in around two minutes or $k = 2000$
in about 22 minutes (for $c > 25$ so that $b$ is real); the main barrier to going higher is memory
consumption, which grows roughly as $N^3 \log N$: we need to remember $O(N^2 \log N)$ numbers
and they need to be kept at $O(N)$ bits of precision due to the increasingly large cancellations
between different $H_{m,n}$, which often reach into the thousands of binary orders of magnitude.

Using a cluster with 128 GB of RAM, we estimate that we could reach order of 6000 in a
few hours. We also find that the coefficients of $q^i$ approach a power law in $i$ well before the
limits of our desktop computation, and expect that a numerical fit for this power law would
be good enough to get higher order coefficients.

At the end of this section, we want to mention an issue about the recursion relation if $b^2$
is a rational number. Notice that the denominator in B.1.9 and the denominator of $R_{m,n}$ in
B.1.7 can be zero:

$$h_{m,n} + mn - h_{m_l,n_l} = 0 \quad \Rightarrow \quad b^2 = \frac{m + m_l}{n - n_l} \text{ or } \frac{m - m_l}{n + n_l} \quad (B.1.10)$$
\[ \lambda_{k,l} = 0 \Rightarrow b^2 = -\frac{k}{l} \quad (B.1.11) \]

Both of these will eventually occur for any rational choice of \( b^2 \). This would appear to preclude numerical computation entirely (since for numerical calculation, \( b \) provided to the computer will always be rational), but actually for almost all rational numbers they will not appear until very high orders in the computation, so they can be ignored as long as the numerator or denominator of \( b^2 \) (as a irreducible fraction) is very large. In this paper, we’ve chose \( \sqrt{c} \) to be irrational (and set \( b \) to be a very high-precision number) to avoid this problem.

**B.2 Technical Details and Extra Plots**

**B.2.1 A Non-Perturbative Differential Equation for the Vacuum Block**

Here we describe the functions appearing in the differential equation (3.4.2). Note that although the equation itself is perturbative, its solution includes non-perturbative corrections to the heavy-light vacuum Virasoro block. The equation was derived [12] by studying the general differential equations satisfied by degenerate operators and then analytically continuing these equations in the integer index \( r \) labeling the degenerate operators. We should also note that although equation (3.4.2) only includes some of the first \( 1/c \) corrections, if one zooms in on the vicinity of the forbidden singularities by holding \( \sqrt{c}(z - z_n) \) fixed at large \( c \), then the equation incorporates all of the leading effects at large \( c \). As discussed in [12], there are both general arguments and consistency checks on the validity of equation (3.4.2).

We identify the parameter \( r = 2\pi iT_H = \sqrt{1 - \frac{24b_H}{c}} \), so that \( T_H \) is the Hawking temperature associated with the heavy operator. We also are using a Euclidean time variable...
\(\tau = -\log(1 - z)\). Then the functions included in equation (3.4.2) are \(g_H \equiv g_{2\pi T_H}\) with

\[
g_r(\tau) = \coth\left(\frac{\tau}{2}\right) - r \coth\left(\frac{r\tau}{2}\right)
\]  \hspace{1cm} \text{(B.2.1)}

and \(\Sigma_H \equiv \Sigma_r + \Sigma_{-r}\) where we define

\[
\Sigma_r(\tau) = -\frac{1}{r \sinh\left(\frac{r\tau}{2}\right)} \left( e^{-\frac{r\tau}{2}} \tilde{B}_r(\tau) + e^{\frac{r\tau}{2}} \tilde{B}_r(-\tau) - 2 \cosh\left(\frac{r\tau}{2}\right) \tilde{B}_r(0) \right).
\]  \hspace{1cm} \text{(B.2.2)}

Finally, we have introduced the function \(\tilde{B}_r(t)\) which can be represented as

\[
\tilde{B}_r(\tau) = -\log(1 - e^\tau) - \frac{e^{r\tau} \, _2F_1(1, r, 1 + r, e^{\tau})}{r}
\]  \hspace{1cm} \text{(B.2.3)}

For derivations and more complete descriptions see [12].
Figure B.3: We have found empirically that the time and height of the maxima of heavy-light Virasoro blocks have a simple dependence on both $h$ and $h_H$. This figure shows data on the parameters $b_{\text{height}}$ and $b_{\text{time}}$ defined in equations (3.3.6) and (3.3.5). These plots both have $c = 10$. Each point is obtained from linear fitting of data points at $\frac{h}{c} = \frac{n}{3}$ for $n = 1, 2, \cdots, 30$. We see explicitly that there is very little dependence on $h_L$, especially at large values of $h_H$.

### B.2.2 Some Extra Plots

In this section we have included some extra plots for readers who might like to see some more details and examples. These include the semiclassical fit to our numerical results for $h, h_L \propto c$ using [9] (figure B.2), the behavior of the $b_{\text{time}}$ and $b_{\text{height}}$ parameters from equations (3.3.6) and (3.3.5) (figure B.3) and a version of figure 3.12 zoomed in on the large $h_H/c$ region (figure B.4), which is rather compressed in that figure.

We also show some plots of the more complicated coefficient behavior which was alluded to in section 3.3.2, with the sign of the coefficients corresponding to the color of plotted points. Figure B.5 illustrates a very common scenario where the coefficients are chaotic at low $c$, but as $c$ increases they coalesce into distinct positive and negative lines. A spike-shaped feature then appears at low order and moves upward, turning the coefficients that it passes positive, until all (visible) coefficients have become positive. The two lines then gradually merge into a single power law similar to those shown in figure 3.14.
Figure B.4: This is a version of figure 3.12 where we have zoomed out to show the small $\frac{h_H}{c}$ region. The zoomed-out points with $\frac{c}{h_L} = (30, 35, 40)$ more closely fit slopes (0.221, 0.233, 0.242), which are shown as solid lines; the (0.521, 0.515, 0.509) fits for large $\frac{h_H}{c}$ are shown as dotted lines.

Figure B.5: This figure shows the coefficients $c_n$ of the $q^{2n}$ expansion of $H$. We plot $|c_n|$ as a function of $n$, with both $n$ and $c_n$ on log scales, for increasing $c$ with $\frac{h_L}{c}$ and $\frac{h_H}{c}$ held constant. The sign of the $c_n$ are illustrated by the color of the points, with blue for positive coefficients and red for negative coefficients.
Appendix C

Appendix to Chapter 4

C.1 Background and Review

Here we collect fairly elementary results that may be of interest to some readers, and that provides some useful background material for the main body of the paper.

C.1.1 Klein-Gordon Equation from the Worldline Path Integral

Here we review that first-quantized particles have propagators that satisfy the Klein-Gordon equation. This follows implicitly from the equivalence between the two-point correlator of a free quantum field and the first-quantized propagator. But we can also understand it more directly.

The first quantized propagator is

\[ K(x_f, x_i) = \int_{x_i}^{x_f} Dx(t) e^{-m \int_t^t \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \]  

(C.1.1)

Since \( K \) propagates wavefunctions in time, it satisfies the Schrodinger equation, and the idea is that this equation is equivalent to the Klein-Gordon equation. For this purpose we need to define a temporal direction for quantization, though we will find that this choice is
irrelevant as the Klein-Gordon equation is covariant. It’s convenient to choose \( t = \log y \) in our AdS case, so that we have a Lagrangian proportional to \( \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} = \sqrt{1 + \dot{x}^i \dot{x}_i} \). Then the canonical momenta are

\[
p^i = -\frac{m \dot{x}^i}{\sqrt{1 + \dot{x}^i \dot{x}_i}} \quad (C.1.2)
\]

and we find that the Hamiltonian is \( H = \sqrt{m^2 - p^ip^j g_{ij}} \). Interpreting the canonical momenta as covariant derivatives \( p_i = \nabla_i \), the square of the Schrödinger equation \( \partial_t^2 K = (\nabla_i \nabla^i + m^2) K \) is the Klein-Gordon equation in our chosen coordinate system. Note that one might try to identify \( p_i = -i\partial_i \) as ordinary derivatives, but this leads to operator ordering ambiguities after quantization since \( g_{ij} \) depends on \( x^i \). The choice \( p_i = -i\nabla_i \) resolves these issues; equivalently, there is a particular choice of ordering of factors of \( g_{ij} \) and \( p_i \) \( \rightarrow -i\partial_i \) in the Hamiltonian that is equivalent to just setting \( p_i = -i\nabla_i \). Presumably, this choice should be correctly determined by a proper treatment of the path integral.

### C.1.2 Geodesics in Euclidean AdS3

We would like to identify the geodesics in pure Euclidean AdS3. The analysis is most elegant using the embedding space coordinates

\[
\begin{align*}
X_0 &= R \frac{\cosh \tau}{\cos \rho} = \frac{1}{2} \left( \frac{y^2 + z \bar{z} + R^2}{y} \right) \\
X_3 &= R \frac{\sinh \tau}{\cos \rho} = \frac{1}{2} \left( \frac{y^2 + z \bar{z} - R^2}{y} \right) \\
X_z &= R \tan \rho e^{i\theta} = \frac{R}{y} z \\
X_{\bar{z}} &= R \tan \rho e^{-i\theta} = \frac{R}{y} \bar{z}
\end{align*}
\quad (C.1.3)
\]

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where we will set the AdS scale $R = 1$. Then the geodesics satisfy $\ddot{X}_A = X_A$ (this equation of motion arises from the action for a point particle in embedding space subject to the constraint $X_A X^A = 1$) which means that

$$X_A(s) = v_A \cosh(s) + u_A \sinh(s) \quad \text{(C.1.4)}$$

for vectors $v_A$ and $u_A$ with $v_A u^A = 0$ and $v_A v^A - u_A u^A = 1$. Note that

$$y = \frac{1}{X_0 - X_3}$$

$$z = \frac{X_z}{X_0 - X_3}$$

$$\bar{z} = \frac{X_{\bar{z}}}{X_0 - X_3} \quad \text{(C.1.5)}$$

so we end up with a simple formula for these coordinates on any geodesic. Note that we have translation symmetry in $z, \bar{z}$ so we may as well set these to zero at a convenient point. One choice is $z = \bar{z} = 0$ at $s = 0$. This means that $v_A$ will have vanishing $z, \bar{z}$ components. A convenient Euclidean parameterization is

$$X_0 = A_0 \cosh(s) + B_0 \sinh(s)$$

$$X_3 = A_3 \sinh(s) + B_3 \cosh(s)$$

$$X_z = A_z \sinh(s)$$

$$X_{\bar{z}} = A_{\bar{z}} \sinh(s) \quad \text{(C.1.6)}$$
We must have $B_3 = \frac{A_0 B_0}{A_3}$ and several other conditions for $B_0$ and $A_3$. Then if we set $A_0 = \frac{y_0}{z_0} + \frac{1}{z_0 y_0}$ then the point $s = z = \bar{z} = 0$ occurs at $y_0$. Thus we find

$$y(s) = y_0 \frac{e^s (y_0^2 + z_0 \bar{z}_0)}{e^{2s} z_0 \bar{z}_0 + y_0^2}$$

$$z(s) = \frac{z_0 (1 - e^{2s}) y_0^2}{y_0^2 + e^{2s} z_0 \bar{z}_0}$$

$$\bar{z}(s) = \frac{\bar{z}_0 (1 - e^{2s}) y_0^2}{y_0^2 + e^{2s} z_0 \bar{z}_0} \tag{C.1.7}$$

Note that at $s = 0$ we have $(y_0, 0, 0)$ while for $s = -\infty$ we have $(0, z_0, \bar{z}_0)$. We can also solve for $s$ in terms of $y$ or $z$, and then re-parameterize. It’s simplest to solve for $s(z)$, which leads to

$$y(z) = \sqrt{1 - \frac{z}{z_0} \sqrt{y_0^2 + z \bar{z}_0}}$$

$$\bar{z}(z) = \frac{\bar{z}_0 z}{z_0} \tag{C.1.8}$$

for geodesics beginning on the boundary at $z_0$ and ending in the bulk at $y_0$ and $z, \bar{z} = 0$.

### C.1.3 Global Reconstruction as a Boundary Operator Expansion

The ideas reviewed in this appendix were briefly explained in [127]. As far as we are aware, the explicit equations in this section were either first obtained by Miguel Paulos, or were derived by us via discussion and collaboration with him. Thus these results should largely be credited to Paulos and the other authors of [127]. A somewhat similar approach was taken in [108]. Ultimately, the point is that the global conformal generators must act on $\phi$ as AdS isometries, and this idea dates back to the beginning of AdS/CFT. Throughout this appendix we will always be discussing the global $\phi$, which we will usually denote as $\phi^g$. 

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C.1.3.1 Global BOE from HKLL Smearing

Here we will show how to recover the global boundary operator expansion (BOE) for a scalar operator \( \phi \)

\[
\phi^g(y, z, \bar{z}) = y^{2h} \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{n!(2h)_n} \left( L_{-1} \bar{L}_{-1} \right)^n \mathcal{O}(z, \bar{z}) \quad \text{(C.1.9)}
\]

from the well-known HKLL [117] smearing procedure.

To obtain a free bulk scalar field from a boundary primary, we ‘smear’ the boundary operator via

\[
\phi^g(y, 0, 0) = \frac{2h - 1}{\pi} \int dz d\bar{z} \left( \frac{y^2 - z\bar{z}}{y} \right)^{2h-2} \mathcal{O}(iz, i\bar{z}) \quad \text{(C.1.10)}
\]

over the Euclidean region \( \bar{z} = z^* \) with \( |z| < y \). We can formally re-write this as

\[
\phi^g(y, 0, 0) = \frac{2h - 1}{\pi} \int_{z\bar{z} < y^2} dz d\bar{z} \left( \frac{y^2 - z\bar{z}}{y} \right)^{2h-2} e^{iz\partial + i\bar{z}\partial} \mathcal{O}(0) \quad \text{(C.1.11)}
\]

As the smearing function depends only on \( z\bar{z} \), and terms with unequal powers of \( z \) and \( \bar{z} \) vanish after angular integration, we can change variables to

\[
\phi^g(y, 0, 0) = (2h - 1) \int_0^{y^2} dx \left( \frac{y^2 - x}{y} \right)^{2h-2} P \left( -x\partial \bar{\partial} \right) \mathcal{O}(0) \quad \text{(C.1.12)}
\]

where \( P(a) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} a^n \). One can do this integral explicitly and find a result with the desired series expansion in \( y^2\partial \bar{\partial} \). One way to see this directly is to perform a rescaling \( x \to xy^2 \) so that

\[
\phi^g(y, 0, 0) = (2h - 1) y^{2h} \int_0^1 dx \left( 1 - x \right)^{2h-2} P \left( -x \left( y^2\partial \bar{\partial} \right) \right) \mathcal{O}(0)
\]

\[
= y^{2h} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2h)_n n!} \left( y^2\partial \bar{\partial} \right)^n \mathcal{O}(0) \quad \text{(C.1.13)}
\]

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which is the desired boundary operator expansion in powers of $y$.

### C.1.3.2 Bulk-Boundary Correlator from BOE

Now we can verify explicitly that we obtain the correct $\langle \phi O \rangle$ correlator from the boundary operator expansion for $\phi$. In fact we will demonstrate a more general result, which makes it possible to compute $\langle \phi^g O T(z_1) \cdots T(z_n) \rangle$:

\[
\langle \phi^g(y, 0, 0)O(z, \bar{z})T(z_1) \cdots T(z_n) \rangle = y^{2h} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2h)_n} \frac{(y^2 \partial_x \partial_{\bar{z}})^n}{(\bar{z} - \bar{x})^{2h}} f(z_i, x, z)
\]

\[
= y^{2h} \frac{1}{\bar{z}^{2h}} \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{y^2}{\bar{z}} \right)^n f(z_i, x, z)
\]

\[
= y^{2h} \bar{z}^{2h} f \left( z_i, -\frac{y^2}{\bar{z}}, z \right) \tag{C.1.14}
\]

where we define $f$ via $\langle O(x)O(z)T(z_1) \cdots T(z_n) \rangle = f(z_i, x, z)(\bar{z} - \bar{x})^{-2h}$, so we have

\[
\langle \phi^g(y)O(z)T(z_1) \cdots T(z_n) \rangle = y^{2h} \left\langle O \left( -\frac{y^2}{\bar{z}} \right) O(z)T(z_1) \cdots T(z_n) \right\rangle \tag{C.1.15}
\]

The simple special case of interest to us is

\[
\langle \phi^g(y, 0, 0)O(z, \bar{z}) \rangle = \left( \frac{y}{y^2 + z\bar{z}} \right)^{2h} \tag{C.1.16}
\]

as expected.

### C.1.3.3 Symmetries of the Global Boundary Operator Expansion

In this section we will show that global conformal symmetry transformations $L_{-1}, L_0, L_1$ act as expected on the global conformally reconstructed $\phi$.

When we regard $\phi$ as a bulk field, the global conformal generators should act on it as the
differential operators

\[ L_{-1} = \partial_z \]
\[ L_0 = z\partial_z + \frac{1}{2}y\partial_y \]
\[ L_1 = z^2\partial_z + y\partial_y - y^2\partial_\bar{z} \] (C.1.17)

So the goal is to show that when the quantum operators \( L_n \) act on equation (C.1.9) in accord with this expectation. In what follows, we will show that an \( L_n \) transformation applied to \( \mathcal{O} \) results in the appropriate differential operator acting on \( \phi \).

The fact that the translation generators act correctly follows easily because \( \partial_z \) commutes with \( (y^2\partial\bar{\theta})^n \). For the dilatation \( L_0 \) note that

\[
\delta\phi^g = y^{2h} \sum_{n=0}^{\infty} \lambda_n y^{2n} (\partial\bar{\theta})^n (z\partial + h) \mathcal{O}(z, \bar{z})
\]
\[
= (z\partial + h) \phi^g + y^{2h} \sum_{n=0}^{\infty} n \lambda_n y^{2n} (\partial\bar{\theta})^n \mathcal{O}(z, \bar{z})
\]
\[
= \left( z\partial + \frac{1}{2}y\partial_y \right) \phi^g \] (C.1.18)

as desired. Note that this is automatic given the structure of expansion, and it does not depend on the form \( \lambda_n = \frac{(-1)^n}{n!(2h)_n} \).

Finally, let us check the special conformal transformation \( L_1 \); we will see that it can only act appropriately if \( \lambda_n \) take the expected form. We need to compute

\[
\delta\phi^g = y^{2h} \sum_{n=0}^{\infty} \lambda_n y^{2n} (\partial\bar{\theta})^n \left( z^2\partial + 2hz \right) \mathcal{O}(z, \bar{z})
\]
\[
= \left( z^2\partial + 2hz \right) \phi^g + y^{2h} \sum_{n=0}^{\infty} \lambda_n y^{2n} \left[ (\partial\bar{\theta})^n, z^2\partial + 2hz \right] \mathcal{O}(z, \bar{z})
\]
\[ z^2 \partial \phi^g + z y \partial_y \phi^g + y^2 y^2 \sum_{n=1}^{\infty} \lambda_n y^{2(n-1)} n (2h + n - 1) \bar{\partial}^n \partial^n O(z, \bar{z}) \]

\[ = z^2 \partial \phi^g + z y \partial_y \phi^g + y^2 \bar{\partial}^2 \sum_{n=0}^{\infty} \lambda_{n+1} (n + 1) (2h + n) y^{2n} \bar{\partial}^{n+1} \partial^n O(z, \bar{z}) \]

\[ = \left( z^2 \partial + z y \partial_y - y^2 \bar{\partial} \right) \phi^g \]

where in the last line, we used

\[ \lambda_n = -\lambda_{n+1}(n + 1)(2h + n). \quad (C.1.19) \]

The same result could also be obtained by demanding that the conformal Casimir acts appropriately on \( \phi^g \), as shown by M. Paulos.

### C.2 Regulation: from Classical Backgrounds to Correlators

In section 4.2, we developed an algorithm to compute the correlators \( \langle T \ldots T \bar{T} \ldots \bar{T} \phi O \rangle \) from the simpler correlator \( \langle \phi O \rangle_{\mu, \bar{\mu}} \) evaluated in states with non-trivial stress tensor vevs:

\[ \langle T(z) \rangle_{\mu, \bar{\mu}} = T_{cl}(\bar{z}), \quad \langle \bar{T}(\bar{z}) \rangle_{\mu, \bar{\mu}} = \bar{T}_{cl}(\bar{z}) \quad (C.2.1) \]

The algorithm was to first view \( \langle \phi O \rangle_{\mu, \bar{\mu}} \) as a functional on the vevs \( T_{cl}(z) \) and \( \bar{T}_{cl}(\bar{z}) \). In a series expansion, this functional takes the general form:

\[ \langle \phi O \rangle_{\mu, \bar{\mu}} = \langle \phi O \rangle_0 \left( 1 + \int dx K_{10}(x) T_{cl}(x) + \int d\bar{x} K_{01}(\bar{x}) \bar{T}_{cl}(\bar{x}) + \int dx K_{11}(x, \bar{x}) T_{cl}(x) \bar{T}_{cl}(\bar{x}) + \ldots \right) \]

\[ = \langle \phi O \rangle_0 \left( \sum_{n, n=0}^{\infty} \prod_{i=1}^{n} dx_i \prod_{i=1}^{n} d\bar{x}_i K_{i,i}(x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n) T_{cl}(x_1) \ldots T_{cl}(x_n) \bar{T}_{cl}(\bar{x}_1) \ldots \bar{T}_{cl}(\bar{x}_n) \right) \quad (C.2.2) \]
Then we compute the vacuum sector of the operator product $\phi O$ that includes all contributions from Virasoro descendants of the vacuum\(^1\), which is done by replacing $T_{cl}$ and $\bar{T}_{cl}$ in $\langle O\phi \rangle_{\mu,\bar{\mu}}$ by quantum operators $T$ and $\bar{T}$.

However, generically operators products of $T$ have short distance singularities when two $T$’s approach each other, which will occur due to the integration over positions in (C.2.2). In [77] we empirically discovered a simple regulator (equation C.10 there) that, when applied to the “quantum” version of (C.2.2), produces the correct OPE block. The correlator between the regulated product of $T$’s, denoted as $[T(x_1) \ldots T(x_n)]$, and external, unregulated $T(z_i)$’s were found to be:

$$\langle T(z_1) \ldots T(z_k)[T(x_1) \ldots T(x_n)]\rangle = 0, \quad n > k$$  \hfill (C.2.3)

$$\langle T(z_1) \ldots T(z_k)[T(x_1) \ldots T(x_n)]\rangle \equiv \sum_{\text{groupings}} \prod_{i=1}^{n} \langle T(z_1) \ldots T(z_k)T(x_i)\rangle, \quad n \leq k$$ \hfill (C.2.4)

The sum is over different groupings of $T(z_i)$’s. Note that since in each correlator there is only one $T(x_i)$, the results never diverge as $x_i \to x_j$. Thus the regulator fully specifies correlators of the OPE block with stress tensors.

To summarize, we proposed that the vacuum sector of the $\phi O$ operator product is:

$$\phi O = \left[ \langle \phi O \rangle_{B|T_{cl} \to T, \bar{T}_{cl} \to \bar{T}} \right] + \ldots$$ \hfill (C.2.5)

where the square bracket represents the regularization applied to all products of $T$ and $\bar{T}$’s. In the current context this regulator is defined by (C.2.3-C.2.4). In [77] and this paper, this proposal survived extensive and non-trivial checks by direct computation.

In this appendix, we would like to provide a general argument for this proposal. In particular, we would like to show that, under fairly general assumptions, it correctly extracts multi-$T$ vacuum correlators such as

\(^1\)All other contributions to $\phi O$ involve quantum operators that are not descendants of the vacuum. Thus they do not contribute to the multi-$T$ correlators that we are computing in this appendix.
\( \langle T(z_1) \ldots T(z_n) \bar{T}(\bar{z}_1) \ldots \bar{T}(\bar{z}_n) \phi \rangle_0 \) from simpler core correlators such as \( \langle \phi \rangle_{\mu, \bar{\mu}} \) on a background with non-trivial source. We also show that this algorithm does not seem to rely on conformal symmetry and may work in a wider range of settings.

Suppose we have a generic field theory containing a bosonic quantum operator \( T \). It is possible to construct a classical source for it, such that \( T \) has a classical vev:

\[
\langle T(x) \rangle_\mu = T_{cl}(x). \tag{C.2.6}
\]

We view this equation as a mapping between functions \( \mu \leftrightarrow T_{cl} \). We will make the assumption this mapping is one-to-one, and \( \mu = 0 \) maps to \( T_{cl} = 0 \). In particular, this assumes that given any \( T_{cl}(x) \), there must exist a unique source configuration \( \mu(x) \) that sets up this vev. Thus we can write the functional \( \mu[T_{cl}] \) as the solution of (C.2.6). Note that the source is defined in the usual way by shifting the action in the Euclidean path integral:

\[
S \to S + \int dz \mu(z) T(z) \tag{C.2.7}
\]

The input of our algorithm is \( \langle X \rangle_{\mu[T_{cl}]} \) as a functional on \( T_{cl} \).

\[
\langle X \rangle_{\mu[T_{cl}]} = \langle X \rangle_{\mu[T_{cl}]} \bigg|_{\mu \to 0} = \int dx_1 \delta T_{cl}(x_1) \langle X \rangle_{\mu[T_{cl}]} \bigg|_{\mu \to 0} \tag{C.2.8}
\]

Once this is known, we should have enough information to determine vacuum multi-T correlators \( \langle XT(z_1) \ldots T(z_n) \rangle_0 \). We first compute the simplest of this family:

\[
\langle XT(z_1) \rangle_{c,0} = \frac{\delta}{\mu(z_1)} \langle X \rangle_\mu \bigg|_{\mu \to 0}
\]

\[
= \int dx_1 \delta T_{cl}(x_1) \frac{\delta}{\mu(z_1)} \frac{\delta}{\delta T_{cl}(x_1)} \langle X \rangle_{\mu[T_{cl}]} \bigg|_{\mu \to 0}
\]

\[
= \langle X \rangle_0 \int dx_1 \langle T(z_1) T(x_1) \rangle_0 \tilde{K}_{10}^X(x_1)
\]
\[
\langle T(z_1) \left[ \langle X \rangle_{\mu[T]} \right]_0 \rangle
\]

In the second step, we used:

\[
\frac{\delta T_{cl}(x_1)}{\mu(z_1)} \bigg|_{\mu \to 0} = \frac{\delta}{\mu(z_1)} \langle T(x_1) \rangle_{\mu} \bigg|_{\mu \to 0} = \langle T(z_1)T(x_1) \rangle_0
\]

\[
\frac{\delta \langle X \rangle_{\mu[T_{cl}]}(x_1)}{\delta T_{cl}(x_1)} \bigg|_{\mu \to 0} = \frac{\delta}{\delta T_{cl}(x_1)} \langle X \rangle_0 \int dx \tilde{K}_1^X(x)T_{cl}(x) \bigg|_{\mu \to 0} = \langle X \rangle_0 \tilde{K}_1^X(x_1)
\]

where we have inserted a series expansion of \( \langle X \rangle_{\mu[T_{cl}]} \) in the style of (C.2.2), which should exist given the non-singular limit \( \langle X \rangle_{\mu[T_{cl}]} \to \rangle = \langle X \rangle_0 \). When we replace \( X \to \phi O \), (C.2.9) is precisely the result predicted by inserting (C.2.5) into \( \langle T(z_1)\phi O \rangle \) and evaluate using (C.2.3-C.2.4). We made this clear in the last step.

A slightly more non-trivial example is \( \langle XTT \rangle \):

\[
\langle X T(z_1) T(z_2) \rangle_{c,0} = \frac{\delta}{\delta \mu(z_1)} \frac{\delta}{\delta \mu(z_2)} \langle X \rangle_{\mu} \bigg|_{\mu \to 0}
\]

\[
= \int dx_2 \frac{\delta}{\delta \mu(z_1)} \left( \frac{\delta T_{cl}(x_2)}{\delta \mu(z_2)} \frac{\delta}{\delta T_{cl}(x_2)} \langle X \rangle_{\mu[T_{cl}]} \right)_{T_{cl} \to 0}
\]

\[
= \int dx_2 \frac{\delta^2 T_{cl}(x_2)}{\delta \mu(z_1) \delta \mu(z_2)} \frac{\delta}{\delta T_{cl}(x_2)} \langle X \rangle_{\mu[T_{cl}]} \bigg|_{T_{cl} \to 0}
\]

\[
+ \int dx_1 dx_2 \frac{\delta T_{cl}(x_2)}{\delta \mu(z_2)} \frac{\delta T_{cl}(x_1)}{\delta \mu(z_1)} \frac{\delta}{\delta T_{cl}(x_2)} \langle X \rangle_{T_{cl}} \bigg|_{T_{cl} \to 0}
\]

\[
= \langle X \rangle_0 \int dx_1 \langle T(z_1)T(z_2)T(x_1) \rangle \tilde{K}_1^X(x_1)
\]

\[
+ \langle X \rangle_0 \int dx_1 dx_2 \langle T(z_1)T(x_1) \rangle_0 \langle T(z_2)T(x_2) \rangle_0 \tilde{K}_1^X(x_1, x_2)
\]

\[
= \langle T(z_1)T(z_2) \left[ \langle X \rangle_{\mu[T]} \right]_0 \rangle
\]

Again this exactly agrees with the result of our OPE block defined with regulator (C.2.4).
It is easy to see why this works to level $n$, $\langle XT(z_1) \ldots T_n(z_n) \rangle$:

$$\langle XT(z_1) \ldots T(z_n) \rangle_0 = \left. \frac{\delta}{\mu(z_1)} \ldots \frac{\delta}{\mu(z_n)} \langle X \rangle_{\mu} \right|_{\mu \to 0} \quad \text{(C.2.13)}$$

Each time we add a $T(z_{n+1})$, the corresponding $\frac{\delta}{\delta \mu(z_{n+1})}$ either act on $\langle OO \rangle_{T_{cl}}$ as $\int dx_{n+1} \frac{\delta T(x_{n+1})}{\delta \mu(z_{n+1})} \frac{\delta}{\delta T(x_{n+1})}$, where it picks up a single $T$ from the OPE block of $OO$, or acts on an existing derivative $\frac{\delta^k T_{cl}(x_{ik})}{\delta \mu(z_{i_1})} \ldots \frac{\delta}{\delta T(x_{i_k})}$, where it adds a point to a existing multi-$T$ correlator. By construction, there are never two $T$’s from the $X$ OPE block appearing in the same vev. Thus, there are no UV divergences. The result is our OPE block defined with regulator (C.2.4).

To summarize, given the correlator of operator product $X$ on non-trivial backgrounds, $\langle X \rangle_{\mu[T_{cl}]}$, we can extract the vacuum correlator between $X$ and any number of $T$ insertions using:

$$\langle T(z_1) \ldots T(z_n) X \rangle_0 = \sum_{\text{groupings } i < n} \prod \int dx_n \langle T(z_{i_1}) \ldots T(z_{i_{kn}}) T(x_i) \rangle \frac{\delta}{\delta T_{cl}(x_i)} \langle X \rangle_{\mu[T_{cl}]} \bigg|_{T_{cl} \to 0}$$

$$= \langle T(z_1) \ldots T(z_n) \left[ \langle X \rangle_{\mu[T]} \right] \rangle_0 \quad \text{(C.2.14)}$$

where in the second line we interpreted the result as computing the correlator between $T(z_1) \ldots T(z_n)$ and the OPE block of the operator product $X$, which is constructed and regulated as given in the first line. This algorithm should work in any field theory as long as the mapping $\langle T \rangle_{\mu} = T_{cl}$ is one-to-one between $\mu$ and $T_{cl}$.

## C.3 Bulk Virasoro Transformations

We would like to find an extension of a boundary Virasoro transformation into the bulk, such that this bulk transformation will preserve the Fefferman-Graham form of the metric. To achieve this, this bulk Virasoro transformation must depend on the initial bulk metric.
In other words, the Virasoro transformations acts in the following way:

\[
(z, \bar{z}, y, f, \bar{f}) \rightarrow \left(\tilde{z}, \tilde{\bar{z}}, \tilde{y}, \tilde{f}, \bar{\tilde{f}}\right) \tag{C.3.1}
\]

The bulk metric is specified by \((f(z), \bar{f}(\bar{z}))\), which determines the vev of stress tensors and the boundary Virasoro transformations back to the uniformizing coordinate (4.2.8), reproduced here:

\[
z_u = f(z) - \frac{2y^2f'^2\bar{f}''}{4f'f'' + y^2f''f''}, \quad \bar{z}_u = \bar{f}(\bar{z}) - \frac{2y^2\bar{f}'^2f''}{4f'f'' + y^2f''f''} \tag{C.3.2}
\]

\[y_u = 4y\frac{(f'\bar{f}'')^{\frac{3}{2}}}{4f'f'' + y^2f''f''} \tag{C.3.3}
\]

Collectively, we may denote \(P = (z, \bar{z}, y, f, \bar{f})\) and the above coordinate map to the uniformizing coordinate is denoted as \(P_u(P)\). Given any Virasoro transformation \((g(z), \bar{g}(\bar{z}))\), the way we obtain its bulk completion on any background metric that preserves the Fefferman-Graham gauge is to first map the original coordinate back to the uniformizing coordinate, and then transform from it to the new coordinate such that the composition is equivalent to \((g(z), \bar{g}(\bar{z}))\) on the boundary. In equations, this means the new point in the \(\tilde{P}\) satisfies

\[P_u\left(\tilde{P}\right) = P_u\left(P\right) \tag{C.3.4}\]

\[
\bar{f}^{-1} \circ f(z) = g(z), \quad \bar{f}^{-1} \circ \bar{f}(\bar{z}) = \bar{g}(\bar{z}) \tag{C.3.5}
\]

We consider a generic background that is specified by \((f(z), \bar{f}(\bar{z}))\). Then we do a small Virasoro transformation generated by \(L_m\) on this background. On the boundary, this transformation is defined as

\[(1 + \epsilon L_m)z = z + \epsilon z^{m+1} \tag{C.3.6}\]
This transformation takes
\[ (f, \tilde{f}) \rightarrow (\tilde{f}, \tilde{f}) \] (C.3.7)

\( \tilde{f} \) is determined by:
\[ f^{-1} \circ \tilde{f}(z) = z - \epsilon z^{m+1} \] (C.3.8)

which means
\[ \tilde{f} = f - \epsilon z^{m+1} f' \equiv f + \epsilon \delta_m f \] (C.3.9)

We then solve
\[ P_u (P + \epsilon \delta_m P) = P_u (P) \] (C.3.10)

The solution is (4.3.13), reproduced here:
\begin{align*}
\delta_m z &= \frac{z^{m-1} \left( (m^2 + m + z^2 S(z)) \bar{S}(\bar{z}) y^4 - 4z^2 \right)}{y^4 S(z) S(\bar{z}) - 4} \quad (C.3.11) \\
\delta_m \bar{z} &= \frac{2m(m+1)y^2 z^{m-1}}{y^4 S(z) S(\bar{z}) - 4} \quad (C.3.12) \\
\delta_m y &= \frac{1}{2} (m+1) y z^m \quad (C.3.13)
\end{align*}

Note that the \( f \) and \( \tilde{f} \) organize themselves exactly to reproduce \( S \) and \( \bar{S} \), where
\[ \bar{S} = \frac{\tilde{f}^{(3)} \tilde{f}' - \frac{3}{2} \tilde{f}^m}{\tilde{f}'^2} = \frac{12}{c} \tilde{T} \] (C.3.14)

Clearly, we see that \( L_m \) with \( m \geq 2 \) will leave points \((y, 0, 0)\) invariant.

For \( L_1 \) this is explicitly not the case. In fact, the action of \( L_1 \) is somewhat non-trivial.

On a background with \( L = 0 \) (correlators \( \langle \phi \bar{O} \bar{T} \cdots \bar{T} \rangle \) without any \( T \)), we have:
\[ L_1 \phi(y, 0, 0) = \left( -y^2 \partial - \frac{6}{c} y^4 \bar{T}(0) \partial \right) \phi(y, 0, 0). \] (C.3.15)
One way to test whether this is correct is to compute

\[\langle O(z, \bar{z}) \bar{T}(\bar{z}_1) L_1 \phi (y, 0, 0) \rangle \overset{?}{=} \langle O(z, \bar{z}) \bar{T}(\bar{z}_1) \left( -y^2 \bar{\partial} - \frac{6}{c} y^4 \bar{T}(0) \partial \right) \phi (y, 0, 0) \rangle \quad (C.3.16)\]

Note that the first term on the RHS, which is the naive transformation of \( \phi \) (it’s the transformation of \( \phi^\text{global} \) under \( L_1 \)), gives a wrong result:

\[- y^2 \langle O(z, \bar{z}) \bar{T}(\bar{z}_1) \bar{\partial} \phi (y, 0, 0) \rangle \quad (C.3.17)\]

\[= 2y^2 \bar{z} \left( y^2 \bar{z} (2(h - 3) \bar{z}_1 \bar{z} - 3(h - 1) \bar{z}_1^2 + 3 \bar{z}^2) - z^2 \bar{z}^2 \bar{z}_1 ((h - 1) \bar{z}_1 + \bar{z}) + 3y^4 \bar{z} (\bar{z} - \bar{z}_1)^2 \right) / (\bar{z} - \bar{z}_1)^2 \bar{z}_1^4 (z \bar{z} + y^2)^2 \]

This is wrong because it has a \( 1/\bar{z}_1 \) pole, which is inconsistent with the condition of equation (4.3.3). But the second term

\[- \frac{6}{c} y^4 \langle O(z, \bar{z}) [\bar{T}(\bar{z}_1) \bar{T}(0)] \partial \phi (y, 0, 0) \rangle = -3y^4 \frac{1}{\bar{z}_1} \langle O(z, \bar{z}) \partial \phi (y, 0, 0) \rangle = - \frac{6y^4 \bar{z}}{\bar{z}_1^4 (z \bar{z} + y^2)} \quad (C.3.18)\]

has precisely the right form to cancel this pole. Then combining these two terms, we have

\[\langle O(z, \bar{z}) \bar{T}(\bar{z}_1) \left( -\frac{6}{c} y^4 \bar{T}(0) \partial - y^2 \bar{\partial} \right) \phi (y, 0, 0) \rangle \]

\[= \langle O(z, \bar{z}) \phi (y, 0, 0) \rangle \frac{2y^2 z \bar{z}^2 \left( h \left( y^2 (2 \bar{z} - 3 \bar{z}_1) - z \bar{z} \bar{z}_1 \right) + z \bar{z} (\bar{z}_1 - \bar{z}) \right) \bar{z}^3 \bar{z}_1^2 (z \bar{z} + y^2)^2}{(\bar{z} - \bar{z}_1)^2 \bar{z}_1^4} \]

\[= - (2hz + z^2 \partial_z) \langle O(z, \bar{z}) \bar{T}(\bar{z}_1) \phi (y, 0, 0) \rangle \quad (C.3.19)\]

\[= - \langle [L_1, O(z, \bar{z})] \bar{T}(\bar{z}_1) \phi (y, 0, 0) \rangle \]

\[= \langle O(z, \bar{z}) \bar{T}(\bar{z}_1) L_1 \phi (y, 0, 0) \rangle \]

Similarly, we checked (C.3.15) also work in the case of \( \langle \bar{T} \bar{T} \partial L_1 \phi \rangle \). In particular, we checked

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\[
\langle \bar{T}(\bar{z}_1)\bar{T}(\bar{z}_2)O(z, \bar{z})L_1\phi(y, 0, 0) \rangle
\]
\[
= \langle \bar{T}(\bar{z}_1)\bar{T}(\bar{z}_2)O(z, \bar{z}) \left( -y^2\partial - \frac{6}{c}\bar{T}(0)y^4\partial \right) \phi(y, 0, 0) \rangle
\]
\[
= -y^2 \langle \bar{T}(\bar{z}_1)\bar{T}(\bar{z}_2)O(z, \bar{z}) \tilde{\partial} \phi(y, 0, 0) \rangle
\]
\[
- \frac{6}{c} y^4 \langle \bar{T}(\bar{z}_1)\bar{T}(\bar{z}_2)\bar{T}(0) \rangle \langle O(z, \bar{z}) \partial \phi(y, 0, 0) \rangle
\]
\[
- \frac{6}{c} y^4 \left( \langle \bar{T}(\bar{z}_1)\bar{T}(0) \rangle \langle \bar{T}(\bar{z}_2)O(z, \bar{z}) \partial \phi(y, 0, 0) \rangle + (z_1 \leftrightarrow z_2) \right)
\]
\[
= - \left( 2hz + z^2\partial_z \right) \langle \bar{T}(\bar{z}_1)\bar{T}(\bar{z}_2)O(z, \bar{z}) \phi(y, 0, 0) \rangle
\]
\[
= - \langle \bar{T}(\bar{z}_1)\bar{T}(\bar{z}_2) [L_1, O(z, \bar{z})] \phi(y, 0, 0) \rangle.
\]

The fact that these work nicely are non-trivial checks for our method.

## C.4 Additional Technical Results

### C.4.1 Gravitational Wilson Line Computations at Higher Orders

In this section, we provide the details to derive the bulk-boundary OPE block kernels up to order \( \frac{1}{c^2} \). First, we need to solve the following equation at large \( c \)

\[
S(f, z) \equiv \frac{f'''(z)f'(z) - \frac{3}{2}(f''(z))^2}{(f'(z))^2} = \frac{12}{c}T(z) \tag{C.4.1}
\]

and determine \( f(z) \) and \( \bar{f}(z) \) as functions of the stress tensor operators \( T(z), \bar{T}(z) \). We’ll do this by expanding \( f(z) \) in terms of large \( c \) as follows

\[
f(z) = f_0(z) + \sum_{n=1}^{\infty} \frac{f_n(z)}{c^n} \tag{C.4.2}
\]
with \( f_0(z) = z \) satisfies \( S(f_0(z), z) = 0 \) at leading order. At order \( \frac{1}{c} \) and \( \frac{1}{c^2} \), \( f_1(z) \) and \( f_2(z) \) are determined by the following differential equations

\[
f_1^{(3)}(z) - 12T(z) = 0, \tag{C.4.3}
\]

\[
2f_1^{(3)}(z)f_1'(z) + 3f_1''(z)^2 - 2f_2^{(3)}(z) = 0.
\]

The first equation is easy to solve and the solution with desired boundary condition is

\[
f_1(z) = 6 \int_0^z dz' (z - z')^2 T(z'). \tag{C.4.4}
\]

Using this solution, the second equation in (C.4.3) becomes

\[
f_2^{(3)} = f_1^{(3)}f_1' + \frac{3}{2}f_1''^2
\]

\[
= 144T(z) \int_0^z dz' (z - z') T(z') + 432 \int_0^z dz' \int_0^{z'} dz'' T(z') T(z'').
\]

And the solution is

\[
f_2(z) = 36 \int_0^z dz'' (z - z'')^2 \left[ 2T(z'') \int_0^{z''} dz' (z'' - z') T(z') + 6 \int_0^{z''} dz' \int_0^{z'} dz''' T(z') T(z''') \right]
\]

\[
= 72 \int_0^z dz' \int_0^{z'} dz'' T(z') T(z'') (z - z')^2 (z - z'').
\]

Now we can turn to the derivation of the bulk-boundary OPE block kernels. Expanding the coordinates transformation (4.2.8) in terms of large \( c \), i.e. using C.4.2 with \( f_0(z) = z \), we have

\[
u = y + \frac{y (\tilde{f}_1'(\bar{z}) + f_1'(z))}{2c} - \frac{y^2 (2y^2 f_1''(z)\tilde{f}_1''(\bar{z}) + (f_1'(z) - \tilde{f}_1'(\bar{z}))^2 - 4 (\tilde{f}_1''(\bar{z}) + f_2''(z)))}{8c^2} + \mathcal{O}(c^{-3})
\]

\[
w = z + \frac{f_1(z) - \frac{1}{2}y^2 \tilde{f}_1''(\bar{z})}{c} + \frac{2f_2(z) - y^2 (2f_1'(z) - \tilde{f}_1'(\bar{z}))}{2c^2} \tilde{f}_1''(\bar{z}) + \tilde{f}_2''(\bar{z})) + \mathcal{O}(c^{-3})
\]
and similar expression for $\bar{w}$. Expanding the bulk-boundary two-point function and using the above result, we find

$$
\log \phi (y, z_f, z_f) \mathcal{O} (z_i, \bar{z}_i)
$$

$$
= 2h \log \left( \frac{u_f \sqrt{w' (z_i) \bar{w}' (\bar{z}_i)}}{u_f^2 + (w_f - w_i) (\bar{w}_f - \bar{w}_i)} \right)
$$

$$
= 2h \log \left( \frac{y}{y^2 + \bar{w} \bar{z}} \right) + \frac{h (z \bar{z} + y^2 f_1 (z) - 2 \bar{z} f_1 (z))}{c (z \bar{z} + y^2)} - \frac{2h y^2 f_1 (z) \bar{f}_1 (\bar{z})}{c^2 (z \bar{z} + y^2)^2} - \frac{h ((z \bar{z} + y^2) (f_1' (z)^2 - 2 f_2' (z)) + 4 \bar{z} f_2 (z)) - 2 \bar{z}^2 f_1 (z)^2}{2c^2 (z \bar{z} + y^2)^2}
$$

$$
+ K_T + K_{TT} + \mathcal{O} (c^{-3})
$$

with $K_T$ and $K_{TT}$ the complex conjugate of $K_T$ and $K_{TT}$ respectively. In the third line of the above equations, we’ve put the two operators at $\phi (y, 0, 0)$ and $\mathcal{O} (z, \bar{z})$.

Plugging in the solutions for $f_n, \bar{f}_n$, we have

$$
K_T = \frac{12h}{c} \int_0^z d\zeta \frac{(y^2 + z' \bar{z}) (z - z')}{z \bar{z} + y^2} T (z')
$$

$$
K_{TT} = \int_0^z d\zeta' \int_0^{z'} d\zeta'' \frac{72h (z - z')^2 (y^2 + z z'')^2}{c^2 (z \bar{z} + y^2)^2} T (z') T (z'')
$$

$$
K_{TT} = -\frac{72h y^2}{c^2 (z \bar{z} + y^2)^2} \int_0^z d\zeta' (z - z')^2 \int_0^{z'} d\zeta'' (z - \zeta'')^2 T (z') T (\zeta')
$$

Sending $y = 0$, we find

$$
K_T \underset{y \to 0}{\approx} \frac{12h}{c} \int_0^z d\zeta \frac{z' (z - z')}{z} T (z')
$$
\[
K_{TT} \overset{y=0}{\rightarrow} \int_0^z dz' \int_0^{z''} d\bar{z}'' \, \frac{72h (z - z')^2 z''^2}{c z^2} T (z') T (z'')
\]

\[
K_{\bar{T}T} \overset{y=0}{\rightarrow} 0
\]

which are exactly the boundary-boundary OPE kernels found in [77].

### C.4.2 Computations Using the Bulk-Boundary OPE Block

In this section, we’ll provide the details for computing \( \langle \phi O T T \rangle \) and \( \langle \phi O \bar{T} T \rangle \) using bulk-boundary OPE block with the regulator proposed in Appendix C.2 of [77] and discussed in details in appendix C.2. The regulator (C.2.3-C.2.4) is basically saying that when computing \( \langle \phi O T_1 \cdots T_n \bar{T}_1 \cdots \bar{T}_m \rangle \), the kernels in the OPE block of \( \phi O \) that will contribute are those whose numbers of \( T \) and \( \bar{T} \) are equal or less than \( n \) and \( m \) respectively.

#### C.4.2.1 \( \langle \phi O T T \rangle \)

Using the regulator (C.2.3-C.2.4), the kernels in the bulk-boundary OPE of \( \phi O \) that contribute to \( \langle \phi O T T \rangle \) are \( K_T K_{\bar{T}} \) and \( K_{TT} \). So \( \langle \phi O T T \rangle \) is given by

\[
\frac{\langle \phi (y, 0, 0) O (z, \bar{z}) T (z_1) \bar{T} (\bar{w}_1) \rangle}{\langle \phi (y, 0, 0) O (z, \bar{z}) \rangle} = \langle e^{K_T + K_{\bar{T}} + K_{TT} + \cdots} T (z_1) \bar{T} (\bar{w}_1) \rangle
\]

\[
= \langle (K_T K_{\bar{T}} + K_{TT}) T (z_1) \bar{T} (\bar{w}_1) \rangle \tag{C.4.5}
\]

The first term is

\[
\langle K_T K_{\bar{T}} T (z_1) \bar{T} (\bar{w}_1) \rangle
\]

\[
= \left( \frac{144 h^2}{c^2} \int_0^z dz' \int_0^z d\bar{z}' \left( \frac{y^2 + z' \bar{z}}{z \bar{z} + y^2} \right) \frac{(z - z') (y^2 + \bar{z}' z) (\bar{z} - \bar{z}')}{z \bar{z} + y^2} \right) \langle [T (z')] \bar{T} (\bar{w}_1) \rangle T (z_1) \bar{T} (\bar{w}_1)
\]

\[
= \frac{h^2 z^2 \bar{z}^2 (y^2 (3 \bar{w}_1 - 2 \bar{z}) + \bar{w}_1 z \bar{z}) (y^2 (3z_1 - 2z) + z_1 z \bar{z})}{z_1^3 \bar{w}_1^3 (z - z_1)^2 (\bar{w}_1 - \bar{z})^2 (z \bar{z} + y^2)^2}
\]
where in the second line we use the regulated four-point function

$$\left\langle \left[T \left( z' \right) \bar{T} \left( \bar{z}' \right) \right] T \left( z_1 \right) \bar{T} \left( \bar{w}_1 \right) \right\rangle = \frac{\epsilon^2}{4} \frac{1}{(z' - z_1)^4 (\bar{z}' - \bar{w}_1)^4}. \quad (C.4.6)$$

The above result is just the contribution from $\langle \phi \mathcal{O} T \rangle \langle \phi \mathcal{O} \bar{T} \rangle$.

The second term in equation (C.4.5) is

$$\langle K_T T \left( z_1 \right) T \left( \bar{w}_1 \right) \rangle$$

$$= -\frac{72 \epsilon^2}{c^2 (z \bar{z} + y^2)^2} \int_0^y d\bar{z}' \int_0^y d\bar{z} \left( z - \bar{z} \right)^2 \left( \bar{z} - \bar{z}' \right)^2 \left\langle \left[T \left( \bar{z}' \right) \bar{T} \left( \bar{z} \right) \right] T \left( z_1 \right) \bar{T} \left( \bar{w}_1 \right) \right\rangle$$

$$= \frac{2 \epsilon^2 z^3 \bar{z}^3}{z_1 \bar{w}_1 (z - z_1) (\bar{w}_1 - \bar{z}) (z \bar{z} + y^2)^2} \quad (C.4.7)$$

So putting these two terms together, we get

$$\frac{\langle \phi \left( y, 0, 0 \right) \mathcal{O} \left( z, \bar{z} \right) T \left( z_1 \right) \bar{T} \left( \bar{w}_1 \right) \rangle}{\langle \phi \left( y, 0, 0 \right) \mathcal{O} \left( z, \bar{z} \right) \rangle} = \frac{h^2 z^2 \bar{z}^2 \left( y^2 \left( 3 \bar{w}_1 - 2 \bar{z} \right) + \bar{w}_1 z \bar{z} \right) \left( y^2 \left( 3 z_1 - 2 z \right) + z_1 z \bar{z} \right)}{z_1 \bar{w}_1 (z - z_1) (\bar{w}_1 - \bar{z}) (z \bar{z} + y^2)^2} + \frac{2 \epsilon^2 z^3 \bar{z}^3}{z_1 \bar{w}_1 (z - z_1) (\bar{w}_1 - \bar{z}) (z \bar{z} + y^2)^2}$$

Sending $y \to 0$, the second term vanishes, and the first term will reduce to the boundary four-point function $\langle \mathcal{O} \left( 0, 0 \right) \mathcal{O} \left( z, \bar{z} \right) T \left( z_1 \right) \bar{T} \left( \bar{w}_1 \right) \rangle = \langle \mathcal{O} \mathcal{O} T \rangle \langle \mathcal{O} \mathcal{O} \bar{T} \rangle = \frac{h^2 z^2 \bar{z}^2}{z^2 \bar{z}^2 (z_1 - z)^2 (\bar{w}_1 - \bar{z})^2}$ as expected.

**C.4.2.2 $\langle \phi \mathcal{O} T T \rangle$**

Using the regulator (C.2.3-C.2.4), the kernels in the bulk-boundary OPE of $\phi \mathcal{O}$ that contribute to $\langle \phi \mathcal{O} T T \rangle$ are the identity, $K_T$, $K_T T$ and $K_T K_T$. So $\langle \phi \mathcal{O} T T \rangle$ is given by

$$\frac{\langle \phi \left( y, 0, 0 \right) \mathcal{O} \left( z, \bar{z} \right) T \left( z_1 \right) T \left( z_2 \right) \rangle}{\langle \phi \left( y, 0, 0 \right) \mathcal{O} \left( z, \bar{z} \right) \rangle} = \langle \epsilon^{K_T + K_T T + \cdots} T \left( z_1 \right) T \left( z_2 \right) \rangle$$

$$= \langle T \left( z_1 \right) T \left( z_2 \right) \rangle + \left( \left(K_T + K_T T + \frac{K_T K_T}{2}\right) T \left( z_1 \right) T \left( z_2 \right) \right). \quad (C.4.9)$$
The first term is trivial and it’s just \( \langle T(z_1) T(z_2) \rangle = \frac{c}{4(z_1 - z_2)^2}. \)

The first two terms in the second braket give the following contribution

\[
\langle K_T T(z_1) T(z_2) \rangle + \langle K_{TT} T(z_1) T(z_2) \rangle
\]
\[
= \frac{12h}{c} \int_0^z dz' \frac{(y^2 + z' \bar{z}) (z - z')}{z \bar{z} + y^2} \langle T(z') T(z_1) T(z_2) \rangle
\]
\[
+ \int_0^z dz' \int_0^{z'} dz'' \frac{72h (z - z')^2 (y^2 + \bar{z} z'')^2}{c^2 (z \bar{z} + y^2)^2} \langle [T(z') T(z'')] T(z_1) T(z_2) \rangle \tag{C.4.10}
\]
\[
= 2hz^2 \left( y^2 z \bar{z} z_1 z_2 (z (z_1 + z_2) - 4z_1 z_2) - \bar{z}^2 z_1 z_2^2 + y^4 (z z_1 z_2 (z_1 + z_2) - 3z_1^2 z_2^2 - z^2 (z_1 - z_2)^2) \right)
\]
\[
(z - z_1) z_1^2 z_2^2 (z_2 - z) (z_2 - z_1)^2 (z \bar{z} + y^2)^2
\]

where in the second line and third line, we used

\[
\langle T(z') T(z_1) T(z_2) \rangle = \frac{c}{(z_1 - z_2)^2 (z_2 - z') (z_1 - z')^2}, \tag{C.4.11}
\]
\[
\langle [T(z') T(z'')] T(z_1) T(z_2) \rangle = \frac{c^2}{4} \left( \frac{1}{(z' - z_1)^4 (z'' - z_2)^4} + \frac{1}{(z' - z_2)^4 (z'' - z_1)^4} \right).
\]

Notice that there is no logarithm in the result of equation (C.4.10). But if one computes \( \langle K_T T(z_1) T(z_2) \rangle \) and \( \langle K_{TT} T(z_1) T(z_2) \rangle \) separately, one can see that they both have logarithmic terms, but they cancel out exactly!

The last term in the second bracket of equation (C.4.9) is

\[
\left\langle \frac{K_T K_T}{2} T(z_1) T(z_2) \right\rangle
\]
\[
= \int_0^z dz' \int_0^{z'} dz'' \left( \frac{72h^2 (y^2 + z' \bar{z}) (z - z') (y^2 + z'' \bar{z}) (z - z'')}{z \bar{z} + y^2} \right) \langle [T(z') T(z'')] T(z_1) T(z_2) \rangle
\]
\[
= \frac{h^2 z^4 (z z_1 \bar{z} + 2y^2 (3z_1 - 2z)) (z z_2 \bar{z} - 2y^2 \bar{z} + 3y^2 z_2)}{(z - z_1)^2 z_1^2 z_2^2 (z_2 - z)^2} \tag{C.4.12}
\]

which is just the contribution from \( \langle \phi OT \rangle \langle \phi OT \rangle \).
So putting everything together, we have

$$\frac{\langle \phi (y, 0, 0) \mathcal{O} (z, \bar{z}) T (z_1) T (z_2) \rangle}{\langle \phi (y, 0, 0) \mathcal{O} (z, \bar{z}) \rangle} = \frac{c}{2 (z_1 - z_2)^4} + \frac{h^2 z^4 (z_1 z \bar{z} + y^2 (3 z_1 - 2 z)) (z_2 z \bar{z} + y^2 (3 z_2 - 2 z))}{z_1^2 \bar{z}_2^2 (z - z_1)^2 (z - z_2)^2 (z \bar{z} + y^2)^2} \ 	ext{(C.4.13)}$$

$$+ \frac{2 h z^2 \left(y^2 z \bar{z} z_1 z_2 (z (z_1 + z_2) - 4 z_1 z_2) - z^2 z \bar{z} z_1^2 z_2^2 + y^4 \left(z z_1 z_2 (z_1 + z_2) - 3 z_1^2 z_2^2 - z^2 (z_1 - z_2)^2 \right)\right)}{(z - z_1) z_1^2 \bar{z}_2^2 (z_2 - z) (z_2 - z_1)^2 (z \bar{z} + y^2)^2}$$

Sending $y \to 0$ the above result does give us $\frac{\langle \mathcal{O}(0,0) \mathcal{O}(z,\bar{z}) T(z_1) T(z_2) \rangle}{\langle \mathcal{O}(0,0) \mathcal{O}(z,\bar{z}) \rangle}$, which is

$$\frac{\langle \mathcal{O} (0, 0) \mathcal{O} (z, \bar{z}) T (z_1) T (z_2) \rangle}{\langle \mathcal{O} (0, 0) \mathcal{O} (z, \bar{z}) \rangle} = \frac{1}{(z_1 - z_2)^4} \left[ \frac{c}{2} + \frac{hu^2 (hu - 2) + 2}{(u - 1)^2} \right] \ 	ext{(C.4.14)}$$

$$= \frac{c}{2 (z_1 - z_2)^4} + \frac{h u^2 \left(h_1^2 + \frac{2 z_1 z_2 (z - z_1) (z - z_2)}{(z_1 - z_2)^2}\right)}{z_1^2 \bar{z}_2^2 (z - z_1)^2 (z - z_2)^2}$$

where $u \equiv \frac{z_1 z_2}{z_1 z_2} = \frac{(z_1 - z_2)}{(z_1 - z_2) z_2}$ is the cross ratio.

### C.4.3 Spinning Bulk Wilson Lines

In this appendix we give the derivation of equation (4.2.28) in the text. To begin, we recall how to write the bulk-to-boundary propagators in the vacuum. The general procedure was described in [188], and takes the form

$$\langle A^{\mu_1 \mu_2 \cdots \mu_\ell} (y, z_1) \mathcal{O}_{h, \bar{h}} (z_2, \bar{z}_2) \rangle = \left(\frac{y}{y^2 + z_1 \bar{z}_1} \right)^{2h} \xi'^{\mu_1} \cdots \xi'^{\mu_\ell}, \ (\pm = \text{sgn} (\ell)), \ 	ext{(C.4.15)}$$

for the case $h - \bar{h} = \ell$ of interest. Here, $(\xi'^y, \xi'^z, \xi'^{\bar{z}}) = (y z_1, z_1^2, -y^2)$ is the Killing vector associated with holomorphic special conformal generators, and $(\xi^y, \xi^z, \xi^{\bar{z}}) = (y \bar{z}_1, -y^2, \bar{z}_1^2)$ for anti-holomorphic ones.

To promote this to an arbitrary background, we perform the transformation (4.2.8).
Because $A_{\mu_1...\mu_\ell}$ is a tensor, this transformation includes factors of

$$
\frac{\partial x'^\mu_f}{\partial x'^\nu_f},
$$

where $x_f$ are the transformed coordinates $(y_f, z_f, \bar{z}_f)$. The transformed coordinates include dependence on the second derivatives of $f, \bar{f}$, and so the above Jacobian factor depends on its third derivatives. These third derivatives $f'''(z_2), \bar{f}'''(\bar{z}_2)$ can be eliminated in terms of the stress tensor $T(z_2), \bar{T}(\bar{z}_2)$ at the point $(z_2, \bar{z}_2)$. Moreover, as before we can eliminate $f'(z_2), f''(z_2)$ in terms of $x_T(z_1)$ and $E_T$. Making such substitutions, we find that

$$
(\xi'_-)_\mu' = \frac{\partial x'^\mu}{\partial x'^\nu_f}(\xi_-)_\mu(u_2, w_2, \bar{w}_2) = t^\mu_\mu' E_T^{-2} \bar{E}^{-2} \bar{f}'(\bar{z}_1)(\xi_-)_\mu(y, z, \bar{z}),
$$

where $t^\mu_\mu'$ is given in (4.2.29). We also have, from massaging (4.2.11) a bit, that

$$
\left(\frac{u_2}{u_2^2 + f_21 f_{21}}\right)^{2h} = E_T^{2h} \bar{E}_T^{2h} (f'(z_1)\bar{f}'(\bar{z}_1))^{-h} \left(\frac{y}{y^2 + x_T(z_1)\bar{x}_T(\bar{z}_1)}\right)^{2h}.
$$

Multiplying by $(\xi'_-)_\mu'\cdots(\xi'_-)_\mu'$ and using (C.4.17), we find

$$
\left(\frac{u_2}{u_2^2 + f_21 f_{21}}\right)^{2h}(\xi'_-)_\mu'\cdots(\xi'_-)_\mu' = E_T^{2h} \bar{E}_T^{2h} (f'(z_1))^{-h} (f'(\bar{z}_1))^{-h}
$$

$$
\times \left(\frac{y}{y^2 + x_T(z_1)\bar{x}_T(\bar{z}_1)}\right)^{2h} t^\mu_1\cdots t^\mu_\ell(\xi_-)_\mu_1\cdots(\xi_-)_\mu_\ell.
$$

Equation (4.2.28) follows by using the fact that the Wilson line factors simply impose the constraint $x \to x_T(z_1), \bar{x} \to x_T(\bar{z}_1)$ and produce factors $E_T^{2h}, \bar{E}_T^{2h}$.

### C.4.4 Bulk Witten Diagram Computation for $\langle \phi OT \rangle$

In this section, we will show that the result we obtained for $\langle \phi OT \rangle$ using bulk-boundary OPE block and the recursion relation agrees with the result of the bulk Witten diagram
computation for $\langle \phi(y,0,0)O(z,\bar{z})T(z_1) \rangle$, shown in Fig. C.1. This should be expected, as the definition of equation (4.2.4) is essentially the first-quantized version of the bulk field theory that leads to the Witten diagram we will discuss. We will first show that the result is exact, using a trick [189] that obviates the need to perform integrals over AdS$_3$. Then we will explicitly evaluate the diagram in the large $h$ limit using saddle point approximation (this will give the same exact result), where we can make direct contact with some of the results from section 4.2.

![Diagram](image)

**Figure C.1:** Dashed (solid) lines are graviton (scalar) propagators.

In order to compute this diagram, we need four ingredients: the scalar bulk-to-boundary propagator, the scalar bulk-to-bulk propagator, the vertex structure associated with the scalar-graviton interaction, and the graviton bulk-to-boundary propagator. The standard prescription is to multiply these propagators together, and integrate over the bulk. There are a variety of conventions for normalizing these objects, so we will mostly ignore the overall numerical prefactors, which can be fixed in any case in terms of operator normalizations and the stress tensor Ward identity.

The bulk-to-bulk propagator, specializing to our coordinate set-up, is given by

$$G_{(y,0,0),(y',z',\bar{z}')} = \frac{e^{-2h\sigma}}{1 - e^{-2\sigma}}, \quad (C.4.20)$$
where $\sigma \equiv \sigma(y', z', \bar{z}', (y, 0, 0))$ is the bulk-bulk geodesic between $(y', z', \bar{z}')$ and $(y, 0, 0)$

$$\sigma(y', z', \bar{z}', (y, 0, 0)) = \log \frac{1 + \sqrt{1 - \xi^2}}{\xi}, \quad \text{with} \quad \xi = \frac{2yy'}{y^2 + y'^2 + z'\bar{z}'}.$$  \hspace{1cm} (C.4.21)

The scalar bulk-to-boundary propagator is given by

$$K(y', z', \bar{z}', (z, \bar{z})) = \left( \frac{y'}{y'^2 + (z - z')(\bar{z} - \bar{z}')} \right)^{2h},$$  \hspace{1cm} (C.4.22)

while can also be written as

$$K(y', z', \bar{z}', (z, \bar{z})) = e^{-2h\sigma(y', z', \bar{z}', (z, \bar{z}))},$$  \hspace{1cm} (C.4.23)

where $\sigma(y', z', \bar{z}', (z, \bar{z})) = \log \frac{y'^2 + (z' - z)(\bar{z}' - \bar{z})}{y'}$ is the regulated bulk-boundary geodesic length.

The vertex structure is given by $h_{\mu\nu}T^\mu_\mu$, where $T^\mu_\mu$ is the bulk matter stress energy tensor. It can be derived from the bulk equations of motion, and is given by [190]

$$T^\mu_\mu = (g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha})\partial_\alpha K\partial_\beta G - g^{\mu\nu}(g^{\rho\alpha}\partial_\rho K\partial_\alpha G + m^2 KG).$$  \hspace{1cm} (C.4.24)

We are interested in the holomorphic part of this tensor object, since the coupling we need is $h_{zz}T^zz_\mu$. In the Fefferman-Graham gauge, it simplifies to

$$T^zz_\mu = 2g^{zz}g^{zz}\partial_z K\partial_z G = -2y^4G\partial_z^2 K.$$  \hspace{1cm} (C.4.25)

Finally, we need the graviton bulk-to-boundary propagator in this gauge. $h_{zz}(y, z, \bar{z})$ is by
definition equal to $-\frac{6T(z)}{c}$, as in equation (5.2.1). So we have

$$\langle h_{zz}(y', z', \bar{z}') T(z_1) \rangle = -\frac{6}{c} \langle T(z') T(z_1) \rangle = \frac{-3}{(z' - z_1)^4} \quad \text{(C.4.27)}$$

Putting these ingredients together, the bulk integral corresponding to fig. C.1 is then

$$\langle \phi(y, z_3, \bar{z}_3) \mathcal{O}(z_2, \bar{z}_2) T(z_1) \rangle = \int_{\text{AdS}_3} \sqrt{g} dz' dz' dy' (-2y'^4)$$

$$\times G_{(y, z_3, \bar{z}_3), (y', z', \bar{z}')}(z_2, \bar{z}_2) \frac{-3}{(z' - z_1)^4}. \quad \text{(C.4.28)}$$

The trick [189] to evaluating this kind of Witten diagram integral is first to simplify the problem as much as possible using global conformal invariance, and second to recall that the bulk scalar Feynman propagator satisfies the Klein-Gordon equation

$$\left(\nabla^2 - m^2\right) G(X, Y) = \delta_{\text{AdS}}(X - Y), \quad \text{(C.4.29)}$$

where $m^2 = 2h(2h - 2)$. This means that if we act with the bulk differential operator $(\nabla^2 - m^2)$ on the Witten diagram that computes $\langle \phi(X) \mathcal{O}(z_2) T(z_1) \rangle$, then we will be left with just the integrand above, with $G$ removed. We can simplify the calculation by shifting $z_2$ to 0 with a translation, then performing an inversion, and finally shifting $z_3 \to 0$ by

---

\[2\] It was also shown [150] using smearing functions that $h_{zz}(y, z, \bar{z})$ is simply given by boundary stress energy tensor $T(z)$:

$$h_{zz}(y, z, \bar{z}) \propto \frac{1}{\pi y^2} \int_{z \leq y^2} dz' z' T_{zz}(z + iz')$$

$$= \frac{1}{\pi y^2} \int_0^y r dr \int_0^{2\pi} d\theta T(z + ire^{i\theta}) \quad \text{(C.4.26)}$$

$$= T(z).$$
The resulting equation of motion is

\[
(\nabla^2 - m^2)A(y, z_3, \bar{z}_3) = -12\Delta(\Delta + 1)\bar{z}_1^4y^\Delta \left(\frac{y}{y^2 + z_3\bar{z}_3}\right)^4,
\]

\[A(y, z_3, \bar{z}_3) \equiv \langle \phi(y', z_3', \bar{z}_3')O(z_2', \bar{z}_2')T(z'_1)\rangle, \tag{C.4.30}\]

where \((y', z'_1)\) are the transformed coordinates. For comparison, the result in (4.2.30) in terms of the transformed coordinates is

\[\langle \phi(y', z_3', \bar{z}_3')O(z_2', \bar{z}_2')T(z'_1)\rangle = \frac{\Delta y^\Delta z_1^4z_3^2(3y^2 + z_3\bar{z}_3)}{2(y^2 + z_3\bar{z}_3)^3} = \frac{\Delta}{2}y^{\Delta-4}z_3^4t^2(1 + 2t), \tag{C.4.31}\]

where \(t \equiv \frac{y^2}{y^2 + z_3\bar{z}_3}\). Taking \(A(y, z_3, \bar{z}_3) = y^{\Delta-4}z^2f(t)\), the equation of motion is simply

\[f''(t) + \frac{(-\Delta + (\Delta - 1)t + 4)f'(t)}{(t - 1)t} + \frac{2(\Delta - 3)f(t)}{(t - 1)t^2} - \frac{3\Delta(\Delta + 1)t^2}{t - 1} = 0. \tag{C.4.32}\]

It is straightforward to check that the result in (4.2.30), i.e. \(f(t) = \frac{1}{2}t^2(1 + 2t)\), satisfies this equation. More constructively, there are two boundary conditions that must be imposed to fix the solution; one of these is that there is no \(y^{2-\Delta}\) piece near the boundary, and the other can be chosen so that the correct \(\langle OOT \rangle\) three-point function is reproduced at \(y \sim 0\); since (4.2.30) manifestly satisfies these conditions, it is the correct solution. Thus our result exactly matches the Witten diagram.

Next, at large \(h\), we can also evaluate the integral (C.4.28) directly using saddle point approximation (the result of this saddle point approximation turns out to be exact) and see how the kernel (4.2.15) emerges. After some manipulations, the bulk integral (C.4.28) can be re-cast into a more suggestive form

\[\langle \phi(y, 0, 0)O(z, \bar{z})T(z_1)\rangle = 12h(2h + 1) \int_{\text{AdS}_4} \frac{dz'dz'dy'}{y'^3} e^{-2hL(y', z', \bar{z}')} \frac{e^{-2\sigma(y', z', \bar{z}', z, \bar{z})}}{1 - e^{-2\sigma(y, 0, 0, y', z', \bar{z}')}} y^2 (z' - z_1)^4. \tag{C.4.33}\]

\(^3\) Because of the presence of the bulk coordinate \(y\), it is not enough to just take \(z_2 \to \infty\), rather, we must actually perform the transformation \((z \to z - z_2\) followed by an inversion) that takes \(z_2 \to \infty\).
The notation $\sigma_{a,b}$ indicates the (regulated) geodesic length between points $a$ and $b$. We have also defined $L(y', z', \bar{z}')$ to be the sum of the lengths of geodesics from $(y, 0, 0)$ to $(y', z', \bar{z}')$ and from $(y', z', \bar{z}')$ to $(z, \bar{z})$, that is

$$L(y', z', \bar{z}') \equiv \sigma_{(y, 0, 0), (y', z', \bar{z}')} + \sigma_{(y', z', \bar{z}'), (z, \bar{z})}. \quad \text{(C.4.34)}$$

In the large $h$ limit, the integral will localize along the geodesics from $(y, 0, 0)$ to $(z, \bar{z})$ to minimize $L$. This geodesic parameterized by $z'$ is given by

$$z' = \frac{\bar{z}}{z} z', \quad y'^2 = \left(1 - \frac{z'}{z}\right) (y^2 + z' \bar{z}), \quad \text{(C.4.35)}$$

so that the saddle point approximation to equation (C.4.33) is

$$\langle \phi OT \rangle \propto \frac{24h^2}{c} e^{-2hL(y,0,0)} \int_0^z dz' \frac{1}{\sqrt{\det \partial^2 L}} \frac{e^{-2\sigma(y',z',\bar{z}'),(z,\bar{z})} 1}{1 - e^{-2\sigma(y,0,0),(y',z',\bar{z}')} y'(z' - z_1)^4}, \quad \text{(C.4.36)}$$

where the determinant is given by

$$\det \partial^2 L = \det \begin{pmatrix} \partial^2_{z'} L & \partial_{z'} \partial_{y'} L \\ \partial_{y'} \partial_{z'} L & \partial^2_{y'} L \end{pmatrix} = \frac{4z^5(z'\bar{z} + y^2)}{z'^2(z' - z)(z\bar{z} + y^2)^4}, \quad \text{(C.4.37)}$$

evaluated along the geodesic (C.4.35). Plugging this in (and neglecting an order 1 numerical factor) and performing the $z'$ integral, we obtain

$$\langle \phi(y, 0, 0) O(z, \bar{z}) T(z_1) \rangle \propto \frac{12h}{c} \langle \phi(y, 0, 0) O(z, \bar{z}) \rangle \int_0^z dz' \frac{2(z - z')(z'\bar{z} + y^2)}{z\bar{z} + y^2} \frac{c}{2(z' - z_1)^4}$$

$$= \langle \phi(y, 0, 0) O(z, \bar{z}) \rangle \frac{h^2}{z_1^3(z_1 - z)^2} \left( z_1 + \frac{2y^2(z_1 - z)}{y^2 + z\bar{z}} \right) \quad \text{(C.4.38)}$$

matching equation 4.2.30 as expected. This demonstrates how the kernel of equation (4.2.15) emerges from a bulk Witten diagram calculation.
C.4.5 Solving for the Quantum Operator $\phi$

C.4.5.1 Solutions to the Conditions of equation (4.3.3) at Level 3 and Level 4

In this section, we provide the solutions to the conditions of equation (4.3.3) at level 3 and level 4.

At level 3, $|\phi\rangle_3 = \lambda_3 \mathcal{L}_{-3} \mathcal{T} \mathcal{L}_{-3} |\mathcal{O}\rangle$ and $\lambda_3 \mathcal{L}_{-3}$ is given by with

$$
\lambda_3 \mathcal{L}_{-3} = (-1)^3 \left( \frac{L_{-1}^3}{|L_{-1} \mathcal{O}|^2} + \frac{L_{-1} \mathcal{L}^{\text{quasi}}_{-2}}{|L_{-1} \mathcal{L}^{\text{quasi}}_{-2} \mathcal{O}|^2} + \frac{\mathcal{L}^{\text{quasi}}_{-3}}{|\mathcal{L}^{\text{quasi}}_{-3}|^2} \right),
$$

(C.4.39)

where $\mathcal{L}^{\text{quasi}}_{-3} = L_{-1}^3 - 2(h+1) L_{-1} L_{-2} + (h+1)(h+2) L_{-3}$ and the norms are

$$
|L_{-1} \mathcal{L}^{\text{quasi}}_{-2} \mathcal{O}|^2 = 2(h+2) |\mathcal{L}^{\text{quasi}}_{-2} \mathcal{O}|^2 = 4 \frac{(2h+1)(h+2)((2h+1)c+2h(8h-5))}{9},
$$

$$
|\mathcal{L}^{\text{quasi}}_{-3}|^2 = 2h(h+1)(h+2) \left( (c-7)h+c+3h^2+2 \right).
$$

At level 4, $|\phi\rangle_4 = \lambda_4 \mathcal{L}_{-4} \mathcal{T} \mathcal{L}_{-4} |\mathcal{O}\rangle$ and $\lambda_4 \mathcal{L}_{-4}$ is given by

$$
\lambda_4 \mathcal{L}_{-4} = \frac{L_{-1}^2 \mathcal{L}^{\text{quasi}}_{-2}}{|L_{-1} \mathcal{L}^{\text{quasi}}_{-2}|^2} + \frac{L_{-1} \mathcal{L}^{\text{quasi}}_{-3}}{|L_{-1} \mathcal{L}^{\text{quasi}}_{-3}|^2} + b_{4,1} \mathcal{L}^{\text{quasi},(4,1)}_{-4} + b_{2,2} \mathcal{L}^{\text{quasi},(2,2)}_{-4}
$$

(C.4.40)

where

$$
\mathcal{L}^{\text{quasi},(4,1)}_{-4} = L_{-1}^4 - \frac{4(2h+3)}{375} \left[ (16h(2h+11) + 267)L_{-4} - 5(6h+9)L_{-2}^2 - 5(16h + 49)L_{-1}L_{-3} + 125L_{-1}^2 L_{-2} \right],
$$

$$
\mathcal{L}^{\text{quasi},(2,2)}_{-4} = L_{-1}^4 + \frac{16}{9} h(h+3)L_{-2}^2 + \left( \frac{8h}{3} + 10 \right) L_{-1}L_{-3} - \frac{4}{3}(2h+3)L_{-1}^2 L_{-2} - 4(h+3)L_{-4}.
$$

$\mathcal{L}^{\text{quasi},(4,1)}_{-4}$ and $\mathcal{L}^{\text{quasi},(2,2)}_{-4}$ are not orthogonal to each other. $\mathcal{L}^{\text{quasi},(4,1)}_{-4}$ becomes a null-state when $c = c_{4,1}(h) = -\frac{8h}{5} - \frac{45}{2h+3} + \frac{53}{5}$, and $\mathcal{L}^{\text{quasi},(2,2)}_{-4}$ becomes a null-state when $c = c_{2,2}(h) = 1 - 8h$. The coefficients of them, $b_{4,1}$ and $b_{2,2}$ are given by

$$
b_{4,1} = \frac{1125(10c + 116h - 81)}{8(2h+3)(2h+5)(8h-3)(8h+27)(5c(2h+3) + 2(h-1)(8h-33))(2ch+c+2h(8h-5))},
$$

$$
333
$$
They are actually the solution to

\[
\begin{pmatrix}
\left\langle \mathcal{O} \right| \mathcal{L}_{-4}^{\text{quasi},(4,1)} \left| \mathcal{O} \right\rangle & \left\langle \mathcal{O} \right| \mathcal{L}_{-4}^{\text{quasi},(4,1)} \left| \mathcal{O} \right\rangle \\
\left\langle \mathcal{O} \right| \mathcal{L}_{-4}^{\text{quasi},(2,2)} \left| \mathcal{O} \right\rangle & \left\langle \mathcal{O} \right| \mathcal{L}_{-4}^{\text{quasi},(2,2)} \left| \mathcal{O} \right\rangle
\end{pmatrix}
\begin{pmatrix}
b_{4,1} \\
b_{2,2}
\end{pmatrix}
= \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

One can show that for non-orthogonal quasi-primaries at higher order, their coefficients will be given by the solutions to the equation corresponding to the above one at that order. And for global descendants of these non-orthogonal quasi-primaries, their coefficients will be given by a similar equation. These equations can be derived using the method similar to the one in section 4.3.2.2.

C.4.5.2 From Vacuum Sector Correlators to φ Via the OPE

We determined the vacuum sector correlators

\[
\langle \phi(X)\mathcal{O}(z)T(z_1)\cdots T(z_n)\bar{T}(\bar{z}_1)\cdots \bar{T}(\bar{z}_m) \rangle
\]

using the bulk-boundary OPE block in section 4.2. Thus we can straightforwardly determine the BOE expansion, expressing \( \phi_N \) in terms of Virasoro descendants of \( \mathcal{O} \) by studying the multi-OPEs of \( \mathcal{O} \) with the various stress tensors.

To perform this analysis explicitly, we start with the \( \langle \phi\mathcal{O} \rangle \) correlator and then add more and more \( T \) and \( \bar{T} \), modifying \( \phi_N \) each time to obtain the correct correlators. We already found that global BOE of equation (4.3.6) produces the correct \( \langle \phi\mathcal{O} \rangle \) correlator (see appendix C.1.3 for details). Thus the next step is to modify the BOE to achieve the correct \( \langle \phi\mathcal{O}T \rangle \) correlators, without disrupting \( \langle \phi\mathcal{O} \rangle \). For this purpose it is useful to compute

\[
\langle \phi_{\text{global}}(y, 0, 0)\mathcal{O}(z)T(z_1) \rangle = \frac{h(z\bar{z} + y^2)^2}{(z_1 - z)^2(z_1\bar{z} + y^2)^2} \langle \phi(y)\mathcal{O}(z) \rangle
\]

(C.4.42)
as shown via a more general argument in appendix C.1.3. Now we can subtract this result from the full correlator in equation (4.2.30) to obtain correlators of $\phi_N$ with the contributions of global conformal descendants of $O$ removed. Expanding to low order in $y$, this is

$$\left(\frac{z\bar{z}}{y}\right)^h \left\langle O(z) T(z_1) \left( \phi - \phi^{\text{global}} \right) \right\rangle = -\frac{3hy^4}{z_1^4z^2} + 2hy^6 \left( \frac{1}{z_1^4z^2z^3} + \frac{2}{z_1^5z^3} \right) + \cdots \quad (C.43)$$

Notice that the expansion only begins at order $y^4$, and that as a function of $z_1$, the location of the stress tensor, each term has a pole at the origin of order 4 or higher. The first observation indicates that the first Virasoro correction occurs in $\phi_2$, while the second confirms that these corrections all involve Virasoro descendants of $O$, i.e., new quasi-primaries like $[TO]$. We can match to the Virasoro descendants at levels 2 and 3, namely the operators $L_{-2}\bar{L}_{-1}O$, $L_{-3}\bar{L}_{-1}O$, and $L_{-1}L_{-2}\bar{L}_{-1}O$, by computing correlators such as

$$\left\langle O(z, \bar{z}) T(z_1) L_{-2}\bar{L}_{-1}O(0) \right\rangle \approx \frac{2h(2h+1)}{z^{2h+2}z_1^{2h}} \frac{c}{2z_1^2} \quad (C.44)$$

where we have neglected terms that are independent of $c$. Comparing this with equation C.4.43 at large $c$, we see that we need to add

$$\delta \phi_2 \approx -\left( \frac{y^4}{2!(2h)_2} \right) \frac{12h}{c} L_{-2}\bar{L}_{-1}O(0) \quad (C.45)$$

to $\phi_2$ at this order. At order $y^6$ we would add a linear combination of $L_{-3}\bar{L}_{-1}O$ and $L_{-1}L_{-2}\bar{L}_{-1}O$.

The second step in the analysis is to go back and ‘fix’ the $\left\langle \phi O \right\rangle$ correlators, as $\delta \phi_2$ above will alter it. To achieve this goal, we simply need to supplement $\delta \phi_2$ to make it proportional to a new level 2 quasi-primary. This leads to

$$\delta \phi_2 \approx \left( \frac{y^4}{2!(2h)_2} \right) \left( L_{-1}^2 - \frac{12h}{c} L_{-2} \right) \bar{L}_{-1}O(0) \quad (C.46)$$
to leading order at large $c$. With this choice, $\delta \phi_2$ will have a vanishing correlator with $\mathcal{O}$, and thus $\langle \phi \mathcal{O} \rangle$ will remain correct.

However, we can determine all of these coefficients more precisely and systematically using the condition of equation (4.3.3), as we’ll do in next subsection.

**C.4.5.3 Solving for $\phi$ at Large $c$**

In this section, we’ll use the definition of $\phi$ to derive the leading order terms of the $\frac{1}{c}$ and $\frac{1}{c^2}$ corrections to $\phi$.

We know that at the leading order of the large $c$ limit, $\phi(y, 0, 0)$ will reduce to $\phi^{\text{global}}(y, 0, 0)$, that is

$$\lim_{c \to \infty} \phi(y, 0, 0) |0\rangle = \phi^{\text{global}}(y, 0, 0) |0\rangle = \phi^{\text{global}}(y, 0, 0).$$

We’ll expand $\phi(y, 0, 0) |0\rangle = \sum_{N=0}^{\infty} y^{2h+2N} \frac{(-1)^N}{N!(2h)^N} \left(L_{-1} \mathcal{L}_{-1}\right)^N |\mathcal{O}\rangle$. (C.4.47)

We’ll expand $\phi(y, 0, 0) |0\rangle = \sum_{N=0}^{\infty} y^{2h+2N} |\phi\rangle_N$ and write $|\phi\rangle_N$ as follows

$$|\phi\rangle_N = \lambda_N \mathcal{L}_{-N} \mathcal{L}_{-N} |\mathcal{O}\rangle.$$ (C.4.48)

And we’ll derive the coefficients of the following terms at order $\frac{1}{c}$ and $\frac{1}{c^2}$ in $\mathcal{L}_{-N}$:

$$\mathcal{L}_{-N} = L_{-1}^N + \frac{1}{c} \sum_{k=2}^{N} \eta_{N,k} L_{-k} L_{-1}^{N-k} + \frac{1}{c^2} \sum_{k_1, k_2=2}^{N} \frac{\kappa_{N,k_1,k_2}}{k_1 \geq k_2} L_{-k_1} L_{-k_2} L_{-1}^{N-k_1-k_2} + \mathcal{O}(c^{-3}).$$ (C.4.49)

To derive $\eta_{N,k}$, we just need to consider the first two terms in the above equation. Using the condition of equation (4.3.3) we have

$$L_m \left[ L_{-1}^N + \frac{1}{c} \sum_{k=2}^{N} \eta_{N,k} L_{-k} L_{-1}^{N-k} \right] |\mathcal{O}\rangle = 0 + \mathcal{O}(c^{-1}), \quad 2 \leq m \leq N.$$ (C.4.50)
The first term can be calculated exactly as follows\(^4\)

\[
L_m L_{-1}^N |O\rangle = (m + 1)! \sum_{i=1}^{N-(m-1)} \left( \begin{array}{c} N - i \\ m - 1 \end{array} \right) (h + i - 1) L_{-1}^{N-m} |O\rangle
\]

\[
= \frac{N!(h (m + 1) + N - m)}{(N - m)!} L_{-1}^{N-m} |O\rangle \quad \text{(C.4.51)}
\]

The second term is easy to calculate at leading order of large \(c\), which is given by

\[
L_m \sum_{k=2}^{N} \frac{1}{c} \eta_{N,k} L_{-k} L_{-1}^{N-k} |O\rangle = \eta_{N,m} \frac{m(m^2 - 1)}{12} L_{-1}^{N-m} |O\rangle + O(c^{-1}) \quad \text{(C.4.52)}
\]

where we used the Virasoro algebra \([L_m, L_n] = (m - n)L_{m+n} + \frac{m(m^2 - 1)c}{12} \delta_{m,-n}\). Equating the RHSs of equation (C.4.51) and equation (C.4.52), and solving for \(\eta_{N,m}\), we find

\[
\eta_{N,m} = -\frac{12(h (m + 1) + N - m)N!}{(N - m)!m(m^2 - 1)} \quad \text{(C.4.53)}
\]

To derive \(\kappa_{N,k_1,k_2}\), we need to use the following conditions,

\[
L_{m_2} L_{m_1} \left( L_{-1}^N + \frac{1}{c} \sum_{k=2}^{N} \eta_{N,k} L_{-k} L_{-1}^{N-k} + \frac{1}{c^2} \sum_{k_1,k_2=2}^{N} \kappa_{N,k_1,k_2} L_{-k_1} L_{-k_2} L_{-1}^{N-k_1-k_2} \right) |O\rangle = 0 + O(c^{-1})
\]

\[
\text{(C.4.54)}
\]

with \(m_1, m_2 \geq 2\) and \(m_1 \geq m_2\), because \(L_{m_2} L_{m_1}\) acting on the \(\frac{1}{c^2}\) terms will contribute to leading order \(O(c^0)\).

\(^4\)Equation (C.4.51) comes from the following procedure. We commute \(L_m\) with \(m L_{-1}\) to get \(L_0\). To do so we need to choose \(m L_{-1}s\) from the \(N L_{-1}s\). If the position of the last \(L_{-1}\) for these \(m L_{-1}s\) is the \(i\)th \(L_{-1}\) in the \(N L_{-1}s\) from the right, then it means that we need to choose \((m - 1) L_{-1}s\) from \((n - i) L_{-1}s\), where there are \(\binom{N-i}{m-1}\) of ways to do so. Commuting \(L_m\) with \(m L_{-1}\) will eventually gives us a \(L_0\) times a factor of \((m + 1)!\). And there are \((i - 1) L_{-1}s\) remained on the right of this \(L_0\), so the eigenvalue of \(L_0\) will be \(h + i - 1\).
We already know that

\[ L_{m_1} \left( L^n_{-1} + \sum_{k=2}^{n} \frac{1}{c} \lambda_{n,k} L_{-k} L^{n-k}_{-1} \right) = 0 + O \left( c^{-1} \right) \] (C.4.55)

so in the following we only need to consider the remaining contribution of the second term, which comes from \( k = m_2 \) and \( k = m_1 + m_2 \),

\[ L_{m_2}L_{m_1} \sum_{k=2 \atop k \neq m_1}^{N} \frac{1}{c} \lambda_{N,k} L_{-k} L^{N-k}_{-1} |O\rangle = \frac{m_2 (m_2^2 - 1)}{12} \left[ \lambda_{N,m_1+m_2} (2m_1 + m_2) \right. \]

\[ + \lambda_{N,m_2} \left( \frac{(N-m_2)! (h(m_1 + 1) + N - m_2 - m_1)}{(N - m_2 - m_1)!} \right) \] \[ L^{N-m_1-m_2}_{-1} |O\rangle \] (C.4.56)

The third term in equation (C.4.54) give the following leading order contribution

\[ L_{m_2}L_{m_1} \sum_{k_1,k_2=2 \atop k_1 \geq k_2}^{N} \frac{1}{c^2} \kappa_{N,k_1,k_2} L_{-k_1} L_{-k_2} L^{n-k_1-k_2}_{-1} |O\rangle \] (C.4.57)

\[ = \left( 1 + \delta_{m_1,m_2} \right) \kappa_{N,m_1,m_2} \frac{m_1 (m_1^2 - 1) m_2 (m_2^2 - 1)}{144} L^{N-m_1-m_2}_{-1} |O\rangle + O \left( c^{-1} \right). \]

So equating the RHSs of the above two equations, and solving for \( \kappa_{N,m_1,m_2} \), we find

\[ \kappa_{N,m_1,m_2} = - \frac{\lambda_{N,m_1+m_2} (2m_1 + m_2) + \lambda_{N,m_2} \frac{(N-m_2)! (h(m_1 + 1) + N - m_2 - m_1)}{(N - m_2 - m_1)!}}{(1 + \delta_{m_1,m_2}) \frac{m_1 (m_1^2 - 1)}{12}}. \] (C.4.58)

So \( L_{-N} \) is by given equation (D.1.15) with \( \eta_{N,k} \) and \( \kappa_{N,k_1,k_2} \) given by equation (C.4.53) and equation (C.4.58).

Notice that the \( \eta_{N,k} \) and \( \kappa_{N,k_1,k_2} \) we derived above are just the leading order results, ie there are \( \frac{1}{c} \) corrections to them. And there are other terms, like \( L^n_{-1} \), at order \( \frac{1}{c} \) and \( \frac{1}{c^2} \). In general, these \( \frac{1}{c} \) corrections should form quasi-primaries and their global descendants, such that \( \langle \phi O \rangle \) will always be given by \( \langle \phi O \rangle = \langle \phi^{\text{global}} O \rangle \), which is just the bulk-boundary propagator in vacuum.
C.4.6 Explicit Form of the Stress-Tensor Correlator Recursion and Calculation

We can document the origin of various terms in the recursion relation from section 4.3.3 very explicitly as

\[
G_{n+1,m} = \left( \begin{array}{c}
- \frac{\partial_z + \sum_{i=1}^n \frac{\partial_{z_i}}{z_{n+1}}}{L_{-1} \phi} - \frac{\frac{\partial_y}{2} + \frac{z}{z_{n+1}}}{L_0 \phi} - \frac{\sum_{i=1}^n \frac{z_i}{z_{n+1}}}{[L_1, \mathcal{O}(z, \bar{z})]} - \frac{\frac{z}{z_{n+1}}}{[L_1, \mathcal{T}(Z_i)]} \\
\frac{h}{(z_{n+1} - z)^2} + \frac{\partial_z}{T(z_{n+1}) \mathcal{O}(z, \bar{z})} + \sum_{i=1}^n \frac{2}{(z_{n+1} - z_i)^2} + \frac{\partial_{z_i}}{T(z_{n+1}) T(z_i)} \\
\sum_{i=1}^n \frac{T(z_1) \cdots T(z_{i-1}) T(Z_i) \cdots T(z_n) \mathcal{T}(w_1) \cdots \mathcal{T}(w_m) \mathcal{O}(z, \bar{z}) \phi(y, 0, 0)}{2 (z_{n+1} - z_i)^4} \\
\end{array} \right) G_{n,m} \\
\right)
\]

One can use the above recursion relation to easily derive \( \langle \phi \mathcal{O} T \rangle \), \( \langle \phi \mathcal{O} T T \rangle \) and \( \langle \phi \mathcal{O} T T \rangle \) that we derived in section C.4.2 using bulk-boundary OPE block. For comparison, we provide these computations here.

For one \( T \) insertion, we have

\[
\langle \phi(y, 0, 0) \mathcal{O}(z, \bar{z}) T(z_1) \rangle = \left( - \frac{\partial_z}{z_1} + \frac{\frac{\partial_y}{2}}{z_1^2} - \frac{\frac{z}{z_1}}{z_1^3} + \frac{h}{(z_1 - z)^2} + \frac{\partial_z}{z_1 - z} \right) \langle \phi \mathcal{O} \rangle \\
= \left( \frac{y}{z \bar{z} + y^2} \right)^{2h} \frac{h z^2 (z_1 (z \bar{z} + 3y^2) - 2y^2 z)}{(z - z_1)^2 z_1^3 (y^2 + z \bar{z})}. \quad (C.459)
\]
For one $T$ and one $T$ insertions, we have

$$
\langle \phi (y, 0, 0) O (z, \bar{z}) T (z_1) T (w_1) \rangle
$$

$$
= \left( -\frac{\partial_z}{w_1} + \frac{\bar{y}}{w_1^2} + \frac{\bar{z} (2h + z \partial_z)}{w_1^3} + \frac{h}{(w_1 - \bar{z})^2} + \frac{\partial_{\bar{z}}}{w_1 - \bar{z}} \right) \langle \phi (y, 0, 0) O (z, \bar{z}) T (z_1) \rangle
$$

$$
= \left( \frac{y}{z \bar{z} + y^2} \right)^2 \left[ \frac{h^2 z^2 \bar{z}^2 (y^2 (3z_1 - 2z_2) + z_1 z \bar{z})}{z_1 w_1^3 (z_1 - z)^2 (w_1 - \bar{z})^2 (z \bar{z} + y^2)^2}
\right]
$$

\begin{align}
& \quad + \frac{2h y \bar{z} z^3 \bar{z}^3}{z_1 w_1^3 (z - z_1) (w_1 - \bar{z}) (z \bar{z} + y^2)^2}.
\end{align}

For two $T$ insertions, we have

$$
\langle \phi (y, 0, 0) O (z, \bar{z}) T (z_1) T (z_2) \rangle
$$

$$
= \left( -\frac{\partial_z + \partial_{z_2}}{z_2} + \frac{\bar{y}}{z_2^2} - \frac{z (2h + z \partial_z)}{z_2^3} - \frac{z_1 (4 + z_1 \partial_{z_1})}{z_2^3} \right) \langle \phi (y, 0, 0) O (z, \bar{z}) T (z_1) \rangle
$$

$$
+ \left( \frac{h}{(z_2 - z)^2} + \frac{\partial_z}{z_2 - z} + \frac{2}{(z_2 - z_1)^2} + \frac{\partial_{z_1}}{z_2 - z_1} \right) \langle \phi (y, 0, 0) O (z, \bar{z}) T (z_1) \rangle
$$

$$
+ \frac{c}{2 (z_2 - z_1)^2} \langle \phi (y, 0, 0) O (z, \bar{z}) \rangle
$$

\begin{align}
& \quad \left[ \frac{c}{2 (z_1 - z_2)^4} + \frac{h^2 z^4 (z_1 z \bar{z} + y^2 (3z_1 - 2z_2)) (z_2 z \bar{z} + y^2 (3z_2 - 2z))}{z_1^3 z_2^3 (z - z_1)^2 (z - z_2)^2 (z \bar{z} + y^2)^2}
\right]
\end{align}

$$
+ \frac{2h z^2 (y^2 z \bar{z} z_1 z_2 (z (z_1 + z_2) - 4z_1 z_2) - z^2 z_2^2 z_1^2 z_2^2 + y^4 \left( z_1 z_2 (z_1 + z_2) - 3z_1^2 z_2^2 - z^2 (z_1 - z_2)^2 \right))}{(z - z_1) z_2^2 (z_2 - z) (z_2 - z_1)^2 (z \bar{z} + y^2)^2}.
$$

One can see that the above results are exactly what we found in section C.4.2.
Appendix D

Appendix to Chapter 5

D.1 Perturbative Computations of the Propagator

In this section we will show that our first order result for the full propagator

\[
\langle \phi \phi \rangle = \frac{\rho^h}{1 - \rho} \left( 1 + \frac{12}{c} \left( \frac{\rho (2h^2(\rho - 1)^2 + h(3\rho - 11) + 2)(\rho - 1) + \rho^2((\rho - 5)\rho + 10)}{(\rho - 1)^4} \right) + 2h \rho^2 \Phi(\rho, 1, 2h + 1) + h \rho^{1-2h} B_\rho(2h + 1, -1) + 2(h - 1)h \log(1 - \rho) \right) + \mathcal{O}\left(\frac{1}{c^2}\right)
\]  

(D.1.1)

follows directly from perturbation theory. Note that this is the full propagator, which receives equal contributions from \( T \) and \( \bar{T} \), and so its \( 1/c \) correction is enhanced by a factor of 2 compared to the purely holomorphic propagator. Primarily, we will be showing how this result matches an AdS\(_3\) gravitational loop calculation (similar calculations in higher dimensions were recently studied in [191], but as far as we know this calculation has not been carried out previously in AdS\(_3\)). However, we will also demonstrate how the important \( h \)-independent \( \frac{1}{c} \) terms arise directly from our definition of \( \phi \) using a unitarity-based argument (an explicit sum over intermediate states). We also provide a comparison with \( U(1) \) Chern-Simons theory at short-distances, which does not display power-law UV/IR mixing.
D.1.1 AdS\(_3\) Gravity at One-Loop

The only non-vanishing contribution to \(\langle \phi(X_1)\phi(X_2) \rangle\) from bulk perturbation theory at order \(1/c\) comes from the diagram of Fig. 5.3, as our regulator sets contact diagrams to zero. In position space in AdS\(_3\), this contributes

\[
\langle \phi(X_1)\phi(X_2) \rangle = \int d^3X d^3Y \, G(X_1,Y_1)G(Y_1,Y_2)H(Y_1,Y_2)G(Y_2,X_2)\, V_1V_2 \tag{D.1.2}
\]

where \(G\) is a scalar propagator, \(H\) is a graviton propagator, and \(V_1\) and \(V_2\) are vertex factors associated with vertices and index contractions at \(Y_1\) and \(Y_2\), which we will specify below.

We can greatly simplify the computation by acting on the correlator with \((\nabla^2 + m^2)\), the Klein-Gordon operator associated with both \(X_1\) and \(X_2\) [189]. This collapses both of the external propagators to delta functions, giving

\[
\left((\nabla_1^2 + m^2)\left((\nabla_2^2 + m^2) \langle \phi(X_1)\phi(X_2) \rangle\right) = V_1H(X_1,X_2)G(X_1,X_2)\, V_2 \tag{D.1.3}
\]

so now there are no integrals to do. The tree-level scalar propagator is simply

\[
G(X_1,X_2) = \frac{\rho^h}{1 - \rho} \tag{D.1.4}
\]

as usual. The graviton propagator in our gauge is identical to the stress tensor correlator \(\langle T(z_1)T(z_2) \rangle\), so it is simply

\[
H(X_1,X_2) = \frac{1}{2c(z_1 - z_2)^4} \to \frac{1}{2c} \frac{\rho^2}{(\sqrt{\rho} - 1)^8} \tag{D.1.5}
\]

where we have re-written \(z_{12}\) in terms of \(\rho = e^{-2\sigma(X_1,X_2)}\) by fixing all of the parameters other than \(z_{12}\). Here we are only computing the holomorphic part, but of course the anti-holomorphic part makes an equal anti-holomorphic contribution. The vertex factors arise
entirely from differentiating $G(X_1, X_2)$ by $\partial_{x_1}^2$ and $\partial_{x_2}^2$, and this leads to

$$V_1 G(X_1, X_2) V_2 = -\frac{16h^4 (\sqrt{\rho} - 1)^3 \rho^h}{(\sqrt{\rho} + 1)^5} + \frac{16h^3 (\sqrt{\rho} - 1)^2 (7\rho + 3)\rho^h}{(\sqrt{\rho} + 1)^6}$$

$$- \frac{4h^2 (\sqrt{\rho} - 1) (\rho(71\rho + 98) + 11)\rho^h}{(\sqrt{\rho} + 1)^7} - \frac{120(\rho + 1)(\rho(\rho + 5) + 1)\rho^{h+1}}{(\sqrt{\rho} - 1) (\sqrt{\rho} + 1)^9}$$

$$+ \frac{4h(\rho(77\rho + 239) + 101) + 3)\rho^h}{(\sqrt{\rho} + 1)^8}$$

when written in terms of $\rho$. Altogether, this means that we should expect

$$V_1 H_1 G_1 H_2 V_2 = -\frac{16h^4 \rho^{h+2}}{(\rho - 1)^5} + \frac{16h^3(7\rho + 3)\rho^{h+2}}{(\rho - 1)^6} - \frac{4h^2 (71\rho^2 + 98\rho + 11)\rho^{h+2}}{(\rho - 1)^7}$$

$$+ \frac{4h(77\rho^3 + 239\rho^2 + 101\rho + 3)\rho^{h+2}}{(\rho - 1)^8} - \frac{120(\rho^3 + 6\rho^2 + 6\rho + 1)\rho^{h+3}}{(\rho - 1)^9}$$

This should be equal to $(\nabla_1^2 + m^2) (\nabla_2^2 + m^2) \langle \phi \phi \rangle$. This quantity can also be re-written in terms of $\rho$; writing $K(\rho)$ for the propagator, we find

$$\left(\nabla_1^2 + m^2\right) \left(\nabla_2^2 + m^2\right) \langle \phi \phi \rangle = 16(h - 1)^2 h^2 K(\rho) + \frac{64 (-h^2 + h + 1) \rho^2 K'(\rho)}{\rho - 1}$$

$$- \frac{32\rho^2 ((h - 1)h(\rho - 1) - 7\rho + 1)K''(\rho)}{\rho - 1}$$

$$+ \frac{64\rho^3 (2\rho - 1)K^{(3)}(\rho)}{\rho - 1} + 16\rho^4 K^{(4)}(\rho)$$

(D.1.8)

Apparently we are faced with the daunting task of solving a 4th order ODE with a complicated source. Fortunately, we already know part of the answer from semiclassical calculations (keeping the $h^2/c$ terms) and also from computations as $h \to 0$ in appendix D.1.2. After inputing these terms and then leaving the remaining terms in $K(\rho)$ as an unknown function, we were able to solve. And given a proposed $K(\rho)$, it is very easy to verify that it is in fact...
valid by inputting it into the differential equation.

Using this method, we find that the full \(1/c\) correction due to the holomorphic gravitons is

\[
\frac{6\rho^h}{1-\rho} \left( \frac{\rho (2h^2(\rho - 1)^2 + h(\rho(3\rho - 11) + 2)(\rho - 1) + \rho^2((\rho - 5)\rho + 10))}{(\rho - 1)^4} + 2h\rho^2\Phi(\rho, 1, 2h + 1) + h\rho^{1-2h}B_\rho(2h + 1, -1) + 2(h - 1)h \log(1 - \rho) \right) \tag{D.1.9}
\]

where \(\Phi\) is a Lurch and \(B\) is the Beta function. The anti-holomorphic gravitons make an equal contribution at order \(1/c\), so we simply need to double this result. Intriguingly, if we expand as \(\rho = e^{-2\sigma}\) then the singular terms are

\[
\frac{9}{8c\sigma^5} - \frac{3(5 + 2(-1 + h)h)}{8c\sigma^3} + \frac{12(-1 + h)h \log(\sigma)}{c\sigma} \tag{D.1.10}
\]

So we see that the AdS mass \(2h(2h - 2)\) appears prominently, and we only have odd powers of \(1/\sigma\) appearing (we have dropped some terms that are simply \(1/\sigma\), as these are no more singular than the free field theory result). Restoring the AdS scale, we have

\[
\langle \phi \phi \rangle \approx \frac{1}{\sigma} \left( \frac{3G_NR^3}{4\sigma^4} - \frac{G_NR(10 + m^2R^2)}{8\sigma^2} + 2G_N m^2R \log \left( \frac{\sigma}{R} \right) \right) \tag{D.1.11}
\]

to leading order at short distances. This makes it clear that the scale \(\sigma \sim \sqrt[4]{G_NR^3}\) has made an explicit appearance.

**Comparison with U(1) Chern-Simons**

The one-loop AdS\(_3\) gravity result displays a surprising UV/IR mixing. To better understand this result, we will briefly compare it with a \(U(1)\) Chern-Simons theory.

The double application of the Klein-Gordon equation in (D.1.8) applies to loop computations of the AdS\(_3\) propagator in other theories. If we re-write this equation in terms of \(\sigma\),
and only keep the terms that dominate at short distances, we find

\[
\left( \nabla_1^2 + m^2 \right) \left( \nabla_2^2 + m^2 \right) f(\sigma) \approx f^{(4)}(\sigma) + \frac{4f^{(3)}(\sigma)}{\sigma}
\]

(D.1.12)

where \( f(\sigma) \) is the propagator at short distances. In a \( U(1) \) Chern-Simons theory, the propagator and vertices will be closely related to those that we found for gravity. We expect

\[
\langle A_z(X_1)A_z(X_2) \rangle \propto \frac{1}{\sigma^2_{12}}
\]

and the vertices can be obtained from \( \partial \bar{z}_1 \partial \bar{z}_2 \) applied to the scalar field propagator. In the short-distance limit, this leads to the differential equation

\[
F^{(4)}_{CS}(\sigma) + \frac{4F^{(3)}_{CS}(\sigma)}{\sigma} \propto \frac{1}{\sigma^3} + \cdots
\]

(D.1.13)

with the solution

\[
\langle \phi \phi \rangle_{CS} \propto -\frac{\log(\sigma)}{6\sigma} + \frac{\kappa}{\sigma} + \cdots
\]

(D.1.14)

to leading order at short distances, where \( \kappa \) is a free parameter (which would be fixed in the full solution by boundary conditions) and the ellipsis denotes less singular terms. Thus we see that unlike \( \text{AdS}_3 \) gravity, in perturbation theory the bulk \( U(1) \) Chern-Simons theory does not exhibit power-law UV/IR mixing at short-distances.

D.1.2 Unitarity-Based Calculation from the Definition of \( \phi \)

In this section we will use the large \( c \) expansion of the \( \mathcal{L}_{-N} \) that define the level \( N \) contribution to \( \phi \) in order to directly compute the \( 1/c \) correction to the propagator as \( h \to 0 \). One can interpret this as a unitarity-based version of the calculation of the previous section, as we are decomposing each \( \phi \) in \( \langle \phi \phi \rangle \) into a sum over the 'double-trace' states in the \( T(z)O(0) \) OPE. We previously computed [3] the first \( 1/c \) corrections to \( \mathcal{L}_{-N} \), which are the coefficients
\[ \eta_{N,k} \text{ of} \]

\[ \mathcal{L}_{-N} = L_{-1}^N + \frac{1}{c} \sum_{k=2}^{N} \eta_{N,k} L_{-k} L_{-1}^{N-k} + \cdots \]  

(D.1.15)

and found that (see appendix D.5.3 of [3])

\[ \eta_{N,k} = -\frac{12(h(k+1)+N-k)}{k(k^2-1)} \frac{N!}{(N-k)!} \]  

(D.1.16)

We can use this result to directly compute the \(1/c\) terms in \(\langle \phi \phi \rangle\). In the rest of this section, we will only keep the effects that survive in the limit \(h \to 0\), which means that we can drop the term above proportional to \(h\).

We will also need the matrix element

\[ \mathcal{M}_{k,p}^{N,M} = \langle L_{-k} L_{-1}^{N-k} \mathcal{O}(z) L_{-p} L_{-1}^{M-p} \mathcal{O}(w) \rangle \]

\[ \approx \frac{hc}{6} \frac{(-1)^{M+N} (k+p-1)! (M+N-k-p-1)!}{(k-2)!(p-2)! (z-w)^{2h+N+M}} \]  

(D.1.17)

where we have only kept the leading term at small \(h\). We need to multiply by \(\eta_{N,k}\) factors and sum over \(k\) and \(p\), giving (setting \(w=0\))

\[ \sum_{p,k=2}^{M,N} \eta_{N,k} \eta_{M,p} \mathcal{M}_{k,p}^{N,M} = \frac{6h(N-2)(N-1)(M-2)(M-1)(M+N-3)!}{cz^{N+M}} \]  

(D.1.18)

To see how to use this result, let us recall the computation to leading order and compare it to the \(1/c\) correction we wish to calculate. The global correlator can be computed from the sums

\[ \langle \phi \phi \rangle_{\text{global}} = \sum_{N,M} \frac{(-1)^{N+M} y_1^{2M} y_2^{2N}}{N! M! (2h)_N (2h)_M} \langle L_{-1}^M \bar{L}_{-1}^M \mathcal{O} L_{-1}^{N} \bar{L}_{-1}^{N} \mathcal{O} \rangle \]  

(D.1.19)
where we have

\[ \langle L^M - 1 \bar{L}^M \rangle = \frac{(2h)^M + N(2h)^M + N}{(z \bar{z})^{M+N}} \]

\approx 4h^2[(M + N - 1)!!]^2

\tag{D.1.20}

One can easily verify directly that these formula agree with \( \langle \phi \phi \rangle_{\text{global}} = \frac{\rho^h}{1 - \rho} \) when \( h \to 0 \).

To obtain the \( 1/c \) correction, we must make the replacement

\[ \langle L^M - 1 \bar{L}^M \rangle \to \langle \bar{L}^M \bar{L}^M \rangle \]

\[ \sum_{p,k=2}^{M,N} \eta_{N,k} \eta_{M,p} M^{N,M} \]

\tag{D.1.21}

Similarly, we find exact agreement between

\[ \langle \phi \phi \rangle_{1/c} = \frac{3}{4} \sum_{N,M} \frac{(-1)^{N+M} y_1^2 y_2^2}{N! M!(2h)^N(2h)_M} \frac{12h^2(N - 2)_2(M - 2)_2(M + N - 3)!}{c(z \bar{z})^{M+N}} \]

in the limit \( h \to 0 \) and our result

\[ 6\rho^3 \frac{(\rho^2 - 5\rho + 10)}{c(1 - \rho)^5} \]

\tag{D.1.22}

for the holomorphic part of the correction to \( \langle \phi \phi \rangle \).

To perform the relevant sums, it is useful to write \( s = M + N \) and first sum over \( M \) with fixed \( s \). Setting \( y_i = 1 \) WLOG and working with the variable \( z \bar{z} \), this leads to

\[ \langle \phi \phi \rangle_{1/c} = \sum_{s=4}^{\infty} \frac{3(-1)^s 4^{s-3}(s - 5)(s - 4)(s - 3)(s - 2)(s - 1)\Gamma \left( s - \frac{5}{2} \right)}{\sqrt{\pi} c \Gamma(s + 1)(z \bar{z})^s} \]

\tag{D.1.23}

The sum over \( s \) can now be performed exactly, and in the small \( z \bar{z} \) limit it gives

\[ \langle \phi \phi \rangle_{1/c} \approx \frac{9}{8c(z \bar{z})^2} + \cdots \]

\tag{D.1.24}
as expected. However, note that the sums defining $\langle \phi \phi \rangle$ provide a long-distance (or near-boundary) expansion, whereas the interesting physics occurs at short distances in the bulk. Thus connecting the two regimes requires an analytic continuation, meaning that we need to perform the full sum to observe the short-distance singularity. Each term in the sums over $M, N$ is more singular than the total.

D.2 Details of the Computation of $F_{n, \bar{n}}$

In this section, we provide the details for computing $\langle \phi^{n, \bar{n} \pi, n, \bar{n} \pi}_{i,j} \rangle$. In Section 5.2.2, we defined $\phi^{n, \bar{n} \pi}_{i,j}$ to be (WLOG, assuming $n \geq \bar{n}$)

$$\phi^{n, \bar{n} \pi}_{i,j} (y, z, \bar{z}) \equiv y^{2h+2n} \sum_{m=0}^{\infty} (-1)^{n+m} y^{2m} \left| L_{-1}^{n+m} \mathcal{O} \right|^2 \frac{L_{m}^{\mathcal{L}^{\text{quasi}, i} \mathcal{L}^{\text{quasi}, j}}}{\left| L_{-1}^{\mathcal{L}^{\text{quasi}, i} \mathcal{O}} \right|^2 \left| L_{-1}^{\mathcal{L}^{\text{quasi}, j} \mathcal{O}} \right|^2} \mathcal{O} (z, \bar{z}), \quad (D.2.1)$$

and $F_{n, \bar{n} \pi} (h)$ to be

$$F_{n, \bar{n} \pi} (h) \equiv \langle \phi^{n, \bar{n} \pi}_{i,j} (y_1, z_1, \bar{z}_1) \phi^{n, \bar{n} \pi}_{i,j} (y_2, z_2, \bar{z}_2) \rangle \left| L_{-n}^{\mathcal{L}^{\text{quasi}, i}} L_{-\bar{n}}^{\mathcal{L}^{\text{quasi}, j}} \mathcal{O} \right|^2 \quad (D.2.2)$$

Since eventually we’ll show that $F_{n, \bar{n} \pi} (h)$ only depends on the level of the quasi-primary $(n, \bar{n})$, we’ll suppress the indexes $(i, j)$. Defining

$$\mathcal{O}^{n, \bar{n} \pi} = L_{-n}^{\mathcal{L}^{\text{quasi}}} L_{-\bar{n}}^{\mathcal{L}^{\text{quasi}}} \mathcal{O} \quad (D.2.3)$$

and using the following identities,

$$\left| \mathcal{O}^{n, \bar{n} \pi} \right|^2 = \left| L_{-n}^{\mathcal{L}^{\text{quasi}}} \mathcal{O} \right|^2 \left| L_{-\bar{n}}^{\mathcal{L}^{\text{quasi}}} \mathcal{O} \right|^2$$

$$\left| L_{-1}^{n+m} \mathcal{O} \right|^2 = (2h)_{n+m} (n+m)! \quad (D.2.4)$$

$$\left| L_{-1}^{m} L_{-n}^{\mathcal{L}^{\text{quasi}}} \mathcal{O} \right|^2 = (2h+2n)_m m! \left| L_{-n}^{\mathcal{L}^{\text{quasi}}} \mathcal{O} \right|^2$$

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we find

$$
\phi^{n,\bar{n}}(y, z, \bar{z}) = \frac{(-1)^n y^{2h+2n}}{|\mathcal{O}^{n,\bar{n}}|^2} \frac{\sum_{m=0}^{\infty} (-1)^m y^{2m} (2h)_{n+m} (n+m)!}{(2h+2n)_m m! (2h+2\bar{n})_{n-\bar{n}+m} (n-\bar{n}+m)!}
$$

$$
\times L_{-1}^m T_{-1}^m \left( T_{-1}^{n,\bar{n}} \mathcal{O}^{n,\bar{n}}(z, \bar{z}) \right) \quad (D.2.5)
$$

For simplicity, we’ll define

$$
h_n \equiv h + n,
$$

$$
h_{\bar{n}} \equiv h + \bar{n},
$$

$$
l \equiv n - \bar{n}.
$$

Then $$\langle \phi^{n,\bar{n}}(y_1, z_1, \bar{z}_1) \phi^{n,\bar{n}}(y_2, z_2, \bar{z}_2) \rangle$$ is given by

$$
\langle \phi^{n,\bar{n}} \phi^{n,\bar{n}} \rangle = \frac{(y_1 y_2)^{2h_n}}{|\mathcal{O}^{n,\bar{n}}|^4} \frac{\sum_{m,m'=0}^{\infty} (-1)^{m+m'} y_1^{2m} y_2^{2m'} (2h)_{n+m} (n+m)!}{(2h_n)_m (2h_{\bar{n}})_{l+m} m! (l+m)!} \frac{(2h)_{n+m'} (n+m')!}{(2h_n)_m' (2h_{\bar{n}}')_{l+m'} m'! (l+m')!}
$$

$$
\times \left( L_{-1}^m T_{-1}^m \left[ T_{-1}^{n,\bar{n}} \mathcal{O}^{n,\bar{n}} \right] L_{-1}^{m'} T_{-1}^{m'} \left[ T_{-1}^{n,\bar{n}} \mathcal{O}^{n,\bar{n}} \right] \right). \quad (D.2.7)
$$

The second line of above equation is given by

$$
\langle L_{-1}^m T_{-1}^{m'} \left[ T_{-1}^{n,\bar{n}} \mathcal{O}^{n,\bar{n}} \right] L_{-1}^m T_{-1}^{m'} \left[ T_{-1}^{n,\bar{n}} \mathcal{O}^{n,\bar{n}} \right] \rangle
$$

$$
= \partial_{z_1} \bar{\partial}_{\bar{z}_1} \partial_{z_2} \bar{\partial}_{\bar{z}_2} \frac{(-1)^{n+\bar{n}} |\mathcal{O}^{n,\bar{n}}|^2}{(z_1 - z_2)^{2h_n} (\bar{z}_1 - \bar{z}_2)^{2h_{\bar{n}}}} \quad (D.2.8)
$$

$$
= \frac{|\mathcal{O}^{n,\bar{n}}|^2 (2h_n)_{m+m'} (2h_{\bar{n}})_{2l+m+m'} \frac{(2h)_{m+m'} (2l+m+m')!}{(2h_n)_{m+m'} (2h_{\bar{n}})_{2l+m+m'}!}}{z_1^{2h_n + m+m'} \bar{z}_1^{2h_{\bar{n}} + 2l+m+m'}},
$$

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where in the second line, we’ve used the fact that the two-point function of the quasi-
primaries is given by

\[
\langle \mathcal{O}^{n,\pi} (z_1, \bar{z}_1) \mathcal{O}^{n,\pi} (z_2, \bar{z}_2) \rangle = \frac{(-1)^{n+\pi} |\mathcal{O}^{n,\pi}|^2}{(z_1 - z_2)^{2h_n} (\bar{z}_1 - \bar{z}_2)^{2h_{\pi}}}. \tag{D.2.9}
\]

and the \((-1)^{n+\pi}\) is canceled by the derivatives acting on \(z_1\) and \(\bar{z}_1\) in the third line of equation \((D.2.8)\). For later convenience, the factor \((2h_{\pi})_{m+m'+2l}\) in the last line of equation \((D.2.8)\) can be written as

\[
(2h_{\pi})_{2l+m+m'} = (2h_{\pi})_{2l} (2h_n)_{m+m'}. \tag{D.2.10}
\]

Now let’s simplify the first line of equation \((D.2.7)\):

\[
\frac{(2h)_{n+m} (n+m)!}{(2h)_m (2h_{\pi})_{l+m} m! (l+m)! (2h_n)_{m'} (2h_{\pi})_{l+m'} m'! (l+m')!} = \left[ \frac{n!}{l! (2h_{\pi})_l} \right]^2 \frac{(2h + n)_m (2h + n)_{m'} (n+1)_m (n+1)_{m'} 1}{(2h_n)_m (2h_{\pi} + l)_m (2h_{\pi} + l)_{m'} (l+1)_m (l+1)_{m'} m! m'!}.
\tag{D.2.11}
\]

Putting everything together, we have

\[
\mathcal{F}_{n,\pi} (h) \equiv \langle \phi^{n,\pi} \phi^{n,\pi} \rangle |\mathcal{O}^{n,\pi}|^2 \tag{D.2.12}
\]

\[
\begin{align*}
&= \left( \frac{y_1 y_2}{z_{12} \bar{z}_{12}} \right)^{2h_n} (2h_{\pi})_{2l} \left[ \frac{n! (2h)_n}{l! (2h_{\pi})_l} \right]^2 \sum_{m,m'=0}^{\infty} \left( \frac{-y_1^2}{z_{12} \bar{z}_{12}} \right)^m \left( \frac{-y_2^2}{z_{12} \bar{z}_{12}} \right)^{m'} m! m'!
&\times \frac{(2h_n)_{m+m'} (2h_n)_{m+m'} (2h + n)_m (2h + n)_{m'} (n+1)_m (n+1)_{m'}}{(2h_n)_m (2h_{\pi} + l)_m (2h_{\pi} + l)_{m'} (l+1)_m (l+1)_{m'}}
&= (Y_1 Y_2)^{2h_n} (2h_{\pi})_{2l} \left[ \frac{n! (2h)_n}{l! (2h_{\pi})_l} \right]^2 \times F_{0,3}^{2,2} \left( \begin{array}{c}
2h_n, 2h_n : 2h + n, 2h + n; n + 1, n + 1; \\
- : 2h_n, 2h_n; 2h_n - l, 2h_n - l; l + 1, l + 1;
\end{array} \right), -Y_1, -Y_2 \right).
\end{align*}
\]

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where $F_{2,2}^{2,2}$ is a Kampe de Feriet series and we’ve defined

$$Y_1 = \frac{y_1^2}{z_{12}\bar{z}_{12}}, \quad Y_2 = \frac{y_2^2}{z_{12}\bar{z}_{12}}. \quad (D.2.13)$$

Now we will discuss several properties of the function $F_{n,\bar{n}}$. First, it is not just a function of the geodesic separation between the two points. In addition, it depends on a parameter encoding the angle between the two points and the $z-\bar{z}$ plane. Most of the results presented in section 5.4.2 are computed when the two points lie on the same constant $y$-plane. We note that if we take the other limit, where the separation is purely in the $y$ direction, the small $-y_i^2$ expansion presented above is not useful. To explore the behavior of $F_{n,\bar{n}}$ in this configuration, we need to re-sum the series. We are not aware of existing results that fully solve this problem. However, we can partially re-sum the series using a Borel style procedure, yielding the following integral representation:

$$F_{n,\bar{n}} = 2R_{h,n,\bar{n}}\eta^{h+n} \int_0^\infty daK_0 \left(2\sqrt{\frac{a}{Y_1}}\right) a^{2(h+n)-1} W(h, n, \bar{n}; a) W(h, n, \bar{n}; \eta a) \quad (D.2.14)$$

where $K_0$ is the modified Bessel function and we’ve defined $\eta = \frac{Y_2}{Y_1}$,

$$W(h, n, \bar{n}; a) = _2F_3(n + 1, 2h + n; 2h + 2n, n - \bar{n} + 1, 2h + n + \bar{n}; -a), \quad (D.2.15)$$

and

$$R_{h,n,\bar{n}} = (2h)_{2(n-\bar{n})} \left(\frac{n!(2h)_n}{\Gamma(2h_n)(n-\bar{n})!(2h_{\bar{n}})_{n-\bar{n}}}\right)^2. \quad (D.2.16)$$

This integral is typically convergent at large values of $Y_1$, making it useful for computing $F_{n,\bar{n}}$ when the two points are only separated on the $y$-direction.

In the case the $\bar{n} = 0$ (or $n = 0$) we have $l = n$, and the expression for $F_{n,0}$ is simplified.

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to be

\[
F_{n,0}(h) = (2h)_{2n} (Y_1 Y_2)^{2hn} \sum_{m,m'=0}^{\infty} \frac{(2h_n)_{m+m'} (2h_n)_{m+m'} (-1)^{m+m'} m! m'!}{(2h_n)_m (2h_n)_{m'}} Y_1^m Y_2^{m'}
\]

\[
= (2h)_{2n} (Y_1 Y_2)^{2hn} F_4 (2h_n, 2h_n, 2h_n, 2h_n, -Y_1, -Y_2)
\]

\[
= (2h)_{2n} \frac{\rho^{h+n}}{1 - \rho}
\]

with \( \rho = \frac{\xi^2}{(1 + \sqrt{1 - \xi^2})}, \xi = \frac{2\sqrt{Y_1 Y_2}}{1 + Y_1 + Y_2} \) and \( F_4 \) is the Appell hypergeometric function.

### D.3 Correlators of Stress Tensors with \( \phi \phi \)

In [3], we used the OPE blocks of \( \phi O \) to compute correlation functions of the form \( \langle \phi O T \cdots T T \cdots T \rangle \).

Similarly, we can derive the OPE block for two bulk operators \( \phi \phi \), and use it to compute the correlation functions of the form \( \langle \phi \phi T \cdots T T \cdots T \rangle \) with the regulator proposed in Appendix B of [3]. Notice that this method will only give the first several terms of the large \( c \) limit of \( \langle \phi \phi T \cdots T T \cdots T \rangle \), up to order \( O(c^0) \), in contrast to the cases in [3], where the correlation functions \( \langle \phi O T \cdots T T \cdots T \rangle \) computed in that paper are exact. This is because this bulk-bulk OPE block does include the gravitational dressing of the \( \phi \) operators.

In the vacuum AdS\(_3\) metric

\[
d s^2 = \frac{d u^2 + d w d \bar{w}}{u^2},
\]

the bulk-bulk propagator is given by

\[
\langle \phi (u_0, w_0, \bar{w}_0) \phi (u_1, w_1, \bar{w}_1) \rangle = \frac{e^{-2h \Sigma}}{1 - e^{-2\Sigma}}
\]

where the geodesic length \( \Sigma \) between the two bulk operators is given by

\[
\Sigma = \log \frac{1 + \sqrt{1 - \Xi^2}}{\Xi}, \quad \Xi = \frac{2u_0 u_1}{u_0^2 + u_1^2 + (w_0 - w_1)(\bar{w}_0 - \bar{w}_1)}
\]
Now, we can view the coordinates \((u, w, \overline{w})\) as the result of an operator valued diffeomorphism from a general vacuum metric of the form

\[
ds^2 = \frac{dy^2 + dzd\overline{z}}{y^2} - \frac{6T(z)}{c}dz^2 - \frac{6\overline{T}(z)}{c}d\overline{z}^2 + y^2 \left( \frac{36T(z)\overline{T}(z)}{c^2} \right)dzd\overline{z}.
\] (D.3.4)

The diffeomorphism [132] is given by

\[
w \to f(z) = \frac{2y^2 (f'(z))^2 \overline{f}'(\overline{z})}{4f'(z)\overline{f}'(\overline{z}) + y^2 f''(z)\overline{f}''(\overline{z})},
\]

\[
\overline{w} \to \overline{f}(\overline{z}) = \frac{2y^2 \left( \overline{f}'(\overline{z}) \right)^2 f''(z)}{4f'(z)\overline{f}'(\overline{z}) + y^2 f''(z)\overline{f}''(\overline{z})},
\]

\[
u \to \frac{4 \left( f'(z)\overline{f}'(\overline{z}) \right)^{\frac{3}{2}}}{4f'(z)\overline{f}'(\overline{z}) + y^2 f''(z)\overline{f}''(\overline{z})}.
\]

And \(T(z)\) (and similar for \(\overline{T}(\overline{z})\)) satisfies

\[
\frac{12T(z)}{c} = \frac{f'''(z)f'(z) - \frac{3}{2} (f''(z))^2}{(f'(z))^2},
\] (D.3.6)

which can be solve order by order in \(\frac{1}{c}\) and the first two terms are

\[
f(z) = z + \frac{f_1(z)}{c} + \mathcal{O} \left( \frac{1}{c^2} \right)
\] (D.3.7)

with

\[
f_1(z) = -6 \int_0^z dz' (z - z')^2 T(z').
\] (D.3.8)

Suppose that the positions of the two operators in the general vacuum background are at \((y, 0, 0)\) and \((y, z, \overline{z})\) \(^1\), that is, \(u_0 = u(y, 0, 0)\) and \(u_1 = u(y, z, \overline{z})\), and similarly for \(w_0, \overline{w}_0, w_1\) and \(\overline{w}_1\), then as in the bulk-boundary case, we can expand the geodesic separation in terms

\(^1\)Here, we consider the case that the two bulk operators are at the same bulk depth \(y\) for simplicity.
of large $c$ as follows

$$\log \frac{\Xi}{1 + \sqrt{1 - \Xi^2}} = \log \frac{\xi}{1 + \sqrt{1 - \xi^2}} + K^b_T + K^b_T + O\left(\frac{1}{c^2}\right) \quad \text{(D.3.9)}$$

where $\xi = \frac{2y^2}{zy^2 + z\bar{z}}$ and

$$K^b_T = \frac{z\bar{z}f'_1(z) - 2z\bar{z}f_1(z) + y^2zf''_1(z)}{2c \sqrt{z\bar{z}(z\bar{z} + 4y^2)}}. \quad \text{(D.3.10)}$$

Here, we use superscibe $b$ in $K^b_T$ to denote that these are the OPE blocks for two bulk operators, in contract to the case in [3], where one of the operators is on the boundary. So plugging in the expression of $f_1$, we get

$$K^b_T = \frac{1}{c} \int_0^z dz' \frac{6(z-z')z' + y^2z}{\sqrt{z\bar{z}(z\bar{z} + 4y^2)}} T(z'). \quad \text{(D.3.11)}$$

When sending $y$ to $0$, $K^b_T$ reduces to the OPE block of the two operators on the boundary [3].

Now, expanding the RHS of equation (D.3.2) in terms of large $c$ using equation (D.3.9), we get the OPE block of two bulk operators

$$\phi(y, 0, 0) \phi(y, z, \bar{z}) \sim \rho^h \left[1 + 2 \left(h + \frac{\rho}{1 - \rho}\right) \left(K^b_T + K^b_T + O\left(\frac{1}{c^2}\right)\right)\right]$$

with $\rho = \frac{\xi^2}{(1 + \sqrt{1 - \xi^2})^2}$. So using $\langle \phi(y, 0, 0) \phi(y, z, \bar{z}) \rangle_{\text{global}} = \frac{\rho^h}{1 - \rho}$, we find

$$\frac{\langle \phi(y, 0, 0) \phi(y, z, \bar{z}) T(z_1) \rangle}{\langle \phi(y, 0, 0) \phi(y, z, \bar{z}) \rangle_{\text{global}}} = 2 \left(h + \frac{\rho}{1 - \rho}\right) \langle K^b_T T(z_1) \rangle \quad \text{(D.3.12)}$$

$$= 2 \left(h + \frac{\rho}{1 - \rho}\right) 12 \int_0^z dz' \frac{z(z - z')z' + zy^2}{2c \sqrt{z\bar{z}(z\bar{z} + 4y^2)}} \langle T(z') T(z_1) \rangle$$

$$= 2 \left(h + \frac{\rho}{1 - \rho}\right) \frac{z^2 [(6y^2 + z\bar{z})z_1(z_1 - z) + 2y^2z^2]}{2z_1^3 (z_1 - z)^3 \sqrt{z\bar{z}(z\bar{z} + 4y^2)}}$$
In Section 3.1, we shown that (equation (5.3.9))

$$\langle \phi (1, 0, 0) \phi (1, 1, 1) T (z_1) \rangle = \frac{\xi (\xi + (2 \xi + 1) (z_1 - 1) z_1)}{2 (z_1 - 1)^3 z_1^3} g' (\xi).$$  

(D.3.13)

This is actually equivalent to equation (D.3.12) if we replace $g(\xi)$ with its leading large $c$ limit, that is

$$\lim_{c \to \infty} g (\xi) = \log \langle \phi \phi \rangle_{\text{global}} = \log \left( \frac{\rho}{1 - \rho} \right).$$  

(D.3.14)

To see this, notice that $\frac{d}{d\xi} \log \left( \frac{\rho}{1 - \rho} \right) = 2 \left( h + \frac{\rho}{1 - \rho} \right) \frac{1}{\xi \sqrt{1 - \xi}}$, so that equation (D.3.12) can be written as

$$\langle \phi (y, 0, 0) \phi (y, z, \bar{z}) T (z_1) \rangle_{\text{global}} = \xi \left[ \frac{d}{d\xi} \log \left( \frac{\rho}{1 - \rho} \right) \right] \frac{z^2 (\xi z^2 + (2 \xi + 1) (z_1 - z) z_1)}{2 (z_1 - z)^3 z_1^3}.$$  

(D.3.15)

Setting $z = 1$, we get exactly equation (D.3.13).

One can continue this procedure to compute correlators with more $T(z)$ (and $\bar{T}(z)$) insertions. But the result will not capture the $O(\frac{1}{c})$ terms of the exact correlator $\langle \phi \phi T \cdots \bar{T} \cdots \bar{T} \rangle$ because it does not include the gravitational dressing of $\phi$.

### D.4 Algorithms for Implementing the Recursion Relations

#### D.4.1 $c$-recursion Algorithm

The $c$-recursion relation is

$$F (h, c) = 1 + \sum_{m \geq 1, n \geq 2} - \frac{\partial c_{m,n} (h)}{\partial h} A_{m,n}^{c_{m,n} (2h) 2mn} c - c_{m,n} (h) \rho^{mn} F (h + mn, c_{m,n} (h))$$  

(D.4.1)
We know that the above recursion will give \( F(h, c) \) as the following expansion

\[
F(h, c) = \sum_{N=0}^{\infty} C_N (2h)_N c^N.
\]

(D.4.2)

The factor \((2h)_{2mn}\) in the residue will eventually give \((2h)_N\), so for now let’s consider how the coefficients \(C_N\) are constructed from the above recursion. Let’s denote the residue without the factor \((2h)_{2mn}\) as \(R_{m,n}(h) = -\frac{\partial c_{m,n}(h)}{\partial h} A_{m,n}^c\).

The recursion (D.4.1) is actually saying that every time we can write \(N\) as a sum of products of integers, i.e.

\[
N = m_1 n_1 + m_2 n_2 + \cdots + m_i n_i,
\]

then we get a contribution to \(C_N\) from the recursion. In the above decomposition, each term represents one iteration of the recursion. Denote the contribution to \(C_N\) from the decomposition whose last term is \(m_i n_i\) as \(C_{N,m_i,n_i}\), then we can write \(C_N\) as the following sum

\[
C_N = \sum_{2 \leq m_i n_i \leq N} C_{N,m_i,n_i}.
\]

(D.4.4)

Then \(C_{N,m_i,n_i}\) will satisfy the following equation

\[
C_{N,m_i,n_i} = \frac{R_{m_i,n_i}(h)}{c - c_{m_i,n}(h)} \delta_{N,m_i,n_i}
\]

\[+ \sum_{2 \leq m_j n_j \leq N - m_i n_i} C_{N-m_i,n_i,m_j,n_j} \frac{R_{m_i,n_i}(h + N - m_i n_i)}{c_{m_j,n_j}(h + N - m_i n_i - m_j n_j) - c_{m_i,n_i}(h + N - m_i n_i)}.
\]

(D.4.5)

The first term can be thought of as the boundary condition, which is just the case that there is only one term in the decomposition (D.4.3). The second term\(^2\) sums over all the contributions from the cases where there are more than one term in (D.4.3) and supposes that the second last term is \(m_j n_j\): \(N = m_1 n_1 + \cdots + m_j n_j + m_i n_i\).

\(^2\)The terms in the parentheses are the arguments of the functions \(R_{m_i,n_i}\) and \(c_{m,n}\), not to be confused as a factor times \(R_{m_i,n_i}\) and \(c_{m,n}\).

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we can first compute all the boundary terms $C_{m,n_i,m_i,n_i} = R_{m_i,n_i}(h)$. Then we increase $n$ from $n = m_i n_i + 2$ to $N$. For each $n$, we compute all the $C_{n,m_i,n_i}$ via equation (D.4.5). We are able to do this because all the information (i.e. $C_{n-m_i,n_i,m_i,n_i}$) needed to compute $C_{n,m_i,n_i}$ has already been computed. The complexity for this algorithm will be roughly $N^4(\log N)^2$.

**D.4.2 h-recursion Algorithm**

The algorithm for implementing $h$-recursion will be faster than the $c$-recursion. The reason is that in the $h$-recursion

$$H(h, c) = 1 + \sum_{m,n} q^{mn} \frac{(2h_{m,n})_{2mn} A_{m,n}^c}{h - h_{m,n}(c)} H(h_{m,n} + mn, c),$$

(D.4.6)

each time each time we only change $h \rightarrow h_{m,n} + mn$, whereas in the $c$-recursion, we change both $h \rightarrow h + mn$ and $c \rightarrow c_{m,n}(h)$. Denoting the coefficients of $q^N$ in $H$ as $H_N$, i.e. $H = 1 + \sum_{N=2}^{\infty} H_N q^N$, then we can write the solution of $H_N$ as in equations (D.4.4) and (D.4.5). But here, we’ll think of the problem in another way. In equation (D.4.4) and (D.4.5), we were working backward from the last step to arrive at $N$ from $N - m_i n_i$. But since in the $h$-recursion, $H(h_{m,n} + mn, c)$ only depends on $m, n$ and $c$, it’s actually easier to consider the problem here forward from the first step, that is, we can write $H_N$ as the following sum (define $\tilde{R}_{m,n} \equiv (2h_{m,n})_{2mn} A_{m,n}^c$)

$$H_N = \sum_{2 \leq mn \leq N} \frac{\tilde{R}_{m,n}}{h - h_{m,n}(c)} H_{m,n}^{(N-mn)}$$

(D.4.7)

\(^3\text{In fact, } F(h + mn, c_{m,n}(h)) \text{ depends on the value of } h + mn, \text{ so it actually depends on the \textit{“history”} of the recursion. For example, in the decomposition (D.4.3), the first term will involve } F(h + m_1 n_1, c_{m_1,n_1}(h)), \text{ but the second term will involve } F(h + m_1 n_1 + m_2 n_2, c_{m_2,n_2}(h + m_1 n_1)).\)
where $H_{m,n}^{(N-mn)}$ is the coefficient of $q^{N-mn}$ in $H(h_{m,n} + mn, c)$. Then it’s easy to see that $H_{m,n}^{(N-mn)}$ is given by

$$H_{m,n}^{(N-mn)} = \sum_{2 \leq m_i, n_i \leq N-mn} \frac{\tilde{R}_{m_i,n_i}}{h_{m,n} + mn - h_{m_i,n_i}} H_{m_i,n_i}^{(N-mn-m_i,n_i)}$$  \hspace{1cm} (D.4.8)

The complexity for the $h$-recursion will be roughly $N^3 (\log N)^2$. We’ve described the algorithm for implementing the $h$-recursion for Virasoro blocks in detail in [2], and the $h$-recursion for $\langle \phi \phi \rangle_{\text{holo}}$ is almost the same (except that the residues are different, which doesn’t affect the algorithm), so we refer the reader to Appendix A of that paper.
Appendix E

Appendix to Chapter 6

E.1 Coordinate Systems

The purpose of this appendix is to explain the relationship between the BTZ black hole in its standard form and in the coordinate system that we use in this paper. We will see that the relation has a surprising feature: real values of the standard AdS-Schwarzschild coordinates \((t_E, r, \theta)\) correspond with complex values for the Fefferman-Graham radial coordinate \(y\). As far as we are aware, this feature has not been noted in the literature. For completeness and perhaps for pedagogical value, we will also make some elementary comments concerning the connection between diffeomorphisms and conformal transformations.

E.1.1 Various Coordinate Relations

The Euclidean BTZ black hole metric is typically written using Schwarzchild coordinates

\[
\begin{align*}
 ds^2 &= \left( \frac{r^2 - r_+^2}{r^2} + \frac{r^2}{r^2 - r_+^2} \right) dt_E^2 + \frac{r^2 dr^2}{\left( \frac{r^2}{r^2 - r_+^2} \right)^2} + r^2 \left( d\theta + \frac{r+r_-}{r^2} dt_E \right)^2 
\end{align*}
\]  

(E.1.1)

where we note that to avoid a conical singularity at the horizon, we must identify \(t_E \sim t_E + \frac{2\pi}{r_+}\), and by definition we identify \(\theta \sim \theta + 2\pi\). As we take \(r \to \infty\) with fixed \(t_E, \theta\) we approach
the boundary cylinder, with metric $ds^2 = dt_E^2 + d\theta^2$. We can easily obtain the Lorentzian BTZ metric via the simultaneous analytic continuations $t_E \to it$ and $r_- \to ir_-$. 

Our exact results are based in a Fefferman-Graham coordinate system, where in the presence of a heavy source the Euclidean metric takes the form

$$ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2} - \frac{6h_H}{cz^2}d\zeta^2 - \frac{6\bar{h}_H}{\bar{c}z^2}\bar{d}\zeta^2 + y^2\frac{36h_H\bar{h}_H}{c^2z^2\bar{z}^2}dzd\bar{z}$$  \hspace{1cm} (E.1.2)

The boundary corresponds to $y \to 0$, and if we take this limit uniformly (without scaling by any function of $z, \bar{z}$) then we obtain a flat boundary metric $ds^2 = dzd\bar{z}$. The heavy sources with conformal weights $(h_H, \bar{h}_H)$ are located at $z = 0$ and $z = \infty$ on the boundary. When the sources are absent, this metric reduces to that of the standard Euclidean Poincaré patch for AdS$_3$.

Throughout, we will use the relation $\alpha = \sqrt{1 - \frac{24h_H}{c}}$, and $\bar{\alpha} = \sqrt{1 - \frac{24\bar{h}_H}{c}}$, and by convention when $\alpha, \bar{\alpha}$ are imaginary we take them to have opposite signs. These parameters are related to the outer and inner horizon radii of the Euclidean black hole via $\alpha = ir_+ - r_-$ and $\bar{\alpha} = -ir_+ - r_-$. We mostly focus on the spherically symmetric case with $r_- = 0$.

Now let us discuss the coordinate relations. First, let us note that equation (E.1.1) does have a simple relationship with a metric that looks superficially like our Fefferman-Graham coordinate system. This is a third distinct form of the metric

$$ds^2 = \frac{dn^2}{n^2} + \frac{\alpha^2}{4}d\xi^2 + \frac{\bar{\alpha}^2}{4}\bar{d}\xi^2 + \left(\frac{1}{n^2} + n^2\frac{\alpha^2\bar{\alpha}^2}{16}\right)\frac{d\xi d\bar{\xi}}{\xi\bar{\xi}}$$  \hspace{1cm} (E.1.3)

Notice that in the absence of sources, when $\alpha = \bar{\alpha} = 1$, this metric does not reduce to the Poincaré patch form of AdS$_3$. Relatedly, when we approach the boundary by taking the limit $n \to 0$, the term $\frac{\alpha^2}{4}d\xi^2$ in the metric has an interpretation as an expectation value for the CFT stress tensor $\langle T \rangle = \frac{\alpha^2}{4\xi}$, and it is non-zero even in the vacuum. Both of these facts follow because in equation (E.1.3), the coordinates $\xi, \bar{\xi}$ parameterize the surface of a cylinder, rather than a flat plane when we take $n \to 0$. This is manifest with $\xi = e^{t_E + i\theta}$ and
\( \bar{\xi} = e^{4E-i\theta} \).

We can relate the metric (E.1.3) and the standard form of BTZ (E.1.1) straightforwardly; we take \( \xi = e^{4E-i\theta} \) and \( \bar{\xi} = e^{4E-i\theta} \) as above, while

\[
r^2 = \frac{(\alpha^2 n^2 - 4) (\bar{\alpha}^2 n^2 - 4)}{16n^2}
\]

This means that the horizon is located at\(^1\)

\[
n_* = \frac{2}{\sqrt{\alpha \bar{\alpha}}} = \frac{2}{\sqrt{r^2_+ + r^2_-}}
\]

in the coordinate system of equation (E.1.3).

Now let us identify a relation between the metric (E.1.3) and the Fefferman-Graham metric (E.1.2) that we are using in this paper. This is more complicated, but it can be achieved by obtaining both metrics as sub-regions of empty Poincaré patch AdS\(_3\). Starting with

\[
ds^2 = \frac{du^2 + dx d\bar{x}}{u^2}
\]

we can obtain any vacuum metric by identifying [132]

\[
u = y \frac{4(f'(z) \bar{f}'(\bar{z}))^2}{4f'(z) f'(\bar{z}) + y^2 f''(z) \bar{f}''(\bar{z})}
\]

\[
x = f(z) - \frac{2y^2 (f'(z))^2 \bar{f}''(\bar{z})}{4f'(z) f'(\bar{z}) + y^2 f''(z) \bar{f}''(\bar{z})}
\]

\[
\bar{x} = \bar{f}(\bar{z}) - \frac{2y^2 (\bar{f}'(\bar{z}))^2 f''(z)}{4f'(z) f'(\bar{z}) + y^2 f''(z) \bar{f}''(\bar{z})}
\]

for suitable \( f, \bar{f} \). To obtain the metric of equation (E.1.3) we use \( f(z) = e^{\alpha z} \) followed

\(^1\)Recall that in the Euclidean region this actually represents a line, rather than a 2d surface, because the thermal Euclidean time circle shrinks to a point at the horizon.
by \( z = \log \xi \), whereas to obtain equation (E.1.2) we directly use \( f(z) = z^\alpha \). These two transformations are subtly different because the derivatives of \( f \) are respect to different variables. The end result is a dictionary between coordinate systems

\[
\begin{align*}
\frac{y}{z} &= \frac{4z \bar{z} \sqrt{\alpha \bar{\alpha} z^{\alpha - 1} \bar{z}^{\bar{\alpha} - 1}}}{4 z \bar{z} + (\alpha - 1)(\bar{\alpha} - 1)y^2} = u = \frac{4n \sqrt{\alpha \bar{\alpha} \xi^\alpha \bar{\xi}^{\bar{\alpha}}}}{4 + \alpha \bar{\alpha} n^2} \\
\frac{z^\alpha}{z} &= \frac{4z \bar{z} - (\alpha + 1)(\bar{\alpha} - 1)y^2}{4 z \bar{z} + (\alpha - 1)(\bar{\alpha} - 1)y^2} = x = \frac{\xi^\alpha (4 - \alpha \bar{n}^2)}{4 + \alpha \bar{\alpha} n^2} \\
\frac{\bar{z}^{\bar{\alpha}}}{z} &= \frac{4z \bar{z} - (\alpha - 1)(\bar{\alpha} + 1)y^2}{4 z \bar{z} + (\alpha - 1)(\bar{\alpha} - 1)y^2} = \bar{x} = \frac{\bar{\xi}^{\bar{\alpha}} (4 - \alpha \bar{n}^2)}{4 + \alpha \bar{\alpha} n^2}
\end{align*}
\]

Notice that at small \( y \) and \( n \), we have \( z \approx \xi, \bar{z} \approx \bar{\xi}, \) and \( \frac{y^2}{z} \approx n \). This means that taking the limit \( y \to 0 \) results in a different boundary metric from \( n \to 0 \); in the former case we obtain a CFT in flat space, whereas in the latter case we obtain the CFT on a cylinder.

One can solve the relation between coordinates explicitly. Defining a discriminant

\[
D^2 \equiv \alpha^4 \bar{\alpha}^4 n^8 - 16 \alpha^2 \bar{\alpha}^2 n^6 + 32(2 \alpha^2 + 2 \bar{\alpha}^2 - \alpha^2 \bar{\alpha}^2)n^4 - 256n^2 + 256 \quad (E.1.9)
\]

we find the results

\[
\begin{align*}
\frac{y^2}{z} &= \frac{\alpha^2 \bar{\alpha}^2 n^4 - 8 n^2 + 16 - D}{2n^2(1 - \alpha^2)(1 - \bar{\alpha}^2)} \\
\frac{z^\alpha}{z} &= \xi^\alpha \left( \frac{\alpha^2 \bar{\alpha}^2 n^4 - 8 \alpha^2 n^2 + 16 + \alpha D}{(1 + \alpha)(16 - \alpha^2 \bar{\alpha}^2 n^4)} \right) \\
\frac{\bar{z}^{\bar{\alpha}}}{z} &= \bar{\xi}^{\bar{\alpha}} \left( \frac{\alpha^2 \bar{\alpha}^2 n^4 - 8 \bar{\alpha}^2 n^2 + 16 + \bar{\alpha} D}{(1 + \alpha)(16 - \alpha^2 \bar{\alpha}^2 n^4)} \right) \quad (E.1.10)
\end{align*}
\]

This makes it possible to connect the standard BTZ metric and our Fefferman-Graham
coordinate system; for completeness note that

\[
n^2 = \frac{2 \left( \alpha^2 + \bar{\alpha}^2 + 4r^2 - \sqrt{(\alpha^2 - \bar{\alpha}^2)^2 + 16r^4 + 8r^2 (\alpha^2 + \bar{\alpha}^2)} \right)}{\alpha^2 \bar{\alpha}^2} \tag{E.1.11}
\]

which allows us to write \(y, z, \bar{z}\) explicitly in terms of \(r, t_E, \theta\). The results simplify somewhat in the spherically symmetric case \(r_\pm = 0\) when we connect directly to the BTZ coordinates. In that case we find

\[
y = \frac{2\sqrt{\xi \bar{\xi}}}{\bar{r}} \left( r - \frac{\sqrt{r^2 - r_+^2} - 1}{r_+^2 + 1} \right) \\
z = \frac{1}{r} \xi \\
\bar{z} = \frac{1}{\bar{r}} \bar{\xi} \tag{E.1.12}
\]

with \(\bar{r} \equiv \left( \frac{r + ir_+ \sqrt{r^2 - r_+^2} - 1}{(1 + ir_+) \sqrt{r^2 - r_+^2}} \right)^{1/r_+^2}\). Recall that we can rewrite these results in terms of \(t_E, \theta\) of the BTZ metric via \(\xi = e^{t_E+i\theta}\) and \(\bar{\xi} = e^{t_E-i\theta}\); note that the BTZ and Fefferman-Graham time coordinates are only identical at the boundary.

However, these expressions imply something unexpected about the 3d real manifold in the \((y, z, \bar{z})\) coordinate systems associated with real \(r, t_E, \theta\) in the standard BTZ metric – the \(y\) coordinate takes complex values when \((r, t_E, \theta)\) are real. This occurs whenever \(r^2 < 1 + r_+^2\). In particular, the horizon corresponds with

\[
y^2 = \frac{4}{(r_+ \pm i)^2 - r_-^2} \tag{E.1.13}
\]

Despite these complex values for \(y\), by definition the line element \(ds^2\) from (E.1.2) will be real when evaluated as a function of real \(t, r, \theta\) (and also after a Lorentzian continuation via \(t_E \to it\) and \(r_\to ir_\)). Nevertheless, these complex values for \(y\) are a feature of the relationships between these coordinate systems.
Eddington-Finkelstein

To study the horizon of a BTZ black hole, it is useful to use coordinates that are well-behaved in its vicinity. Thus we can use the Eddington-Finkelstein coordinate

\[ v = t - \frac{1}{r_+} \tanh^{-1} \left( \frac{r}{r_+} \right) + i \frac{\pi}{2r_+} \]  

(E.1.14)

which we have written in terms of the Lorentzian BTZ time coordinate \((t = -it_E)\) and radius. In the spherically symmetric case, this produces a metric

\[ ds^2 = -(r^2 - r_+^2)dv^2 + 2dvdr + r^2d\theta^2 \]  

(E.1.15)

which is non-singular through the horizon.

Holomorphic Limit

The relationship between the \(\frac{y^2}{z\bar{z}}\) and \(r\) coordinates simplifies in the holomorphic limit, where \(\bar{h}_H = 0\) and \(\bar{\alpha} = 1\). In that case we simply find that

\[ \frac{y^2}{z\bar{z}} = \frac{4}{4r^2 + \alpha^2 - 1} \]  

(E.1.16)

where the Fefferman-Graham coordinates are on the left hand side. We see that even in the case of deficit angles (with real \(\alpha\)), when \(r\) becomes sufficiently small we must analytically continue to complex values of \(y\). However, the relationship between \(z, \bar{z}\) and \(r\) remains quite complicated.

E.1.2 Bulk-boundary Vacuum Block in BTZ Coordinates

In Section 6.4, we’ve developed several methods to compute the bulk-boundary vacuum block \(\mathcal{V}_0(y, z, \bar{z})\) in the following configuration: \(\langle \mathcal{O}_H(\infty) \mathcal{O}_H(1) \mathcal{O}_L(z, \bar{z}) \phi(y, 0, 0) \rangle\), where the heavy sources \(\mathcal{O}_H\) are at \(z = 1\) and \(z = \infty\). However, the Euclidean BTZ metric (E.1.2)
in the Fefferman-Graham coordinate system has heavy sources located at $z = 0$ and $z = \infty$ on the boundary. To make the physics more transparent in that metric, we can move the heavy operator $\mathcal{O}_H$ at $z = 1$ to $z = 0$, by using a conformal transformation that takes $\infty \to \infty$, $1 \to 0$, $z \to 1$. This uniquely fixes the conformal transformation to be

$$
x \to \frac{1 - x}{1 - z}.
$$

(E.1.17)

Under this transformation, the bulk position $(y, 0, 0)$ transforms as

$$
(y, 0, 0) \to \left( \frac{y}{\sqrt{(1 - z)(1 - \bar{z})}}, \frac{1}{1 - z}, \frac{1}{1 - \bar{z}} \right) \equiv (y', z', \bar{z}'),
$$

(E.1.18)

and we find

$$
\langle \mathcal{O}_H (\infty) \mathcal{O}_H (0) \mathcal{O}_L (1) \phi (y', z', \bar{z}') \rangle \equiv (1 - z)^{h_L} (1 - \bar{z})^{h_L} \langle \mathcal{O}_H (\infty) \mathcal{O}_H (1) \mathcal{O}_L (z, \bar{z}) \phi (y, 0, 0) \rangle.
$$

(E.1.19)

Now we can use equation (E.1.12) to map $(y', z', \bar{z}')$ to the usual BTZ coordinates $(r, t_E, \theta)$, i.e.

$$
y' = 2 \tilde{r} \left( \frac{r - \sqrt{r^2 - r^2_+} - 1}{r^2_+ + 1} \right) e^{t_E}, \quad z' = \frac{1}{r} e^{t_E + i\theta}, \quad \bar{z}' = \frac{1}{\tilde{r}} e^{t_E - i\theta},
$$

(E.1.20)

with $\tilde{r} \equiv \left( \frac{r + ir_+ \sqrt{r^2 - r^2_+} - 1}{(1 - ir_+) \sqrt{r^2 - r^2_+}} \right)^{1/r^2_+}$. Using the relationship between $(y', z', \bar{z}')$ and $(y, z, \bar{z})$, that is, equation (E.1.18), we find that to map the bulk-boundary vacuum block $\mathcal{V}_0 (y, z, \bar{z})$ of Section 6.4 from $(y, z, \bar{z})$ to the BTZ coordinates $(r, t_E, \theta)$, we need to use

$$
y = 2 \frac{r - \sqrt{r^2 - r^2_+} - 1}{r^2_+ + 1},
$$

$$
z = 1 - \tilde{r} e^{-t_E - i\theta},
$$

(E.1.21)
$\tilde{z} = 1 - \tilde{r}e^{-t+i\theta}$.

### E.1.3 Inversion Symmetry

In this section, we give two examples of inversion symmetry discussed in section 6.3.3.3. In Feffereman-Graham gauge, the AdS$_3$ metric

$$ds^2 = \frac{dy^2 + dzd\tilde{z}}{y^2} - \frac{S(z)}{2}dz^2 - \frac{\tilde{S}(\tilde{z})}{2}d\tilde{z}^2 + y^2 S(z) \tilde{S}(\tilde{z}) \frac{dzd\tilde{z}}{4}$$  \hspace{1cm} (E.1.22)

can be obtained from the pure Poincare metric $ds^2 = \frac{du^2 + dx d\bar{x}}{u^2}$ with transformations (E.1.7), where $S(z)$ and $\tilde{S}(\tilde{z})$ are given by Schwarzian derivatives $S(z) = \{f(z), z\}, \quad \tilde{S}(\tilde{z}) = \{\tilde{f}(\tilde{z}), \tilde{z}\}$. This metric has an inversion symmetry, because the same metric can be obtained by the same functions $f, \tilde{f}$, but with inverse arguments, i.e. $f\left(\frac{1}{\bar{z}}\right), \tilde{f}\left(\frac{1}{\tilde{z}}\right)$. Specifically, the inversion corresponds to the identification between unprimed and primed coordinates through the relations

$$\tilde{u}\left(y, f(z), \tilde{f}(\tilde{z})\right) = u\left(y', f\left(\frac{1}{\bar{z}'\prime}\right), \tilde{f}\left(\frac{1}{\tilde{z}'\prime}\right)\right)$$

$$x\left(y, f(z), \tilde{f}(\tilde{z})\right) = x\left(y', f\left(\frac{1}{\bar{z}'\prime}\right), \tilde{f}\left(\frac{1}{\tilde{z}'\prime}\right)\right)$$  \hspace{1cm} (E.1.23)

$$\bar{x}\left(y, f(z), \tilde{f}(\tilde{z})\right) = \bar{x}\left(y', f\left(\frac{1}{\bar{z}'\prime}\right), \tilde{f}\left(\frac{1}{\tilde{z}'\prime}\right)\right)$$

The solutions to these equations (i.e. $(y, z, \tilde{z})$ in terms of $(y', z', \tilde{z}')$) are often rather complicated. Here, we give two examples: the Poincare AdS$_3$ and BTZ black holes.
Poincare AdS$_3$

The Poincare metric $ds^2 = \frac{dy^2+dzd\bar{z}}{y^2}$ can be simply obtained by $f(z) = z, \tilde{f}(\bar{z}) = \bar{z}$. So we have $f\left(\frac{1}{\bar{z}}\right) = \frac{1}{\bar{z}}$ and $\tilde{f}\left(\frac{1}{z}\right) = \frac{1}{z}$. And equations (E.1.23) become

$$\tilde{u}(y, z, \bar{z}) = u(y, 1, 1, \bar{z}) = \frac{y'}{y'^2 + z'^2}$$

$$x(y, z, \bar{z}) = z = x(y, 1, 1, \bar{z}) = \frac{z'}{y'^2 + z'^2}$$

$$\bar{x}(y, z, \bar{z}) = \bar{z} = \bar{x}(y, 1, 1, \bar{z}) = \frac{z'}{y'^2 + z'^2}$$

where the relations between $(y, z, \bar{z})$ and $(y', z', \bar{z'})$ are manifest.

BTZ black holes

The BTZ black hole case is more relevant to this work; it is also more complicated. To obtain the BTZ black hole metric (E.1.2) in terms of $(y, z, \bar{z})$, we used $f(z) = z^{ir+}, \tilde{f}(\bar{z}) = \bar{z}^{-ir+}$, with $r_+ = \sqrt{\frac{24h\mu}{c}} - 1$. So we have $f\left(\frac{1}{\bar{z}}\right) = \bar{z}^{-ir+}, \tilde{f}\left(\frac{1}{z}\right) = z^{ir+}$. Then equations (E.1.23) become

$$u(y, z^{ir+}, \bar{z}^{-ir+}) = \frac{4r_+ y z^{\frac{1}{2}+ir+} \bar{z}^{-\frac{1}{2}-ir+}}{4z \bar{z} + (r_+^2 + 1) y^2} = u(y', z'^{ir+}, z'^{-ir+}) = \frac{4r_+ y' z'^{\frac{1}{2}+ir+} \bar{z}'^{-\frac{1}{2}-ir+}}{4z' \bar{z}' + (r_+^2 + 1) y'^2}$$

$$x(y, z^{ir+}, \bar{z}^{-ir+}) = \frac{z^{ir+} (4z \bar{z} - (r_+ - i)^2 y^2)}{4z \bar{z} + (r_+^2 + 1) y^2} = x(y', z'^{ir+}, z'^{-ir+}) = \frac{z'^{ir+} (4z' \bar{z}' - (r_+ + i)^2 y'^2)}{4z' \bar{z}' + (r_+^2 + 1) y'^2}$$

$$\bar{x}(y, z^{ir+}, \bar{z}^{-ir+}) = \frac{\bar{z}^{-ir+} (4z \bar{z} - (r_+ + i)^2 y^2)}{4z \bar{z} + (r_+^2 + 1) y^2} = \bar{x}(y', \bar{z}'^{ir+}, \bar{z}'^{ir+}) = \frac{\bar{z}'^{-ir+} (4z' \bar{z}' - (r_+ - i)^2 y'^2)}{4z' \bar{z}' + (r_+^2 + 1) y'^2}$$

where the solution gives the coordinate relations after an inversion:

$$y = \frac{2}{\sqrt{z' \bar{z}' r^2}} \left( r - \sqrt{r^2 - r_+^2 - 1} \right)$$

$$z = \frac{1}{\bar{z}' r^2}$$

(E.1.26)
\[
\tilde{z} = \frac{1}{z' \tilde{r}^2}
\]

with \( \tilde{r} = \left( \frac{r + ir_+ \sqrt{r^2 - r_+^2 - 1}}{(1 + ir_+) \sqrt{r^2 - r_+^2}} \right)^{1/2} \) and \( r = \frac{(r_+^2 + 1) y'^2 + 4 \sqrt{z' \bar{z}'}}{4y' \sqrt{z' \bar{z}'}} \). We emphasize that although the above solution looks complicated, in terms of the BTZ metric (6.2.1) in coordinates \((r, t_E, \phi)\), this just corresponds to the time reversal symmetry \( t_E \rightarrow -t_E \). One can also check that expanding the above solution in small \( h_H \), the leading term are indeed given by the inversion solution (E.1.24) for the pure Poincare metric.

### E.1.4 Elementary Note on Diffeomorphisms and Conformal Symmetries

Here we will make some very elementary comments about bulk diffeomorphisms and boundary conformal transformations. These ideas are probably well-known among experts, but they are rarely stated explicitly, so for completeness we will briefly review them. The ultimate point is to contrast the diffeomorphism (E.1.7) with a different and more naive procedure for implementing conformal transformations in AdS/CFT. First, let us remind ourselves of a trivial point concerning the definition of conformal transformations.

Consider a CFT\(_2\) in the metric \( ds^2 = dzd\bar{z} \). If we introduce new coordinate labels via \( z \equiv f(\zeta) \), then we obtain a new expression for the metric, so \( ds^2 = dzd\bar{z} = f' \tilde{f}' d\zeta d\bar{\zeta} \). This is the same physical metric; we have just re-written it using a different set of labels for the points. However, if we now perform a Weyl transformation and multiply our metric by \( \frac{1}{f' \tilde{f}'} \), then we obtain a physically distinct metric \( ds'^2 = d\zeta d\bar{\zeta} \). This metric once again appears flat, but distances between points have clearly changed as a consequence of the Weyl factor. The key point is that the metrics are physically different because we have fixed the relation \( z \equiv f(\zeta) \).

Now let us consider the transformation rule for a primary operator. When we transform from \( \mathcal{O}(z) \) to \( \mathcal{O}(\xi) \), what we really mean is that we define \( z \equiv f(\xi) \) and we change the metric
from $dzd\bar{z} \to d\xi d\bar{\xi}$. Then the transformation rule is

$$(d\xi)^h \mathcal{O}(\xi) = (dz)^h \mathcal{O}(z) \quad \text{(E.1.27)}$$

Since we have that $z = f(\xi)$ this means that as usual

$$\mathcal{O}(\xi) = (f'(\xi))^h \mathcal{O}(z). \quad \text{(E.1.28)}$$

For example, we can verify the standard result for $f(\xi) = e^\xi$ that

$$\langle \mathcal{O}(\xi_1) \mathcal{O}(\xi_2) \rangle = \left( e^{\xi_1} e^{\xi_2} \right)^h \left( \frac{1}{(e^{\xi_1} - e^{\xi_2})^2} \right)^h = \left( \frac{1}{2 \sinh \left( \frac{\xi_1 - \xi_2}{2} \right)} \right)^{2h} \quad \text{(E.1.29)}$$

providing a quick check of the logic.

Now we can see why the diffeomorphism of equation (E.1.7) implements a general conformal transformation in the CFT$_2$. Under this transformation, the boundary metric $ds^2 = dxd\bar{x}$ corresponding to the limit $n \to 0$ becomes a new boundary metric $ds^2 = dzd\bar{z}$ when we take the (different) limit $y \to 0$. Though these boundary metrics appear identical, they are physically distinct, since by definition $x = f(z)$.

We have utilized the bulk diffeomorphism (E.1.7) to move the CFT from one spacetime metric to another via a (Virasoro) conformal transformation. We can distinguish this operation from another kinematical procedure, which appears to function in any number of spacetime dimensions, and is often discussed in the context of the null cone embedding of AdS/CFT. In this procedure we write

$$ds^2 = \frac{dy^2 + dx_i^2}{y^2} \quad \text{(E.1.30)}$$
and then take \( y = \epsilon F(x_i) \) followed by \( \epsilon \to 0 \), resulting in a boundary metric

\[
ds^2 = \frac{1}{F^2(x_i)} dx_i^2
\] (E.1.31)

that differs from the flat metric by a completely general Weyl factor.

While this procedure appears to correctly implement the transformation rule for primary operators, it is purely kinematical. In this sense it is somewhat misleading, as knowledge of CFT\(_d\) correlators in flat spacetime does not determine the correlators in the general metric of equation (E.1.31). For example, this procedure does not account for effects such as the expectation value of the stress tensor in the new background metric, which arises automatically (as a Schwarzian derivative of the conformal transformation) when we use the diffeomorphism (E.1.7) in the context of AdS\(_3\)/CFT\(_2\). Thus the diffeomorphism (E.1.7) correctly implements conformal transformations in CFT\(_2\); the fact that no equivalent diffeomorphism exists in higher dimensions reflects the physical fact that the conformal group is finite dimensional, and cannot be used to implement non-constant Weyl transformations.

### E.2 Bulk-boundary Vacuum Block via OPE Blocks

In this section, we will use the OPE block formalism developed in [3, 77] to compute the vacuum bulk-boundary block \( \mathcal{V}_0 = \langle \mathcal{O}_H (\infty) \mathcal{O}_H (1) \mathcal{P}_0 \mathcal{O}_L (z, \bar{z}) \phi_L (y, 0, 0) \rangle \) up to order \( 1/c^2 \). Here, we are considering the large \( c \) limit, with \( h_L \) and \( h_H \) fixed.

The vacuum bulk-boundary OPE block for \( \mathcal{O}_L \phi_L \) is given by

\[
\frac{\phi_L (y, 0, 0) \mathcal{O}_L (z, \bar{z})}{\langle \phi_L (y, 0, 0) \mathcal{O}_L (z, \bar{z}) \rangle} = e^{K_{T}^{\text{bulk}} + K_{TT}^{\text{bulk}} + K_{TT}^{\text{bulk}} + K_{TT}^{\text{bulk}} + \cdots}
\] (E.2.1)

with

\[
K_T^{\text{bulk}} = \frac{12 h_L}{c} \int_0^z dz' \left( \frac{y^2 + z' \bar{z}}{z \bar{z} + y^2} \right) T(z') ,
\]
\[ K_{TT}^{\text{bulk}} = \frac{72h_L}{c^2(y^2 + z\bar{z})^2} \int_0^z dz' \int_0^{z'} dz'' (z - z')^2 \left( y^2 + z\bar{z}'\right)^2 T(z') T(z''), \quad (E.2.2) \]

\[ K_{TT}^{\text{bulk}} = -\frac{72y^2 h_L}{c^2(y^2 + z\bar{z})^2} \int_0^z dz' (z - z')^2 \int_0^z d\bar{z}' (\bar{z} - \bar{z}')^2 T(z') \bar{T}(\bar{z}'). \]

\[ K_{TT}^{\text{bulk}} \text{ is anti-holomorphic version of } K_{TT}^{\text{bulk}}, \text{ that is, } K_{TT}^{\text{bulk}} \text{ with } T \rightarrow \bar{T} \text{ and } z \leftrightarrow \bar{z}, \text{ and similarly } K_{TT}^{\text{bulk}} \text{ is the anti-holomorphic version of } K_{TT}^{\text{bulk}}. \]

The OPE block of \( \mathcal{O}_H \mathcal{O}_H \) factorizes and we have

\[ \mathcal{O}_H (\infty) \mathcal{O}_H (1) = e^{K_{TT}^{\text{bdy}} + K_{TT}^{\text{bdy}} + \ldots} e^{K_{TT}^{\text{bdy}} + K_{TT}^{\text{bdy}} + \ldots} \]

with

\[ K_{TT}^{\text{bdy}} = \frac{12h_L}{c} \int_1^\infty dz' (z' - 1) T(z'), \quad (E.2.3) \]

\[ K_{TT}^{\text{bdy}} = \frac{72y^2 h_L}{c^2} \int_1^\infty dz' \int_1^z dz'' (z'' - 1)^2 T(z') T(z''). \]

and the anti-holomorphic \( K_{TT}^{\text{bdy}} \) and \( K_{TT}^{\text{bdy}} \). The superscript “bdy” means “boundary”.

To obtain the vacuum block \( \mathcal{V}_0 \), we need to compute the correlation functions of OPE blocks of \( \phi_L \mathcal{O}_L \) and \( \mathcal{O}_H \mathcal{O}_H \). The one-holomorphic-graviton-exchange contribution is

\[ \left\langle K_{TT}^{\text{bulk}} K_{TT}^{\text{bdy}} \right\rangle = \left[ \frac{12h_L}{c} \int_0^z dz' \frac{(y^2 + z'\bar{z}) (z - z')}{z\bar{z} + y^2} T(z') \frac{12h_L}{c} \int_1^\infty dz'' (z'' - 1) T(z'') \right] \]

\[ = - \frac{12h_L h_H (z(2\bar{z} + y^2) + \log(1 - z)) (y^2 - (z - 2)\bar{z}))}{(z\bar{z} + y^2)}, \quad (E.2.4) \]

and the one-anti-holomorphic-graviton-exchange contribution \( \left\langle K_{TT}^{\text{bulk}} K_{TT}^{\text{bdy}} \right\rangle \) is simplify \( \left\langle K_{TT}^{\text{bulk}} K_{TT}^{\text{bdy}} \right\rangle \) with \( z, \bar{z} \) exchanged, ie \( z \leftrightarrow \bar{z} \).

The one-graviton-exchange contribution computed above is order \( 1/c \), and two-graviton-exchanges will contribution at order \( 1/c^2 \). There are three types of two-graviton-exchanges,

\[ K_1 \equiv \left\langle \left( K_{TT}^{\text{bulk}} + K_{TT}^{\text{bulk}} \right) \right\rangle, \quad (E.2.5) \]
\( K_2 \equiv \left( \frac{K_{T_{\text{bulk}}}}{2} + K_{TT_{\text{bulk}}} \right) \left( \frac{K_{T_{\text{bdy}}}}{2} + K_{TT_{\text{bdy}}} \right), \quad (E.2.6) \)

\( K_3 \equiv \left( \frac{K_{T_{\text{bulk}}}}{2} + K_{TT_{\text{bulk}}} \right) \left( \frac{K_{T_{\text{bdy}}}}{2} + K_{TT_{\text{bdy}}} \right), \quad (E.2.7) \)

where we’ve grouped them by contributions from different types of gravitons. \( K_3 \) is simply the anti-holomorphic version of \( K_2 \), ie, \( K_2 \) with \( z, \bar{z} \) exchanged, so we will focus on \( K_1 \) and \( K_2 \).

\( K_1 \) is contribution from exchanges of one holomorphic graviton and one anti-holomorphic graviton. The first term in \( K_1 \) is

\[
\langle K_{TT} \left( K_{T_{\text{bdy}}} K_{T_{\text{bdy}}} \right) \rangle \quad (E.2.8)
\]

\[
= - \frac{10368 h_L^2 h_T^4 y^2}{c^4 (y^2 + \bar{z} z)^2} \int_0^z dz' \int_0^{\bar{z}} d\bar{z}' \int_1^{\infty} dz'' \int_1^{\infty} d\bar{z}'' (z - z')^2 (\bar{z} - \bar{z}')^2 (z'' - 1)(\bar{z}'' - 1)
\]

\[
\times \left( \langle T(z') \bar{T}(\bar{z}') T(z'') \bar{T}(\bar{z}'') \rangle \right)
\]

\[
= - \frac{72 h_L^2 h_T^2 y^2}{c^2 (y^2 + \bar{z} z)^2} ((z - 2)z + 2(z - 1) \log(1 - z)) ((\bar{z} - 2)\bar{z} + 2(\bar{z} - 1) \log(1 - \bar{z})) \]

where we’ve used \( \langle T(z') \bar{T}(\bar{z}') T(z'') \bar{T}(\bar{z}'') \rangle = \frac{c^2}{4} \frac{1}{(z'' - z')^4} \frac{1}{(z'' - \bar{z}')^4} \). The second term in \( K_1 \) is

\[
\langle K_T K_{\bar{T}} \left( K_{T_{\text{bdy}}} K_{\bar{T}_{\text{bdy}}} \right) \rangle \quad (E.2.9)
\]

\[
= \frac{20736 h_L^2 h_T^2}{c^4 (y^2 + \bar{z} z)^2} \int_0^z dz' \int_0^{\bar{z}} d\bar{z}' \int_1^{\infty} dz'' \int_1^{\infty} d\bar{z}'' \left( y^2 + z'\bar{z} \right) \left( y^2 + \bar{z}'z \right) (z - z') \left( y^2 + \bar{z}'\bar{z} \right) (\bar{z} - \bar{z}')
\]

\[
\times (z'' - 1) (\bar{z}'' - 1) \left( \langle T(z') \bar{T}(\bar{z}') T(z'') \bar{T}(\bar{z}'') \rangle \right)
\]

\[
= \frac{144 h_L^2 h_T^2 (\bar{z} (y^2 + 2z) + (y^2 - z (\bar{z} - 2)) \log(1 - \bar{z})) (z (2\bar{z} + y^2) + (y^2 - (z - 2)\bar{z}) \log(1 - z))}{c^2 (y^2 + \bar{z} z)^2}.
\]

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And the sum of these two terms gives us

\[ \mathcal{K}_1 = \frac{72h^2 h_L}{c^2 (y^2 + z \bar{z})^2} \left[ y^2 ((2 - z)z + 2(1 - z) \log(1 - z)) ((\bar{z} - 2) \bar{z} + 2 (\bar{z} - 1) \log (1 - \bar{z})) + 2h_L \left( z (2\bar{z} + y^2) + \log(1 - z) \left( y^2 - (z - 2)\bar{z} \right) \right) \left( \bar{z} \left( y^2 + 2\bar{z} \right) + \left( y^2 + 2z - z\bar{z} \right) \log (1 - \bar{z}) \right) \right]. \]

Similarly, \( \mathcal{K}_2 \) is given by

\[ \mathcal{K}_2 = \frac{72h_H h_L}{c^4 (y^2 + z \bar{z})^2} \int_0^\infty dz' \int_0^{z'} dz'' \left[ 2h_L \left( y^2 + z'\bar{z} \right) (z - z') \left( y^2 + z''\bar{z} \right) (z - z'') + (z - z')^2 \left( y^2 + z''\bar{z} \right)^2 \right] \times \int_1^\infty dz''' \int_1^{z'''} dz'''' \left[ 2h_H \left( z''' - 1 \right) \left( z'''' - 1 \right) + (z''' - 1)^2 \right] \left[ \langle T(z') T(z'') \rangle \left[ T(z''') T(z''''') \right] \right] \]

\[ = \frac{72h^2 h_L}{c^2 (y^2 + z \bar{z})^2} \left[ \log^2(1 - z) \left( h_H \left( h_L (y^2 - (z - 2)\bar{z})^2 - 2(z - 1)\bar{z} (\bar{z} + y^2) \right) + (1 - z)\bar{z} \left( 2h_L (\bar{z} + y^2) - z\bar{z} + \bar{z} \right) \right) - 2L \log(1 - z) \left( 2y^2 \bar{z} - (z - 2)\bar{z}^2 + y^4 \right) + \frac{1}{2} (2y^2 \bar{z} (6h_H (z (4h_L - 1) - 6) + 6z + 34) + 16z^2 (3h_H (h_L - 1) - 3h_L + 2) + g^2 (2h_L (z^2 (6h_H + 1) + 6z - 24) + z^2 (6h_H + 1) + z (10 - 30h_H) + 24)) \]

\[ + \frac{1}{4} \left( \log(1 - z) \left( -2y^2 \bar{z} \left( 6h_H \left( z^2 h_L - 4h_L z + 2z + 3 \right) - 12z h_L + 18h_H + 6z - 5 \right) - (z - 2)^2 z^2 (6h_H (4h_L - 1) - 6h_L - 1) + y^4 \left( 6h_H \left( 2z^2 h_H + 3z - 4 \right) + 12z^2 h_H - 18z h_H - 13z + 12 \right) \right) \right]. \]

The four-point function of \( T \) in the second line is regularized as in \([3, 77]\) and it’s given by

\[ \langle [T(z') T(z'')] [T(z''') T(z''''')] \rangle = \frac{c^2}{4} \left( \frac{1}{(z' - z'')^4} \right) + \mathcal{O}(c). \]

Note that the above equation, in the first term of the first line, we’ve made use of the symmetry between \( z' \) and \( z'' \) to change the integration range from \( \int_0^1 dz' \int_0^{z'} dz'' \) to \( \frac{1}{2} \int_0^1 dz' \int_0^{z'} dz'' \), so that both terms of the first line have the same integration ranges. And similarly for the second line.

Adding up all the above contributions, we obtain the result for the bulk-boundary vacuum block \( V_0 \) up to order \( 1/c^2 \), and it’s given by

\[ V_0 = \left( \frac{y}{y^2 + z \bar{z}} \right)^{2h_L} \left[ 1 + \langle K_T^{\text{bulk}} K_T^{\text{bdy}} \rangle + \langle K_T^{\text{bdy}} K_T^{\text{bdy}} \rangle + \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + \mathcal{O}(\frac{1}{c^3}) \right]. \]

(E.2.10)

In Section 6.3.3.3 and Appendix E.1.3, we discussed the inversion symmetry, i.e. the symmetry under \( t_E \rightarrow -t_E \). Here, we would to comment that the above large \( c \) expansion of \( V_0 \) is symmetric under \( t_E \rightarrow -t_E \) at order \( 1/c \) but not \( 1/c^2 \). The order \( 1/c \) terms of the
above result is just the same as the $1/c$ terms of the semiclassical result when expanded at large $c$. But at $1/c^2$, there are quantum-correction terms in the above result that are not included in the semiclassical result, which breaks this symmetry if we use the naive semiclassical transformation (E.1.21) and expand to order $1/c^2$.

### E.3 Details of the Recursion Relation and Algorithm

In this section, we will analyze the structure of the proto-field $\phi(y, z, \bar{z})$ and bulk-boundary blocks in more details and explain why the recursion relation of Section 7.4 works.

In the main text, we’ve written the proto-field as a sum over descendant levels $N$ as follows

$$
\phi(y, z, \bar{z}) = y^{2h} \sum_{n=0}^{\infty} (-1)^n y^{2n} \lambda_n^{(h)} \mathcal{L}_{-n} \bar{\mathcal{L}}_{-n} \mathcal{O}(z, \bar{z}), \quad \lambda_n^{(h)} \equiv \frac{1}{(2h)_m m!}.
$$

(E.3.1)

with $\mathcal{L}_{-n}$ and $\bar{\mathcal{L}}_{-n}$ uniquely determined by the bulk primary condition (6.2.6) and the normalization condition (6.2.7). As shown in [3], we can solve these conditions and write the proto-field $\phi$ as a sum over quasi-primaries and their global descendants. That is, we can write $\phi$ as

$$
\phi(y, z, \bar{z}) = \sum_{n, \bar{n}} \sum_{i,j} \phi_{i,j}^{n,\bar{n}}
$$

(E.3.2)

where $\phi_{i,j}^{n,\bar{n}}$ means the contribution to $\phi$ from the $i$th level $n$ holomorphic quasi-primary and $j$th level $\bar{n}$ anti-holomorphic quasi-primary and their global descendants. And in the above sum, we sum over all quasi-primaries. Here we’ve assume that all the quasi-primaries are orthogonal. It can be shown that $\phi_{i,j}^{n,\bar{n}}$ is given by [3]

$$
\phi_{i,j}^{n,\bar{n}}(y, z, \bar{z}) \equiv y^{2h+2n} \sum_{m=0}^{\infty} \frac{(-1)^{n+m} y^{2m}}{\lambda_{n+m}^{(h)}} \left( \frac{L_{-1}^m \mathcal{L}_{-n}^{\text{quasi},i}}{[L_{-1}^m \mathcal{L}_{-n}^{\text{quasi},i} \mathcal{O}^{\bar{z}}]} \right) \left( \frac{\bar{L}_{-1}^{m-n-\bar{n}} \bar{\mathcal{L}}_{-\bar{n}}^{\text{quasi},j} \mathcal{O}^{\bar{z}}}{[\bar{L}_{-1}^{m-n-\bar{n}} \bar{\mathcal{L}}_{-\bar{n}}^{\text{quasi},j} \mathcal{O}^{\bar{z}}]} \right) \mathcal{O}(z, \bar{z}).
$$

(E.3.3)

In writing down the above equation, we’ve assumed that $n \geq \bar{n}$, but the case with $n < \bar{n}$
is similar. One interesting fact about the above equation is that the contribution to \( \phi_{i,j}^{n,\bar{n}} \)
from each descendant of \( \mathcal{O} \) is normalized by its norm with other factors independent of
the central charge \( c \). As we will see, this is a feature that also holds for the the Virasoro
projection operator. And this is one of the reason that we can use the \( c \)-recursion to compute
the bulk-boundary blocks.

Similar to \( \phi \), the holomorphic part of the proto-field \( \tilde{\phi}_{h}^{\text{holo}} \) defined in equation (6.4.10),
i.e
\[
\tilde{\phi}_{h}^{\text{holo}} (y, z, \bar{z}) = y^{2h} \sum_{n=0}^{\infty} \lambda^{(h)} y^{2n} \mathcal{L}_{-n} \mathcal{O}_{h} (z, \bar{z}).
\]  
(E.3.4)
can be written as a sum over contributions from different quasi-primaries as
\[
\tilde{\phi}_{h}^{\text{holo}} (y, z, \bar{z}) = \sum_{n=0}^{\infty} \phi_{i}^{n}
\]  
(E.3.5)
with
\[
\phi_{i}^{n} (y, z, \bar{z}) = y^{2h+2n} \left[ \frac{L_{m_{1}}^{\text{quasi},i} \mathcal{O}}{L_{-m}^{\text{quasi},i} \mathcal{O}} \right]^{2} \sum_{m_{1}=0}^{\infty} \lambda^{(h+m)} y^{2m} L_{-1}^{m_{1}} L_{-n}^{\text{quasi},i} \mathcal{O} (z, \bar{z}).
\]  
(E.3.6)

In order to make the structure of the holomorphic bulk-boundary block \( \mathcal{V}_{\text{holo}} (h_{1}, h_{2}, c) \) more
transparent, we can also write the holomorphic Virasoro projection operator in terms of
quasi-primaries and their global descendants:
\[
\mathcal{P}_{h_{1}}^{\text{holo}} = \sum_{m=0}^{\infty} \sum_{j} \sum_{m_{1}=0}^{\infty} \left[ L_{-1}^{m_{1}} L_{-m}^{\text{quasi},j} \mathcal{O}_{h_{1}} \right] \left[ L_{-1}^{m_{1}} L_{-m}^{\text{quasi},j} \mathcal{O}_{h_{1}} \right]^{2}
\]  
(E.3.7)
\[
= \sum_{m=0}^{\infty} \sum_{j} \left[ L_{-m}^{\text{quasi},j} \mathcal{O}_{h_{1}} \right]^{2} \left[ \sum_{m_{1}=0}^{\infty} \lambda^{(h_{1}+m)} \left[ L_{-1}^{m_{1}} L_{-m}^{\text{quasi},j} \mathcal{O}_{h_{1}} \right] \left[ L_{-1}^{m_{1}} L_{-m}^{\text{quasi},j} \mathcal{O}_{h_{1}} \right] \right].
\]

Plugging equation (E.3.6) and (E.3.7) into the definition of \( \mathcal{V}_{\text{holo}} (h_{1}, h_{2}, c) \), we obtain
\[
\mathcal{V}_{\text{holo}} (h_{1}, h_{2}, c) \equiv \left\langle \mathcal{O}_{H} \mathcal{O}_{H} \mathcal{P}_{h_{1}}^{\text{holo}} \mathcal{O}_{L} \tilde{\phi}_{h_{2}}^{\text{holo}} (y, 0, 0) \right\rangle
\]
\[ y^{2h_2} \sum_{m,n=0}^{\infty} \sum_{j} \lambda_{m_1}^{(h_1+m)} \left( L_{-m} \mathcal{O}_{h_1} \right)^2 \lambda_{m_2}^{(h_2+n)} y^{2m_2} \]

\[ \times \left( \mathcal{O}_m \mathcal{O}_n \right) \left( L_{-m} \mathcal{L}_{-n} \mathcal{O}_{h_2} \right) \]

The two factors in the last line can be simplified to be

\[ \langle \mathcal{O}_m \mathcal{O}_n (1) | L_{-m} \mathcal{L}_{-n} \mathcal{O}_{h_1} \rangle = (h_1 + m)_m \langle \mathcal{O}_m \mathcal{O}_n (1) | L_{-m} \mathcal{O}_{h_1} \rangle, \quad (E.3.9) \]

and

\[ \langle L_{-m} \mathcal{L}_{-n} \mathcal{O}_{h_1} | \mathcal{O}_L (z) | L_{-m} \mathcal{L}_{-n} \mathcal{O}_{h_2} (0,0) \rangle \]

\[ = s_{m_1, m_2} (h_1 + m, h_L, h_2 + n) z^{h_1 + m + h_2 - m_2} \langle \mathcal{L}_{-m} \mathcal{O}_{h_1} | \mathcal{L}_{-n} \mathcal{O}_{h_2} \rangle, \quad(E.3.10) \]

with \( s_{m_1, m_2} (h_1 + m, h_L, h_2 + n) \) given in (6.4.24).

Now we can separate the factors in \( \mathcal{V}_{\text{holo}} \) that depend on \( c \) (i.e. those terms that involve \( \mathcal{L}_{\text{quasi}} \)) from those that don’t depend on \( c \), and write \( \mathcal{V}_{\text{holo}} \) as a sum over global blocks

\[ \mathcal{V}_{\text{holo}} (h_1, h_2, c) = \sum_{m,n=0}^{\infty} C_{m,n} G (h_1 + m, h_2 + n). \quad (E.3.11) \]

with

\[ C_{m,n} = \sum_{i,j} \frac{\langle \mathcal{O}_m \mathcal{O}_n | \mathcal{L}_{-m} \mathcal{L}_{-n} \mathcal{O}_{h_1} \rangle \langle \mathcal{L}_{-m} \mathcal{O}_{h_1} | \mathcal{O}_L (1) | \mathcal{L}_{-n} \mathcal{O}_{h_2} \rangle}{\mathcal{L}_{-m} \mathcal{O}_{h_1}^2 | \mathcal{L}_{-n} \mathcal{O}_{h_2}^2} \quad (E.3.12) \]

and the global blocks are

\[ G (h_1, h_2) = z^{h_1} \left( \frac{y^2}{z} \right)^{h_2} \sum_{m_1, m_2=0}^{\infty} \frac{(h_1)_m_1 s_{m_1, m_2} (h_1, h_L, h_2)}{(2h_1)_m_1 m_1! (2h_2)_m_2 m_2!} \left( \frac{y^2}{z} \right)^{m_2}. \quad (E.3.13) \]

It’s easily seen from the above derivation that \( G (h_1 + m, h_2 + n) \) are the contributions from the global descendants of quasi-primaries of dimension \( h_1 + m \) and \( h_2 + n \). The sum over \( i,j \)
in equation (E.3.12) is summing over the level-\(m\) quasi-primaries of \(\mathcal{O}_{h_1}\) and level-\(n\) quasi-primaries of \(\mathcal{O}_{h_2}\). So \(C_{m,n}\) is the sum of the product of 3-pt functions of quasi-primaries with primaries normalized by the norms of the quasi-primaries, at specific levels.

A detail derivation of the recursion can then be obtained along the line of [148]. Basically, the function \(A_{m,n}^c\) in (6.4.19) encode the information about the norms of the states in the denominator of (E.3.12), and \(P_{m,n}^c\) in (6.4.20) encodes the 3-pt functions of one quasi-primaries with two primaries. In (E.3.12), we have a 3-pt function with 2 quasi-primaries, but at the residues of (6.4.15), one of the quasi-primaries becomes a primary, that’s why \(P_{m,n}^c\) can be used to compute this 3-pt function. The reason that the Zamolodchikov recursion relation can be modified to compute the bulk-boundary Virasoro blocks is that the the structure of the proto-field \(\phi\) is very similar to the structure of the projection operator, i.e. the proto-field is built up of descendant states of \(\mathcal{O}\) normalized by their norms.

**Algorithm for Solving the Recursion**

Solving the recursion (6.4.15) (reproduced here for convenience)

\[
\mathcal{V}_{\text{holo}}(h_1, h_2, c) = \mathcal{V}_{\text{holo}}(h_1, h_2, c \to \infty) \\
+ \sum_{m \geq 2, n \geq 1} \frac{R_{m,n}(h_1, h_2)}{c - c_{m,n}(h_1)} \mathcal{V}_{\text{holo}}(h_1 \to h_1 + mn, h_2, c \to c_{mn}(h_1)) \\
+ \sum_{m \geq 2, n \geq 1} \frac{S_{m,n}(h_1, h_2)}{c - c_{m,n}(h_2)} \mathcal{V}_{\text{holo}}(h_1, h_2 \to h_2 + mn, c \to c_{mn}(h_2)),
\]

will give us the coefficients \(C_{M,N}\) (here we use \(M, N\) instead of \(m, n\) for clarity). The basic idea of the algorithm for obtaining \(C_{M,N}\) is similar to that of the algorithm for computing \(\langle \phi \phi \rangle_{\text{holo}}\) in [4] using the \(c\)-recursion relation, as described in detail in Appendix D of that paper. Here, we briefly describe the algorithm for this more complicated recursion.

From recursion (E.3.14), we know that \(C_{M,N}\) get contribution from every decomposition
of $M, N$ in the following forms

\[ M = m_1 \tilde{m}_1 + \ldots + m_k \tilde{m}_k \ldots + m_\ell \tilde{m}_\ell, \quad N = n_1 \tilde{n}_1 + \ldots + n_l \tilde{n}_l \ldots + n_j \tilde{n}_j \]  

(E.3.15)

where $m_k, \tilde{m}_k, n_l, \tilde{n}_l$ are integers with $m_k, n_l \geq 2$ and $\tilde{m}_k, \tilde{n}_l \geq 1$ and the orders of the products in the sums matter. We can imaging obtaining $(M, N)$ from $(0, 0)$ step by step, where at each step, we either choose $m_k \tilde{m}_k$ or $n_l \tilde{n}_l$. Different ways of arriving at $(M, N)$ give different contributions to $C_{M, N}$. Denoting the contribution to $C_{M, N}$ whose last step is $m_i \tilde{m}_i$ as $C_{(M, m_i, \tilde{m}_i), (N, n_j, \tilde{n}_j), 1}$ and the contribution to $C_{M, N}$ whose last step is $n_j \tilde{n}_j$ as $C_{(M, m_i, \tilde{m}_i), (N, n_j, \tilde{n}_j), 2}$. Then we have

\[ C_{M, N} = \sum_{2 \leq m_i, \tilde{m}_i \leq M} \sum_{2 \leq n_j, \tilde{n}_j \leq N} \left[ C_{(M, m_i, \tilde{m}_i), (N, n_j, \tilde{n}_j), 1} + C_{(M, m_i, \tilde{m}_i), (N, n_j, \tilde{n}_j), 2} \right]. \]  

(E.3.16)

$C_{(M, m_i, \tilde{m}_i), (N, n_j, \tilde{n}_j), 1}$ and $C_{(M, m_i, \tilde{m}_i), (N, n_j, \tilde{n}_j), 2}$ are computed as follows (from special and simple to more general cases):

1. The simplest case is

\[ C_{(m_1 \tilde{m}_1, m_1, \tilde{m}_1), (0, 0, 0), 1} = \frac{R_{m_1, \tilde{m}_1} (h_1, h_2)}{c - c_{m_1, \tilde{m}_1} (h_1)} \]  

(E.3.17)

\[ C_{(0, 0, 0), (n_1 \tilde{n}_1, n_1, \tilde{n}_1), 2} = \frac{S_{n_1, \tilde{n}_1} (h_1, h_2)}{c - c_{n_1, \tilde{n}_1} (h_2)} \]

where the recursion is only used once.

2. For the case with $N = 0$ and $M - m_i \tilde{m}_i \geq 2$, we have

\[ C_{(M, m_i, \tilde{m}_i), (0, 0, 0), 1} = \sum_{2 \leq m_k \tilde{m}_k \leq M - m_i \tilde{m}_i} \frac{R_{m_i, \tilde{m}_i} (h_1 + M - m_i \tilde{m}_i, h_2) C_{(M, m_i, m_k \tilde{m}_k), (0, 0, 0), 1}}{c_{m_i, \tilde{m}_i} (h_1 + M - m_i \tilde{m}_i - m_k \tilde{m}_k) - c_{m_i, \tilde{m}_i} (h_1 + M - m_i \tilde{m}_i)} \]

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Similarly for $M = 0$ and $N - n_j \tilde{n}_j \geq 2$, we have

\[ C_{(0,0),(N,n_j,\tilde{n}_j),2} = \sum_{2 \leq n_k \tilde{n}_k \leq N - n_j \tilde{n}_j} \frac{S_{n_j,\tilde{n}_j} (h_1, h_2 + N - n_j \tilde{n}_j) C_{(0,0),(N-n_j,\tilde{n}_j),2}}{c_{n_k,\tilde{n}_k} (h_2 + N - n_j \tilde{n}_j - n_k \tilde{n}_k) - c_{n_j,\tilde{n}_j} (h_2 + N - n_j \tilde{n}_j)} \]

3. For terms with $M - m_i \tilde{m}_i \geq 2$ and $N = n_1 \tilde{n}_1$, we have

\[ C_{(M,m_i,\tilde{m}_i),(1,n_1,1,n_1,1),1} \]

\[ = \sum_{2 \leq m_k \tilde{m}_k \leq M - m_i \tilde{m}_i} R_{m_i,\tilde{m}_i} (h_1 + M - m_i \tilde{m}_i, h_2 + n_1 \tilde{n}_1) \times \left[ \frac{C_{(M-m_i \tilde{m}_i,m_k,\tilde{m}_k),(1,n_1,1,n_1,1),1}}{c_{m_k,\tilde{m}_k} (h_1 + M - m_i \tilde{m}_i - m_k \tilde{m}_k) - c_{m_i,\tilde{m}_i} (h_1 + M - m_i \tilde{m}_i)} + \frac{C_{(M,m_i,\tilde{m}_i,m_k,\tilde{m}_k),(1,n_1,1,n_1,1),2}}{c_{n_1,\tilde{n}_1} (h_2) - c_{m_i,\tilde{m}_i} (h_1 + M - m_i \tilde{m}_i)} \right] \]

and

\[ C_{(M,m_i,\tilde{m}_i),(1,n_1,1,n_1,1),2} = \frac{S_{n_1,\tilde{n}_1} (h_1 + M, h_2) C_{(M,m_i,\tilde{m}_i),(0,0,0),1}}{c_{m_i,\tilde{m}_i} (h_1 + M - m_i \tilde{m}_i) - c_{n_1,\tilde{n}_1} (h_2)}. \]

And similarly for the case with $N - n_j \tilde{n}_j \geq 2$ and $M = m_1 \tilde{m}_1$.

4. For the general case with $m - m_i \tilde{m}_i \geq 2$ and $n - n_j \tilde{n}_j \geq 2$, we have

\[ C_{(M,m_i,\tilde{m}_i),(N,n_j,\tilde{n}_j),1} \]

\[ = \sum_{2 \leq m_k \tilde{m}_k \leq M - m_i \tilde{m}_i} R_{m_i,\tilde{m}_i} (h_1 + M - m_i \tilde{m}_i, h_2 + N) \times \left[ \frac{C_{(M-m_i \tilde{m}_i,m_k,\tilde{m}_k),(N,n_j,\tilde{n}_j),1}}{c_{m_k,\tilde{m}_k} (h_1 + M - m_i \tilde{m}_i - m_k \tilde{m}_k) - c_{m_i,\tilde{m}_i} (h_1 + M - m_i \tilde{m}_i)} + \frac{C_{(M,m_i,\tilde{m}_i,m_k,\tilde{m}_k),(N,n_j,\tilde{n}_j),2}}{c_{n_j,\tilde{n}_j} (h_2 + N - n_j \tilde{n}_j) - c_{m_i,\tilde{m}_i} (h_1 + M - m_i \tilde{m}_i)} \right] \]

and

\[ C_{(M,m_i,\tilde{m}_i),(N,n_j,\tilde{n}_j),2} \]

\[ = \sum_{2 \leq n_k \tilde{n}_k \leq N - n_j \tilde{n}_j} S_{n_j,\tilde{n}_j} (h_1 + M, h_2 + N - n_j \tilde{n}_j) \times \left[ \frac{C_{(M,m_i,\tilde{m}_i,m_k,\tilde{m}_k),(N,n_j,\tilde{n}_j),2}}{c_{n_j,\tilde{n}_j} (h_2 + N - n_j \tilde{n}_j) - c_{m_i,\tilde{m}_i} (h_1 + M - m_i \tilde{m}_i)} \right] \]
Using the above equations, we can compute \( C_{(M,m_i,\tilde{m}_i),(N,n_j,\tilde{n}_j)}^{1,1} \) and \( C_{(M,m_i,\tilde{m}_i),(N,n_j,\tilde{n}_j)}^{1,2} \) from small \((M,N)\) to larger \((M,N)\) up to the order we want for \( C_{M,N} \). The Mathematica code for this algorithm is attached with this paper.

### E.4 Multi-Trace Contributions and Bulk Fields

In this appendix, we will discuss some of the differences between correlators of the proto-field \( \phi \) and a full bulk field \( \varphi \) that can be seen within perturbation theory in a low-energy EFT description. In particular, consider a bulk theory with only \( \varphi \) and gravity as low-energy fields:

\[
S = \int d^3x \sqrt{g} \left( M_p R + \frac{1}{2} (\nabla \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - 2\Lambda \right). \tag{E.4.1}
\]

The bulk field \( \varphi \) will contain contributions from multi-trace \( \mathcal{O}^n \) operators in the CFT due to gravitational interactions, even at tree-level. This occurs because \( \varphi \) is dressed by the bulk gravitational field \( h_{\mu\nu} \), which in turn picks up contributions from multi-trace operators at the boundary. It is easier to see this effect on \( h_{\mu\nu} \), which is what we will calculate in this section.

Fortunately, the main content of the necessary computations were done in [189]. In the presence of two boundary scalar operators \( \mathcal{O}(x_1) \) and \( \mathcal{O}(x_3) \), the bulk field \( h_{\mu\nu} \) is given by

\[
h_{\mu\nu} = |x_{13}|^{-2\Lambda} \frac{1}{(w^2)^2} J_{\mu\lambda}(w) J_{\nu\rho}(w) I_{\lambda\rho}(w' - x'_{13}),
\]

\[
I_{\mu\nu}(w' - x'_{13}) = f(t) \left( \frac{1}{1 - d g_{\mu\nu}} + \frac{\delta_{\mu\nu}}{w'^2} \right) + \ldots, \tag{E.4.2}
\]

where \( f(t) \) is the solution to a differential equation to be presented below. The \ldots here
are pure diffeomorphism terms, which depend on the choice of gauge; when $h_{\mu\nu}$ is part of an internal graviton line in a bulk correlator, e.g. $\langle \varphi \mathcal{O} \mathcal{O} \mathcal{O} \rangle$, these gauge-dependent contributions vanish, and we will neglect them. The notation of the above equation is that $w = (w_0, \vec{w})$ is the bulk position of $h_{\mu\nu}$, where $w_0$ is the radial Poincaré patch direction. Translation invariance has been used to set the operators at $\mathcal{O}(0)$ and $\mathcal{O}(x_{13})$. The primes denote inversion,

$$w' = \frac{w}{w^2} = \frac{w}{w_0^2 + \vec{w}^2}, \quad x'_{13} = \frac{x_{13}}{x_{13}^2}, \quad (E.4.3)$$

and $g'_{\mu\nu} \equiv \frac{\delta_{\mu\nu}}{w_0^2}$. The argument $t$ is

$$t = \frac{w_0^2}{w_0^2 + (\vec{w} - \vec{x}_{13})^2}. \quad (E.4.4)$$

The polarization vectors are

$$J_{\mu\nu}(w) = \delta_{\mu\nu} - 2 \frac{w_\mu w_\nu}{w^2}. \quad (E.4.5)$$

The boundary operator content of $h_{\mu\nu}$ can be read off by taking the limit $x_{13} \to 0$. Using the OPE, we have

$$\langle h_{\mu\nu} \mathcal{O}(x_1) \mathcal{O}(x_3) \rangle \overset{x_1, x_3 \to 0}{\sim} \langle h_{\mu\nu} \rangle x_{13}^{-2\Delta} + \langle h_{\mu\nu} \mathcal{O}^2 \rangle + \ldots \quad (E.4.6)$$

In the limit $x_1, x_3 \to 0$, we have $x'_{13} \to \infty$, so

$$t \to \frac{w_0^2}{x_{13}^2} = \frac{x_{13}^2 w_0^2}{(w^2)^2} \sim 0 \quad (E.4.7)$$

. The differential equation determining the function $f(t)$ is

$$4t(1-t)f'(t) - 2(d-2)f(t) = 2\Delta t^\Delta, \quad (E.4.8)$$
and its solution in $d = 2$ is

$$f(t) = c_1 + \frac{t^\Delta(\Delta + \Delta t_2 F_1(1, \Delta + 1; \Delta + 2; t) + 1)}{2(\Delta + 1)}$$  \hfill (E.4.9)

where $c_1$ is fixed by an appropriate boundary condition. We are interested in the limit $t \to 0$, where

$$f(t) \sim c_1 + \frac{t^\Delta}{2} + \mathcal{O}(t^{\Delta+1})$$  \hfill (E.4.10)

Therefore,

$$I_{\mu\nu} \approx \left(c_1 + \frac{1}{2} \left(\frac{x_{13}^{2\Delta} w_0^{-2\Delta}}{(w^2)^{2\Delta}}\right)\right) \left(-\frac{\delta_{\mu\nu}}{w_0^2} + \frac{\delta_{0\mu}\delta_{0\nu}}{w_0^2}\right)$$  \hfill (E.4.11)

and

$$\langle h_{\mu\nu}\mathcal{O}(x_1)\mathcal{O}(x_2) \rangle \approx |x_{13}|^{-2\Delta} \frac{1}{(w^2)^2} J_{\mu\lambda}(w) J_{\nu\rho}(w) \left(c_1 + \frac{1}{2} \left(\frac{x_{13}^{2\Delta} w_0^{-2\Delta}}{(w^2)^{2\Delta}}\right)\right) \left(-\frac{\delta_{\lambda\rho}}{w_0^2} + \frac{\delta_{0\lambda}\delta_{0\rho}}{w_0^2}\right)$$  \hfill (E.4.12)

Comparing to (E.4.6), we read off from the leading term at small $x_{13}$ that $c_1$ is the vacuum expectation value of the stress tensor, and so should be taken to vanish. From the subleading term, we obtain the double-trace $\mathcal{O}^2$ content of the bulk gravitational field:

$$\langle h_{\mu\nu}\mathcal{O}^2(0) \rangle \approx \left(\frac{w_0}{(w^2)}\right)^{2\Delta} \left(-\frac{\delta_{\mu\nu} + J_{\mu0}J_{\nu0}}{w_0^2}\right).$$  \hfill (E.4.13)

Near the boundary, $w_0 \to 0$, this contribution vanishes,\(^2\) as it must since the bulk field $w_0^{-2}h_{\mu\nu}$ becomes the boundary stress tensor in this limit. However, it is clearly nonzero at $w_0 > 0$, and this effect implies that each bulk graviton that dresses a bulk field $\varphi$ brings (at least) two boundary $\mathcal{O}$s along with it. For instance, in the tree-level diagram with one-graviton exchange for $\langle \varphi\mathcal{O}\mathcal{O}\mathcal{O} \rangle$, this effect produces a contribution from $\mathcal{O}^3$ to the bulk field.

\(^2\)For $\Delta < 2$, the computation should be modified to use an alternate boundary condition for $\varphi$.  

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E.5 Non-Holomorphic Bulk Monodromy Method

In this appendix, we describe how to apply the monodromy method to the full bulk block for $\langle \phi_L \mathcal{O}_L \mathcal{O}_H \mathcal{O}_H \rangle$. The analysis is complicated by the fact that both $T$ and $\bar{T}$ get contributions from the heavy background, and so both the holomorphic and the anti-holomorphic Schrodinger equations for $\psi, \bar{\psi}$ are difficult to solve and must be solved simultaneously. We will again work only to first order in $h_L/c$, and show how to solve the monodromy equation order-by-order in a small $y$ expansion. It would be much preferable to have a method to solve directly at any $y$. However, we will see that it is already somewhat nontrivial that the monodromy method contains enough information to solve for the bulk block, so the fact that it can be solved order-by-order in $y$ is a useful proof of principle.

The potential $T$ for the Schrodinger equation is again derived using the singular terms of the $T \times \mathcal{O}$ and the $T \times \phi$ OPEs, for the latter see (6.3.12). Since we are expanding in $h_L/c$, we divide the potential $T(z)$ of the Schrodinger equation for $\psi$ into a “heavy” piece and a “light” piece:

$$T(z) = T_H(z) + T_L(z),$$

(E.5.1)

where the heavy piece is just $T_H(z) = \frac{h_L}{z^2}$, the stress tensor in the heavy state background. In the $T \times \phi$ OPE, at leading order in $h_L/c$, only $T_H$ contributes on the RHS of (6.3.12), so ambiguities related to the singularities of $T_L(z)$ at the location of the light operators do not arise at this order. We use conformal invariance again by demanding that $T(z)$ and $\bar{T}(\bar{z})$ decay like $z^{-4}, \bar{z}^{-4}$ at large $z, \bar{z}$. In this section, it will be more convenient to work with the configuration

$$z_1 = \infty, \quad z_2 = 0, \quad z_4 = 1.$$  

(E.5.2)
After making these simplifications and performing some straightforward but tedious manipulations, the light piece \( T_L(z) \) is

\[
T_L(z) = \frac{h_L}{z(1-z)^2} + \frac{1}{2} c_y y_3 \left( \frac{1}{z(1-z)} + \frac{1}{(z-z_3)^2} \right) + \frac{c_{z_3}(1-z_3)z_3}{(1-z)z(z-z_3)} - \frac{c_{z_3} y_3^2}{(z-z_3)^3}
\]

\[
- \frac{y_3^4 (c_{z_3} + c_{\bar{z}_3}y_3^2) \frac{T_H(z_3)}{\epsilon} \frac{6T_H(\bar{z}_3)}{\epsilon}}{(z-z_3)^2(1-y_3^4 \frac{36}{\epsilon^2} T_H(z_3) T_H(\bar{z}_3))} + \ldots,
\]

(E.5.3)

where \( \ldots \) are higher order in \( h_L/c \), coming from the evaluation of \( T(z_3), \bar{T}(\bar{z}_3) \) inside the \( T \times \phi \) OPE. As before, \( c_X \equiv \partial_X g \). At zeroth order in \( h_L/c \), only \( T_H \) contributes to the Schrodinger euqation, and the solutions for \( \psi \) are

\[
\psi^{(1)}(z) = z^{\frac{1-\alpha_H}{2}}, \quad \psi^{(2)}(z) = z^{\frac{1+\alpha_H}{2}}.
\]

(E.5.4)

At next order, we apply the method of separation of variables, which ultimately gives the monodromy matrix \( M \) as the residues of a matrix \( m_{ij} \)

\[
m_{ij} = \frac{T_L(z)}{\psi^{(2)}(z)\psi^{(1)}(z) - \psi^{(2)}(z)\psi^{(1)}(z)} \psi^{(i)}(z) \bar{\psi}^{(j)}(z),
\]

(E.5.5)

in terms of which \( M \) is just

\[
M_{ij} = 2\pi i (\text{res}_{z \to 1} m_{ij} + \text{res}_{z \to z_3} m_{ij}).
\]

(E.5.6)

The diagonal components vanish, and the off-diagonal components are

\[
M_{12} = -i\pi z_3^{-\alpha_H-1}(y_3^2 (\alpha_H-1)\alpha_H c_{z_3}+ y_3 z_3 c_{y_3} (\alpha_H + z_3^{\alpha_H-1} + 2 z_3^2 c_{z_3} (z_3^{\alpha_H-1} + 2 h_L \alpha_H z_3^{\alpha_H+1})) \]

\[
M_{21} = M_{12}(\alpha_H \to -\alpha_H)
\]

(E.5.7)

The eigenvalues vanishing requires \( M_{12} M_{21} = 0 \), i.e. either \( M_{12} \) or \( M_{21} \) must vanish.
To solve for the “action” $g$ order-by-order in $y_3$, we take

$$
\frac{c}{6} g = 2 h_L \log(y_3) + h_L \sum_{n=0}^{\infty} y_3^{2n} g_{2n}(z_3, \bar{z}_3)
$$

(E.5.8)

If we demand that $M_{12}$ and $\bar{M}_{12}$ vanish, we find two differential equations for $g_0$:

$$
g^{(0,1)}_0(z_3, \bar{z}_3) = -\frac{\alpha_H z_3^{\alpha_H} - \bar{z}_3^{\alpha_H} - \alpha_H + 1}{z_3 \left(\bar{z}_3^{\alpha_H} - 1\right)}
$$

$$
g^{(1,0)}_0(z_3, \bar{z}_3) = -\frac{\alpha_H - \alpha_H z_3^{\alpha_H} - z_3^{\alpha_H} + 1}{z_3 \left(\bar{z}_3^{\alpha_H} - 1\right)}
$$

(E.5.9)

These are solved by

$$
g_0(z_3, \bar{z}_3) = 2 \log \left(\frac{\alpha_H}{(z_3^{\alpha_H} - 1) (\bar{z}_3^{\alpha_H} - 1)}\right) + (\alpha_H - 1) \log (z_3 \bar{z}_3)
$$

(E.5.10)

which just reproduces the boundary block in this large $c$, small $h_L/c$ limit.

Next, we solve for $g_2$. At this order, the equations we find for $g_2$ reduce to

$$
g^{(1,0)}_2(z_3, \bar{z}_3) = \frac{(\alpha_H - 1) \alpha_H ((\alpha_H + 1) z_3^{\alpha_H} + \alpha_H - 1) - 2 z_3 \bar{z}_3 (\alpha_H + z_3^{\alpha_H} - 1) g_2(z_3, \bar{z}_3) (\bar{z}_3^{\alpha_H} - 1)}{2 z_3 \bar{z}_3 (z_3^{\alpha_H} - 1) (\bar{z}_3^{\alpha_H} - 1)}
$$

$$
g^{(0,1)}_2(z_3, \bar{z}_3) = \frac{(\alpha_H - 1) \alpha_H (\alpha_H + (\alpha_H + 1) z_3^{\alpha_H} - 1) - 2 z_3 \bar{z}_3 (z_3^{\alpha_H} - 1) g_2(z_3, \bar{z}_3) (\bar{z}_3^{\alpha_H} + \alpha_H - 1)}{2 z_3 \bar{z}_3 (z_3^{\alpha_H} - 1) (\bar{z}_3^{\alpha_H} - 1)}
$$

(E.5.11)

Each of these equations can be viewed as an ordinary differential equation, with $z_3$ or $\bar{z}_3$ treated as a constant. Solving these ODEs, one therefore obtains two equations for $g_2$, one with an integration function of $z_3$, one with an integration function of $\bar{z}_3$:

$$
g_2(z_3, \bar{z}_3) = \frac{-2 z_3 c_1(\bar{z}_3) z_3^{\alpha_H} (\bar{z}_3^{\alpha_H} - 1) + (\alpha_H - 1) ((\alpha_H + 1) \bar{z}_3^{\alpha_H} + \alpha_H - 1)}{2 z_3 \bar{z}_3 (z_3^{\alpha_H} - 1) (\bar{z}_3^{\alpha_H} - 1)}
$$

$$
g_2(z_3, \bar{z}_3) = \frac{-2 z_3 c_1(z_3) (z_3^{\alpha_H} - 1) \bar{z}_3^{\alpha_H} + (\alpha_H - 1) (\alpha_H + (\alpha_H + 1) \bar{z}_3^{\alpha_H} - 1)}{2 z_3 \bar{z}_3 (z_3^{\alpha_H} - 1) (\bar{z}_3^{\alpha_H} - 1)}
$$
We can solve for \( c_1(\bar{z}_3) \) in terms of \( c_1(z_3) \):

\[
c_1(\bar{z}_3) = \frac{z_3^{-\alpha_H} \left( 2c_1(z_3) z_3^{\alpha_H+1}\bar{z}_3^{\alpha_H} - 2z_3 c_1(z_3) \bar{z}_3^{\alpha_H} - \alpha_H^2 \bar{z}_3^{\alpha_H} + \alpha_H^2 \bar{z}_3^{\alpha_H} - z_3^{\alpha_H} \right)}{2\bar{z}_3 (\bar{z}_3^{\alpha_H} - 1)}
\]

(E.5.13)

Since the RHS cannot depend on \( z_3 \), we can take \( z_3 \) to be any value we want. Naively, we can just take \( z_3 = 1 \), but this is too fast since this causes \( c_1(z_3) \) to be multiplied by \( z_3^{\alpha_H+1} - 1 \to 0 \), and in the correct answer \( c_1 \) has a singularity at \( z_3 = 1 \). In the correct answer, the singularity cancels the zero, but to extract the correct answer will define a new function as

\[
c_1(t) \equiv \frac{b_1(t)}{t(t^{\alpha_H} - 1)}
\]

(E.5.14)

Now, \( b_1(t) \) is regular at \( t = 1 \). The above equation for \( c_1(\bar{z}_3) \) in terms of \( c_1(z_3) \) becomes

\[
b_1(\bar{z}_3) = \frac{1}{2} z_3^{-\alpha_H} \left( 2b_1(z_3) \bar{z}_3^{\alpha_H} + \left( \alpha_H^2 - 1 \right) (z_3^{\alpha_H} - \bar{z}_3^{\alpha_H}) \right)
\]

(E.5.15)

Setting \( z_3 = 1 \), we obtain an equation for \( b_1(\bar{z}_3) \) in terms of \( b_1(1) \). The solution is

\[
b_1(t) = \frac{\beta_1 t^{\alpha_H} + \alpha_H^2 - 1}{2}
\]

(E.5.16)

where \( \beta_1 \) is an undetermined integration constant.

We fix \( \beta_1 \) by substituting back into \( g_2(z_3, \bar{z}_3) \) and demanding that the result be a holomorphic times antiholomorphic function. We compute

\[
\partial_{z_3} \partial_{\bar{z}_3} \log g_2(z_3, \bar{z}_3) = \frac{(\alpha_H - 1)^2 \alpha_H^2 ((\alpha_H + 1)^2 - \beta_1) z_3^{\alpha_H-1}\bar{z}_3^{\alpha_H-1}}{(z_3^{\alpha_H} (\beta_1 z_3^{\alpha_H} + \alpha_H^2 - 1) - (\alpha_H - 1) ((\alpha_H + 1) z_3^{\alpha_H} + \alpha_H - 1))^2}
\]
and therefore

\[ \beta_1 = (1 + \alpha_H)^2 \] (E.5.18)

This procedure can be continued recursively to any order in \( y \), and we have explicitly checked that it works up to and including \( g_4 \).
Appendix F

Appendix to Chapter 7

F.1 Bulk Primary Conditions as Mirage Translations

Bulk operators must leave an imprint in boundary correlators representing their conserved charges and energies. This imprint manifests as singularities in correlators involving conserved currents $J(z)$ (see section 7.2.3) or the stress tensor $T(z)$. In this section we develop some formalism for displacing the singularities associated with local charge or energy in a CFT$_2$. This will allow us to alter the ‘gravitational dressing’ of bulk operators. We will also identify an alternative explanation for the bulk primary condition [3]. In appendix F.1.2 we provide a review of the singularity structure of $T(z)$ correlators with CFT primary operators as derived from the bulk.

F.1.1 Mirage Translations

In a translation-invariant theory such as a CFT, we use the momentum generator $P_\mu$ to move local operators around. In a CFT$_2$ this means that

$$\mathcal{O}(z, \bar{z}) = e^{zL_{-1} + \bar{z}\bar{L}_{-1}} \mathcal{O}(0)e^{-zL_{-1} - \bar{z}\bar{L}_{-1}}$$

(F.1.1)
since $L_{-1}, \bar{L}_{-1}$ are the holomorphic and anti-holomorphic momentum generators.

Now let us assume that the CFT has a holomorphic $U(1)$ current $J(z)$. Correlators with the current such as

$$\langle J(z_1) \mathcal{O}^\dagger(z, \bar{z}) \mathcal{O}(0) \rangle = q \frac{z}{(z - z_1)z_1} \frac{1}{z^{2h}\bar{z}^{2\bar{h}}} \quad (F.1.2)$$

have singularities in $z_1$ when $J$ collides with charged operators, which indicate the presence of charge localized at 0 and $z$. We will pose the following question: *can we find an operator that moves local charge without moving the associated primary operators?* Or equivalently, can we move the primary operators while leaving its charge in place?

We can sharpen these questions into precise criteria for correlators. We would like to find a modified translation operator $G_{h,q}(z_f)$ that can appear in correlators as\(^1\)

$$\langle \mathcal{O}^\dagger(z) J(z_k) \cdots J(z_1) [G_{h,q}(z_f) \mathcal{O}(0)] \rangle \quad (F.1.3)$$

We wish to choose $G_{h,q}$ so that $\mathcal{O}^\dagger(z, \bar{z})$ only has an OPE singularity with $[G_{h,q}(z_f) \mathcal{O}(0)]$ when $z - z_f$ vanish, but the currents $J(z_j)$ have OPE singularities with $[G_{h,q}(z_f) \mathcal{O}(0)]$ when $z_j \to 0$. So the non-local object $[G_{h,q} \mathcal{O}]$ behaves like a mirage, present at both 0 and $z_f$.

Conventional translation operators automatically satisfy the first condition. They also satisfy the second condition when the charge of $\mathcal{O}$ is $q = 0$, suggesting that $G_{h,0}(z_f, 0) = e^{z_f L_{-1}}$. So let us modify the translation generator and define

$$G_{h,q}(z_f) = \sum_{n=0}^{\infty} z_f^n \frac{J_{-n}}{n!} \quad (F.1.4)$$

where we have $J_{-n} = L_{-1}^n + O(q)$, so that $J_{-n}$ implicitly depends on $h$ and $q$. Our criterion

\(^1\)We are implicitly assuming we can separate $z_i$ and $w_j$ from 0 and $z_f$ and perform radial quantization about the non-local object $[G_{h,q} \mathcal{O}]$. 

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require an OPE

\[ J(z_1) \left[ G_{h,q}(z_f) \mathcal{O}(0) \right] = \frac{q}{z_1} [G_{h,q}(z_f,0)\mathcal{O}(0)] + \cdots \]  \hspace{1cm} (F.1.5)

where the ellipsis denotes finite terms, so that we are demanding that the only singularity is a simple pole at \( z_1 = 0 \). This condition will automatically be satisfied if

\[ [J_m, \mathcal{J}_{-n}\mathcal{O}(0)] = 0 \quad \text{for all} \quad m \geq 1 \]  \hspace{1cm} (F.1.6)

and for any \( n \). These conditions uniquely determine \( \mathcal{J}_{-n} \) up to an overall factor. These overall factors will be fixed as in equation (F.1.4) by the requirement that \( G_{h,q}(z_f) \) acts on \( \mathcal{O} \) as a conventional translation in its two-point function with \( \mathcal{O}^\dagger \).

We can repeat this exercise and replace charge with energy-momentum, and \( J(z) \) with the CFT\(_2\) energy-momentum tensor \( T(z) \). In that case, we could write

\[ G_h(z_f) = \sum_{n=0}^{\infty} \frac{z_f^n}{n!} \mathcal{L}_{-n} \]  \hspace{1cm} (F.1.7)

where \( \mathcal{L}_{-n} \) implicitly depends on the dimension \( h \) of the primary \( \mathcal{O} \) to which we are applying \( G_h \).

We must have \( \mathcal{L}_{-1} = L_{-1} \) simply because there are no other level-one combinations of Virasoro generators. This means that the OPE \( T(z_1) \left[ G_h(z_f)\mathcal{O}(0) \right] \) necessarily contains a third-order pole \( \frac{1}{z_1^3} \), but it needn’t have any higher order singularities. The absence of any further singularities at \( z_1 \to 0 \) or anywhere else in the complex plane implies that the \( \mathcal{L}_{-n} \) must satisfy the bulk primary conditions

\[ [L_m, \mathcal{L}_{-n}\mathcal{O}(0)] = 0 \quad \text{for all} \quad m \geq 2, \]  \hspace{1cm} (F.1.8)

which were previously discovered in the context of bulk reconstruction. Here we see them
appearing in the answer to a question concerning the CFT alone.

F.1.2 Singularity Structure of $\langle T(z) \rangle$ from Einstein’s Equations

In this section, we generalize the discussion of section 7.2.3 to gravity, showing how Einstein’s equations in the presence of a massive source on the boundary dictate the singularity structure of correlators with the CFT stress tensor. This is elementary, as in essence it amounts to Gauss’s law for AdS$_3$ gravity [192]. But we review the argument to emphasize the connection between gravitational dressing and singularities.

We wish to establish that the OPE of a scalar primary with the CFT stress tensor has $\frac{1}{z^2}$ singularity by using the bulk equations of motion. A scalar primary inserted at the origin will create a scalar particle propagating in the bulk. We assume that the particle is sufficiently heavy to model its wavefunction with a worldline.

In global AdS$_3$ with metric $ds^2 = (r^2 + 1) dt_E^2 + \frac{1}{r^2 + 1} dr^2 + r^2 d\theta^2$, the only non-vanishing component of the bulk energy-momentum tensor of this particle is

$$T^{tt}_B = \frac{m}{2\pi r} \delta(r), \quad (F.1.9)$$

where we denote the bulk energy-momentum tensor with a subscript “B” to avoid confusion ($T, \bar{T}$ will still be the boundary bulk energy-momentum tensor). We are interested in describing the above particle in Poincare patch $ds^2 = \frac{dy^2 + dz d\bar{z}}{y^2}$. The coordinate maps that connect these two metrics are

$$y = \frac{e^{t_E}}{\sqrt{r^2 + 1}}, \quad z = \frac{r e^{t_E} e^{i\theta}}{\sqrt{r^2 + 1}}, \quad \bar{z} = \frac{r e^{t_E} e^{-i\theta}}{\sqrt{r^2 + 1}}. \quad (F.1.10)$$

The trajectory of the particle, which is simply $r = 0$ in the global coordinates, is corresponding to $(y, z, \bar{z}) = (e^{t_E}, 0, 0)$. Since we want to study the singularity of the boundary stress tensor $T(z)$ as it approaches the source, we need to localize to a small neighborhood around $r = 0$, that is, we’ll take the limit $z, \bar{z} \to 0$ in the following calculations. The full backreacted
metric will take the form of equation (7.3.2), where here we will interpret $T$ and $\bar{T}$ as components of a classical gravitational field. The delta function in the bulk energy-momentum tensor (F.1.9) can be transformed to the Poincare patch via

$$
\frac{1}{2\pi r} \delta(r) \to \frac{1}{y\sqrt{|g|}} \delta^2(z, \bar{z})
$$

where the Jacobian accounts for $y = e^t$ at $r = 0$. Thus, the covariant bulk energy-momentum tensor in Poincare patch will be given by

$$
T_{\mu\nu}^{B} = m v^\mu v^\nu \frac{1}{y\sqrt{|g|}} \delta^2(z, \bar{z})
$$

where the velocity of the particle following a geodesic is $v^\mu = (\dot{y}, \dot{z}, \dot{\bar{z}}) = (y, z, \bar{z})$ with constant $r$ and $\theta$. We’ll assume $T(z)$ to be more singular than $\frac{1}{z}$ and similarly for $\bar{T}(\bar{z})$. In the small $z, \bar{z}$ limit, we have $\sqrt{|g|} \approx \frac{18y\bar{T}}{c^{2}}$ and $v^\alpha v^\beta g_{\alpha\beta} \approx \frac{36y^{2}z\bar{z}T}{c^{2}}$. The resulting simplified form of the stress tensor is

$$
T_{\mu\nu}^{B} \approx \frac{mc^{4}v^\mu v^\nu}{648y^{4}z\bar{z}T^{2}\bar{T}^{2}} \delta^2(z, \bar{z})
$$

We apply the same limit to the LHS of Einstein equation to find the $y\bar{z}$ component to be

$$
G^{y\bar{z}} - g^{y\bar{z}} \approx \frac{c^{3}\partial_{z}T}{54y^{3}T^{2}\bar{T}^{2}}.
$$

So the $y\bar{z}$ component of the Einstein’s equation $G^{\mu\nu} - g^{\mu\nu} = 8\pi G_{N}T_{\mu\nu}^{B}$ is

$$
\partial_{z}T = \frac{\pi m}{z} \delta^2(z, \bar{z}),
$$

where we’ve used $G_{N} = \frac{3}{2c}$. So we find

$$
T(z) = \frac{m}{2z^{2}}.
$$
where we’ve used $\partial_{\bar{z}} \frac{1}{\bar{z}} = \pi \delta^2 (z, \bar{z})$. Similarly, from the $yz$ component of the Einstein’s equation, we can get $\bar{T} (\bar{z}) = \frac{m}{2\pi}$. Other components of the Einstein’s equation are trivially satisfied once we substitute these solutions for $T(z)$ and $\bar{T} (\bar{z})$.

So in the large mass approximation $m \approx 2h$ we can conclude that $T = \frac{h}{z}$ in the presence of a source localized at the origin. Bulk fields must be leave a similar imprint on the boundary $T$ correlators.

### F.2 Solving the Charged Bulk Primary Conditions

In this appendix we solve the charged bulk primary conditions for the operators $J_{-n}$, first exactly for the first few $n$ in appendix F.2.1, and then in appendix F.2.2, we study the large $k$ limit and obtain all the terms at order $1/k$ in $J_{-n}$ for all $n$.

#### F.2.1 Exact Solutions

We’ll expand the bulk charged field as

$$
\phi (y, z, \bar{z}) = \sum_{n=0}^{\infty} y^{2h+2n} \lambda_n J_{-n} \bar{L}_1^n \mathcal{O} (z, \bar{z}),
$$

with $\lambda_n \equiv \frac{(-1)^n}{n! (2h)_n}$

where we’ve factored out $\lambda_n$ in $\phi$ for later convenience. Now our task is to solve for $J_{-n}$s, which satisfies the following two conditions

$$
J_m J_{-n} \mathcal{O} = 0, \quad \text{for } m \geq 1,
$$

$$
L_1^n J_{-n} \mathcal{O} = n! (2h)_n \mathcal{O}, \quad \text{(F.2.2)}
$$

where the first one is simply the bulk-primary condition (7.2.16), and the second one just is to ensure that $\phi$ has the correct bulk-boundary propagator with $\mathcal{O}(w, \bar{w})$, i.e. $\langle \phi(y, z, \bar{z}) \mathcal{O}(w, \bar{w}) \rangle = \ldots$
\[
\left(\frac{y}{y^2 + (z-w)(z-w)}\right)^{2h}. \quad \text{One can also understand the second one as giving a normalization condition for } J_{-n}. \text{ It can be shown that the above two conditions uniquely fix } J_{-n}.
\]

At each level \(n\), we simply write \(J_{-n} \mathcal{O}\) as a sum over all possible level \(n\) descendant operators with unknown coefficients, and use the above equations to fix the coefficients. There will be equal number of unknown coefficients and independent equations at each level \(n\). The solutions for \(n\) up to 4 are given by

\[
J_{-1} = \frac{1}{1 - \frac{q^2}{2hk}} \left( L_{-1} - \frac{q}{k} J_{-1} \right),
\]

\[
J_{-2} = \frac{1}{1 - \frac{4h+1}{2h(2h+1)} \frac{q^2}{k} + \frac{1}{2h(2h+1)} \frac{q^4}{k^2}} \left( L_{-1}^2 - \frac{q}{k} (J_{-2} + 2J_{-1}L_{-1}) + \frac{q^2}{k^2} J_{-1}^2 \right), \tag{F.2.3}
\]

\[
J_{-3} = \frac{1}{1 - \frac{6h^2+6h+1}{2h(h+1)(2h+1)} \frac{q^2}{k} + \frac{3}{4h(h+1)} \frac{q^4}{k^2} - \frac{1}{4h(h+1)(2h+1)} \frac{q^6}{k^3}} \times \left( L_{-1}^2 - \frac{q}{k} \left( 2J_{-3} + 3J_{-2}L_{-1} + 3J_{-1}L_{-1}^2 \right) + \frac{3q^2}{k^2} (J_{-2}J_{-1} + J_{-1}J_{-1}L_{-1}) - \frac{q^3}{k^3} J_{-1}^3 \right),
\]

and

\[
J_{-4} = \frac{1}{1 - \frac{(16h^3+30h^2+22h+3)q^2}{2h(2h+3)(2h^2+3h+1)k} + \frac{(24h^2+36h+11)q^4}{4h(2h+3)(2h^2+3h+1)k^2} - \frac{(4h+3)q^6}{2h(2h+3)(2h^2+3h+1)k^3} + \frac{q^8}{4h(2h+3)(2h^2+3h+1)k^4}} \times \left[ L_{-1}^4 - \frac{q}{k} \left( 6J_{-4} + 8J_{-3}L_{-1} + 6J_{-2}L_{-1}^2 + 4J_{-1}L_{-1}^3 \right) \right.
\]

\[
+ \frac{q^2}{k^2} \left( 8J_{-3}J_{-1} + 3J_{-2}J_{-2} + 12J_{-2}J_{-1}L_{-1} + 6J_{-1}L_{-1}^2 \right) \]

\[
- \frac{6q^3}{k^3} \left( 6J_{-2}J_{-1}^2 + 4J_{-1}^3L_{-1} \right) + \frac{q^4}{k^4} J_{-1}^4 \right].
\]

In principle, one can continue this calculation up to arbitrarily large \(n\). However, as \(n\) increase, the number of descendant operators at level \(n\) will increase very fast and the calculation becomes very complicated.
One useful way of organizing the terms in $J_{-n}O$ is to separate the contribution of the global descendants of $O$ from that of quasi-primary operators and their global descendants, i.e.

$$J_{-n}O = L_{-1}^n O + \text{quasi-primaries plus their global descendants.} \quad (F.2.4)$$

For example, we can rewrite $J_{-1}O$ as

$$J_{-1}O = L_{-1}O + \frac{q^2}{2\hbar k - q^2} \left( L_{-1} - \frac{2\hbar}{q} J_{-1} \right) O \quad (F.2.5)$$

where $\left( L_{-1} - \frac{2\hbar}{q} J_{-1} \right) O$ is a quasi-primary operator since it satisfies $L_1 \left( L_{-1} - \frac{2\hbar}{q} J_{-1} \right) O = 0$.

Actually, we can be more precise about the statement (F.2.4). Similar to the case of gravity considered in section 3.2.2 of [3], one can show that $J_{-n}O$ can be written as

$$J_{-n}O = L_{-1}^n O + \sum_{j=0}^{n} \sum_{i} \frac{L_{-1}^{n-j} O_{h+j}^{(i)}}{\left| L_{-1}^{n-j} O_{h+j}^{(i)} \right|^2} \quad (F.2.6)$$

where the $O_{h+j}^{(i)}$ represent the $i$th quasi-primary at level $j$ (so they satisfy $L_1 O_{h+j}^{(i)} = 0$) and we’ve assumed that the quasi-primaries are orthogonal to each other. The denominator in each term is the norm of the corresponding operator. The simplest example of the above expression is given by $J_{-1}O$ in (F.2.5), which can be written as

$$J_{-1}O = L_{-1} + 2\hbar \left( L_{-1} - \frac{2\hbar}{q} J_{-1} \right) \frac{O}{\left| \left( L_{-1} - \frac{2\hbar}{q} J_{-1} \right) O \right|^2} \quad (F.2.7)$$

where we’ve used the fact that $\left| \left( L_{-1} - \frac{2\hbar}{q} J_{-1} \right) O \right|^2 = \frac{2\hbar (2\hbar k - q^2)}{q^2}$.

Note that the quasi-primaries $O_{h+j}^{(i)}$ must include at least one $J$ generator in them (like the $J_{-1}$ in $\left( L_{-1} - \frac{2\hbar}{q} J_{-1} \right) O$), so their norms must be at least order $k$ in the large $k$ limit. This means that in the large $k$ limit, we have

$$\lim_{k \to \infty} J_{-n} = L_{-1}^n.$$
as expected.

The decomposition of $J_{-n}O$ in (F.2.4) is useful is because we can make use of this fact when computing some correlation function. For example, since the two-point functions of $O^\dagger$ with quasi-primaries vanish, the terms that will contribute to $\langle \phi O^\dagger \rangle$ are those global descendants terms $L_{-1}^n O$. Therefore, we have

$$\langle \phi (y, z, \bar{z}) O (z_1, \bar{z}_1) \rangle = \sum_{n=0}^{\infty} y^{2h+2n} \lambda_n \langle L_{-1}^n \bar{L}_{-1}^n O (z, \bar{z}) O^\dagger (z_1, \bar{z}_1) \rangle = \left( \frac{y}{y^2 + (z - z_1)(\bar{z} - \bar{z}_1)} \right)^{2h}$$ (F.2.8)

### F.2.2 Large $k$ Expansion

In this section, we are going to solve the charged bulk primary conditions to next to leading order of the large $k$ limit. We can assume the ansatz for $J_{-n}O$ up to this order to be

$$J_{-n} = \left( 1 + \frac{a_0^{(n)}}{k} \right) L_{-1}^n + \sum_{i=1}^{n} \frac{a_i^{(n)}}{k} J_{-i} L_{-1}^{n-i} + O \left( \frac{1}{k^2} \right)$$ (F.2.9)

where $O \left( \frac{1}{k^2} \right)$ includes all the higher order terms. For the exact $J_{-n}O$, we have $J_m J_{-n}O = 0$ for $m \geq 1$. This means that we must have

$$J_m \left[ \left( 1 + \frac{a_0^{(n)}}{k} \right) L_{-1}^n + \sum_{i=1}^{n} \frac{a_i^{(n)}}{k} J_{-i} L_{-1}^{n-i} \right] O = O \left( \frac{1}{k} \right),$$ (F.2.10)

since $J_m$ acting on some of the order $\frac{1}{k^2}$ terms may give order $\frac{1}{k}$ result. So the leading order contribution to the LHS of the above equation should vanish,

$$J_m \left( \left( 1 + \frac{a_0^{(n)}}{k} \right) L_{-1}^n O + \sum_{i=1}^{n} \frac{a_i^{(n)}}{k} J_{-i} L_{-1}^{n-i} \right) O \approx \binom{n}{m} m! q L_{-1}^{n-m} + m a_m^{(n)} L_{-1}^{n-m} O = 0$$ (F.2.11)
where we discarded the order $\frac{1}{k}$ terms on the RHS of the approximation symbol and we’ve assumed $1 \leq m \leq n$ in the above equation. Solving the equation on the RHS for $a_{m}^{(n)}$, we get

$$a_{m}^{(n)} = -\frac{n!}{(n-m)!m^q}$$  \hspace{1cm} (F.2.12)

Now using the condition that $L_{1}^{n}J_{-n}\mathcal{O} = n!(2h)_{n}\mathcal{O}$, we can solve for $a_{0}^{(n)}$. We have

$$L_{1}^{n}J_{-n}\mathcal{O} = \left[1 + \frac{a_{0}^{(n)}}{k}\right]n!(2h)_{n} + \sum_{i=1}^{n} \frac{a_{i}^{(n)}}{k^i} \left(\begin{array}{c} n \\ i \end{array}\right) (n-i)!(2h)_{n-i} \mathcal{O} + O\left(\frac{1}{k^2}\right),$$  \hspace{1cm} (F.2.13)

Requiring the RHS to be equal to $n!(2h)_{n}\mathcal{O}$ up to order $\frac{1}{k^2}$, we get

$$a_{0}^{(n)} = -\frac{1}{(2h)_{n}} \sum_{i=1}^{n} a_{i}^{(n)} (2h)_{n-i} = -q(\psi^{(0)}(1-2h) - \psi^{(0)}(-2h-n+1))$$  \hspace{1cm} (F.2.14)

where $\psi^{(0)}$ is the digamma function.

In summary, the order $\frac{1}{k}$ terms in $J_{-n}$ are given by (F.2.9) with $a_{i}^{(n)}$ and $a_{0}^{(n)}$ given by (F.2.12) and (F.2.14), respectively. In appendix F.3.1, we are going to use this result to compute the $\frac{1}{k}$ correction to the bulk propagator $\langle \phi^\dagger \phi \rangle$.

### F.3 Bulk Correlation Functions in $U(1)$ Chern-Simons Theory

In this appendix we first compute the $1/k$ corrections to the bulk propagator $\langle \phi^\dagger \phi \rangle$ using CFT techniques (i.e., using the result we obtained in last section for the $1/k$ corrections in $\phi$). Then in appendix F.3.2, we compute $\langle \phi \mathcal{O}^\dagger J \rangle$ and $\langle \phi^\dagger \phi \rangle$ using Witten diagrams, and get exactly the same results as the CFT calculations. These calculations provide a non-trivial check of our definition of a bulk charged scalar field using the bulk primary condition.
(7.2.16).

**F.3.1 CFT Calculation of \( \langle \phi^\dagger \phi \rangle \)**

We can compute \( \langle \phi^\dagger (y_1, z_1, \bar{z}_1) \phi (y_2, \bar{z}_2, \bar{z}_2) \rangle \), where \( \phi^\dagger \) is \( \phi \) with \( q \rightarrow -q \), using the result we obtained in last section for the \( 1/k \) correction terms in \( \phi \). Up to order \( 1/k \), \( \phi \) is given by

\[
\phi (y, z, \bar{z}) = y^{2h} \sum_{n=0}^{\infty} y^{2n} \lambda_n \mathcal{J}_{-n} \bar{L}_{-1}^n \mathcal{O} (z, \bar{z}), \quad \lambda_n = \frac{(-1)^n}{n! (2h)_n} \tag{F.3.1}
\]

with

\[
\mathcal{J}_{-n} = \left( 1 + a_0^{(n)} \right) L_{-1}^n + \sum_{i=1}^{n} a_i^{(n)} \frac{J_{-i} L_{-1}^{n-i}}{k} + O \left( \frac{1}{k^2} \right). \tag{F.3.2}
\]

and \( a_i^{(n)} = -q \frac{n!}{(n-i)!} \) and \( a_0^{(n)} = -q \phi(0)(1-2h) - \psi(0)(-2h-n+1) \), as computed in appendix **F.2.2**. We can compute \( \langle \phi (y_1, z_1, \bar{z}_1) \phi (y_2, \bar{z}_2, \bar{z}_2) \rangle \) by directly inserting the above expansion of \( \phi \) into \( \langle \phi^\dagger \phi \rangle \) and summing over all the contributions. So up to order \( 1/k \), we have

\[
\langle \phi^\dagger (y_1, z_1, \bar{z}_1) \phi (y_2, \bar{z}_2, \bar{z}_2) \rangle = (y_1 y_2)^{2h} \sum_{n_1, n_2=0}^{\infty} y_1^{2n_1} y_2^{2n_2} \lambda_{n_1} \lambda_{n_2} (A + B + C + D) \tag{F.3.3}
\]

with

\[
A \equiv \left( 1 + a_0^{(n_1)} + a_0^{(n_2)} \right) \langle L_{-1}^{n_1} \bar{L}_{-1}^{n_1} \mathcal{O}^\dagger (z_1, \bar{z}_1) L_{-1}^{n_2} \bar{L}_{-1}^{n_2} \mathcal{O} (z_2, \bar{z}_2) \rangle, \tag{F.3.4}
\]

\[
B \equiv \sum_{i_2=1}^{n_2} \frac{a_i^{(n_2)}}{k} \langle L_{-1}^{n_1} \bar{L}_{-1}^{n_1} \mathcal{O}^\dagger (z_1, \bar{z}_1) J_{-i_2} L_{-1}^{n_2-i_2} \bar{L}_{-1}^{n_2} \mathcal{O} (z_2, \bar{z}_2) \rangle, \tag{F.3.5}
\]

\[
C \equiv \sum_{i_1=1}^{n_1} \frac{a_i^{(n_1)}}{k} \langle J_{-i_1} L_{-1}^{n_1-i_1} \bar{L}_{-1}^{n_1} \mathcal{O}^\dagger (z_1, \bar{z}_1) L_{-1}^{n_2} \bar{L}_{-1}^{n_2} \mathcal{O} (z_2, \bar{z}_2) \rangle, \tag{F.3.6}
\]

\[
D \equiv \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \frac{a_i^{(n_1)} a_i^{(n_2)}}{k^2} \langle J_{-i_1} L_{-1}^{n_1-i_1} \bar{L}_{-1}^{n_1} \mathcal{O}^\dagger (z_1, \bar{z}_1) J_{-i_2} L_{-1}^{n_2-i_2} \bar{L}_{-1}^{n_2} \mathcal{O} (z_2, \bar{z}_2) \rangle. \tag{F.3.7}
\]

where we’ve only kept the terms that may contribute to order \( 1/k \).
The $\frac{1}{k}$ term in $A$ will cancel the $\frac{1}{k}$ terms in $B$ and $C$, so the sum of $A + B + C$ will be

$$A + B + C = \left\langle L_{-1} L_{-1} \tilde{L}_{-1} \tilde{O}^\dagger (z_1, \bar{z}_1) L_{-1} L_{-1} \tilde{O} (z_2, \bar{z}_2) \right\rangle. \quad (F.3.9)$$

Since $L_{-1}$ and $\tilde{L}_{-1}$ are simply derivatives $\partial_z$ and $\partial_{\bar{z}}$, this is simply computed to be

$$A + B + C = \frac{[(2h)_{n_1+n_2}]^2}{z_{12}^{2h+n_1+n_2}}. \quad (F.3.10)$$

The sum of this term in equation (F.3.3) will give the free field limit of $\langle \phi \phi \rangle$, which is

$$\langle \phi_0 \phi_0 \rangle = \frac{\rho^h}{1-\rho}, \quad \text{with } \rho = \left(\frac{\xi}{1 + \sqrt{1-\xi^2}}\right)^2, \quad \xi = \frac{2y_1y_2}{y_1^2 + y_2^2 + z_{12} \bar{z}_{12}} \quad (F.3.11)$$

where $\phi_0$ is given in (F.3.8).

Now the only term left to compute is $D$. The anti-holomorphic part can be computed simply, and we have

$$\left\langle J_{-i_1} L_{-1}^{n_1-i_1} \tilde{L}_{-1}^{n_1} \tilde{O}^\dagger (z_1, \bar{z}_1) J_{-i_2} L_{-1}^{n_2-i_2} \tilde{L}_{-1}^{n_2} \tilde{O} (z_2, \bar{z}_2) \right\rangle \quad (F.3.12)$$

$$= \frac{(2h)_{n_1+n_2} (-1)^{n_1}}{z_{12}^{2h+n_1+n_2}} \left\langle J_{-i_1} L_{-1}^{n_1-i_1} \tilde{O} (z_1) J_{-i_2} L_{-1}^{n_2-i_2} \tilde{O} (z_2) \right\rangle$$

Since we only care about the order $\frac{1}{k}$ terms in $D$, we only need to compute the order $k$ term

$^2$This can be checked by direct calculation. On the other hand, we can also see that this must be true by separating $\phi$ into two parts: $\phi = \phi_0 + \phi_{q,p}$, where $\phi_0$ is the free-field

$$\phi_0 = y^{2h} \sum_{n=0}^{\infty} y^{2n} \lambda_n L_{-1}^{n} \tilde{L}_{-1}^{n} \tilde{O} \quad (F.3.8)$$

and $\phi_{q,p}$ includes the contributions from quasi-primaries and their global descendants. We can do this separation because of the property of $J_{-n}$ in (7.2.18). The point here is that $\langle \phi_0 \phi_{q,p} \rangle$ must be zero, since the two-point function $\tilde{O}$ with a quasi-primary is zero and this is also true for the two-points of their global descendants. So the $1/k$ corrections can only come from $\langle \phi_{q,p} \phi_{q,p} \rangle$, that is, the terms in $D$ of (F.3.7).
in the two-point function on the RHS of the above equation, which is

\[ \langle J_{-i_1} L_{1}^{n_1 - i_1} \mathcal{O}(z_1) \, J_{-i_2} L_{2}^{n_2 - i_2} \mathcal{O}(z_2) \rangle \]  

\[ = \frac{1}{2\pi i} \oint_{z_1} \frac{dz}{z - z_1} \frac{1}{(z - z_2)^{i_1 + 1}} \langle L_{1}^{n_1 - i_1} \mathcal{O}(z_1) \, J_{i_2} J_{-i_2} L_{1}^{n_2 - i_2} \mathcal{O}(z_2) \rangle + \ldots \]

\[ = \frac{1}{2\pi i} \oint_{z_1} \frac{dz}{z - z_1} \frac{i_2 k}{(z - z_2)^{i_1} (z - z_2)^{i_2 + 1}} \langle L_{1}^{n_1 - i_1} \mathcal{O}(z_1) \, L_{1}^{n_2 - i_2} \mathcal{O}(z_2) \rangle + \ldots \]

\[ = k (-1)^{n_1 - 1} (2h)_{n_1 + n_2} (i_2)_{i_1} \left( \frac{n_1}{i_1 - 1} \right) + \ldots \]

where we’ve only kept the leading order terms.

Putting everything together, we have

\[ \langle \phi^\dagger \phi \rangle = \langle \phi_0 \phi_0 \rangle = \frac{q^2}{k} (X_1 X_2)^h \]  

\[ \times \sum_{n_1, n_2=0}^{\infty} \frac{(-1)^{n_1 + n_2} (2h)_{n_1 + n_2} (2h)_{n_1} (2h)_{n_2}}{(n_1 - i_1)!(n_2 - i_2)!(i_1 - 1)!(i_2 - 1)!} X_1^{n_1} X_2^{n_2} + O \left( \frac{1}{k^2} \right) \]

where we’ve defined \( X_1 \equiv \frac{y_1^2}{z_{12} z_{12}}, \) \( X_2 \equiv \frac{y_2^2}{z_{12} z_{12}}. \) Mathematica is not able to perform the above sums, but it can be checked that the following expression

\[ \langle \phi^\dagger \phi \rangle = \frac{\rho^h}{1 - \rho} \left[ 1 - \frac{q^2}{k} \left( \rho^2 F_1(1, 2h + 1; 2(h + 1); \rho) + \frac{\rho}{2h - \log(1 - \rho)} \right) \right] + O \left( \frac{1}{k^2} \right) \]

\[ \text{(F.3.15)} \]

gives the same expansion when expanded in small \( X_1 \) and \( X_2. \) In appendix F.3.2.2, we’ll show that the above expression is the result of the bulk Witten diagram calculation. Thus this provides a non-trivial check of our definition of a bulk charged proto-field in \( U(1) \) Chern-Simons theory.
F.3.2 Bulk Witten Diagram Calculation for $\langle J\phi^\dagger \rangle$ and $\langle \phi^\dagger \phi \rangle$

In this section, we are going to compute $\langle J\phi^\dagger \rangle$ and $\langle \phi^\dagger \phi \rangle$ using Witten diagrams. The calculations here will be very similar to that of $\langle T\phi \phi \rangle$ in [3] and that of $\langle \phi \phi \rangle$ in [5], except that the $\langle A_z A_z \rangle$ two-point function and the bulk three-point vertex are different, but actually simpler.

The action of the $U(1)$ Chern-Simons theory in Poincare AdS$_3$ is given by

$$I = \int d^3x \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \sqrt{|g|} \left( \nabla_\mu \phi (\nabla^\mu \phi)^\dagger + m^2 \phi \phi^\dagger \right)$$  \hspace{1cm} (F.3.16)

with $\nabla_\mu \phi = (\partial_\mu + iq A_\mu) \phi$ and $(\nabla_\mu \phi)^\dagger = (\partial_\mu - iq A_\mu) \phi^\dagger$. The Poincare AdS$_3$ metric and its inverse are

$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{y^2} & 0 & 0 \\ 0 & 0 & \frac{1}{2y^2} \\ 0 & \frac{1}{2y^2} & 0 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} y^2 & 0 & 0 \\ 0 & 0 & 2y^2 \\ 0 & 2y^2 & 0 \end{pmatrix}$$  \hspace{1cm} (F.3.17)

in the coordinate system $(y, z, \bar{z})$. From the Chern-Simons action (F.3.16), we can see that the photon two point function is given by

$$H (Y_1, Y_2) \equiv \langle A_z (y_1, z_1, \bar{z}_1) A_z (y_2, z_2, \bar{z}_2) \rangle = \frac{1}{k (z_1 - z_2)^2}. \hspace{1cm} (F.3.18)$$

Here and in some other parts of this paper, we’ll use $X$ or $Y$ to denote $(y, z, \bar{z})$ for convenience.

The scalar two point function is just the usual bulk-bulk propagator as in (F.3.11)

$$G (Y_1, Y_2) \equiv \langle \phi^\dagger (y_1, z_1, \bar{z}_1) \phi (y_2, z_2, \bar{z}_2) \rangle = \frac{\rho^h}{1 - \rho}, \hspace{1cm} (F.3.19)$$

with $m^2 = 4h (h - 1)$. The photon-scalar-scalar three point vertex is given by

$$i q g^{\mu\nu} A_\mu \left( \phi \partial_\mu \phi^\dagger - \phi^\dagger \partial_\mu \phi \right). \hspace{1cm} (F.3.20)$$
And since we’ll only be interested in the \( z \) component of \( A_z \), the above vertex becomes

\[
i 2qy^2 A_z \left( \phi \partial_z \phi^\dagger - \phi^\dagger \partial_z \phi \right)
\]  

where we’ve used \( g^{zz} = 2y^2 \). We’ll assume\(^3\) that we are free to perform an integration by parts, so that the above vertex can be written as \(-4iqy^2 \partial_z \phi \). We’ll also need to bulk-boundary propagator for the charged scalar field and the photon field. These are given by

\[
K (Y_1, (z_2, \bar{z}_2)) \equiv \langle \phi (y_1, z_1, \bar{z}_1) \phi^\dagger (z_2, \bar{z}_2) \rangle = \left( \frac{y_1}{y_1^2 + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)^2} \right)^{2h}
\]  

and

\[
\langle A_z (Y_1) J (z_2) \rangle = \frac{1}{(z_1 - z_2)^2},
\]

where we’ve used the fact that \( J(z) = \frac{A_k}{k} \).

\[
\text{F.3.2.1 Witten Diagram Calculation for } \langle J O^\dagger \phi \rangle
\]

The Witten diagram for \( \langle J O^\dagger \phi \rangle \) is given by figure F.1. We’ll use the saddle point approximation to evaluate this diagram. The idea is that instead of integrating the bulk vertex point over \( \text{AdS}_3 \), we only integrate along the geodesic connecting \( \phi \) and \( O^\dagger \). In appendix D.4 of [3], we showed that the saddle point approximation for \( \langle \phi O T \rangle \) actually gives the exact result. We expect the same thing to happen here.

The bulk-boundary three-point function \( \langle J (z_1) O^\dagger (z_2, \bar{z}_2) \phi (X_1) \rangle \) (with \( X_1 = (y, 0, 0) \)) is computed as usual by

\[
\langle J O^\dagger \phi \rangle = \int_{\text{AdS}_3} \sqrt{g} d^3X' \frac{1}{(z' - z_1)^2} 2i q y'^2 \left[ G_{z'} - G_{\bar{z}'z} \right] \]

where we’ve denoted \( X' = (y', z', \bar{z}') \), and also used the bulk-boundary propagator for

\[^{3}\text{There may be some subtleties about this, which we explain in the calculation of } \langle \phi O J \rangle \text{ in next subsection.}\]
the photons (F.3.23). Explicitly, the coordinate dependence of $G$ and $K$ are $G = G(X_1, X')$ and $K = K(X', (z_2, \bar{z}_2))$. Now as mentioned above, we integrate by parts\(^4\) to find

$$\langle J \phi \rangle = -\int_{\text{AdS}_3} \sqrt{g} d^3 x' \frac{4i q y'^2}{(z' - z_1)^2} G \partial z' K$$ (F.3.25)

Now we can do a saddle point approximation as in appendix D.4 of [3]. The idea is as follows. We can write

$$G(X_1, X') = \frac{e^{-2h\sigma(X_1, X')}}{1 - e^{-2\sigma(X_1, X')}}$$ (F.3.26)

where $\rho \equiv e^{-2\sigma}$ and $\sigma_{(X_1, X')} = \rho_{(X_1, X')} = \log \frac{y'^2 + (z' - z_2)(z' - \bar{z}_2)}{y'}$ is the (regularized) bulk-boundary geodesic length between $X_1 = (y, 0, 0)$ and $X' = (y', z', \bar{z}')$, whose expression can be obtained from (F.3.11). Similarly, the bulk boundary propagator can be written as

$$K(X', (z_2, \bar{z}_2)) = e^{-2h\sigma_{(X', (z_2, \bar{z}_2)}}$$ (F.3.27)

where $\sigma_{(X', (z_2, \bar{z}_2))} = \log \frac{y'^2 + (z' - z_2)(z' - \bar{z}_2)}{y'}$ is the (regularized) bulk-boundary geodesic length between $X' = (y', z', \bar{z}')$ and $(z_2, \bar{z}_2)$. So the integral (F.3.25) after simplification can be\(^4\)In fact, when performing an integration by parts, there is a delta function term coming from $\partial_{z'} \frac{1}{z' - z_1} = \pi \delta^2(z' - z_1, \bar{z}' - \bar{z}_1)$. Such terms would contaminate pure CFT correlators (as well as bulk correlators) and violate Ward identities, and so we have dropped them. This can be viewed as a choice of regulator. It would be very interesting to better understand regulation, and the role of these terms, in future work.

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\(^4\)In fact, when performing an integration by parts, there is a delta function term coming from $\partial_{z'} \frac{1}{z' - z_1} = \pi \delta^2(z' - z_1, \bar{z}' - \bar{z}_1)$. Such terms would contaminate pure CFT correlators (as well as bulk correlators) and violate Ward identities, and so we have dropped them. This can be viewed as a choice of regulator. It would be very interesting to better understand regulation, and the role of these terms, in future work.
written in the following form

\[
\langle J^\dagger \phi \rangle = 8i h q \int \frac{dz' dz'' dy'}{y''^2} e^{-2hL(y',z',z'')} \frac{e^{-\sigma(y',z',z'')(z' - z_2)}}{1 - e^{-2\sigma(y,0,0),(y',z',z'')}(z' - z_1)^2} (z' - z_2)
\]

(F.3.28)

where \( L(y',z',z'') \equiv \sigma(X_1,X') + \sigma(X',(z_2,\bar{z}_2)) \). Now we can take the large \( h \) limit and the integration will be dominated by the line integral along the geodesic from \( X_1 = (y,0,0) \) to \( (z_2,\bar{z}_2) \). This geodesic parameterized by \( z' \) is given by

\[
z' = \frac{\bar{z}}{z} z', \quad y'^2 = \left(1 - \frac{z'}{z}\right) \left(y^2 + z'\bar{z}\right),
\]

(F.3.29)

so that the saddle point approximation to equation (F.3.25) is

\[
\langle J^\dagger \phi \rangle = 8\pi i q e^{-2hL(y,0,0)} \int_0^{z_2} dz' \frac{1}{\sqrt{\det \partial^2 L}} \frac{e^{-2\sigma(x',(z_2,\bar{z}_2))} 1 (z' - z_2)}{y'^2 (z' - z_1)^2},
\]

(F.3.30)

where the determinant is given by

\[
\det \partial^2 L = \det \begin{pmatrix}
\partial^2_z L & \partial_z \partial_{y'} L \\
\partial_{y'} \partial_z L & \partial^2_{y'} L
\end{pmatrix} = \frac{4 z_2^2 (z'\bar{z}_2 + y'^2)}{z'^2 (z' - z_2) (z_2\bar{z}_2 + y'^2)^2}.
\]

(F.3.31)

Plugging this in (F.3.30), we have

\[
\langle J(z_1) O^\dagger (z_2, \bar{z}_2) \phi(y,0,0) \rangle = q \langle \phi O^\dagger \rangle \int_0^{z_2} dz' \frac{1}{(z_1 - z')^2} = \frac{q z_2}{(z_2 - z_1)^2 z_1} \langle \phi O^\dagger \rangle
\]

(F.3.32)

where we’ve used the fact that \( \langle \phi(y,0,0) O^\dagger (z_2, \bar{z}_2) \rangle = e^{-2hL(y,0,0)} = \left(\frac{y}{y^2 + z_2^2}\right)^{2h} \) and we neglected a numerical constant in obtaining the above result. The above result is exactly the same as the result (7.2.22) obtained via assuming \( \phi \) to be a charged bulk field defined by the bulk primary condition (7.2.16).
F.3.2.2 One-loop Correction to $\langle \phi^\dagger \phi \rangle$

In this subsection, we are going to compute one loop correction to $\langle \phi^\dagger \phi \rangle$. The Witten diagram is given in figure F.2, and it’s computed as follows

$$
\langle \phi^\dagger (X_1) \phi (X_2) \rangle_{1 \text{ loop}} = -16q^2 \int \sqrt{|g|} d^3X' \int \sqrt{|g|} d^3X'' y'^2 y''^2 
\times G(X_1, X') \partial \bar{z}_1 G(X', X'') H(X', X'') \partial \bar{z}_2 G(X'', X_2)
$$

(F.3.33)

Acting on this two-point function with the Klein-Gordon operator twice, we have

$$
\left( \nabla_1^2 + m^2 \right) \left( \nabla_2^2 + m^2 \right) \langle \phi^\dagger (X_1) \phi (X_2) \rangle_{1 \text{ loop}} = -16q^2 H(X_1, X_2) y_1^2 y_2^2 \bar{z}_1 \bar{z}_2 G(X_1, X_2)
$$

(F.3.34)

By using the photon propagator (F.3.18) and the bulk-bulk propagator (F.3.19), one can show that the RHS of the above equation is

$$
-\frac{32q^2 \rho^{k+1} (2h^2(1-\rho)^2 + h (-5\rho^2 + 4\rho + 1) + 3\rho(\rho + 1))}{k(1-\rho)^5}.
$$

(F.3.35)

Assuming $P(\rho) \equiv \langle \phi^\dagger (X_1) \phi (X_2) \rangle_{1 \text{ loop}}$ only depends on the $\rho$, equation (F.3.34) becomes

$$
16 (h - 1)^2 \rho^2 P'(\rho) + \frac{64}{\rho - 1} \left( -h^2 + h + 1 \right) \rho^2 P' (\rho)
$$

(F.3.36)
\[
- \frac{32 \rho^2 ((h - 1) h (\rho - 1) - 7 \rho + 1) P''(\rho)}{\rho - 1} + \frac{64 \rho^3 (2 \rho - 1) P^{(3)}(\rho)}{\rho - 1} + 16 \rho^4 P^{(4)}(\rho)
\]

\[
= - \frac{32 q^2 h^{\rho + 1} (2 h^2 (1 - \rho)^2 + h (-5 \rho^2 + 4 \rho + 1) + 3 \rho (\rho + 1))}{k (1 - \rho)^5}
\]

Luckily, Mathematica is able to solve the above fourth order differential equation. Fixing the integration constants, we get

\[
P(\rho) = - \frac{q^2}{k} \frac{\rho^h}{1 - \rho} \left[ \frac{\rho^2 F_1(1, 2 h + 1; 2(h + 1); \rho)}{2 h + 1} + \frac{\rho}{2 h} - \log(1 - \rho) \right], \tag{F.3.37}
\]

which is exactly the same as the \(\frac{1}{k}\) correction to the two-point function of the bulk charged proto-field computed in appendix F.3.1 using pure CFT technics.

### F.4 Computations of \(\langle \phi \mathcal{O} T \rangle\) in Global AdS\(_3\)

In this section, we use two methods to calculate \(\langle \phi \mathcal{O} T \rangle\) in global AdS\(_3\). These two methods are essentially the same. Both of them are based on the idea of bulk-boundary OPE blocks (as we called them in [3] and [5]) or the bulk-boundary bi-local operator \(\phi \mathcal{O}\) (F.4.2) (a generalization of the boundary bi-local operator as defined in [168]). The first method is to expand the vacuum channel of the bulk-boundary bi-local operator in terms of the \(\epsilon\) operator defined in section 6 of [168], and then use the \(\epsilon\) propagator, which was derived from the Alekseev-Shatashvilli theory of boundary gravitons in that paper, to compute \(\langle \phi \mathcal{O} T \rangle\).

The second method is to expand the vacuum channel of the bi-local in terms of the energy-momentum tensor \(T\) and use the two-point function of \(T\) to compute \(\langle \phi \mathcal{O} T \rangle\). Both of them give the same result as (7.3.31) obtained using the properties of the bulk proto-field.

#### F.4.1 \(\langle \phi \mathcal{O} T \rangle\) in Global AdS\(_3\) from Alekseev-Shatashvilli

The idea here very similar to that of section 7.5.2, where we computed the \(1/c\) corrections to the bulk-boundary propagator in a heavy background. The difference is that here we are
considering bulk-boundary bi-local operator $\phi \mathcal{O}$ in the vacuum.

As in section 7.3.3.2, we use $f(z) = e^z$ and $\bar{f}(\bar{z}) = e^{\bar{z}}$ to obtain the global AdS$_3$ metric (7.3.28). But now we are going to include perturbation as follows

$$f(z) = e^{z + i\epsilon(z)/c}, \quad \text{and} \quad \bar{f}(\bar{z}) = e^{\bar{z} - i\bar{\epsilon}(\bar{z})/c} \quad (F.4.1)$$

(with $\epsilon(z)$ and $\bar{\epsilon}(z)$ promoted to operators). Specifically, the vacuum contribution to the bulk-boundary bi-local operator is given by

$$\phi(y_1, z_1, \bar{z}_1) \mathcal{O}(z_2, \bar{z}_2)|_{\text{vac}} = \left(f'(z_2) \bar{f}'(\bar{z}_2)\right)^h \left(\frac{u_1}{u_1^2 + (x_1 - f(z_2))(\bar{x}_1 - f(\bar{z}_2))}\right)^{2h}. \quad (F.4.2)$$

The $(u_1, x_1, \bar{x}_1)$ here are functions of $(y_1, z_1, \bar{z}_1)$ given by (7.3.10) with $f(z)$ and $\bar{f}(\bar{z})$ given by (F.4.1). One can see that the above expression becomes the boundary bi-local defined in [168] when we send $\phi$ to the boundary by taking $y_1 \to 0$. Written in terms of the coordinates $(y, z, \bar{z})$ and expanded in large $c$, the above vacuum block becomes

$$\frac{\phi(y_1, z_1, \bar{z}_1) \mathcal{O}(z_2, \bar{z}_2)|_{\text{vac}}}{\langle \phi(y_1, z_1, \bar{z}_1) \mathcal{O}(z_2, \bar{z}_2) \rangle_{\text{global AdS}_3}} = 1 + \frac{1}{c} \left(\frac{B\epsilon_1 + C\epsilon'_1 + D\epsilon''_1 + E\epsilon_2 + F\epsilon'_2}{A} + O\left(\frac{1}{c^2}\right)\right) \quad (F.4.3)$$

where $\epsilon_i \equiv \epsilon(z_i)$ and the derivatives are with respect to $z_i$. The bulk-boundary propagator $\langle \phi(y_1, z_1, \bar{z}_1) \mathcal{O}(z_2, \bar{z}_2) \rangle_{\text{global AdS}_3}$ in global AdS$_3$ is given in (7.3.32) (up to a factor of $(\xi_2 \bar{\xi}_2)^h$ coming from the $\left(f'(z_2) \bar{f}'(\bar{z}_2)\right)^h$ in (F.4.2)), i.e.

$$\langle \phi \mathcal{O} \rangle_{\text{global AdS}_3} = \left(\frac{4y_1 \sqrt{\xi_1 \xi_2 \bar{\xi}_1 \bar{\xi}_2}}{(\xi_1 \xi_1 + \xi_2 \bar{\xi}_2)(y_1^2 + 4) + (\xi_1 \xi_2 + \bar{\xi}_2 \bar{\xi}_1)(\bar{y}_1^2 - 4)}\right)^{2h}, \quad (F.4.4)$$

where we’ve defined $\xi_i \equiv e^{zi}$ and $\bar{\xi}_i = e^{\bar{z}i}$. The denominator and the coefficients in the numerator of (F.4.3) are given by

$$A = i \left((y_1^2 - 4) \left(\xi_2 \bar{\xi}_1 + \xi_1 \bar{\xi}_2\right) + \left(y_1^2 + 4\right) \left(\xi_1 \bar{\xi}_1 + \xi_2 \bar{\xi}_2\right)\right),$$
\[B = \left( y_1^2 - 4 \right) \left( \xi_1 \bar{\xi}_2 - \xi_2 \bar{\xi}_1 \right) + \left( y_1^2 + 4 \right) \left( \xi_1 \bar{\xi}_1 - \xi_2 \bar{\xi}_2 \right),\]

\[C = \left( y_1^2 + 4 \right) \left( \xi_2 \bar{\xi}_1 + \xi_1 \bar{\xi}_2 \right) + \left( y_1^2 - 4 \right) \left( \xi_1 \bar{\xi}_1 + \xi_2 \bar{\xi}_2 \right),\]  \hspace{1cm} (F.4.5)

\[D = -2y_1^2 \left( \xi_1 - \xi_2 \right) \left( \bar{\xi}_1 + \bar{\xi}_2 \right),\]

\[E = \left( y_1^2 + 4 \right) \left( \xi_2 \bar{\xi}_2 - \xi_1 \bar{\xi}_1 \right) + \left( y_1^2 - 4 \right) \left( \xi_2 \bar{\xi}_1 - \xi_1 \bar{\xi}_2 \right),\]

\[F = - \left( y_1^2 - 4 \right) \left( \xi_2 \bar{\xi}_1 + \xi_1 \bar{\xi}_2 \right) - \left( y_1^2 + 4 \right) \left( \xi_1 \bar{\xi}_1 + \xi_2 \bar{\xi}_2 \right).\]

We’ve only kept the holomorphic \(\epsilon\) terms in (F.4.3), since the anti-holomorphic \(\bar{\epsilon}\) will not contribute to \(\langle \phi OT \rangle\).

To compute \(\langle \phi OT \rangle\), we also need to write the energy-momentum tensor \(T\) in terms of the \(\epsilon\) operators. As explained in section 2 of [3], \(T\) is simply given by the Schwarzian derivative as follows:

\[T(z) = \frac{c}{12} \left[ \frac{f'''(z)f'(z) - \frac{3}{2} (f''(z))^2}{(f'(z))^2} \right] = -\frac{c}{24} - \frac{i}{12} \left( \epsilon'(z) - \epsilon^{(3)}(z) \right) + O \left( \frac{1}{c} \right).\]  \hspace{1cm} (F.4.6)

The last piece of information we need to compute \(\langle \phi OT \rangle\) is the \(\epsilon\) propagator \(\langle \epsilon \epsilon \rangle\). The action obeyed by \(\epsilon\) and \(\bar{\epsilon}\) is the saddle point quadratic action derived from the Alekseev-Shatashvilli action. And the \(\epsilon\) propagator is given in section 6 of [96],

\[\langle \epsilon(z) \epsilon(0) \rangle = 6c \left( 1 - 1 + \frac{3\xi}{2} - \frac{(1 - \xi)^2}{\xi} \ln (1 - \xi) \right), \quad \xi \equiv e^z\]  \hspace{1cm} (F.4.7)

Note that we have a different normalization for the \(\epsilon\)s.
Eventually, $\langle \phi (y_1, z_1, \bar{z}_1) \mathcal{O} (z_2, \bar{z}_2) T (z) \rangle$ is given by\(^5\)

$$\langle \phi \mathcal{O} T \rangle = \frac{h (\xi_1 - \xi_2)^2 \xi^2}{(\xi - \xi_1)^2 (\xi - \xi_2)^2} \left[ \xi - \xi_1 + \frac{4y_1^2 \xi_1 (\xi - \xi_2) (\xi_1 + \xi_2)}{(\xi_1 \xi_1 + \xi_2 \xi_2) (y_1^2 + 4) + (\xi_1 \xi_2 + \xi_2 \xi_1) (y_1^2 - 4)} \right] - \frac{c}{24} \quad (F.4.8)$$

which agrees with (7.3.31) up to a trivial transformation for $\mathcal{O}$ and $T$ from the complex plane $\left( \xi, \bar{\xi} \right)$ to the cylinder $\left( z, \bar{z} \right)$.

### F.4.2 $\langle \phi \mathcal{O} T \rangle$ in Global AdS\(_3\) from $\langle TT \rangle$

We can also compute $\langle \phi \mathcal{O} T \rangle$ in global AdS\(_3\) without using the $\epsilon$ propagator as we did in last subsection. Instead, we’ll use the two-point function of the energy-momentum tensor $T$. This was the method that we used in [3] to compute $\langle \phi \mathcal{O} T \rangle$ (equation (7.3.26)) in Poincaré AdS\(_3\), and also in appendix B of [5] to compute the $1/c$ corrections to the vacuum block of all-light bulk-boundary correlator $\langle \mathcal{O}_2 \mathcal{O}_2 \phi_1 \mathcal{O}_1 \rangle$ (with $\phi_1$ the proto-field of section 7.3.1) up to order $1/c^2$.

The idea is actually very similar to that of last section. Instead of denoting the perturbation via $\epsilon$, we expand $f (z)$ in the large $c$ limit as

$$f (z) = e^z + \frac{1}{c} f_1 (z) + \ldots , \quad (F.4.9)$$

and similarly for $\bar{f} (\bar{z})$. Then similar to last subsection, the energy-momentum tensor $T (z)$ is given by

$$T (z) = \frac{c}{12} \left[ f''' (z) f' (z) - \frac{3}{2} (f'' (z))^2 \right] = -\frac{c}{24} + \frac{1}{12} e^{-z} \left( 2 f_1' - 3 f_1'' + f_1^{(3)} \right) + O \left( \frac{1}{c} \right) . \quad (F.4.10)$$

The energy-momentum tensor $T (\xi)$ on the complex plane with coordinates $\left( \xi = e^z, \bar{\xi} = e^{\bar{z}} \right)$

\(^5\)There could be $1/c$ corrections to $\langle \phi \mathcal{O} T \rangle$ coming from the higher order terms in (F.4.3) and (F.4.6). But since the result for $\langle \phi \mathcal{O} T \rangle$ is exact without any $1/c$ correction (as computed using the properties of the bulk proto-field in 7.3.3), such higher order have been dropped by the regulator, as in [77, 3, 5].
is related to $T(z)$ as usual by

$$T(z) = -\frac{c}{24} + \xi^2 T(\xi), \quad (F.4.11)$$

so from (F.4.10), we have

$$T(e^z) = \frac{1}{12} e^{-3z} \left( 2 f'_1 - 3 f''_1 + f''(3) \right) \quad (F.4.12)$$

We can now solve for $f_1(z)$ in terms of $T(e^z)$, and the solution is given by

$$f_1(z) = 12 \int_0^z \left( \int_0^{z'} e^{z'+z''} (e^{z''} - e^{z''}) T(e^{z''}) \, dz'' \right) \, dz'. \quad (F.4.13)$$

The bulk-boundary OPE block or the bulk boundary bi-local operator (F.4.2) can then be expanded in large $c$,

$$\frac{\langle \phi(y,0,0) \mathcal{O}(z_2,\bar{z}_2) \rangle_{\text{vac}}}{\langle \phi(y,0,0) \mathcal{O}(z_2,\bar{z}_2) \rangle_{\text{global AdS}_3}} = 1 + \frac{1}{c} \left( \frac{h f'_1 (z_2)}{\xi_2} - \frac{2 h \left( (y^2 + 4) \bar{\xi}_2 y^2 - 4 \right) f_1 (z_2)}{(y^2 + 4) (1 + \xi_2 \xi_2) + (y^2 - 4) (\xi_2 + \bar{\xi}_2)} \right) + O \left( \frac{1}{c^2} \right) \quad (F.4.14)$$

Using $\langle T(e^z) T(e^{z_2}) \rangle = \frac{c}{(e^z - e^{z_2})^4}$ on the complex plane, we find

$$\langle f_1(z_2) T(e^z) \rangle = \frac{c (\xi_2 - 1)^3}{(\xi - 1)^3 (\xi - \xi_2)} \quad (F.4.15)$$

$$\langle f'_1(z_2) T(e^z) \rangle = -\frac{c (\xi_2 - 1)^2 \xi_2 (1 - 3 \xi + 2 \xi_2)}{(\xi - 1)^3 (\xi - \xi_2)^2}$$

and $\langle \phi(y,0,0) \mathcal{O}(z_2,\bar{z}_2) T(z) \rangle$ (after transforming $T$ to the cylinder) is given by

$$\frac{\langle \phi \mathcal{O} T \rangle}{\langle \phi \mathcal{O} \rangle_{\text{global AdS}_3}} = \frac{h (1 - \xi_2) \xi^2}{(\xi - 1)^3 (\xi - \xi_2)^2} \left[ \xi - 1 + \frac{4 \xi_2^2 (\xi - \xi_2) (1 + \xi_2)}{(1 + \xi_2 \xi_2) (y^2 + 4) + (\xi_2 + \bar{\xi}_2) (y^2 - 4)} \right] - \frac{c}{24} \quad (F.4.16)$$

which is exactly the result obtained in last subsection (F.4.8) with $y_1 = y, z_1 = \bar{z}_1 = 0$. 

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Biography

Hongbin Chen grew up in the coastal city Shantou in the south of China. He received his B.S. in physics from the University of Science and Technology of China in 2014. In the same year, he began his Ph.D. study in theoretical high energy physics at the Johns Hopkins University.

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