CRYSTALLINE LIFTS OF GALOIS REPRESENTATIONS

by

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Abstract

Crystalline lifts of Galois representations

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Let G be a connected split reductive group over $\mathbb{Z}_p$, and let K be a p-adic field. We show continuous homomorphisms $\bar{r} : G_K \to G(\bar{\mathbb{F}}_p)$ admit crystalline lifts when they are G-completely reducible.

We also show when $p > 3$ and G is the exceptional group G_2, $\bar{r}$ admits crystalline lifts.

Advisor: David Savitt

For my family.

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Contents

Chapter 1

Introduction

In this thesis, we present various techniques for lifting local Galois representations valued in reductive groups, culminating in a complete resolution of the problem for the exceptional group G_2 in characteristic $p > 3$.

Let $\mathbb{Q}_p$ be the field of p-adic integers. One of the most notable achievements of number theory in the new millennium is the classification of irreducible admissible smooth modular representations of the p-adic group $\mathrm{GL}_2(\mathbb{Q}_p)$ in terms of 2-dimensional modular representations of the absolute Galois group $G_{\mathbb{Q}_p}$. Since then, people have been working on generalizing this connection between modular smooth representations (representation-theoretic side) and modular Galois representations (Galois side) to other p-adic reductive groups. The representation-theoretic side of the picture is still very unclear. In recent years, people have sought to understand the geometry of the moduli space of Galois representations with the hope to gain new insights of this speculated connection.

An important tool for studying mod p Galois representations is to first lift them to characteristic

0. For example, in order to get a uniform ramification bound of mod p Galois representations, [CL11] first lift them to characteristic 0 where powerful tools of p-adic Hodge theory are available. When studying a local Galois representation, a very successful technique is to first realize it as the local component of a global Galois representation, and then apply powerful global methods such as the Taylor-Wiles patching. When the local Galois representation is in characteristic p, the preliminary step of globalization is to construct characteristic 0 lifts (see [EG14]).

Recently, in [EG19], the problem of lifting mod p local Galois representations valued in GL_n is fully settled. In their work, the existence of lifts is used to understand the geometry of the moduli stack of mod p Galois representations. Roughly speaking, we single out a special class of characteristic 0 Galois representations, the so-called crystalline representations, whose moduli stack is equidimensional; by showing any mod p Galois representation admits a crystalline lift, we find that the map between the moduli of crystalline representations and the moduli of mod p Galois representations is surjective. As a consequence, the moduli stack of mod p Galois representations is Noetherian, and has an equidimensional reduced substack.

In this thesis, we consider more general reductive groups. We will follow the strategy of [EG19] and do it in two steps. The first step is to construct lifts for semi-simple Galois representations, and the second step is to lift extension classes. Many things that are straightforward for GL_n can be subtle for general reductive groups. For this strategy to made sense, we will have to answer the following questions:

Q1. How to classify semi-simple mod p Galois representations?

Q2. How to lift semi-simple mod p Galois representations using the classification theorem?

Q3. Is an extension of crystalline lifts still crystalline?

For GL_n, the answers are simple. To answer Q1, we just need to realize semi-simple mod p Galois representations are always direct sums of inductions of characters; for Q2, note that inductions of crystalline characters lift semi-simple mod p Galois representations; Q3 is slightly more involved and by computing Galois cohomology involving Fontaine's period rings, we get an affirmative answer under an ordinarity assumption. Now let us turn to general reductive groups. To classify semi-simple mod p Galois representations, we have to make sense of what "induction of characters" means. It turns out the correct analogue of "character" is a Galois representation valued in a maximal torus, and "induction" should be interpreted as a choice of a Weyl group element. For GL_n, the Weyl group is the permutation group S_n and thus there is a unique way (up to conjugacy) of inducing an irreducible representation, namely, choosing the longest permulation $(123\cdots n)$. For general reductive groups, there are multiple ways of inducing irreducible representations from a "character". To prove this gives a complete classification, we will have to prove a Steinberg-Winter style theorem for maximal tori. We explain in section 3.1 how dynamic methods can be used to attack Steinberg-Winter type questions. Q2 now becomes a linear algebra question: we want to construct a solution of a system of linear equations lifting a fixed solution of the linear system modulo some ideal (the coefficient ring being the group ring of the Weyl group). Q3 now reduces to a non-abelian Galois cohomology question. We will carry out these non-abelian Galois cohomology computations in chapter 2 and answer Q3 affirmatively. We prove the following theorem:

Theorem 1 (3.5, 4.2.3, 3.8)**.** *Let $\mathbb{F}$ be a finite field of characteristic p. Let $K/\mathbb{Q}_p$ be a finite extension, and let G be a split reductive group over $W(\mathbb{F})$. Write K^{ur} for the maximal unramified extension of K inside a fixed algebraic closure.*

Let $\bar{\rho} : G_K \to G(\mathbb{F})$ be a group homomorphism whose image is a G-completely reducible subgroup.

- *There exists a characteristic* 0 *lift* $\rho : G_K \to G(W(\mathbb{F}))$ *of* $\bar{\rho}$*;*

- *There exists a Hodge-Tate regular crystalline lift* $\rho : G_K \to G(K^{\mathrm{ur}})$ *of* $\bar{\rho}$*.*

Now we can proceed to the second step of this strategy. The major challenge is, since the maximal parabolics of general reductive groups dont't have an abelian unipotent radical, the extension class is valued in *non-abelian Galois cohomology*. The arguments of [EG19] are geometric, and make use of the sheaf of the second Galois cohomology module over the moduli stack of mod p Galois representations. Non-abelian cohomology certainly does not nicely live in families, and even if one can make sense of a non-abelian "sheaf" of obstructions, we need brand new ideas to make the strategy work in a non-abelian setup.

To explain our approach, we take a step back and look at the first non-trivial example GL_3. We observe that, as a consequence of local Tate duality, the obstruction of lifting an extension class valued in the Heisenberg group vanishes if the obstruction of lifting the extension class modulo the center of the Heisenberg group vanishes. A linear algebraic group is Heisenberg if it is abelian modulo its center and its center is the affine line $\mathbb{G}_a$. It turns out that this observation is useful for reductive groups of type B, C, D, and G_2. For example, unipotent radicals of maximal parabolics of GSp_4 are either abelian or Heisenberg. For higher rank groups, the unipotent radicals might not be Heisenberg but one can replace them by their Heisenberg quotient.

Now we know for many reductive groups we can replace non-abelian Galois cohomology by a certain form of abelian Galois cohomology, and it makes it possible to adapt the arguments of [EG19]. Let us briefly review the geometric arguments of [EG19]. By choosing a suitable cochain complex computing Galois cohomology, one can reduce the problem of constructing crystalline lifts to the following linear algebra problem:

Principle 1. *Let $A_0 x = 0$ be a system of linear equations with a solution α. Let $A(t)$, $t \in T$ be deformations of the coefficient matrix A_0, parameterized by a space T. Assume $A(t)$ is "generically of maximal rank" over T. Then there exists $t \in T$ such that $A(t)x = 0$ admits a solution which is a deformation of α.*

When we say $A(t)$ is "generically of maximal rank" over T, we mean the locus in T where $\operatorname{corank} A(t) \geq t$ has codimension greater than t (in particular, the maximal rank locus is dense). [EG19] used geometric arguments to reduce the "generically of maximal rank" check to a combinatorial problem. So to adapt the proof in [EG19], we need to do the following things:

T1. Show that the Heisenberg trick works for higher rank groups (not just GL_3);

T2. Solve the combinatorial problem.

I used the theory of Demuškin groups, and explicit generators and relators to work out T1 for p not too small. (To be more precise, p is assumed to be coprime to the cardinality of the Weyl groups of proper Levi subgroups of G. This assumption is a byproduct of a prime-to-p descent argument, and can be removed if we know more about the integral structure of cup products on Galois cohomology.) It is the most technical and non-conceptual part of the proof. The combinatorial problem is also much more complicated for general groups, and we will only work out the G_2-case in this thesis.

In the next section, we will elaborate on how the non-abelian issue is addressed.

1. Obstruction theory

Let K be a p-adic field, and let G be a connected reductive group over $\mathbb{Z}_p$.

If $\bar{\rho}(G_K)$ is an irreducible subgroup of $G(\bar{\mathbb{F}}_p)$ (that is, it is not contained in any proper parabolic subgroup of $G(\bar{\mathbb{F}}_p)$), then by Theorem 1, $\bar{\rho}$ admits a crystalline lift. So we assume $\bar{\rho}$ factors through a proper maximal parabolic P. Let $P = L \rtimes U_P$ be the Levi decomposition. Let $\bar{r} : G_K \to L(\bar{\mathbb{F}}_p)$ be the Levi factor of $\bar{\rho}$. Then $\bar{\rho}$ defines a 1-cocycle $[\bar{c}] \in H^1(G_K, U_P(\bar{\mathbb{F}}_p))$. What we will actually do is to construct a lift $[c] \in H^1(G_K, U_P(\bar{\mathbb{Z}}_p))$ of $[\bar{c}]$.

In the GL_N-case, all maximal parabolics have abelian unipotent radical, so it suffices to consider abelian cohomology. When G is not GL_N, parabolic subgroups with abelian unipotent radical are rare. For example, when G is the exceptional group G_2, all parabolics have non-abelian unipotent radical.

In this thesis, we consider the case where U_P admits a quotient U such that

- The adjoint group $U^{\mathrm{ad}} := U/Z(U)$ is abelian;
- The center $Z(U)$ is isomorphic to $\mathbb{G}_a$; and
- The Galois action descends to U and there is a bijection of obstructions "$H^2(G_K, U_P(\bar{\mathbb{F}}_p))$" $\cong$ "$H^2(G_K, U(\bar{\mathbb{F}}_p))$".

We call U a Heisenberg quotient of U_P. When G is of type B, C, D or G_2, it is always possible to choose a parabolic P whose unipotent radical admits a Heisenberg quotient (see subsection 1.2).

Let $\operatorname{Spec} R$ be an irreducible component of a crystalline lifting ring $\operatorname{Spec} R_{\bar{r}}^{\mathrm{crys},\underline{\lambda}}$ (Definition 4.0.1) of $\bar{r}$. Let $r^{\mathrm{univ}} : G_K \to L(R)$ be the universal family. The Levi factor group acts on U via conjugation $\phi : L \to \mathrm{Aut}(U)$. Write $\phi^{\mathrm{ad}} : L \to \mathrm{GL}(U^{\mathrm{ad}})$ and $\phi^z : L \to \mathrm{GL}(Z(U))$ for the graded pieces of ϕ.

The theorem we prove is:

Theorem 2 (4.2.1)**.** *Let $[\bar{c}] \in H^1(G_K, U(\mathbb{F}))$ be a characteristic p cocycle.*

Assume

[1] $H^2(G_K, \phi^{\mathrm{ad}}(r^{\mathrm{univ}}))$ *is sufficiently generically regular (Definition 4.1.1);*

[2] $p \neq 2$*;*

[3] There exists a finite Galois extension K'/K *of prime-to-p degree such that* $\phi(\bar{r})|_{G_{K'}}$ *is Lyndon-Demuškin (Definition 1.0.2); and*

[4] There exists a $\bar{\mathbb{Z}}_p$*-point of* $\operatorname{Spec} R$ *which is mildly regular (Definition 2.0.1) when restricted to* $G_{K'}$*.*

Then there exists a $\bar{\mathbb{Z}}_p$*-point of* $\operatorname{Spec} R$ *which gives rise to a Galois representation* $r^\circ : G_K \to L(\bar{\mathbb{Z}}_p)$ *such that if we endow* $U(\bar{\mathbb{Z}}_p)$ *with the* G_K*-action* $G_K \xrightarrow{r^\circ} L(\bar{\mathbb{Z}}_p) \xrightarrow{\phi} \mathrm{Aut}(U(\bar{\mathbb{Z}}_p))$*, the cocycle* $[\bar{c}]$ *has a characteristic* 0 *lift* $[c] \in H^1(G_K, U(\bar{\mathbb{Z}}_p))$*.*

Remark [3] is automatically satisfied if p is sufficiently large; and [4] should be regarded as an induction hypothesis.

1.0.1. Example: $G = \mathrm{GL}_3$

Let $\bar{\rho} : G_K \to \mathrm{GL}_3(\bar{\mathbb{F}}_p)$ be a Galois representation. There are two ways of encoding the data of $\bar{\rho}$ as a 1-cocycle in Galois cohomology.

(I) Use the fact $\bar{\rho}$ factors through a maximal parabolic

$$P = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix} \ltimes \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} = L \ltimes A$$

where $A \cong \mathbb{G}_a^{\oplus 2}$ is a rank-2 abelian group. Let $\bar{r} : G_K \xrightarrow{\bar{\rho}} P(\bar{\mathbb{F}}_p) \to L(\bar{\mathbb{F}}_p)$ be the Levi factor of $\bar{\rho}$. The information of $\bar{\rho}$ is encoded in a 1-cocycle $[\bar{c}] \in H^1(G_K, \phi(\bar{r})) =: H^1(G_K, A(\bar{\mathbb{F}}_p))$. We first construct a lift $r^\circ : G_K \to (\mathrm{GL}_2 \times \mathrm{GL}_1)(\bar{\mathbb{Z}}_p)$ of $\bar{r}$. Then we construct a lift $[c] \in H^1(G_K, A(\bar{\mathbb{Z}}_p))$ of $[\bar{c}]$.

(II) Use the fact $\bar{\rho}$ factors through a Borel (minimal parabolic)

$$B = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \ltimes \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} = T \ltimes H$$

where the Levi group T is a maximal torus, and the unipotent radical H is the Heisenberg group. Let $\bar{r} : G_K \to T(\bar{\mathbb{F}}_p)$ be the Levi factor of $\bar{\rho}$. To reconstruct $\bar{\rho}$ from $\bar{r}$, we only need the information of a 1-cocycle $[\bar{c}] \in H^1(G_K, H(\bar{\mathbb{F}}_p))$. We first construct a lift of $\bar{r}$, and then construct a lift of $\bar{c}$. Now $H^1(G_K, H(\bar{\mathbb{F}}_p))$ is non-abelian Galois cohomology.

We make use of the graded structure of $\mathrm{Lie}\, H$ when we construct a lift of $[\bar{c}]$. We have a short exact sequence

$$1 \to \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to H \to \begin{bmatrix} 1 & * & \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \to 1.$$

We will first construct a lift modulo $Z(H)$, and then extend the lift modulo $Z(H)$ to a cocycle on the whole unipotent radical H.

Theorem 2 applies in this situation, so we have a new proof for the group GL_3. For higher ranks groups, the inductive step requires the full strength of the Emerton-Gee stacks, so we don't get a new proof for GL_N, $N > 3$.

1.0.2. We have a short exact sequence of groups $0 \to Z(U) \to U \to U^{\mathrm{ad}} \to 0$. Since $Z(U)$ is a central, normal subgroup, we have a long exact sequence of pointed sets

$$H^1(G_K, Z(U)) \to H^1(G_K, U) \to H^1(G_K, U^{\mathrm{ad}}) \xrightarrow{\delta} H^2(G_K, Z(U)).$$

Note that δ is a quadratic form, and there is an associated bilinear form

$$\cup : H^1(G_K, U^{\mathrm{ad}}) \times H^1(G_K, U^{\mathrm{ad}}) \to H^2(G_K, Z(U))$$

defined by $x \cup y = (\delta(x+y) - \delta(x) - \delta(y))/2$.

The technical heart of our method is an analysis of $\cup$ on the cochain/cocycle level. So we need a finite cochain complex computing Galois cohomology which interacts nicely with the bilinear form $\cup$. Thanks to the theory of Demuškin groups, there is an explicitly defined cochain complex (the so-called Lyndon-Demuškin complex) which computes $H^\bullet(G_{K'}, U^{\mathrm{ad}})$ and $H^\bullet(G_{K'}, Z(U))$ after a finite Galois extension K'/K. When $[K' : K]$ is prime to p, we can fully understand cup products on the cochain/cocycle level via Lyndon-Demuškin complexes endowed with $G_K/G_{K'}$-action.

We have the following nice obstruction theory:

Theorem 3 (3.3.4)**.** *Assume $p \neq 2$. Let L be a reductive group over $\mathcal{O}_E$ and fix an algebraic group homomorphism $L \to \mathrm{Aut}(U)$. Let $r : G_K \to L(\mathcal{O}_E)$ be a Galois representation.*

If there exists a finite Galois extension K'/K of prime-to-p degree such that $r|_{G_{K'}}$ is Lyndon-Demuškin and mildly regular, then there is a short exact sequence of pointed sets

$$H^1(G_K, U(\bar{\mathbb{Z}}_p)) \to H^1(G_K, U(\bar{\mathbb{F}}_p)) \xrightarrow{\delta} H^2(G_K, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$$

where δ has a factorization $H^1(G_K, U(\bar{\mathbb{F}}_p)) \to H^1(G_K, U^{\mathrm{ad}}(\bar{\mathbb{F}}_p)) \to H^2(G_K, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$.

1.1. The existence of crystalline lifts for G_2

The exceptional group G_2 has (up to conjugacy) two maximal parabolics: the short root parabolic, and the long root parabolic. When $\bar{\rho} : G_K \to G_2(\bar{\mathbb{F}}_p)$ factors through the short root parabolic, we can directly apply Theorem 2 to construct a crystalline lift of $\bar{\rho}$. When $\bar{\rho} : G_K \to G_2(\bar{\mathbb{F}}_p)$ factors through the long root parabolic, we can apply Theorem 2 to construct a lift modulo the center of the unipotent radical U_P. If we assume furthermore that $\bar{\rho}$ does not factor through the short root parabolic, then by Tate local duality, $H^2(G_K, Z(U_P)(\mathbb{F})) = 0$, and it is unobstructed to extend the lift to the whole unipotent radical. Putting everything together, we have the following theorem:

Theorem 4 (6.2)**.** *Assume $p > 3$. Every mod ϖ Galois representation valued in the exceptional group G_2*

$$\bar{\rho} : G_K \to G_2(\bar{\mathbb{F}}_p)$$

admits a crystalline lift $\rho^\circ : G_K \to G_2(\bar{\mathbb{Z}}_p)$.

Moreover, if $\bar{\rho}$ factors through a maximal parabolic $P = L \ltimes U_P$ and the Levi factor $\bar{r}_{\bar{\rho}} : G_K \to L(\bar{\mathbb{F}}_p)$ of $\bar{\rho}$ admits a Hodge-Tate regular and crystalline lift $r_1 : G_K \to L(\bar{\mathbb{Z}}_p)$ such that the adjoint representation $G_K \xrightarrow{r_1} L(\bar{\mathbb{Z}}_p) \to \mathrm{GL}(\mathrm{Lie}(U_P(\bar{\mathbb{Z}}_p)))$ has Hodge-Tate weights slightly less than $\underline{0}$, then ρ° can be chosen such that it factors through the maximal parabolic P and its Levi factor r_{ρ° lies on the same irreducible component of the spectrum of the crystalline lifting ring that r_1 does.

1.2. Crystalline lifting for classical groups

A maximal parabolic P of a (split) classical group $G = G(V, \langle , \rangle)$ of type B_l, C_l or D_l is the stablizer of an isotropic subspace $F \subset V$.

Let U_P be the unipotent radical of P. The Levi factor L of P acts on U_P by conjugation. Let λ be the similitude character of G. Fix an isomorphism $\iota : F \cong V/F^{\perp}$ such that $\langle x, \iota(x) \rangle = 1$ for all x. Then $\iota \otimes \lambda^{-1}$ is L-equivariant.

It is easy to see $U_P^{\mathrm{ad}} := U_P/Z(U_P) \cong \mathrm{Hom}_{\text{Vector space scheme}}(F^{\perp}/F, F)$ is an abelian group. Moreover, there exists an L-equivariant isomorphism

$$Z(U_P) = \mathrm{Hom}_{\text{Vector space scheme}}(V/F^{\perp}, F) \otimes \lambda^{-1}.$$

Note that ι induces an abelian group scheme morphism $\mathrm{tr}_{\iota} : Z(U_P) \to \mathbb{G}_a \otimes \lambda^{-1}$ which is L-equivariant. The quotient group $U := U_P/\ker \mathrm{tr}_{\iota}$ is a Heisenberg quotient, and thus our obstruction theory applies to it.

Let $\bar{\rho} : G_K \to P(\mathbb{F})$ be a mod ϖ Galois representation with Levi factor $\bar{r} : G_K \to L(\mathbb{F})$. We only need to consider the case where $\bar{\rho}$ is irreducible when restricted to the isotropic subspace F (otherwise replace F by a smaller isotropic subspace). In this case, tr_{ι} induces an isomorphism

$$H^2(G_K, Z(U_P)(\mathbb{F})) \cong H^2(G_K, \lambda^{-1} \otimes \mathbb{F})$$

by Schur's lemma and local Tate duality. By the long exact sequence of Galois cohomology, we have a bijection of obstructions $H^2(G_K, U_P(\bar{\mathbb{F}}_p)) \cong H^2(G_K, U(\bar{\mathbb{F}}_p))$ and thus the existence of crystalline lifts of $\bar{\rho}$ is equivalent to the existence of crystalline lifts of $\bar{\rho}$ modulo $\ker \mathrm{tr}_{\iota}$.

So Theorem (2) and Theorem (3) suffice for classical groups. To establish the existence of crystalline lifts of mod ϖ representations valued in classical groups, we only need to establish a codimension estimate on the moduli stack of (ϕ, Γ)-modules valued in classical groups in the manner of section 4.5. Our method requires p to be coprime to the cardinality of the Weyl groups of proper Levi subgroups of G (due to Lemma 2.2.2.1, although the assumption on p can possibly be relaxed by Jannsen-Wingberg theory).

Chapter 2

Extension of crystalline representations

1. Preliminaries

1.1. ***p*-adic Hodge theory** Let $E/\mathbb{Q}_p$ be a p-adic field with ring of integers $\mathcal{O}$. Let G be a smooth connected group over $\mathcal{O}$. Let $\rho : G_K \to G(\mathcal{O})$ be a group homomorphism. We say ρ is *crystalline* if for all algebraic representations $G \to GL_N$, the composition $G_K \to GL_N(\mathcal{O})$ is crystalline in the usual sense.

1.1.1. Theorem The representation ρ is crystalline if and only if for some faithful embedding $G \to \mathrm{GL}_N$ and some finite extension L/E, the map $I_K \to G_K \xrightarrow{\rho} G(\mathcal{O}) \to G(\mathcal{O}_L) \to \mathrm{GL}_N(\mathcal{O}_L)$ is crystalline.

Proof. By [Le13, 5.3.2], we only need to check a single faithful embedding $G \hookrightarrow \mathrm{GL}_N$. By [BC08, 9.3.1], we only have to look at the inertia. □

1.1.2. Lemma Let T be a smooth connected subgroup of G. Assume $\rho : G_K \to G(\mathcal{O})$ factors through $T(\mathcal{O})$. Then ρ is crystalline as a G-valued representation if and only if it is crystalline as a T-valued representation.

Proof. Choose an embedding $G \hookrightarrow \mathrm{GL}_N$. The lemma follows by applying the theorem above twice. $\square$

1.2. Exact $\otimes$-filtrations We review notions that are necessary for our general construction.

Let $\mathscr{C}$ be an ind-tannakian category ([SN72, III 1.1.1]) over a ring A. Let Vect_A be the category of projective A-modules. Let $\omega : \mathscr{C} \to \mathrm{Vect}_A$ be an exact tensor functor. For $X \in \mathrm{Vect}_A$, a filtration of X indexed by $\mathbb{Z}$ is an tuple $(\mathrm{Fil}^n X)_{n\in\mathbb{Z}}$ where $\mathrm{Fil}^n X \in \mathrm{Vect}_A$, $\mathrm{Fil}^n X \supset \mathrm{Fil}^{n+1} X$, $\cap \mathrm{Fil}^n X = 0$, and $\cup \mathrm{Fil}^n X = X$. Let $\mathrm{Fil}\,\mathrm{Vect}_A$ be the category of filtered (indexed by $\mathbb{Z}$) projective A-modules.

An *exact $\otimes$-filtration* F on ω is a factorization

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "\omega"] \arrow[dr, "\tilde{\omega}"'] & & \mathrm{Vect}_A \\ & \mathrm{Fil}\,\mathrm{Vect}_A \arrow[ur, "\text{forget}"'] & \end{tikzcd}$$

such that the following are satisfied:

(FE 1) For $X \in \mathscr{C}$, $\mathrm{Fil}^n \tilde{\omega}(X)$ is a direct summand of $\omega(X)$;

(FE 2) The associated graded functor $\mathrm{gr}_F(\tilde{\omega})$ is exact.

(FE 3) For all $n \in \mathbb{Z}$, $X, Y \in \mathscr{C}$,

$$\mathrm{Fil}^n \tilde{\omega}(X \otimes Y) = \sum_{i+j=n} \mathrm{Fil}^i \tilde{\omega}(X) \otimes \mathrm{Fil}^j \tilde{\omega}(Y).$$

Let ω, ω' be exact tensor functors with exact $\otimes$-filtrations F, F', respectively. Denote by

$$\text{Isom-fil}^{\otimes}((\omega, F), (\omega', F'))$$

the functor of tensor isomorphisms inducing an isomorphism of filtrations. Set

$$\text{Aut}^{\otimes}(F) := \text{Isom-fil}^{\otimes}((\omega, F), (\omega, F))$$

for simplicity of notation. Denote by $\text{Aut}^{\otimes !}(F)$ the subfunctor of $\text{Aut}^{\otimes}(F)$ which induces the identity of the associated grading.

1.3. Splitting of exact $\otimes$-filtrations Let $\text{gr}\,\text{Vect}_A$ be the category of graded vector spaces. An exact tensor functor $\omega : \mathscr{C} \to \text{gr}\,\text{Vect}_A$ induces a canonical exact $\otimes$-filtration F_{can}, which is defined as $\text{Fil}^n(\omega(X)) := \sum_{n' \geq n} \text{gr}_{n'}\,\omega(X)$ for all $X \in \mathscr{C}$.

An exact $\otimes$-filtration is said to be *splittable* if it is isomorphic to the canonical exact $\otimes$-filtration associated to a graded exact tensor functor.

2. Extension of weakly admissible filtered ϕ-G-torsors

2.1. Filtered ϕ-G-torsors and crystalline representations

2.1.1. Definition Let $K/\mathbb{Q}_p$ be a finite extension. Let $K_0 = W(k)[1/p]$ where k is the residue field of K. Let E be a sufficiently large coefficient field (admitting an embedding of the normal closure of K). A *filtered ϕ-module with coefficients in E* is a triple (D, ϕ_D, θ_D) where

- D is a finite free module over $K_0 \otimes_{\mathbb{Q}_p} E$;

- $\phi_D : (\phi \otimes 1)^* D \to D$ is an isomorphism of $K_0 \otimes_{\mathbb{Q}_p} E$-modules;

- θ_D is a filtration on $D_K := D \otimes_{K_0} K$ such that $\theta_D^j D_K = 0$ if $j \gg 0$, and $\theta_D^j D_K = D_K$ if $j \ll 0$.

Here $\phi \otimes 1 : K_0 \otimes E \to K_0 \otimes E$ sends $x \otimes y$ to $\phi(x) \otimes y$.

2.1.2. Definition A *filtered ϕ-G-torsor with coefficients in E* is a triple (T, ϕ_T, θ_T) such that

- T is a G-torsor over $\operatorname{Spec} K_0 \otimes E$;

- $\phi_T : (\phi \otimes 1)^* T \to T$ is a G-equivariant isomorphism over $\operatorname{Spec} K_0 \otimes_{\mathbb{Q}_p} E$;

- θ_T is an exact $\otimes$-filtration on $T_K := T \underset{\operatorname{Spec} K_0 \otimes_{\mathbb{Q}_p} E}{\times} \operatorname{Spec} K \otimes_{\mathbb{Q}_p} E$.

More precisely, θ_T is an exact $\otimes$-filtration on the functor $\operatorname{Rep}(G) \to \operatorname{Vect}_{K \otimes E}$ defined by $V \mapsto T_K \times^G V$. By Tannakian theory, a G-torsor always comes from a rigid exact $\otimes$-functor $\operatorname{Rep}(G) \to \operatorname{Vect}_{K \otimes E}$, so we don't distinguish them.

2.1.3. Remark (1) We can define the notion of filtered ϕ-G-torsor with coefficient in E for any smooth E-group scheme G.

(2) By Tannakian theory and the functoriality of the twisted product $- \times^G -$, a filtered ϕ-G-torsor (T, ϕ_T, θ_T) is nothing but a rigid exact tensor functor from $\operatorname{Rep}_G(E)$ to the category of filtered ϕ-modules with coefficients in E.

For ease of notation, we write $- \otimes -$ for $- \otimes_{\mathbb{Q}_p} -$.

2.1.4. Pushforward Let $f : G \to H$ be a group scheme morphism.

Let (T, ϕ_T, θ_T) be a filtered ϕ-G-torsor. Define $T' := T \times^G H := T \times H / \{(t,h) \sim (g^{-1} \cdot t, f(g) \cdot h), g \in G, t \in T, h \in H\}$. Then T' is an H-torsor with H-action defined by $h \cdot (t, h') = (t, hh')$. Since $(\phi \otimes 1)^* T \times^G H \cong (\phi \otimes 1)^*(T \times^G H)$ canonically, we can define $\phi_{T'} := \phi \times^G H : (\phi \otimes 1)^* T' \to T'$. Recall that θ_T is a functor $\mathrm{Rep}(G) \to \mathrm{Vect}_{K \otimes E}$. Define $\theta_{T'} := f_*(\theta_T)$ to be the composite $\mathrm{Rep}(H) \xrightarrow{f^*} \mathrm{Rep}(G) \to \mathrm{Vect}_{K \otimes E}$. The triple $(T', \phi_{T'}, \theta_{T'})$ is a filtered ϕ-H-torsor. We write $f_*(T, \phi_T, \theta_T)$ for $(T', \phi_{T'}, \theta_{T'})$.

2.1.5. Framing Let (T, ϕ_T, θ_T) be a filtered ϕ-G-torsor. Suppose the underlying G-torsor T is a trivial G-torsor, there exists a canonical embedding

$$\iota : T(K_0 \otimes E) \hookrightarrow T(K_0 \otimes E) \times_{\{\mathrm{pt}\}, \phi \otimes 1} \{\mathrm{pt}\} = (\phi \otimes 1)^* T(K_0 \otimes E), \quad \{\mathrm{pt}\} = (\mathrm{Spec}\, K_0 \otimes E)(K_0 \otimes E)$$

A *framing* of T is an element $\xi \in T(K_0 \otimes E)$. Since T is a G-torsor, there exists a unique element $X_\xi \in G(K_0 \otimes E)$ such that $\phi_T(\iota(\xi)) = X_\xi \cdot \xi$.

Let $g \in G(K_0 \otimes E)$. Now we change the framing from ξ to $g \cdot \xi$. We have $\phi_T(\iota(g \cdot \xi)) = \phi(g) \phi_T(\iota(\xi)) = \phi(g) X_\xi \cdot \xi = \phi(g) X_\xi g^{-1} g \cdot \xi$. Therefore

$$X_{g \cdot \xi} = \phi(g) X_\xi g^{-1}.$$

Let $f : G \to H$ be a group scheme homomorphism. Let $\xi \in T(K_0 \otimes E)$ be a framing. Then

$f_*(\xi) \in (T \times^G H)(K_0 \otimes E)$ is a framing of $f_*(T, \phi_T, \theta_T)$. It is easy to see that

$$X_{f_*\xi} = f(X_\xi).$$

2.1.6. Weak admissibility For simplicity, we define the weak admisibility of a filtered ϕ-G-torsor via Tannakian theory. A filtered ϕ-G-torsor T is *weakly admissible* if for any algebraic representation $G \to GL(V)$, the twisted product $T \times^G V$ is a weakly admissible filtered ϕ-module.

2.1.7. Crystalline representations Since the covariant Fontaine's functors V_{cris} and D_{cris} are rigid exact tensor functors (see the paragraph before [C11, 9.1.9]), the category of weakly admissible filtered ϕ-G-torsors is equivalent to the category of crystalline representations valued in G. We also denote by V_{cris} and D_{cris} the equivalences of categories in the G-valued case.

2.2. Parabolic liftings

Let P be a parabolic subgroup of G with unipotent radical U and Levi factor L. Let $\pi_L : P \to L$ be the quotient map. Let $(\bar{T}, \phi_{\bar{T}}, \theta_{\bar{T}})$ be a fixed filtered ϕ-L-torsor with coefficients in E.

Define

$$\begin{aligned} \mathrm{Lift}(\bar{T}) &= \mathrm{Lift}(\bar{T}, \phi_{\bar{T}}, \theta_{\bar{T}}) \\ &= \{(T, \phi_T, \theta_T) : \textit{filtered } \phi\textit{-}P\textit{-torsors valued in } E \textit{ such that } \ (\pi_L)_* T = \bar{T}\}/\sim \end{aligned}$$

where the equivalence relation $\sim$ is defined to be isomorphisms of filtered ϕ-P-torsors respecting $(\pi_L)_*$.

2.2.1. Throughout this section, we assume $\bar{T}$ admits a framing $\bar{\xi} \in \bar{T}(K_0 \otimes E)$. By fixing $\bar{\xi}$, we also fixed a framing of $\iota_* \bar{T}$ for various sections $\iota : L \hookrightarrow P$.

In particular, for two different sections $\iota_1, \iota_2 : L \to P$, the two P-torsors $\bar{T} \times^{L,\iota_1} P$ and $\bar{T} \times^{L,\iota_2} P$ are identified without further mention.

Moreover, since the base scheme is a disjoint union of spectra of perfect fields, any element of $\mathrm{Lift}(\bar{T})$ admits a framing ([Se02, Proposition 6, III.2.1]).

2.2.2. Lemma Let $T \in \mathrm{Lift}(\bar{T})$. Then there exists a section $\iota : L_{K\otimes E} \hookrightarrow P_{K\otimes E}$ such that

$$\theta_T = \iota_*(\theta_{\bar{T}})$$

(ι is a group scheme morphism and $\pi_L \circ \iota = \mathrm{id}$.)

Proof. Since G is smooth and the coefficient ring is of characteristic 0, the exact $\otimes$-filtration θ_T is Zariski-locally splitable on $\operatorname{Spec} K \otimes E$ ([SN72, IV.2.4]). Since $K \otimes E$ is a direct sum of fields, θ_T is splittable. So θ_T is the canonical filtration associated to a graded tensor functor, or equivalently a cocharacter $\omega : (\mathbb{G}_{\mathrm{m}})_{K\otimes E} \to P_{K\otimes E}$ ([SN72, IV.1.3]). We choose an arbitrary embedding $L_{K\otimes E} \subset P_{K\otimes E}$. The image of ω is contained in a maximal torus, and hence contained in a conjugate of $L_{K\otimes E}$ (say $L_{K\otimes E} = \coprod_{i:K\hookrightarrow E} L_{E_i}$, the image of $\omega \otimes E_i$ is contained in a conjugate of L_{E_i}).

Choose a section $\iota : L_{K\otimes E} \to P_{K\otimes E}$ such that $\omega(\mathbb{G}_{\mathrm{m}}) \subset \iota(L_{K\otimes E})$. We have $\iota_*(\pi_L)_*(\omega) = \omega$. Now it is clear that $\theta_{\bar{T}}$ is the canonical exact $\otimes$-filtration associated to $(\pi_L)_*(\omega)$, and $\theta_T = \iota_*(\theta_{\bar{T}})$ $\square$.

2.2.3. The adjoint filtered ϕ-module Recall that the upper central series of U defines a filtration

$$\{1\} = U_s \subset U_{s-1} \subset \cdots \subset U_0 = U$$

such that each of $\mathrm{gr}_i\, U := U_i/U_{i+1}$ is abelian. We have

$$\mathrm{Lie}\, U = \bigoplus_{i=1}^{s} \mathrm{gr}_i\, U = \mathrm{gr}_\bullet\, U$$

Since $P = L \ltimes U$, a section $L \to P$ induces an (adjoint) action $L \curvearrowright U$. Let $\mathrm{ad} : L \to \mathrm{Aut}(U)$ be the induced group scheme homomorphism. Note that the abelianization $\mathrm{gr}^\bullet(\mathrm{ad}) : L \to \mathrm{Aut}(\mathrm{Lie}\, U)$ does not depend on the choice of $L \to P$.

Define

$$\mathrm{gr}^\bullet(\mathrm{ad})\bar{T} := \mathrm{gr}^\bullet(\mathrm{ad})_*(\bar{T}, \phi_{\bar{T}}, \theta_{\bar{T}})$$

2.2.4. Lemma If $\bar{T}$ is weakly admissible, then so is $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$.

Proof. By Tannakian theory (more precisely by the fact that the Tannakian category is generated by $\rho \otimes \rho^*$ where ρ is any faithful representation of the Tannakian group), a Galois representation $G_K \to \mathrm{Aut}(\mathrm{Lie}\, U)(E)$ is crystalline if and only if for some faithful algebraic representation $\mathrm{Aut}(\mathrm{Lie}\, U) \to \mathrm{GL}(V)$ the representation $G_K \to \mathrm{GL}(V(E))$ is crystalline. Since the composition $L \to \mathrm{Aut}\,\mathrm{Lie}(U) \to \mathrm{GL}(V)$ is an algebraic representation, $G_K \to \mathrm{GL}(V(E))$ is crystalline if $G_K \to L(E)$ is crystalline. The first claim is proved by passing to the category of crystalline representations via V_{cris}. □

2.3. G-ordinarity

2.3.1. Newton polygon of isocrystals Let $\breve{K}$ be the p-adic completion of the maximal unramified extension of K_0. By the Diedonné-Manin classification, the category of isocrystals over $\breve{K}$ is a semisimple category. The simple objects can be classified by rational numbers s/r, where r is a positive integer and s is an integer coprime to r. Denote by $D_{r,s}$ the simple object labeled by the rational number s/r. $D_{r,s}$ has dimension r, and we call s/r the *slope* of $D_{r,s}$.

Let (D,ϕ) be an isocrystal over K_0. Then $\breve{D} = \breve{K} \otimes_{K_0} D$ is a direct sum of simple objects D_{r_i,s_i}. We call the numbers s_i/r_i that appear in the direct sum decomposition the slopes of D. Say D has slopes $\{\alpha_0 < \cdots < \alpha_n\}$ with multiplicities $\{\mu_0, \cdots, \mu_n\}$. The *Newton polygon* of D is the convex polygon with leftmost endpoint $(0,0)$, and having μ_i consecutive segments of horizontal distance 1 and slope α_i.

2.3.1.1 Lemma If all slopes of D are positive numbers, then for any lattice $\mathcal{L} \subset D$, we have

$$\lim_{n\to\infty} \phi^n \mathcal{L} = \{0\}$$

(in the sense that the diameter of the bounded sets $\phi^n\mathcal{L}$ converges to 0.) Note that $\mathcal{L}$ is not assumed to be ϕ-stable.

Proof. Let N be the product of the denominator of the slopes of D. By the Diedonné-Manin classification, there is a basis $\{x_1, .., x_t\}$ of $\breve{K} \otimes_{K_0} D$ such that $\phi^N x_i = p^{S_i} x_i$ for positive integers S_i, $1 \le i \le t$. The p-adic topology on D can be defined by

$$|\lambda_1 x_1 + \cdots + \lambda_t x_t| = \max_{1\le i\le t}(|\lambda_i|)$$

where $\lambda_i \in \breve{K}$, $1 \leq i \leq t$. For any $x \in D$, $|\phi^N x| < \frac{1}{p}|x|$. Therefore for any lattice $\mathcal{L}$, we have $\lim_{n\to\infty} \phi^{nN}\mathcal{L} = \{0\}$. Replacing $\mathcal{L}$ by $\phi^k\mathcal{L}$, $1 \leq k < N$, we have $\lim_{n\to\infty} \phi^{k+nN}\mathcal{L} = \{0\}$. Combining these, we have $\lim_{n\to\infty} \phi^n\mathcal{L} = \{0\}$. □

2.3.1.2 Corollary If the slopes of D are either all positive numbers or all negative numbers, the map $1 - \phi : D \to D$ is invertible.

Proof. If the slopes of D are all positive numbers, then $1 + \phi + \phi^2 + \cdots$ converges and is an inverse of $1 - \phi$.

If the slopes of D are all negative numbers, then the slopes of the dual isocrystal $D^\vee$ are all positive numbers. By choosing a basis of D, the matrix of $\phi^\vee$ is the transpose inverse of that of ϕ, and $(1 - \phi^{-t}) = -\phi^{-t}(1 - \phi^t)^{-1} = -\phi^{-t}(1 + \phi^t + \phi^{2t} + \cdots)$ is invertible. □

2.3.2. Definition Let (T, ϕ_T, θ_T) be a filtered ϕ-P-torsor. $\bar{T} = (\pi_L)_*T$ is a filtered ϕ-L-torsor. Note that since $\mathrm{Aut}(\mathrm{Lie}\, U)$ is a general linear group, $\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})$ is an filtered isocrystal with coefficients.

We say θ_T, T, or $\bar{T}$ is *G-ordinary* if the filtration $\theta_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}$ on the vector space $\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})$ satisfies $\theta^0_{\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}}(\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}) = 0$. In other words, the Hodge polygon of $\theta_{\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}}$ lies below the x-axis except for the left endpoint.

2.3.2.1 Lemma A weakly admissible filtered ϕ-G torsor T is G-ordinary if and only if all Hodge-Tate weights of the crystalline representation $V_{\mathrm{cris}}(\mathrm{gr}^\bullet(\mathrm{ad})\bar{T})$ are negative integers, where V_{cris} is the covariant Fontaine functor.

Proof. It is a standard p-adic Hodge theory calculation. See for example [C11, section 8.3]. □

2.3.3. Dynamic methods We need some results from [Crd11, Section 4.1]. Let X be a scheme over a base scheme S, and fix a $\mathbb{G}_{\mathrm{m}}$-action $m : \mathbb{G}_{\mathrm{m}} \times X \to X$ on X. For each $x \in S(S)$, we say

$$\lim_{t \to 0} m(t, x) \text{ exists,}$$

if the morphism $\mathbb{G}_{\mathrm{m}} \to X$, $t \mapsto m(t, x)$ extends a a morphism $\mathbb{A}^1 \to X$.

Let λ be a cocharacter of a reductive group G. Define the following functor on the category of $K \otimes E$-algebras $P_G(\lambda)(k) = \{g \in G(k) | \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists.}\}$ where k is a general $K \otimes E$-algebra.

$P_G(\lambda)$ is a smooth subgroup of G, and all parabolic subgroups of G are of the form $P_G(\lambda)$ for some λ.

Define $U_G(\lambda)(k) = \{g \in G(k) | \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} = 1\}$. Then $U_G(\lambda) \subset P_G(\lambda)$ is the unipotent radical.

Denote by $L_G(\lambda)$ the quotient $P_G(\lambda)/U_G(\lambda)$.

Let $f : G \to H$ be a group scheme morphism. We have induced group scheme morphisms $P_G(f) : P_G(\lambda) \to P_H(f_* \lambda)$ and $U_G(f) : U_G(\lambda) \to U_H(f_* \lambda)$ ([Crd11, Theorem 4.1.7]).

A cocharacter λ of G induces a filtration $F(\lambda)$ on the trivial G-torsor ([SN72, IV 2.1.5]).

2.3.3.1 Theorem Consider the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$. We have

$$\mathrm{Lie}\,\mathrm{Aut}^{\otimes}(F(\lambda)) = F(\lambda)^0(\mathrm{Lie}(G))$$

and

$$\mathrm{Lie}\,\mathrm{Aut}^{\otimes !}(F(\lambda)) = F(\lambda)^1(\mathrm{Lie}(G)).$$

As a consequence, we have $P_G(\lambda) = \mathrm{Aut}^{\otimes}(F(\lambda))$ and $U_G(\lambda) = \mathrm{Aut}^{\otimes !}(F(\lambda))$.

Proof. The first paragraph is a special case of [SN72, IV 2.1.4.1] where $\alpha = 0, 1$. The second paragraph follows from [Crd11, Theorem 4.1.7(4)]. □

2.3.4. Suppose $P_{K\otimes E} = P_G(\lambda)$ for some cocharacter λ of G. The cocharacter λ induces a filtration $F(\lambda)$ on $G_{K\otimes E}$.

Let (T, ϕ_T, θ_T) be a G-ordinary filtered ϕ-P-torsor whose underlying G-torsor is a trivial G-torsor. By Lemma 2.2.2, there exists an embedding $\iota : L_{K\otimes E} \to P_{K\otimes E}$ such that $\theta_T = \iota_*(\theta_{\bar{T}})$. We'll explicitly construct ι when T is G-ordinary and show that such an embedding is unique. Write i for the embedding $P_{K\otimes E} \to G_{K\otimes E}$.

2.3.4.1 Proposition There exists a unique embedding $\iota : L_{K\otimes E} \to P_{K\otimes E}$ such that $\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E} \subset \iota(L_{K\otimes E})$.

Proof. The intersection of two parabolics of a reductive group always contains a maximal torus ([M18, 19.33]). Let $S \subset \mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E}$ be a maximal torus. Let $S_0 \subset S$ be the maximal subtorus such that the centralizer $Z(S_0)$ is (an embedding of) the Levi factor $L_{K\otimes E}$ of $P_{K\otimes E}$.

Let $U_{K\otimes E}$ be the unipotent radical of $P_{K\otimes E}$. We have the Levi decomposition

$$\mathrm{Lie}\,P_{K\otimes E} = \mathrm{Lie}\,Z(S_0) \oplus \mathrm{Lie}(U_{K\otimes E}) = \mathrm{Lie}\,Z(S_0) \oplus \bigoplus_{\alpha \in \Phi^+(S_0, G)} \mathfrak{g}_\alpha.$$

where $\mathfrak{g}_\alpha$ is the α-weight space and $\Phi^+(S_0, G)$ is the set of weights occurring in $\mathrm{Lie}(U_{K\otimes E})$.

By Theorem 1.1, $\mathrm{Lie}\,\mathrm{Aut}^{\otimes}(i_*\theta_T) = (i_*\theta_T)^0(\mathrm{Lie}\,G)$. Since T is G-ordinary, we have $\theta^0_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}(\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})) = 0$. It is clear that the filtration $i_*\theta_T$ on $\mathrm{Lie}\,G$ and the filtration $\theta_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}$ on $\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T}) = \mathrm{Lie}(U_{K\otimes E})$ are compatible. So $i_*\theta^0_T(\mathrm{Lie}\,G) \cap \mathrm{Lie}(U_{K\otimes E}) = 0$.

Consider the S_0-weight decomposition of $\mathrm{Lie}\,\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap \mathrm{Lie}\,P_{K\otimes E}$. By the previous paragraph, there is no positive S_0-weights, and therefore $\mathrm{Lie}\,\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap \mathrm{Lie}\,P_{K\otimes E} \subset \mathrm{Lie}\,Z(S_0)$. So we've shown $\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E} \subset Z(S_0)$.

It remains to show the uniqueness of ι. Let $g \in P(K\otimes E)$ and suppose $\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E} \subset gZ(S_0)g^{-1}$. The Proposition follows from the fact that $S_0 \subset gZ(S_0)g^{-1}$ implies $gZ(S_0)g^{-1} = Z(S_0)$. □

2.3.4.2 Lemma If $\theta_T = \iota_*\theta_{\bar{T}}$, then $\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E} \subset \iota(L_{K\otimes E})$.

Proof. Choose a splitting ω of $\theta_{\bar{T}}$. Choose a maximal torus S of the centralizer of ω. By Theorem 1.1 and Theorem [Crd11, 4.1.7(4)], we have

$$\mathrm{Aut}^{\otimes}(i_*\theta_T) = \mathrm{Lie}\,P_G(\omega) = \bigoplus_{\langle\alpha,\omega\rangle \geq 0} \mathfrak{g}_\alpha$$

where α ranges from all S-roots of G, and $\mathfrak{g}_\alpha$ is the S-weight space of weight α. Meanwhile, $P_{K\otimes E} = P_G(\lambda)$ for some cocharacter $\lambda : \mathbb{G}_{\mathrm{m}} \to S$. Since T is G-ordinary, $\theta^0_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}(\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})) = 0$, which implies for any root $\alpha \in \Phi(S, G)$ such that $\langle\alpha,\lambda\rangle > 0$ we have $\langle\alpha,\omega\rangle < 0$. This lemma now follows from the S-weight decomposition of the Lie algebra of $\mathrm{Aut}^{\otimes}(i_*\theta_T) \cap P_{K\otimes E}$. □

2.3.5. Lemma If T is G-ordinary, the map $1 - \phi_{\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}}$ is invertible.

Proof. By Corallary 2.3.1, it suffices to show all slopes of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$ are negative numbers.

The G-ordinarity condition guarantees the Hodge polygon of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$ lies below the x-axis (except for the left endpoint which is the origin). Weak admissibility of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$ implies the Newton polygon of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$ and the Hodge polygon of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$ have the same right endpoint, and the Newton polygon of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$ lies on or above the Hodge polygon of $\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$. In particular, the largest slope of the Newton polygon is smaller or equal to the largest slope of the Hodge polygon. In other words, all slopes of the Newton polygon are negative numbers. □

2.3.6. Lemma Let T be a filtered ϕ-P-torsor. Assume T is weakly admissible and G-ordinary.

A section $\iota : L_{K_0\otimes E} \hookrightarrow P_{K_0\otimes E}$ induces a projection $P_{K_0\otimes E} \xrightarrow{\pi_U} U_{K_0\otimes E}$ and a decomposition $P_{K_0\otimes E} = L_{K_0\otimes E}U_{K_0\otimes E}$.

There exists a unique section ι such that $\phi_T = \iota_*(\phi_{\bar{T}})$. (See 2.2.1.)

Proof. For ease of notation, write L, P, U for $L_{K_0\otimes E}$, $P_{K_0\otimes E}$, and $U_{K_0\otimes E}$, respectively. Fix a framing ξ of T. Write $\bar{T}$ for $(\pi_L)_*T$.

We have fixed a framing $\bar{\xi} \in L(K_0 \otimes E)$ of $\bar{T}$. Choose a section $\iota_0 : L \hookrightarrow P$, which induces a projection $P \xrightarrow{\pi_{U,0}} U$. There exists a unique isomorphism of P-torsors $(\iota_0)_*\bar{T} \cong T$ under which $(\iota_0)_*\bar{\xi}$ is identified with ξ. We identify $(\iota_0)_*\bar{T}$ and T via this isomorphism. By remark 2.1.4, $\phi_T = (\iota_0)_*(\phi_{\bar{T}})$ if and only if $X_\xi = \iota_0(X_{\bar{\xi}})$. Or equivalently, $\pi_{U,0}(X_\xi) = 1$.

Set $M_0 := \pi_{U,0}(X_\xi)$, and $A_0 := \iota_0(\pi_L(X_\xi))$. Then $X_\xi = A_0M_0$ (we identify U as a normal subgroup of P). Let $\iota : L \hookrightarrow P$ be another section with induced projection $P \xrightarrow{\pi_U} U$. Set $A := \iota(\pi_L(X_\xi))$ and $M := \pi_U(X_\xi)$. Note that there exists $N \in U(K_0 \otimes E)$ such that $A = NA_0N^{-1}$ ([SN72, IV 2.2.5.3]). Since

$$AM = A_0M_0 = X_\xi$$

we have $NA_0N^{-1}M = A_0(A_0^{-1}NA_0N^{-1}M) = A_0M_0$, and thus

$$M_0 = \mathrm{Ad}_{A_0^{-1}}(N)N^{-1}M.$$

For ease of notation, we write $\phi_{\mathrm{Ad}} := \mathrm{Ad}_{A_0^{-1}}$, and $M_0 = \phi_{\mathrm{Ad}}(N)N^{-1}M$.

For an integer $1 \leq i \leq s$, write gr^i for the projection $U_{i-1} \to U_{i-1}/U_i$. We use additive notation when working with abelian groups.

We have

$$\mathrm{gr}_1 M = (1 - \phi_{\mathrm{Ad}})\,\mathrm{gr}_1(N) + \mathrm{gr}_1 M_0.$$

Note that $(1 - \phi_{\mathrm{Ad}})\,\mathrm{gr}_1(N) = (1 - \phi^{-1}_{\mathrm{gr}_\bullet\,\mathrm{ad}\,\bar{T}})\,\mathrm{gr}_1(N)$. By Lemma 2.3.5, $(1 - \phi_{\mathrm{Ad}}) : \mathrm{Lie}\,U \to \mathrm{Lie}\,U$ is invertible. We choose $N \in U$ such $\mathrm{gr}_1(N) = (\phi_{\mathrm{Ad}} - 1)^{-1}M_0$ (choosing ι is equivalent to choosing N). Hence we can arrange it so that $\mathrm{gr}_1 M = 0$.

Now we assume $\mathrm{gr}_1 M_0 = 0$, that is, $M_0 \in U_1$. We choose ι such that $N \in U_1$. We have

$$\mathrm{gr}_2 M = (1 - \phi_{\mathrm{Ad}})\mathrm{gr}_2(N) + \mathrm{gr}_2 M_0.$$

We can kill $\mathrm{gr}_2(M)$ in a similar manner. We repeat this process, and will ultimately kill M.

The uniqueness of ι is a byproduct of the proof of the existence part. In each step of the above algorithm, the choice is unique. $\square$

Denote by $\mathrm{Scin}(P)$ the set of sections $L \hookrightarrow P$. Note that $\mathrm{Scin}(P)$ is a U-torsor ([SN72, IV 2.2.5.3]).

2.3.7. Let $(\bar{T}, \phi_{\bar{T}}, \theta_{\bar{T}})$ be a filtered ϕ-L-torsor which is weakly admissible and G-ordinary (with respect to the parabolic P).

The following map

$$\delta : \mathrm{Scin}(P_{K_0\otimes E}) \times \mathrm{Scin}(P_{K\otimes E}) \to \mathrm{Lift}(\bar{T})$$

$$(\iota_\phi, \iota_\theta) \mapsto (T, (\iota_\phi)_*\phi_{\bar{T}}, (\iota_\theta)_*\theta_{\bar{T}})$$

is a surjection by Lemma 2.3.6 and Lemma 2.2.2.

Note that $U(K_0 \otimes E)$ acts diagonally on $\mathrm{Scin}(P_{K_0\otimes E}) \times \mathrm{Scin}(P_{K\otimes E})$.

2.3.7.1 Theorem We have bijections

$$\begin{aligned} \mathrm{Lift}(\bar{T}) &\cong \{\text{Orbits of } \mathrm{Scin}(P_{K_0\otimes E}) \times \mathrm{Scin}(P_{K\otimes E}) \text{ under } U(K_0 \otimes E) \text{ action}\} \\ &\cong \mathrm{Scin}(P_{K\otimes E}) \end{aligned}$$

Proof. We only need to show two liftings of $\bar{T}$ are equivalent if and only if they lie in the same $U_{K_0\otimes E}$-orbit.

Let $\delta(\iota^1_\phi, \iota^1_\theta)$ and $\delta(\iota^2_\phi, \iota^2_\theta)$ be two liftings. Let $h : \delta(\iota^1_\phi, \iota^1_\theta) \cong \delta(\iota^2_\phi, \iota^2_\theta)$ be an isomorphism of filtered ϕ-G-torsors whose pushforward along π_L is the identity map. Identify the underlying trivial $P_{K_0\otimes E}$-torsor with $P_{K_0\otimes E}$. Then h is just conjugation by an element u of the unipotent radical $U(K_0 \otimes E)$. In particular,

$$(\iota^1_\phi)_*\phi_{\bar{T}} = (u\iota^2_\phi u^{-1})_*\phi_{\bar{T}}, \quad (\iota^1_\theta)_*\theta_{\bar{T}} = (u\iota^2_\theta u^{-1})_*\theta_{\bar{T}}.$$

By Proposition 2.3.4 and Lemma 2.3.4, two different sections $\iota_\theta : L_{K\otimes E} \hookrightarrow P_{K\otimes E}$ gives two different filtrations $(\iota_\theta)_*\theta_{\bar{T}}$. Therefore we have $\iota_\theta^1 = u\iota_\theta^2 u^{-1}$. Similarly, by Lemma 2.3.6, we have $\iota_\phi^1 = u\iota_\phi^2 u^{-1}$. So $(\iota_\phi^1, \iota_\theta^1)$ and $(\iota_\phi^2, \iota_\theta^2)$ are in the same U-orbit, as desired. $\square$

The theorem above is reminiscent of the double complex computing the cohomology of filtered ϕ-modules.

2.3.8. Recall

$$\{1\} = U_s \subset U_{s-1} \subset \cdots \subset U_0 = U$$

is the upper central series of U.

2.3.8.1 Corollary Let T_i be a filtered ϕ-P/U_i-torsor for some $1 \leq i \leq s$, which can be lifted to a filtered ϕ-P-torsor. Assume $\bar{T} := T_i \mod U/U_i$ is a G-ordinary and weakly admissible filtered ϕ-L-torsor.

The set of filtered ϕ-P/U_{i+1}-torsors which lifts T_i and admits a lifting to to a filtered ϕ-P-torsor is an $\mathbb{Q}_p$-affine space isomorphic to $U_i(K \otimes E)/U_{i+1}(K \otimes E)$.

2.4. Crystallinity of parabolic liftings

2.4.1. Enough gaps in Hodge-Tate weights We say a crystalline representation $\rho : G_K \to L(E)$ has enough gaps in Hodge-Tate weights with respect to P if the adjoint representation

$$G_K \xrightarrow{\rho} L(E) \to \mathrm{Aut}(\mathrm{Lie}\, U)(E)$$

has labelled Hodge-Tate weights slightly less then $\underline{0}$ in the sense of [EG19, 6.3].

We remark that having enough gaps in Hodge-Tate weights is strictly stronger than being G-ordinary. More precisely, G-ordinarity does not require one of the inequalities in [EG19, 6.3] to be strict.

2.4.2. Proposition A filtered ϕ-P-torsor T is weakly admissible if the filtered ϕ-L-torsor $(\pi_L)_*(T)$ is weakly admissible with respect to P.

Proof. We first remark this proposition for general linear groups is a reformulation of the standard fact that the category of weakly admissible filtered ϕ-modules is an abelian category (see, for example, [C11, Proposition 8.2.10]).

Write $\bar{T} = (\pi_L)_*(T)$. The parabolic P of G is defined by a cocharacter λ of G.

Let $f : G \to \mathrm{GL}(V)$ be an algebraic representation. Then we have induced maps

$$P_G(\lambda)(f) : P = P_G(\lambda) \to P_{\mathrm{GL}(V)}(f_*\lambda) =: P'$$

and

$$L_G(\lambda)(f) : L = L_G(\lambda) \to L_{\mathrm{GL}(V)}(f_*\lambda) =: L'.$$

The filtered ϕ-L'-torsor $(L_G(\lambda))_*(\bar{T})$ is weakly admissible because $\bar{T}$ is weakly admisible. Write i for the embedding $P \subset G$. Since

$$(L_G(\lambda)(f))_*(\pi_L)_*T = (\pi_{L'})_*(P_G(\lambda)(f))_*T$$

using the result for $\mathrm{GL}(V)$ it follows that i_*T is weakly admissible as a filtered ϕ-G-torsor by standard Tannakian theory arguments. Again by Tannakian theory or more precisely the fact that the Tannakian category of representations of P are generated by a single faithful embedding, T is weakly admissible as a filtered ϕ-P-torsor. $\square$

Theorem 2.3.7 and Proposition 2.4.2 together give a complete description of parabolic liftings of G-ordinary crystalline representations, which allow us to prove Theorem (A) with the help of some Galois cohomology arguments.

2.4.3. Let P be a parabolic of G with the unipotent radical U and Levi factor L. Let $\rho : G_K \to L(E)$ be a Galois representation. A parabolic lifting is a commutative diagram

$$\begin{tikzcd} G_K \arrow[rr, "\tilde{\rho}"] \arrow[dr, "\rho"] & & P(E) \arrow[dl] \\ & L(E) & \end{tikzcd}$$

2.4.3.1 Theorem If ρ is crystalline with enough gaps in Hodge-Tate weights with respect to P, any parabolic lifting $\tilde{\rho} : G_K \to P(E)$ is crystalline.

2.4.3.2 We'll prove the theorem by inductively constructing weakly admissible filtered ϕ-P/U_i-torsors which corresponds to $\tilde{\rho}$ mod U_i via Fontaine's functors V_{cris} and D_{cris}.

Proof. Since crystallinity is insensitive to base change, we assume the filtered ϕ-L-torsor $D_{\text{cris}}(\rho)$ has a trivial underlying L-torsor, by possibly enlarging the coefficient field E.

By [EG19, Lemma 6.3.1], having enough gaps in Hodge-Tate weights implies for all i,

$$H^1_f(G_K, U_i/U_{i+1}) = H^1(G_K, U_i/U_{i+1})$$

where H^1_f is the subgroup of crystalline extensions. Here U_i/U_{i+1} is endowed with the adjoint action $G_K \xrightarrow{\rho} L(E) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(U_i/U_{i+1})$. Write ρ_i for $\tilde{\rho}$ mod U_i.

We argue by induction. Assume $\rho_i : G_K \to P/U_i$ is crystalline (and admits a lifting to P). By Corollary 2.3.8.1 and Proposition 2.4.2, the set of crystalline representations $G_K \to P/U_{i+1}$ which lifts ρ_i and admits a lifting to P is an affine space isomorphic to

$$U_i(K \otimes E)/U_{i+1}(K \otimes E)$$

which has the same $\mathbb{Q}_p$-dimension as $H^1_f(G_K, U_i/U_{i+1})$. On the other hand, the set of all liftings (not necessarily crystalline) is an $H^1(G_K, U_i/U_{i+1})$-torsor. An injective, affine map of an affine space into another affine space

$$\begin{aligned} &\{\text{crystalline representations valued in } P/U_{i+1} \text{ lifting } \rho_i \text{ and admits a lifting to } P\} \\ \hookrightarrow &\{\text{representations valued in } P/U_{i+1} \text{ lifting } \rho_i\} \end{aligned}$$

of the same dimension is an isomorphism. By comparing the dimension, we conclude ρ_{i+1} is crystalline. $\square$

2.4.4. Remark We remark that proving Theorem (A) using the strategy of the proof of Proposition 2.4.2 will not work. This is because *G-ordinarity is not preserved by* $P_G(\lambda)(f)$. (Hint: consider the simplest example $\mathrm{Sym}^2 : \mathrm{GL}(V) \to \mathrm{GL}(\mathrm{Sym}^2(V))$.)

2.5. Extensions of anti-G-ordinary filtered ϕ-L-torsors

We give a more complete picture of the theory of parabolic extensions by working out the anti-G-ordinary case.

2.5.1. Definition Let (T, ϕ_T, θ_T) be a filtered ϕ-P-torsor. $\bar{T} = (\pi_L)_* T$ is a filtered ϕ-L-torsor. We say θ_T, T, or $\bar{T}$ is *anti-G-ordinary* if the filtration $\theta_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}$ on the vector space $\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})$ satisfies $\theta^1_{\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}}(\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}) = \mathrm{gr}^\bullet(\mathrm{ad})\bar{T}$. In other words, the Hodge polygon of $\theta_{\mathrm{gr}^\bullet(\mathrm{ad})\bar{T}}$ lies above the x-axis except for the left endpoint.

2.5.2. Proposition Let (T, ϕ_T, θ_T) be a weakly admissible, anti-G-ordinary filtered ϕ-P-torsor.

(1) There is a unique section $\iota : L_{K_0 \otimes E} \hookrightarrow P_{K_0 \otimes E}$ such that $\phi_T = \iota_*((\pi_L)_* \phi_T)$.

(2) For any section $\iota : L_{K \otimes E} \hookrightarrow P_{K \otimes E}$, $\iota_*((\pi_L)_* \theta_T) = \theta_T$.

(3) For any section $\iota : L_{K_0 \otimes E} \hookrightarrow P_{K_0 \otimes E}$, $(T, \phi_T, \theta_T) \cong \iota_*((\pi_L)_*(T, \phi_T, \theta_T))$.

Proof. (1) The proof of Lemma 2.3.6 works verbatim.

(2) We adapt the arguments of subsection 2.3.4. Choose any section $\iota : L_{K\otimes E} \hookrightarrow P_{K \otimes E}$. Let $S \subset \mathrm{Aut}^{\otimes}(i_* \theta_T) \cap P_{K \otimes E}$ be a maximal torus. Let $S_0 \subset S$ be the maximal subtorus such that the centralizer $Z(S_0)$ is (an embedding of) the Levi factor $L_{K \otimes E}$ of $P_{K \otimes E}$. Let $U_{K \otimes E}$ be the unipotent radical of $P_{K \otimes E}$. We have the Levi decomposition

$$\mathrm{Lie}\, P_{K \otimes E} = \mathrm{Lie}\, Z(S_0) \oplus \mathrm{Lie}(U_{K \otimes E}) = \mathrm{Lie}\, Z(S_0) \oplus \bigoplus_{\alpha \in \Phi^+(S_0, G)} \mathfrak{g}_\alpha.$$

By Theorem 1.1, $\mathrm{Lie}\,\mathrm{Aut}^{\otimes}(i_* \theta_T) = (i_* \theta_T)^0(\mathrm{Lie}\, G)$. Since T is anti-G-ordinary, we have

$$\theta^1_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}(\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})) = \mathrm{gr}^\bullet(\mathrm{ad})(\bar{T}).$$

It is clear that the filtration $i_*\theta_T$ on $\mathrm{Lie}\, G$ and the filtration $\theta_{\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T})}$ on $\mathrm{gr}^\bullet(\mathrm{ad})(\bar{T}) = \mathrm{Lie}(U_{K\otimes E})$ are compatible. So $i_*\theta_T^0(\mathrm{Lie}\, G) \cap \mathrm{Lie}(U_{K\otimes E}) \supset i_*\theta_T^1(\mathrm{Lie}\, G) \cap \mathrm{Lie}(U_{K\otimes E}) = \mathrm{Lie}(U_{K\otimes E})$. Thus we have

$$(\dagger) \qquad U_{K\otimes E} \subset \mathrm{Aut}^{\otimes}(i_*\theta_T).$$

By Lemma 2.2.2, $\iota'_*((\pi_L)_*\theta_T) = \theta_T$ for some ι'. There exists an $N \in U(K \otimes E)$ such that $\iota' = N\iota N^{-1}$. By $(\dagger)$, $N\theta_T N^{-1} = \theta_T$, and therefore $\iota_*((\pi_L)_*\theta_T) = \theta_T$.

(3) is a consequence of (1) and (2). $\square$

2.5.3. Corollary Let P be a parabolic of G with the unipotent radical U and Levi factor L. Let $\rho : G_K \to L(E)$ be a Galois representation. If ρ is crystalline and anti-G-ordinary with respect to P, there is one and only one crystalline parabolic lifting $\tilde{\rho} : G_K \to P(E)$ of ρ.

Proof. By the previous lemma, up to isomorphism, there exists a unique parabolic extension of the filtered ϕ-P-torsor which lifts $D_{\mathrm{cris}}(\rho)$. The corollary follows from the equivalence of categories explained in 2.1.7. $\square$

Chapter 3

G-completely reducible mod p Galois representations: the classification and lifts

1. A variant of Steinberg-Winter theorem

The key tool in this section is dynamic methods.

1.1. Dynamic methods We review [Crd14, Section 4.1]. Let X be a scheme over a base scheme S, and fix a $\mathbb{G}_{\mathrm{m}}$-action $m : \mathbb{G}_{\mathrm{m}} \times X \to X$ on X. For each $x \in X(S)$, we say

$$\lim_{t \to 0} m(t, x) \quad \text{exists},$$

if the morphism $\mathbb{G}_{\mathrm{m}} \to X$, $t \mapsto m(t,x)$ extends a a morphism $\mathbb{A}^1 \to X$. If the limit exists, the origin $0 \in \mathbb{A}^1(S)$ maps to a unique element $\alpha \in X(S)$; we write $\lim_{t\to 0} m(t,x) = \alpha$.

Let λ be a cocharacter of a reductive group G over a field k. Define the following functor on the category of k-algebras

$$P_G(\lambda)(A) = \{g \in G(A) | \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists.}\}$$

where A is a general k-algebra.

Define

$$U_G(\lambda)(A) = \{g \in G(A) | \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} = 1\},$$

and denote by $Z_G(\lambda)$ the centralizer of λ in G.

Since G is a reductive group over a field, $P_G(\lambda)$ is a parabolic subgroup of G, $U_G(\lambda)$ is the unipotent radical of $P_G(\lambda)$, and $Z_G(\lambda)$ is a Levi subgroup of $P_G(\lambda)$.

The following proposition is the first application of dynamic methods in this section, and motivates us to consider G-compete reducibility in Steinberg-Winter type questions.

1.2. Proposition Let M be a connected reductive group over a field k. Let $\bar{k}$ be the algebraic closure of k. Let $F_M : M \to M$ be an automorphism of M which can be realized as conjugation by an element $g \in G(k)$ after an embedding $M \hookrightarrow G$.

If g is semisimple, then g and $Z_M(M)^\circ$ generate a G-completely reducible subgroup.

Proof. Let $P \subset G_{\bar{k}}$ be a parabolic subgroup of G which contains both g and $Z_M(M)^\circ$. We want to show a Levi subgroup of P also contains both g and $Z_M(M)^\circ$.

Put $L := Z_P(Z_M(M)^\circ)$, the centralizer of $Z_M(M)^\circ$ in P. Note that conjugation by g fixes L.

We claim L contains a maximal torus of G. Since $Z_M(M)^\circ$ is a torus, it is contained in a maximal torus of P. A maximal torus of P is also a maximal torus of G. Any maximal torus containing $Z_M(M)^\circ$ is in the centralizer of $Z_M(M)^\circ$ because of commutativity of tori.

By Steinberg-Winter, there exists a maximal torus $T \subset L$ which is fixed by g. By the previous paragraph, T is also a maximal torus of G.

By dynamic methods, there exists a cocharacter $\lambda : \mathbb{G}_\mathrm{m} \to T$ such that $P = P_G(\lambda)$. The two cocharacters $\lambda, g\lambda g^{-1} : \mathbb{G}_\mathrm{m} \to T$ lie in the same maximal torus, and can be regarded as elements of the cocharacter lattice $X_*(G,T)$. Since $g \in P$, $g(P_G(\lambda))g^{-1} = P_G(g\lambda g^{-1}) = P_G(\lambda)$. So $\lambda, g\lambda g^{-1} \in X_*(G,T)$ are in the same Weyl chamber. Since $g \in N_G(T)$, $g\lambda g^{-1}$ and λ are in the same Weyl orbit, and thus we must have $\lambda = g\lambda g^{-1}$. So $g \in Z_G(\lambda)$ and $Z_M(M)^\circ \subset T \subset Z_G(\lambda)$. Since $Z_G(\lambda)$ is a Levi subgroup of P, we are done. $\square$

1.3. A generalization of dynamic methods Dynamic methods allow us to prove theorems over general base schemes by doing mathematical analysis. To do so, we need to generalize the functors $P_G(-)$.

Let $f : \mathbb{G}_\mathrm{m} \to G$ be a *k-scheme morphism*. Define the following functor on the category of k-algebras

$$P_G(f)(A) = \{g \in G(A) | \lim_{t \to 0} f(t)gf(t)^{-1} \text{ exists.}\}$$

where A is a general k-algebra. We call f a *fake cocharacter*. Here "a limit exists" means the scheme morphism $\mathbb{G}_\mathrm{m} \to G$, defined by $t \mapsto f(t)gf(t)^{-1}$, extends to a scheme morphism $\mathbb{A}^1 \to G$. Note that $P_G(f)$ is not representable in general. We define similarly $U_G(f)$.

1.4. Lemma Let G be a connected reductive group over a field k. Let $\lambda, \mu : \mathbb{G}_{\mathrm{m}} \to G$ be cocharacters of G. Assume $P_G(\lambda) = P_G(\mu) =: B$ is a Borel subgroup of G. Let U be the unipotent radical of B.

(1) The functor $P_G(\mu\lambda)$ is representable by a Borel subgroup. In fact, we have $P_G(\mu\lambda) = P_G(\mu) = P_G(\lambda)$.

(2) The limit

$$\lim_{t \to 0} \lambda(t)\mu(t)\lambda(t)^{-1}\mu(t)^{-1}$$

exists in the sense of subsection 1.1, and lies in U.

(3) Let u be an element of U. The limit

$$\lim_{t \to 0} \lambda(t)u\mu(t)u^{-1}\lambda(t)^{-1}\mu(t)^{-1}$$

exists in the sense of subsection 1.1 and lies in U.

(4) Now assume λ is a product of cocharacters $\lambda_1, \ldots, \lambda_s$ such that $P_G(\lambda_i) = B$ for all i. Then $P_G(\lambda) = B$, and the limits in (2) and (3) still exist and lie in U.

Moreover, for any embedding $G \hookrightarrow H$ of connected reductive groups, $P_H(\lambda)$ is representable by a parabolic subgroup of H.

Proof. (1) Since all maximal tori in B are conjugate to each other, there exists an element $x \in U_G(\lambda) = U_G(\mu)$ such that conjugation by x maps the maximal torus containing λ to the maximal torus containing μ. In particular, $(x\lambda x^{-1})\mu = \mu(x\lambda x^{-1})$. Write ξ for $x\lambda x^{-1}$. We have

$$\begin{aligned}\lim_{t\to 0}\mu(t)\lambda(t)g\lambda(t)^{-1}\mu(t)^{-1} &= \lim_{t\to 0}\mu(t)x^{-1}\xi(t)xgx^{-1}\xi(t)^{-1}x\mu(t)^{-1} \\ &= \lim_{t\to 0}(\mu(t)x^{-1}\mu(t)^{-1})\cdot(\mu(t)\xi(t)xgx^{-1}\xi(t)^{-1}\mu(t)^{-1})\cdot(\mu(t)x\mu(t)^{-1}) \\ &= \lim_{t\to 0}\mu(t)\xi(t)xgx^{-1}\xi(t)^{-1}\mu(t)^{-1}\end{aligned}$$

Note that the last step is because $x \in U_G(\mu)$, and $\lim_{t\to 0}\mu(t)x\mu(t)^{-1} = 1$. So we have $P_G(\mu\lambda) = x^{-1}P_G(\mu\xi)x$. Since $\mu\xi$ is a genuine cocharacter, $P_G(\mu\xi)$ is representable by a parabolic.

Since $\mu\xi = \xi\mu$, we can regard μ and ξ as elements in a cocharacter lattice $X_*(G,T)$ where T is a maximal torus containing μ and ξ. Since $P_G(\mu) = P_G(\lambda) = P_G(\xi)$, μ and ξ lie in the (interior of the) same Weyl chamber. The cocharacter $\mu\xi$ is the sum of μ and ξ in the cocharacter lattice $X_*(G,T)$, and lies in the same Weyl chamber. So $P_G(\mu\xi) = P_G(\mu) = P_G(\lambda)$. Since $x \in B$, we have $P_G(\mu\lambda) = x^{-1}P_G(\mu\xi)x = P_G(\mu) = P_G(\lambda)$.

(2) Since all maximal tori of B are conjugate to each other, there exists an element $g \in U$ such that $gZ_G(\lambda)g^{-1} = Z_G(\mu)$. Write $\xi := g\lambda g^{-1}$, and we have $\xi\mu = \mu\xi$. By part (1), $P_G(\xi\mu) = B$. By the dynamic description of the Borel B, the limits

$$\lim_{t\to 0}\xi(t)g\xi(t)^{-1} = 1,$$

$$\lim_{t\to 0}\xi(t)\mu(t)g^{-1}\mu(t)^{-1}\xi(t)^{-1} = 1, \text{ and}$$

$$\lim_{t\to 0}\mu(t)g\mu(t)^{-1} = 1$$

all exist. The expression

$$\begin{aligned}\lambda(t)\mu(t)\lambda(t)^{-1}\mu(t)^{-1} &= g^{-1}\xi(t)g\mu(t)g^{-1}\xi(t)^{-1}g\mu(t)^{-1} \\ &= g^{-1}\cdot(\xi(t)g\xi(t)^{-1})\cdot(\xi(t)\mu(t)g^{-1}\mu(t)^{-1}\xi(t)^{-1})\cdot(\mu(t)g\mu(t)^{-1})\end{aligned}$$

has a limit as $t \to 0$.

(3) We have

$$\lambda(t)u\mu(t)u^{-1}\lambda(t)^{-1}\mu(t)^{-1} = (\lambda(t)u\mu(t)u^{-1}\lambda(t)^{-1}u\mu(t)^{-1}u^{-1})(u\mu(t)u^{-1}\mu(t)^{-1}).$$

So (3) follows from (2).

(4) The method is the same but notations are more complicated. We define inductively cocharacters $\xi_i$that commute with each other, and elements u_i of U. Our induction assumption is $P_G(\lambda_1\cdots\lambda_j) = P_G(\xi_1\cdots\xi_j) = B$ for all $j < s$. Define $\xi_1 := \lambda_1$ and $u_1 := 1$. Let u_i be an element of U such that $\xi_i := u_i\lambda_i u_i^{-1}$ commutes with $\xi_1\cdots\xi_{i-1}$. Write ζ_j for $\xi_1\xi_2\cdots\xi_j$, and write v_j for u_j/u_{j-1} (set $u_0 = 1$). We have, for $g \in G$,

$$\begin{aligned}\lambda(t)g\lambda(t)^{-1} = &(\zeta_1(t)v_2\zeta_1(t)^{-1})(\zeta_2(t)v_3\zeta_2(t)^{-1})\cdots \\ &(\zeta_s(t)u_sgu_s^{-1}\zeta_s(t)^{-1}) \\ &(\zeta_{s-1}(t)v_s\zeta_{s-1}(t)^{-1})^{-1}\cdots(\zeta_1(t)v_2\zeta_1(t)^{-1})^{-1} \qquad (\dagger)\end{aligned}$$

which has a limit if and only if $g \in B$. Similarly,

$$\begin{aligned}\lambda(t)\mu(t)\lambda(t)^{-1}\mu(t)^{-1} =& (\zeta_1(t)v_2\zeta_1(t)^{-1})(\zeta_2(t)v_3\zeta_2(t)^{-1})\cdots\\ &(\zeta_s(t)u_s\mu(t)u_s^{-1}\zeta_s(t)^{-1}\mu(t)^{-1}\\ &\mu(t)(\zeta_{s-1}(t)v_s\zeta_{s-1}(t)^{-1})^{-1}\cdots(\zeta_1(t)v_2\zeta_1(t)^{-1})^{-1}\mu(t)^{-1}\end{aligned}$$

By (1), $P_G(\mu\zeta_j) = B$ for all j, and therefore each of the factors $\mu(t)(\zeta_j(t)v_{j+1}\zeta_j(t)^{-1})^{-1}\mu(t)^{-1}$ admits a limit 1. So $\lambda(t)\mu(t)\lambda(t)^{-1}\mu(t)^{-1}$ admits a limit in U by (3).

Next we consider the "moreover" part. (†) holds for $g \in H$ as well. So $P_H(\lambda) = u_s^{-1}P_H(\zeta_s)u_s$ is a parabolic subgroup of H. □

1.5. Lemma Let $F : M \to M$ be an automorphism of a connected reductive group. Let $B \subset M$ be a Borel subgroup fixed by F, with unipotent radical U. There exists a cocharacter μ of M, a positive integer d and an element u of U such that $\mu = uF^d(\mu)u^{-1}$ and $B = P_M(\mu)$.

Proof. By replacing M by its derived subgroup, we can and do assume M is semi-simple. Let μ be a cocharacter of M such that $B = P_M(\mu)$.

Let $i \geq 0$ be an integer. There exists a maximal torus T_i of B such that $F^i(\mu) \subset T_i$. Since all maximal tori of B are conjugate by an element of U, there exists an element u_i of U such that $T_0 = u_iT_iu_i^{-1}$.

So $u_i^{-1}F^i(\mu)u_i \subset T_0$, and we can regard it as an element x_i of the cocharacter lattice $X_*(M, T_0)$. Since μ is a regular cocharacter, its centralizer $Z_M(\mu)$ is a maximal torus of M, and thus is just T_0. Since automorphisms of M send the centralizers to the centralizers, $u_i^{-1}F^iu_i : M \to M$ fixes T_0. Recall that $\mathrm{Aut}(M) \subset \mathrm{Inn}(M) \rtimes \mathrm{Aut}(\mathrm{Dynkin}(\Phi(M, T_0)))$, that is, after fixing a pinning, an automorphism of M comes from an automorphism of its Dynkin diagram. Since $u_i^{-1}F^iu_i$ fixes T_0 and B, it induces an isomorphism of the Dynkin diagram of M and thus induces an isometry of the coroot lattice of M. Since M is semi-simple, its coroot lattice and its cocharacter lattice span the same $\mathbb{R}$-vector space, and thus $u_i^{-1}F^iu_i$ induces an isometry of $X_*(M,T_0)\otimes_{\mathbb{Z}}\mathbb{R}$. In particular, the set $\{x_i\}$ is bounded and thus finite. So $x_{i_0} = x_{i_0+d}$ for some $i_0 \geq 0$ and $d > 0$. We have $u_{i_0}^{-1}F^{i_0}(\mu)u_{i_0} = u_{i_0+d}^{-1}F^{i_0+d}(\mu)u_{i_0+d}$. Thus $\mu = u_{i_0}u_{i_0+d}^{-1}F^d(\mu)u_{i_0+d}u_{i_0}^{-1}$. □

Recall a subgroup $\Gamma \subset G(\bar{k})$ is said to be G-irreducible if Γ is not contained in any proper parabolic subgroup of $G(\bar{k})$.

1.6. Theorem Let M be a connected reductive group over a field k. Let $\bar{k}$ be the algebraic closure of k. Let $F_M : M \to M$ be an automorphism of M which can be realized as conjugation by an element $g \in G(k)$ after an embedding $M \hookrightarrow G$.

If g and $Z_M(M)$ generate a G-irreducible subgroup, then M is a torus.

Proof. One of the key ingredients is the results of Steinberg on endormorphisms of linear algebraic groups. By [St68, Theorem 7.2], any automorphism of a linear algebraic group fixes a Borel subgroup. Let $B_M \subset M$ be a Borel fixed by F_M.

There exists a cocharacter $\lambda : \mathbb{G}_{\mathrm{m}} \to M$ such that $B_M = P_M(\lambda)$. Let U_M be the unipotent radical of B_M. By the previous lemma, there exists $d > 0$ and an element u of U_M such that $F_M^d(\lambda) = u\lambda u^{-1}$. Consider the fake cocharacter $\mu : \mathbb{G}_{\mathrm{m}} \to M$, defined by

$$\mu := F_M^{d-1}(\lambda)F_M^{d-2}(\lambda)\cdots F_M(\lambda)\lambda.$$

Note that $F_M(\mu) = (u\lambda u^{-1})\mu\lambda^{-1}$.

By Lemma 1.4, we have

(i) $P_G(\mu)$ is representable by a parabolic subgroup of G;

(ii) $P_M(\mu) = P_M(\lambda) = M \cap P_G(M)$.

1.6.0.1 Claim $g \in P_G(\mu)$.

Proof. We verify this using the definition of $P_G(\mu)$. We have

$$\begin{aligned}
\lim_{t\to 0} \mu(t)g\mu(t)^{-1} &= \lim_{t\to 0} \mu(t)g\mu(t)^{-1}g^{-1}g \\
&= \lim_{t\to 0} \mu(t)F_M(\mu)(t)^{-1}g \\
&= \lim_{t\to 0} \mu(t)\lambda(t)\mu(t)^{-1}u\lambda(t)^{-1}u^{-1}g \\
&= \lim_{t\to 0}(\mu(t)\lambda(t)\mu(t)^{-1}\lambda(t)^{-1})(\lambda(t)u\lambda(t)^{-1})u^{-1}g
\end{aligned}$$

The claim follows from Lemma 1.4 (4). □

Note that since μ is valued in M, $Z_M(M) \subset Z_G(\mu)$.

Let Γ be the subgroup of G generated by $Z_M(M)^\circ$ and g. As a consequence of the claim, we have $\Gamma \subset P_G(\mu)$. By Lemma 1.4 (1), $P_M(\mu) = P_M(\lambda)$ is a Borel subgroup of M. By the dynamic description of Borel subgroups, we have $P_G(\mu) \cap M = P_M(\mu)$. So $P_G(\mu)$ is a proper parabolic subgroup of G if $P_M(\mu)$ is a proper parabolic subgroup of M. Since Γ is assumed to be G-irreducible, we must have $P_M(\mu) = M$. Since $M = P_M(\mu) = B_M$ is chosen to be a Borel subgroup of M, M is forced to be a torus. □

1.7. Corollary Let M be a connected reductive group over a field k. Let $\bar{k}$ be the algebraic closure of k. Let $F_M : M \to M$ be an automorphism of M which can be realized as conjugation by an element $g \in G(k)$ after an embedding $M \hookrightarrow G$.

Assume

(i) g and $Z_M(M)^\circ$ generate a G-completely reducible subgroup;

(ii) $\operatorname{rk} M = \operatorname{rk} G$ and $\operatorname{char} k \neq 2, 3$; and

(iii) M has a connected center.

Then F_M fixes a maximal torus T of $M_{\bar{k}}$.

Proof. Let Γ be the subgroup of G generated by $Z_M(M)$ and g. If Γ is G-irreducible, we are done because of Theorem 1.6. So we assume there exists a proper parabolic subgroup P of $G_{\bar{k}}$ such that $\Gamma \subset P$.

By Borel-de Siebenthal theory (see [Pep15] or [Gil10, Theorem 0.1]), when k $\neq 2, 3$, $\operatorname{rk} M = \operatorname{rk} G$ implies $M = Z_G(Z_M(M))^\circ$.

We will prove a slightly stronger version of the corollary. We claim F_M fixes a maximal torus of $M_{\bar{k}}$ assuming (i), (ii), and

(iii') There exists a torus Z of M such that $M = Z_G(Z)^\circ$.

Since Γ is G-completely reducible, there exists a Levi subgroup $L \subset P$ such that $\Gamma \subset L$. Note that $(M \cap L)^\circ = Z_L(Z)^\circ$, which is a reductive subgroup (see [Gil10, Lemma 0.2(1)]) of L fixed by g. We claim $(M \cap L)^\circ$ is of maximal rank. Let S be any maximal torus of L containing Z. Since S is commutative and connected, we have $S \subset Z_L(Z)^\circ = (M \cap L)^\circ$. Thus $\operatorname{rk}(M \cap L)^\circ = \operatorname{rk} S = \operatorname{rk} L = \operatorname{rk} G$.

We apply induction on the dimension of G. Since $Z_L(Z)^\circ = (M \cap L)^\circ$, assumption (iii') is satisfied by $(M \cap L)^\circ$; assumption (i) is also satisfied because L is a Levi subgroup of G. Since $\dim L < \dim G$, by induction there exists a maximal torus T of $(M \cap L)^\circ$ which is fixed by F_M. Since $\mathrm{rk}(M \cap L)^\circ = \mathrm{rk}\, G$, T is also a maximal torus of G. $\square$

1.8. We explain how our methods can possibly be used to establish a stronger form of Steinberg-Winter, at least for groups having connected center. Dynamic methods are very well behaved for disconnected linear algebraic groups. We similarly define G-complete reducibility for general linear algebraic groups by replacing parabolics by pseudo-parabolics. Let $F : M \to M$ be an automorphism which can be realized as conjugation by an element g of G after an embedding $M \hookrightarrow G$. Let H be the scheme-theoretic closure of the (abstract) group generated by M and g. Note that H is a disconnected reductive group, and $\mathrm{rk}\, H = \mathrm{rk}\, M$. Let Γ be the subgroup of H generated by $Z_M(M)^\circ$ and g. We expect that the H-complete reducibility of Γ implies the existence of a fixed maximal torus.

2. The structure of G-completely reducible mod ϖ Galois representations

In this section, we give a complete description of all G-completely reducible mod ϖ Galois representations valued in split reductive groups.

The first step is to show G-complete reducibility implies tame ramification, reducing the classification of mod ϖ Galois representations to the question of classification of (certain) solvable subgroups of derived length 2 of reductive groups.

2.1. Lemma Let P_K be the wild inertia of G_K. If $\bar{\rho} : G_K \to G(\bar{\mathbb{F}}_p)$ is G-completely reducible, $\bar{\rho}(P_K) = \{\mathrm{id}\}$.

Proof. Let $P_K \subset G_K$ be the wild inertia. The image $\bar{\rho}(P_K) \subset G(\bar{\mathbb{F}}_p)$ is a p-group, and thus consists of unipotent elements. By [BT71, Corollaire 3.9], there exists a parabolic subgroup P of $G_{\bar{\mathbb{F}}_p}$ with unipotent radical $R_u(P)$ such that

- $\bar{\rho}(P_K) \subset R_u(P)(\bar{\mathbb{F}}_p)$, and
- $N(\bar{\rho}(P_K)) \subset P(\bar{\mathbb{F}}_p)$;

here $N(\bar{\rho}(P_K))$ is the normalizer of $\bar{\rho}(P_K)$. Since P_K is a normal subgroup of G_K, $\bar{\rho}(G_K) \subset N(\bar{\rho}(P_K)) \subset P(\bar{\mathbb{F}}_p)$. Since $\bar{\rho}$ is G-completely reducible, $\bar{\rho}(G_K)$ is contained in a Levi subgroup L of P. So $\bar{\rho}(P_K) \subset L(\bar{\mathbb{F}}_p) \cap R_u(P)(\bar{\mathbb{F}}_p) = \{\mathrm{id}\}$. $\square$

2.2. Definition We say $\bar{\rho} : G_K \to G(\bar{\mathbb{F}}_p)$ is *strongly semi-simple* if there exists a maximal torus T of $G(\bar{\mathbb{F}}_p)$ such that $\bar{\rho}(I_K) \subset T(\bar{\mathbb{F}}_p)$ and $\bar{\rho}(G_K) \subset N_G(T(\bar{\mathbb{F}}_p))$.

2.3. Theorem If $\bar{\rho} : G_K \to G(\bar{\mathbb{F}}_p)$ is G-completely reducible, then $\bar{\rho}$ is strongly semisimple.

Moreover, if $\bar{\rho}$ is G-irreducible, there exists a *unique* maximal torus T of $G(\bar{\mathbb{F}}_p)$ containing $\bar{\rho}(I_K)$. Consequently, if $\bar{\rho}(G_K) \subset G(\mathbb{F})$, T has a model defined over the ring of Witt vectors $W(\mathbb{F})$.

Proof. By induction on the dimension of G, we can reduce the general case to the case where $\bar{\rho}$ is G-irreducible. Recall that $\bar{\rho}$ is G-irreducible if it does not factor through any proper parabolic of

G. If $\bar{\rho}$ does factor through a proper parabolic of G, the G-complete reducibility forces $\bar{\rho}$ to factor through a proper Levi subgroup of G, which is a reductive group of strictly smaller dimension.

So we assume $\bar{\rho}$ is G-irreducible in the rest of the proof. By Lemma 2.1, $\bar{\rho}(I_K)$ is a finite cyclic group generated by elements of order prime to p. Write M for $Z^{\circ}_{G(\bar{\mathbb{F}}_p)}(\bar{\rho}(I_K))$, the neutral component of the centralizer of $\bar{\rho}(I_K)$ in G. Since $\bar{\rho}(I_K)$ consists of semi-simple elements of $G(\mathbb{F})$, M is a reductive subgroup of G. Let $\Phi_K \in G_K$ be a topological generator of G_K/I_K. Since I_K is a normal subgroup of G_K, the conjugation by $\bar{\rho}(\Phi_K)$ action induces an automorphism of M, which we denote by $F_M : M \to M$.

Next we show $\bar{\rho}(I_K) \subset Z_M(M)$. Since G is connected, a semisimple element of G is contained in a maximal torus. Since $\bar{\rho}(I_K)$ is a cyclic group consisting of semi-simple elements, there exists a maximal torus T containing $\bar{\rho}(I_K)$. Since a torus is connected, we have $T \subset M$, and thus $\bar{\rho}(I_K) \subset M(\bar{\mathbb{F}}_p)$. It is immediate from the definition of M that $\bar{\rho}(I_K) \subset Z_M(M)$.

By Theorem 1.6, M is a torus. Let T be any maximal torus of G containing $\bar{\rho}(I_K)$. Since T is commutative and connected, we have $T \subset Z_G(\bar{\rho}(I_K))^{\circ} = M$. So M is the unique maximal torus containing $\bar{\rho}(I_K)$. Now consider the "moreover" part. For $\sigma \in \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F})$, $\sigma(M)$ is also a maximal torus containing $\bar{\rho}(I_K)$. So $\sigma(M) = M$, and thus by Galois descent M is defined over $\mathbb{F}$. By [Crd14, B.3.5], T has a model over $W(\mathbb{F})$. $\square$

2.4. Example We illustrate the technical proof using a very concrete example. Let $G = \mathrm{GL}_4$. Let $\bar{\rho} : G_K \to \mathrm{GL}_4(\bar{\mathbb{F}}_p)$ be a semi-simple Galois representation. We decompose $V = V_{\chi_1} \oplus V_{\chi_2}$ into I_K-isotropic subspaces. Here χ_1, $\chi_2 : I_K \to \bar{\mathbb{F}}_p^{\times}$ are distinct charatcers such that for $v \in V_{\chi_i}$ and

$\sigma \in I_K$, $\bar{\rho}(\sigma)v = \chi_i(\sigma)v$, $i = 1, 2$.

$$\bar{\rho}|_{I_K} = \begin{bmatrix} \chi_1 & & & \\ & \chi_1 & & \\ & & \chi_2 & \\ & & & \chi_2 \end{bmatrix}$$

There are two possibilities: either both V_i are $\bar{\rho}(\Phi_K)$-stable, or $\bar{\rho}(\Phi_K)$ sends V_i to V_{3-i}, $i = 1, 2$. The first case is simple: $V = V_{\chi_1} \oplus V_{\chi_2}$ as a G_K-module. Now we consider the latter case. By Steinberg's theorem [St68, Theorem 7.2], we can assume $\bar{\rho}(\Phi_K)$ fixes a Borel

$$P_M = \begin{bmatrix} * & * & & \\ & * & & \\ & & * & * \\ & & & * \end{bmatrix}$$

of $M = \mathrm{GL}_2 \times \mathrm{GL}_2 \subset \mathrm{GL}_4$ and thus we must have

$$\bar{\rho}(\Phi_K) = \begin{bmatrix} & & a & b \\ & & & c \\ d & f & & \\ & e & & \end{bmatrix}$$

for some $a, b, c, d, e, f \in \bar{\mathbb{F}}_p$. The Borel subgroup P_M is of shape $P_M(\lambda)$ for

$$\lambda(t) = \begin{bmatrix} t^\alpha & * & & \\ & t^\beta & & \\ & & t^\gamma & * \\ & & & t^\delta \end{bmatrix}$$

for $\alpha > \beta$ and $\gamma > \delta$. We have

$$\bar{\rho}(\Phi_K)\lambda(t)\bar{\rho}(\Phi_K)^{-1} = \begin{bmatrix} t^\gamma & * & & \\ & t^\delta & & \\ & & t^\alpha & * \\ & & & t^\beta \end{bmatrix}$$

and thus

$$\bar{\rho}(\Phi_K)\lambda(t)\bar{\rho}(\Phi_K)^{-1}\lambda(t) = \begin{bmatrix} t^{\alpha+\gamma} & * & & \\ & t^{\beta+\delta} & & \\ & & t^{\alpha+\gamma} & * \\ & & & t^{\beta+\delta} \end{bmatrix}$$

Since $\alpha + \gamma > \beta + \delta$, we have

$$P_{\mathrm{GL}_4}(\bar{\rho}(\Phi_K)\lambda\bar{\rho}(\Phi_K)^{-1}\lambda) = \begin{bmatrix} * & * & * & * \\ & * & & * \\ * & * & * & * \\ & * & & * \end{bmatrix}$$

and finally we observe $\bar{\rho}(\Phi_K) \in P_{\mathrm{GL}_4}(\bar{\rho}(\Phi_K)\lambda\bar{\rho}(\Phi_K)^{-1}\lambda)$. In general, by Lemma 1.5, there exists an integer d such that $\prod_{i=d-1}^{0} \bar{\rho}(\Phi_K)^i\lambda(t)\bar{\rho}(\Phi_K)^{-i}$ gives the desired parabolic.

3. Crystalline lifts of irreducible mod ϖ Galois representations

Write κ for the residue field of K. Fix a coefficient field E with ring of integers $\mathcal{O}$ and uniformizer ϖ. Write $\mathbb{F}$ for the residue field $\mathcal{O}/\varpi$. Assume $\kappa \subset \mathbb{F}$. Let $\Phi_K \in G_K$ be a (lift of a) topological generator of G_K/I_K. Fix an algebraic closure $\bar{K}$ of K.

In this section, we assume G is a split group since we are primarily interested in Galois representations valued in L-groups. The L-group of a connected reductive group is split, albeit possibly disconnected.

3.1. For each maximal torus T of G, write $M_{T,\mathrm{cris}}$ for the set of representations $I_K \to T(\mathcal{O})$, which can be extended to a crystalline representation $G_{K'} \to T(\mathcal{O})$ for some finite unramified extension K'/K inside $\bar{K}$. Let $W(G,T)$ be the Weyl group of G with respect to T. Since the union of two finite unramified extensions inside $\bar{K}$ is still a finite unramified extension, $M_{T,\mathrm{cris}}$ is an abelian group.

The abelian group $M_{T,\mathrm{cris}}$ has a $\mathbb{Z}[W(G,T)]$-module structure, defined by $wv := (\sigma \mapsto wv(\sigma)w^{-1})$, for $w \in W(G,T)$ and $v \in M_{T,\mathrm{cris}}$.

The abelian group $M_{T,\mathrm{cris}}$ also has a $\mathbb{Z}[G_K/I_K]$-module structure, defined by $\alpha v := (\sigma \mapsto v(\alpha^{-1}\sigma\alpha))$ for $\alpha \in G_K$ and $v \in M_{T,\mathrm{cris}}$.

The following lemma is clear.

3.1.0.1 Lemma and Definition The $\mathbb{Z}[W(G,T)]$-module structure and the $\mathbb{Z}[G_K/I_K]$-module structure on $M_{T,\mathrm{cris}}$ commute with each other, and therefore $M_{T,\mathrm{cris}}$ is a $\mathbb{Z}[W(G,T)]\otimes\mathbb{Z}[G_K/I_K]$-module.

Similarly, write $M_{T,\mathbb{F}}$ for the abelian group of mod ϖ representations $I_K \to T(\mathbb{F})$. The abelian group $M_{T,\mathbb{F}}$ has a $\mathbb{Z}[W(G,T)]\otimes\mathbb{Z}[G_K/I_K]$-module structure.

3.2. Lemma Write $\zeta : N_G(T) \to W(G,T)$ for the quotient map.

(1) Let w be an element of $N_G(T)(\mathcal{O})$ of finite order. An element $v \in M_{T,\mathrm{cris}}$ extends to a continuous representation $\rho : G_K \to N_G(T)(\mathcal{O})$ by setting $\rho(\Phi_K) = w^{-1}$ and $\rho|_{I_K} = v$ if and only if

$$v \in \ker(M_{T,\mathrm{cris}} \xrightarrow{\zeta(w)\otimes 1 - 1\otimes\Phi_K} M_{T,\mathrm{cris}}).$$

(2) Let $\bar{w}$ be an element of $N_G(T)(\mathbb{F})$. An element $v \in M_{T,\mathbb{F}}$ extends to a representation $\bar{\rho} : G_K \to N_G(T)(\mathbb{F})$ by setting $\bar{\rho}(\Phi_K) = \bar{w}^{-1}$ and $\bar{\rho}|_{I_K} = v$ if and only if

$$v \in \ker(M_{T,\mathbb{F}} \xrightarrow{\zeta(\bar{w})\otimes 1 - 1\otimes\Phi_K} M_{T,\mathbb{F}}).$$

Proof. (1) Since w is of finite order, it suffices to show $v \in M_{T,\mathrm{cris}}$ extends to a representation $\rho : W_K \to N_G(T)(\mathcal{O})$ of the Weil group $W_K \cong I_K \rtimes \mathbb{Z}$ by setting $\rho(\Phi_K) = w^{-1}$ and $\rho|_{I_K} = v$ if and only if $v \in \ker(M_{T,\mathrm{cris}} \xrightarrow{\zeta(w)\otimes 1 - 1\otimes\Phi_K} M_{T,\mathrm{cris}})$. If v is extendable to ρ, then for all $\sigma \in I_K$

$$\rho(\Phi_K^{-1}\sigma\Phi_K) = w\rho(\sigma)w^{-1};$$

the left hand side restricted to I_K is $(1\otimes\Phi_K)v$, and the right hand side restricted to I_K is $(\zeta(w)\otimes 1)v$.

So $(1\otimes\Phi_K)v = (\zeta(w)\otimes 1)v$. Conversely, if $(1\otimes\Phi_K)v = (\zeta(w)\otimes 1)v$, then $v(\Phi_K^{-1}\sigma\Phi_K) = wv(\sigma)w^{-1}$ for all $\sigma \in I_K$. Define $\rho(\sigma\Phi^n) := v(\sigma)w^{-n}$ for all $\sigma \in I_K$ and $n \in \mathbb{Z}$. It is clear ρ is well-defined on W_F, and extends to G_F uniquely by continuity.

(2) is similar to (1). □

Note that by [Crd14, 5.1.6], the Weyl group $W(G,T)$ is a constant group scheme when T is split.

3.2.1. Definition For an element of the Weyl group $w \in W(G,T) = W(G,T)(\mathbb{F}) = W(G,T)(\mathcal{O})$, define

$$M_{T,w,\mathrm{cris}} := \ker(M_{T,\mathrm{cris}} \xrightarrow{w\otimes 1 - 1\otimes\Phi_K} M_{T,\mathrm{cris}}), \text{ and}$$

$$M_{T,w,\mathbb{F}} := \ker(M_{T,\mathbb{F}} \xrightarrow{w\otimes 1 - 1\otimes\Phi_K} M_{T,\mathbb{F}}).$$

The following simple lemma is essentially how we construct crystalline lifts.

3.3. Lemma Let $\mathbb{Z}[X]$ be the polynomial ring. Let $a(X), b(X) \in \mathbb{Z}[X]$ be two polynomials. Let n and N be integers. Assume $a(n)b(n) = 0$.

Let $\widetilde{M}$ be a $\mathbb{Z}[X]/(a(X)b(X) - N)$-module. Write M for $\widetilde{M} \otimes_{\mathbb{Z}} \mathbb{Z}/N$.

If $\widetilde{M}$ has a torsion-free, finitely generated underlying abelian group, the sequence

$$0 \to a(X)M \to M \xrightarrow{\cdot b(X)} b(X)M \to 0$$

is short exact.

Proof. Pick $\bar{v} \in \ker(M \to b(X)M)$. Let $v \in \widetilde{M}$ be a lifting of $\bar{v}$. We have $b(X)v \mapsto 0$ in M. Since $M = \widetilde{M} \otimes \mathbb{Z}/N$, $b(X)v = Nu$ for some $u \in \widetilde{M}$. Multiply both sides by $a(X)$, we get $a(X)b(X)v = Nv = Na(X)u$. Since $\widetilde{M}$ is $\mathbb{Z}$-torsion-free, we have $v = a(X)u$, as desired. □

3.4. Proposition If $w^{[\mathbb{F}:\kappa]} = 1$ and E contains K, the map

$$M_{T,w,\mathrm{cris}} \to M_{T,w,\mathbb{F}}$$

is surjective.

Proof. Write $f := [\mathbb{F} : \kappa]$. Let K_f be the unramified extension of K of degree f.

We single out a $\mathbb{Z}[W(G,T)] \otimes \mathbb{Z}[G_K/I_K]$-submodule $M^0_{T,\mathrm{cris}} \subset M_{T,\mathrm{cris}}$ which consists of elements that can be extended to a representation $G_{K_f} \to T(\mathcal{O})$. Note that $M^0_{T,\mathrm{cris}} \to M_{T,\mathbb{F}}$ is surjective because the fundamental character of niveau f admits a crystalline lift, namely, the Lubin-Tate character of the field K_f. Put $M^0_{T,w,\mathrm{cris}} := M^0_{T,\mathrm{cris}} \cap M_{T,w,\mathrm{cris}}$.

Note that on both $M^0_{T,\mathrm{cris}}$ and $M_{T,\mathbb{F}}$, we have $(w \otimes 1)^f = (1 \otimes \Phi_K)^f = \mathrm{id}$, where Φ_K is the fixed topological generator of G_K/I_K.

Put

$$\Xi := \sum_{i=0}^{f-1} w^i \otimes \Phi_K^{f-1-i}.$$

Commutativity of $w \otimes 1$ and $1 \otimes \Phi_K$ implies $(w \otimes 1 - 1 \otimes \Phi_K)\Xi = (w \otimes 1)^f - (1 \otimes \Phi_K)^f$. In particular, the inclusion $\Xi M^0_{T,\mathrm{cris}} \to M^0_{T,\mathrm{cris}}$ factors through $M^0_{T,w,\mathrm{cris}}$ (which is the arrow at the top of the diagram below).

Consider the commutative diagram

$$\begin{tikzcd}
\Xi M^0_{T,\mathrm{cris}} \arrow[r] \arrow[d] & M^0_{T,\boldsymbol{w},\mathrm{cris}} \arrow[r] \arrow[d] & M^0_{T,\mathrm{cris}} \arrow[r, "\boldsymbol{w}\otimes 1-1\otimes\Phi_K"] \arrow[d] & M^0_{T,\mathrm{cris}} \arrow[d] \\
\Xi M_{T,\mathbb{F}} \arrow[r] \arrow[d] & M_{T,\boldsymbol{w},\mathbb{F}} \arrow[r] & M_{T,\mathbb{F}} \arrow[r, "\boldsymbol{w}\otimes 1-1\otimes\Phi_K"] & M_{T,\mathbb{F}} \\
0 & & &
\end{tikzcd}$$

It is clear that $\Xi M^0_{T,\mathrm{cris}} \to \Xi M_{T,\mathbb{F}}$ is surjective. So it suffices to show

$$\Xi M_{T,\mathbb{F}} \hookrightarrow M_{T,\boldsymbol{w},\mathbb{F}}$$

is surjective.

Let $\bar{\chi} : I_K \to \mathbb{F}^\times$ be a fundamental character of niveau f. Note that $\bar{\chi}$ generates the abelian group $M_{\mathbb{G}_\mathrm{m},\mathbb{F}}$. Indeed, there is an abelian group isomorphism $\iota_{\bar{\chi}} : \mathbb{Z}/(q^f-1) \xrightarrow{\cong} M_{\mathbb{G}_\mathrm{m},\mathbb{F}}$, sending 1 to $\bar{\chi}$. We have $M_{T,\mathbb{F}} \cong M_{\mathbb{G}_\mathrm{m},\mathbb{F}} \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathrm{GrpSch}}(\mathbb{G}_\mathrm{m}, T)$. Note that the Weyl group element $\boldsymbol{w}$ acts on $\mathrm{Hom}_{\mathrm{GrpSch}}(\mathbb{G}_\mathrm{m}, T)$ via conjugation $v \mapsto \boldsymbol{w} v \boldsymbol{w}^{-1}$.

We specialize Lemma 3.3 as follows:

- Set $\widetilde{M} = \mathrm{Hom}_{\mathrm{GrpSch}}(\mathbb{G}_\mathrm{m}, T)$, and regard it as a $\mathbb{Z}[X]$-module where X acts by $\boldsymbol{w}$;
- Set $M = M_{T,\mathbb{F}}$, and regard M as a $\mathbb{Z}[X]$-module via $X \mapsto \boldsymbol{w} \otimes 1$;
- Set $N = q^f - 1$;
- Set $n = q$;
- Set $a(X) = \sum_{i=0}^{f-1-i} X^i q^{f-1-i}$;

- Set $b(X) = q - X$;

We can identify M with $\widetilde{M} \otimes_{\mathbb{Z}} \mathbb{Z}/(q^f - 1)$ via the map $\iota_{\bar{\chi}} : \mathbb{Z}/(q^f - 1) \xrightarrow{\cong} M_{\mathbb{G}_{\mathrm{m}},\mathbb{F}}$.

Here are a few things to check:

(i) $\widetilde{M}$ is finitely generated and torsion-free over $\mathbb{Z}$.

(ii) $(a(X)b(X) - q^f + 1)$ kills $\widetilde{M}$;

(iii) $\widetilde{M} \otimes_{\mathbb{Z}} \mathbb{Z}/(q^f - 1) \cong M$ as abelian groups;

(iv) $a(q)b(q) = 0$.

Items (i), (iii) and (iv) are clear. For item (ii), notice that $a(X)b(X) = q^f - X^f$. Since we assumed $w^f = 1$, $a(X)b(X) = q^f - 1$. □

The goal of the rest of this section is to prove the following theorem:

3.5. Theorem Let κ be the residue field of K. Let $\mathbb{F}/\kappa$ be a finite extension. Let K^{ur} be the maximal unramified extension of K with ring of integers $\mathcal{O}_{K_{\mathrm{ur}}}$.

Let $\bar{\rho} : G_K \to G(\mathbb{F})$ be a strongly semi-simple (see Definition 2.2) representation.

(1) There exists a crystalline representation $\rho : G_K \to G(\mathcal{O}_{K^{\mathrm{ur}}})$ lifting $\bar{\rho}$.

(2) Assume G admits a simply-connected derived subgroup and $\bar{\rho}$ is G-irreducible. Let $\mathbb{F}_{\bar{\rho}}$ be the splitting field of $\bar{\rho}|_{I_K}$, that is, the smallest field extension $\mathbb{F}_{\bar{\rho}}$ of $\mathbb{F}$ such that $\bar{\rho}|_{I_K} : I_K \to G(\mathbb{F})$ factors through the $\mathbb{F}_{\bar{\rho}}$-points of a split torus of G. Then ρ can be chosen to have image in $G(\mathcal{O}_{K_{\bar{\rho}}})$ where $K_{\bar{\rho}}$ is the unramified extension of K with residue field $\mathbb{F}_{\bar{\rho}}$.

3.6. The strategy is as follows: the first step is to choose a lift of $\bar{\rho}|_{I_K}$ which admits an extension to the whose Galois group G_K. This is already done in Proposition 3.4. The second step is to choose a lift of all Frobenius elements. The continuity of the lift is free because we'll only use finite order lifts (modulo the image of I_K) of Frobenius elements.

3.7. Lemma There exists a finite subgroup $\widetilde{N} \subset N_G(T)(W(\mathbb{F}))$ such that $\widetilde{N} \to N_G(T)(\mathbb{F})$ is surjective.

Proof. The key ingredient is Tits' theory of extended Weyl groups.

By [Ti66], there exists a subgroup $\widetilde{W} \subset N_G(T)(W(\mathbb{F}))$ which is isomorphic to the extension of the Weyl group $W(G,T)$ by $(\mathbb{Z}/2)^{\otimes l}$ for some $l \geq 0$, and generates the whole Weyl group. Write $[-] : T(\mathbb{F}) \to T(W(\mathbb{F}))$ for the Teichmüller lift.

3.7.0.1 Fact The Teichmüller lift is the unique p-adic continuous multiplicative section of $T(W(\mathbb{F})) \to T(\mathbb{F})$.

Proof. We include a proof here because it is short. It is well-known for $T = \mathbb{G}_{\mathrm{m}}$.

In general, choose a faithful representation $i : T \to \mathrm{GL}_N \subset \mathrm{Mat}_{N\times N}$. Let $s, t : T(\mathbb{F}) \to T(\mathcal{O})$ be two multiplicative sections. We have $i(s(x)) - i(t(x)) \equiv 1 \mod p^f$ for all $x \in T(\mathbb{F})$; $(i(s(x)) - i(t(x)))^{p^{nf}} \equiv 1 \mod p^{(n+1)f}$; and $i(s(x)) - i(t(x)) = i(s(x^{p^{nf}})) - i(t(x^{p^{nf}})) \equiv (i(s(x)) - i(t(x)))^{p^{nf}} \equiv 1 \mod p^{nf}$ for all n. □

For each $w \in \widetilde{W}$ and $x \in T(\mathbb{F})$, $x \mapsto w^{-1}[wxw^{-1}]w$ is a continuous section of $T(W(\mathbb{F})) \to T(\mathbb{F})$ and must be equal to the Techmüller lift. Let $\widetilde{N}$ be the composite $\widetilde{W} \cdot [T(\mathbb{F})]$. Since for all

$w, w' \in \widetilde{W}$ and $x, x' \in T(\mathbb{F})$, we have $w[x]w'[x'] = ww'[w'^{-1}xw'x']$, $\widetilde{N}$ is a finite order subgroup of $N_G(T)(W(\mathbb{F}))$, as desired. $\square$

The existence of $\widetilde{N}$ has the following immediate consequence:

3.8. Corollary Let $\bar{\rho} : G_K \to G(\mathbb{F})$ be a G-completely reducible representation. There exists a lift $\rho : G_K \to G(W(\mathbb{F}))$ of $\bar{\rho}$.

Indeed, for any lift v of $\bar{\rho}|_{I_K}$ to $G(\mathcal{O})$ that can be extended to the whole Galois group G_K, there exists a lift $\bar{\rho}$ to $G(\mathcal{O})$ whose inertia is v.

Proof. We first prove the first paragraph. Set $\rho|_{I_K}$ to be the Teichmüller lift of $\bar{\rho}|_{I_K}$. Let $\Phi_K \in G_K$ be a lift of the topological generator of G_K/I_K. Choose an element $n \in \widetilde{N}$ which lifts $\bar{\rho}(\Phi_K)$. Set $\rho(\Phi_K) = n$. Write $n = wt$ where w is an element of Tits' extended Weyl group $\widetilde{W}$ and t lies in the Teichmüller lift of $T(\mathbb{F})$. Let σ be an element of I_K. Write x for $\bar{\rho}(\sigma)$. We have $\rho(\Phi_K\sigma\Phi_K^{-1}) = [\bar{\rho}(\Phi_K\sigma\Phi_K^{-1})] = [wxw^{-1}]$. By the proof of the previous lemma, $[wxw^{-1}] = w[x]w^{-1} = w\rho(\sigma)w^{-1} = n\rho(\sigma)n^{-1}$, and thus ρ extends uniquely to a continuous homomorphism $G_K \to G(W(\mathbb{F}))$.

Now we prove the "indeed" part. It is an immediate consequence of Lemma 3.2 and Lemma 3.7. $\square$

3.9. Lemma Let $\bar{\rho} : G_K \to G(\mathbb{F})$ be a G-irreducible Galois representaion. By Theorem 2.3, there exists a unique maximal torus T of G such that $\bar{\rho}(G_K) \subset N_G(T)(\mathbb{F})$.

Let κ be the residue field of K. Let $\mathbb{F}_0 \subset \mathbb{F}$ be the smallest subfield of $\mathbb{F}$ containing κ such that $\bar{\rho}(I_K) \subset T(\mathbb{F}_0)$. (Recall that G is a Chevalley group and has a $\mathbb{Z}$-model.) Let $\Phi_K \in G_K$ be a lift of a topological generator of G_K/I_K. The map $G_K \to N_G(T)(\mathbb{F}) \to W(G,T)(\mathbb{F})$ maps Φ_K to an element w of the Weyl group $W(G,T)(\mathbb{F})$.

If G admits a simply-connected derived subgroup, then $w^{[\mathbb{F}_0:\kappa]} = 1$ in $W(G,T)(\mathbb{F})$.

Proof. Write $f_0 := [\mathbb{F}_0 : \kappa]$. Let $s \in \bar{\rho}(I_K)$ be a generator. By the proof of Theorem 2.3, $T = Z_G(s)^\circ$ is the connected centralizer of s. Since $\bar{\rho}(I_K) \subset T(\mathbb{F}_0)$, we have $\bar{\rho}(\Phi_K)^{f_0} s \bar{\rho}(\Phi_K)^{-f_0} = s$. So $\bar{\rho}(\Phi_K)^{f_0} \in Z_G(s) \cap N_G(T)$. Since G has a simply-connected derived subgroup, $Z_G(s) = Z_G(s)^\circ$. So $\bar{\rho}(\Phi_K)^{f_0} \in T$, that is, $w^{[\mathbb{F}_0:\kappa]} = 1$ in $W(G,T)(\mathbb{F})$. $\square$

Proof of Theorem 3.5. (1) We choose a sufficiently large coefficient field E (which is unramified over K) such that the cardinality of the Weyl group $W(G,T)$ divides $[\mathbb{F} : \kappa]$. The assumption of Proposition 3.4 is satisfied. So there exists a crystalline lift $v : I_K \to T(\mathcal{O})$ such that $v = \bar{\rho}|_{I_K}$ mod ϖ. By Lemma 3.2, v can be extended to G_K.

(2) For ease of notation, replace $\mathbb{F}$ by $\mathbb{F}_{\bar{\rho}}$. Write $\mathcal{O}$ for $\mathcal{O}_{K_{\bar{\rho}}}$. We choose the field $\mathbb{F}_0$ as in Lemma 3.9. Note that the maximal torus in Lemma 3.9 is split: let S be a maximal split torus over $\mathbb{F}$ such that $\bar{\rho}(I_K) \subset S(\mathbb{F})$; since $T = Z_G(\bar{\rho}(I_K))^\circ$, we have $T \supset S$; now since G is a split group, we must have $S = T$. Let K_{f_0} be the unramified extension of K of degree $[K_{f_0} : K] = [\mathbb{F}_0 : \kappa]$. Let $\mathcal{O}_0$ be the ring of integers of K_{f_0}. We have $\mathcal{O}_0 \subset \mathcal{O}$. By the previous Lemma, Proposition 3.4 is applicable, and thus there exists a lift $v : I_K \to T(\mathcal{O}_0)$ such that $v = \bar{\rho}|_{I_K}$ mod ϖ and v admits an extension to a representation $G_K \to N_G(T)(\mathcal{O}_0)$. By Corollary 3.8, v admits an extension to G_K which lifts $\bar{\rho}$.

Fix $\Phi_K \in G_K$, a lift of a topological generator of G_K/I_K. By Lemma 3.7, we choose a finite order lift $X \in \widetilde{N} \subset N_G(T)(\mathcal{O})$ of $\bar{\rho}(\Phi_K)$. Since the Weyl group scheme is a constant group scheme, any two lifts of $\bar{\rho}(\Phi_K)$ have the same conjugation action on the maximal torus $N_G(T)(\mathcal{O})$, and therefore we can extend v to a representation $G_K \to N_G(T)(\mathcal{O})$ by setting $\Phi_K \mapsto X$. $\square$

4. Hodge-Tate theory for Galois representations valued in reductive groups

The Hodge-Tate theory for GL_N is reviewed in Appendix 3.5. In this section, we discuss Hodge-Tate theory for general reductive groups, and show G-irreducible mod ϖ Galois representations admit Hodge-Tate regular crystalline lifts.

4.1. First properties of Hodge-Tate cocharacters

4.1.1. Definitions Fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. Let $K, E \subset \overline{\mathbb{Q}_p}$ be finite extensions of $\mathbb{Q}_p$. The field E will serve as the coefficient field. To define colabeled Hodge-Tate gradings, we assume K is a subfield of E and therefore G_E as a subgroup of G_K.

Let $\mathbb{C} := \mathbb{C}_K$ be the completed algebraic closure of K. Let $\sigma : E \hookrightarrow \mathbb{C}$ be an embedding. Let (ρ, V) be a Hodge-Tate representation of G_K. Then one can define *the σ-colabeled Hodge-Tate grading* on $\mathbb{C} \otimes_{\sigma,E} V$ by setting the i-th graded piece to be

$$\mathrm{Im}((\mathbb{C}(i) \otimes_{\sigma,E} V)^{G_E} \otimes_E \mathbb{C}(-i) \to \mathbb{C} \otimes_{\sigma,E} V)$$

which is compatible with tensor product and duality.

Let G be a reductive group over E. A G-valued representation is Hodge-Tate if for all representations $G \to \mathrm{GL}(V)$, V is a Hodge-Tate G_K-module. Let $\rho : G_K \to G(E)$ be a Hodge-Tate G-valued representation. Consider $G(\sigma) \circ \rho : G_K \to G(\mathbb{C})$. By Tannakian theory, there is a cocharacter $\mathcal{HT}(\rho)^\sigma : \mathbb{G}_\mathrm{m} \to G_\mathbb{C}$, such that for any faithful representation $i : G \to \mathrm{GL}_N$, the composition $i(\mathcal{HT}(\rho)^\sigma)$ recovers the Hodge-Tate grading on $i(G(\sigma) \circ \rho) : G_K \to \mathrm{GL}_N(\mathbb{C})$.

Set $\mathcal{HT}(\rho) := (\mathcal{HT}(\rho)^\sigma)_{\sigma:E\hookrightarrow\mathbb{C}} \in \prod_{E\hookrightarrow\mathbb{C}} X_*(G_\mathbb{C})$. We call $\mathcal{HT}(\rho)$ the *co-labeled Hodge-Tate cocharacter* of ρ.

The formation of co-labeled Hodge-Tate cocharacters is clearly functorial in G.

4.1.2. Lemma Let $f : G \to H$ be a morphism of reductive groups over E. If $\rho : G_K \to G(E)$ is a Hodge-Tate representation, we have $\mathcal{HT}(f \circ \rho) = f(\mathcal{HT}(\rho))$.

Proof. It follows immediately from Tannakian theory. □

4.1.3. Regular cocharacter Let H be a reductive group with maximal torus S. A cocharacter $x \in X_*(H, S)$ is said to be *regular* if it is not killed by any root of H (with respect to S).

We say ρ is *Hodge-Tate regular* if for all $\sigma : E \hookrightarrow \mathbb{C}$, the cocharacter $\mathcal{HT}(\rho)^\sigma$ of $G_\mathbb{C}$ is regular.

When $G = \mathrm{GL}_N$, we can also define *labeled Hodge-Tate weights* (see Appendix 3.5). It turns out labeled Hodge-Tate regularity is equivalent to colabeled Hodge-Tate regularity. So our definition coincides with the usual notion of Hodge-Tate regularity in the literature.

4.1.4. Lemma Assume $G = \mathrm{GL}_N$. Assume E admits an embedding of the Galois closure of K. Then ρ is Hodge-Tate regular if and only if the *labeled Hodge-Tate weight* $\mathbf{k} = (k_\tau)_{\tau:K\hookrightarrow E}$ is regular in the sense that each $k_\tau \in \mathbb{Z}^N$ contains distinct numbers.

Proof. It follows from Proposition 5.1.4. □

4.1.5. Lemma Let K'/K be a finite field extension such that $K' \subset E$. Let $\rho : G_K \to G(E)$ be a Hodge-Tate G-valued representation. We have $\mathcal{HT}(\rho|_{G_{K'}}) = \mathcal{HT}(\rho)$.

Proof. Note that the Definition 4.1.1 only makes use of G_E and does not depend on K. □

4.1.6. Lemma Let $\rho_1, \rho_2 : G_K \to G(E)$ be two Hodge-Tate representations whose image is abelian and consists of semisimple elements. If $\rho_1\rho_2 = \rho_2\rho_1$, then $\mathcal{HT}(\rho_1\rho_2) = \mathcal{HT}(\rho_1)\mathcal{HT}(\rho_2)$.

Proof. By the previous lemma, it is harmless to shrink G_K and thus we can assume ρ_1, ρ_2 both factor through a maximal torus T of G. By descent, we can assume T is split. Write $i : T \hookrightarrow G$ for the embedding of the maximal torus T. Let $t_1, t_2 : G_K \to T(E)$ be representations such that $i(t_1) = \rho_1$ and $i(t_2) = \rho_2$.

We have $\mathcal{HT}(\rho_1) = i(\mathcal{HT}(t_1))$ and $\mathcal{HT}(\rho_2) = i(\mathcal{HT}(t_2))$ by functoriality (Lemma 4.1.2). So it suffices to show $\mathcal{HT}(t_1t_2) = \mathcal{HT}(t_1)\mathcal{HT}(t_2)$. Since T is a split torus, the general case follows from the special case $T = \mathbb{G}_{\mathrm{m}}$. The Hodge-Tate cocharacter of $t_1t_2 : G_K \to \mathbb{G}_{\mathrm{m}}(E)$ is completely decided by the Hodge-Tate weight of t_1t_2. The lemma follows because the Hodge-Tate weight of t_1t_2 is the sum of the Hodge-Tate weight of t_1 and the Hodge-Tate weight of t_2. □

We use the following lemma to construct Hodge-Tate regular cocharacters.

4.1.7. Lemma Assume $E = K_f$ is the unramified extension of K of degree f inside the fixed algebraic closure $\overline{K} := \overline{\mathbb{Q}_p}$ of K. Fix a maximal split torus T of G. Write $i : T \to G$ for the embedding.

For each colabel $\sigma_0 : K_f \hookrightarrow \mathbb{C}$, and each cocharacter $\lambda \in X_*(G(\mathbb{C}), T(\mathbb{C}))$, there exists a crystalline representation $t : G_{K_f} \to T(K_f)$ such that

$$\mathcal{HT}(i(t))^\sigma = \begin{cases} \lambda & \text{if} \quad \sigma = \sigma_0, \\ \text{the trivial cocharacter} & \text{if otherwise.} \end{cases}$$

Proof. Let $\chi_{\mathrm{LT}} : G_{K_f} \to \mathcal{O}_{K_f}^*$ be a Lubin-Tate character. Choose an isomorphism $T \cong \mathbb{G}_{\mathrm{m}}{}^{\times r}$, $r = \operatorname{rk} T$.

The field K_f is a subfield of $\overline{K}$ by its choice. The composite $K_f \hookrightarrow \overline{K} \hookrightarrow \mathbb{C}$ defines a canonical embedding of K_f in $\mathbb{C}$. Since K_f/K is a Galois extension, there exists a unique $\iota \in \mathrm{Gal}(K_f/K)$ such that $\sigma_0 \circ \iota$ is the canonical embedding $K_f \hookrightarrow \mathbb{C}_K$.

Put $t = \iota(\chi_{\mathrm{LT}}^{h_1}, \cdots, \chi_{\mathrm{LT}}^{h_r})$, $h_1, \cdots, h_r \in \mathbb{Z}$. By Lemma 4.1.2 and Lemma 5.3.5, $\mathcal{HT}(i(t))^\sigma$ is the trivial cocharacter if $\sigma \neq \sigma_0$. Since the co-labeled Hodge-Tate weights of the Lubin-Tate character is $(1, 0 \cdots, 0)$, if we let the tuple $(h_1, \cdots, h_r)$ range over all $\mathbb{Z}^r$, then $\mathcal{HT}(i(\iota(\chi_{\mathrm{LT}}^{h_1}, \cdots, \chi_{\mathrm{LT}}^{h_r})))^{\sigma_0}$ ranges over all cocharacters in $X_*(G(\mathbb{C}), T(\mathbb{C}))$. So we can choose $(h_1, \cdots, h_r)$ so that $\mathcal{HT}(i(\iota(\chi_{\mathrm{LT}}^{h_1}, \cdots, \chi_{\mathrm{LT}}^{h_r})))^{\sigma_0} = \lambda$. $\square$

4.2. Hodge-Tate regular lifts of strongly semisimple mod ϖ Galois representations

In many applications, we need Hodge-Tate regular crystalline representations. For example, crystalline deformation rings of regular Hodge-Tate weights have the largest dimension, which is exploited in the work [EG19].

The following lemma shows as long as a crystalline lift exists, Hodge-Tate regular lifts also exist.

We will specialize to the case where $E = K_f$, the unramified extension of K of degree f.

4.2.1. Local class field theory Let $\mathrm{Art}_K : K^\times \to G_K^{\mathrm{ab}}$ be the local Artin map, which we normalize so that a uniformizer corresponds to a geometric Frobenius element.

Note that Art_K induces an isomorphism

$$\mathrm{Art}_K^{-1} : \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{ur}}) \xrightarrow{\cong} \mathcal{O}_K^\times$$

See the paragraph after the proof of [Iw86, 6.2] for a reference. Denote by r_K the induced map $I_K \to \mathcal{O}_K^\times$.

4.2.1.1 Theorem [Iw86, 6.11] Let $\sigma : K \to K'$ be an isomorphism of fields. Then the following diagram is commutative:

$$\begin{tikzcd} K^\times \arrow[r, "\mathrm{Art}_K"] \arrow[d, "\sigma"] & G_K^{\mathrm{ab}} \arrow[d, "\sigma^*"] \\ K'^\times \arrow[r, "\mathrm{Art}_{K'}"] & G_{K'}^{\mathrm{ab}} \end{tikzcd}$$

Here $\sigma^* : \tau \mapsto \sigma\tau\sigma^{-1}$.

4.2.2. Corollary Let $\sigma : K \to K$ be a continuous field automorphism. Then $r_K(\sigma\tau\sigma^{-1}) = \sigma(r_K(\tau))$ for all $\tau \in I_K$.

Proof. It is an immediate consequence of Theorem 4.2.1.1. □

4.2.3. Theorem Let $\bar{\rho} : G_K \to G(\mathbb{F})$ be a G-completely reducible representation. Let κ be the residue field of K. Assume $\kappa \subset \mathbb{F}$.

(1) There exists a Hodge-Tate regular crystalline lift $\rho : G_K \to G(\mathcal{O}_{K_f})$ for some positive integer f.

(2) If G has a simply connected derived subgroup and $\mathbb{F}$ is the splitting field of $\bar{\rho}|_{I_K}$ (see Theorem 3.5), then f can be taken as $[\mathbb{F} : \kappa]$.

Proof. Write $i : T \hookrightarrow G$ for the embedding of the maximal torus T.

We will show that as long as a crystalline lift exists, a Hodge-Tate regular crystalline also exists with the same coefficient field. The existence of crystalline lifts is Theorem 3.5.

We keep notations used in the proof of Proposition 3.4. We set $\mathcal{O} := \mathcal{O}_{K_f}$. Recall that $\Xi := \sum_{i=0}^{f-1} w^i \otimes \Phi_K^{f-1-i}$, where $\Phi_K \in G_K/I_K$ is a generator of G_K/I_K, and $w \in W(G,T)$ is the Weyl group element which corresponds to $\bar{\rho}(\Phi_K)^{-1}$. Recall that the submodule $M^0_{T,\mathrm{cris}} \subset M_{T,\mathrm{cris}}$ consists of representations $I_K \to T(\mathcal{O})$ which are extendable to G_{K_f}. For each element of $u \in M^0_{T,\mathrm{cris}}$, choose an extension $t_u : G_{K_f} \to T(\mathcal{O})$. The Hodge-Tate cocharacter $\mathcal{HT}(i(t_u))$ does not depend on the choice of t_u. It makes sense to write $\mathcal{HT}(u)$ for $\mathcal{HT}(i(t_u))$ (where t_u is any choice of extension).

In the proof of Proposition 3.4, we've shown that there exists $v \in \Xi M^0_{T,\mathrm{cris}} \subset M^0_{T,w,\mathrm{cris}}$ which is a lift of $\bar{\rho}|_{I_K}$.

Fix a colabel $\sigma_0 : K_f \hookrightarrow \mathbb{C}$. By Lemma 4.1.7, there exists a crystalline representation $t : G_{K_f} \to T(\mathcal{O})$ such that $\mathcal{HT}(i(t))^\sigma$ is a regular cocharacter in $X_*(G(\mathbb{C}), T(\mathbb{C}))$ if $\sigma = \sigma_0$, and is the trivial cocharacter if $\sigma \neq \sigma_0$.

The restriction $t|_{I_K}$ defines an element $v_0 \in M^0_{T,\mathrm{cris}}$. By Lemma 4.1.2, we have

$$\mathcal{HT}((w \otimes 1)v_0) = w\mathcal{HT}(v_0)w^{-1}.$$

By Lemma 5.3.5 and Corollary 4.2.2, we have

$$\mathcal{HT}((1 \otimes \Phi_K)v_0)^{\sigma} = \mathcal{HT}(v_0)^{\sigma\circ\Phi_K^{-1}}.$$

Summing up, we have

$$\begin{aligned}
\mathcal{HT}(\Xi v_0)^{\sigma_0\circ\Phi_K^{-1-i+f}} &= \mathcal{HT}(\sum_{j=0}^{f-1} w^j \otimes \Phi_K^{f-1-j}v_0)^{\sigma_0\circ\Phi_K^{-1-i+f}} \\
&= \prod_{j=0}^{f-1} \mathcal{HT}(w^j \otimes \Phi_K^{f-1-j}v_0)^{\sigma_0\circ\Phi_K^{-1-i+f}} \\
&= \prod_{j=0}^{f-1} w^j\mathcal{HT}(1 \otimes \Phi_K^{f-1-j}v_0)^{\sigma_0\circ\Phi_K^{-1-i+f}} w^{-j} \\
&= \prod_{j=0}^{f-1} w^j\mathcal{HT}(v_0)^{\sigma_0\circ\Phi_K^{-1-i+f}\circ\Phi_K^{1+j-f}} w^{-j} \\
&= w^i\mathcal{HT}(v_0)^{\sigma_0} w^{-i}
\end{aligned}$$

By Definition 4.1.3, Ξv_0 is Hodge-Tate regular.

Let C be a very large positive integer. Write N for the cardinality of $\mathbb{F}^\times$. Define $v' := v + CN\Xi v_0$. Since $M_{T,\mathbb{F}}$ is N-torsion, v' is a lift of $\bar{\rho}|_{I_K}$. We have $\mathcal{HT}(v') = \mathcal{HT}(v)\mathcal{HT}(\Xi v_0)^{CN}$. Since $\mathcal{HT}(\Xi v_0)$ is a regular cocharacter, $\mathcal{HT}(v')$ is also a regular cocharacter if $C \gg 0$.

Since $\Xi M^0_{T,\mathrm{cris}} \subset M^0_{T,w,\mathrm{cris}}$, we have $v + \Xi v_0 \in M_{T,w,\mathrm{cris}}$. By Corollary 3.8, v' extends to a representation $G_K \to G(\mathcal{O})$ which is a crystalline representation lifting $\bar{\rho}$. $\square$

5. Appendix: Hodge-Tate theory with coefficients

Let $K/\mathbb{Q}_p$, $E/\mathbb{Q}_p$ be finite extensions. Assume E admits an embedding of the Galois closure of K. Fix an embedding $K \hookrightarrow E$. Let V be a finite dimensional E-vector space. Let $\rho : G_K \to \mathrm{GL}(V)$ be a continuous representation. Assume ρ is Hodge-Tate. Let $\mathbb{C} := \mathbb{C}_K$ be the completed algebraic closure of K. Let $\mathbb{B}_{\mathrm{HT}} := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}(n)$ be the Hodge-Tate period ring. Then $\mathbb{B}_{\mathrm{HT}} \otimes V := \mathbb{B}_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V$ is a $\mathbb{C} \otimes E$-module with G_K-action.

Let σ be an embedding $E \hookrightarrow \mathbb{C}$. Define

$$\begin{aligned} V_\sigma &:= \{\sum x_i \otimes y_i \in \mathbb{B}_{\mathrm{HT}} \otimes V \,|\, \sum \sigma(a)x_i \otimes y_i = \sum x_i \otimes a y_i \text{ for all } a \in E\} \\ &= \bigcap_{a\in E} \mathrm{Ker}(l_{1\otimes a} - l_{\sigma(a)\otimes 1}) \qquad \text{(where } l_x \text{ is scalar multiplication by } x) \end{aligned}$$

It is easy to see that

5.0.1. Lemma Let $L_\sigma \subset \mathbb{C}$ be the subfield generated by K and $\sigma(E)$.

(i) V_σ is a G_{L_σ}-stable $\mathbb{C} \otimes E$-submodule of $\mathbb{B}_{\mathrm{HT}} \otimes V$;

(ii) V_σ is isomorphic to $\mathbb{B}_{\mathrm{HT}} \otimes_{\sigma,E} V$ as a G_{L_σ}-semi-linear $\mathbb{C}$-module;

(iii) $\mathbb{B}_{\mathrm{HT}} \otimes V = \bigoplus_{\sigma:E\hookrightarrow\mathbb{C}} V_\sigma$.

Let L be the Galois closure of L_σ in $\mathbb{C}$. Write $D_\sigma(V) := V_\sigma^{G_L}$. By (iii),

$$\bigoplus_{\sigma:E\hookrightarrow\mathbb{C}} D_\sigma(V) = (\mathbb{B}_{\mathrm{HT}} \otimes V)^{G_L} = D_{\mathrm{HT}}(V) \otimes_K L$$

The Hodge-Tate grading on $D_{\mathrm{HT}}(V)$ induces a grading on each of $D_\sigma(V)$. So $D_\sigma(V_\sigma)$ is a graded L-vector space. We denote by $\mathrm{HT}^\sigma(V)$ the multiset of integers n in which n occurs with multiplicity $\dim_L \mathrm{gr}^n D_\sigma(V_\sigma)$, and call it the *$\sigma$-co-labeled Hodge-Tate weights of V*. [1]

5.1. Labeled Hodge-Tate weights Let $\tau : K \hookrightarrow E$ be an embedding. Define

$$\begin{aligned} \tilde{V}_\tau &:= \{\sum x_i \otimes y_i \in \mathbb{B}_{\mathrm{HT}} \otimes V | \sum a x_i \otimes y_i = \sum x_i \otimes \tau(a) y_i \text{ for all } a \in K\} \\ &= \bigcap_{a\in K} \mathrm{Ker}(l_{a\otimes 1} - l_{1\otimes\tau(a)}) \end{aligned}$$

5.1.1. Lemma We have

$$\tilde{V}_\tau = \bigoplus_{\sigma:E\hookrightarrow\mathbb{C},\sigma|_{\tau K}=\tau^{-1}} V_\sigma$$

Proof. Unravel the definitions. □

While V_σ is only G_{L_σ}-stable, $\tilde{V}_\tau$ is G_K-stable! Write $\tilde{D}_\tau(V) := (\tilde{V}_\tau)^{G_K}$. We want to remind readers the usual definition of τ-labeled Hodge-Tate weights (for example, the definition in [GK14, 1.1]).

[1] This is a non-standard terminology.

5.1.2. Definition The multiset $\mathrm{HT}_\tau(V)$ is as follows: an integer n appears with multiplicity

$$\dim_E \mathrm{gr}^n(D_{\mathrm{HT}}(V) \otimes_{E\otimes_{\mathbb{Q}_p} K,\tau} E)$$

5.1.3. Lemma We have $\dim_E \mathrm{gr}^n(D_{\mathrm{HT}}(V) \otimes_{E\otimes_{\mathbb{Q}_p} K,\tau} E) = \dim_E \mathrm{gr}^n(\tilde{D}_\tau(V))$.

Proof. It is easy to see (by unravelling the definitions) that the natural map

$$\tilde{V}_\tau^{G_K} \hookrightarrow D_{\mathrm{HT}}(V) \twoheadrightarrow D_{\mathrm{HT}}(V) \otimes_{E\otimes_{\mathbb{Q}_p} K,\tau} E$$

is injective, and E-linear. So it must be an E-isomorphism because of the direct sum decomposition.

$\square$

When we divide a multiset by an integer s, we divide the multiplicity of all members of the multiset by s. For example $\frac{1}{2}\{1,1,2,2,2,2\} = \{1,2,2\}$.

5.1.4. Proposition We have $\mathrm{HT}_\tau(V) = \frac{1}{[E:K]} \bigcup_{\sigma:E\hookrightarrow\mathbb{C},\sigma|_{\tau K}=\tau^{-1}} \mathrm{HT}^\sigma(V)$.

Proof. Let L be as before. We have

$$\tilde{D}_\tau(V) \otimes_K L \;=\; \tilde{V}_\tau^{G_L} = \bigoplus_{\sigma:E\hookrightarrow\mathbb{C},\sigma|_{\tau K}=\tau^{-1}} V_\sigma^{G_L} = \bigoplus_{\sigma:E\hookrightarrow\mathbb{C},\sigma|_{\tau K}=\tau^{-1}} D_\sigma(V)$$

as graded modules. So

$$\dim_E(\tilde{D}_\tau(V)) = \frac{1}{[E:K]} \dim_K(\tilde{D}_\tau(V)) = \frac{1}{[E:K]} \dim_L(\tilde{D}_\tau(V) \otimes_K L) = \frac{1}{[E:K]} \sum_{\sigma:E\hookrightarrow\mathbb{C},\sigma|_{\tau K}=\tau^{-1}} \dim_L D_\sigma(V)$$

$\square$

Thus the multiset of τ-labeled Hodge-Tate weights is the average of certain multisets of σ-co-labeled Hodge-Tate weights.

5.2. Galois twist The following is a convenient observation.

5.2.1. Lemma Let K, E be arbitrary finite extensions of $\mathbb{Q}_p$. Let L/E be a field extension. Let $\sigma : E \hookrightarrow \mathbb{C}$ be an embedding. Let $\tilde{\sigma} : L \hookrightarrow \mathbb{C}$ be an embedding extending σ. Let K'/K be a finite extension. Then

(1) $\mathrm{HT}^{\sigma}(\mathrm{Res}_{G_K}^{G_{K'}} V) = \mathrm{HT}^{\sigma}(V)$;

(2) $\mathrm{HT}^{\sigma}(V) = \mathrm{HT}^{\tilde{\sigma}}(V \otimes_E L)$.

Assume moreover that E admits an embedding of the Galois closure of K. Let $\tau : K \hookrightarrow E$ be an embedding. Then

(3) $\mathrm{HT}_{\tau}(V) = \mathrm{HT}_{\tau}(V \otimes_E L)$.

Proof. (1), (3): unravel definitions; (2): $\mathbb{B}_{\mathrm{HT}} \otimes_{L,\tilde{\sigma}} (V \otimes_E L) = (\mathbb{B}_{\mathrm{HT}} \otimes_{L,\tilde{\sigma}} L) \otimes_E V = \mathbb{B}_{\mathrm{HT}} \otimes_{E,\sigma} V$. $\square$

5.2.2. Corollary Assume E contains the Galois closure of K. Let $\theta \in \mathrm{Aut}(E/\mathbb{Q}_p)$. Let $\tau : K \hookrightarrow E$ be an embedding. Then

(1) $\mathrm{HT}^{\sigma}(V \otimes_{E,\theta} E) = \mathrm{HT}^{\sigma\circ\theta}(V)$.

(2) $\mathrm{HT}_{\tau}(V \otimes_{E,\theta} E) = \mathrm{HT}_{\theta^{-1}\circ\tau}(V)$.

Proof. (1) It is a special case of Lemma 5.2.1(2).

(2) By Proposition 5.1.4,

$$\begin{aligned}
\mathrm{HT}_\tau(V \otimes_{E,\theta} E) &= \frac{1}{[E:K]} \sum_{\sigma: E \hookrightarrow \mathbb{C}, \sigma|_{\tau K} = \tau^{-1}} \mathrm{HT}^\sigma(V \otimes_{E,\theta} E) \\
&= \frac{1}{[E:K]} \sum_{\sigma: E \hookrightarrow \mathbb{C}, \sigma|_{\tau K} = \tau^{-1}} \mathrm{HT}^{\sigma\circ\theta}(V) \\
&= \frac{1}{[E:K]} \sum_{\sigma: E \hookrightarrow \mathbb{C}, \sigma\circ\theta^{-1}|_{\tau K} = \tau^{-1}} \mathrm{HT}^\sigma(V) \\
&= \mathrm{HT}_{\theta^{-1}\circ\tau}(V) \quad \square
\end{aligned}$$

5.3. Lubin-Tate characters In this subsection, we want to rewrite some results of [Se89, III.A.1-III.A.5] using the language we just developed.

5.3.1. Remark Proposition B.2 of [C11, Appendix B] contains a result more general than this subsection.

5.3.2. Note that *the cyclotomic character has Hodge-Tate weight* -1.

5.3.3. Lubin-Tate characters of Galois extensions of $\mathbb{Q}_p$ We start with the simpliest case. Let $E = K/\mathbb{Q}_p$ be a finite Galois extension. Let π be a uniformizer of K. Let F_π be the Lubin-Tate formal group associated to K and π. Let $\chi_K := \chi_{K,\pi} : G_K \to \mathcal{O}_E^\times$ be the Tate module of F_π, as is the notation of [Se89]. Then $\chi_K|_{I_K} = r_K^{\otimes -1}$ (see subsection 4.2.2). (So r_K is crystalline.)

5.3.3.1 Lemma Let $\sigma_1 \in \mathrm{Gal}(K/\mathbb{Q}_p)$. Then a σ-co-labeled Hodge-Tate weight of $\sigma_1 \circ \chi_K$ is -1 if $\sigma = \sigma_1^{-1}$, and 0 if otherwise.

Proof. See [Se89, Thm 2, III.A.5] and [Se89, Prop III.A.4]. Note that

- Serre's K and E are reversed,
- Galois hypothesis is required by [Se89, III.A.3(b)],
- Serre's W_σ is our $\mathrm{gr}^0 V_\sigma$.

□

5.3.4. Lemma Now suppose $E = K/\mathbb{Q}_p$ is not necessarily Galois. A σ-co-labeled Hodge-Tate weight of χ_K is -1 if $\sigma = \mathrm{id}$ [2], and 0 if otherwise.

Proof. Choose a Galois closure L of K over $\mathbb{Q}_p$. Consider

$$\begin{tikzcd} G_L^{\mathrm{ab}} \arrow[r] \arrow[d, "\mathrm{rec}"] & G_K^{\mathrm{ab}} \arrow[d, "\mathrm{rec}"] \\ L^\times \arrow[r, "N_{L/K}"] & K^\times \end{tikzcd}$$

By local class field theory, $\chi_K|_{G_L} = N_{L/K} \circ \chi_L = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma \circ \chi_L$. By Lemma 5.3.3, for $\tau \in \mathrm{Gal}(L/\mathbb{Q}_p)$,

$$\mathrm{HT}^\tau(\chi_K|_{G_L}) = \begin{cases} -1 & \text{if } \tau \text{ fixes } K \\ 0 & \text{if otherwise} \end{cases}$$

Now apply Lemma 5.2.1(1), (2) to conclude. □

[2]More precisely the tautological embedding of E in $\mathbb{C}$

5.3.5. Lemma Let $K/\mathbb{Q}_p$ be a finite extension, and let $E/\mathbb{Q}_p$ be a finite extension admitting $\iota : K \hookrightarrow E$.

(1) For each $\sigma : E \hookrightarrow \mathbb{C}$, the σ-co-labeled Hodge-Tate weight of $\iota \circ \chi_K$ is -1 if $\sigma \circ \iota = \mathrm{id}_K$, and 0 if otherwise.

(2) Suppose further E admits an embedding of the normal closure of K. Then for each $\sigma : K \to E$, the σ-labeled Hodge-Tate weight of $\iota \circ \chi_K$ is -1 if $\sigma = \iota$, and 0 if otherwise.

Proof. (1) We have

$$\begin{aligned} \mathrm{HT}^{\sigma}(\iota \circ \chi_K) &= \mathrm{HT}^{\sigma \circ \iota}(\chi_K) && \text{By Lemma 5.2.1(2)} \\ &= \begin{cases} -1 & \text{if } \sigma \circ \iota = \mathrm{id}_K \\ 0 & \text{if otherwise} \end{cases} && \text{By Lemma 5.3.4} \end{aligned}$$

(2) Follows from Proposition 5.1.4 and (1). □

5.3.6. Lemma Let $K/\mathbb{Q}_p$ be a finite extension. Let L/K be an unramified extension in $\mathbb{C}$. Let L' be the Galois closure of L over $\mathbb{Q}_p$. Let $\iota : K \hookrightarrow L'$ be the tautological embedding. Let $\Phi_K \in G_K$ be a lift of a topological generator of G_K/I_K. Let $d = [L : K]$. Then

(1) Let $\sigma : L \hookrightarrow \mathbb{C}$. Then

$$\mathrm{HT}^{\sigma}(\mathrm{Ind}_{G_L}^{G_K}(\chi_L)) = \mathrm{HT}^{\sigma}(\chi_L) \cup \mathrm{HT}^{\sigma}(\Phi_K \circ \chi_L) \cup \cdots \cup \mathrm{HT}^{\sigma}(\Phi_K^{d-1} \circ \chi_L)$$

(2) Let $\tau : K \hookrightarrow \mathbb{C}$. Then

$$\mathrm{HT}_\tau(\mathrm{Ind}_{G_L}^{G_K}(\iota \circ \chi_L)) = \begin{cases} \{0, \ldots, 0, -1\} & \text{if } \tau \text{ is the canonical embedding, and} \\ \{0, \ldots, 0, 0\} & \text{if otherwise.} \end{cases}$$

Proof. (1) Follows from Lemma 5.2.1 and Corollary 4.2.2.

(2) We have

$$\begin{aligned}
\mathrm{HT}_\tau(\mathrm{Ind}_{G_L}^{G_K}(\iota \circ \chi_L)) &= \frac{1}{[L' : K]} \bigcup_{\tilde{\sigma}: L' \hookrightarrow \mathbb{C}, \tilde{\sigma}|_{\tau K} \circ \tau = id} \mathrm{HT}^{\tilde{\sigma}}(\mathrm{Ind}_{G_L}^{G_K}(\iota \circ \chi_L)) \\
&= \frac{1}{[L' : K]} \bigcup_{\tilde{\sigma}: L' \hookrightarrow \mathbb{C}, \tilde{\sigma}|_{\tau K} \circ \tau = id, \sigma := \tilde{\sigma}|_L} \mathrm{HT}^{\sigma}(\mathrm{Ind}_{G_L}^{G_K}(\chi_L)) \\
&= \frac{1}{[L : K]} \bigcup_{\sigma: L \hookrightarrow \mathbb{C}, \sigma|_{\tau K} \circ \tau = id} \mathrm{HT}^{\sigma}(\mathrm{Ind}_{G_L}^{G_K}(\chi_L)) \\
&= \frac{1}{[L : K]} \bigcup_{\sigma: L \hookrightarrow \mathbb{C}, \sigma|_{\tau K} \circ \tau = id} \mathrm{HT}^{\sigma}(\chi_L) \cup \mathrm{HT}^{\sigma}(\Phi_K \circ \chi_L) \cup \cdots \cup \mathrm{HT}^{\sigma}(\Phi_K^{d-1} \circ \chi_L) \\
&= \frac{1}{[L : K]} \bigcup_{\sigma: L \hookrightarrow \mathbb{C}, \sigma|_{\tau K} \circ \tau = id} \bigcup_{k=0}^{d-1} \delta_{\sigma, \iota \circ \Phi_K^k}
\end{aligned}$$

Here $\delta_{X,Y}$ is $\{-1\}$ if $X = Y$ and is $\{0\}$ if otherwise. Since Φ_K^k is the identity when restricted on K, the last line is 0 unless τ is the canonical embedding; in this case, the last line becomes $\frac{1}{[L:K]} \bigcup_{j=0}^{d-1} \bigcup_{k=0}^{d-1} \delta_{\iota \circ \Phi_K^j, \iota \circ \Phi_K^k} = \{0, \ldots, 0, -1\}$. $\square$

Chapter 4

Lifting extension classes

1. Lyndon-Demuškin theory

Assume $p \neq 2$.

Let $K/\mathbb{Q}_p$ be a finite extension containing the p-th roots of unity. The maximal pro-p quotient of the absolute Galois group G_K has a very nice description. The following well-known theorem can be found, for example, in [Se02, Section II.5.6].

1.0.1. Theorem Let $G_K(p)$ be the maximal pro-p quotient of G_K. Then $G_K(p)$ is the pro-p completion of the following one-relator group

$$\Gamma^{\text{disc}} := \langle x_0, \cdots, x_{n+1} | x_0^q (x_0, x_1)(x_2, x_3) \ldots (x_n, x_{n+1}) \rangle$$

where $n = [K : \mathbb{Q}_p]$, $(x, y) = x^{-1}y^{-1}xy$, and $q = p^s$ is the largest power of p such that K contains the q-th roots of unity.

1.0.2. Definition A continuous G_K-module A is said to be *Lyndon-Demuškin* if G_K acts on A through $G_K(p)$.

1.1. Discrete group cohomology of Demuškin groups

The main reference of this subsection is [Ly50].

1.1.0.1 Derivations A derivation of a group G is a left G-module M, together with a map $D : G \to M$ such that $D(uv) = Du + uDv$.

Say F is a free group with generators $x_1, \ldots x_m$. Denote by dFJ the module of universal derivations. Then dFJ is the free $\mathbb{Z}[F]$-module with basis $\{dx_i | i = 1, \ldots, m\}$.

Let $u \in F$. We can write $du \in dFJ$ as a linear combination of the basis elements: $du = \sum \frac{\partial u}{\partial x_i} dx_i$ where $\frac{\partial u}{\partial x_i} \in \mathbb{Z}[F]$.

1.1.0.2 Theorem ([Ly50, Corollary 11.2]) Let $G = \langle x_1, \ldots, x_m | R\rangle$ be a one-relator group where $R = Q^q$ for no $q > 1$. Let K be any left G-module. Then

$$H^2(G, K) \cong K/(\frac{\partial R}{\partial x_1}, \ldots \frac{\partial R}{\partial x_m})K$$

and $H^n(G, K) = 0$ for all $n > 2$.

1.1.0.3 Corollary We have $H^2(\Gamma^{\mathrm{disc}}, \mathbb{F}_p) = \mathbb{F}_p$.

Proof. We have the following computation:

$$\begin{aligned}
\frac{\partial R}{\partial x_0} &= 1 + x_0 + \cdots + x_0^{q-2} + x_0^{q-1}x_1^{-1} \\
\frac{\partial R}{\partial x_1} &= x_0^{q-1}x_1^{-1}(x_0 - 1) \\
\frac{\partial R}{\partial x_2} &= x_0^{q}(x_0, x_1)x_2^{-1}(x_3^{-1} - 1) \\
\frac{\partial R}{\partial x_3} &= x_0^{q}(x_0, x_1)x_2^{-1}x_3^{-1}(x_2 - 1) \\
\vdots & \quad \vdots \\
\frac{\partial R}{\partial x_{2k}} &= x_0^{q}(x_0, x_1)\cdots(x_{2k-2}, x_{2k-1})x_{2k}^{-1}(x_{2k+1}^{-1} - 1) \\
\frac{\partial R}{\partial x_{2k+1}} &= x_0^{q}(x_0, x_1)\cdots(x_{2k-2}, x_{2k-1})x_{2k}^{-1}x_{2k+1}^{-1}(x_{2k} - 1) \\
\vdots & \quad \vdots
\end{aligned}$$

Since $H^2(\Gamma^{\mathrm{disc}}, \mathbb{F}_p) = \frac{\mathbb{F}_p}{(\partial R/\partial x_0, \cdots, \partial R/x_{n+1})}$, it suffices to show

$$\frac{\partial R}{\partial x_0}\mathbb{F}_p = \cdots = \frac{\partial R}{\partial x_{n+1}}\mathbb{F}_p = 0.$$

Since $\mathbb{F}_p$ is a trivial G_K-module, it is clear $\frac{\partial R}{\partial x_1}\mathbb{F}_p = \ldots \frac{\partial R}{\partial x_{n+1}}\mathbb{F}_p = 0$. We also have $\frac{\partial R}{\partial x_0} = 1 + 1 + \cdots + 1 = q = 0 \bmod p$. □

1.2. Comparing cohomology of Demuškin groups and Galois cohomology

Let $K/\mathbb{Q}_p$ be a p-adic field containing the group of p-th root of unity. Let A be a Lyndon-Demuškin G_K-module. We want to compare the usual group cohomology $H^\bullet(\Gamma^{\mathrm{disc}}, A)$ and the continuous profinite cohomology $H^\bullet(G_K, A)$.

Note that there is a functorial map

$$(\dagger)\ \ H^\bullet(G_K, A) \to H^\bullet(\Gamma^{\mathrm{disc}}, A)$$

induced from the forgetful functor $\mathrm{Mod}_{\mathrm{cont}}(G_K(p)) \to \mathrm{Mod}(\Gamma^{\mathrm{disc}})$.

1.2.1. Lemma Let $\mathbb{F}_p$ be the G_K-module with trivial G_K-action. Then $(\dagger)$ induces isomorphisms:

(1) $H^1(G_K, \mathbb{F}_p) = H^1(\Gamma^{\mathrm{disc}}, \mathbb{F}_p)$;

(2) $H^2(G_K, \mathbb{F}_p) = H^2(\Gamma^{\mathrm{disc}}, \mathbb{F}_p)$;

Proof. (1) We have

$$H^1(G_K, \mathbb{F}_p) = \mathrm{Hom}_{\mathrm{cont}}(G_K, \mathbb{F}_p);$$

$$H^1(\Gamma^{\mathrm{disc}}, \mathbb{F}_p) = \mathrm{Hom}(\Gamma^{\mathrm{disc}}, \mathbb{F}_p).$$

Since $H^1(G_K, \mathbb{F}_p)$ classifies continuous extension classes of two trivial G_K-modules, $(\dagger)$ is injective. By local Euler characteristic, $\dim H^1(G_K, \mathbb{F}_p) = [K : \mathbb{Q}_p] + \dim H^0(G_K, \mathbb{F}_p) + \dim H^2(G_K, \mathbb{F}_p) = n + 2 = \dim H^1(\Gamma^{\mathrm{disc}}, \mathbb{F}_p)$. So $(\dagger)$ is an isomorphism.

(2) We have a commutative diagram

$$\begin{tikzcd}
H^1(G_K, \mathbb{F}_p) \times H^1(G_K, \mathbb{F}_p) \arrow[r, "\cup"] \arrow[d] & H^2(G_K, \mathbb{F}_p) \arrow[d] \\
H^1(\Gamma^{\mathrm{disc}}, \mathbb{F}_p) \times H^1(\Gamma^{\mathrm{disc}}, \mathbb{F}_p) \arrow[r, "\cup"] & H^2(\Gamma^{\mathrm{disc}}, \mathbb{F}_p)
\end{tikzcd}$$

Note that the first row is a non-degenerate pairing, and $H^2(G_K, \mathbb{F}) \cong \mathbb{F}$ by local Tate duality. By Lyndon's theorem or Corollary 1.1.0.3, we have $H^2(\Gamma^{\mathrm{disc}}, \mathbb{F}) \cong \mathbb{F}$. So it remains to show the cup product of the second row is non-trivial. Let $[c_0], [c_1] \in H^1(\Gamma^{\mathrm{disc}}, \mathbb{F}_p)$. $[c_0] \cup [c_1] = 0$ if and only if there exists a group homomorphism

$$\Gamma^{\mathrm{disc}} \to \begin{bmatrix} 1 & c_0 & * \\ & 1 & c_1 \\ & & 1 \end{bmatrix}$$

for some $*$. Define $c_i : \Gamma^{\mathrm{disc}} \to \mathbb{F}_p$ by sending x_i to 1 and other generators to 0, $i = 0, 1$. Then $[c_0] \cup [c_1] \neq 0$. □

1.2.2. Corollary Let A be a finite $\mathbb{F}_p$-vector space endowed with Lyndon-Demuškin G_K-action. Then (†) is an isomorphism $H^\bullet(G_K, A) \cong H^\bullet(\Gamma^{\mathrm{disc}}, A)$.

Proof. By Theorem 1.1.0.2, the cohomology groups $H^k(\Gamma^{\mathrm{disc}}, A) = 0$ for $k > 2$. So it remains to compare the cohomology groups of degree ≤ 2.

Let $G_K(p)$ be the maximal pro-p quotient of G_K. Then A is a $G_K(p)$-module. Since $G_K(p)$ is a pro-p group, A must contain the trivial representation $\mathbb{F}_p$. In particular, there is a short exact sequence

$$0 \to \mathbb{F}_p \to A \to A' \to 0$$

which induces the long exact sequence

$$\begin{tikzcd} H^0(G_K, A') \arrow[r] \arrow[d] & H^1(G_K, \mathbb{F}_p) \arrow[r] \arrow[d] & H^1(G_K, A) \arrow[r] \arrow[d] & H^1(G_K, A') \arrow[r] \arrow[d] & H^2(G_K, \mathbb{F}_p) \arrow[d] \\ H^0(\Gamma^{\mathrm{disc}}, A') \arrow[r] & H^1(\Gamma^{\mathrm{disc}}, \mathbb{F}_p) \arrow[r] & H^1(\Gamma^{\mathrm{disc}}, A) \arrow[r] & H^1(\Gamma^{\mathrm{disc}}, A') \arrow[r] & H^2(\Gamma^{\mathrm{disc}}, \mathbb{F}_p) \end{tikzcd}$$

We apply induction on the length of A. By the Five Lemma, we have $H^1(G_K, A) = H^1(\Gamma^{\mathrm{disc}}, A)$.

We also have the long exact sequence

$$\begin{tikzcd}
H^1(G_K, A') \arrow[r] \arrow[d] & H^2(G_K, \mathbb{F}_p) \arrow[r] \arrow[d] & H^2(G_K, A) \arrow[r] \arrow[d] & H^2(G_K, A') \arrow[r] \arrow[d] & H^3(G_K, \mathbb{F}_p) \arrow[d] \\
H^1(\Gamma^{\text{disc}}, A') \arrow[r] & H^2(\Gamma^{\text{disc}}, \mathbb{F}_p) \arrow[r] & H^2(\Gamma^{\text{disc}}, A) \arrow[r] & H^2(\Gamma^{\text{disc}}, A') \arrow[r] & H^3(\Gamma^{\text{disc}}, \mathbb{F}_p)
\end{tikzcd}$$

By Lyndon's theorem, $H^3(\Gamma^{\text{disc}}, \mathbb{F}_p) = 0$. By local Tate duality, $H^3(G_K, \mathbb{F}_p) = 0$. Again by the Five Lemma, we have $H^2(G_K, A) = H^2(\Gamma^{\text{disc}}, A)$. $\square$

By induction on the order of A, (†) is an isomorphism for any finite p-power torsion group A.

1.2.3. Corollary Let A be a finite $\mathbb{Z}_p$-module endowed with Lyndon-Demuškin G_K-action. Then there is a canonical isomorphism $H^\bullet(G_K, A) = H^\bullet(\Gamma^{\text{disc}}, A)$.

Proof. We have a short exact sequence for each $k > 0$,

$$0 \to \varprojlim_i{}^1 H^{k-1}(G_K, A/p^i A) \to H^k(G_K, A) \to \varprojlim_i H^k(G_K, A/p^i A) \to 0.$$

The first term is 0 due to the finiteness of the cohomology of torsion G_K-modules. So $H^k(G_K, A) = \varprojlim_i H^k(G_K, A/p^i A)$, and the corollary is reduced to the p-power torsion case. By [Ly50, Theorem 11.1], $H^\bullet(\Gamma^{\text{disc}}, A/p^i A)$ is a finite set. So we have $H^k(\Gamma^{\text{disc}}, A) = \varprojlim_i H^k(\Gamma^{\text{disc}}, A/p^i)$ by a similar argument. $\square$

The lemma above tells us that, for our purposes, the cohomology groups of $G_K(p)$ can be computed via the discrete model. So we can make use of the fine machineries of combinatorial group theory.

1.3. Lyndon-Demuškin Complex

1.3.1. Abelian coefficient case Let A be a G_K-module whose underlying abelian group is a finitely generated $\mathbb{Z}_p$-module such that the the action of G_K factors through $G_K(p)$.

Then there is an explicit co-chain complex computing the Galois cohomology $H^\bullet(G_K, A)$.

Define $C^\bullet_{\mathrm{LD}}(A) = [C^0_{\mathrm{LD}}(A) \xrightarrow{d^1} C^1_{\mathrm{LD}}(A) \xrightarrow{d^2} C^2_{\mathrm{LD}}(A)]$ as the following cochain complex supported on degrees [0,2]

$$A \xrightarrow{\begin{bmatrix} 1-x_0 \\ \cdots \\ 1-x_{n+1} \end{bmatrix}} A^{\oplus(n+2)} \xrightarrow{\begin{bmatrix} \partial R/\partial x_0 \\ \cdots \\ \partial R/\partial x_{n+1} \end{bmatrix}^T} A.$$

Then by [Ly50, Section 11] (the proof of [Ly50, Theorem 11.1]),

$$H^\bullet(C^\bullet_{LD}(A)) = H^\bullet(G_K, A).$$

The idea of Lyndon Demuškin complex is simple. A 1-cochain $c \in C^1_{\mathrm{LD}}(A)$ is simply a set-theoretical function

$$c : \{x_0, \dots, x_{n+1}\} \to A.$$

We can extend c to be a function on the free group

$$c : \langle x_0, \dots, x_{n+1} \rangle \to A$$

by setting $c(gh) := c(g) + g \cdot c(h)$ for any g, h in the free group with $(n+2)$ generators; equivalently,

c is the map which makes the following diagram commute

$$\begin{tikzcd}
\langle x_0,\dots,x_{n+1}\rangle \arrow[r, "(\mathrm{id},c)"] \arrow[dr, "\mathrm{id}"'] & \langle x_0,\dots,x_{n+1}\rangle \ltimes A \arrow[d] \\
& \langle x_0,\dots,x_{n+1}\rangle
\end{tikzcd}$$

Let

$$R = x_0^q(x_0,x_1)(x_2,x_3)\dots(x_n,x_{n+1})$$

be the single relation defining the Demuškin group. The differential operator $d^2 : C^1_{\mathrm{LD}}(A) \to C^2_{\mathrm{LD}}(A)$ is nothing but the evaluation of the extended map c at the relation R, that is, $d^2(c) = c(R)$. So a 1-cochain c is a 1-cocycle if and only if its evaluation at R is 0; equivalently, the map c factors through $\langle x_0,\dots,x_{n+1}|R\rangle$ and the following diagram commutes.

$$\begin{tikzcd}
\langle x_0,\dots,x_{n+1}|R\rangle \arrow[r, "(\mathrm{id},c)"] \arrow[dr, "\mathrm{id}"'] & \langle x_0,\dots,x_{n+1}|R\rangle \ltimes A \arrow[d] \\
& \langle x_0,\dots,x_{n+1}|R\rangle
\end{tikzcd}$$

Before we proceed to nilpotent coefficient case, we review cupproducts in group cohomology.

1.4. General cup products in group cohomology

Let V be a unipotent algebraic group of class 2 over $\mathcal{O}_E$. Let Γ be an abstract group, together with a homomorphism $\theta : \Gamma \to \mathrm{Aut}(V)(\mathcal{O}_E)$. By the Lie correspondence, $\mathrm{Aut}(\mathrm{Lie}\, V) \cong \mathrm{Aut}(V)$, and thus θ induces a $\mathcal{O}_E$-linear Γ-action on $\mathrm{Lie}\, V$ which respects Lie brackets.

We fix a grading $\mathrm{Lie}\, V = V_1 \oplus V_2$ such that $[V_1,V_1] \subset V_2$, and $[V,V_2] = 0$. We will write V for $V(\mathcal{O}_E)$ for simplicity.

Let $f : \Gamma \to V$ be a crossed homomorphism. By definition, for any $g_1, g_2 \in \Gamma$, $f(g_1g_2) = f(g_1)g_1f(g_2)$. Write $c = c_1 + c_2$ for $\log(f)$, where c_1 values in V_1 and c_2 values in V_2. By the Baker-Campbell-Hausdorff formula, we have

$$
\begin{aligned}
(*) \quad c(gh) &= c(g) + gc(h) + [c(g), gc(h)]/2 \\
&= (c_1(g) + gc_1(h)) + (c_2(g) + gc_2(h)) + [c_1(g), gc_1(h)]/2
\end{aligned}
$$

1.4.1. Lemma Let $a, b \in H^1(\Gamma, V_1)$ be two crossed homomorphisms. The 2-cochain $B(a, b) : (g, h) \mapsto [a(g), gb(h)]$ is a 2-cocycle.

Proof. By definition, we have

$$
\begin{aligned}
d^2(B(a,b))(g_1, g_2, g_3) &= g_1[a(g_2), g_2b(g_3)] - [d^1a(g_1, g_2), g_1g_2b(g_3)] \\
&\quad + [a(g_1), g_1d^1b(g_2, g_3)] + [a(g_1), g_1b(g_2)] \\
&= g_1[a(g_2), g_2b(g_3)] - [a(g_1) + g_1a(g_2), g_1g_2b(g_3)] \\
&\quad + [a(g_1), g_1b(g_2) + g_1g_2b(g_3)] + [a(g_1), g_1b(g_2)] \\
&= 0
\end{aligned}
$$

□

For crossed homomorphisms $a \in H^1(\Gamma, V_1)$, define $Q(a) := B(a, a)$.

Define $a \cup b := (Q(a+b, a+b) - Q(a) - Q(b))/2 = (B(a, b) + B(b, a))/2$. We have $a \cup b \in H^2(\Gamma, V_2)$.

1.4.2. Lemma Let $\Gamma' \subset \Gamma$ be a normal subgroup of finite index. Write Δ for Γ/Γ'.

The cup product $\cup : H^1(\Gamma', V_1) \times H^1(\Gamma', V_1) \to H^2(\Gamma', V_2)$ is Δ-equivariant.

Proof. Let $a, b \in H^1(\Gamma^1, V_1)$, and let $\sigma \in \Gamma$. We have by definition $\sigma \cdot a(g) = \sigma a(\sigma^{-1} g \sigma)$, and

$\sigma \cdot B(a,b)(g,h) = \sigma B(a,b)(\sigma^{-1}g\sigma, \sigma^{-1}h\sigma)$ (see [Se02, Section I.5.8]). We immediately have $\sigma \cdot B(a,b) = B(\sigma \cdot a, \sigma \cdot b)$. $\square$

1.4.3. Nilpotent coefficients Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_E$, residue field $\mathbb{F}$, and uniformizer ϖ.

Let U be a unipotent (smooth connected) linear algebraic group over $\operatorname{Spec} \mathcal{O}_E$. Write

$$1 = U_0 \subset U_1 \cdots \subset U_k = U$$

for the upper central series of U.

Assume $p > k$. There is a canonical isomorphism of schemes $U \cong \operatorname{Lie} U$ sending $g \mapsto \log g$. To define the logarithm function, it is convenient to choose an embedding $U \hookrightarrow \mathrm{GL}_N$, and define log using the commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \mathrm{GL}_N \arrow[d, "\log_{\leq k}"] \\ \operatorname{Lie} U \arrow[r] & \mathrm{Mat}_{N\times N} \end{tikzcd}$$

where $\log_{\leq k}$ is the truncated logorithm function.

We assume $k = 2$ from now on because it suffices for our applications.

Fix a Galois action $G_K \to \operatorname{Aut}(U)(\mathcal{O}_E)$ such that the image group is a pro-p subgroup of $\operatorname{Aut}(U)(\mathcal{O}_E)$.

Let A be an $\mathcal{O}_E$-algebra. Recall that a *non-abelian crossed homomorphism* valued in $U(A)$ is a map $c : G_K \to U(A)$ such that

$$c(gh) = c(g)(g \cdot c(h))$$

for all $g, h \in G_K$. Set $\mathfrak{c} := \log(c) : G_K \to \operatorname{Lie} U(A)$. By the Baker-Campbell-Hausdorff formula,

$$(\dagger) \quad \mathfrak{c}(gh) = \mathfrak{c}(g) + g \cdot \mathfrak{c}(h) + \frac{1}{2}[\mathfrak{c}(g), g \cdot \mathfrak{c}(h)].$$

Our definition of the Lyndon-Demuškin cochain complex is motivated by (†).

1.4.3.1 Definition Let A be an $\mathcal{O}_E$-algebra. The Lyndon-Demuškin complex with nilpotent coefficients is defined to be the following cochain complex $C^\bullet_{\mathrm{LD}}(U(A))$ supported on degrees [0,2]:

$$\operatorname{Lie} U(A) \xrightarrow{d^1} (\operatorname{Lie} U(A))^{\oplus n+2} \xrightarrow{d^2} \operatorname{Lie} U(A)$$

where d^1 is defined by

$$d^1(v) = \left(-v + x_i \cdot v + \frac{1}{2}[-v, x_i \cdot v]\right)_{i=0,\ldots,n+1}.$$

We need some preparations before we define d^2. An element $c = (\alpha_0, \cdots, \alpha_{n+1}) \in C^1_{\mathrm{LD}}(U(A))$ can be regarded as a function on the free group with $(n+2)$ generators

$$c : \langle x_0, \cdots, x_{n+1} \rangle \to \operatorname{Lie} U(A)$$

by setting $c(x_i) = \alpha_i$ for each i and extending it to the whole free group by

$$c(gh) := c(g) + g \cdot c(h) + \frac{1}{2}[c(g), g \cdot c(h)]$$

(To see it is well defined, note that $\exp(c)$ is a crossed homomorphism for a free group.) We define d^2 as

$$d^2(c) := c(R) = c(x_0^q(x_0, x_1)(x_2, x_3)\dots(x_n, x_{n+1})).$$

For, $c = d^1(v)$, we have $c(x_i) = -v + x_i \cdot v + [-v, x_i \cdot v]/2$ and $c(x_j) = -v + x_j \cdot v + [-v, x_j \cdot v]/2$. So $c(x_i x_j) = c(x_i) + x_i \cdot c(x_j) + [c(x_i), x_i \cdot c(x_j)]/2 = -v + x_i x_j \cdot v + [-v, x_i x_j \cdot v]/2$. Therefore $(d^1(v))(g) = -v + g \cdot v + [-v, g \cdot v]/2$ for all g in the free group. In particular, $d^2(d^1(v)) = -v + R \cdot v = 0$.

1.4.3.2 Remark (1) When U is an abelian group, then we recover the definition (for the cohomology of $U(A)$) in the previous section;

(2) The main reason we define $C^\bullet_{\mathrm{LD}}(U(A))$ this way is because we want to compare it with $C^\bullet_{\mathrm{LD}}(\mathrm{Lie}\, U(A))$.

Note that $C^\bullet_{\mathrm{LD}}(\mathrm{Lie}\, U(A))$ and $C^\bullet_{\mathrm{LD}}(U(A))$ have the same underlying group, but their differential $d^\bullet$ is different.

(3) Note that $d^2(c) = 0$ if and only if c defines a crossed homomorphism $\mathfrak{c} : G_K \to \mathrm{Lie}\, U(A)$ in the sense of (†).

A crossed homomorphism $\mathfrak{c} : G_K \to \mathrm{Lie}\, U(A)$ can be interpreted as a group homomorphism $G_K \to U(A) \rtimes G_K$, $g \mapsto \exp(\mathfrak{c}(g)) \rtimes g$. On the other hand, an element $c \in C^1_{\mathrm{LD}}(U(A))$ can be interpreted as a group homomorphism $r_c : \langle x_0, \cdots, x_{n+1}\rangle \to U(A) \rtimes \langle x_0, \cdots, x_{n+1}\rangle$, $g \mapsto \exp(c(g)) \rtimes g$. So c defines a crossed homomorphism if and only if $r_c(R) = 1$, which is exactly $d^2(c) = 0$.

(4) The differential maps are generally non-linear.

1.4.3.3 Definition We define $Z^i_{\mathrm{LD}} := (d^{i+1})^{-1}(0)$, and $B^i_{\mathrm{LD}} := d^i(C^{i-1}_{\mathrm{LD}})$ for $i = 0, 1, 2$.

1.4.3.4 Proposition For any $\mathcal{O}_E$-alegbra A, we have

$$H^0(G_K, U(A)) \cong Z^0_{\mathrm{LD}}(U(A))$$

and a surjection of pointed sets

$$Z^1_{\mathrm{LD}}(U(A)) \to H^1(G_K, U(A)).$$

Proof. The second claim is explained in remark 1.4.3.2(3). The first claim follows from the definitions. $\square$

$\operatorname{Lie} U$ has a lower central series filtration. Let $Z(U)$ be the center of U. Write U^{ad} for $U/Z(U)$. Since U is nilpotent of class 2, $\operatorname{Lie} U$ is isomporphic to its graded Lie algebra $\operatorname{Lie} U \cong \mathrm{gr}^\bullet(\operatorname{Lie} U)$. We will fix a grading $\operatorname{Lie} U \cong Z(U) \oplus U^{\mathrm{ad}}$ of the Lie algebra $\operatorname{Lie} U$ once for all. In particular, we fixed a projection $\mathrm{pr} : \operatorname{Lie} U \to Z(U)$.

1.4.4. Cup products Let $c \in C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(A))$. Let $\widetilde{c} \in C^1_{\mathrm{LD}}(U(A))$ be the (unique) lift of c such that $\mathrm{pr}(\widetilde{c}(x_0)) = \ldots \mathrm{pr}(\widetilde{c}(x_{n+1})) = 0$. Define

$$Q(c) := \mathrm{pr}(d^2(\widetilde{c})) = \mathrm{pr}(\widetilde{c}(R)) \in C^2_{\mathrm{LD}}(Z(U)(A)).$$

By expanding $Q(-)$ using the Baker-Campbell-Hausdorff formula, we can show it is a quadratic form. By induction on length for any word $w \in \langle x_0, \ldots, x_{n+1} \rangle$, we have $c(w) = \prod_i \alpha_i c(x_i) +$

$\sum_{i,j} \beta_{ij}[c(x_i), \gamma_{ij} c(x_j)]$ with $\alpha_i, \beta_{ij}, \gamma_{ij} \in \mathbb{Z}[\langle x_0, \ldots, x_{n+1} \rangle]$. Hence $\mathrm{pr}\, \widetilde{c}(w) = \mathrm{pr}(\sum_{i,j} \beta_{ij}[c(x_i), \gamma_{ij} c(x_j)])$ is a quadratic form.

We define

$$\begin{aligned} C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(A)) \times C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(A)) &\overset{\cup}{\to} C^2_{\mathrm{LD}}(Z(U)(A)) \\ x \cup y &:= \frac{1}{2}(Q(x+y) - Q(x) - Q(y)) \end{aligned}$$

which is a symmetric bilinear form.

Remark Alternatively, we can choose an arbitrary lift $\widetilde{c}$ of c. Now $\mathrm{pr}(d^2(\widetilde{c}))$ is an inhomogeneous polynomial of degree two. We recover Q by taking the homogeneous part of degree two.

1.4.4.1 Lemma Under the identification $C^1_{\mathrm{LD}}(U(A)) = C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(A)) \oplus C^1_{\mathrm{LD}}(Z(U)(A))$, we have

$$Z^1_{\mathrm{LD}}(U(A)) = \{(x,y) \in C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(A)) \oplus C^1_{\mathrm{LD}}(Z(U)(A)) | d^2 x = 0,\ x \cup x + d^2 y = 0\}.$$

Proof. It is obvious from the definition of d^2 and Q. The projection of $d^2(x,y)$ to $C^2_{\mathrm{LD}}(U^{\mathrm{ad}}(A))$ is $d^2 x$; and the projection of $d^2(x,y)$ to $C^2_{\mathrm{LD}}(Z(U)(A))$ is $x \cup x + d^2 y$. $\square$

Write $H^1_{\mathrm{LD}}(U^{\mathrm{ad}})(A)$ for $Z^1_{\mathrm{LD}}(U^{\mathrm{ad}})(A)/B^1_{\mathrm{LD}}(U^{\mathrm{ad}})(A)$.

1.4.4.2 Lemma The pairing $\cup$ on the cochain level induces a symmetric pairing on the cohomology level

$$H^1_{\mathrm{LD}}(U^{\mathrm{ad}}(A)) \times H^1_{\mathrm{LD}}(U^{\mathrm{ad}}(A)) \overset{\cup}{\to} H^2_{\mathrm{LD}}(Z(U)(A)).$$

Proof. It suffices to show for all $x \in Z^1_{\mathrm{LD}}(U^{\mathrm{ad}})(A)$ and $y \in B^1_{\mathrm{LD}}(U^{\mathrm{ad}})(A)$, $Q(x+y) - Q(x) \in B^2_{\mathrm{LD}}(Z(U)(A))$ (taking $x = 0$ gives $Q(y) \in B^2_{\mathrm{LD}}(Z(U)(A))$).

Let $\widetilde{x} \in C^1_{\mathrm{LD}}(U(A))$ be the unique extension of x such that $\mathrm{pr}\,\widetilde{x} = 0$. The cocycle $\widetilde{x}$ represents a group homomorphism $\rho_{\widetilde{x}} : \langle x_0, \cdots, x_{n+1}\rangle \to U(A) \rtimes \langle x_0, \cdots, x_{n+1}|R\rangle$ such that $\rho_{\widetilde{x}}(R) \in Z(U)(A)$. There exists $n \in U(A)$ such that $n\rho_{\widetilde{x}}n^{-1}$ is represented by a cocycle $(x+y, f)$ extending $x+y$. We have $n\rho_{\widetilde{x}}(R)n^{-1}\rho_{\widetilde{x}}(R)^{-1} = 1 \in U(A) \rtimes \langle x_0, \cdots, x_{n+1}|R\rangle$ since $\rho_{\widetilde{x}}(R)$ lies in the center of $U(A)$. Since $Q(x+y) - d^2(f) = n\rho_{\widetilde{x}}(R)n^{-1}$ and $Q(x) = \rho_{\widetilde{x}}(R)$, we have $Q(x+y) - Q(x) = d^2 f \in B^2_{\mathrm{LD}}(Z(U)(A))$. $\square$

We now compare it with group cohomology.

1.4.4.3 Lemma There exists an isomorphism of $\mathbb{F}$-vector spaces $H^2_{\mathrm{LD}}(Z(U)(\mathbb{F})) \xrightarrow{\cong} H^2(G_K, Z(U)(\mathbb{F}))$ such that the following diagram commutes

$$\begin{tikzcd}
H^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \times H^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \arrow[r, "\cup"] \arrow[d] & H^2_{\mathrm{LD}}(Z(U)(\mathbb{F})) \arrow[d, "\cong"] \\
H^1(G_K, U^{\mathrm{ad}}(\mathbb{F})) \times H^1(G_K, U^{\mathrm{ad}}(\mathbb{F})) \arrow[r, "\cup"] & H^2(G_K, Z(U)(\mathbb{F}))
\end{tikzcd}$$

Proof. Say $\dim_{\mathbb{F}} H^1(G_K, U^{\mathrm{ad}}(\mathbb{F})) = d$. The cup product (which is symmetric bilinear) can be thought of as a quadratic polynomial in $\mathbb{F}[X_1, \cdots, X_d]$. To show the two cup products differ by a unit, it suffices to show they have the same zero locus in the affine space $\mathbb{A}^d$. On the group-theoretic side, for $[x] \in H^1(G_K, U^{\mathrm{ad}}(\bar{\mathbb{F}}_p))$, $[x] \cup [x] = [0]$ if and only if $[x]$ extends to a cocycle in $H^1(G_K, U(\bar{\mathbb{F}}_p))$. So it remains to show the same thing is true on the Lyndon-Demuškin side. This has been proved in Lemma 1.4.4.1. Let $[x] \in H^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F}))$. Assume $x \cup x \in B^2_{\mathrm{LD}}(Z(U)(\bar{\mathbb{F}}_p))$. Say $x \cup x = dy$, $y \in C^1_{\mathrm{LD}}(Z(U)(\bar{\mathbb{F}}_p))$. Then $d^2(x, -y) = 0$, and thus $(x, -y) \in Z^1_{\mathrm{LD}}(U(\bar{\mathbb{F}}_p))$. Conversely if $(x,y) \in Z^1_{\mathrm{LD}}(U(\bar{\mathbb{F}}_p))$ for some y, then $[x] \cup [x] = [-d^2 y] = 0$. $\square$

Recall $Z^1_{\mathrm{LD}}(U(A))$ (Lyndon-Demuškin complex with non-abelian coefficients) and $Z^1_{\mathrm{LD}}(\mathrm{Lie}\, U(A))$ (Lyndon-Demuškin complex with abelian coefficients) are both subsets of $C^1_{\mathrm{LD}}(U(A))$.

1.4.4.4 Lemma If $Z(U)(\mathbb{F}) \cong \mathbb{F}$, then

$$Z^1_{\mathrm{LD}}(U(\mathbb{F})) \subset Z^1_{\mathrm{LD}}(\mathrm{Lie}\, U(\mathbb{F}))$$

that is, the non-abelian cocycles with $U(\mathbb{F})$-coefficients are automatically abelian cocycles with $(\mathrm{Lie}\, U(\mathbb{F}))$-coefficients.

Proof. We have remarked in 1.4.3.2(2) that $C^1_{\mathrm{LD}}(U(\mathbb{F}))$ and $C^1_{\mathrm{LD}}(\mathrm{Lie}\, U(\mathbb{F}))$ have the same underlying space. By Lemma 1.4.4.1, an element of $Z^1_{\mathrm{LD}}(U(\mathbb{F}))$ is a pair (x,y) such that $d^2 x = 0$ and $x \cup x + d^2 y = 0$. By our assumption that G_K acts on $\mathbb{F}$ trivially and Corollary 1.1.0.3, $C^2_{\mathrm{LD}}(Z(U)(\mathbb{F})) = H^2(G_K, Z(U)(\mathbb{F}))$ and thus $B^2_{\mathrm{LD}}(Z(U)(\mathbb{F})) = 0$. So $d^2 y = 0$ automatically, and (x,y) defines an element of $Z^1_{\mathrm{LD}}(\mathrm{Lie}\, U(\mathbb{F}))$. $\square$

2. An analysis of cup products

Let E be a p-adic field with ring of integers $\mathcal{O}_E$, residue field $\mathbb{F}$ and uniformizer ϖ.

Let U be a smooth connected unipotent group of class 2 over $\mathrm{Spec}\, \mathcal{O}_E$, with center $Z(U) \cong \mathbb{G}_a$. Write U^{ad} for $U/Z(U)$.

2.0.1. Definition Let K' be a p-adic field. A Lyndon-Demuškin action $G_{K'} \to \mathrm{Aut}(U)(\mathcal{O}_E)$ is said to be *mildly regular* if the following are satisfied:

(MR1) $H^0(G_{K'}, U^{\mathrm{ad}}(E)) = 0$;

(MR2) The bilinear pairing

$$\cup_{\mathbb{F}} : C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \times C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \to C^2_{\mathrm{LD}}(Z(U)(\mathbb{F}))$$

is non-degenerate.

2.0.1.1 Remark In practice U is the unipotent radical of a parabolic subgroup of a reductive group and (MR2) is equivalent to "p being not too small". We worked out the G_2-case in Appendix 4.7, and showed that if $p > 5$, (MR2) always holds. The same proof but with more complicated notation should work for general reductive groups.

2.0.1.2 Remark In general, (MR2) can be checked by computer algebra systems because it is a finite field vector space question for a finite number of small p's. We include an algorithm (written in SageMath) in Appendix 4.8.

The following proposition is a summary of Appendix 4.7:

Proposition If U is the unipotent radical of the short root parabolic of G_2 or the quotient of the unipotent radical of the long root parabolic of G_2 by its center, then (MR2) is true when $p \geq 5$.

2.0.2. Definition Given a tuple of labeled Hodge-Tate weights $\underline{\lambda}$, we say $\underline{\lambda}$ is slightly less than 0 if for each $\sigma : K' \hookrightarrow \bar{\mathbb{Q}}_p$, λ_σ consists of non-positive integers, and for at least one σ, λ_σ consists of negative integers. (The cyclotomic character has Hodge-Tate weight -1.)

2.0.3. Proposition Assume $p \geq 5$. If U is the unipotent radical of the short root parabolic of G_2 or the quotient of the unipotent radical of the long root parabolic of G_2 by its center, then $G_{K'} \to \mathrm{Aut}(U)(\mathcal{O}_E)$ is mildly regular if $U^{\mathrm{ad}}(E)$ is Hodge-Tate of labeled Hodge-Tate weights slightly less then 0.

Proof. Write $\underline{\lambda}$ for the labeled Hodge-Tate weights of the Galois module $U^{\mathrm{ad}}(E)$. If $H^0(G_{K'}, U^{\mathrm{ad}}(E)) \neq 0$, then for all embedding $\sigma : K' \hookrightarrow \bar{\mathbb{Q}}_p$, $0 \in \lambda_\sigma$.

The proposition now follows from Remark 2.0.1.2. $\square$

2.1. Cup products mod ϖ

2.1.1. Lemma The image of $Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E)) \to C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F}))$ has codimension at most $\dim_E U^{\mathrm{ad}}(E)$.

Proof. Say $\dim_{\mathbb{F}} C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) = \mathrm{rank}_{\mathcal{O}_E} C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E)) = N$.

Since $Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))$ is the kernel of $C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E)) \to C^2_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))$, and $\mathrm{rank}_{\mathcal{O}_E} C^2_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E)) = \dim_E U^{\mathrm{ad}}(E)$, we have

$$\mathrm{rank}_{\mathcal{O}_E} Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E)) \geq N - \dim_E U^{\mathrm{ad}}(E).$$

Since $C^2_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))$ is torsion-free, $Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))$ is saturated in $C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))$, and is thus a direct summand. In particular, the image of $Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))$ in $C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F}))$ has dimension $\geq N - \dim_E U^{\mathrm{ad}}(E)$. $\square$

2.1.2. Lemma If

$$\cup_{\mathbb{F}} : C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \times C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \to C^2_{\mathrm{LD}}(Z(U)(\mathbb{F}))$$

is non-degenerate, then the kernel of

$$\cup'_{\mathbb{F}} : Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi \times Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi \to C^2_{\mathrm{LD}}(Z(U)(\mathbb{F}))$$

has dimension at most $\dim_E U^{\mathrm{ad}}(E)$.

Remark Note that $Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \neq Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi$ in general.

Proof. For ease of notation, write C for $C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F}))$, and write Z for the image of $Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))$ in C. Note that $Z \cong Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi$ by the proof of the above lemma.

Let $K \subset Z$ be the kernel of $\cup'_{\mathbb{F}}$. Since the cup product on C is non-degenerate, there exists a subspace $F \subset C$ of dimension equal to that of K, such that the restriction of the cup product to $(F + K)$ is also non-degenerate. Since $F \cap Z = 0$, $\dim C \geq \dim(F + Z) = \dim Z + \dim F = \dim Z + \dim K$. The lemma now follows from the previous lemma. $\square$

We also record the following lemma whose proof is similar.

2.1.3. Lemma (1) The image of $Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \to C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F}))$ has codimension at most $\dim_E U^{\mathrm{ad}}(E)$.

(2) If

$$\cup_{\mathbb{F}} : C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \times C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \to C^2_{\mathrm{LD}}(Z(U)(\mathbb{F}))$$

is non-degenerate, then the kernel of

$$\cup_{\mathbb{F}} : Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \times Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \to C^2_{\mathrm{LD}}(Z(U)(\mathbb{F}))$$

has dimension at most $\dim_E U^{\mathrm{ad}}(E)$.

2.2. Nontriviality of cup products

2.2.1. Example: the completely split case In this paragraph we analyze the special case where the $G_{K'}$ action on $U^{\mathrm{ad}}(\mathbb{F})$ is trivial and $H^2(G_{K'}, Z(U)(\mathbb{F})) = Z(U)(\mathbb{F}) = \mathbb{F}$. It will be used in the proof of Theorem 2.2.2. We keep the notations from subsection 1.4.

Since the center of $\mathrm{Lie}\, U$ is one-dimensional, the Lie bracket

$$U^{\mathrm{ad}}(\mathbb{F}) \times U^{\mathrm{ad}}(\mathbb{F}) \xrightarrow{[-,-]} Z(U)(\mathbb{F})$$

is a non-degenerate, alternating pairing. Choose a basis $\{e_1, \cdots, e_k, e'_1, \cdots, e'_k\}$ of $U^{\mathrm{ad}}(\mathbb{F})$ such that $[e'_i, e'_j] = [e_i, e_j] = 0$ and $[e_i, e'_j] = -[e'_i, e_j] = \delta_{i,j}$. Since by assumption the $G_{K'}$-action on $U^{\mathrm{ad}}(\mathbb{F})$ is trivial, the cup product

$$\cup : H^1(G_{K'}, U^{\mathrm{ad}}(\mathbb{F})) \times H^1(G_{K'}, U^{\mathrm{ad}}(\mathbb{F})) \to H^2(G_{K'}, Z(U)(\mathbb{F}))$$

is isomorphic to the (exterior) direct sum of cup products

$$\cup_i : H^1(G_{K'}, \mathbb{F}e_i \oplus \mathbb{F}e'_i) \times H^1(G_{K'}, \mathbb{F}e_i \oplus \mathbb{F}e'_i) \to H^1(G_{K'}, \mathbb{F})$$

Write $\wedge$ for the usual cup product $H^1(G_{K'}, \mathbb{F}) \times H^1(G_{K'}, \mathbb{F}) \to H^2(G_{K'}, \mathbb{F})$ which appears in local

Tate duality. By definition, for $a, b \in H^1(G_{K'}, \mathbb{F})$ we have

$$\begin{aligned} Q(ae_i + be_i') &= B(ae_i + be_i', ae_i + be_i') \\ &= ((g,h) \mapsto [a(g)e_i + b(g)e_i', a(h)e_i + b(h)e_i']) \\ &= ((g,h) \mapsto (a(g)b(h) - b(g)a(h)) \\ &= a \wedge b - b \wedge a \\ &= 2a \wedge b \end{aligned}$$

and thus for $a_1, b_1, a_2, b_2 \in H^1(G_{K'}, \mathbb{F})$

$$B(a_1 e_i + b_1 e_i', a_2 e_i + b_2 e_i') = 2(a_1 \wedge b_2 + a_2 \wedge b_2)$$

Since $\wedge$ is a non-degenerate pairing, B is also a non-degenerate pairing.

2.2.2. Theorem Let K'/K be a finite Galois extension of p-adic fields of prime-to-p degree. Let $r : G_K \to \mathrm{Aut}(U)(\mathcal{O}_E)$ be a continuous group homomorphism.

If $r|_{G_{K'}}$ is Lyndon-Demuškin and mildly regular, then one of the following are true:

(i) $H^2(G_K, Z(U)(\mathbb{F})) = 0$, or

(ii) the symmetric bilinear pairing

$$H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E)) \otimes \mathbb{F} \times H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E)) \otimes \mathbb{F} \to H^2(G_K, Z(U)(\mathcal{O}_E)) \otimes \mathbb{F}$$

is non-trivial.

Remark Note that $H^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E)) \cong H^1(G_{K'}, U^{\mathrm{ad}}(\mathcal{O}_E))$, and $H^1(G_{K'}, U^{\mathrm{ad}}(\mathcal{O}_E))^{G_K} = H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))$. The symmetric pairing in the theorem is the restriction to $H^1(G_{K'}, U^{\mathrm{ad}}(\mathcal{O}_E))$ of the symmetric pairing defined in Lemma 1.4.4.2.

Proof. Assume $H^2(G_K, Z(U)(\mathbb{F})) \neq 0$. Consider the diagram

$$\begin{tikzcd}
H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E)) \times H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E)) \arrow[r] \arrow[d, hook] & H^2(G_K, Z(U)(\mathcal{O}_E)) \arrow[d, "\cong"] \\
H^1(G_{K'}, U^{\mathrm{ad}}(\mathcal{O}_E)) \times H^1(G_{K'}, U^{\mathrm{ad}}(\mathcal{O}_E)) \arrow[r] & H^2(G_{K'}, Z(U)(\mathcal{O}_E)) \\
Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E)) \times Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E)) \arrow[r] \arrow[u] & C^2(Z(U)(\mathcal{O}_E)) \arrow[u]
\end{tikzcd}$$

By Lemma 2.1.2, the kernel of

$$H^1(G_{K'}, U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi \times H^1(G_{K'}, U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi \to H^2(G_{K'}, Z(U)(\mathbb{F}))$$

has $\mathbb{F}$-dimension at most $\dim_E U^{\mathrm{ad}}(E)$. Write Δ for $G_K/G_{K'}$, which acts on $H^1(G_{K'}, U^{\mathrm{ad}}(\mathcal{O}_E))$ with fixed-point subspace $H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))$.

By an averaging argument (explained below), the kernel of

$$H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi \times H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi \to H^2(G_K, Z(U)(\mathbb{F}))$$

is contained in the kernel of

$$H^1(G_{K'}, U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi \times H^1(G_{K'}, U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi \to H^2(G_{K'}, Z(U)(\mathbb{F}))$$

and thus has $\mathbb{F}$-dimension at most $\dim_E U^{\mathrm{ad}}(E)$. (Let $[c] \in H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi$ and suppose $[c] \cup [d] = 0$ for all $[d] \in H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi$. Let $[c'] \in H^1(G_{K'}, U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi$. Then $\sum_{\sigma \in \Delta} \sigma([c] \cup [c']) = [c] \cup \sum_{\sigma \in \Delta} \sigma([c']) = 0$. Since $H^2(G_K, Z(U)(\mathbb{F})) \neq 0$ and $H^2(G_{K'}, Z(U)(\mathbb{F}))$ is 1-dimensional, we have $H^2(G_K, Z(U)(\mathbb{F})) = H^2(G_{K'}, Z(U)(\mathbb{F}))$ and thus $\sum_{\sigma \in \Delta} \sigma([c] \cup [c']) = \#\Delta\sigma([c] \cup [c'])$.)

By the local Euler characteristic,

$$\begin{aligned} \dim_E H^1(G_K, U^{\mathrm{ad}}(E)) &= \dim_E H^2(G_K, U^{\mathrm{ad}}(E)) + \dim_E H^0(G_K, U^{\mathrm{ad}}(E)) + \dim_E U^{\mathrm{ad}}(E)[K : \mathbb{Q}_p] \\ &\geq \dim_E H^2(G_K, U^{\mathrm{ad}}(E)) + \dim_E U^{\mathrm{ad}}(E). \end{aligned}$$

We will now consider two possibilities: $H^2(G_K, U^{\mathrm{ad}}(\mathbb{F})) \neq 0$ and $H^2(G_K, U^{\mathrm{ad}}(\mathbb{F})) = 0$.

Case $H^2(G_K, U^{\mathrm{ad}}(\mathbb{F})) \neq 0$. Since $H^2(G_K, U^{\mathrm{ad}}(\mathbb{F})) \neq 0$, $H^2(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))$ is non-trivial. So either we have $\dim_E H^2(G_K, U^{\mathrm{ad}}(E)) > 0$, or $H^2(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))$ has non-trivial torsion. If $H^2(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))$ has non-trivial torsion, then again by the local Euler characteristic (mod ϖ version), $H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))$ also has non-trivial torsion. In either case, $\dim_{\mathbb{F}} H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi \geq \dim_E U^{\mathrm{ad}}(E) + 1$. So the kernel of the cup product is a proper subspace of $H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))/\varpi$.

If $K \neq \mathbb{Q}_p$, then

$$\begin{aligned} \dim_E H^1(G_K, U^{\mathrm{ad}}(E)) &= \dim_E H^2(G_K, U^{\mathrm{ad}}(E)) + \dim_E H^0(G_K, U^{\mathrm{ad}}(E)) + \dim_E U^{\mathrm{ad}}(E)[K : \mathbb{Q}_p] \\ &\geq 2\dim_E U^{\mathrm{ad}}(E) \end{aligned}$$

and (iii) must be true.

Case $H^2(G_K, U^{\mathrm{ad}}(\mathbb{F})) = 0$. By Nakayama's Lemma, $H^2(G_K, U^{\mathrm{ad}}(\mathcal{O}_E)) = 0$. By [EG19], there exists a perfect $\mathcal{O}_E$-complex $[C^0 \to C^1 \to C^2]$ concentrated in degrees $[0, 2]$ which computes

$H^\bullet(G_K, U^{\mathrm{ad}}(\mathcal{O}_E))$. By the universal coefficient theorem, there exists a short exact sequence

$$0 \to H^1(C^\bullet) \otimes \mathbb{F} \to H^1(C^\bullet \otimes \mathbb{F}) \to Tor_1^{\mathcal{O}_E}(H^2(C^\bullet), \mathbb{F}) \to 0$$

So $H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E)) \otimes_{\mathcal{O}_E} \mathbb{F} = H^1(G_K, U^{\mathrm{ad}}(\mathbb{F}))$. We assume (i) and (ii) are false, and try to get a contradiction. The kernel of

$$H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E)) \otimes \mathbb{F} \times H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E)) \otimes \mathbb{F} \to H^2(G_K, Z(U)(\mathcal{O}_E)) \otimes \mathbb{F}$$

has dimension $h^1 := \dim_{\mathbb{F}} H^1(G_K, U^{\mathrm{ad}}(\mathbb{F}))$. By the local Euler characteristic,

$$(*) \quad h^1 = \dim_E U^{\mathrm{ad}}(E)[K : \mathbb{Q}_p] + \dim_{\mathbb{F}} H^0(G_K, U^{\mathrm{ad}}(\mathbb{F})).$$

By Lemma 2.1.3, the kernel k_Z of

$$Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \times Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \to H^2(G_{K'}, Z(U)(\mathbb{F}))$$

has dimension at most $\dim_E U^{\mathrm{ad}}(E)$. Since the cup product is trivial on $H^1(G_K, U^{\mathrm{ad}}(\mathbb{F}))$, we have

$$(**) \quad \dim k_Z \geq \dim H^1(G_K, U^{\mathrm{ad}}(\mathbb{F})) + \dim B^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) = h^1 + \dim B^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})).$$

Combining $(*)$ and $(**)$, we have

$$\dim_E U^{\mathrm{ad}}(E) \geq \dim_{\mathbb{F}} k_Z \geq \dim_E U^{\mathrm{ad}}(E)[K : \mathbb{Q}_p] + \dim_{\mathbb{F}} H^0(G_K, U^{\mathrm{ad}}(\mathbb{F})) + \dim B^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F}))$$

So we conclude that

$$1 = [K : \mathbb{Q}_p]$$
$$0 = H^0(G_K, U^{\mathrm{ad}}(\mathbb{F}))$$
$$0 = B^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F}))$$

In particular, we have $H^0(G_{K'}, U^{\mathrm{ad}}(\mathbb{F})) = U^{\mathrm{ad}}(\mathbb{F})$, and the cup product on $H^1(G_{K'}, U^{\mathrm{ad}}(\mathbb{F}))$ has dimension exactly $\dim_E U^{\mathrm{ad}}(E)$. However, by Example 2.2.1, the cup product on $H^1(G_{K'}, U^{\mathrm{ad}}(\mathbb{F}))$ is non-degenerate by local Tate duality. □

Theorem 2.2.2 is used in the following scenerio.

2.2.2.1 Lemma Let L be a split reductive group over $\mathbb{F}$. Let $r : G_K \to L(\mathbb{F})$ be a Galois representation valued in L. Let r^{ss} be the semi-simplification of r. Write $G_{K'}$ for the kernel of r^{ss}. Then the degree $[K' : K]$ divides $(q-1)^r \# W_L$ where

- r is the rank of L,
- q is a power of p, and
- $\# W_L$ is the cardinality of the Weyl group of L.

Proof. By Theorem 2.3, r^{ss} is tamely ramified and factors through the normalizer of a maximal torus of L (after possibly extending the base field). □

In particular, if $L = G_2$ and $p > 3$, the kernel of r^{ss} defines a Galois extension K'/K of prime-to-p degree; and $r|_{G_{K'}}$ is Lyndon-Demuškin since it has trivial semi-simplification.

3. Non-abelian obstruction theory via Lyndon-Demuškin cocycle group with external Galois action

Let $K/\mathbb{Q}_p$ be a p-adic field. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_E$, residue field $\mathbb{F}$, and uniformizer ϖ.

Let L be a split reductive group over $\mathcal{O}_E$. Fix a Galois representation

$$r^\circ : G_K \to L(\mathcal{O}_E)$$

throughout this section.

Let U be a unipotent group over $\mathcal{O}_E$ whose adjoint group is abelian. Let $Z(U)$ be the center of U. The adjoint group U^{ad} is defined to be $U/Z(U)$.

Fix a group scheme homomorphism $\phi : L \to \mathrm{Aut}(U)$ throughout this section. In particular, there is a Galois action $\phi(r^\circ) : G_K \xrightarrow{r^\circ} L(\mathcal{O}_E) \xrightarrow{\phi(\mathcal{O}_E)} \mathrm{Aut}(U)(\mathcal{O}_E)$. We will talk about non-abelian Galois cohomology $H^\bullet(G_K, U(\mathcal{O}_E))$ and $H^\bullet(G_K, U(\mathbb{F}))$ using this Galois action throughout this section.

Let K'/K be a prime-to-p, finite Galois extension of K containing the group of p-th root of unity, such that $r^\circ(G_{K'}) \subset L(\mathcal{O}_E)$ is a pro-p group. Write Δ for $\mathrm{Gal}(K'/K)$. Set $\Gamma := G_K$, and $H := G_{K'}$.

3.1. Non-abelian inflation-restriction Let R be either $\mathcal{O}_E$ or $\mathbb{F}$. For ease of notation, write U for $U(R)$.

3.1.0.1 Non-abelian Galois cohomology We recall a few facts about the non-abelian version of Galois cohomology. Let Γ be a (profinite) group Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of groups with continuous Γ-action. If $A \to B$ is *central*, that is, A is contained in the center of B, then we have a long exact sequence ([Se02, Proposition 43, 5.7])

$$\begin{aligned} 1 &\to A^\Gamma \to B^\Gamma \to C^\Gamma \\ &\to H^1(\Gamma, A) \to H^1(\Gamma, B) \to H^1(\Gamma, C) \\ &\xrightarrow{\delta} H^2(\Gamma, A) \end{aligned}$$

of pointed sets. Let $H \subset G$ be a closed normal subgroup. Then there is an exact sequence ([Se02, 5.8])

$$1 \to H^1(\Gamma/H, A^H) \to H^1(\Gamma, A) \to H^1(H, A)^{\Gamma/H}.$$

If A is an abelian group, then the sequence above can be upgraded to the inflation-restriction exact sequence:

$$1 \to H^1(\Gamma/H, A^H) \to H^1(\Gamma, A) \to H^1(H, A)^{\Gamma/H} \to H^2(\Gamma, A^H).$$

3.1.0.2 Theorem [Ko02, Theorem 3.15] Let Γ be a profinite group, H a normal subgroup of finite index, and A an (abelian) G-module whose elements have finite order coprime to $(\Gamma : H)$. Then

$$H^n(\Gamma/H, A^H) = 0$$

for all $n \geq 1$, and the restriction

$$H^n(\Gamma, A) \to H^n(H, A)^{\Gamma/H}$$

is an isomorphism.

The fact above gives the following diagram with exact columns

$$\begin{tikzcd}
H^1(\Gamma, Z(U)) \arrow[r, "\cong", "\mathrm{res}"'] \arrow[d] & H^1(H, Z(U))^{\Delta} \arrow[d] \\
H^1(\Gamma, U) \arrow[r, hook, "\mathrm{res}"'] \arrow[d, "\alpha_1"] & H^1(H, U)^{\Delta} \arrow[d, "\alpha_2"] \\
H^1(\Gamma, U^{\mathrm{ad}}) \arrow[r, "\cong", "\mathrm{res}"'] \arrow[d, "\delta_1"] & H^1(H, U^{\mathrm{ad}})^{\Delta} \arrow[d, "\delta_2"] \\
H^2(\Gamma, Z(U)) \arrow[r, hook] & H^2(H, Z(U))
\end{tikzcd}$$

3.1.0.3 Proposition The restriction map of non-abelian 1-cocycles

$$H^1(\Gamma, U) \to H^1(H, U)^{\Delta}$$

is a bijection.

Proof. Let $[c] \in H^1(H, U)^{\Delta}$. Since $\delta_1(\mathrm{res}^{-1}(\alpha_2[c])) = \delta_2(\alpha_2[c]) = 0$, there exists $[b] \in H^1(\Gamma, U)$ such that $\alpha_1(\mathrm{res}([b])) = \alpha_2([c])$. Since $\alpha_2^{-1}(\alpha_2([c]))$ is a $H^1(H, Z(U))^{\Delta}$-torsor, we can twist $[b]$ to make $\mathrm{res}([b]) = [c]$. $\square$

3.1.0.4 Representation-theoretic interpretation of non-abelian 1-cocycles Let $\mathcal{P}$ be a group which is a semi-direct product $\mathcal{L} \ltimes \mathcal{U}$. Let $q_{\mathcal{L}} : \mathcal{P} \to \mathcal{L}$ be the quotient map. Fix a section $\mathcal{L} \to \mathcal{P}$ of $q_{\mathcal{L}}$, which allows us to identify (set-theoretically) $\mathcal{P}$ with $\mathcal{U} \times \mathcal{L}$. Write $q_{\mathcal{U}} : \mathcal{P} \to \mathcal{U}$ be the projection determined by the fixed section $\mathcal{L} \to \mathcal{P}$. For $g \in \mathcal{P}$, write $g = g_{\mathcal{U}} g_{\mathcal{L}}$ such that $g_{\mathcal{U}} \in \mathcal{U} \times \{1\}$ and $g_{\mathcal{L}} \in \{1\} \times \mathcal{L}$. Let Γ be a profinite group. Let $\bar{\tau} : \Gamma \to \mathcal{L}$ be a group homomorphism. Let $\tau : \Gamma \to \mathcal{P}$ be a lift of $\bar{\tau}$. Set $c := q_{\mathcal{U}} \circ \tau : \Gamma \to \mathcal{U}$. Then

$$\begin{aligned} c(gh) &= q_{\mathcal{U}}(\tau(g)\tau(h)) = q_{\mathcal{U}}(\tau(g)_{\mathcal{U}}\tau(g)_{\mathcal{L}}\tau(h)_{\mathcal{U}}\tau(h)_{\mathcal{L}}) \\ &= q_{\mathcal{U}}(\tau(g)_{\mathcal{U}}\tau(g)_{\mathcal{L}}\tau(h)_{\mathcal{U}}\tau(g)_{\mathcal{L}}^{-1}\tau(gh)_{\mathcal{L}}) \quad (q_{\mathcal{L}} \text{ is a group homomorphism}) \\ &= c(g)(\tau(g)_{\mathcal{L}}c(h)\tau(g)_{\mathcal{L}}^{-1}) \\ &=: c(g)(\tau(g)_{\mathcal{L}} \cdot c(h)) \end{aligned}$$

is a (non-abelian) crossed homomorphism. So $H^1(\Gamma, \mathcal{U})$ classifies liftings τ of $\bar{\tau}$ up to equivalence.

3.1.1. Lifting characteristic p cocycles via inflation-restriction

Let $[\bar{c}] \in H^1(\Gamma, U(\mathbb{F}))$ be a characteristic p cocycle. Assume the restriction $[\bar{c}|_H] \in H^1(H, U(\mathbb{F}))$ has a characteristic 0 lift $[c_h] \in H^1(H, U(\mathcal{O}_E))$. We want to build a lift $[c] \in H^1(\Gamma, U(\mathcal{O}_E))$ of $[\bar{c}]$ using $[c_h]$.

Note that when U is an abelian group, this can be easily achieved by taking the average

$$[c] := \frac{1}{\#\Delta} \sum_{g \in \Delta} g \cdot [c_h].$$

Here we identify $H^1(\Gamma, U(\mathcal{O}_E))$ with a subset of $H^1(H, U(\mathcal{O}_E))$ via Proposition 3.1.0.3.

Such a trick does not work anymore when U is non-abelian. Nonetheless, we have the following:

3.1.1.1 Lemma If there exists $[c_h] \in H^1(H, U(\mathcal{O}_E))$ and $[d] \in H^1(\Gamma, U^{\mathrm{ad}}(\mathcal{O}_E))$ such that

- $\alpha_2([c_h]) = \mathrm{res}([d])$ and
- $[c_h|_H] \bmod \varpi = [\bar{c}|_H]$

then there exists $[c] \in H^1(\Gamma, U(\mathcal{O}_E))$ which is a lifting of $[\bar{c}]$.

$$\begin{tikzcd}
& H^1(\Gamma, Z(U)(\mathcal{O}_E)) \arrow[r, hook, "\mathrm{res}"'] \arrow[d] & H^1(H, Z(U)(\mathcal{O}_E)) \arrow[d] & \\
& H^1(\Gamma, U(\mathcal{O}_E)) \arrow[r, hook, "\mathrm{res}"'] \arrow[d, "\alpha_1"] & H^1(H, U(\mathcal{O}_E)) \arrow[d, "\alpha_2"] & \ni [c_h] \\
{[d] \in} & H^1(\Gamma, U^{\mathrm{ad}}(\mathcal{O}_E)) \arrow[r, "\mathrm{res}"'] \arrow[d, "\delta_1"] & H^1(H, U^{\mathrm{ad}}(\mathcal{O}_E)) \arrow[d, "\delta_2"] & \\
& H^2(\Gamma, Z(U)(\mathcal{O}_E)) \arrow[r, hook] & H^2(H, Z(U)(\mathcal{O}_E)) &
\end{tikzcd}$$

Proof. Since

$$\delta_1([d]) = \delta_2(\alpha_2([c_h])) = 0,$$

$[d] = \alpha_1([c'])$ for some $[c'] \in H^1(\Gamma, U(\mathcal{O}_E))$. Since $\mathrm{res}([c'])$ and $[c_h] \in H^1(H, U(\mathcal{O}_E))$ have the same image in $H^1(H, U^{\mathrm{ad}}(\mathcal{O}_E))$ (via α_2), it makes sense to talk about the difference $\mathrm{res}([c']) - [c_h] \in$

$H^1(H, Z(U)(\mathcal{O}_E))$. [1] Consider the following diagram

$$\begin{tikzcd}
H^1(\Gamma, Z(U)(\mathcal{O}_E)) \arrow[r] \arrow[d, hook, "\mathrm{res}"] & H^1(\Gamma, Z(U)(\mathbb{F})) \arrow[r, "\delta"] \arrow[d, hook, "\mathrm{res}"] & H^2(\Gamma, Z(U)(\mathcal{O}_E)) \arrow[d, hook] \\
H^1(H, Z(U)(\mathcal{O}_E)) \arrow[r] & H^1(H, Z(U)(\mathbb{F})) \arrow[r, "\delta"] & H^2(H, Z(U)(\mathcal{O}_E))
\end{tikzcd}$$

Let $[\bar{c}'] \in H^1(\Gamma, Z(U)(\mathbb{F}))$ be the reduction mod ϖ of $[c']$. Since $\mathrm{res}([\bar{c}']) - [\bar{c}_h]$ has a lift,

$$\delta(\mathrm{res}([\bar{c}'] - [\bar{c}_h])) = 0 \in H^2(H, Z(U)(\mathbb{F})).$$

Therefore

$$\delta([\bar{c}'] - [\bar{c}]) = \delta(\mathrm{res}([\bar{c}'] - [\bar{c}])) = \delta(\mathrm{res}([\bar{c}'] - [\bar{c}_h])) = 0.$$

and $[\bar{c}'] - [\bar{c}]$ has a characertistic 0 lift $[x]$, and $[c] := [c'] - [x]$ is a lift of $[\bar{c}]$. $\square$

The purpose of the whole Section 4.3 is to prove Theorem 3.3.2 which extends the above lemma.

3.2. External Galois action on the Lyndon-Demuškin cocycle group

The earlier subsection shows there is an identification

$$H^1(\Gamma, U(\mathcal{O}_E)) \cong H^1(H, U(\mathcal{O}_E))^{\Delta}.$$

The goal of this subsection is to upgrade this identification to the cochain level.

Since the Galois action

$$\phi(r^\circ)|_{G_{K'}} : G_{K'} \to U(\mathcal{O}_E)$$

[1] $H^1(H, U(\mathcal{O}_E))$ is a $H^1(H, Z(U)(\mathcal{O}_E))$-principle homogeneous space.

is Lyndon-Demuškin, we have a Lyndon-Demuškin complex $C^\bullet_{\mathrm{LD}}(U(\mathcal{O}_E))$ computing $H^\bullet(H, U(\mathcal{O}_E))$. Recall that in paragraph 1.4.3.1, we explained a 1-cochain $c \in C^1_{\mathrm{LD}}(U(\mathcal{O}_E))$ is the same as a function

$$\mathfrak{c} : \langle x_0, \cdots, x_{n+1}\rangle \to (\mathrm{Lie}\, U)(\mathcal{O}_E)$$

such that

$$\mathfrak{c}(gh) = \mathfrak{c}(g) + g \cdot \mathfrak{c}(h) + \frac{1}{2}[\mathfrak{c}(g), g \cdot \mathfrak{c}(h)]$$

for all g, h; or, equivalently, a function

$$c : \langle x_0, \cdots, x_{n+1}\rangle \to U(\mathcal{O}_E)$$

such that

$$c(gh) = c(g)(g \cdot c(h))$$

for all g, h.

A cochain $c : \langle x_0, \cdots, x_{n+1}\rangle \to U(\mathcal{O}_E)$ lies in $Z^1_{\mathrm{LD}}(U(\mathcal{O}_E))$ if and only if it factors through the (discrete) Demuškin group $\langle x_0, \cdots, x_{n+1}|R\rangle$ (see Remark 1.4.3.2(3)).

Let $c \in Z^1_{\mathrm{LD}}(\mathcal{O}_E)$, regarded as a function $\langle x_0, \ldots, x_{n+1}|R\rangle \to U(\mathcal{O}_E)$. Since $U(\mathcal{O}_E)$ is a pro-p group, the crossed homomorphism necessarily factors through the pro-p completion, that is, we have a commutative diagram

$$\begin{tikzcd}
 & \langle x_0, \ldots, x_{n+1}|R\rangle \arrow[r, "c"] \arrow[d, "\pi"] & U(\mathcal{O}_E) \\
G_{K'}(p) \arrow[r, equal] & \widehat{\langle x_0, \ldots, x_{n+1}|R\rangle}^{p} \arrow[ur, "\widehat{c}"] &
\end{tikzcd}$$

Since we have identified the pro-p quotient of $G_{K'}$ with the pro-p completion of $\langle x_0, \cdots, x_{n+1} | R\rangle$, we can define, for each $g \in G_K$, an automorphism α_g of $Z^1_{\mathrm{LD}}(U(\mathcal{O}_E))$ via

$$\alpha_g(c) := (h \mapsto g \cdot \widehat{c}(g^{-1}\pi(h)g)).$$

So we defined an action of G_K on $Z^1_{\mathrm{LD}}(U(\mathcal{O}_E))$.

For ease of notation, write $g \cdot c$ for $\alpha_g(c)$. Note that $(g \cdot c)(h) = (\alpha_g(c))(h)$ is different from $g \cdot c(h)$. We apologize for the confusing notation.

3.2.0.1 Remark We don't know whether or not we can define a G_K-action on the whole cochain group $C^1_{\mathrm{LD}}(U(\mathcal{O}_E))$. It seems to involve some subtle combinatorial group theory.

3.2.0.2 Digression We conjecture that the cup product

$$\cup : Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E)) \times Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E)) \to C^2_{\mathrm{LD}}(Z(U(\mathcal{O}_E)))$$

is compatible with G_K-action.

This conjecture would hold, for example, if for each $g \in G_K$, the conjugation by g

$$\phi_g : G_{K'} \to G_{K'}$$

can be lifted to an automorphism of free pro-p groups on $(n+2)$-generators

$$\phi_g : \langle x_0, \cdots, x_{n+1}\rangle \to \langle x_0, \cdots, x_{n+1}\rangle.$$

This is closely related to the so-called *Dehn-Nielsen* theorem. Classically, Dehn-Nielsen is saying all automorphism of the fundamental group of the genus g closed surface M_g are induced by a homeomorphism. The algebraic version of Dehn-Nielsen can be formualted as, under the usual presentation of $F = \langle a_1, b_1, \cdots, a_g, b_g \rangle \to \langle a_1, b_1, \cdots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] \rangle \cong \pi_1(M_g)$, all automorphism of $\pi_1(M_g)$ is induced from an automorphism of the free group F.

Conjecture Dehn-Nielsen holds for $\operatorname{Spec} K$.

Assume $Z(U(\mathcal{O}_E)) \cong \mathcal{O}_E$ from now on.

3.3. Constructing non-abelian cocycles

Recall that $H^1(H, U^{\mathrm{ad}})^\Delta = H^1(G_K, U^{\mathrm{ad}})$ where $H = G_{K'}$ and K'/K is a normal extension of prime-to-p degree. Define

$$\begin{aligned} (Z^1_{LD})^\Delta &:= \{x \in Z^1_{\mathrm{LD}} | \text{image of } x \text{ in } H^1 \text{ is contained in } (H^1)^\Delta\} \\ &= \{x \in Z^1_{\mathrm{LD}} | g \cdot x - x \in B^1_{\mathrm{LD}} \text{ for all } g \in G_K\} \end{aligned}$$

Since $Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))^\Delta$ is a submodule of a finite flat $\mathcal{O}_E$-module, it is finite $\mathcal{O}_E$-flat.

We keep all notations from the previous subsections.

Assume $Z(U)(\mathcal{O}_E) = \mathcal{O}_E$ from now on.

We fix some notation. The quotient $U \to U/Z(U) = U^{\mathrm{ad}}$ induces maps $\mathrm{ad} : Z^1_{\mathrm{LD}}(U(\mathcal{O}_E)) \to Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))$.

3.3.1. Lemma Assume $p \neq 2$ and the cup product

$$(\dagger) \qquad \cup : H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E)) \otimes \mathbb{F} \times H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E)) \otimes \mathbb{F} \to H^2(G_K, Z(U)(\mathbb{F}))$$

is non-trivial.

Let $(\bar{c}, \bar{f}) \in Z^1_{\mathrm{LD}}(U(\mathbb{F}))$ (using Lemma 1.4.4.1). Assume $\bar{c} \in Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F}))^{\Delta}$. If $\bar{c}$ admits a characteristic 0 lift $c' \in Z^1_{\mathrm{LD}}(U^{\mathrm{ad}})(\mathcal{O}_E)$, then $(\bar{c}, \bar{f})$ admits a lift $(c, f) \in Z^1_{\mathrm{LD}}(U(\bar{\mathbb{Z}}_p))$ such that $c \in Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))^{\Delta}$.

Proof. Pick an arbitrary lift $f \in C^1_{\mathrm{LD}}(Z(U)(\mathcal{O}_E))$ of $\bar{f}$. Choose a system of representatives $\{g_i\} \subset G_K$ of Δ. Since $[\bar{c}] = \frac{1}{\#\Delta} \sum g_i \cdot [\bar{c}] \in H^1(G_{K'}, U^{\mathrm{ad}}(\mathbb{F}))$, $c' - \frac{1}{\#\Delta} \sum g_i \cdot c' \in B^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$. By replacing c' by the Δ-average $\frac{1}{\#\Delta} \sum g_i \cdot c'$ + some coboundary, we assume $c' \in Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))^{\Delta}$.

Let $\lambda \in \bar{\mathbb{Z}}_p^{\times}$ be a scalar.

Since the symmetric bilinear pairing $(\dagger)$ is non-trivial, there exists $y \in Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E))^{\Delta}$ such that $y \cup y \neq 0 \mod \varpi$. Consider

$$(c' + \lambda y) \cup (c' + \lambda y) + d^2(f) = c' \cup c' + d^2(f) + 2\lambda c' \cup y + \lambda^2 y \cup y \in C^2(Z(U)(\mathcal{O}_E)) \cong \mathcal{O}_E$$

which is a degree two polynomial in λ whose Newton polygon has vertices

$$(0, v_p(y \cup y)), (1, v_p(c' \cup y)), (2, 0).$$

Since $v_p(y \cup y) > 0$, the polynomial above has at least one solution λ_0 with positive p-adic valuation. Set $(c, f) := (c' + \lambda_0 y, f)$.

We have $(c, f) \in Z^1_{\mathrm{LD}}(U(\bar{\mathbb{Z}}_p))$ by Lemma 1.4.4.1 and $c \in Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))^{\Delta}$. $\square$

3.3.2. Theorem Assume $p \neq 2$ and the cup product

$$\cup : H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E)) \otimes \mathbb{F} \times H^1(G_K, U^{\mathrm{ad}}(\mathcal{O}_E)) \otimes \mathbb{F} \to H^2(G_K, Z(U)(\mathbb{F}))$$

is non-trivial.

Let $[(\bar{c}, \bar{f})] \in H^1(G_K, U(\mathbb{F}))$ be a characteristic p cocycle. If $[\bar{c}|_{G_{K'}}] \in H^1(G_{K'}, U^{\mathrm{ad}}(\mathbb{F}))$ admits a characteristic 0 lift in $H^1(G_{K'}, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$, then $[(\bar{c}, \bar{f})]$ admits a characteristic 0 lift $[(c, f)] \in H^1(G_K, U(\bar{\mathbb{Z}}_p))$.

Proof. We choose a cocycle $(\bar{c}, \bar{f}) \in Z^1_{\mathrm{LD}}(U(\mathbb{F}))$ which defines the cohomology class $[(\bar{c}, \bar{f})]$. Clearly $\bar{c} \in Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F}))^\Delta$. Say $[d] \in H^1(G_{K'}, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$ is a lift of $[\bar{c}]$, which is defined by $d \in Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$. Write $\bar{d}$ for the image of d in $Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\bar{\mathbb{F}}_p))$. By changing d by a coboundary, we can assume $\bar{d} = \bar{c}$.

Lemma 3.3.1 produces $(c, f) \in Z^1_{\mathrm{LD}}(U(\bar{\mathbb{Z}}_p))$ such that $c \in Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))^\Delta$. Now the theorem follows from Lemma 3.1.1.1. □

Theorem 3.3.2 is saying when U is a nilpotent group of class 2 with 1-dimensional center, there exists a short exact sequence of pointed sets

$$H^1(G_K, U(\bar{\mathbb{Z}}_p)) \to H^1(G_K, U(\bar{\mathbb{F}}_p)) \xrightarrow{\delta} H^2(G_{K'}, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$$

under technical assumptions.

Combining Theorem 3.3.2 and Theorem 2.2.2, we have very nice obstruction theory for lifting mod ϖ cohomology classes in the mildly regular case.

3.3.3. Theorem Assume $p \neq 2$. Let $r : G_K \to L(\mathcal{O}_E)$ be a continuous group homomorphism. Let K'/K be a finite Galois extension of prime-to-p degree such that $r|_{G_{K'}}$ is Lyndon-Demuškin and mildly regular.

There is a short exact sequence of pointed sets

$$H^1(G_K, U(\bar{\mathbb{Z}}_p)) \to H^1(G_K, U(\bar{\mathbb{F}}_p)) \xrightarrow{\delta} H^2(G_{K'}, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$$

where δ has a factorization $H^1(G_K, U(\bar{\mathbb{F}}_p)) \xrightarrow{p} H^1(G_K, U^{\mathrm{ad}}(\bar{\mathbb{F}}_p)) \to H^2(G_{K'}, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$.

Proof. Write Δ for $G_K/G_{K'}$. By the moreover part of Theorem 2.2.2, there are three cases to consider.

Case I: the cup product $(\dagger)$ $H^1(G_K, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p)) \otimes \mathbb{F} \times H^1(G_K, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p)) \otimes \mathbb{F} \to H^2(G_K, Z(U)(\bar{\mathbb{Z}}_p)) \otimes \mathbb{F}$ is non-trivial. This is a corollary of Theorem 3.3.2.

Case II: $H^2(G_K, Z(U)(\mathbb{F})) = 0$. By Nakayama's lemma, $H^2(G_K, Z(U)(\bar{\mathbb{Z}}_p)) = 0$.

Let $[(\bar{c}, \bar{f})] \in H^1(G_K, U(\bar{\mathbb{F}}_p))$ be a cohomology class defined by $(\bar{c}, \bar{f}) \in Z^1_{\mathrm{LD}}(U(\bar{\mathbb{F}}_p))$.

Set $\delta : H^1(G_K, U(\bar{\mathbb{F}}_p)) \to H^2(G_{K'}, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$ to be the composite

$$H^1(G_K, U(\bar{\mathbb{F}}_p)) \xrightarrow{[(\bar{c}, \bar{f})] \mapsto [\bar{c}]} H^1(G_K, U^{\mathrm{ad}}(\bar{\mathbb{F}}_p)) \to H^2(G_{K'}, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p)).$$

If $\delta([(\bar{c}, \bar{f})]) = 0$, then there exists a lift $c \in Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$ of $\bar{c}$. By replacing c by the Δ-average of c, we assume $c \in Z^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))^{\Delta}$. Since $H^2(G_K, Z(U)(\bar{\mathbb{Z}}_p)) = 0$, $[c \cup c] = 0$ and thus there exists $g \in C^1_{\mathrm{LD}}(Z(U)(\bar{\mathbb{Z}}_p))^{\Delta}$ such that $c \cup c = -d^2(g)$. Write $\bar{g}$ for the image of g in $C^1_{\mathrm{LD}}(Z(U)(\bar{\mathbb{F}}_p))$. We have $\bar{g} - \bar{f} \in Z^1_{\mathrm{LD}}(Z(U)(\bar{\mathbb{F}}_p))^{\Delta}$. Since $H^2(G_K, Z(U)(\bar{\mathbb{Z}}_p)) = 0$, there exists a lift $h \in Z^1_{\mathrm{LD}}(Z(U)(\bar{\mathbb{Z}}_p))^{\Delta}$ of $\bar{f} - \bar{g}$. It is clear that $[(c, g+h)] \in H^1(G_K, U(\bar{\mathbb{Z}}_p))$ is a lift of $[(\bar{c}, \bar{f})]$. $\square$

3.3.4. Corollary Assume $p \neq 2$. Let $r : G_K \to L(\mathcal{O}_E)$ be a continuous group homomorphism.

If there exists a finite Galois extension K'/K of prime-to-p degree such that $r|_{G_{K'}}$ is Lyndon-Demuškin and mildly regular, then there is a short exact sequence of pointed sets

$$H^1(G_K, U(\bar{\mathbb{Z}}_p)) \to H^1(G_K, U(\bar{\mathbb{F}}_p)) \xrightarrow{\delta} H^2(G_K, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$$

where δ has a factorization $H^1(G_K, U(\bar{\mathbb{F}}_p)) \xrightarrow{p} H^1(G_K, U^{\mathrm{ad}}(\bar{\mathbb{F}}_p)) \to H^2(G_K, U^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$.

Proof. It is an immediate consequence of Theorem 3.3.3. □

4. The Machinery for lifting non-abelian cocycles

Let $K/\mathbb{Q}_p$ be a p-adic field. Let $E/\mathbb{Q}_p$ be the coefficient field with ring of integers $\mathcal{O}_E$, residue field $\mathbb{F}$ and uniformizer ϖ.

4.0.1. Crystalline lifting rings Let L be a connected reductive group over $\mathcal{O}_E$, and $\bar{r} : G_K \to L(\mathbb{F})$ be a mod ϖ representation. Let $\underline{\lambda}$ be a Hodge type. The crystalline lifting ring $R_{\bar{r}}^{\mathrm{crys},\underline{\lambda},\mathcal{O}}$ of $\bar{r}$ of p-adic Hodge type $\underline{\lambda}$ is constructed in [BG19, Theorem 3.3.8]. It is an $\mathcal{O}$-flat quotient of the universal lifting ring, and has generic fiber equidimensional of dimension $\dim_E L + \dim_E \mathrm{Res}_{K\otimes E/K} L/P_{\underline{\lambda}}$ where $P_{\underline{\lambda}}$ is the parabolic subgroup determined by the p-adic Hodge type $\underline{\lambda}$. If $\underline{\lambda}$ is a regular p-adic Hodge type, $P_{\underline{\lambda}}$ is a Borel subgroup.

4.1. A geometric argument of Emerton-Gee

4.1.1. Definition Let $\mathcal{F}$ be a coherent sheaf over a scheme X. We say $\mathcal{F}$ is *sufficiently generically regular* (= SGR) if for each $s \geq 0$, the locus

$$X_s := \{x \in \operatorname{Spec} R \mid \dim \kappa(x) \otimes_R \mathcal{F} \geq s\}$$

has codimension $\geq s+1$ in $\operatorname{Spec} R$.

4.1.2. Theorem Let $X = \operatorname{Spec} R$ be an irreducible component of a crystalline lifting ring of $\bar{r}$. Let $r^{\text{univ}} : G_K \to L(R)$ be the universal family of Galois representations on X. Assume $X[1/p] \neq \emptyset$. Let $F : L \to \operatorname{GL}(V)$ be an algebraic representation where V is a vector space scheme over $\mathcal{O}_E$.

Assume $H^2(G_K, F(r^{\text{univ}}))$ is SGR. Given any $[\bar{c}] \in H^1(G_K, F(\bar{r}))$, there exists a $\bar{\mathbb{Z}}_p$-point of X giving rise to a Galois representation $r^\circ : G_K \to L(\bar{\mathbb{Z}}_p)$, such that the 1-cocycle $[\bar{c}]$ admits a lift $[c] \in H^1(G_K, F(r^\circ))$.

Proof. The proof is almost identical to that of [EG19, Theorem 6.3.2].

Instead of repeating their argument, we would like to explain the main ideas behind the proof, and why we need the sufficiently generically regular condition.

We have a complex of finitely generated projective R-modules concentrated on degree [0, 2]

$$C^0 \to C^1 \xrightarrow{d} C^2$$

which computes the Galois cohomology $H^\bullet(G_K, F(r^{\text{univ}}))$. Let $Z^1 := \ker(d)$ and $B^2 := \operatorname{Im}(d)$. A mod ϖ cocycle $[\bar{c}]$ is represented by an element $\bar{c}$ in the kernel of $C^1/\varpi \to C^2/\varpi$. We fix an arbitrary lift $\widetilde{c} \in C^1$ of $\bar{c}$. We can do a formal blowup $\operatorname{Spec} \widetilde{R} \to \operatorname{Spec} R$, so that the pull-back of

B^2 on $\operatorname{Spec}\widetilde{R}$ a locally free sheaf. To make the exposition short, we simply assume B^2 is locally free over $\operatorname{Spec} R$, but we should not think of $\operatorname{Spec} R$ as a local ring anymore, because after formal blow-up, there are more points in the special fiber. Now we have a sequence of locally free sheaf of modules

$$C^1 \to B^2 \to C^2.$$

The key here is we want to regard this as a sequence of vector bundles instead of sheaf of modules. Write $\mathscr{V}(\mathcal{F})$ for $\underline{\operatorname{Spec}}(\operatorname{Sym}\mathcal{F}^\vee)$, the vector bundle associated to the coherent sheaf $\mathcal{F}$. So we have a sequence of scheme morphisms

$$\begin{tikzcd}
\mathscr{V}(C^1) \arrow[r, "f"] \arrow[rr, bend left, "d"] & \mathscr{V}(B^2) \arrow[r] & \mathscr{V}(C^2) \\
& \operatorname{Spec} R \arrow[ul, "s"] \arrow[u, "f\circ s"] \arrow[ur, "d\circ s"'] &
\end{tikzcd}$$

The element $\widetilde{c}$ of C^1 defines a section $s : \operatorname{Spec} R \to \mathscr{V}(C^1)$ such that the section $d \circ s : \operatorname{Spec} R \to \mathscr{V}(C^2)$ intersects with the identity section $e_{\mathscr{V}(C^2)} : \operatorname{Spec} R \to \mathscr{V}(C^2)$.

It turns out $\bar{c} \in \ker(C^1/\varpi \to C^2/\varpi)$ admits a lift in Z^1, as long as the section $f \circ s$ intersects with the identity section $e_{\mathscr{V}(B^2)}$ of $\mathscr{V}(B^2)$. The intersection $(d \circ s) \cap e_{\mathscr{V}(C^2)}$ should occur above a codimension 1 locus of $\operatorname{Spec} R$. If the support of $H^2 = C^2/B^2$ is small (that is, has big codimension), then the intersection should happen at some point $x \in \operatorname{Spec} R$ outside of the support of H^2, and we are done. Of course, we oversimplified the situation, see [EG19] for a complete account. $\square$

4.2. A non-abelian lifting theorem

4.2.1. Theorem Let U be a unipotent linear algebraic group of class 2 whose center is isomorphic to $\mathbb{G}_a$. Write $Z(U)$ for the center of U and U^{ad} for $U/Z(U)$. Fix an algebraic group homomorphism $\phi : L \to \mathrm{Aut}(U)$ with graded pieces $\phi^{\mathrm{ad}} : L \to \mathrm{GL}(U^{\mathrm{ad}})$ and $\phi^{z} : L \to \mathrm{GL}(Z(U))$.

Fix a mod ϖ representation $\bar{r} : G_K \to L(\mathbb{F})$. Let $[\bar{c}] \in H^1(G_K, U(\mathbb{F}))$ be a characteristic p cocycle.

Let $\operatorname{Spec} R$ be an irreducible component of a crystalline lifting ring of $\bar{r}$.

Assume

[1] $H^2(G_K, \phi^{\mathrm{ad}}(r^{\mathrm{univ}}))$ is SGR;

[2] $p \neq 2$;

[3] There exists a finite Galois extension K'/K of prime-to-p degree such that $\phi(\bar{r})|_{G_{K'}}$ is Lyndon-Demuškin; and

[4] There exists a $\bar{\mathbb{Z}}_p$-point of $\operatorname{Spec} R$ which is mildly regular when restricted to $G_{K'}$. (In particular, $\operatorname{Spec} R[1/p] \neq 0$.)

Then there exists a $\bar{\mathbb{Z}}_p$-point of $\operatorname{Spec} R$ which gives rise to a Galois representation $r^\circ : G_K \to L(\bar{\mathbb{Z}}_p)$ such that if we endow $U(\bar{\mathbb{Z}}_p)$ with the G_K-action $G_K \xrightarrow{r^\circ} L(\bar{\mathbb{Z}}_p) \xrightarrow{\phi} \mathrm{Aut}(U)(\bar{\mathbb{Z}}_p)$, the cocycle $[\bar{c}]$ has a characteristic 0 lift $[c] \in H^1(G_K, U(\bar{\mathbb{Z}}_p))$.

Proof. Combine Theorem 4.1.2 and Corollary 3.3.4. $\square$

We explain how the above theorem will be used. Let G be a connected reductive group over $\mathcal{O}_E$. Let $\bar{\rho} : G_K \to G(\mathbb{F})$ be a mod ϖ representation. Assume $\bar{\rho}$ factors through a parabolic $P \subset G$, with Levi decomposition $P = L \ltimes U$. Denote by $\phi : L \to \mathrm{Aut}(U)$ the conjugation action. We assume U is nilpotent of class 2, so U^{ad} is an abelian group. Write $\bar{r}$ for the Levi factor of $\bar{\rho}$.

$$\begin{tikzcd} & P(\bar{\mathbb{F}}_p) \arrow[d] \\ G_K \arrow[ur, "\bar{\rho}"] \arrow[r, "\bar{r}"] & L(\bar{\mathbb{F}}_p) \end{tikzcd}$$

Then $\bar{\rho}$ defines a cohomology class $[\bar{c}] \in H^1(G_K, \phi(\bar{r}))$, and the theorem above can be used to lift $[\bar{c}]$.

4.3. An unobstructed lifting theorem

The following result will be used in the proof of the main theorem.

4.3.1. Proposition Let V be a unipotent linear algebraic group such that $V(\bar{\mathbb{Z}}_p)$ is equipped with a continuous G_K-action. Let $[\bar{c}] \in H^1(G_K, V(\bar{\mathbb{F}}_p))$ be a characteristic p cocycle. Let $Z(V)$ be the center of V, and write V^{ad} for $V/Z(V)$. The quotient $V \to V^{\mathrm{ad}}$ induces a map $\mathrm{ad} : H^1(G_K, V) \to H^1(G_K, V^{\mathrm{ad}})$. Assume $H^2(G_K, Z(V)(\bar{\mathbb{F}}_p)) = 0$.

If $\mathrm{ad}([\bar{c}])$ admits a lift in $H^1(G_K, V^{\mathrm{ad}}(\bar{\mathbb{Z}}_p))$, then $[\bar{c}]$ admits a lift in $H^1(G_K, V(\bar{\mathbb{Z}}_p))$.

Proof. By [Se02, Proposition 43], since $Z(V)$ is a central normal subgroup of V, there exists a long exact sequence of pointed sets

$$\begin{tikzcd} H^1(G_K, V(\bar{\mathbb{Z}}_p)) \arrow[r, "\mathrm{ad}"] \arrow[d] & H^1(G_K, V^{\mathrm{ad}}(\bar{\mathbb{Z}}_p)) \arrow[r, "\delta"] \arrow[d] & H^2(G_K, Z(V)(\bar{\mathbb{Z}}_p)) \arrow[d] \\ H^1(G_K, V(\bar{\mathbb{F}}_p) \arrow[r, "\mathrm{ad}"] & H^1(G_K, V^{\mathrm{ad}}(\bar{\mathbb{F}}_p)) \arrow[r] & H^2(G_K, Z(V)(\bar{\mathbb{F}}_p)) \end{tikzcd}$$

By Nakayama's Lemma, we have $H^2(G_K, Z(V)(\bar{\mathbb{Z}}_p)) = 0$. In particular, there exists $[c'] \in$

$H^1(G_K, V(\bar{\mathbb{Z}}_p)$ such that $\mathrm{ad}([\bar{c}]) = \mathrm{ad}([c'])$ mod ϖ. Write $[\bar{c}']$ for $[c']$ mod ϖ. Say $[\bar{c}] = [\bar{c}'] + [\bar{f}]$ for some $[\bar{f}] \in H^1(G_K, Z(V)(\bar{\mathbb{F}}_p))$ (recall that $H^1(G_K, V)$ is a $H^1(G_K, Z(V))$-torsor). Since $H^1(G_K, Z(V)(\bar{\mathbb{Z}}_p)) = 0$, there exists a lift $[f]$ of $\bar{f}$. The cocycle $[c] := [c'] + [f]$ is a lift of $[\bar{c}]$. $\square$

5. Codimension estimates of loci cut out by H^2

Assume $p > 3$. Let $K/\mathbb{Q}_p$ be a finite extension. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_E$, uniformizer ϖ, and residue field $\mathbb{F}$.

5.1. The Emerton-Gee stack

We follow the notation of [EG19]. For each $d > 0$, [EG19] constructed the moduli stack $\mathcal{X}_d = \mathcal{X}_{K,d}$ of projective étale (ϕ, Γ_K)-modules of rank d.

We prove a mild generalization of [EG19, Proposition 5.4.4(1)].

Let T be a reduced finite type $\bar{\mathbb{F}}_p$-scheme. Let $f : T \to (\mathcal{X}_{a,\mathrm{red}})_{\bar{\mathbb{F}}_p} \times (\mathcal{X}_{d,\mathrm{red}})_{\bar{\mathbb{F}}_p}$ be a morphism. By functoriality, there is a morphism

$$\eta : (\mathcal{X}_{a,\mathrm{red}})_{\bar{\mathbb{F}}_p} \times (\mathcal{X}_{d,\mathrm{red}})_{\bar{\mathbb{F}}_p} \to (\mathcal{X}_{ad,\mathrm{red}})_{\bar{\mathbb{F}}_p}$$

sending a pair of (ϕ, Γ)-modules M, N to their hom module $\mathrm{Hom}_{\phi,\Gamma}(M, N)$. The morphism $\eta(f)$ corresponds to a family $\bar{\rho}_T$ of rank-ad Galois representations over T. We assume $H^2(G_K, \bar{\rho}_{\eta(t)})$ is of constant rank for all $t \in T(\bar{\mathbb{F}}_p)$. By [EG19, Lemma 5.4.1], the coherent sheaf $H^2(G_K, \bar{\rho}_T)$ is locally free of rank r as an $\mathcal{O}_E$-module.

By [EG19, Theorem 5.1.29], we can choose a complex of finite rank locally free $\mathcal{O}_E$-modules

$$C_T^0 \to C_T^1 \to C_T^2$$

computing $H^\bullet(G_K, \bar{\rho}_T)$. Since $H^2(G_K, \bar{\rho}_T)$ is a locally free sheaf, the truncated complex

$$C_T^0 \to Z_T^1$$

is again a complex of locally free $\mathcal{O}_T$-modules. The vector bundle $\mathscr{V}(Z_T^1) := \underline{Spec}(\mathrm{Sym}(Z_T^1)^\vee)$ associated to the locally free sheaf Z_T^1 parameterizes all extensions

$$0 \to \bar{\rho}_{\eta(t)} \to ? \to \bar{\mathbb{F}}_p \to 0, \quad t \in T(\bar{\mathbb{F}}_p)$$

of the trivial G_K-representation $\bar{\mathbb{F}}_p$ by $\bar{\rho}_{\eta(t)}$. There are two projection morphisms

$$()_1 : (\mathcal{X}_{a,\mathrm{red}})_{\bar{\mathbb{F}}_p} \times (\mathcal{X}_{d,\mathrm{red}})_{\bar{\mathbb{F}}_p} \to (\mathcal{X}_{a,\mathrm{red}})_{\bar{\mathbb{F}}_p}$$

and

$$()_2 : (\mathcal{X}_{a,\mathrm{red}})_{\bar{\mathbb{F}}_p} \times (\mathcal{X}_{d,\mathrm{red}})_{\bar{\mathbb{F}}_p} \to (\mathcal{X}_{d,\mathrm{red}})_{\bar{\mathbb{F}}_p}$$

For each $t \in T(\bar{\mathbb{F}}_p)$, $f(t)_1 \in (\mathcal{X}_{a,\mathrm{red}})(\bar{\mathbb{F}}_p)$ corresponds to a rank-a Galois representation $\bar{\rho}_{t_1}$, and $f(t)_2 \in (\mathcal{X}_{d,\mathrm{red}})(\bar{\mathbb{F}}_p)$ corresponds to a rank-d Galois representation $\bar{\rho}_{t_2}$. We have $\bar{\rho}_{\eta(t)} = \mathrm{Hom}_{G_K}(\bar{\rho}_{t_1}, \bar{\rho}_{t_2})$. So we can also regard $\mathscr{V}(Z_T^1)$ is a scheme parametrizing all extensions

$$0 \to \bar{\rho}_{t_1} \to ? \to \bar{\rho}_{t_2} \to 0, \quad t \in T(\bar{\mathbb{F}}_p)$$

and we have a morphism sending extension classes to equivalence classes of G_K-representations

$$g : \mathscr{V}(Z^1_T) \to (\mathcal{X}_{a+d,\mathrm{red}})_{\bar{\mathbb{F}}_p}.$$

5.1.0.1 Lemma Let e denote the dimension of the scheme-theoretic image of T in $(\mathcal{X}_{a,\mathrm{red}})_{\bar{\mathbb{F}}_p} \times (\mathcal{X}_{d,\mathrm{red}})_{\bar{\mathbb{F}}_p}$. Then the scheme-theoretic image of $V = \mathscr{V}(Z^1_T)$ in $(\mathcal{X}_{a+d,\mathrm{red}})_{\bar{\mathbb{F}}_p}$ has dimension at most

$$e + r + ad[K : \mathbb{Q}_p].$$

Proof. Without loss of generality, we assume T (and hence V) is irreducible. The proof is a routine calculation using stacks. We follow the proof of [EG19, Proposition 5.4.4] closely.

Let $v \in V(\bar{\mathbb{F}}_p)$. Write t for the composite $\operatorname{Spec} \bar{\mathbb{F}}_p \xrightarrow{v} V \to T$. Write $f(t)$ for the composite $f \circ t$ Write $g(v)$ for the composite $g \circ v$. Define

$$T_{f(t)} := T \underset{f,(\mathcal{X}_{a,\mathrm{red}})_{\bar{\mathbb{F}}_p} \times (\mathcal{X}_{d,\mathrm{red}})_{\bar{\mathbb{F}}_p}, f(t)}{\times} \operatorname{Spec} \bar{\mathbb{F}}_p$$

$$V_{g(v)} := V \underset{g,(\mathcal{X}_{a,\mathrm{red}})_{\bar{\mathbb{F}}_p} \times (\mathcal{X}_{d,\mathrm{red}})_{\bar{\mathbb{F}}_p}, g(v)}{\times} \operatorname{Spec} \bar{\mathbb{F}}_p$$

$$V_{f(t),g(v)} := V_{g(v)} \underset{(\mathcal{X}_{a,\mathrm{red}})_{\bar{\mathbb{F}}_p} \times (\mathcal{X}_{d,\mathrm{red}})_{\bar{\mathbb{F}}_p}, f(t)}{\times} \operatorname{Spec} \bar{\mathbb{F}}_p.$$

Note that $V_{f(t),g(v)} \cong T_{f(t)} \times_T V_{g(v)}$.

By [stacks-project, Tag 0DS4], it suffices to show, for v lying in some dense open subset of V,

$$\dim V_{f(t),g(v)} \geq \dim V - (e + r + ad[K : \mathbb{Q}_p]).$$

Let $\bar{\rho}_{f(t)_1}$ denote the Galois representation corresponding to $f(t)_1 : \operatorname{Spec} \bar{\mathbb{F}}_p \to (\mathcal{X}_{a,\mathrm{red}})_{\bar{\mathbb{F}}_p}$. Let

$\bar{\rho}_{f(t)_2}$ denote the Galois representation corresponding to $f(t)_2 : \operatorname{Spec} \bar{\mathbb{F}}_p \to (\mathcal{X}_{d,\mathrm{red}})_{\bar{\mathbb{F}}_p}$. Say $G_{t_1} := \operatorname{Aut}(\bar{\rho}_{f(t)_1})$, and $G_{t_2} := \operatorname{Aut}(\bar{\rho}_{f(t)_2})$. The morphism $f(t)$ factors through a monomorphism

$$[\operatorname{Spec} \bar{\mathbb{F}}_p / G_{t_1}] \times [\operatorname{Spec} \bar{\mathbb{F}}_p / G_{t_2}] \hookrightarrow (\mathcal{X}_{a,\mathrm{red}})_{\bar{\mathbb{F}}_p} \times (\mathcal{X}_{d,\mathrm{red}})_{\bar{\mathbb{F}}_p}$$

which induces a monomorphism

$$([\operatorname{Spec} \bar{\mathbb{F}}_p / G_{t_1}] \times [\operatorname{Spec} \bar{\mathbb{F}}_p / G_{t_2}]) \underset{(\mathcal{X}_{a,\mathrm{red}})_{\bar{\mathbb{F}}_p} \times (\mathcal{X}_{d,\mathrm{red}})_{\bar{\mathbb{F}}_p}}{\times} V_{g(v)} \hookrightarrow V_{g(v)}.$$

So it suffices to show

$$(\dagger) \qquad \dim V_{f(t),g(v)} \geq \dim V - (e + r + ad[K : \mathbb{Q}_p]) + \dim G_{t_1} + \dim G_{t_2}$$

for v lying in a dense open of V.

There exists an étale cover S of $(T_{f(t)})_{\mathrm{red}}$ such that the pull-back family $\bar{\rho}_S$ is a trivial family with fiber $\bar{\rho}_t$.

Let $C^0_S \to Z^1_S$ denote the pullback family of $C^0_T \to Z^1_T$ to S. $C^0_S \to Z^1_S$ is also the pullback family of the fiber $C^0_t \to Z^1_t$ to S. Write W for the affine scheme associated to $H^1(G_K, \bar{\rho}^\vee_{f(t)_1} \otimes \bar{\rho}_{f(t)_2})$. By the isomorphism

$$H^1(G_K, \bar{\rho}^\vee_{f(t)_1} \otimes \bar{\rho}_{f(t)_2}) \cong \operatorname{Ext}_{G_K}(\bar{\rho}_{f(t)_1}, \bar{\rho}_{f(t)_2})$$

there is a morphism $W \to (\mathcal{X}_{a+d,\mathrm{red}})_{\bar{\mathbb{F}}_p}$. Denote by w the image of v in w. We have

$$S \times_T V_{g(v)} = S \times_T V \times_W W_{h(w)}.$$

Let V' be the kernel of $S \times_T V \to S \times_{\bar{\mathbb{F}}_p} W$, which is a trivial vector bundle over S. We have

$$\begin{aligned}
\dim V_{f(t),g(v)} &= \dim S \times_T V_{g(v)} \\
&= \operatorname{rank} V' + \dim S + \dim W_{h(w)} \\
&= \operatorname{rank} Z^1_T - \dim H^1(G_K, \bar{\rho}^\vee_{f(t)_1} \otimes \bar{\rho}_{f(t)_2}) + \dim S + \dim W_{h(w)}
\end{aligned}$$

Note that $\dim V - \dim T = \operatorname{rank} Z^1_T$, and by local Euler characteristic $H^0(G_K, \bar{\rho}^\vee_{f(t)_1} \otimes \bar{\rho}_{f(t)_2}) - H^1(G_K, \bar{\rho}^\vee_{f(t)_1} \otimes \bar{\rho}_{f(t)_2}) + r = -ad[K : \mathbb{Q}_p]$. We can replace T by a dense open of T where $e = \dim T - \dim T_{f(t)} = \dim T - \dim S$. Combine all these equalities, (†) becomes

$$\dim W_{h(w)} \geq \dim H^0(G_K, \bar{\rho}^\vee_{f(t)_1} \otimes \bar{\rho}_{f(t)_2}) + \dim G_{t_1} + \dim G_{t_2}$$

which follows from the fact that

$$H^0(G_K, \bar{\rho}^\vee_{f(t)_1} \otimes \bar{\rho}_{f(t)_2}) \rtimes (G_{t_1} \times G_{t_2}) \subset \operatorname{Aut}(\bar{\rho}_w)$$

and $\dim W_{h(w)} \geq \dim \operatorname{Aut}(\bar{\rho}_w)$. □

We recall some terminology from [EG19]. Denote by $\mathrm{ur}_x : \mathbb{G}_\mathrm{m} \to \mathcal{X}_1$ the family of unramified characters of G_K. Let T be a reduced finite type $\mathbb{F}$-scheme. Let $T \to \mathcal{X}$ be a morphism, corresponding to a family $\bar{\rho}_T$ of G_K-representations over T. We can construct the family of unramified twisting $\bar{\rho}_T \boxtimes \mathrm{ur}_x$ over $T \times \mathbb{G}_\mathrm{m}$. $\bar{\rho}_T$ is said to be *twistable* if whenever $\bar{\rho}_t \cong \bar{\rho}_{t'} \otimes \mathrm{ur}_a$ for $t, t' \in T(\bar{\mathbb{F}}_p)$ and $a \in \bar{\mathbb{F}}_p^\times$, we have $a = 1$. $\bar{\rho}_T$ is said to be *essentially twistable* if for each $t \in T(\bar{\mathbb{F}}_p)$, the set of $a \neq 1$ for which $\bar{\rho}_t \cong \bar{\rho}_{t'} \otimes \mathrm{ur}_a$ is finite.

We say $\bar{\rho}_T$ is *untwistable* if $\bar{\rho}$ is not essentially twistable.

From now on, write $\mathcal{X} = (\mathcal{X}_{2,\mathrm{red}})_{\bar{\mathbb{F}}_p}$ for the moduli stack parameterizing (ϕ, Γ)-modules of rank 2.

Let $\bar{r}^{\mathrm{univ}}$ be the universal family of (ϕ, Γ)-modules over $\mathcal{X}$.

5.2. Loci cut out by $H^2(G_K, \mathrm{sym}^3 / \det^2)$

Write H^2 for $H^2(G_K, \frac{\mathrm{sym}^3(\bar{r}^{\mathrm{univ}})}{\det(\bar{r}^{\mathrm{univ}})^2})$. Let $x \in \mathcal{X}(\bar{\mathbb{F}}_p)$ with corresponding Galois representation $\bar{r}_x : G_K \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$.

5.2.0.1 Lemma If $\bar{r}_x$ is irreducible, then

$$h_x^2 := \dim_{\bar{\mathbb{F}}_p} H^2(G_K, \frac{\mathrm{sym}^3(\bar{r}_x)}{\det(\bar{r}_x)^2}) \leq 2.$$

Proof. An irreducible mod ϖ representation is of the shape $\mathrm{Ind}_{G_{K_2}}^{G_K} \bar{\chi}$ for some character $\bar{\chi}$ of the degree-2 unramified extension K_2 of K. A direct computation shows

$$\mathrm{sym}^3(\bar{r}_x) = \mathrm{Ind}(\bar{\chi}^3) \oplus \mathrm{Ind}(\bar{\chi} \det \bar{r}_x).$$

Both $H^2(G_K, \frac{\mathrm{Ind}(\bar{\chi}^3)}{\det(\bar{r}_x)^2})$ and $H^2(G_K, \frac{\mathrm{Ind}(\bar{\chi} \det \bar{r}_x)}{\det(\bar{r}_x)^2})$ has dimension at most 1. This is because the induction of a character can't be a direct sum of two isomorphic characters (when $p \neq 2$). $\square$

5.2.0.2 Corollary The locus of $\bar{r}_x$ in $\mathcal{X}$ where $\bar{r}_x$ is irreducible and

$$H^2(G_K, \frac{\mathrm{sym}^3(\bar{r}_x)}{\det(\bar{r}_x)^2}) \geq r$$

is locally closed and of dimension at most $[K:\mathbb{Q}_p]-r$.

Proof. Up to unramified twist, there are only finitely many irreducible representations. The automorphism group of an irreducible representation is 1-dimensional. By Lemma 5.2.0.1, we have $h^2_x \leq 2$ when $\bar{r}_x$ is irreducible.

We first consider the locus where $h^2_x = 2$. An irreducible representation $\bar{r}_x$ corresponds to a morphism $x : \operatorname{Spec}\bar{\mathbb{F}}_p \to \mathcal{X}$ which factors through an immersion $[\operatorname{Spec}\bar{\mathbb{F}}_p/G_x] \hookrightarrow \mathcal{X}$ where $G_x = \mathbb{G}_\mathrm{m}$ is the automorphism group of $\bar{r}_x$. So the scheme-theoretic image of x is (-1)-dimensional. This locus consists of the scheme-theoretic image of finitely many x's, and thus has dimension -1.

Then we consider the locus where $h^2_x \leq 1$. This locus consists of the unramified twists of finitely many irreducible G_K-representations. By [EG19, 3.8], a morphism $x : \operatorname{Spec}\bar{\mathbb{F}}_p \to \mathcal{X}$ can be upgraded to a morphism $x \boxtimes \mathrm{ur} : \operatorname{Spec}\bar{\mathbb{F}}_p \times \mathbb{G}_\mathrm{m} \to \mathcal{X}$ whose scheme-theoretic image consists of the unramified twists of $\bar{r}_x$. By [EG19, Lemma 5.3.2], the scheme-theoretic image of $x \boxtimes \mathrm{ur}$ is 0-dimensional.

In either case, dim of locus $\leq [K:\mathbb{Q}_p] - h^2_x$. □

5.2.0.3 Lemma If $\bar{r}_x$ is a direct sum of distinct characters, then

$$H^2(G_K, \frac{\operatorname{sym}^3(\bar{r}_x)}{\det(\bar{r}_x)^2}) \leq 2.$$

Proof. Say $\bar{r}_x \sim \begin{bmatrix} \bar{\chi}_1 & \\ & \bar{\chi}_2 \end{bmatrix}$. We have

$$\frac{\operatorname{sym}^3(\bar{r}_x)}{\det(\bar{r})^2} \cong \bar{\chi}_1\bar{\chi}_2^{-2} \oplus \bar{\chi}_2^{-1} \oplus \bar{\chi}_1^{-1} \oplus \bar{\chi}_2\bar{\chi}_1^{-2}.$$

If $\bar{\chi}_1 \neq \bar{\chi}_2$, then the multiset $\{\bar{\chi}_1\bar{\chi}_2^{-2}, \bar{\chi}_2^{-1}, \bar{\chi}_1^{-1}, \bar{\chi}_1^{-2}\bar{\chi}_2\}$ contains at most 2 isomorphic characters.

$\square$

5.2.0.4 Corollary The locus of $\bar{r}_x$ in $\mathcal{X}$ where $\bar{r}_x$ is a direct sum of distinct characters and

$$H^2(G_K, \frac{\mathrm{sym}^3(\bar{r}_x)}{\det(\bar{r}_x)^2}) \geq r$$

is locally closed and of dimension at most $[K : \mathbb{Q}_p] - r$.

Proof. By Lemma 5.2.0.3, we have $h_x^2 \leq 2$ when $\bar{x} = \alpha \oplus \beta$ is a direct sum of distinct characters.

In the locus where $h_x^2 = 2$, we must have $\pm\alpha = \pm\beta = \mathbb{F}(-1)$. The morphism $x : \operatorname{Spec} \bar{\mathbb{F}}_p \to \mathcal{X}$ factors through $[\operatorname{Spec} \bar{\mathbb{F}}_p / G_x] \hookrightarrow \mathcal{X}$ and $\dim G_x = 2$. So this locus is closed of dimension -2.

In the locus where $h_x^2 \geq 1$, we have one of the following: (i) $\alpha = \mathbb{F}(-1)$, (ii) $\beta = \mathbb{F}(-1)$, (iii) $\alpha = \beta^2(-1)$, (iv) $\beta = \alpha^2(-1)$. By symmetry, we can only consider case (i) and (iii). In either case, α is determined by β and β, up to unramified twists, has only finitely many choices. Let $y : \operatorname{Spec} \bar{\mathbb{F}}_p \to (\mathcal{X}_1)_{\mathrm{red}}$ be the point corresponding to β. Let $\xi : (\mathcal{X}_1)_{\mathrm{red}} \to \mathcal{X}$ be the morphism sending β to $\bar{\mathbb{F}}_p(-1) \oplus \beta$ (or $\beta^2(-1) \oplus \beta$). We claim the scheme-theoretic image Z of $\xi(y \boxtimes \mathrm{ur})$ has dimension at most -1. By [stacks-project, 0DS4], there is a dense open $T \subset \mathbb{G}_{\mathrm{m}}$ such that for all $\bar{\mathbb{F}}_p$-point t of T, $\dim Z = \dim T - \dim T \times_{\mathcal{X},t} \operatorname{Spec} \bar{\mathbb{F}}_p$. Write T_t for $T \times_{\mathcal{X},t} \operatorname{Spec} \bar{\mathbb{F}}_p$. Write G_x for $\mathrm{Aut}(\bar{r}_x)$. Note that T_t is a G_x-torsor over $T \times_{\mathcal{X},t} [\operatorname{Spec} \bar{\mathbb{F}}_p/G_x]$ (a closed subscheme of T). So $\dim T_t \geq \dim G_x = 2$, and thus $\dim Z = \dim T - \dim T_t \leq 1 - 2 = -1$.

Finally consider the locus where $h_x^2 \geq 0$. Up to unramified twists, both α and β have finitely many choices. Let $y_1 : \operatorname{Spec} \bar{\mathbb{F}}_p \to (\mathcal{X}_1)_{\mathrm{red}}$ and $y_2 : \operatorname{Spec} \bar{\mathbb{F}}_p \to (\mathcal{X}_1)_{\mathrm{red}}$ be the points corresponding to α and β. Let $\Xi : (\mathcal{X}_1)_{\mathrm{red}} \times (\mathcal{X}_1)_{\mathrm{red}} \to \mathcal{X}$ be the morphism sending $(\alpha, \beta) \mapsto \alpha \oplus \beta$. Let

Z be the scheme-theoretic image of $\Xi((y_1 \boxtimes \mathrm{ur}) \boxtimes (y_2 \boxtimes \mathrm{ur}))$. By a similar argument, $\dim Z \leq \dim(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}) - \dim \mathrm{Aut}(\bar{r}_x) = 2 - 2 = 0$. $\square$

5.2.0.5 Lemma If $\bar{r}_x$ is a direct sum of isomorphic characters, then

$$H^2(G_K, \frac{\mathrm{sym}^3(\bar{r}_x)}{\det(\bar{r}_x)^2}) \leq 4.$$

Proof. This is trivial because the underlying $\bar{\mathbb{F}}_p$-vector space is 4-dimensional. $\square$

5.2.0.6 Corollary The locus of $\bar{r}_x$ in $\mathcal{X}$ where $\bar{r}_x$ is a direct sum of isomorphic characters and

$$H^2(G_K, \frac{\mathrm{sym}^3(\bar{r}_x)}{\det(\bar{r}_x)^2}) \geq r$$

is locally closed and of dimension at most $[K : \mathbb{Q}_p] - r$.

Proof. The automorphism group is 4-dimensional. So the locus in the moduli stack has dimension $\dim \mathbb{G}_{\mathrm{m}} - \dim \mathrm{Aut}(\bar{r}_x) = 1 - 4 = -3$. $\square$

5.2.0.7 Lemma If $\bar{r}_x$ is a non-trivial extension of two characters, then

$$h^2_x := \dim H^2(G_K, \frac{\mathrm{sym}^3(\bar{r}_x)}{\det(\bar{r}_x)^2}) \leq 1$$

and when the equality holds, the quotient character of $\bar{r}_x$ is a character whose third power is $\bar{\mathbb{F}}_p(1)$.

Proof. This is where we make use of the assumption $p > 3$. Say $\bar{r}_x \sim \begin{bmatrix} \bar{\chi}_1 & \bar{c} \\ & \bar{\chi}_2 \end{bmatrix}$. We have

$$\mathrm{sym}^3(\bar{r}_x) \sim \begin{bmatrix} \bar{\chi}_1^3 & \bar{\chi}_1^2\bar{c} & * & * \\ & \bar{\chi}_1^2\bar{\chi}_2 & 2\bar{\chi}_1\bar{\chi}_2\bar{c} & * \\ & & \bar{\chi}_1\bar{\chi}_2^2 & 3\bar{\chi}_2^2\bar{c} \\ & & & \bar{\chi}_2^3. \end{bmatrix}$$

We claim $\mathrm{sym}^3(\bar{r}_x)$ has a unique G_K-invariant quotient line. Let $\{e_1, e_2\}$ be a basis of the representation space of $\bar{r}_x$ such that e_1 is an invariant line. Then $\{e_1^3, e_1^2e_2, e_1e_2^2, e_2^3\}$ is a basis of the representation space of $\mathrm{sym}^3(\bar{r}_x)$. By duality, the claim is equivalent to saying that $\mathrm{sym}^3(\bar{r}_x)$ has a unique invariant line. Clearly $\{e_1^3\}$ defines an invariant line. Assume there is another invariant line $\mathrm{span}(v)$. We quotient $\mathrm{sym}^3(\bar{r}_x)$ by $\mathrm{span}(e_1^3)$. The quotient representation has a unique invariant line generated by the image of $e_1^2e_2$. So $v \in \mathrm{span}(e_1^3, e_1^2e_2)$. But then we must have $v \in \mathrm{span}(e_1^3)$, since $[\bar{c}]$ is a non-trivial extension class. $\square$

5.2.0.8 Corollary The locus of $\bar{r}_x$ in $\mathcal{X}$ where $\bar{r}_x$ is a non-trivial extension of two characters and

$$H^2(G_K, \frac{\mathrm{sym}^3(\bar{r}_x)}{\det(\bar{r}_x)^2}) \geq r$$

is locally closed and of dimension at most $[K : \mathbb{Q}_p] - r$.

Proof. The locus of non-trivial extensions is the complement of all previous loci, and is thus locally closed.

Say $\bar{r}_x$ is the extension of $\bar{\beta}$ by $\bar{\alpha}$. By Lemma 5.2.0.7, we have $h_x^2 \leq 1$ when $\bar{r}_x$ is a nontrivial extension of characters. So the locus where $\bar{r}_x$ is a non-trivial extension of characters consists of four sub-loci:

(i) $h_x^2 = 1$ and $\mathrm{Ext}^2(\beta, \alpha) = 0$;

(ii) $h_x^2 = 1$ and $\mathrm{Ext}^2(\beta, \alpha) \neq 0$;

(iii) $h_x^2 = 0$ and $\mathrm{Ext}^2(\beta, \alpha) = 0$; and

(iv) $h_x^2 = 0$ and $\mathrm{Ext}^2(\beta, \alpha) \neq 0$;

By [stacks-project, 0BDI], each sub-locus is a locally closed subset.

In each of the four cases, write $T \subset (\mathcal{X}_{1,\mathrm{red}})_{\bar{\mathbb{F}}_p} \times (\mathcal{X}_{1,\mathrm{red}})_{\bar{\mathbb{F}}_p}$ for the locus of the pairs (α, β) satisfying the corresponding condition. Say $\dim T = e$, and $\dim \mathrm{Ext}^2(\beta, \alpha) = r$. By Lemma 5.1.0.1, each sub-locus has dimension at most

$$e + r + [K : \mathbb{Q}_p].$$

In sub-locus (i), β has only finitely many choices, so $e = -1$, $r = 0$; in sub-locus (ii), both β and α have only finitely many choices, so $e = -2$, $r = 1$; in sub-locus (iii), both β and α can vary in a dense open of $(\mathcal{X}_{1,\mathrm{red}})_{\bar{\mathbb{F}}_p}$, so $e = 2\dim(\mathcal{X}_{1,\mathrm{red}})_{\bar{\mathbb{F}}_p} = 0$, $r = 0$; in sub-locus (iv), when α is chosen, β has only finitely many choices, so $e = -1$, $r = 1$. We can verify that in each case $e + r + [K : \mathbb{Q}_p] \leq \dim \mathcal{X} - h_x^2 = [K : \mathbb{Q}_p] - h_x^2$. $\square$

5.2.0.9 Theorem The locus of $\bar{r}_x$ in $\mathcal{X}$ for which

$$H^2(G_K, \frac{\mathrm{sym}^3(\bar{r}_x)}{\det(\bar{r}_x)^2}) \geq r$$

is of dimension at most $[K : \mathbb{Q}_p] - r$.

Proof. This theorem follows immediately from Lemma 5.2.0.1, Lemma 5.2.0.7, Lemma 5.2.0.3, Lemma 5.2.0.5, and their corollaries. $\square$

Fix a mod ϖ representation $\bar{r} : G_K \to GL_2(\mathbb{F})$. Let $\underline{\lambda}$ be a Hodge type. Let R be an irreducible component of the crystalline lifting ring $R_{\bar{r}}^{\mathrm{crys},\underline{\lambda},\mathcal{O}_E}$. Assume $\operatorname{Spec} R[1/p] \neq \emptyset$. Let r^{univ} be the universal family of Galois representations on R.

Since $H^2(G_K, \frac{\mathrm{sym}^3(r^{\mathrm{univ}})}{\det(r^{\mathrm{univ}})^2})$ is a coherent sheaf, by the semicontinuity theorem, the locus $X_s := \{x \in \operatorname{Spec} R | \dim \kappa(x) \otimes_R H^2 \geq s\}$ is locally closed, and has a reduced induced scheme structure.

5.2.0.10 Theorem Let R be an irreducible component of the crystalline lifting ring with regular labeled Hodge-Tate weights. If $H^2(G_K, \frac{\mathrm{sym}^3(r^{\mathrm{univ}})}{\det(r^{\mathrm{univ}})^2})$ is ϖ-torsion, the locus

$$\{x \in \operatorname{Spec} R | \dim \kappa(x) \otimes_R H^2(G_K, \frac{\mathrm{sym}^3(r^{\mathrm{univ}})}{\det(r^{\mathrm{univ}})^2}) \geq s\}$$

for $s \leq 1$ has codimension $\geq s + 1$ in $\operatorname{Spec} R$.

Proof. The proof is identical to that of [EG19, Theorem 6.1.1] if we use Theorem 5.2.0.9 instead of [EG19, Theorem 5.5.12]. $\square$

6. The existence of crystalline lifts for the exceptional group G_2

6.1. Parabolics of G_2

Let G_2 be the Chevalley group over $\mathcal{O}_E$ of type G_2.

Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_E$, residue field $\mathbb{F}$ and uniformizer ϖ.

We remind the reader of the root system of G_2:

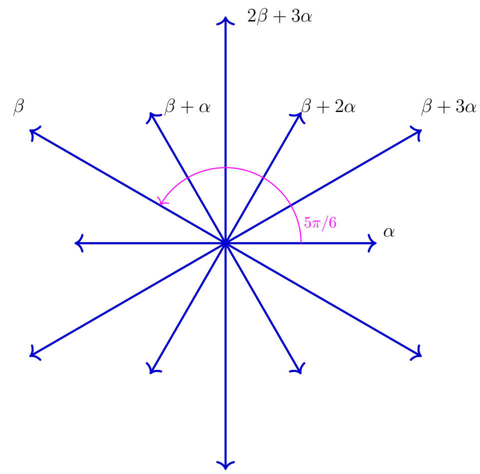

Figure 4.1: Root system of G_2

6.1.1. The short root parabolic Let $P \subset G_2$ be the short root parabolic, which admits a Levi decomposition $P = L \ltimes U$. The Levi factor L is a copy of GL_2 and the unipotent radical U is a nilpotent group of class 2. Write U^{ad} for $U/Z(U)$.

Fix an isomorphism $\mathrm{std} : L \cong \mathrm{GL}_2$. We have

- $Z(U) \cong \mathbb{G}_a$, and
- $U^{\mathrm{ad}} \cong \mathbb{G}_a^{\oplus 4}$.

Write $\operatorname{Lie} U = Z(U) \oplus U^{\mathrm{ad}}$. The Levi factor acts on U by conjugation. We have an isomorphism of L-modules

$$\operatorname{Lie} U \cong \frac{1}{\det^2} \operatorname{sym}^3(\mathrm{std}) \oplus \frac{1}{\det}$$

where $\det : L \to \mathbb{G}_{\mathrm{m}}$ is the determinant character, and $\mathrm{std} : L \xrightarrow{\cong} \mathrm{GL}_2$ is the fixed isomorphism. The above short exact sequence can be upgraded to a short exact sequence of groups with L-actions

$$0 \to \frac{1}{\det} \to U \to \frac{1}{\det^2} \operatorname{sym}^3(\mathrm{std}) \to 0.$$

6.1.2. The long root parabolic Let $Q \subset G_2$ be the long root parabolic, which admits a Levi decomposition $Q = L' \ltimes V$ where $L' \cong \mathrm{GL}_2$ and V is a nilpotent group of class 3. Fix an isomorphism $\mathrm{std} : L' \xrightarrow{\cong} \mathrm{GL}_2$. Write det for the composition $L' \xrightarrow{\mathrm{std}} \mathrm{GL}_2 \xrightarrow{\det} \mathrm{GL}_1$.

Write U' for $V/Z(V)$. Then U' is a nilpotent group of class 2 whose center is isomorphic to $\mathbb{G}_a$. The conjugation action of L' on U' is given by $U'/Z(U') \cong \mathrm{std}$, and $Z(U') \cong \det$, as L'-modules.

6.2. Theorem Assume $p > 3$. Let $K/\mathbb{Q}_p$ be a p-adic field. Let $\bar{\rho} : G_K \to G_2(\bar{\mathbb{F}}_p)$ be a mod ϖ Galois representation. Then $\bar{\rho}$ admits a crystalline lift $\rho^\circ : G_K \to G_2(\bar{\mathbb{Z}}_p)$ of $\bar{\rho}$.

Moreover, if $\bar{\rho}$ factors through a maximal parabolic and the Levi factor $\bar{r}_{\bar{\rho}}$ of $\bar{\rho}$ admits a Hodge-Tate regular and crystalline lift r_1 such that the adjoint representation $\phi^{\mathrm{Lie}}(r_1)$ has Hodge-Tate weights slightly less than $\underline{0}$, then ρ° can be chosen such that it factors through the same maximal parabolic and its Levi factor r_{ρ° lies on the same irreducible component of the spectrum of the crystalline lifting ring that r_1 does.

Proof. If $\bar{\rho}$ is irreducible, then $\bar{\rho}$ admits a crystalline lift by Theorem 1.

The exceptional group G_2 has two maximal parabolic subgroups: the short root parabolic, and the long root parabolic.

If $\bar{\rho}$ is reducible, then it factors through either parabolic subgroups.

6.2.1. The short root parabolic case

Let $P \subset G_2$ be the short root parabolic. Recall that P has a Levi decomposition $P = L \ltimes U$. Fix an isomorphism $L \cong \mathrm{GL}_2$.

By Lemma 2.2.2.1, there exists a finite Galois extension K'/K, of prime-to-p degree such that $\bar{r}|_{K'}$ is Lyndon-Demuškin.

Write $Z(U)$ for center of U, and write U^{ad} for $U/Z(U)$. Write $\phi : L \to \mathrm{Aut}(U)$ for the conjugation action, with graded pieces $\phi^{\mathrm{ad}} : L \to \mathrm{GL}(U^{\mathrm{ad}})$ and $\phi^{z} : L \to \mathrm{GL}(Z(U))$. Write ϕ^{Lie} for $\phi^{\mathrm{ad}} \oplus \phi^{z}$.

6.2.1.1 Lemma Assume $p > 2$. There exists a Hodge-Tate regular crystalline lifting $r^\circ : G_K \to L(\bar{\mathbb{Z}}_p)$ of the Levi factor $\bar{r}$, such that the adoint representation $\phi^{\mathrm{Lie}}(r^\circ) : G_K \xrightarrow{r^\circ} L(\bar{\mathbb{Z}}_p) \to \mathrm{GL}(\mathrm{Lie}\, U(\bar{\mathbb{Z}}_p))$ has labeled Hodge-Tate weights slightly less that $\underline{0}$.

Proof. Since $L = \mathrm{GL}_2$, it is well-known Hodge-Tate regular crystalline lifts of $\bar{r}$ exists. We have $\phi^{\mathrm{Lie}}(r^\circ) = \frac{1}{\det r^{\circ 2}} \mathrm{sym}^3(r^\circ) \oplus \frac{1}{\det r^\circ}$. So by replacing r° by a Tate twist, we can ensure $\phi^{\mathrm{Lie}}(r^\circ)$ labeled Hodge-Tate weights slightly less that $\underline{0}$.

$\square$

Let $\mathrm{Spec}\, R$ be an irreducible component (with non-empty generic fiber) of a crystalline lifting ring $R_{\bar{r}}^{\mathrm{crys},\underline{\lambda}}$ of regular labeled Hodge-Tate weights $\underline{\lambda}$ such that the labeled Hodge-Tate weights $\phi^{\mathrm{Lie}}(\underline{\lambda})$ are slightly less 0. By the lemma above, such a $\mathrm{Spec}\, R$ exists.

Let $r^{\mathrm{univ}} : G_K \to L(R)$ be the universal Galois representation.

The mod ϖ Galois representation $\bar{r}$ defines a Galois action $\phi(\bar{r}) : G_K \to \mathrm{Aut}(U(\bar{\mathbb{F}}_p))$ on $U(\bar{\mathbb{F}}_p)$. By 3.1.0.4, the datum of $\bar{\rho} : G_K \to G_2(\bar{\mathbb{F}}_p)$ is encoded in a non-abelian cocycle $[\bar{c}] \in H^1(G_K, U(\bar{\mathbb{F}}_p))$.

The strategy for lifting $\bar{\rho}$ is as follows. We choose a suitable $\bar{\mathbb{Z}}_p$-point x of $\operatorname{Spec} R$ which defines a lift $r_x : G_K \to L(\bar{\mathbb{Z}}_p)$ of $\bar{r}$, and endow $U(\bar{\mathbb{Z}}_p)$ with the Galois action $\phi(r_x) : G_K \xrightarrow{r_x} L(\bar{\mathbb{Z}}_p) \to \mathrm{Aut}(U(\bar{\mathbb{Z}}_p))$. There is a map of pointed set $H^1(G_K, U(\bar{\mathbb{Z}}_p)) \to H^1(G_K, U(\bar{\mathbb{F}}_p))$. If the cohomology class $[\bar{c}]$ admits a lift $[c] \in H^1(G_K, U(\bar{\mathbb{Z}}_p))$, then $\bar{\rho}$ admits a lift $\rho : G_K \to G_2(\bar{\mathbb{Z}}_p)$ whose datum is encoded in $[c]$. Such a lift ρ is crystalline by the main result of Theorem 2.4.3.1, since $\phi^{\mathrm{Lie}}(r^\circ)$ has labeled Hodge-Tate weights slightly less than $\underline{0}$.

By Theorem 4.2.1, to lift the non-abelian 1-cocycle $[\bar{c}]$, it suffices to verify the following:

[1] $H^2(G_K, \mathrm{sym}^3(r^{\mathrm{univ}})/\det^2(r^{\mathrm{univ}}))$ is SGR;

[2] $p \neq 2$;

[3] There exists a finite Galois extension K'/K of prime-to-p degree such that $\phi(\bar{r})|_{G_{K'}}$ is Lyndon-Demuškin; and

[4] There exists a $\bar{\mathbb{Z}}_p$-point of $\operatorname{Spec} R$ which is mildly regular when restricted to $G_{K'}$.

[1] is verified by Theorem 5.2.0.10. Note that since the Hodge type of $\operatorname{Spec} R$ is chosen so that $\mathrm{sym}^3(r_x)/\det(r_x)^2$ has labeled Hodge-Tate weights slightly less than $\underline{0}$, $H^2(G_K, \mathrm{sym}^3(r_x)/\det(r_x)^2)$ is torsion for any characteristic 0 point x of $\operatorname{Spec} R$. [3] follows from Lemma 2.2.2.1, and [4] follows from Proposition 2.0.3.

6.2.2. The long root parabolic case

Let $Q \subset G_2$ be the long root parabolic. Q has a Levi decomposition $Q = L' \ltimes V$. Fix an isomorphism $\mathrm{std} : L' \xrightarrow{\cong} \mathrm{GL}_2$. Write det for the composition $L' \xrightarrow{\mathrm{std}} \mathrm{GL}_2 \xrightarrow{\det} \mathrm{GL}_1$.

Let $\{1\} = V_0 \subset V_1 \subset V_2 \subset V_3 = V$ be the upper central series of V. Then the conjugation action of L' on each graded piece is given by

- $V_3/V_2 \cong \det \otimes \mathrm{std}$;
- $V_2/V_1 \cong \det$;
- $V_1 \cong \mathrm{std}$.

Suppose $\bar{\rho}$ factors through the long root parabolic Q, but not the short root parabolic P. Then the Levi factor

$$\bar{r} : G_K \xrightarrow{\bar{\rho}} Q(\bar{\mathbb{F}}_p) \to L'(\bar{\mathbb{F}}_p)$$

is necessarily an irreducible representation. If we endow each graded piece of $V(\bar{\mathbb{F}}_p)$ with the Galois action $G_K \xrightarrow{\bar{r}} L(\bar{\mathbb{Z}}_p) \to \mathrm{GL}(V_{i+1}(\bar{\mathbb{F}}_p)/V_i(\bar{\mathbb{F}}_p))$, then we have, by local Tate duality,

$$H^2(G_K, V_3(\bar{\mathbb{F}}_p)/V_2(\bar{\mathbb{F}}_p)) = H^2(G_K, \bar{r} \otimes \det \bar{r}) = 0$$

$$H^2(G_K, V_1(\bar{\mathbb{F}}_p)) = H^2(G_K, \bar{r}) = 0$$

So the only cohomological obstruction occurs in the second graded piece.

The datum of $\bar{\rho}$ is encoded in a non-abelian cocycle $[\bar{c}] \in H^1(G_K, V(\bar{\mathbb{F}}_p))$. Just as is done in the short root parabolic case, it suffices to lift the cocycle $[\bar{c}]$. By Proposition 4.3.1, since the only cohomological obstruction lies in the second graded piece, it suffices to lift $\mathrm{ad}([\bar{c}]) \in H^1(G_K, (V/V_1)(\bar{\mathbb{F}}_p))$.

Write U' for V/V_1. Recall that U' is a nilpotent group of class 2 with rank-1 center, and we can directly appeal to Theorem 4.2.1. We repeat the procedure worked out in the short root case 6.2.1.

Let r° be a lift of $\bar{r}$ such that r° is Hodge-Tate regular and crystalline and the Hodge-Tate weights of r° are strictly less than $\underline{0}$.

Let $\operatorname{Spec} R$ be the irreducible component of the crystalline lifting ring of $\bar{r}$ containing r°. Write $r^{\text{univ}} : G_K \to \mathrm{GL}_2(R)$ for the universal family.

Write $Z(U')$ for the center of U', and write $U'^{\,\text{ad}}$ for $U'/Z(U')$. Write ϕ^{ad} for the conjugate action $L' \to \operatorname{Aut}(U'^{\,\text{ad}})$ and write ϕ^z for the conjugate action $L' \to \operatorname{Aut}(Z(U'))$.

Note that $\phi^{\text{ad}}(r^{\text{univ}}) = r^{\text{univ}}$ and $\phi^z(r^{\text{univ}}) = \det r^{\text{univ}}$.

We have the following check list:

[1] $H^2(G_K, \det(r^{\text{univ}})r^{\text{univ}})$ is SGR;

[2] $p \neq 2$;

[3] There exists a finite Galois extension K'/K of prime-to-p degree such that $\phi(\bar{r})|_{G_{K'}}$ is Lyndon-Demuškin; and

[4] There exists a $\bar{\mathbb{Z}}_p$-point of $\operatorname{Spec} R$ which is mildly regular when restricted to $G_{K'}$.

By the assumption $H^2(G_K, \det(r^{\text{univ}})r^{\text{univ}}) = 0$. [3] follows from Lemma 2.2.2.1, and [4] follows from Proposition 2.0.3. $\square$

7. Appendix: Non-denegeracy of mod ϖ cup product for G_2

Let $\mathbb{F}$ be a finite field of characteristic $p > 3$. Write G_2 for the Chevalley group over $\mathbb{F}$ of type G_2.

Let P be the short root parabolic of G_2. Let $P = L \ltimes U$ be the Levi decomposition. Let $\bar{r} : G_K \to L(\mathbb{F})$ be a Galois representation which is Lyndon-Demuškin. Since $L \cong \mathrm{GL}_2$, $\bar{r}$ is the extension of two trivial characters.

Denote by $\phi : L \to \mathrm{Aut}(U)$ the conjugation action.

G_K acts on U via the conjugate action $G_K \xrightarrow{r^\circ} L \xrightarrow{\phi} \mathrm{Aut}(U)$.

We set up a computational framework to prove various claims. Let $\{x_0, \cdots, x_n, x_{n+1}\}$ be the Demuškin generators.

Let $\{e_1, e_2\}$ be a basis of the representation space of $\bar{r}$ such that r° is upper-triangular with respect to this basis. Without loss of generality, assume $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Say for $i = 0, \cdots, n+1$, $\bar{r}(x_i) = \begin{bmatrix} 1 & l_i \\ & 1 \end{bmatrix}$.

The set $\{e_1^3, e_1^2 e_2, e_1 e_2^2, e_2^3\}$ is a basis of the representation space $\mathrm{sym}^3(\bar{r})$, which is identified with $U^{\mathrm{ad}}(\mathbb{F})$.

In diagram 6.1, α is the short root, and β is the short root. Each root x generates a root group $U_x \subset U$. The short root parabolic P has 7 root groups: the 5 root groups

$$\{U_\beta, U_{\beta+\alpha}, U_{\beta+2\alpha}, U_{\beta+3\alpha}, U_{2\beta+3\alpha}\}$$

lying above the x-axis generates the unipotent radical U, the two root groups $\{U_\alpha, U_{-\alpha}\}$ lying on the x-axis are the root groups of the Levi factor group L. Say under the identification $\mathrm{std} : L \cong \mathrm{GL}_2$, the matrices $\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$ are identified with the root group U_α. Now that we have identifications

$$\begin{aligned} \operatorname{span} e_1^3 &\sim U_\beta \\ \operatorname{span} e_1^2 e_2 &\sim U_{\beta+\alpha} \\ \operatorname{span} e_1 e_2^2 &\sim U_{\beta+2\alpha} \\ \operatorname{span} e_2^3 &\sim U_{\beta+3\alpha} \end{aligned}$$

For ease of notation, write $E_0 := e_1^3$, $E_1 := e_1^2 e_2$, $E_2 := e_1 e_2^2$, $E_3 := e_2^3$. A basis of

$$C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathcal{O}_E)) \cong \{\langle x_0, \cdots, x_{n+1}\rangle \to U_\beta(\mathcal{O}_E) \oplus U_{\beta+\alpha}(\mathcal{O}_E) \oplus U_{\beta+2\alpha}(\mathcal{O}_E) \oplus U_{\beta+3\alpha}(\mathcal{O}_E)\}$$

is given by

$$\mathscr{B} = \begin{Bmatrix} x_0^* E_0, x_1^* E_0, & \dots, & x_{n+1}^* E_0, \\ x_0^* E_1, x_1^* E_1, & \dots, & x_{n+1}^* E_1, \\ x_0^* E_2, x_1^* E_2, & \dots, & x_{n+1}^* E_2, \\ x_0^* E_3, x_1^* E_3, & \dots, & x_{n+1}^* E_3 \end{Bmatrix}$$

where $x_i^* E_j$ is the cochain $c : \langle x_0, \cdots, x_{n+1}\rangle$ such that $c(x_k) = \delta_{ik} E_j$, where δ_{ik} is the Kronecker delta. For any $c \in C^1_{\mathrm{LD}}(U^{\mathrm{ad}})$, we can write down the $\mathscr{B}$-coordinates $[c]_{\mathscr{B}} := (c_v)_{v\in\mathscr{B}}$ of c.

7.0.1. Lemma The cup products on cochains

$$\cup_{\mathbb{F}} : C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \times C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \to C^2_{\mathrm{LD}}(Z(U)(\mathbb{F}))$$

is non-degenerate.

Ideas We compute the cup products $v \cup w$ for $v, w \in \mathscr{B}$. The matrix $[\cup_{\mathbb{F}}]_{\mathscr{B}}$ is anti-lower-triangular, (that is, of the shape

$$\begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ 0 & * & * & * \\ * & * & * & * \end{bmatrix}$$

whose anti-diagonal blocks are constant invertible matrices), and thus non-degenerate.

To help the reader better understand what's going on, we attached SageMath code in the Appendix 4.8.

Proof. Recall the relator of the Lyndon-Demuškin group is

$$R = x_0^q (x_0, x_1)(x_2, x_3) \dots (x_n, x_{n+1}).$$

Since we are working mod ϖ, we have for any $p > 5$, any $g \in G_{K'}$, $\phi(\bar{r}(g))^p \equiv \mathrm{id} \bmod \varpi$ (See Appendix 4.8 for the verification). In particular, the relator R reduces to

$$(x_0, x_1) \dots (x_n, x_{n+1})$$

when we compute mod ϖ. (When $p = 5$, things are still good, and can be confirmed by running the SageMath code in the appendix.)

We regard cochains in $C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F}))$ as a $(U^{\mathrm{ad}}(\mathbb{F}))$-valued function on the free group with generators $\{x_0, \dots, x_{n+1}\}$, Now we let c be the "universal" mod ϖ 1-cochain. That is, we let

$$\begin{Bmatrix} \lambda_{0,0}, & \lambda_{1,0}, & \dots, & \lambda_{n+1,0}, \\ \lambda_{0,1}, & \lambda_{1,1}, & \dots, & \lambda_{n+1,1}, \\ \lambda_{0,2}, & \lambda_{1,2}, & \dots, & \lambda_{n+1,2}, \\ \lambda_{0,3}, & \lambda_{1,3}, & \dots, & \lambda_{n+1,3} \end{Bmatrix}$$

be indeterminants, and set

$$c := \sum \lambda_{i,j} x_i^* E_j \in C^1_{\mathrm{LD}}(U^{\mathrm{ad}}(\mathbb{F})) \otimes \mathbb{Z}[\lambda_{i,j}].$$

The cup product

$$c \cup c = Q(c) \in C^2_{\mathrm{LD}}(Z(U)(\mathbb{F})) \otimes \mathbb{Z}[\lambda_{i,j}] = Z(U)(\mathbb{F}) \otimes \mathbb{Z}[\lambda_{i,j}] \cong \mathbb{F}[\lambda_{i,j}]$$

will be a quadratic form in variables $\{\lambda_{i,j}\}$, and the matrix of this quadratic form is nothing but the matrix $[\cup_{\mathbb{F}}]_{\mathscr{B}}$. Recall that $c \cup c = Q(c)$ is defined to be the projection of $\widetilde{c}(R)$ onto the center of the Lie algebra $\mathrm{Lie}\, U$, where $\widetilde{c} \in C^1_{\mathrm{LD}}(U(\mathbb{F}))$ is the unique extension of c to a $U(\mathbb{F})$-valued cochain as is explained in Section 4.1.

Write $[\cup_{\mathbb{F}}]_{\mathscr{B}}$ as a block matrix

$$[\cup_{\mathbb{F}}]_{\mathscr{B}} = \begin{array}{cc} & \begin{array}{cccc} \beta & \beta+\alpha & \beta+2\alpha & \beta+3\alpha \end{array} \\ \begin{array}{c} \beta \\ \beta+\alpha \\ \beta+2\alpha \\ \beta+3\alpha \end{array} & \left(\begin{array}{c|c|c|c} M_{11} & M_{12} & M_{13} & M_{14} \\ \hline M_{21} & M_{22} & M_{23} & M_{24} \\ \hline M_{31} & M_{32} & M_{33} & M_{34} \\ \hline M_{41} & M_{42} & M_{43} & M_{44} \end{array}\right) \end{array}$$

where each M_{ij} is an $(n+2) \times (n+2)$ matrix. We say the blocks M_{24}, M_{33}, M_{34}, M_{42}, M_{43}, M_{44} are *strictly below the anti-diagonal*, and we call M_{41}, M_{32}, M_{23} and M_{14} the anti-diagonal blocks.

$$\begin{array}{cc} & \begin{array}{cccc} \beta & \beta+\alpha & \beta+2\alpha & \beta+3\alpha \end{array} \\ \begin{array}{c} \beta \\ \beta+\alpha \\ \beta+2\alpha \\ \beta+3\alpha \end{array} & \left(\begin{array}{c|c|c|c} & & & \\ \hline & & & M_{24} \\ \hline & & M_{33} & M_{34} \\ \hline & M_{42} & M_{43} & M_{44} \end{array}\right) \end{array}$$

Figure 4.2: Strictly below anti-diagonal

$$\begin{array}{cc} & \begin{array}{cccc} \beta & \beta+\alpha & \beta+2\alpha & \beta+3\alpha \end{array} \\ \begin{array}{c} \beta \\ \beta+\alpha \\ \beta+2\alpha \\ \beta+3\alpha \end{array} & \left(\begin{array}{c|c|c|c} & & & M_{14} \\ \hline & & M_{23} & \\ \hline & M_{32} & & \\ \hline M_{41} & & & \end{array}\right) \end{array}$$

Figure 4.3: Anti-diagonal blocks

Sublemma Let $g = g_1 g_2 \dots g_s$. Write ϕ_i for $\phi(\bar{r}(g_1, \dots, g_{i-1}))$. We have

$$\widetilde{c}(g) = \sum \phi_i \widetilde{c}(g_i) + \frac{1}{2} \sum_{i<j} [\phi_i \widetilde{c}(g_i), \phi_j \widetilde{c}(g_j)].$$

Proof. An immediate consequence of the Baker–Campbell–Hausdorff formula. $\square$

Note that $\phi(\bar{r}((x_i, x_j)) = \mathrm{id}$, so

$$\begin{aligned}\widetilde{c}(R) &= \widetilde{c}(x_0^q(x_0, x_1)(x_2, x_3)\dots(x_n, x_{n+1})) \\ &= \sum \widetilde{c}((x_{2k}, x_{2k+1})) + \frac{1}{2}\sum_{j<k}[\widetilde{c}((x_{2j}, x_{2j+1}), \widetilde{c}((x_{2k}, x_{2k+1})]\end{aligned}$$

We have

$$\widetilde{c}((x_{2k}, x_{2k+1})) = -\phi(x_{2k}^{-1})(\phi(x_{2k+1}) - 1)\widetilde{c}(x_{2k}) + \phi(x_{2k}^{-1}x_{2k+1}^{-1})(\phi(x_{2k}) - 1)\widetilde{c}(x_{2k+1}) + Z_k = Y_k + Z_k$$

where Z_k is a sum of Lie brackets (see below), and lies in the center of the $\mathrm{Lie}\, U$. Note that $[Y_j, Y_k]$ only contributes to the part of $[\cup_{\mathbb{F}}]_{\mathscr{B}}$ which lies strictly below the anti-diagonal, because $(\phi(x_{2k}) - 1)$ and $(\phi(x_{2k+1}) - 1)$ moved the appearance of the inderterminant $\lambda_{i,j}$ from the root group $U_{\beta+j\alpha}$ to the root group $U_{\beta+(j+1)\alpha}$.

So it remains to analyze $\sum Z_k$. We have

$$\begin{aligned}2Z_k &= [-\phi(x_{2k}^{-1})\widetilde{c}(x_{2k}), -\phi(x_{2k}^{-1}x_{2k+1}^{-1})\widetilde{c}(x_{2k+1})] \\ &+ [-\phi(x_{2k}^{-1})\widetilde{c}(x_{2k}), +\phi(x_{2k}^{-1}x_{2k+1}^{-1})\widetilde{c}(x_{2k})] \\ &+ [-\phi(x_{2k}^{-1})\widetilde{c}(x_{2k}), +\phi(x_{2k}^{-1}x_{2k+1}^{-1}x_{2k})\widetilde{c}(x_{2k+1})] \\ &+ [-\phi(x_{2k}^{-1}x_{2k+1}^{-1})\widetilde{c}(x_{2k+1}), +\phi(x_{2k}^{-1}x_{2k+1}^{-1})\widetilde{c}(x_{2k})] \\ &+ [-\phi(x_{2k}^{-1}x_{2k+1}^{-1})\widetilde{c}(x_{2k+1}), +\phi(x_{2k}^{-1}x_{2k+1}^{-1}x_{2k})\widetilde{c}(x_{2k+1})] \\ &+ [\phi(x_{2k}^{-1}x_{2k+1}^{-1})\widetilde{c}(x_{2k}), +\phi(x_{2k}^{-1}x_{2k+1}^{-1}x_{2k})\widetilde{c}(x_{2k+1})]\end{aligned}$$

Write

$$
\begin{aligned}
2Z_k' \quad := \quad & [-\widetilde{c}(x_{2k}), -\widetilde{c}(x_{2k+1})] \\
+ \quad & [-\widetilde{c}(x_{2k}), \widetilde{c}(x_{2k})] \\
+ \quad & [-\widetilde{c}(x_{2k}), \widetilde{c}(x_{2k+1})] \\
+ \quad & [-\widetilde{c}(x_{2k+1}), \widetilde{c}(x_{2k})] \\
+ \quad & [-\widetilde{c}(x_{2k+1}), \widetilde{c}(x_{2k+1})] \\
+ \quad & [\widetilde{c}(x_{2k}), \widetilde{c}(x_{2k+1})]
\end{aligned}
$$

Z_k' is obtained by replacing all Galois action in Z_k by the trivial action. $Z_k - Z_k'$ only contributes to the part of $[\cup_{\mathbb{F}}]_{\mathscr{B}}$ with lies strictly below the anti-diagonal for a similar reason (a "shifting" effect). It is easy to see that

$$Z_k' = [\widetilde{c}(x_{2k}), \widetilde{c}(x_{2k+1})] = \pm\lambda_{2k,0}\lambda_{2k+1,3} \pm \lambda_{2k+1,0}\lambda_{2k,3} \pm 3\lambda_{2k,1}\lambda_{2k+1,2} \pm 3\lambda_{2k+1,2}\lambda_{2k,1}.$$

As a consequence of these computations, we see that each of the anti-diagonal blocks of $[\cup]_{\mathscr{B}}$ are constant matrices:

$$\pm M_{41} = \pm M_{14} = \begin{bmatrix} \begin{bmatrix} & -1/2 \\ 1/2 & \end{bmatrix} & & & \\ & \begin{bmatrix} & -1/2 \\ 1/2 & \end{bmatrix} & & \\ & & \cdots & \\ & & & \begin{bmatrix} & -1/2 \\ 1/2 & \end{bmatrix} \end{bmatrix}$$

and

$$\pm M_{32} = \pm M_{23} = \begin{bmatrix} \begin{bmatrix} & -3/2 \\ 3/2 & \end{bmatrix} & & & \\ & \begin{bmatrix} & -3/2 \\ 3/2 & \end{bmatrix} & & \\ & & \cdots & \\ & & & \begin{bmatrix} & -1/2 \\ 1/2 & \end{bmatrix} \end{bmatrix}.$$

So $[\cup_{\mathbb{F}}]_{\mathscr{B}}$ is an invertible matrix. □

The long root parabolic case is much simpler.

8. Appendix: Sagemath code

8.0.1. Proposition Let $V \subset B$ be the unipotent radical of the Borel of G_2. Let $g \in V(\bar{\mathbb{Z}}_p)$. If $p > 5$, then $g^p = \mathrm{id} \bmod \varpi$.

Proof. Let $P \supset B$ be the short root parabolic. Let $P = L \ltimes U$ be the Levi decomposition. Let $\pi : P \to L$ be the quotient. Say $\pi(g) = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}$. Fix a projection $P \to U$. Also fix a projection $U \to Z(U)$. Say the projection of g onto $U/Z(U) \cong \mathbb{A}^4$ via $P \to U \to U/Z(U)$ is (u_0, u_1, u_2, u_3). Say the projection of g onto $Z(U) \cong \mathbb{A}^1$ via $P \to U \to Z(U)$ is u_4.

For simplicity, we write $g = (l; u_0, u_1, u_2, u_3; u_4)$. We have, for any integer q,

$$\begin{aligned} g^q = \ & (ql; qu_0, -\frac{1}{2}q(q-1)u_0 l + qu_1, \\ & -\frac{1}{6}q(q-1)(2q-1)u_0 l^2 + q(q-1)u_1 l + qu_2, \\ & -\frac{1}{4}q^2(q-1)^2 u_0 l^3 + \frac{1}{2}q(q-1)(2q-1)u_1 l^2 + \frac{3}{2}q(q-1)u_2 l + qu_3, qu_4; \\ & \frac{1}{120}(q-1)q(q+1)(3q^2-2)u_0^2 l^3 - \frac{1}{2}(q-1)q(q+1)(u_1^2 + u_0 u_2)l) \end{aligned}$$

It can be computed by hand, and can be verified by computer algebra system. The Proposition follows from the above computation immediately. □

The following is the SageMath source code for computing cup product.

```
# Generate basis vectors of of C^1_{LD}(U)

def generate_LU(i):
```

```
    Ai = var("A%d"%i)
    Bi = var("B%d"%i)
    Ci = var("C%d"%i)
    Di = var("D%d"%i)
    Ei = var("E%d"%i)
    li = var("l%d"%i)
    gi = var("g%d"%i)
    hi = var("h%d"%i)
    Li = matrix(SR, [[gi,li],[0,hi]])
    ui = vector(SR, [Ai,Bi,Ci,Di,Ei])
    u_i = -phi(Li.inverse())*ui
    return {"L": Li, "L_": Li.inverse(), "U": ui, "U_": u_i}

def generate_LU_pair(i):
    LUi = generate_LU(i)
    LUi1 = generate_LU(i+1)
    return [
        [LUi["L_"], LUi["U_"]],
        [LUi1["L_"], LUi1["U_"]],
        [LUi["L"], LUi["U"]],
        [LUi1["L"], LUi1["U"]],
    ]
```

```
# The matrix for r(x_i)

def Levi(i):
    hi = eval("h%d"%i)
    li = eval("l%d"%i)
    gi = eval("g%d"%i)
    return matrix(SR, [[gi, li], [0, hi]])

# Conjugation action of the Levi factor on Lie U

def phi(m):
    a=m[0,0]
    b=m[0,1]
    c=m[1,0]
    d=m[1,1]
    dt = det(m)
    return matrix(SR, [
        [a*a*a,a*a*b,a*b*b,b*b*b,0],
        [3*a*a*c,a*a*d+2*a*c*b,2*a*b*d+b*b*c,3*b*b*d,0],
        [3*a*c*c,2*a*c*d+b*c*c,2*b*c*d+a*d*d,3*b*d*d,0],
        [c*c*c,c*c*d,c*d*d,d*d*d,0],
        [0,0,0,0,dt]])/dt/dt
```

```
# Lie bracket on Lie U

def brkt(X,Y):
    return -vector(SR, [0,0,0,0,
        X[0]*Y[3]+3*X[1]*Y[2]-Y[0]*X[3]-3*Y[1]*X[2]])

# The differential
#     d^2: C^1 -> C^2

def general_cup_product(pairs):
    X_list = []
    levi = identity_matrix(SR, 5)
    for pr in pairs:
        X_list.append(levi*pr[1])
        levi = levi*phi(pr[0])
    ret = 0
    for X in X_list:
        ret += X
    for i in range(0, len(X_list)):
        for j in range(i + 1, len(X_list)):
            ret += (1/2) * brkt(X_list[i], X_list[j])
    return ret
```

```
def differential_degree_2(p = 5, g = 5, n = 4):
    pairs = []
    LU01 = generate_LU_pair(0)
    for i in range(0, g - 1):
        pairs.append(LU01[2])
    pairs.append(LU01[1])
    pairs.append(LU01[2])
    pairs.append(LU01[3])
    i = 2
    while i < n + 1:
        for pr in generate_LU_pair(i):
            pairs.append(pr)
        i += 2
    return general_cup_product(pairs)

# Generate the basis of C1

def basis_of_C1(n):
    basis = []
    for letter in "ABCD":
        for i in range(0, n + 2):
            basis.append(eval("%s%d"%(letter, i)))
    return basis
```

```
# Organize the boundary map
#                   delta: C^1(U^ad) -> C^2(Z(U))
# as a quadratic form
#                   cup: C^1 x C^1 -> C^2(Z(U))

def as_quad_form(expr, variables):
    coef = {}
    remain = expr.expand()
    for i1 in range(0, len(variables)):
        for i2 in range(i1, len(variables)):
            var1 = variables[i1]
            var2 = variables[i2]
            coef[(var1*var2).__repr__()] =
                remain.coefficient(var1*var2).expand().factor()
            remain -= coef[(var1*var2).__repr__()]*var1*var2
    BL_arr = []

    for i1 in range(0, len(variables)):
        BL_arr.append([])
        for i2 in range(0, len(variables)):
            var1 = variables[i1]
            var2 = variables[i2]
```

```
            if i1 < i2:
                var_s = (var1 * var2).__repr__()
            else:
                var_s = (var2 * var1).__repr__()
            coef_s = coef[var_s].__repr__()
            if i1 == i2:
                BL_arr[-1].append(coef[var_s].expand())
            else:
                BL_arr[-1].append((1/2)* coef[var_s].expand())

    BL_mat = matrix(SR, BL_arr)
    return coef, BL_mat

# Cup product

def cup_product(p = 5, g = 5, n = 4):
    cR = differential_degree_2(p, g, n)
    return as_quad_form(cR[4], basis_of_C1(n), False)

# Matrix of cup product mod p

def matrix_substitute(mat, subdict, bring=QQ):
    new_arr = []
```

```
    for i in range(0, mat.nrows()):
        new_arr.append([])
        for j in range(0, mat.ncols()):
            entry = 0 + mat[i][j]
            entry = entry.subs(subdict)
            new_arr[-1].append(entry)
    return matrix(bring, new_arr)


def cup_product_mod_p(p=5, g=5, n=4):
    coef, BLm = cup_product(p, g, n)
    sub_dict = {}
    for i in range(0, n+2):
        sub_dict[eval("g%d"%i)] = 1
        sub_dict[eval("h%d"%i)] = 1
    BLmod = matrix_substitute(BLm, sub_dict, SR)
    return BLmod
```

If we compute `cup_product_mod_p(5,4,4)` in SageMath notebook, we'll get an anti-lower-triangular matrix in the sense of Lemma 7.0.1.

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