# A TREATISE ON THE METHOD OF LEAST SQUARES,

OR THE

APPLICATION OF THE THEORY OF PROBABILITIES IN THE COMBINATION OF OBSERVATIONS.

BY

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# NOTE.

THE following pages are printed from the stereotype plates of the Appendix to the author's *Manual of Spherical and Practical Astronomy*, without any change in the numbering of the pages or of the tables. The text, therefore, commences with p. 469 and ends with p. 566; and the tables are numbered IX., IX. A., X. and X. A., as they stand in the original work.

As the Method of Least Squares is applicable in almost all the physical sciences where numerical results are to be deduced from observations, and is here treated from fundamental and general principles, it is thought that this separate publication of the Appendix will supply the want that has for a long time been felt of a suitable text-book on this subject for the use of students of practical science generally, and more especially of classes in our scientific schools.

ST. LOUIS, January 1, 1868.

# APPENDIX.

## METHOD OF LEAST SQUARES.*

1. A NUMBER of observations being taken for the purpose of determining one or more unknown quantities, and these observations giving discordant results, it is an important problem to determine the *most probable* values of the unknown quantities. The method of least squares may be defined to be that method of treating this general problem which takes as its fundamental principle, that *the most probable values are those which make the sum of the squares of the residual errors a minimum.* But, to understand this definition, some degree of acquaintance with the method itself is necessary.

* The first published application of the method is to be found in LEGENDRE, *Nouvelles méthodes pour la détermination des orbites des comètes*, Paris, 1806. The development, however, from fundamental principles is due to GAUSS, who declared that he had used the method as early as 1795. See his *Theoria Motus Corporum Cœlestium*, 1809, Lib. II. Sec. III.; *Disquisitio de elementis ellipticis Palladis*, 1811; *Bestimmung der Genauigkeit der Beobachtungen* (v. LINDENAU und BOHNENBERGER'S *Zeitschrift*, 1816, I. s. 185); *Theoria combinationis observationum erroribus minimis obnoxiæ*, 1823; *Supplementum theoriæ combinationis*, &c., 1826: all of which have been rendered quite accessible through a French translation by J. BERTRAND, *Méthode des moindres carrées. Mémoires sur la combinaison des observations*, par CH. FR. GAUSS, Paris, 1855.

For a digest of the preceding, together with the results of the labors of BESSEL and HANSEN, see ENCKE, *Ueber die Methode der kleinsten Quadrate*, Berliner Astron. Jahrbuch for 1834, 1835, 1836; in connection with which must be mentioned especially the practical work of GERLING, *Die Ausgleichungsrechnungen der practischen Geometrie*, Hamburg, 1843.

See also LAPLACE, *Théorie analytique des probabilités*, Liv. II. Chap. IV.; POISSON, *Sur la probabilité des résultats moyens des observations*, in the Connaissance des Temps for 1827; ENCKE, in the Berlin Jahrbuch for 1853; BESSEL, in *Astron. Nach.*, Nos. 358, 359, 399; HANSEN, in *Astron. Nach.*, Nos. 192, 202 et seq.; PEIRCE, in the *Astron. Journal* (Cambridge, Mass.), Vol. II. No. 21; LIAGRE, *Calcul des probabilités et théorie des erreurs*, Bruxelles, 1852.

ERRORS TO WHICH OBSERVATIONS ARE LIABLE.

2. Every observation which is a *measure*, however carefully it may be made, is to be regarded as subject to error; for experience teaches that repeated measures of the same quantity, *when the greatest precision is sought*,* do not give uniformly the same result. Two kinds of errors are to be distinguished.

*Constant* or *regular errors* are those which in all measures of the same quantity, made under the same circumstances, obtain the same magnitude; or whose magnitude is dependent upon the circumstances according to any determinate law. The causes of such errors must be the subject of careful preliminary search in all physical inquiries, so that their action may be altogether prevented or their effect removed by calculation. For example, among the constant errors may be enumerated refraction, aberration, &c.; the effect of the temperature of rods used in measuring a base line in a survey; the error of division of a graduated instrument when the same division is used in all the measures; any peculiarity of an instrument which affects a particular measurement always by the same amount, such as inequality of the pivots of a transit instrument, defective adjustment of the collimation, imperfections of lenses, defects of micrometer screws, &c., to which must be added constant peculiarities of the observer, who, for example, may always note the passage of a star over a thread of a transit instrument too soon, or too late, by a constant quantity, or who, in attempting to bisect a star with a micrometer thread, constantly makes the upper or the lower portion the greater; or who, in observing the contact of two images (in sextant measures, for instance), assumes for a contact a position in which the images are really at some constant small distance, or a position in which the images are really overlapped, &c. &c.

Thus, we have three kinds of constant errors:

1st. *Theoretical*, such as refraction, aberration, &c., whose effects, when their causes are once thoroughly understood, may be calculated *a priori*, and which thenceforth cease to exist as errors.

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* The qualification, "when the greatest precision is sought," is important; for if, *e.g.*, we were to determine the latitude of a place by repeated measures of the meridian altitude of the same fixed star with a sextant divided only to whole degrees, all our measures might give the same degree. The accordance of observations is, therefore, not to be taken as an infallible evidence of their accuracy. It is especially when we approach *the limits of our measuring powers* that we become sensible of the discrepancies of observations.

The detection of a constant error in a certain class of observations very commonly leads to investigations by which its cause is revealed, and thus our physical theories are improved.

2d. *Instrumental*, which are discovered by an examination of our instruments, or from a discussion of the observations made with them. These may also be removed when their causes are fully understood, either by a proper mode of using the instrument, or by subsequent computation.

3d. *Personal*, which depend upon peculiarities of the observer, and in delicate inquiries become the subject of special investigation under the name of "personal equations."

We are to assume that, in any inquiry, all the sources of constant error have been carefully investigated, and their effects eliminated as far as practicable. When this has been done, however, we find by experience that there still remain discrepancies, which must be referred to the next following class.

*Irregular* or *accidental errors* are those which have irregular causes, or whose effects upon individual observations are governed by no fixed law connecting them with the circumstances of the observations, and, therefore, can never be subjected *a priori* to computation. Such, for example, are errors arising from tremors of a telescope produced by the wind; errors in the refraction produced by anomalous changes of density of the strata of the atmosphere; from unavoidable changes in the several parts of an instrument produced by anomalous variations of temperature, or anomalous contraction and expansion of the parts of an instrument even at known temperatures; but, more especially, errors arising from the imperfection of the senses, as the imperfection of the eye in measuring very small spaces, of the ear in estimating small intervals of time, of the touch in the delicate handling of an instrument, &c.

This distinction between constant and irregular errors is, indeed, to a certain extent, rather relative than absolute, and depends upon the sense, more or less restricted, in which we consider observations to be of the *same nature* or made under the *same circumstances*. For example, the errors of division of an instrument may be regarded as constant errors when the same division comes into all measures of the same quantity, but as irregular when in every measure a different division is used, or when the same quantity is measured repeatedly with different instruments.

After a full investigation of the constant or regular errors, it is the next business of the observer to diminish as much as possible the irregular errors by the greatest care in the observations; and finally, when the observations are completed, there remains the important operation of combining them, so that the outstanding, unavoidable, irregular errors may have the least probable effect upon the results. For this combination we invoke the aid of the method of least squares, which may be said to have for its object the restriction of the effect of irregular errors within the narrowest limits according to the theory of probabilities, and, at the same time, to determine from the observations themselves the errors to which our results are probably liable. It is proper to observe here, however, to guard against fallacious applications, that the theory of the method is grounded upon the hypothesis that we have taken a large number of observations, or, at least, a number sufficiently large to determine the errors to which the observations are liable.

## CORRECTION OF THE OBSERVATIONS.

3. When no more observations are taken than are sufficient to determine one value of each of the unknown quantities sought, we have no means of judging of the correctness of the results, and, in the absence of other information, are compelled to accept these results as true, or, at least, as the most probable. But when additional observations are taken, leading to different results, we can no longer unconditionally accept any one result as true, since each must be regarded as contradicting the others. The results cannot all be true, and are all probably, in a strict sense, false. The absolutely true value of the quantity sought by observation must, in general, be regarded as beyond our reach; and instead of it we must accept a value which may or may not agree with any one of the observations, but which is rendered *most probable* by the existence of these observations.

The condition under which such a probable value is to be determined, is that *all contradiction among the observations is to be removed.* This is a logical necessity, since we cannot accept for truth that which is contradictory or leads to contradictory results.

The contradiction is obviously to be removed by applying to the several observations (or conceiving to be applied) probable *corrections*, which shall make them agree with each other, and which we have reason to suppose to be equivalent in amount to the accidental errors severally. But let us here remark that we do not in this statement by any means imply that an observer is to *arbitrarily* assume a system of corrections which will produce accordance: on the contrary, the method we are about to consider is designed to remove, as far as possible, every arbitrary consideration, and to furnish a set of principles which shall always guide us to the most probable results. The conscientious observer, having taken every care in his observation, will set it down, however discrepant it may appear to him, as a portion of the testimony collected, out of which the truth, or the nearest approximation to it, is to be sifted.

Admitting, therefore, that the observations give us the best, as indeed the only, information we can obtain respecting the desired quantities, we must find a system of corrections which shall not only produce the desired accordance, but which shall also be the *most probable* corrections, and further *be rendered most probable by these observations themselves.*

## THE ARITHMETICAL MEAN.

4. In order to discover a principle which may serve as a basis for the investigation, let us examine first the case of direct observations made for the purpose of determining a single unknown quantity.

Let the quantity to be determined by direct observation be denoted by $x$. (Suppose, for example, to fix our ideas, that this quantity is the linear distance between two fixed terrestrial points.) If but one measure of $x$ is taken and the result is $a$, we must accept as the only and, therefore, the most probable value, $x = a$. Let a second observation, taken under the same or precisely equivalent circumstances, and with the same degree of care, so that there is no reason for supposing it to be more in error than the first, give the value $b$. Then, since there is no reason for preferring one observation to the other, the value of $x$ must be so taken that the differences $x - a$, $x - b$ shall be numerically equal; and this gives

$$x = \tfrac{1}{2}(a + b)$$

This result must be regarded as the only one that can be inferred from the two observations consistently with our definition of accidental errors; for positive and negative accidental errors of equal absolute magnitude are to be regarded as equal errors and as equally probable, since, from the care bestowed on the observations and the supposed similarity of the circumstances under which they are made, there is no reason *a priori* for assuming either a positive or a negative error to be the more probable.

Now let a third observation be added, giving the value $c$. Since the three observations are of equal reliability, or, as we shall hereafter say, of *equal weight*, we must so combine $a$, $b$, and $c$ that each shall have a like influence upon the result; in other words, $x$ must be a symmetrical function of $a$, $b$, and $c$. If we first consider $a$ and $b$ alone, then $a$ and $c$, then $b$ and $c$, we shall find the values

$$\tfrac{1}{2}(a+b), \qquad \tfrac{1}{2}(a+c), \qquad \tfrac{1}{2}(b+c),$$

with each of which the additional observation $c$, $b$, or $a$ is to be combined. Each combination must result in the same symmetrical function, which, whatever it may be, can be denoted by the functional symbol $\psi$. We must, therefore, have

$$\begin{aligned} x &= \psi\left[\tfrac{1}{2}(a+b), c\right] \\ &= \psi\left[\tfrac{1}{2}(a+c), b\right] \\ &= \psi\left[\tfrac{1}{2}(b+c), a\right] \end{aligned}$$

Introducing the sum of $a$, $b$, and $c$, or putting

$$s = a + b + c$$

these become

$$\begin{aligned} x &= \psi\left[\tfrac{1}{2}(s-c), c\right] = \psi\left[s, c\right] \\ &= \psi\left[\tfrac{1}{2}(s-b), b\right] = \psi\left[s, b\right] \\ &= \psi\left[\tfrac{1}{2}(s-a), a\right] = \psi\left[s, a\right] \end{aligned}$$

But $s$ is already a symmetrical function of $a$, $b$, and $c$, and therefore these equations cannot all result in the same symmetrical function unless $c$, $b$, $a$, in the respective developments of the functions, disappear and leave only $s$. Hence we must have

$$x = \psi(s)$$

Now, to determine $\psi$, we observe that, as it must be general, its nature may be learned from any special but known case. Such a case is that in which the three observations give three equal values, or $a = b = c$; and in that case we have, as the only value, $x = a$, or

$$a = \psi(3a)$$

and, consequently, the symbol $\psi$ signifies here the division by 3. Hence, generally,

$$x = \frac{a + b + c}{3}$$

In the same manner, if it had been previously shown that for $m$ equally good observations the most probable value is

$$x = \frac{a + b + c + \ldots\ldots + n}{m}$$

it would follow that for an additional observation $p$ we must have

$$x = \frac{a + b + c + \ldots\ldots + n + p}{m + 1}$$

for, putting $s = a + b + c + \ldots + n + p$, we shall have

$$x = \psi \left[\frac{1}{m}(s - p), p\right] = \psi\,[s, p] = \psi\,(s), \text{ \&c.}$$

But we have shown that the form is true for three observed values: hence, it is true for four; and since it is true for four values it is true for five; and thus generally for any number.*

The principle here demonstrated, that the arithmetical mean of a number of equally good observations is the most probable value of the observed quantity, is that which has been universally adopted as the most simple and obvious, and might well be received as axiomatic. The above demonstration is chiefly valuable as exhibiting somewhat more clearly the nature of the assumption that underlies the principle, which is that, under strictly similar circumstances, positive and negative errors of the same absolute amount are equally probable.

5. If now $n', n'', n''' \ldots\ldots n^{(m)}$ are the $m$ observed values of a required quantity $x$, and if $x_0$ denotes their arithmetical mean, the assumption of $x_0$ as the most probable value of $x$ gives $n' - x_0$, $n'' - x_0$, $n''' - x_0$, &c., as the most probable system of corrections (subtractive from the observed values) which produce the required accordance. But the equation

$$x_0 = \frac{n' + n'' + n''' + \ldots\ldots + n^{(m)}}{m} \tag{1}$$

* ENCKE, Berliner Astron. Jahrbuch for 1834, p. 262.

may also be put under the form

$$(n' - x_0) + (n'' - x_0) + (n''' - x_0) + \ldots\ldots (n^{(m)} - x_0) = 0$$

that is, *the algebraic sum of the corrections is zero.*

This is, however, not the only characteristic of the system of corrections resulting from the use of the arithmetical mean. Let us examine the sum of the squares of the corrections. For brevity, let us denote the corrections, or, as they will be hereafter called, the *residuals*, by the symbol $v$: so that

$$v' = n' - x_0, \qquad v'' = n'' - x_0, \qquad v''' = n''' - x_0, \text{ \&c.}$$

and also denote the sums of quantities of the same kind by enclosing the common symbol in rectangular brackets: so that

$$[v] = v' + v'' + v''' + \text{\&c.}$$
$$[vv] = v'v' + v''v'' + v'''v''' + \text{\&c.}$$

a notation usually employed throughout the method of least squares. We have

$$[v] = 0 \tag{2}$$

and

$$[vv] = (n' - x_0)^2 + (n'' - x_0)^2 + (n''' - x_0)^2 + \ldots\ldots$$
$$= [nn] - 2\,[n]\,x_0 + mx_0^2$$

But since we have also

$$x_0 = \frac{[n]}{m}$$

this equation becomes

$$\left.\begin{aligned} [vv] &= [nn] - 2\,[n]\,\frac{[n]}{m} + m\,\frac{[n]^2}{m^2} \\ &= [nn] - \frac{[n]^2}{m} \end{aligned}\right\} \tag{3}$$

Let $x_1$ be any assumed value of $x$, giving the residuals

$$v_1 = n' - x_1 \qquad v_2 = n'' - x_1 \qquad v_3 = n''' - x_1, \text{ \&c.}$$

then, as above,

$$[v_1v_1] = [nn] - 2\,[n]\,x_1 + mx_1^2$$

Substituting in this the value of $[nn]$ given by (3), we find

$$\left.\begin{aligned} [v_1v_1] &= [vv] + \frac{[n]^2}{m} - 2\,[n]\,x_1 + mx_1^2 \\ &= [vv] + m\left(\frac{[n]}{m} - x_1\right)^2 \\ &= [vv] + m\,(x_0 - x_1)^2 \end{aligned}\right\} \tag{4}$$

This equation determines the sum of the squares of the residuals for any assumed value of $x$. Since the last term is always positive, we see that this sum for any value of $x$ differing from the arithmetical mean $x_0$ is always greater than $[vv]$. Hence it is a second characteristic of the arithmetical mean, that it makes *the sum of the squares of the residuals a minimum.*

6. Observations may be not only *direct*, that is, made directly upon the quantity to be determined, but also *indirect*, that is, made upon some quantity which is a function of one or more quantities to be determined. Indeed, the greater part of the observations in astronomy, and in physical science generally, belong to the latter class. Thus, let $x, y, z \ldots\ldots$ be the quantities to be determined, and $M$ a function of them denoted by $f$, or

$$M = f(x, y, z \ldots.) \tag{5}$$

and let us suppose an observation to be made upon the value of $M$. We then have but a single equation between $x, y, z \ldots.$ and the observed quantity $M$, and the problem is as yet indeterminate. Various systems of values may be found to satisfy the equation, either exactly or approximately. Let us, however, suppose that the most probable system (as yet unknown) is expressed by $x = p$, $y = q$, $z = r \ldots.$, and let the value of the function, when these values are substituted in it, be denoted by $V$, or put

$$V = f(p, q, r \ldots.) \tag{6}$$

then $M - V$ is the residual error of the observation. In like manner, if a number of observations of the same kind be taken, in which the observed quantities $M', M'', M''' \ldots$ are functions determined by the same elements $p, q, r, \ldots.$, and if $V', V'', V''' \ldots.$ are the values of these functions when $p, q, r \ldots.$ are substituted in them, then $M' - V'$, $M'' - V''$, $M''' - V''' \ldots.$ are the residual errors of the observations. If there are $\mu$ unknown quantities and also $\mu$ observations, and no more, there will be $\mu$ equations between the known and unknown quantities, which will fully determine the values of these unknown quantities: so that the probable values $p, q, r \ldots.$ are, in that case, those determinate values which *exactly* satisfy all the equations, and, consequently, reduce every one of the residuals $M' - V'$, $M'' - V''$, &c. to zero. But, if there are more than $\mu$ observations, the determinate values found from $\mu$ equations alone will not necessarily satisfy the remaining equations, in consequence of accidental errors in the observations. The problem, then, is *to determine from* ALL *the observations*, or *from all the equations, the most probable system of values of the unknown quantities*, or, which is the same thing, *the most probable system of residual errors.* In the case of direct observations, we have seen that the most probable value of the unknown quantity was that which made the algebraic sum of the residuals zero; but this principle followed from taking the arithmetical mean of the *same* quantity, and is obviously inapplicable in the present case. The second principle, that the most probable value is that which makes the sum of the squares of the residuals a minimum, is of a more general character, and might be assumed at once, as at least a *plausible* principle, to serve as the basis of the solution of our problem; but it will be more satisfactory to justify its adoption by the calculus of probabilities.

### THE PROBABILITY CURVE.

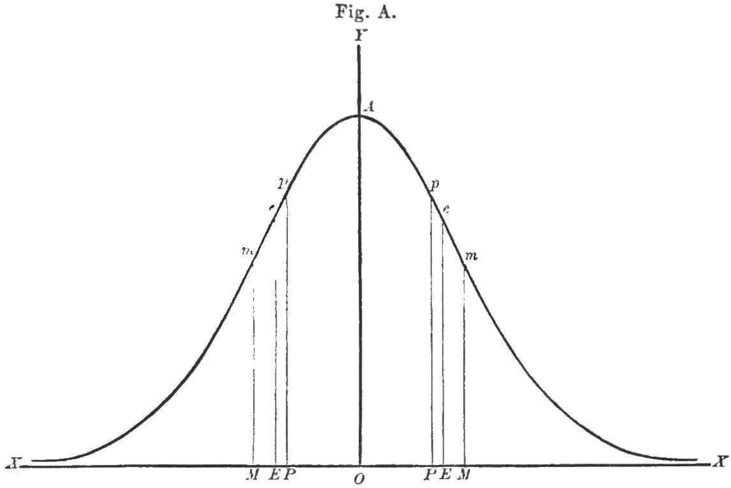
7. Although accidental errors would seem at first sight to be of a capricious and irregular nature which would exclude them from the domain of mathematics, yet, upon examination from theoretical considerations, confirmed, as will be shown, by experience, we shall find that they are subject to remarkably precise laws. In the first place, we remark that they are subject to the following fundamental laws: 1st. Errors in excess and in defect —*i.e.* positive and negative, but of equal absolute value—are equally probable, and in a large number of observations are equally frequent. 2d. In every species of observations, there is a limit of error which the greatest accidental errors do not exceed: thus, if $l$ denotes the absolute magnitude of this limit, all the positive errors are comprised between 0 and $+ l$, and all the negative errors between 0 and $- l$, and, consequently, all the errors are distributed over the interval $2l$. 3d. The errors are not distributed uniformly over this interval $2l$, but the smaller errors are more frequent than the larger ones.

Thus the frequency of an error of a given magnitude may be regarded as a function of the error itself: so that, if we denote an error of a certain magnitude by $\Delta$, and its relative frequency in a given large number of observations by $\varphi\Delta$, this function should obtain its maximum value for $\Delta = 0$, and become zero when $\Delta = \pm l$. If, then, we denote the *probability** of an error $\Delta$ by $y$, or put

$$y = \varphi\Delta \qquad (7)$$

we may regard this as the equation of a curve, taking $\Delta$ as the abscissa and $y$ as the ordinate. The nature of this curve will be accurately defined when we have discovered the form of the function $\varphi\Delta$, but we can see in advance that a curve such as Fig. A is required to satisfy the conditions already imposed upon

Fig. A.




this function. For its maximum ordinate must correspond to $\Delta = 0$; it must be symmetrical with reference to the axis of $y$, since equal errors with opposite signs have equal probabilities; and it must approach very near to the axis of abscissæ for values of $\Delta$ near the extreme limits, although the impossibility of assigning such extreme limits of error with precision must prevent us from fixing the point at which the curve will finally meet the axis.

8. The number of possible errors in any class of observations is, strictly speaking, finite; for there is always a limit of accuracy to the observations, even when we employ the most refined instruments, in consequence of which there is a numerical succession in our results. Thus, if 1″ is the smallest measure in a

* That is, if the error $\Delta$ occurs $n$ times in $m$ observations, $y = \phi\Delta = \frac{n}{m}$.

given case, the possible errors, arranged in their order of magnitude, can only differ by 1″ or an integral number of seconds. Hence, our geometrical representation should strictly consist of a number of isolated points; but, as these points will be more and more nearly represented by a continuous curve as we increase the accuracy of the observations, and thus diminish the intervals between the successive ordinates, we may, without hesitation, adopt such a continuous curve as expressing the law of error. We shall, therefore, regard $\Delta$ as a continuous variable, and $\varphi\Delta$ as a continuous function of it.

Now, by the theory of probabilities, if $\varphi\Delta$, $\varphi\Delta'$, $\varphi\Delta''$ ....... are the respective probabilities of all the possible errors $\Delta$, $\Delta'$, $\Delta''$ ...... we have*

$$\varphi\Delta + \varphi\Delta' + \varphi\Delta'' + \ldots\ldots = 1$$

when the number of possible errors is finite. But the assumed continuity of our curve requires that we consider the difference between successive values of $\Delta$ as infinitesimal, and thus the number of values of $\varphi\Delta$ is infinite, and the probability of any one of these errors is an infinitesimal. To meet this difficulty, let us observe that if a finite series of errors $\Delta$, $\Delta'$, $\Delta''$ .... be expressed in the smallest unit employed in the observations, these errors, arranged in the order of their magnitude, will be a series of consecutive integral numbers; the probability of the error $\Delta$ may be regarded as the same as the probability that the error falls between $\Delta$ and $\Delta + 1$; and the probability of an error between $\Delta$ and $\Delta + i$ will be the sum of the probabilities of the errors $\Delta$, $\Delta + 1$, $\Delta + 2$, ..... $\Delta + (i - 1)$. If $i$ is small, the probability of each of the errors from $\Delta$ to $\Delta + i$ will be nearly the same as that of $\Delta$: so that their sum will differ but little from $i\varphi\Delta$. As the interval between the successive errors diminishes, this expression becomes more accurate; and hence when we take $d\Delta$, the infinitesimal, instead of $i$, we have $\varphi\Delta \,.\, d\Delta$ as the rigorous expression of the probability that an error falls between $\Delta$ and $\Delta + d\Delta$. Hence, it follows, in general, that the probability that an error falls between any given limits $a$ and $b$ is the sum of all

* For if there are $n$ errors equal to $\Delta$, $n'$ equal to $\Delta'$, &c., and the whole number of errors is $m$, the probabilities of the errors are respectively $\phi\Delta = \frac{n}{m}$, $\phi\Delta' = \frac{n'}{m}$, &c., and the sum of these is $\frac{n + n' + \ldots\ldots}{m} = \frac{m}{m} = 1$.

the elements of the form $\varphi\Delta \,.\, d\Delta$ between these limits, or the integral

$$\int_a^b \varphi\Delta \,.\, d\Delta$$

and this integral, taken between the extreme limits of error, and thus embracing all the possible errors, will be

$$\int_{-l}^{+l} \varphi\Delta \,.\, d\Delta = 1$$

We have heretofore assumed that the function $\varphi\Delta$ is to be zero for $\Delta = \pm l$. It must also be added that, since the probability of any error greater than $\pm l$ is also zero, we should have to determine this function in such a manner that it would be zero for all values of $\Delta$ from $+ l$ to $+ \infty$ and from $- l$ to $- \infty$. The obvious impossibility of determining such a function leads us to extend the limits $\pm l$ to $\pm \infty$, and to take

$$\int_{-\infty}^{+\infty} \varphi\Delta \,.\, d\Delta = 1 \tag{8}$$

This will evidently be allowable if the integral taken from $\pm l$ to $\pm \infty$ is so small as to be practically insignificant. Besides, the extreme limits of error can never be fixed with precision, and it will suffice if the function $\varphi\Delta$ is such that it becomes very small for those errors which are regarded as very large.

9. Returning now to the general case of indirect observations, Art. 6, in which we suppose a quantity $M = f(x, y, z, \ldots.)$ to be observed, let $\Delta$, $\Delta'$, $\Delta''$ .... be the errors of the several observed values of $M$, and $\varphi\Delta$, $\varphi\Delta'$, $\varphi\Delta''$ .... their respective probabilities; then, the probability that these errors occur at the same time in the given series being denoted by $P$, we have, by a theorem of the calculus of probabilities,*

$$P = \varphi\Delta \,.\, \varphi\Delta' \,.\, \varphi\Delta'' \ldots\ldots \tag{9}$$

The most probable system of values of the unknown quantities

* If a single action of a cause can produce the effects $a$, $a'$, $a''$, .... with the respective probabilities $p$, $p'$, $p''$, .... the probability that two successive independent actions of the cause will produce the effects $a$ and $a'$ is $pp'$; and similarly for any number of effects. Thus, if an urn contains 2 white balls, 3 red ones, and 5 black ones, the probability that in two successive drawings (the original number of balls being the same at each drawing) one ball will be white and the other red is $\frac{2}{10} \times \frac{3}{10}$.

$x, y, z \ldots.$ will be that which makes the probability $P$ a maximum. Consequently, since $x, y, z \ldots.$ are here supposed to be independent,* the derivative of $P$ relatively to each of these variables must be equal to zero; or, since $\log P$ varies with $P$, the derivatives of $\log P$ must satisfy this condition, and we shall have

$$\frac{1}{P}\cdot\frac{dP}{dx}=0, \qquad \frac{1}{P}\cdot\frac{dP}{dy}=0, \text{ \&c.}$$

which, since

$$\log P=\log \varphi\varDelta+\log \varphi\varDelta'+\log \varphi\varDelta''+\ldots..$$

give the equations

$$\left.\begin{array}{l}\varphi'\varDelta\,\dfrac{d\varDelta}{dx}+\varphi'\varDelta'.\dfrac{d\varDelta'}{dx}+\varphi'\varDelta''.\dfrac{d\varDelta''}{dx}+\ldots..=0\\[2ex] \varphi'\varDelta.\dfrac{d\varDelta}{dy}+\varphi'\varDelta'.\dfrac{d\varDelta'}{dy}+\varphi'\varDelta''.\dfrac{d\varDelta''}{dy}+\ldots..=0\\[2ex] \varphi'\varDelta.\dfrac{d\varDelta}{dz}+\varphi'\varDelta'.\dfrac{d\varDelta'}{dz}+\varphi'\varDelta''.\dfrac{d\varDelta''}{dz}+\ldots..=0\\ \qquad\text{\&c.}\qquad\quad\text{\&c.}\end{array}\right\} \quad (10)$$

in which we have put

$$\varphi'\varDelta=\frac{d\varphi\varDelta}{\varphi\varDelta\,.\,d\varDelta} \quad (11)$$

The number of equations in (10) being the same as that of the unknown quantities, these equations will serve to determine the unknown quantities when we have discovered the value of the function $\varphi'\varDelta$, as will be shown hereafter.

Since the functions $\varphi\varDelta$ and $\varphi'\varDelta$ are supposed to be general, and therefore applicable whatever the number of unknown quantities, we may determine them by an examination of the special case in which there is but one unknown quantity, or that in which the observed values $M, M', M'' \ldots.$ belong to the same quantity. In that case, the hypothesis that $x$ is the value of this quantity gives the errors

$$\varDelta=M-x, \qquad \varDelta'=M'-x, \qquad \varDelta''=M''-x\ldots..$$

* That is, subject to no restrictions except that they shall satisfy the observations, or the equations $M=f(x, y, z, \ldots.)$. For the case of "conditioned" observations, see Art. 53 of this Appendix.

whence

$$\frac{d\Delta}{dx} = \frac{d\Delta'}{dx} = \frac{d\Delta''}{dx} \cdots\cdots = -1$$

and the first equation of (10) becomes

$$\varphi'(M-x) + \varphi'(M'-x) + \varphi'(M''-x) + \cdots\cdots = 0 \quad (12)$$

This being general for any number $m$ of observations, and for any observed values $M, M', M'' \ldots\ldots$, let us suppose the special case

$$M' = M'' \ldots\ldots = M - mN$$

Since the arithmetical mean of the observed quantities is here the most probable value of $x$, we have

$$x = \frac{1}{m}(M + M' + M'' + \cdots\cdots)$$
$$= \frac{1}{m}[M + (m-1)(M - mN)]$$
$$= M - (m-1)N$$

whence

$$M - x = (m-1)N$$
$$M' - x = M'' - x \ldots\ldots = -N$$

and, consequently, (12) becomes

$$\varphi'[(m-1)N] + (m-1)\varphi'(-N) = 0$$

or,

$$\frac{\varphi'[(m-1)N]}{(m-1)N} = \frac{\varphi'(-N)}{-N}$$

That is, for all values of $m$, and therefore for all values of $(m-1)N$, we have $\frac{\varphi'[(m-1)N]}{(m-1)N}$ equal to the *same* quantity $\frac{\varphi'(-N)}{-N}$. Hence we have generally $\frac{\varphi'\Delta}{\Delta}$ equal to a constant quantity, and, denoting this constant by $k$, we have

$$\varphi'\Delta = k\Delta$$

or, by (11),

$$\frac{d\varphi\Delta}{\varphi\Delta} = k\Delta \,.\, d\Delta$$

Integrating,

$$\log \varphi\Delta = \tfrac{1}{2}k\Delta^2 + \log \varkappa$$

whence

$$\varphi\Delta = \varkappa\, e^{\frac{1}{2}k\Delta\Delta}$$

in which $e$ is the base of the Napierian system of logarithms.

Since $\varphi\Delta$ must decrease as $\Delta$ increases, $\frac{1}{2}k$ must be essentially negative: representing it, therefore, by $-h^2$, our function becomes

$$\varphi\Delta = \varkappa\, e^{-hh\Delta\Delta}$$

To determine the constant $\varkappa$, let this value be substituted in (8), which gives

$$\int_{-\infty}^{+\infty} \varkappa\, e^{-hh\Delta\Delta}\, d\Delta = 1$$

Putting

$$t = h\Delta \tag{13}$$

this integral becomes

$$\frac{\varkappa}{h}\int_{-\infty}^{+\infty} e^{-tt}\, dt = 1$$

The known value of the definite integral in the first member is $\sqrt{\pi}$ (see Vol. I. p. 153); whence

$$\varkappa = \frac{h}{\sqrt{\pi}}$$

and the complete expression of $\varphi\Delta$ becomes

$$\varphi\Delta = \frac{h}{\sqrt{\pi}}\, e^{-hh\,\Delta\Delta} \tag{14}$$

The constant $h$ must depend upon the nature of the observations, and will be particularly examined hereafter. If we here take it as the unit of abscissæ in the curve of probability, the equation (7) becomes

$$y = \frac{1}{\sqrt{\pi}}\, e^{-\Delta\Delta}$$

by which the curve may be constructed. The values of $y$ for a few values of $\Delta$ are as follows:

| $\Delta$ | $y$ | Diff. | $\Delta$ | $y$ | Diff. |
|---|---|---|---|---|---|
| 0.0 | 0.5642 | | 1.6 | 0.0436 | |
| | | — .0221 | | | — .0215 |
| 0.2 | 0.5421 | | 1.8 | 0.0221 | |
| | | — .0613 | | | — .0118 |
| 0.4 | 0.4808 | | 2.0 | 0.0103 | |
| | | — .0872 | | | — .0058 |
| 0.6 | 0.3936 | | 2.2 | 0.0045 | |
| | | — .0961 | | | — .0027 |
| 0.8 | 0.2975 | | 2.4 | 0.0018 | |
| | | — .0899 | | | — .0011 |
| 1.0 | 0.2076 | | 2.6 | 0.0007 | |
| | | — .0739 | | | — .0005 |
| 1.2 | 0.1337 | | 2.8 | 0.0002 | |
| | | — .0542 | | | — .0001 |
| 1.4 | 0.0795 | | 3.0 | 0.0001 | |
| | | — .0359 | | | |
| 1.6 | 0.0436 | | $\infty$ | 0.0000 | |

The curve, Fig. A, in Art. 7, is constructed from this table; but, to exhibit its character more distinctly, the scale of the ordinates is four times that of the abscissæ (which, indeed, corresponds to the case of $h = 2$). We see that the curve approaches very near to the axis for moderate values of $\varDelta$, and that the assumption of $\pm \infty$ instead of finite limits of $\varDelta$ can involve no practical error. It is evident that the axis $XX$ is an asymptote to the curve.

The differences in the above table indicate that the curve approaches the axis most rapidly at a point whose abscissa is between 0.6 and 0.8. The exact position of this point, which is a point of inflexion, is found by putting the second differential coefficient of $y$ equal to zero, which gives

$$\frac{d^2y}{d\varDelta^2} = -\frac{2}{\sqrt{\pi}}e^{-\Delta\Delta} + \frac{4\varDelta\varDelta}{\sqrt{\pi}}e^{-\Delta\Delta} = 0$$

whence

$$\varDelta = \frac{1}{\sqrt{2}} = 0.7071$$

The ordinate $Mm$ is drawn at this point. We shall have occasion to refer to it again hereafter.

### THE MEASURE OF PRECISION.

10. The constant $h$ requires special consideration. Since the exponent of $e$ in (14) must be an abstract number, $\frac{1}{h}$ must be a concrete quantity of the same kind as $\varDelta$. In a class of observations in which $\varDelta$ is small for a given probability $\varphi\varDelta$, $\frac{1}{h}$ will be small, and $h$ will be large. Thus, $h$ will be the greater the more precise the nature of the observations, and is, therefore, called by GAUSS the *measure of precision*. If in one system of observations the probability of an error $\varDelta$ is expressed by

$$\frac{h}{\sqrt{\pi}}e^{-hh\Delta\Delta}$$

and in another, more or less precise, by

$$\frac{h'}{\sqrt{\pi}}e^{-h'h'\Delta\Delta}$$

the probability that in one observation of the first system the error committed will be comprised between the limits $-\delta$ and $+\delta$ will be expressed by the integral

$$\int_{-\delta}^{+\delta} \frac{h}{\sqrt{\pi}} e^{-hh\Delta\Delta}\, d\Delta$$

and, in like manner, the probability that the error of an observation in the second system will be comprised between $-\delta'$ and $+\delta'$ will be expressed by

$$\int_{-\delta'}^{+\delta'} \frac{h'}{\sqrt{\pi}} e^{-h'h'\Delta\Delta}\, d\Delta$$

These integrals are evidently equal when we have $h\delta = h'\delta'$. If, for example, we have $h' = 2h$, the integrals will be equal when $\delta = 2\delta'$; that is, the double error will be committed in the first system with the same probability as the simple error in the second, or, in the usual mode of expression, the second system will be twice as precise as the first. We shall presently see how the value of $h$ can be found for any given observations.

## THE METHOD OF LEAST SQUARES.

11. The preceding discussion leads directly to important practical results. We have seen (Art. 9) that to find the most probable values of $x, y, z \ldots.$ from the observed values of $M = f(x, y, z, \ldots.)$ we are to render the probability $P = \varphi\Delta \,.\, \varphi\Delta' \,.\, \varphi\Delta'' \ldots.$ a maximum, that is, by (14),

$$P = h^m \,\pi^{-\frac{1}{2}m}\, e^{-hh\,(\Delta\Delta + \Delta'\Delta' + \Delta''\Delta'' + \ldots.)} \tag{15}$$

must be a maximum; and this requires that the quantity $\Delta\Delta + \Delta'\Delta' + \Delta''\Delta'' + \ldots.$ should be a minimum. Thus, the principle that *the most probable values of the unknown quantities are those which make the sum of the squares of the residual errors a minimum,* is not limited to the case of direct observations, but is entirely general.

The principle is readily extended to observations of unequal precision. For if the degree of precision of the observations $M, M', M'' \ldots.$ be respectively $h, h', h'' \ldots.$, and we compare these observed quantities with the values $V, V', V'' \ldots.$, computed with the most probable values of $x, y, z \ldots.$, whereby we obtain the residual errors $M - V = \Delta$, $M' - V' = \Delta' \ldots.$, it is the same thing as if we had taken observations of equal precision (represented by 1) upon the quantities $hM$, $h'M'$, $h''M'' \ldots.$, and had compared them with the computed quantities $hV$, $h'V'$, $h''V''$...., whereby we should have found the errors $hM - hV = h\Delta$, $h'M' - h'V' = h'\Delta'$...., in which case we should have to reduce to a minimum the quantity

$$h^2\Delta^2 + h'^2\Delta'^2 + h''^2\Delta''^2 + \ldots.$$

that is, *each error being multiplied by its measure of precision, and thereby reduced to the same degree of precision, the sum of the squares of the reduced errors must be a minimum.*

In what precedes is involved the whole theory of the method of least squares. I proceed to develop its practical features.

### THE PROBABLE ERROR.

12. From the preceding articles it follows that the probability that the error of an observation falls between $\Delta$ and $\Delta + d\Delta$ is expressed by

$$\frac{h}{\sqrt{\pi}} e^{-hh\Delta\Delta} d\Delta$$

and the probability that it falls between the limits 0 and $a$ is expressed by

$$\frac{h}{\sqrt{\pi}} \int_{\Delta=0}^{\Delta=a} e^{-hh\Delta\Delta} d\Delta$$

and this integral expresses the number of errors that we should expect to find between the limits 0 and $a$ when the whole number of errors is put = 1 [equation (8)]. If we put $t = h\Delta$, the integral takes the form

$$\frac{1}{\sqrt{\pi}} \int_{t=0}^{t=ah} e^{-tt} dt$$

The whole number of errors, both positive and negative, whose numerical magnitude falls between the given limits is twice this integral, or

$$\frac{2}{\sqrt{\pi}} \int_{t=0}^{t=ah} e^{-tt} dt \qquad (16)$$

The value of this integral (which may be computed by the methods of Vol. I. Art. 113) is given in Table IX. The number of errors between any two given limits will be found by taking the difference between the tabular numbers corresponding to these limits. Since the total number of errors is taken as unity in the table, the required number of errors in any particular case is to be found by multiplying the tabular numbers by the actual number of observations. Thus, if there are 1000 observations, we find that

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| between | $t = 0$ | and | $t = 0.5$ | there | are | 520 | errors. |
| " | $t = 0.5$ | " | $t = 1.0$ | " | " | 322 | " |
| " | $t = 1.0$ | " | $t = 1.5$ | " | " | 123 | " |
| " | $t = 1.5$ | " | $t = 2.0$ | " | " | 29 | " |
| " | $t = 2.0$ | " | $t = \infty$ | " | " | 5 | " |

13. The degrees of precision of different series of observations may be compared together either by comparing the values of $h$, or by comparing the errors which are committed with equal facility in the two systems. The errors to be compared must occupy in the two systems a like position in relation to the extreme errors, and we may select for this purpose in each system *the error which occupies the middle place in the series of errors arranged in the order of their magnitude, so that the number of errors which are less than this assumed error is the same as the number of errors which exceed it.* The error which satisfies this condition is that for which the value of the integral (16) is 0.5. Denoting the corresponding value of $t$ by $\rho$, we find, by interpolation from Table IX.,

$$\rho = 0.47694$$

and we have

$$\frac{2}{\sqrt{\pi}} \int_0^{\rho} e^{-tt}\, dt = \frac{1}{2} \tag{17}$$

If then we denote by $r$ the error which, in any system of observations whose degree of precision is $h$, corresponds to the value $t = \rho$, or put

$$\rho = hr \qquad\qquad h = \frac{\rho}{r} \tag{18}$$

there will be a probability of $\frac{1}{2}$ that the error of any single observation in that system will be less than $r$, and the same probability that it will be greater than $r$; which is sometimes expressed by saying that *it is an even wager that the error will be less than r.* Hence $r$ is called the *probable error.*

We may, therefore, compare different series of observations by comparing their probable errors, their degrees of precision being, by (18), inversely proportional to these errors.

14. In order to apply Table IX. in determining the number of errors in a given class of observations, we must know the

measure of precision $h$, or the probable error $r$: thus, if we wish the number of errors less than $a$, we enter the table with the argument $t = ah$, or $t = \frac{a\rho}{r}$

For greater convenience, we can employ Table IX.A, which gives the same function with the argument $\frac{a}{r}$. For example, if there are 1000 observations whose probable error is $r = 2''$, and we wish to know the number of errors less than $a = 1''$, we take from Table IX.A, with the argument $\frac{a}{r} = 0.5$, the number 0.26407, which multiplied by 1000 gives 264 as the required number.

The following example from the *Fundamenta Astronomiæ* of Bessel will serve to show how far the preceding theory is sustained by experience. In 470 observations made by Bradley upon the right ascension of *Sirius* and *Altair*, Bessel found the probable error of a single observation to be

$$r = 0''.2637$$

Hence, for the number of errors less than 0''.1 the argument of Table IX.A will be $\frac{0.1}{0.2637} = 0.3792$; and for 0.''2, 0''.3, &c., the successive multiples of 0.3792. Thus, we find from the table

| | | | | | |
|---|---|---|---|---|---|
| for | 0''.1 | with arg. | 0.3792 | the number | 0.20187 |
| " | 0 .2 | " | 0.7584 | " | 0.39102 |
| " | 0 .3 | " | 1.1376 | " | 0.55710 |
| " | 0 .4 | " | 1.5168 | " | 0.69372 |
| " | 0 .5 | " | 1.8960 | " | 0.79904 |
| " | 0 .6 | " | 2.2752 | " | 0.87511 |
| " | 0 .7 | " | 2.6544 | " | 0.92661 |
| " | 0 .8 | " | 3.0336 | " | 0.95926 |
| " | 0 .9 | " | 3.4128 | " | 0.97866 |
| " | 1 .0 | " | 3.7920 | " | 0.98946 |
| | | | $\infty$ | " | 1.00000 |

Subtracting each number from the following one, and multiplying the remainder by 470, the number of observations, there were found

| Between | No. of errors by the theory. | No. of errors by experience. |
|---|---|---|
| 0″.0 and 0″.1 | 95 | 94 |
| 0 .1 " 0 .2 | 89 | 88 |
| 0 .2 " 0 .3 | 78 | 78 |
| 0 .3 " 0 .4 | 64 | 58 |
| 0 .4 " 0 .5 | 50 | 51 |
| 0 .5 " 0 .6 | 36 | 36 |
| 0 .6 " 0 .7 | 24 | 26 |
| 0 .7 " 0 .8 | 15 | 14 |
| 0 .8 " 0 .9 | 9 | 10 |
| 0 .9 " 1 .0 | 5 | 7 |
| over 1 .0 | 5 | 8 |

The agreement between the theory and experience, though not absolute, is remarkably close. The number of large errors by experience exceeds that given by the theory, and this has been found in other cases of a similar kind; which shows at least that the extension of the limits of error to $\pm\infty$ has not introduced any error. The discrepancy rather indicates a source of error of an abnormal character, and calls for some criterion by which such abnormal observations may be excluded from our discussions and not permitted to vitiate our results. Such a criterion has been proposed by Prof. PEIRCE, and will be considered hereafter.

#### THE MEAN OF THE ERRORS, AND THE MEAN ERROR.

15. The selection of the probable error as the term of comparison between different series of observations is arbitrary, although it seems to be naturally designated by its middle position in the series of errors. There are two other errors which have been used for the same purpose.

The first is the *mean of the errors*, these being all taken with the positive sign. In order to find its relation to the probable error, let us first consider a finite series of errors

$$\Delta,\ \Delta',\ \Delta'',\ldots\ldots$$

with the respective probabilities

$$\frac{2a}{m},\qquad \frac{2a'}{m},\qquad \frac{2a''}{m},\ldots$$

so that in $m$ observations there will be $2a$ errors (numerically) equal to $\Delta$, $2a'$ equal to $\Delta'$, &c., the probability of a positive error $\Delta$ being $\frac{a}{m}$. The mean of all these errors, each being repeated a number of times proportional to its probability, is

$$\frac{2a\Delta + 2a'\Delta' + 2a''\Delta'' + \ldots}{m} = 2\Delta \cdot \frac{a}{m} + 2\Delta' \cdot \frac{a'}{m} + 2\Delta'' \cdot \frac{a''}{m} + \ldots$$

When the number of errors is infinite, the probability of an error $\Delta$ is to be understood as the probability that it falls between $\Delta$ and $\Delta + d\Delta$, which is $\varphi\Delta \,.\, d\Delta$ (Art. 8), and the above formula for the mean of the errors becomes the sum of an infinite number of terms of the form $2\Delta\varphi\Delta \,.\, d\Delta$. Hence, putting

$$\eta = \text{the mean of the errors,}$$

we have

$$\eta = \int_0^{\infty} \frac{2h}{\sqrt{\pi}} \Delta e^{-hh\Delta\Delta}\, d\Delta = \frac{1}{h\sqrt{\pi}} \tag{19}$$

or, by (18),

$$\left.\begin{aligned} \eta &= \frac{r}{\rho\sqrt{\pi}} = 1.1829\, r \\ r &= 0.8453\, \eta \end{aligned}\right\} \tag{20}$$

Another error, very commonly employed in expressing the precision of observations, is that which has received the appellation of the *mean error* (*der mittlere Fehler* of the Germans), which is not to be confounded with the above mean of the errors. Its definition is, *the error the square of which is the mean of the squares of all the errors.* Hence, putting

$$\varepsilon = \text{the mean error,}$$

we have

$$\varepsilon^2 = \int_{-\infty}^{+\infty} \frac{h}{\sqrt{\pi}} \Delta^2 e^{-hh\Delta\Delta}\, d\Delta = \frac{1}{2h^2} \tag{21}$$

or, by (18),

$$\left.\begin{aligned} \varepsilon &= \frac{r}{\rho\sqrt{2}} = 1.4826\, r \\ r &= 0.6745\, \varepsilon \end{aligned}\right\} \tag{22}$$

When we put $h = 1$, we have $\varepsilon = \sqrt{\tfrac{1}{2}}$. The mean error is, therefore, the abscissa of the point of inflection of the curve of probability (Art. 9). In the figure, p. 479, $OM$ is the mean error,

$OP$ the probable error, $OE$ the mean of the errors, and $Mm$, $Pp$, $Ee$, their respective probabilities.

### THE PROBABLE ERROR OF THE ARITHMETICAL MEAN.

16. The error above denoted by $r$ is the probable error of any one of the observed values of the unknown quantity $x$. We are next to determine the relation between this and the probable error $r_0$ of the arithmetical mean of these values.

If $\Delta, \Delta', \Delta'' \ldots.$ are the errors of the observed values, the most probable value of $x$ is that which renders the probability

$$P = h^m \pi^{-\frac{1}{2}m} e^{-hh(\Delta\Delta + \Delta'\Delta' + \Delta''\Delta'' + \ldots)}$$

a maximum (Art. 11), and, consequently, the sum $\Delta\Delta + \Delta'\Delta' + \ldots.$ a minimum. But this sum is rendered a minimum by the assumption of the arithmetical mean $x_0$ as the most probable value (Art. 5), and hence the quantity $P$ expresses the probability of the arithmetical mean if $\Delta, \Delta', \Delta'' \ldots.$ are the errors of the observations when compared with this mean. The probability of any other value of $x$, as $x_0 + \delta$, will be

$$P' = h^m \pi^{-\frac{1}{2}m} e^{-hh\{(\Delta-\delta)^2 + (\Delta'-\delta)^2 + \ldots.\}}$$
$$= h^m \pi^{-\frac{1}{2}m} e^{-hh\{[\Delta\Delta] - 2[\Delta]\delta + m\delta\delta\}}$$

Since $[\Delta] = \Delta + \Delta' + \Delta'' + \ldots. = 0$ (Art. 5), and $[\Delta\Delta] = m\varepsilon\varepsilon$ (Art. 15), this expression may be put under the form

$$P' = h^m \pi^{-\frac{1}{2}m} e^{-mhh(\epsilon\epsilon + \delta\delta)}$$

and at the same time we have

$$P = h^m \pi^{-\frac{1}{2}m} e^{-mhh\epsilon\epsilon}$$

so that

$$P : P' = 1 : e^{-mhh\delta\delta}$$

that is, the probability of the error zero in the arithmetical mean is to that of the error $\delta$ as $1 : e^{-mhh\delta\delta}$. For a single observation, the probability of the error zero is to that of the error $\delta$ as $1 : e^{-hh\delta\delta}$. Hence the measure of precision (Art. 10) of the single observation being $h$, that of the arithmetical mean of $m$ such observations is $h\sqrt{m}$; from which follows the important theorem that *the precision of the mean of a number of observations increases as the square root of their number.**

If, then, $r$ is the probable error of a single observation, and $r_0$ that of the arithmetical mean, we must have

$$r_0 = \frac{r}{\sqrt{m}} \tag{23}$$

and from the constant relation between the mean and the probable error (22),

$$\varepsilon_0 = \frac{\varepsilon}{\sqrt{m}} \tag{24}$$

### DETERMINATION OF THE MEAN AND PROBABLE ERRORS OF GIVEN OBSERVATIONS.

17. The principles now explained will enable us to determine the mean errors of any given series of directly observed quantities. Let $n, n', n'' \ldots$ be the observed values; $x_0$ their arithmetical mean; $v, v', v'' \ldots$ the residuals found by subtracting $x_0$ from each observed value: so that

$$v = n - x_0, \qquad v' = n' - x_0, \qquad v'' = n'' - x_0, \text{ \&c.}$$

If $x_0$ were certainly the true value of $x$, so that $v, v', v'' \ldots$ were the actual or (as we may say) the *true* errors, and, consequently, identical with $\Delta, \Delta', \Delta'' \ldots$, we should have, according to the above, $m\varepsilon\varepsilon = [\Delta\Delta] = [vv]$, and hence

$$\varepsilon = \sqrt{\left(\frac{[vv]}{m}\right)}$$

and this must always give a close approximation to the value of $\varepsilon$. But the relation $m\varepsilon\varepsilon = [\Delta\Delta]$ was deduced from a consideration of an infinite series of errors which would reduce the mean error of $x_0$ to an infinitesimal, according to the principles assumed, and thus make $v, v', v'' \ldots$ identical with $\Delta, \Delta', \Delta'' \ldots$ A better approximation to the value of $\varepsilon$, where the series is limited, is to be obtained by considering the mean error of $x_0$ itself, and consequently, also, the mean errors of the residuals $v, v', v'' \ldots$ If then we suppose the true value of $x$ to be $x_0 + \delta$, we shall have the true errors

$$\Delta = v - \delta, \qquad \Delta' = v' - \delta, \qquad \Delta'' = v'' - \delta, \text{ \&c.}$$

---

* See, in connection, Arts. 21 and 25.

whence, observing that $[v] = 0$,

$$[\varDelta\varDelta] = m\varepsilon\varepsilon = [vv] - 2\,[v]\,\delta + m\delta^2$$
$$= [vv] + m\delta^2$$

Thus the approximate value $m\varepsilon\varepsilon = [vv]$ requires the correction $m\delta^2$, the value of which depends upon the value we may ascribe to $\delta$. As the best approximation, we may assume it to be the mean error $\varepsilon_0$: so that, by (24),

$$m\delta^2 = m\varepsilon_0^{\,2} = m\,\frac{\varepsilon\varepsilon}{m} = \varepsilon\varepsilon$$

which gives

$$m\varepsilon\varepsilon = [vv] + \varepsilon\varepsilon$$

whence

$$\varepsilon\varepsilon = \frac{[vv]}{m-1} \qquad \varepsilon = \sqrt{\left(\frac{[vv]}{m-1}\right)} \tag{25}$$

and consequently, also, by (22),

$$r = q\sqrt{\left(\frac{[vv]}{m-1}\right)} \qquad q = 0.6745 \tag{26}$$

Thus from the actual residuals the mean and the probable error of a single observed value are found. Hence, by (23) and (24), the mean and probable errors of the arithmetical mean will be found by the formulæ

$$\varepsilon_0 = \sqrt{\left(\frac{[vv]}{m\,(m-1)}\right)} \qquad r_0 = q\sqrt{\left(\frac{[vv]}{m\,(m-1)}\right)} \tag{27}$$

Example.—Let us take the following measures of the outer diameter of Saturn's ring observed by Bessel at the Königsberg Observatory with the heliometer, in the years 1829–1831.* The measures, denoted by $n$, are all reduced to the mean distance of Saturn from the sun, and are here assumed to have the same degree of precision.

* *Astron. Nach.*, Vol. XII. p. 169.

| $n$ | $v$ | $vv$ | $n$ | $v$ | $vv$ |
|---|---|---|---|---|---|
| 38″.91 | − 0″.40 | 0.1600 | 39″.41 | + 0″.10 | 0.0100 |
| 39 .32 | + 0 .01 | .0001 | 39 .40 | + 0 .09 | .0081 |
| 38 .93 | − 0 .38 | .1444 | 39 .36 | + 0 .05 | .0025 |
| 39 .31 | 0 .00 | .0000 | 39 .20 | − 0 .11 | .0121 |
| 39 .17 | − 0 .14 | .0196 | 39 .42 | + 0 .11 | .0121 |
| 39 .04 | − 0 .27 | .0729 | 39 .30 | − 0 .01 | .0001 |
| 39 .57 | + 0 .26 | .0676 | 39 .41 | + 0 .10 | .0100 |
| 39 .46 | + 0 .15 | .0225 | 39 .43 | + 0 .12 | .0144 |
| 39 .30 | − 0 .01 | .0001 | 39 .43 | + 0 .12 | .0144 |
| 39 .03 | − 0 .28 | .0784 | 39 .36 | + 0 .05 | .0025 |
| 39 .35 | + 0 .04 | .0016 | 39 .02 | − 0 .29 | .0841 |
| 39 .25 | − 0 .06 | .0036 | 39 .01 | − 0 .30 | .0900 |
| 39 .14 | − 0 .17 | .0289 | 38 .86 | − 0 .45 | .2025 |
| 39 .47 | + 0 .16 | .0256 | 39 .51 | + 0 .20 | .0400 |
| 39 .29 | − 0 .02 | .0004 | 39 .21 | − 0 .10 | .0100 |
| 39 .32 | + 0 .01 | .0001 | 39 .17 | − 0 .14 | .0196 |
| 39 .40 | + 0 .09 | .0081 | 39 .60 | + 0 .29 | .0841 |
| 39 .33 | + 0 .02 | .0004 | 39 .54 | + 0 .23 | .0529 |
| 39 .28 | − 0 .03 | .0009 | 39 .45 | + 0 .14 | .0196 |
| 39 .62 | + 0 .31 | .0961 | 39 .72 | + 0 .41 | .1681 |
| | | | $x_0 =$ 39 .308 | | $[vv] =$ 1.5884 |

Hence, since $m = 40$, we have, by (25) and (26),

$$\varepsilon = \sqrt{\left(\frac{1.5884}{39}\right)} = 0''.202$$

$$r = 0''.202 \times 0.6745 = 0''.136$$

and consequently, by (23) and (24), or (27),

$$\varepsilon_0 = \frac{0''.202}{\sqrt{(40)}} = 0''.032, \qquad r_0 = \frac{0''.136}{\sqrt{(40)}} = 0''.022$$

That is, the probable error of a single observation was $0''.136$, and that of the final result $x_0 = 39''.308$ was only $0''.022$.

18. The preceding method of finding the probable error from the squares of the residuals is that which is most commonly employed; but when the number of observations is very great, it is desirable to abridge the labor, if possible. A sufficient approximation can be obtained by the use of the first powers of the residuals as follows.

The number of observations being very great, we shall probably have as many positive as negative residuals. If $v'$, $v''$,

$v'''\ldots$ are the positive and $v_1, v_2, v_3\ldots$ the negative residuals, and if the true value of $x$ is $x_0+\delta$, the true errors will be $v'-\delta, v''-\delta, v'''-\delta\ldots.$, and $-v_1-\delta, -v_2-\delta, -v_3-\delta, \ldots.$. *If they are all taken with the positive sign only*, the errors are, therefore,

$$v'-\delta,\ v''-\delta,\ v'''-\delta,\ \ldots. \quad \text{and} \quad v_1+\delta,\ v_2+\delta,\ v_3+\delta,\ \ldots.$$

the mean of which, upon the hypothesis of an equal number of positive and negative residuals, is the same as that of the series

$$v',\ v'',\ v'''\ \ldots. \qquad v_1,\ v_2,\ v_3\ \ldots.$$

Hence, denoting the sum of the *numerical values* of the residuals by $[v]$, and the mean of the actual errors by $\eta$, as in Art. 15, we have

$$\eta=\frac{[v]}{m}$$

and hence, by (20),

$$r=0.8453\,\frac{[v]}{m} \tag{28}$$

and consequently, also, by (22),

$$\varepsilon=1.2533\,\frac{[v]}{m} \tag{29}$$

In the example of the preceding article we find the mean of the residuals taken with the positive sign to be $0''.1555$, which by (28) gives $r=0''.1555\times0.8453=0''.131$, which is perhaps a sufficient approximation to the value found above. In this example, however, we have 22 positive residuals, 17 negative ones, and 1 zero: so that the hypothesis upon which the formula (28) was founded is not strictly applicable. In a larger number of observations we should expect a closer agreement with the hypothesis, and more accordant results.

We may, however, employ the first powers of the residuals more strictly according to the theory of probabilities. In a limited series each residual is to be regarded as liable to a probable error $r'$, and their mean is to be regarded as the mean of the errors of the residuals themselves, rather than as the mean of the errors of the observations. Hence the formula

$$r'=0.8453\,\frac{[v]}{m}$$

gives the probable error of a residual. The relation between $r'$ and $r$ (= the probable error of an observed quantity $n$) may be found as follows. Each observed $n$ may be supposed to be the result of observing the mean quantity $x_0$ increased by an observed error $v$. The probable error of $n = x_0 + v$ is, therefore (by a principle hereafter to be proved),

$$r = \sqrt{(r_0^2 + r'^2)} = \sqrt{\left(\frac{r^2}{m} + r'^2\right)}$$

whence

$$r = r' \sqrt{\frac{m}{m-1}}$$

or

$$r = 0.8453 \frac{[v]}{\sqrt{[m\,(m-1)]}} \tag{30}$$

which agrees with the formula given by C. A. F. PETERS.* According to this formula, we find in the above example $r = 0''.133$.

### DETERMINATION OF THE MEAN AND PROBABLE ERRORS OF FUNCTIONS OF INDEPENDENT OBSERVED QUANTITIES.

19. Suppose, first, the most simple function of two independent observed quantities $x$ and $x_1$, namely, their sum or difference

$$X = x \pm x_1$$

and let the given mean errors of $x$ and $x_1$ be $\varepsilon$ and $\varepsilon_1$. Although the number of observations by which $x$ and $x_1$ have been found may not be given, we may assume it to have been any large number $m$, and the same for each of the quantities; the degrees of precision of the two series being inversely proportional to $\varepsilon$ and $\varepsilon_1$. The true errors of the assumed observations may be assumed to be—

for $x$, $\Delta, \Delta', \Delta'' \ldots\ldots$
for $x_1$, $\Delta_1, \Delta_1', \Delta_1'' \ldots\ldots$

and the errors of $X$, consequently,

$$\Delta \pm \Delta_1, \qquad \Delta' \pm \Delta_1', \qquad \Delta'' \pm \Delta_1'', \ldots\ldots$$

Denoting the mean error of $X$ by $E$, we have, by the definition,

$$\begin{aligned} mE^2 &= (\Delta \pm \Delta_1)^2 + (\Delta' \pm \Delta_1')^2 + (\Delta'' \pm \Delta_1'')^2 + \ldots\ldots \\ &= [\Delta\Delta] \pm 2\,[\Delta\Delta_1] + [\Delta_1\Delta_1] \end{aligned}$$

---

* *Astron. Nach.*, Vol. XLIV. p. 32.

In a great number of observations there must be as many positive as negative products of the form $\Delta\Delta_1$, and such that we shall probably have $[\Delta\Delta_1] = 0$; and since we also have $m\varepsilon^2 = [\Delta\Delta]$, $m\varepsilon_1^2 = [\Delta_1\Delta_1]$, this equation gives

$$E^2 = \varepsilon^2 + \varepsilon_1^2 \tag{31}$$

If we have

$$X = x \pm x_1 \pm x_2$$

and the mean errors of $x$, $x_1$, $x_2$ are $\varepsilon$, $\varepsilon_1$, $\varepsilon_2$, we have by the preceding equation the mean error of $x \pm x_1 = \sqrt{(\varepsilon^2 + \varepsilon_1^2)}$, and by a second application of the same equation, considering $x \pm x_1$ as a single quantity, the mean error of $X$ will be found by the formula

$$E^2 = \varepsilon^2 + \varepsilon_1^2 + \varepsilon_2^2 \tag{31*}$$

and the same principle may be thus extended to the algebraic sum of any number of observed quantities.

In consequence of the constant relation (22), if $r$, $r_1$, $r_2$ .... are the *probable* errors of $x$, $x_1$, $x_2$ .... and $R$ the probable error of $X = x \pm x_1 \pm x_2 \ldots\ldots$, we shall have

$$R^2 = r^2 + r_1^2 + r_2^2 + \ldots\ldots \tag{32}$$

EXAMPLE 1.—The zenith distance of a star observed in the meridian is

$$\zeta = 21°\ 17'\ 20''.3 \quad \text{with the mean error } \varepsilon = 2''.3$$

and the declination of the star is given

$$\delta = 19°\ 30'\ 14''.8 \quad \text{with the mean error } \varepsilon_1 = 0''.8$$

Required the mean error $E$ of the latitude of the place of observation, found by the formula $\varphi = \zeta + \delta$. We have, by (31),

$$E = \sqrt{[(2.3)^2 + (0.8)^2]} = 2''.44$$

Hence

$$\varphi = 40°\ 47'\ 35''.1 \quad \text{with the mean error } E = 2''.44$$

EXAMPLE 2.—The latitude of a place has been found with the mean error $\varepsilon = 0''.25$, and the meridian zenith distance of stars observed at that place with a certain instrument has been found to be subject to the mean error $\varepsilon_1 = 0''.62$: what is the mean error $E$ of the declinations of the stars deduced by the formula $\delta = \varsigma - \zeta$? We have

$$E = \sqrt{[(0.25)^2 + (0.62)^2]} = 0''.67$$

20. Let us next consider the function

$$X = ax$$

and suppose $x$ has been observed with the mean error $\varepsilon$, and $a$ is a given constant. Every observation of $x$ with the error $\pm \Delta$ gives $X$ with the error $\pm a\Delta$: so that the mean error of $X$ must be

$$E = a\varepsilon$$

In general, by combining this with the preceding principle, if we have

$$X = ax + a_1x_1 + a_2x_2 + \ldots.$$

and if the mean errors of $x$, $x_1$, $x_2$ .... are $\varepsilon$, $\varepsilon_1$, $\varepsilon_2$, ...., and $E$ that of $X$, we shall have

$$E^2 = a^2\varepsilon^2 + a_1^2\varepsilon_1^2 + a_2^2\varepsilon_2^2 + \ldots. = [a^2\varepsilon^2] \qquad (33)$$

and the same form may be used for probable errors.

EXAMPLE.—As an example illustrating the application of both the preceding principles, suppose that in order to find the rate of a chronometer we find at the time $t$ its correction $+ 12^m\ 13^s.2$ with the mean error $0^s.3$, and at the time $t'$ the correction $- 12^m\ 21^s.4$ with the same mean error $0^s.3$, and the interval $t' - t = 10$ days. The rate in the whole interval is

$$12^m\ 21^s.4 - 12^m\ 13^s.2 = + 8^s.2$$

with the mean error, according to Art. 19,

$$\sqrt{[(0.3)^2 + (0.3)^2]} = 0^s.42$$

The mean daily rate is then

$$+ \frac{8^s.2}{10} = + 0^s.82$$

with the mean error, according to Art. 20,

$$\frac{0^s.42}{10} = 0^s.042$$

21. If $x$, $x_1$, $x_2$ .... are the several observed values of the *same* quantity, their arithmetical mean being

$$x_0 = \frac{1}{m}(x + x_1 + x_2 + \ldots)$$

and if $r$ is the probable error of each observation, what is the probable error $r_0$ of $x_0$? By Art. 19, the probable error of the sum $x + x_1 + x_2 + \ldots.$ is

$$\sqrt{(r^2 + r^2 + r^2 + \ldots)} = \sqrt{(mr^2)} = r\sqrt{m}$$

and the probable error of $\frac{1}{m}$th of the sum is, by Art. 20,

$$r_0 = \frac{1}{m} \times r\sqrt{m} = \frac{r}{\sqrt{m}}$$

as has been otherwise proved in Art. 16.

22. Let us now take the general case in which $X$ is any function whatever of the observed quantities $x$, $x_1$, $x_2$, .... expressed by

$$X = f(x, x_1, x_2, \ldots.)$$

Let the variables be expressed in the form

$$x = a + x', \qquad x_1 = a_1 + x_1', \qquad x_2 = a_2 + x_2', \ldots.$$

$a$, $a_1$, $a_2$ ... being arbitrarily assumed very nearly equal to $x$, $x_1$, $x_2$ ... respectively, and such that $x'$, $x_1'$, $x_2'$ ..... may be so small that their squares will be insensible. The given mean errors $\varepsilon$, $\varepsilon_1$, $\varepsilon_2$ ... may then be regarded as the mean errors of $x'$, $x_1'$, $x_2'$ .... The function $X$ developed by Taylor's theorem is

$$X = f(a, a_1, a_2 \ldots) + \frac{dX}{dx}x' + \frac{dX}{dx_1}x_1' + \frac{dX}{dx_2}x_2' + \ldots.$$

and the mean error of $X$ will be that of the quantity

$$\frac{dX}{dx}x' + \frac{dX}{dx_1}x_1' + \frac{dX}{dx_2}x_2' + \ldots.$$

or, by (33),

$$E^2 = \left(\frac{dX}{dx}\right)^2 \varepsilon^2 + \left(\frac{dX}{dx_1}\right)^2 \varepsilon_1^2 + \left(\frac{dX}{dx_2}\right)^2 \varepsilon_2^2 + \ldots \qquad (34)$$

or, if $r, r_1, r_2 \ldots$ are the probable errors of $x, x_1, x_2 \ldots$, and $R$ that of $X$,

$$R^2 = \left(\frac{dX}{dx}\right)^2 r^2 + \left(\frac{dX}{dx_1}\right)^2 r_1^2 + \left(\frac{dX}{dx_2}\right)^2 r_2^2 + \ldots \qquad (34^*)$$

This formula is, indeed, but approximative, since we have neglected the terms involving the higher powers in the development of $X$; but the mean errors of these small terms will be insensible if we suppose that the errors $\varepsilon, \varepsilon_1, \varepsilon_2 \ldots$ are so small that the differences between the observed values $x, x_1, x_2 \ldots$ and the true values are of the same order as the quantities $x', x_1', x_2' \ldots$, which will always be the case where proper care has been taken to reduce the accidental errors of observation to their smallest amount. If the given function is implicit, as

$$0 = f\,(X, x, x_1, x_2 \ldots)$$

we should still by differentiation obtain the differential coefficients, and then find the mean error of $X$ by (34).

Example.—The local apparent time at a place in latitude $\varphi = 38° \; 58' \; 53''$ was found (Vol. I. Art. 145) from the sun's zenith distance $\zeta = 73° \; 12' \; 25''$, when the declination was $\delta = -\,22° \; 50' \; 27''$, to be $t = 2^h \; 47^m \; 39^s.4$. What is the probable error of this result, supposing the probable errors of the data to be—

| | | |
|---|---|---|
| Probable error of | $\varphi = r$ | $= 0''.5$ |
| " " | $\delta = r_1$ | $= 0\;.6$ |
| " " | $\zeta = r_2$ | $= 3\;.5$ |

The formula

$$0 = -\cos\zeta + \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos t$$

expresses $t$ as an implicit function of $\varphi$, $\delta$, and $\zeta$. We find (Vol. I. Art. 35)

$$\frac{dt}{d\varphi} = -\frac{1}{\cos\varphi\tan A}$$

$$\frac{dt}{d\delta} = \frac{1}{\cos\delta\tan q}$$

$$\frac{dt}{d\zeta} = \frac{1}{\cos\varphi\sin A}$$

where $A$ is the azimuth and $q$ the parallactic angle. We find from the data $A = + 40° \ 1'$, $q = 32° \ 51'$, whence

$$\frac{dt}{d\varphi} = - 1.532, \qquad \frac{dt}{d\delta} = 1.680, \qquad \frac{dt}{d\zeta} = + 2.001$$

and the probable error of $t$ is, by (34*)

$$R = \sqrt{[(0.5 \times 1.532)^2 + (0.6 \times 1.680)^2 + (3.5 \times 2.001)^2]} = 7''.12$$

or, in seconds of time,

$$R = 0^s.47$$

23. To complete this branch of our subject, it is to be observed that the preceding demonstrations apply only to the case where the quantities entering into combination are independent; but when they are merely different functions of the same observed quantities, the above formulæ are incomplete. Let us suppose that we have $X$ and $X'$, different functions of the same observed quantities $x, x_1, x_2, \ldots$, or

$$X = f\ (x, x_1, x_2, \ldots)$$
$$X' = f'(x, x_1, x_2, \ldots)$$

the mean errors of $x, x_1, x_2 \ldots$ being $\varepsilon, \varepsilon_1, \varepsilon_2 \ldots$; and that we wish to find the mean error $E$ of the function,

$$Y = F(X, X')$$

If any single observation of $x, x_1, x_2 \ldots$ is affected by an error $\delta, \delta_1, \delta_2, \ldots$ respectively, the corresponding errors in $X$ and $X'$ will be—

$$\text{Error in } X, \ \Delta = a\delta + a_1\delta_1 + a_2\delta_2 + \ldots$$
$$\text{“} \quad X', \ \Delta' = a'\delta + a_1'\delta_1 + a_2'\delta_2 + \ldots$$

in which $a, a_1, a_2 \ldots$ are the differential coefficients of $X$, and $a', a_1', a_2' \ldots$ the differential coefficients of $X'$, with reference to $x, x_1, x_2, \ldots$ The corresponding error in $Y$ will be

$$\Delta'' = A\Delta + A'\Delta'$$

in which $A$ and $A'$ are the differential coefficients of $Y$ with reference to $X$ and $X'$. The square of the mean error $E$ will be the mean of the squares of all the values of $\Delta''$ which result from all the possible values of $\delta, \delta_1, \delta_2 \ldots.$

Substituting the values of $\Delta$ and $\Delta'$, we have

$$\Delta'' = (Aa + A'a')\,\delta + (Aa_1 + A'a_1')\,\delta_1 + \ldots.$$

which we may briefly express as follows:

$$\Delta'' = \alpha\delta + \beta\delta_1 + \gamma\delta_2 + \ldots.$$

If the number of values of $\Delta''$ is denoted by $m$, the mean of all the values of $\Delta''^2$ will be

$$\frac{[\Delta''^2]}{m} = \alpha^2\frac{[\delta^2]}{m} + \beta^2\frac{[\delta_1^2]}{m} + \gamma^2\frac{[\delta_2^2]}{m} + \ldots$$

$$+ 2\alpha\beta\frac{[\delta\delta_1]}{m} + 2\alpha\gamma\frac{[\delta\delta_2]}{m} + \ldots.$$

In consequence of the various signs of $\delta\delta_1$, $\delta\delta_2$, &c., the mean value of each of these quantities will be zero; and the mean values of $\delta^2$, $\delta_1^2$, &c. are $\varepsilon^2$, $\varepsilon_1^2$, &c. Hence the formula becomes simply

$$E^2 = (Aa + A'a')^2\varepsilon^2 + (Aa_1 + A'a_1')^2\varepsilon_1^2 + \ldots.$$

or

$$\left.\begin{array}{l} E^2 = A^2(a^2\varepsilon^2 + a_1^2\varepsilon_1^2 + \ldots.) + A'^2(a'^2\varepsilon^2 + a_1'^2\varepsilon_1^2 + \ldots.) \\ \quad + 2AA'(aa'\varepsilon^2 + a_1a_1'\varepsilon_1^2 + \ldots.) \end{array}\right\} \quad (35)$$

To illustrate by a very simple example, let

$$X = 2x \qquad X' = 3x$$

and suppose $\varepsilon = 0.1$; then, to find the mean error $E$ of

$$Y = X + X'$$

we cannot take $E = \sqrt{[(0.2)^2 + (0.3)^2]}$ as we should if $X$ and $X'$ were independent, but by the above formula we must take

$$E = \sqrt{[(0.2)^2 + (0.3)^2 + 2\times2\times3\times(0.1)^2]} = 0.5$$

as in fact we find directly, in this simple case, by first substituting in $Y$ the values of $X$ and $X'$

### WEIGHT OF OBSERVATIONS.

24. Observations of the same kind are said to have the same or different weight according as they have the same or different mean (or probable) errors. We assume *a priori* that observations will have the same weight when they are made under precisely the *same circumstances*, including under this designation every thing that can affect the observations; but whether this condition has in any case been realized can only be learned, *a posteriori*, from the mean errors revealed by the observations themselves.

In order to obtain a numerical expression of the weight, let us suppose all our observations to be compared with a standard fictitious observation the mean error of which is any assumed quantity $\varepsilon_1$. Let the actual observations be subject to the mean error $\varepsilon$. Let it require a number $p$ of standard observations to be combined in order to reduce the mean error of their arithmetical mean to that of an actual observation, that is, to $\varepsilon$; or, according to (24), let

$$\varepsilon = \frac{\varepsilon_1}{\sqrt{p}} \qquad \text{or} \qquad p\varepsilon^2 = \varepsilon_1^{\,2} \tag{36}$$

then one of our actual observations is as good, that is, has the same weight, as $p$ standard observations, and the number $p$ may be used to denote that weight. If, in like manner, other observations of the same kind are subject to the mean error $\varepsilon'$, and we have

$$p'\varepsilon'^2 = \varepsilon_1^{\,2}$$

one of these observations has the weight of $p'$ standard observations, and the weights of the observations of the two actual series may be compared by means of the numbers $p$ and $p'$. The weight of the fictitious observation is here the *unit of weight;* but this unit is altogether arbitrary, since it is only the *relative* weights of actual determinations that are to be considered.

It follows immediately, since we have

$$\varepsilon_1^{\,2} = p\varepsilon^2 = p'\varepsilon'^2$$

or

$$\frac{p}{p'} = \frac{\varepsilon'^2}{\varepsilon^2} \tag{37}$$

that *the weights of two observations are reciprocally proportional to the squares of their mean errors.*

The measure of precision (Art. 10) and the weight are to be distinguished from each other: the former varies inversely as the mean error, the latter inversely as the square of this error.

25. *To find the most probable mean of a number of observations of different weights.*—Let $n'$, $n''$, $n'''$ .... be the given observed values; $p'$, $p''$, $p'''$ .... their respective weights. By the preceding definition of the weight, the quantity $n'$ may be considered as the mean of $p'$ observations of the weight unity, $n''$ as the mean of $p''$ observations of the weight unity, &c. We may, therefore, conceive the given series of observed quantities resolved into a series of standard observations, all of equal weight, and then apply to the latter series the principle of the arithmetical mean. The whole number of equivalent standard observations will be $p' + p'' + p''' + \ldots.$; the sum of the $p'$ standard observations will be $p'n'$; the sum of the $p''$ standard observations will be $p''n''$, &c.: hence the desired mean $x_0$ will be

$$x_0 = \frac{p'n' + p''n'' + p'''n''' + \ldots.}{p' + p'' + p''' + \ldots} \tag{38}$$

or, more briefly,

$$x_0 = \frac{[pn]}{[p]} \tag{38*}$$

This formula shows that although the above demonstration implies that $p'$, $p''$, $p'''$ .... are whole numbers, yet any numbers, whole or fractional, may be used which are in the same proportion; for $f$ being any arbitrary factor, whole or fractional, we may write for (38) the following:

$$x_0 = \frac{fp'n' + fp''n'' + fp'''n''' + \ldots.}{fp' + fp'' + fp''' + \ldots.}$$

and then $fp'$, $fp''$, $fp'''$ .... may be regarded as the weights.

The value of $x_0$ is here an arithmetical mean only in the conventional sense implied in the substitution of fictitious observations with uniform weights for the given observations. It may be called the *general mean* or the *probable mean.*

The weight of this general mean, referred to the unit of $p'$, $p''$, .... is $= p' + p'' + p''' + \ldots.$

The mean error of the general mean will be expressed by

$$\varepsilon_0 = \frac{\varepsilon_1}{\sqrt{(p' + p'' + p''' + \ldots)}} = \frac{\varepsilon_1}{\sqrt{[p]}}$$

where $\varepsilon_1$ is the mean error corresponding to the unit of weight.

If $\varepsilon_1$ is not given, we shall have to find it from the observations themselves. Taking the difference between $x_0$ and each of the given quantities, we have the residuals

$$v' = n' - x_0, \qquad v'' = n'' - x_0, \qquad v''' = n''' - x_0, \ldots$$

If $\varepsilon', \varepsilon'', \varepsilon''' \ldots$ are respectively the mean errors of $n', n'', n''', \ldots.$, we shall have, as in Art. 17,

$$\varepsilon'^2 = v'v' + \varepsilon_0^2$$

whence

$$p'\varepsilon'^2 = \varepsilon_1^2 = p'v'v' + p'\varepsilon_0^2$$

and, in like manner,

$$\varepsilon_1^2 = p''v''v'' + p''\varepsilon_0^2$$
$$\varepsilon_1^2 = p'''v'''v''' + p'''\varepsilon_0^2$$
&c.

The number of given values $n', n'' \ldots$ being $= m$, the sum of these equations is

$$m\varepsilon_1^2 = [pvv] + [p]\,\varepsilon_0^2$$

which combined with the above value of $\varepsilon_0$ gives

$$\varepsilon_1 = \sqrt{\left(\frac{[pvv]}{m-1}\right)} \tag{39}$$

and consequently, also,

$$\varepsilon_0 = \sqrt{\left(\frac{[pvv]}{(m-1)\,[p]}\right)} \tag{40}$$

Example.—Let us suppose that the observations of Saturn's ring in Art. 17 had been given as in the following table, where the mean of the first seven observations of Art. 17 is given $= 39''.179$ with the weight $= 7$, the mean of the next following four $= 39''.285$ with the weight $= 4$, &c.

| $p$ | $n$ | $v$ | $vv$ | $pvv$ |
|---|---|---|---|---|
| 7 | 39″.179 | −0″.129 | .016641 | .1165 |
| 4 | .285 | −0 .023 | 529 | 21 |
| 5 | .294 | −0 .014 | 196 | 10 |
| 4 | .407 | +0 .099 | 9801 | 392 |
| 1 | .410 | +0 .102 | 10404 | 104 |
| 3 | .320 | +0 .012 | 144 | 4 |
| 3 | .377 | +0 .069 | 4761 | 143 |
| 4 | .310 | +0 .002 | 4 | 0 |
| 3 | .127 | −0 .181 | 32761 | 983 |
| 6 | .448 | +0 .140 | 19600 | 1176 |
| $[p] = 40$ | $x_0 = 39$ .308 | | | $[pvv] = .3998$ |

Here the general mean $x_0$ found by (38) of course agrees with that found before. For the mean error corresponding to the unit of weight (which in this case is that of an observation as given in Art. 17), we have, by (39), since $m = 10$,

$$\varepsilon_1 = \sqrt{\left(\frac{.3998}{9}\right)} = 0''.211$$

and for the mean error of $x_0$, by (40),

$$\varepsilon_0 = \sqrt{\left(\frac{.3998}{9 \times 40}\right)} = 0''.033$$

which agree sufficiently well with the former values. A perfect agreement in the mean errors is not to be expected, since our formulæ are based upon the supposition that we have taken a sufficient number of observations to exhibit the several errors to which they are subject in the proportion of their respective probabilities; and this would require a very large number of observations.

26. In the application of the preceding formulæ, it must be observed that when the weights of different determinations of the same quantity are inferred from their mean errors, we must be certain that there are no constant errors (that is, constant during the observations which compose a single determination) before we can combine them together according to these weights, unless the constant errors are known to affect all the determina-

tions equally and with the same sign. For example, if ten measures of the zenith distance of a star are made at one culmination, giving a mean error of $0''.4$, and five measures at another, giving a mean error of $0''.8$, the weights according to these errors would be as 4 to 1. But if it is known that the errors *peculiar to a culmination* (and affecting equally all the individual observations at that culmination) exceed $1''$, it would be better to regard the observations as of the same weight, since there would be a greater probability of eliminating such peculiar errors by taking the simple arithmetical mean. If, however, the observer, from considerations independent of the observations, can estimate the weight of determinations made under different circumstances, then it is evident that these weights will serve for the combination, if the mean accidental errors of the several determinations are sensibly equal.

But if from the different circumstances we have deduced weights for the several determinations, and at the same time the mean errors (deduced from a discussion of the discrepancies of the observations composing each determination) are widely different, it is not easy to assign any general rule for reducing the weights which shall not be subject to some exceptions. In such cases, practical observers and computers have resorted to empirical formulæ, involving some arbitrary considerations, more or less plausible.

In many cases we can proceed satisfactorily as follows. Let

$\varepsilon =$ the mean accidental error of a single observation,
$\eta =$ the mean error peculiar to a determination which rests upon $m$ such observations,
$e =$ the total mean error of such a determination,

then, $\varepsilon$ and $\eta$ being supposed to be independent, we shall have

$$e^2 = \frac{\varepsilon^2}{m} + \eta^2 \tag{41}$$

If then $\eta$ can be obtained from independent considerations, this formula will give the value of $e$, and, consequently, the weight for each determination, and the combination may then be made by (38). For an example of a discussion according to these principles, see Vol. I. Art. 236.

## INDIRECT OBSERVATIONS.

27. I proceed now to the application of the method of least squares to the solution of the general problem of determining the most probable values of any number of unknown quantities of which the observed quantities are functions. The observations are then said to be *indirect*. The particular case of direct observations, already considered, is, however, included in this general problem; being the case in which the number of unknown quantities is reduced to one, and this one is directly observed.

The general problem embraces two classes of problems, which must be distinguished from each other. In the first class, the unknown quantities are *independent*, in the sense that they are subject to no conditions except those established by the observations: so that, *before taking the observations*, any assumed system of values of these quantities has the same probability as any other system. In the second class, there are assigned, *a priori*, certain conditions which the unknown quantities must satisfy at the same time that they satisfy (as nearly as possible) the conditions established by the observations. Thus, for example, if the three angles of a plane triangle are to be determined from observations of any kind, we have, *a priori*, the condition that the sum of these angles must be equal to two right angles, and all the systems of values which do not satisfy this condition are excluded at the outset. This class will be briefly considered hereafter, under the head of "conditioned observations;" but our attention will be chiefly directed to the first class, which includes most of the problems occurring in astronomical inquiries.

Again, the equations which the observations are to satisfy may be *linear* or *non-linear;* the observed quantities may be *explicit* or *implicit* functions of the required quantities; but, for simplicity, we consider first the case of linear equations, to which all the others may always be reduced.

## EQUATIONS OF CONDITION FROM LINEAR FUNCTIONS.

28. Let us suppose the equations between the known and unknown quantities are of the form

$$ax + by + cz + \ldots\ldots + l = V$$

in which $a, b, c, \ldots. l$ are known quantities given by theory for each observation, $V$ is the quantity observed, and $x, y, z \ldots.$ are the quantities to be determined. For each observation, we have a similar equation, and thus a system such as the following:

$$\left.\begin{array}{l} a'x + b'y + c'z + \ldots\ldots + l' = V' \\ a''x + b''y + c''z + \ldots\ldots + l'' = V'' \\ a'''x + b'''y + c'''z + \ldots\ldots + l''' = V''' \\ \quad \&c. \qquad\qquad\qquad \&c. \end{array}\right\} \quad (42)$$

the number of these equations being greater than that of the unknown quantities (Art. 6). If our observations were perfect, all these equations would be satisfied by the same system of values of $x, y, z\ldots$; but, being imperfect, let $M', M'', M'''\ldots\ldots$ denote the values obtained by observation for $V', V'', V'''\ldots.$ When these values are substituted in the second members of (42), there will, in general, be no system of values of $x, y, z\ldots.$ which satisfies all the equations at the same time, and we can only determine that system which is rendered most probable by the observations. Let us therefore denote by $N', N'', N'''\ldots.$ the values which the first members of our equations obtain when any hypothetical or assumed system of values of $x, y, z\ldots.$ is substituted in them; and put

$$v' = N' - M', \qquad v'' = N'' - M'', \qquad v''' = N''' - M''', \ldots.$$

then $v', v'', v'''\ldots.$ are the errors of the observations according to this hypothesis. Finally, let us put

$$n' = l' - M', \qquad n'' = l'' - M'', \qquad n''' = l''' - M''', \ldots.$$

then our equations may be thus expressed:

$$\left.\begin{array}{l} a'x + b'y + c'z + \ldots. + n' = v' \\ a''x + b''y + c''z + \ldots. + n'' = v'' \\ a'''x + b'''y + c'''z + \ldots. + n''' = v''' \\ \quad \&c. \qquad\qquad\qquad \&c. \end{array}\right\} \quad (43)$$

If our observations were perfect, we should be able to find values of $x, y, z\ldots.$ which would reduce all the quantities $v', v''$, $v'''\ldots.$ to zero. It is usual, therefore, to write zero in the second members:

$$\left.\begin{array}{l} a'x + b'y + c'z + \ldots. + n' = 0 \\ a''x + b''y + c''z + \ldots. + n'' = 0 \\ a'''x + b'''y + c'''z + \ldots. + n''' = 0 \\ \quad \&c. \qquad\qquad\qquad \&c. \end{array}\right\} \quad (43^*)$$

and these are called the *equations of condition*, since they express the conditions which the unknown quantities are required to satisfy as nearly as possible. We may, however, with more rigor regard (43) as our equations of condition, and treat them as expressing the general condition that the unknown quantities shall be such as to give the most probable system of errors $v', v'', v''' \ldots$.

Now, according to Art. 11, the most probable system of values of $x, y, z \ldots$ (and, consequently, the most probable system of errors) is that which makes the sum of the squares of the errors a minimum: thus, we are to reduce to a minimum the function

$$[vv] = v'v' + v''v'' + v'''v''' + \ldots .$$

Regarding $[vv]$ as a function of the variables $x, y, z \ldots$ (which we must remember are here independent), the condition of minimum requires that its derivatives taken with reference to each variable shall each be zero; that is,

$$\frac{d\,[vv]}{dx} = 0, \qquad \frac{d\,[vv]}{dy} = 0, \qquad \frac{d\,[vv]}{dz} = 0, \ldots$$

or

$$\left.\begin{array}{l} v'\dfrac{dv'}{dx} + v''\dfrac{dv''}{dx} + v'''\dfrac{dv'''}{dx} + \ldots . = 0 \\ v'\dfrac{dv'}{dy} + v''\dfrac{dv''}{dy} + v'''\dfrac{dv'''}{dy} + \ldots . = 0 \\ v'\dfrac{dv'}{dz} + v''\dfrac{dv''}{dz} + v'''\dfrac{dv'''}{dz} + \ldots . = 0 \\ \qquad\qquad \&c. \end{array}\right\} \qquad (44)$$

(which we might have obtained directly from (10) by substituting $\varphi'\Delta = k\Delta = kv$, and dividing by the constant $k$). But, by differentiating the equations (43) with reference to $x, y, z \ldots$ successively, we have

$$\begin{array}{lll} \dfrac{dv'}{dx} = a', & \dfrac{dv'}{dy} = b', & \dfrac{dv'}{dz} = c', \ldots \\ \dfrac{dv''}{dx} = a'', & \dfrac{dv''}{dy} = b'', & \dfrac{dv''}{dz} = c'', \ldots . \\ \quad \&c. & \quad \&c. & \quad \&c. \end{array}$$

so that (44) are the same as the following:

$$\left.\begin{array}{l} a'v' + a''v'' + a'''v''' + \ldots . = 0 \\ b'v' + b''v'' + b'''v''' + \ldots . = 0 \\ c'v' + c''v'' + c'''v''' + \ldots . = 0 \\ \quad \&c. \end{array}\right\} \quad (44^*)$$

The number of these equations is the same as that of the unknown quantities; and if we now substitute in them the values of $v'$, $v''$, $v'''$ ... from (43), we have the final or, as we shall call them, the *normal* equations, which determine the most probable values of $x$, $y$, $z$....

NORMAL EQUATIONS.

29. We see by (44*) that to form the first normal equation we multiply each of the equations of condition (43) or (43*) by the coefficient of $x$ in that equation, and then form the sum of all the equations thus multiplied. The resulting equation is called the normal equation in $x$.* The sum of the equations of condition severally multiplied by the coefficients of $y$ is the normal equation in $y$, &c. To abbreviate the expression of these sums, we put

$$\begin{array}{l} [aa] = a'a' + a''a'' + a'''a''' + \ldots . \\ [ab] = a'b' + a''b'' + a'''b''' + \ldots . \\ [ac] = a'c' + a''c'' + a'''c''' + \ldots . \\ \quad \&c. \qquad \&c. \end{array}$$

then the normal equations are

$$\left.\begin{array}{l} [aa]\, x + [ab]\, y + [ac]\, z + \ldots . + [an] = 0 \\ [ab]\, x + [bb]\, y + [bc]\, z + \ldots . + [bn] = 0 \\ [ac]\, x + [bc]\, y + [cc]\, z + \ldots . + [cn] = 0 \\ \quad \&c. \qquad \&c. \end{array}\right\} \quad (45)$$

30. The formation of such normal equations is one of the most laborious parts of the computations involved in the method of least squares, especially when the number of equations is very great. It is important to have a means of verification, or "control," to insure their accuracy, before proceeding with the next important process of elimination. A very simple and effective control is the following.

* The "normal equation in $x$" is so called because it is the equation which determines the most probable value of $x$ when the other variables are reduced to zero, or when $x$ is the only unknown quantity; and so of the others.

Form the sums of the coefficients of the unknown quantities in the several equations, namely,

$$\left.\begin{array}{l} a' + b' + c' + \ldots\ldots = s' \\ a'' + b'' + c'' + \ldots\ldots = s'' \\ a''' + b''' + c''' + \ldots\ldots = s''' \\ \&c. \end{array}\right\} \qquad (46)$$

If we multiply each of these by its $n$, and add the products, we have

$$[an] + [bn] + [cn] + \ldots\ldots = [sn] \qquad (47)$$

Also, multiplying each of (46) by its $a$, and adding, then each by its $b$, and adding, and so on, we have

$$\left.\begin{array}{l} [aa] + [ab] + [ac] + \ldots\ldots = [as] \\ [ab] + [bb] + [bc] + \ldots\ldots = [bs] \\ [ac] + [bc] + [cc] + \ldots\ldots = [cs] \\ \&c. \end{array}\right\} \qquad (48)$$

The equations (47) must be satisfied when the absolute terms of the normal equations are correct, and (48) when the coefficients of the unknown quantities are correct.

31. The normal equations will give determinate values of $x, y, z \ldots\ldots$, provided they are really independent. If, however, any two of them become identical by the multiplication of either of them by a constant, the number of independent equations is, in fact, one less than that of the unknown quantities, and the problem becomes indeterminate. This difficulty does not arise from the method by which the normal equations are formed, but from the nature of the given equations of condition. In any such case, additional observations are necessary, for which the coefficients have such varied values as to lead to independent equations. Even when two equations cannot be reduced precisely to a single one by the introduction of a constant factor, if they can be made very nearly identical, the problem is still practically indeterminate. The indetermination will become evident in the actual elimination in practice when any one of the unknown quantities comes out with so small a coefficient that small errors in the observations would greatly change this coefficient. (See Art. 52.)

32. By whatever method the elimination is performed, we shall necessarily arrive at the same final values of the unknown quantities; but, when the number of equations is considerable, the method of substitution, with GAUSS's convenient notation, is universally followed; but, for the present, leaving the reader to choose his method, I proceed to explain the principles by which the mean errors of the values of $x$, $y$, $z$ .... are determined.

### MEAN ERRORS AND WEIGHTS OF THE UNKNOWN QUANTITIES.

33. Since we have put $n' = l' - M'$, $n'' = l'' - M''$, &c. (Art. 28), the mean error of $n', n'', n'''$.... is also that of $M', M'', M'''$,....; that is, the mean error of $n', n'', n'''$.... is to be regarded as the mean error of an observation. If the elimination of the normal equations were fully carried out, each unknown quantity would be finally expressed as a linear function of $n'$, $n''$, $n'''$,...., and the mean errors of the latter being given, those of the unknown quantities would follow by the principle of Art. 20. It results, however, from the symmetry of the normal equations that several forms may be obtained for computing directly the weights of the unknown quantities, and from these weights the mean errors can afterwards be found.

34. *First method of computing the weights of the unknown quantities.*—For simplicity, let us first suppose all the observations to be of equal weight, or the mean errors of $n'$, $n''$, $n'''$ to be equal. Let

$\varepsilon =$ the mean error of an observation,
$\varepsilon_x =$ the mean error of the value of $x$ found from the normal equations,
$p_x =$ the weight of the value of $x$, the weight of an observation being unity;

then (Art. 24)

$$p_x = \frac{\varepsilon^2}{\varepsilon_x^2}$$

Now, let us suppose the elimination to be performed by the method of indeterminate coefficients. Let the first equation of (45) be multiplied by $Q$, the second by $Q'$, the third by $Q''$, &c., and the products added. Then let the factors $Q$, $Q'$, $Q''$.... (whose number is the same as that of the unknown quantities) be supposed to be determined so that in this final equation the coefficients of all the unknown quantities shall be zero, except that of $x$, which shall be unity. The conditions for determining these factors are, therefore,

$$\left.\begin{array}{l}[aa]\,Q + [ab]\,Q' + [ac]\,Q'' + \ldots. = 1 \\ [ab]\,Q + [bb]\,Q' + [bc]\,Q'' + \ldots. = 0 \\ [ac]\,Q + [bc]\,Q' + [cc]\,Q'' + \ldots. = 0 \\ \quad \&c. \qquad\qquad\qquad \&c.\end{array}\right\} \quad (49)$$

and the final equation in $x$ is

$$x + [an]\,Q + [bn]\,Q' + [cn]\,Q'' + \ldots. = 0 \qquad (50)$$

Comparing (45) and (49), we see that the coefficients of $Q, Q', Q''\ldots.$ are the same as those of $x, y, z\ldots.$, but that the absolute terms are $-1$ in (49) instead of $[an]$ in (45), and zero instead of $[bn]$, $[cn]$, &c. Hence, if the elimination of (45) were carried out, and the values of $x, y, z\ldots.$ determined in terms of $n', n'', n'''\ldots.$, the values of $Q, Q', Q''\ldots.$ would be found from these by merely putting $[an] = -1$, and $[bn] = [cn]$, &c. $= 0$. This is also evident from (50). I shall now show that $Q$ is the reciprocal of the required weight of $x$.

The final value of $x$ being a linear function of $n', n'', n'''\ldots.$, the equation (50) may be supposed to be developed in the form

$$x + \alpha' n' + \alpha'' n'' + \alpha''' n''' + \ldots. = 0 \qquad (51)$$

in which $\alpha', \alpha'', \alpha'''\ldots.$ are functions of $a', b', \ldots., a'', b'', \ldots.$, &c.; and these functions are immediately found by developing $[an]$, $[bn]$, &c., in (50); for we then have, by comparing the coefficients of (50) and (51),

$$\left.\begin{array}{l}\alpha' = a'\,Q + b'\,Q' + c'\,Q'' + \ldots. \\ \alpha'' = a''\,Q + b''\,Q' + c''\,Q'' + \ldots. \\ \alpha''' = a'''\,Q + b'''\,Q' + c'''\,Q'' + \ldots. \\ \qquad \&c. \qquad\qquad \&c.\end{array}\right\} \quad (52)$$

Multiplying each of these equations by its $a$, and adding all the products, we obtain, by (49),

$$a'\alpha' + a''\alpha'' + a'''\alpha''' + \ldots. = 1$$

Multiplying each of (52) by its $b$, and adding, we obtain, by (49),

$$b'\alpha' + b''\alpha'' + b'''\alpha''' + \ldots. = 0$$

and so on for as many equations as there are unknown quantities. These relations are briefly expressed thus:

$$[a\alpha] = 1 \qquad [b\alpha] = 0 \qquad [c\alpha] = 0, \&c. \qquad (53)$$

If, then, each of (52) is multiplied by its $\alpha$, and the results are added, we find, by (53),

$$[\alpha\alpha] = \alpha'^2 + \alpha''^2 + \alpha'''^2 + \ldots . = Q \qquad (54)$$

But, by Art. 20, when $\varepsilon$ is the mean error of each of the quantities $n', n'', n''', \ldots .$, the mean error of $x$ found by (51) is

$$\varepsilon_x = \varepsilon \sqrt{[\alpha\alpha]}$$

Hence

$$p_x = \frac{\varepsilon^2}{\varepsilon_x^2} = \frac{1}{[\alpha\alpha]} = \frac{1}{Q} \qquad (55)$$

as was to be proved.

Hence we have a first method of finding the weights. *In the first normal equation write* $-1$ *for the absolute term* $[an]$, *and in the other equations zero for each of the absolute terms* $[bn]$, $[cn]$, &c.; *the value of* $x$ *then found from these equations will be the reciprocal of the weight of the value of* $x$ *found by the general elimination.*

This rule is to be applied to each of the unknown quantities in succession, so that the reciprocal of the weight of $y$ is that value of $y$ which will be found by putting $[bn] = -1$, and $[an] = [cn] = \text{\&c.} = 0$; the reciprocal of the weight of $z$ is that value of $z$ which will be found by putting $[cn] = -1$, and $[an] = [bn]$, &c. $= 0$; &c.

It is evident, moreover, that although we have deduced the rule by the use of indeterminate multipliers, it must hold good whatever method of elimination is adopted.

35. *Second method of computing the weights of the unknown quantities.*—If we write the normal equations thus,

$$\begin{aligned}
[aa]\, x + [ab]\, y + [ac]\, z + \ldots . + [an] &= A \\
[ab]\, x + [bb]\, y + [bc]\, z + \ldots . + [bn] &= B \\
[ac]\, x + [bc]\, y + [cc]\, z + \ldots . + [cn] &= C \\
\text{\&c.} \qquad\qquad\qquad & \text{\&c.}
\end{aligned}$$

and perform the elimination, we shall obtain $x, y, z \ldots .$ in terms of $[aa]$, $[ab]$, &c., and of $A$, $B$, $C$, &c.; and if in the general values thus found we make $A = B = C$, &c. $= 0$, these values will be reduced to those which would be found by carrying out the elimination with zero in the second members of the normal equations. If we suppose the elimination performed by means of the indeterminate factors $Q$, $Q'$, $Q''$.... already employed, the final equation for determining $x$ will be

$$x + [an]Q + [bn]Q' + [cn]Q'' + \dots\dots = QA + Q'B + Q''C + \dots\dots$$

where the coefficient of $A$ is the reciprocal of the required weight of $x$. But, whatever method of elimination is employed, the coefficient of $A$ in this general value of $x$ will necessarily be the same: and hence we derive the second method of determining the weights: *Write $A$, $B$, $C$, &c., instead of 0, in the second members of the normal equations, and carry out the elimination* (by any method at pleasure); *then the final values of $x$, $y$, $z$.... are those terms in the general values which are independent of $A$, $B$, $C$....; the weight of $x$ is the reciprocal of the coefficient of $A$ in the general value of $x$; the weight of $y$ is the reciprocal of the coefficient of $B$ in the general value of $y$*; &c.

36. *Third method of computing the weights of the unknown quantities.*—Let us suppose the elimination to be performed by the method of substitution, still retaining $A$, $B$, $C$.... in the second members, as in the preceding article. The final equation in $x$, according to this method, is found by substituting in the first normal equation the values of $y$, $z$.... given by the other equations. These substitutions do not affect the coefficient of $A$, which remains unity, so long as no reduction is made after the substitutions. Thus, the final equation in $x$ is of the form

$$Rx = T + A + \text{terms in } B, C, \dots\dots$$

in which $T$ is the sum of all the absolute quantities resulting from the substitution, and is a function of $[aa]$, $[ab]$,....$[an]$. Hence the value of $x$ is

$$x = \frac{T}{R} + \frac{A}{R} + \text{terms in } B, C, \dots\dots$$

in which $\frac{T}{R}$ is the final value of $x$ which results when $A = B = C \dots = 0$, and $\frac{1}{R}$ is necessarily the quantity denoted by $Q$ in the preceding articles. Therefore $R$ is the weight of $x$, and hence we have a third method of finding the weights: *Let the first normal equation* (the equation in $x$, Art. 29) *be taken as the final equation for determining $x$, and substitute in it the values of $y$, $z$.... in* *terms of x as found from the remaining equations; then, before freeing the equation of fractions or introducing any reduction factor, the coefficient of x in this equation is the weight of the value of x. In the same manner, substitute in the second normal equation* (the equation in $y$) *the values of x, z.... in terms of y as found from the other equations; the coefficient of y is then the weight of the value of y; and so proceed for each unknown quantity.*

According to this method we determine each unknown quantity, *together with its weight*, by a separate elimination carried through all the equations, in each case changing the order of elimination, until every unknown quantity has been made to come out the last. The algorithm of this process, with GAUSS's convenient system of notation, will be given hereafter (Art. 45).

37. *To find the mean error of observation.*—The weight of $x$ being found, we have the ratio of $\varepsilon_x$ to $\varepsilon$, but we have yet to determine $\varepsilon$, which, in general, cannot be assigned *a priori*, but must be deduced *a posteriori*, that is, from the observations, and consequently from the equations of condition. The residuals $v', v'', v'''$...., in (43), are those which result when the most probable values of $x, y, z$.... (namely, those resulting from the normal equations) are substituted in the first members. The actual or *true* errors (Art. 17) of observation are, however, those values of the first members of (43) which result when the *true* values of $x, y, z$,.... are substituted.

Let $x + \Delta x$, $y + \Delta y$, $z + \Delta z$,.... be the true values which, substituted in the equations of condition, give the true residuals $u', u'', u'''$....; so that we have

$$\left.\begin{array}{l} a'\,(x+\Delta x) + b'\,(y+\Delta y) + c'\,(z+\Delta z) + \ldots . n' = u' \\ a''\,(x+\Delta x) + b''\,(y+\Delta y) + c''\,(z+\Delta z) + \ldots . n'' = u'' \\ a'''(x+\Delta x) + b'''(y+\Delta y) + c'''(z+\Delta z) + \ldots . n''' = u''' \\ \qquad\qquad \&c. \qquad\qquad\qquad\qquad \&c. \end{array}\right\} \quad (56)$$

If these equations be multiplied by $a', a'', a'''$...., respectively, the sum of the products is

$$\left.\begin{array}{l} [aa]\,x + [ab]\,y + [ac]\,z + \ldots . + [an] \\ + [aa]\,\Delta x + [ab]\,\Delta y + [ac]\,\Delta z + \ldots . \end{array}\right\} = [au]$$

which by the first of (45) is reduced to

$$[aa]\,\Delta x + [ab]\,\Delta y + [ac]\,\Delta z + \ldots . - [au] = 0$$

In the same manner, multiplying each of the equations (56) by its $b$, $c$, &c., successively, we form the other equations of the following group:

$$\left.\begin{array}{l}[aa]\,\Delta x + [ab]\,\Delta y + [ac]\,\Delta z + \ldots. - [au] = 0 \\ [ab]\,\Delta x + [bb]\,\Delta y + [bc]\,\Delta z + \ldots. - [bu] = 0 \\ [ac]\,\Delta x + [bc]\,\Delta y + [cc]\,\Delta z + \ldots. - [cu] = 0 \\ \quad\&c. \qquad\qquad \&c.\end{array}\right\} \qquad (57)$$

These being of the same form as the normal equations (45), we see that the value of $\Delta x$ resulting from them will be of the same form as that of $x$ resulting from (45), with only the substitution of $-u$ for $n$: hence, by (51),

$$\Delta x - \alpha' u' - \alpha'' u'' - \alpha''' u''' - \ldots. = 0 \qquad (58)$$

Again, multiplying (56) by $v', v'', v'''\ldots.$, respectively, the sum of the products is, by (44*), reduced to

$$[vn] = [vu]$$

and in the same manner, from (43),

$$[vn] = [vv]$$

whence

$$[vu] = [vv] = [vn] \qquad (59)$$

The sum of the products obtained by multiplying the equations (43) respectively by $u', u'', u'''\ldots.$ is

$$[au]\,x + [bu]\,y + [cu]\,z + \ldots. + [nu] = [vu] = [vv]$$

and from (56), in the same manner,

$$\left.\begin{array}{l}[au]\,x \;\; + [bu]\,y \;\; + [cu]\,z \;\; + \ldots. + [nu] \\ + [au]\,\Delta x + [bu]\,\Delta y + [cu]\,\Delta z + \ldots.\end{array}\right\} = [uu]$$

which two equations give

$$[uu] = [vv] + [au]\,\Delta x + [bu]\,\Delta y + [cu]\,\Delta z + \ldots. \qquad (60)$$

Now, $[uu]$ being the sum of the squares of the true errors of the observations, its value is, as in Art. 17, $= m\varepsilon\varepsilon$, if we put

$m =$ the number of observations,
$\quad =$ the number of equations of condition.

Consequently, if we could assume $\Delta x$, $\Delta y$.... to vanish, we should have

$$\varepsilon\varepsilon = \frac{[vv]}{m}$$

and this will usually give a close approximation to the value of $\varepsilon$, but it will give the true value only in the exceedingly improbable case in which the values of $x$, $y$, $z$.... are absolutely true, whereas they are to be regarded only as the most probable ones furnished by the observations. This formula, then, must always give too small a value of $\varepsilon$, since it ascribes too high a degree of precision to the observations. We must, therefore, add to $[vv]$ the quantities $[au]\Delta x$, $[bu]\Delta y$, &c., as in (60); but, as we cannot assign any other than approximate values of these quantities, let us assume for them their mean values as found by the theory of mean errors. The mean value of $[au]\Delta x$ will be found by multiplying together

$$[au] = a'u' + a''u'' + a'''u''' + \ldots.$$

and

$$\Delta x = \alpha'u' + \alpha''u'' + \alpha'''u''' + \ldots.$$

observing that the errors $u', u'', u'''$...., when we consider only their mean values, are to be regarded as having the double sign $\pm$; so that the mean value of the product will contain only the terms $a'\alpha'u'u'$, $a''\alpha''u''u''$, &c. Hence we take

$$[au]\,\Delta x = a'\alpha'u'u' + a''\alpha''u''u'' + a'''\alpha'''u'''u''' + \ldots.$$

and substituting in this the mean value of $u'u'$, $u''u''$, &c., which in each case is $\varepsilon\varepsilon$, we have

$$[au]\,\Delta x = (a'\alpha' + a''\alpha'' + a'''\alpha''' + \ldots.)\,\varepsilon\varepsilon$$

or, finally, by (53),

$$[au]\,\Delta x = \varepsilon\varepsilon$$

In the same manner, it must follow that $\varepsilon\varepsilon$ is the mean value of each of the terms $[bu]\Delta y$, $[cu]\Delta z$, &c. If then we put

$$\mu = \text{the number of unknown quantities,}$$

the equation (60) becomes

$$m\varepsilon\varepsilon = [vv] + \mu\varepsilon\varepsilon$$

whence

$$\varepsilon\varepsilon = \frac{[vv]}{m - \mu} \qquad \varepsilon = \sqrt{\frac{[vv]}{m - \mu}} \tag{61}$$

It is to be observed that when there is but one unknown quantity, or $\mu = 1$, this general form is reduced to the simple one (25), already given for direct observations.

Finally, $p_x$, $p_y$, $p_z$.... denoting the weights of $x, y, z$.... found by any of the preceding methods, we have

$$\varepsilon_x = \frac{\varepsilon}{\sqrt{p_x}} \qquad \varepsilon_y = \frac{\varepsilon}{\sqrt{p_y}}, \text{ \&c.} \tag{62}$$

38. Example.—Let us suppose the following very simple equations of condition to be given :*

$$\begin{aligned} x - y + 2z - 3 &= 0 \\ 3x + 2y - 5z - 5 &= 0 \\ 4x + y + 4z - 21 &= 0 \\ - x + 3y + 3z - 14 &= 0 \end{aligned}$$

If but the first three of these equations had been given, the problem would have been determinate. We should find from them $x = \frac{18}{7}$, $y = \frac{23}{7}$, $z = \frac{13}{7}$, and we should have to accept these values as final ones, with no means of judging of their accuracy, or of that of the observations upon which the equations are supposed to depend. A fourth observation having given us our fourth equation, we find that the values of $x, y, z$ derived from the first three will not satisfy it, for when they are substituted in it the first member becomes $-\frac{8}{7}$, instead of zero. If we determine the values of $x$, $y$, and $z$ from any three of the equations, and substitute these values in the fourth, we shall find a residual. Each one of the four systems of values of the unknown quantities thus found satisfies three equations exactly, and the fourth approximately; but, all the observations being subject to error, the most probable system of values can seldom satisfy any one of the equations exactly. Hence the necessity of a principle of computation which shall lead as directly as possible to such a probable system of values; and this principle is furnished by the method of least squares.

* Gauss, *Theoria Motus*, Art. 184.

We are, then, by Art. 29, to deduce from these four equations three normal equations, and the values of $x$, $y$, $z$ which exactly satisfy these are to be regarded as the most probable values.

To form the first normal equation, we multiply the first of the above equations of condition by 1 $(=a')$, the second by 3 $(=a'')$, the third by 4 $(=a''')$, and the fourth by $-1$ $(=a^{iv})$, and add the products. We thus find $[aa]=27$, $[ab]=6$, $[ac]=0$, and $[an]=-88$.

To form the second normal equation, we multiply the first equation of condition by $-1$ $(=b')$, the second by 2 $(=b'')$, the third by 1 $(=b''')$, and the fourth by 3 $(=b^{iv})$, and add the products. We thus find $[ab]=6$, $[bb]=15$, $[bc]=1$, $[bn]=-70$.

The third normal equation is formed by multiplying the first equation of condition by 2 $(=c')$, the second by $-5$ $(=c'')$, the third by 4 $(=c''')$, and the fourth by 3 $(=c^{iv})$, and adding the products. We find $[ac]=0$, $[bc]=1$, $[cc]=54$, $[cn]=-107$.

Hence our normal equations are

$$\begin{aligned} 27x + 6y \qquad - 88 &= 0 \\ 6x + 15y + z - 70 &= 0 \\ y + 54z - 107 &= 0 \end{aligned}$$

the solution of which gives, as the most probable values,

$$x = \frac{49154}{19899} = 2.470$$

$$y = \frac{2617}{737} = 3.551$$

$$z = \frac{12707}{6633} = 1.916$$

In order to determine the mean, and hence also the probable, errors of these values, let us first determine their weights according to the preceding methods.

*First.* By the method of Art. 34, we first write $-1, 0, 0$, for the absolute terms of the three normal equations, and we have the three equations for determining the weight of $x$,

$$\begin{aligned} 27x' + 6y' - 1 &= 0 \\ 6x' + 15y' + z' &= 0 \\ y' + 54z' &= 0 \end{aligned}$$

in which accents are employed to distinguish the particular values from the above general ones. These give

$$x' = \frac{809}{19899}$$

which is the reciprocal of the required weight. Hence,

$$p_x = \frac{19899}{809} = 24.597$$

In a similar manner, to find the weight of $y$, we take the equations

$$\begin{aligned} 27x'' + 6y'' \quad\quad\quad\quad &= 0 \\ 6x'' + 15y'' + z'' - 1 &= 0 \\ y'' + 54z'' \quad\quad &= 0 \end{aligned}$$

and find

$$y'' = \frac{54}{737}$$

whence

$$p_y = \frac{737}{54} = 13.648$$

And to find the weight of $z$, the equations

$$\begin{aligned} 27x''' + 6y''' \quad\quad\quad\quad &= 0 \\ 6x''' + 15y''' + z''' \quad\quad &= 0 \\ y''' + 54z''' - 1 &= 0 \end{aligned}$$

which give

$$z''' = \frac{41}{2211}$$

and

$$p_z = \frac{2211}{41} = 53.927$$

*Secondly.* By the method of Art. 35, we write our normal equations thus:

$$\begin{aligned} 27x + 6y \quad\quad\quad - 88 &= A \\ 6x + 15y + z - 70 &= B \\ y + 54z - 107 &= C \end{aligned}$$

and, carrying out the elimination as if $A$, $B$, and $C$ were known quantities, we find

$$\begin{aligned} 19899x &= 49154 + (809)A - 324B + 6C \\ 737y &= 2617 - 12A + (54)B - C \\ 6633z &= 12707 + 2A - 9B + (123)C \end{aligned}$$

and, therefore,

$$x = \frac{49154}{19899} \text{ with the weight } p_x = \frac{19899}{809}$$

$$y = \frac{2617}{737} \quad \text{"} \quad \text{"} \quad \text{"} \quad p_y = \frac{737}{54}$$

$$z = \frac{12707}{6633} \quad \text{"} \quad \text{"} \quad \text{"} \quad p_z = \frac{6633}{123}$$

the same as by the first method.

*Thirdly*. By the method of Art. 36, to find $x$ and its weight we eliminate $y$ and $z$ from the equation in $x$ (the first normal equation) by means of the other equations, employing successive substitutions. The last normal equation gives

$$z = -\frac{1}{54}y + \frac{107}{54}$$

which being substituted in the second gives

$$6x + \frac{809}{54}y - \frac{3673}{54} = 0$$

The value of $y$ from this, namely,

$$y = -\frac{324}{809}x + \frac{3673}{809}$$

being substituted in the first normal equation, and no reduction being made, gives

$$\frac{19899}{809}x - \frac{49154}{809} = 0$$

where the coefficient of $x$ is the weight, and the value of $x$ is the same as before found.

To find $y$ and its weight, we make the second the final equation. From the first and third we find

$$x = -\frac{6}{27}y + \frac{88}{27}$$

$$z = -\frac{1}{54}y + \frac{107}{54}$$

which substituted in the second give

$$\frac{737}{54}y - \frac{2617}{54} = 0$$

where the coefficient of $y$ is its weight.

Finally, to find $z$ with its weight, we make the third normal equation the final one. From the first two we find

$$y = -\frac{9}{123}z + \frac{454}{123}$$

which substituted in the third gives

$$\frac{6633}{123}z - \frac{12707}{123} = 0$$

where the coefficient of $z$ is its weight, and its value is the same as was before found.

By a little attention, it will be perceived that the three methods involve essentially the same numerical operations.

We are next to find the mean errors of $x$, $y$, and $z$; for which purpose we must first find the mean error of an observation, assuming here, for the sake of illustration, that the absolute terms of the given equations of condition are the observed quantities, and that they are subject to the same mean error. Substituting in these equations the above found values of $x$, $y$, and $z$, we obtain the residuals as follows:

| No. | $v$ | $vv$ |
|---|---|---|
| 1 | − 0.249 | 0.0620 |
| 2 | − 0.068 | .0046 |
| 3 | + 0.095 | .0090 |
| 4 | − 0.069 | .0048 |

$$m = 4, \ \mu = 3, \quad [vv] = 0.0804$$

$$\frac{[vv]}{m - \mu} = 0.0804$$

Hence, by (61),

$$\varepsilon = \sqrt{0.0804} = 0.284$$

which is the mean error of an observation, so far as this error can be inferred from so small a number of observations. (See the next article.) Consequently, the mean errors of $x$, $y$, and $z$ are as follows:

$$\varepsilon_x = \frac{\varepsilon}{\sqrt{p_x}} = 0.057$$

$$\varepsilon_y = \frac{\varepsilon}{\sqrt{p_y}} = 0.077$$

$$\varepsilon_z = \frac{\varepsilon}{\sqrt{p_z}} = 0.039$$

Multiplying these errors by the constant 0.6745, we shall have (Art. 15) the probable errors as follows:

| | | | |
|---|---|---|---|
| Probable | error of an observation | | = 0.192 |
| " | " | $x$ | = 0.038 |
| " | " | $y$ | = 0.052 |
| " | " | $z$ | = 0.026 |

39. It has already been remarked in the foregoing pages, and the remark is especially important in the present connection, that the method of least squares supposes in general a great number of observations to have been taken, or a number sufficiently great to determine approximately the errors to which the observations are liable. Theoretically, the greater the number of observations the more nearly will the series of residuals express the series of actual errors, and, consequently, the more correct will be the value of $\varepsilon$ inferred from these residuals. In practice, therefore, no dependence should be placed upon the mean or probable errors deduced from so small a number of observations as we have employed, for the sake of brevity and clearness, in the preceding example. Nevertheless, the method is, even in this case, the best adapted for determining the most probable values of the unknown quantities deducible from the given observations, and also their relative degree of precision. Thus, in this example, the degrees of precision (denoted by $h$, Art. 10) of $x$, $y$, and $z$, being inversely proportional to the mean errors, or directly proportional to the square roots of the weights, are nearly as the numbers 5, 3.7, and 7.3, so that from the four given observations $z$ is about twice as accurately found as $y$, while the precision of $x$ falls between that of $y$ and $z$. But we can place but little dependence upon the result which assigns 0.284 as the mean error of observation, and 0.057, 0.077, 0.039 as the mean errors of $x$, $y$, and $z$, because this result is derived from too small a number of observations.

### EQUATIONS OF CONDITION FROM NON-LINEAR FUNCTIONS.

40. Let the relation between the observed quantities $V'$, $V''$, $V'''$.... and the unknown quantities $X$, $Y$, $Z$.... be, for the observations severally,

$$\left.\begin{array}{l} f'\,(V',\ X, Y. Z, \ldots\ldots) = 0 \\ f''\,(V'',\ X, Y, Z, \ldots\ldots) = 0 \\ f'''(V''', X, Y, Z, \ldots\ldots) = 0 \\ \qquad\qquad \&c. \end{array}\right\} \qquad (63)$$

Let the values of $V'$, $V''$, $V'''$ ...., found by observation, be $M'$, $M''$, $M'''$ .... These values being substituted, we shall have the equations

$$\left.\begin{array}{l} f' \ (M', \ X, Y, Z, \dots) = 0 \\ f'' \ (M'', \ X, Y, Z, \dots) = 0 \\ f''' \ (M''', X, Y, Z, \dots) = 0 \\ \qquad \&c. \end{array}\right\} \quad (64)$$

from which the values of $X$, $Y$, $Z$.... are to be found. But, as we cannot effect the direct solution of these equations according to the method of least squares so long as they are not linear, we resort to the following indirect process, by which linear equations of condition are formed. Let *approximate* values of $X$, $Y$, $Z$.... be found, either by some independent method or from a sufficient number of the equations (64) treated by any suitable process, and denote these approximate values by $X_0$, $Y_0$, $Z_0$.... Let the most probable values be

$$X = X_0 + x, \qquad Y = Y_0 + y, \qquad Z = Z_0 + z, \dots\dots$$

then $x$, $y$, $z$.... are the corrections required to reduce our approximate values to the most probable values; in other words, $x$, $y$, $z$.... are the most probable corrections of the approximate values, and the method of least squares is now to be applied in finding these corrections.

Substitute the approximate values $X_0$, $Y_0$, $Z_0$... in (63), and find, by resolving the equations, the corresponding values of $V'$, $V''$.... which denote by $V_0'$, $V_0''$.... These will be functions which may be thus generally expressed:

$$\begin{array}{l} V_0' = F' \ (X_0, Y_0, Z_0 \dots) \\ V_0'' = F'' \ (X_0, Y_0, Z_0 \dots) \\ \qquad \&c. \end{array}$$

Now, the values of $V'$, $V''$.... which result when the most probable values $X_0 + x$, $Y_0 + y$, $Z_0 + z$ are substituted, and which are yet unknown, being denoted by $N'$, $N''$.... we have

$$\begin{array}{l} N' = F' \ (X_0 + x, \ Y_0 + y, Z_0 + z, \dots) \\ N'' = F'' \ (X_0 + x, \ Y_0 + y, Z_0 + z, \dots) \\ \qquad \&c. \end{array}$$

and by TAYLOR'S Theorem, when we neglect the higher powers of $x$, $y$, $z$ .... which are supposed to be very small quantities, we have

$$N' = V_0' + \frac{d V_0'}{d X_0} x + \frac{d V_0'}{d Y_0} y + \frac{d V_0'}{d Z_0} z + \ldots.$$
$$N'' = V_0'' + \frac{d V_0''}{d X_0} x + \frac{d V_0''}{d Y_0} y + \frac{d V_0''}{d Z_0} z + \ldots.$$
$$\&c. \qquad \&c.$$

where $\frac{d V_0'}{d X_0}$, $\frac{d V_0''}{d X_0}$, &c., $\frac{d V_0'}{d Y_0}$, $\frac{d V_0''}{d Y_0}$, &c. are simply the values of the derivatives of $V'$, $V''$ .... found by differentiating (63) with reference to each of the variables, and afterwards substituting $X_0$, $Y_0$, &c. for $X$, $Y$, .... &c.

If now we denote the derivatives of $V'$, $V''$ .... with reference to $X$ by $a'$, $a''$ ....; their derivatives with reference to $Y$ by $b'$, $b''$ .... &c.: so that

$$N' = V_0' + a'x + b'y + c'z + \ldots.$$
$$N'' = V_0'' + a''x + b''y + c''z + \ldots.$$
$$\&c. \qquad \&c.$$

and then also put

$$v' = N' - M', \qquad v'' = N'' - M'', \ \&c.$$
$$n' = V_0' - M', \qquad n'' = V_0'' - M'', \ \&c.$$

our equations become

$$a'x + b'y + c'z + \ldots + n' = v'$$
$$a''x + b''y + c''z + \ldots + n'' = v''$$
$$a'''x + b'''y + c'''z + \ldots + n''' = v'''$$
$$\&c. \qquad \&c.$$

in which $a'$, $b'$ ... $a''$, $b''$ ... $n'$, $n''$ ... are all known quantities; and $v'$, $v''$ .... are the residual errors of observation. These equations of condition are precisely like those already treated, and, being solved by the same method, give the most probable values of $x$, $y$, $z$ ...., and hence, also, the most probable values of $X$, $Y$, $Z$ ....

This process rests upon the assumption that the approximate values $X_0$, $Y_0$, $Z_0$ ... are already so nearly correct that the squares of $x$, $y$, $z$ ... may be neglected. But should the values found for $x$, $y$, $z$ ... show that this assumption was not admissible, the computation is to be repeated, starting with the last found values $X_0 + x$, $Y_0 + y$, $Z_0 + z$ ... as the approximate values; and then the corrections which these last require will generally be so small that their higher powers may be neglected without sensible error. However, should this still not be the case, successive approximations, commencing always with the last found values, will at length lead to values which require only corrections suitably small.

Even when the given function is already linear, it is mostly expedient to follow the general method just given: namely, to substitute approximate values and form equations of condition to determine their corrections. This reduces $x$, $y$, $z$ ... to small quantities, greatly simplifies the computations, and diminishes the chance of error.

## TREATMENT OF EQUATIONS OF CONDITION WHEN THE OBSERVATIONS HAVE DIFFERENT WEIGHTS.

41. The process above explained assumes that all the observations are subject to the same mean error, and hence are all of the same weight. The more general case, in which the observations are of different weights, is easily reduced to this simple case. For, let

$$a'x + b'y + c'z + \ldots . + n' = v'$$

be an equation of condition of the weight $p'$; that is, one formed for an observation of the weight $p'$. The mean error of an observation of the weight unity being $\varepsilon_1$, the mean error of the actual observation, and, therefore, also of $n'$, is $\varepsilon' = \frac{\varepsilon_1}{\sqrt{p'}}$. Hence the mean error of $n'\sqrt{p'}$ is, by Art. 20, equal to $\varepsilon'\sqrt{p'}$, that is, equal to $\varepsilon_1$. If, therefore, we multiply the equation by $\sqrt{p'}$, so that we have

$$a'\sqrt{p'} \, . \, x + b'\sqrt{p'} . \, y + c'\sqrt{p'} . \, z + \ldots + n'\sqrt{p'} = v'\sqrt{p'}$$

it becomes an equation in which the mean error of the absolute term is the mean error of an observation of the weight unity. Hence we have only to multiply each equation of condition by the square root of its weight in order to reduce them all to the same unit of weight; after which the normal equations will be found as in other cases.

The mean error of observation, found by (61) from the equations of condition thus transformed, will be that of an observation of the weight unity, and the weights of the unknown quantities will come out with reference to the same unit.

ELIMINATION OF THE UNKNOWN QUANTITIES FROM THE NORMAL EQUATIONS BY THE METHOD OF SUBSTITUTION, ACCORDING TO GAUSS.

42. By means of a peculiar notation proposed by GAUSS, the elimination by substitution is carried on so as to preserve throughout the symmetry which exists in the normal equations. In order to explain this method, it will be expedient to suppose a limited number of unknown quantities. I shall take but *four*, but shall give the process in so general a form that it may readily be extended to any number.

The unknown quantities will be denoted by

$$x, y, z, w,$$

and their coefficients in the equations of condition by

$$a, b, c, d,$$

respectively, with sub-numerals denoting the number of the equation or observation upon which it depends, and by

$$n_1, n_2, n_3, \&c.$$

the absolute terms of the 1st, 2d, 3d, &c. equations respectively: so that the $m$ equations of condition (here supposed to be reduced to the same weight by Art. 41) will be

$$\left.\begin{array}{l} a_1x + b_1y + c_1z + d_1w + n_1 = 0 \\ a_2x + b_2y + c_2z + d_2w + n_2 = 0 \\ a_3x + b_3y + c_3z + d_3w + n_3 = 0 \\ \cdots \qquad \cdots \qquad \cdots \\ \cdots \qquad \cdots \qquad \cdots \\ a_mx + b_my + c_mz + d_mw + n_m = 0 \end{array}\right\} \quad (65)$$

and the four normal equations formed from these are

$$\left.\begin{array}{l} [aa]\,x + [ab]\,y + [ac]\,z + [ad]\,w + [an] = 0 \\ [ab]\,x + [bb]\,y + [bc]\,z + [bd]\,w + [bn] = 0 \\ [ac]\,x + [bc]\,y + [cc]\,z + [cd]\,w + [cn] = 0 \\ [ad]\,x + [bd]\,y + [cd]\,z + [dd]\,w + [dn] = 0 \end{array}\right\} \quad (66)$$

The value of $x$ from the first equation is

$$x = -\frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z - \frac{[ad]}{[aa]}w - \frac{[an]}{[aa]}$$

If this is substituted in the other three equations, we shall preserve the symmetry of the result by the following notation:

$$[bb] - \frac{[ab]}{[aa]}[ab] = [bb.1] \qquad [dd] - \frac{[ad]}{[aa]}[ad] = [dd.1]$$

$$[bc] - \frac{[ab]}{[aa]}[ac] = [bc.1] \qquad [bn] - \frac{[ab]}{[aa]}[an] = [bn.1]$$

$$[bd] - \frac{[ab]}{[aa]}[ad] = [bd.1] \qquad [cn] - \frac{[ac]}{[aa]}[an] = [cn.1]$$

$$[cc] - \frac{[ac]}{[aa]}[ac] = [cc.1] \qquad [dn] - \frac{[ad]}{[aa]}[an] = [dn.1]$$

$$[cd] - \frac{[ac]}{[aa]}[ad] = [cd.1]$$

The three equations thus become

$$\left.\begin{array}{l} [bb.1]y + [bc.1]z + [bd.1]w + [bn.1] = 0 \\ [bc.1]y + [cc.1]z + [cd.1]w + [cn.1] = 0 \\ [bd.1]y + [cd.1]z + [dd.1]w + [dn.1] = 0 \end{array}\right\} \qquad (67)$$

The presence of the numeral 1 is all that distinguishes these from original normal equations in $y$, $z$, and $w$. The elimination of $y$ will, therefore, be effected in the same manner as that of $x$. Thus, from the first, we have

$$y = -\frac{[bc.1]}{[bb.1]}z - \frac{[bd.1]}{[bb.1]}w - \frac{[bn.1]}{[bb.1]}$$

the substitution of which in the other two equations leads to the following notation:

$$[cc.1] - \frac{[bc.1]}{[bb.1]}[bc.1] = [cc.2] \qquad [cn.1] - \frac{[bc.1]}{[bb.1]}[bn.1] = [cn.2]$$

$$[cd.1] - \frac{[bc.1]}{[bb.1]}[bd.1] = [cd.2] \qquad [dn.1] - \frac{[bd.1]}{[bb.1]}[bn.1] = [dn.2]$$

$$[dd.1] - \frac{[bd.1]}{[bb.1]}[bd.1] = [dd.2]$$

and the resulting equations are

$$\left.\begin{array}{l}[cc\,.\,2]\,z + [cd\,.\,2]\,w + [cn\,.\,2] = 0\\ [cd\,.\,2]\,z + [dd\,.\,2]\,w + [dn\,.\,2] = 0\end{array}\right\}\qquad(68)$$

From the first of these we have

$$z = -\frac{[cd\,.\,2]}{[cc\,.\,2]}\,w - \frac{[cn\,.\,2]}{[cc\,.\,2]}$$

which, substituted in the second, leads to the following notation:

$$[dd\,.\,2] - \frac{[cd\,.\,2]}{[cc\,.\,2]}\,[cd\,.\,2] = [dd\,.\,3] \quad\Big|\quad [dn\,.\,2] - \frac{[cd\,.\,2]}{[cc\,.\,2]}\,[cn\,.\,2] = [dn\,.\,3]$$

and the resulting equation is

$$[dd\,.\,3]\,w + [dn\,.\,3] = 0 \qquad (69)$$

whence

$$w = -\frac{[dn\,.\,3]}{[dd\,.\,3]}$$

Having thus found $w$, we substitute its value in the first of (68), and deduce $z$. Then the values of $z$ and $w$ being substituted in the first of (67), we deduce $y$; and finally, substituting the values $y$, $z$, and $w$ in the first of (66), we deduce $x$. These latter substitutions are made in the numerical computation, but it is not necessary to write out here the formulæ which result from the literal substitutions, as it would not facilitate the computation.

It may be observed that all the auxiliaries $[bb\,.\,1]$, $[bc\,.\,1]$, $[cc\,.\,2]$, &c., may be expressed by the general formula

$$[\beta\gamma\,.\,\mu] - \frac{[\alpha\beta\,.\,\mu]}{[\alpha\alpha\,.\,\mu]}\,[\alpha\gamma\,.\,\mu] = [\beta\gamma\,.\,(\mu+1)]$$

$\alpha$, $\beta$, $\gamma$ denoting any three letters, and $\mu$ any numeral.

For the convenience of reference, the final equations employed in the actual computation are brought together as follows, the coefficient of that unknown quantity which is found from each after the substitution of the values of the others being reduced to unity:

$$\left.\begin{aligned}
x + \frac{[ab]}{[aa]}y + \frac{[ac]}{[aa]}z + \frac{[ad]}{[aa]}w + \frac{[an]}{[aa]} &= 0 \\
y + \frac{[bc.1]}{[bb.1]}z + \frac{[bd.1]}{[bb.1]}w + \frac{[bn.1]}{[bb.1]} &= 0 \\
z + \frac{[cd.2]}{[cc.2]}w + \frac{[cn.2]}{[cc.2]} &= 0 \\
w + \frac{[dn.3]}{[dd.3]} &= 0
\end{aligned}\right\} \quad (70)$$

As the number of unknown quantities increases, the number of auxiliaries to be found increases very rapidly. If we include the coefficients and absolute terms of the normal equations, the whole number of auxiliaries is shown in the following scheme:*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| No. of unknown quantities ......... | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| No. of auxiliaries ..................... | 2 | 7 | 16 | 30 | 50 | 77 | 112 | 156 |

43. For the purpose of verification, it is expedient to repeat the elimination in inverse order, commencing with the last normal equation and ending with the first, which will bring out $x$. It will not be necessary to write out the formulæ for this inverse elimination, since when the form for computation has been once prepared, it suffices to place in it the coefficients of the normal equations in inverse order, and then to proceed with the numerical operations precisely as in the first elimination. The unknown quantities coming out in the first elimination in the order $w$, $z$, $y$, $x$, they will in the second come out in the order $x$, $y$, $z$, $w$.

This inversion has also the advantage of giving the weights of all the unknown quantities with the greatest facility, as will hereafter be shown.

44. A very complete final verification, or "control," is obtained as follows. Substitute the values of $x$, $y$, $z$, $w$ in the equations of condition, and thus find the residuals $v_1$, $v_2$, $v_3$ .... $v_m$, or the values which the first members assume. Form the sum

$$[vv] = v_1v_1 + v_2v_2 + v_3v_3 + \ldots + v_mv_m$$

* The number of auxiliaries will be, in general,

$$\frac{i(i+1)(i+5)}{2.3}$$

where $i$ denotes the number of unknown quantities.

which is also required in finding the mean error of observation by (61). Also form the following new auxiliaries:

$$[nn] = n_1 n_1 + n_2 n_2 + n_3 n_3 + \ldots\ldots + n_m n_m$$

$$[nn] - \frac{[an]^2}{[aa]} = [nn.1] \qquad\qquad [nn.2] - \frac{[cn.2]^2}{[cc.2]} = [nn.3]$$

$$[nn.1] - \frac{[bn.1]^2}{[bb.1]} = [nn.2] \qquad\qquad [nn.3] - \frac{[dn.3]^2}{[dd.3]} = [nn.4]$$

then, if the whole computation, both of the normal equations themselves and of the subsequent elimination, is correct, we must have

$$[vv] = [nn.4] \tag{71}$$

To demonstrate this, we observe first that we have already, by (59),

$$[vv] = [vn]$$

If now we go back to the equations of condition, and multiply each by its $n$, the sum of the products is

$$[an]\,x + [bn]\,y + [cn]\,z + [dn]\,w + [nn] = [vn] = [vv]$$

If this equation be annexed as a fifth normal equation to the group (66), and the successive substitutions are made in it as in the others, beginning with $x$, it evidently becomes, successively,

$$\begin{aligned}
[bn.1]\,y + [cn.1]\,z + [dn.1]\,w + [nn.1] &= [vv] \\
[cn.2]\,z + [dn.2]\,w + [nn.2] &= [vv] \\
[dn.3]\,w + [nn.3] &= [vv] \\
[nn.4] &= [vv]
\end{aligned}$$

which last is the same as (71).

### DETERMINATION OF THE WEIGHTS OF THE UNKNOWN QUANTITIES WHEN THE ELIMINATION HAS BEEN EFFECTED BY THE METHOD OF SUBSTITUTION.

45. By the general method explained in Art. 36, the elimination would have to be performed as many times as there are unknown quantities. It is desirable to have more direct methods. When there are but four unknown quantities, we can find their weights from the auxiliaries occurring in two successive eliminations in inverse order. In the first elimination, according to the order $a$, $b$, $c$, $d$, we find $w$ by substitution in the last normal

equation, and, the coefficient of $w$ being then $[dd.3]$, it follows, by Art. 36, that the weight of the value of $w$ is

$$p_w = [dd.3]$$

In the inverse elimination, in the order $d$, $c$, $b$, $a$, the coefficient of $x$ in the final equation, which would be denoted by $[aa.3]$, will be the weight of $x$, or

$$p_x = [aa.3]$$

Now, if a third elimination were carried out in the order $x$, $y$, $w$, $z$, or $a$, $b$, $d$, $c$ (the third normal equation now taking the last place), we should have the same auxiliaries as in the first elimination, so far as those denoted by the numerals 1 and 2; and the equations (68) would still be the same, but in the following order:

$$\begin{aligned}[dd.2]\,w + [cd.2]\,z + [dn.2] = 0\\ [cd.2]\,w + [cc.2]\,z + [cn.2] = 0\end{aligned}$$

The value of $w$ given by the first of these is

$$w = -\frac{[cd.2]}{[dd.2]}\,z - \frac{[dn.2]}{[dd.2]}$$

which, substituted in the second, gives for the coefficient of $z$,

$$[cc.3] = [cc.2] - \frac{[cd.2]}{[dd.2]}\,[cd.2] = [dd.3] \times \frac{[cc.2]}{[dd.2]}$$

Therefore we have

$$p_z = [cc.2]\,\frac{[dd.3]}{[dd.2]}$$

In the fourth supposed elimination, in the order $d$, $c$, $a$, $b$, the auxiliaries denoted by 1 and 2 would be the same as in our actually performed second elimination; but in the final equation in $y$ we should have for the coefficient of $y$ the quantity

$$[bb.3] = [bb.2] - \frac{[ab.2]}{[aa.2]}\,[ab.2] = [aa.3] \times \frac{[bb.2]}{[aa.2]}$$

and, therefore,

$$p_y = [bb.2]\,\frac{[aa.3]}{[aa.2]}$$

Thus, when the elimination has been once inverted, we have found the weights of two of the unknown quantities directly, and the weights of the other two in terms of the auxiliaries previously used, and in a form adapted for logarithmic computation.

46. In order to give the above method greater generality, so that the reader may be enabled to extend it to a greater number of unknown quantities, we remark that the product of the form

$$P = [aa]\,[bb\,.\,1]\,[cc\,.\,2]\,[dd\,.\,3]\,\ldots\ldots$$

has the same value whatever order may be followed in the elimination. This is the same as saying that it is a symmetrical function of $a$, $b$, $c$, $d$ . . . which is, consequently, not affected in value by the permutation of these letters.* Suppose, then, four orders of elimination, in which each unknown quantity in turn becomes the last, while the order of the remaining three quantities remains the same; and, to distinguish the auxiliaries which occur in each elimination, let the letter which occurs in the last auxiliary be annexed to each of the others; the above constant product may thus be expressed in the following four forms:

$$\begin{aligned} P &= [aa]_d\,[bb\,.\,1]_d\,[cc\,.\,2]_d\,[dd\,.\,3] \\ &= [aa]_c\,[bb\,.\,1]_c\,[dd\,.\,2]_c\,[cc\,.\,3] \\ &= [aa]_b\,[cc\,.\,1]_b\,[dd\,.\,2]_b\,[bb\,.\,3] \\ &= [bb]_a\,[cc\,.\,1]_a\,[dd\,.\,2]_a\,[aa\,.\,3] \end{aligned}$$

Now, it is evident that each time a new unknown quantity is made the last, we do not change *all* the auxiliaries, but only those which involve the letter which has become the last in the new order. It is readily seen, therefore, that if we annex a letter to those auxiliaries only which have a different value from that which is denoted by the same symbol in the first elimination, we shall have, simply,

$$\begin{aligned} P &= [aa]\,[bb\,.\,1]\,[cc\,.\,2]\,[dd\,.\,3] \\ &= [aa]\,[bb\,.\,1]\,[dd\,.\,2]\,[cc\,.\,3] \\ &= [aa]\,[cc\,.\,1]\,[dd\,.\,2]_b\,[bb\,.\,3] \\ &= [bb]\,[cc\,.\,1]_a\,[dd\,.\,2]_a\,[aa\,.\,3] \end{aligned}$$

* The quantity $P$ is, in fact, nothing more than the common denominator of the values of $x$, $y$, $z$, $w$, when these values are reduced to functions of the known quantities and in the form of simple fractions; and this common denominator must evidently have the same value whatever order of elimination is followed.

from which we deduce

$$\left.\begin{aligned} p_w &= [dd.3] \\ p_z &= [cc.3] = [cc.2]\cdot\frac{[dd.3]}{[dd.2]} \\ p_y &= [bb.3] = [bb.1]\cdot\frac{[cc.2]}{[cc.1]}\cdot\frac{[dd.3]}{[dd.2]_b} \\ p_x &= [aa.3] = [aa]\cdot\frac{[bb.1]}{[bb]}\cdot\frac{[cc.2]}{[cc.1]_a}\cdot\frac{[dd.3]}{[dd.2]_a} \end{aligned}\right\} \quad (72)$$

If this method is applied in the case of six unknown quantities, we shall in each of two eliminations have the weights of three of the unknown quantities by computing each time but one new auxiliary, and, therefore, the weights of all six when the second elimination is the inverse of the first. In the case of but four unknown quantities, by inverting the elimination we can find the weights of $z$ and $y$ twice, and thus verify our work.

47. If we have but three unknown quantities, the weights are determined at the same time with $x$, $y$, and $z$ themselves, by a single elimination in the order $a$, $b$, $c$, in which $z$ comes out first with the weight

$$p_z = [cc.2]$$

and then $y$ and $z$, with the weights

$$p_y = [bb.2] = [bb.1]\cdot\frac{[cc.2]}{[cc.1]}$$

$$p_x = [aa.2] = [aa]\cdot\frac{[bb.1]}{[bb]}\cdot\frac{[cc.2]}{[cc.1]_a}$$

in which

$$[cc.1]_a = [cc] - \frac{[bc]}{[bb]}[bc]$$

### INDEPENDENT DETERMINATION OF EACH UNKNOWN QUANTITY AND ITS WEIGHT, ACCORDING TO GAUSS.

48. Let the four equations (70) be multiplied respectively by 1, $A'$, $A''$, $A'''$, and let these factors be determined by the condition that in the sum of the products the coefficients of $y$, $z$, and $w$ shall be zero. Also, let the last three equations of (70) be multiplied respectively by 1, $B''$, $B'''$, and let these factors

be determined by the condition that in the sum of the products the coefficients of $z$ and $w$ shall be zero. Finally, let the last two equations of (70) be multiplied respectively by 1, $C'''$, and let $C'''$ be determined by the condition that in the sum of the products the coefficient of $w$ shall be zero. The conditions which determine these factors are then

$$\left.\begin{aligned}
0 &= \frac{[ab]}{[aa]} + A' \\
0 &= \frac{[ac]}{[aa]} + \frac{[bc.1]}{[bb.1]}A' + A'' \\
0 &= \frac{[ad]}{[aa]} + \frac{[bd.1]}{[bb.1]}A' + \frac{[cd.2]}{[cc.2]}A'' + A''' \\
0 &= \frac{[bc.1]}{[bb.1]} + B'' \\
0 &= \frac{[bd.1]}{[bb.1]} + \frac{[cd.2]}{[cc.2]}B'' + B''' \\
0 &= \frac{[cd.2]}{[cc.2]} + C'''
\end{aligned}\right\} \quad (73)$$

and the final values of $x$, $y$, $z$, $w$, in terms of these factors, are given as follows:

$$\left.\begin{aligned}
-x &= \frac{[an]}{[aa]} + \frac{[bn.1]}{[bb.1]}A' + \frac{[cn.2]}{[cc.2]}A'' + \frac{[dn.3]}{[dd.3]}A''' \\
-y &= \frac{[bn.1]}{[bb.1]} + \frac{[cn.2]}{[cc.2]}B'' + \frac{[dn.3]}{[dd.3]}B''' \\
-z &= \frac{[cn.2]}{[cc.2]} + \frac{[dn.3]}{[dd.3]}C''' \\
-w &= \frac{[dn.3]}{[dd.3]}
\end{aligned}\right\} \quad (74)$$

49. As the equations (73) are above arranged, all the factors $A$ are determined from the first system of three equations; the factors $B$ from the second system of two equations, &c.; in each case, by successive substitution. This method then enables us to find each unknown quantity independently of the others.

Another form may be given to the computation of the auxiliary factors. Since in the formation of the equations (74) we have regarded [an], [bn], [cn], &c. as independent, we must still so regard them when we invert the process and recompose the equations (70) from (74). If, then, we multiply the equations (74) respectively by 1, $\frac{[ab]}{[aa]}$, $\frac{[ac]}{[aa]}$, $\frac{[ad]}{[aa]}$, and add the products in order to recompose the first of (70), the coefficient of $[an]$ will be $\frac{1}{[aa]}$, but the coefficients of $[bn.1]$, $[cn.2]$, &c. must severally be equal to zero. The same principle will apply when we recompose the second equation of (70) from the last three of (74), &c. Hence we have

$$\left.\begin{aligned}
0 &= A' + \frac{[ab]}{[aa]} \\
0 &= A'' + \frac{[ab]}{[aa]} B'' + \frac{[ac]}{[aa]} \\
0 &= A''' + \frac{[ab]}{[aa]} B''' + \frac{[ac]}{[aa]} C''' + \frac{[ad]}{[aa]} \\
0 &= B'' + \frac{[bc.1]}{[bb.1]} \\
0 &= B''' + \frac{[bc.1]}{[bb.1]} C''' + \frac{[bd.1]}{[bb.1]} \\
0 &= C''' + \frac{[cd.2]}{[cc.2]}
\end{aligned}\right\} \quad (75)$$

According to this scheme, we first find $A'$, $B''$, $C'''$ from the equations in which they occur singly; then, with these factors, we find the values of $A''$, $B'''$, from the equations involving two factors, &c.

50. Again, let us write the 3d, 5th, and 6th equations of (75) in the following order:

$$\begin{aligned}
A''' + \frac{[ab]}{[aa]} B''' + \frac{[ac]}{[aa]} C''' + \frac{[ad]}{[aa]} &= 0 \\
B''' + \frac{[bc.1]}{[bb.1]} C''' + \frac{[bd.1]}{[bb.1]} &= 0 \\
C''' + \frac{[cd.2]}{[cc.2]} &= 0
\end{aligned}$$

Comparing these with the first three of (70), we at once infer that $A'''$, $B'''$, $C'''$ are those values of $x$, $y$, $z$, respectively, which we should obtain from our first three normal equations by putting

$w = 1$ and omitting the terms in $n$; or, going back to (66), that $A'''$, $B'''$, $C'''$ may be determined by the following conditions:

$$[aa]\,A''' + [ab]\,B''' + [ac]\,C''' + [ad] = 0$$
$$[ab]\,A''' + [bb]\,B''' + [bc]\,C''' + [bd] = 0$$
$$[ac]\,A''' + [bc]\,B''' + [cc]\,C''' + [cd] = 0$$

If now we multiply the normal equations (66) by $A'''$, $B'''$, $C'''$, and 1, respectively, and add the products, the conditions just given will cause $x$, $y$, and $z$ to disappear, and the resulting equation in $w$ must be identical* with (69): so that $A'''$, $B'''$, $C'''$ must also satisfy the following condition:

$$[an]\,A''' + [bn]\,B''' + [cn]\,C''' + [dn] = [dn\,.\,3] \qquad (76)$$

The second and fourth equations of (75) being written as follows,

$$A'' + \frac{[ab]}{[aa]}\,B'' + \frac{[ac]}{[aa]} = 0$$
$$B'' + \frac{[bc\,.\,1]}{[bb\,.\,1]} = 0$$

and compared with the first two of (70), we infer that $A''$, $B''$ are those values of $x$ and $y$ which we obtain from the first two normal equations by putting $z = 1$, $w = 0$, and omitting the terms in $n$; that is, $A''$ and $B''$ must satisfy the conditions

$$[aa]\,A'' + [ab]\,B'' + [ac] = 0$$
$$[ab]\,A'' + [bb]\,B'' + [bc] = 0$$

Therefore, if we multiply the first three normal equations (66) by $A''$, $B''$, 1, respectively, and add the products, $x$ and $y$ will disappear, and, the resulting equation being identical with the first of (68), we must also have

$$[an]\,A'' + [bn]\,B'' + [cn] = [cn\,.\,2] \qquad (77)$$

Lastly, it is evident that $A'$ must also satisfy the condition

$$[an]\,A' + [bn] = [bn\,.\,1] \qquad (78)$$

From these relations we readily infer general formulæ for the weights of the unknown quantities.

---

* The equation (69) is the last normal equation, unchanged except by the substitution of *equivalents* for $x$, $y$, and $z$; and in the present article we eliminate $x$, $y$, and $z$ by the use of factors, but do not change the last normal equation, since we multiply it by unity.

According to Art. 34, the reciprocal of the weight of $x$ is that value which we obtain for $x$ if we put $[an] = -1$ and $[bn] = [cn] = [dn] = 0$. But, under these conditions, the equations (76), (77), (78) give

$$[dn.3] = -A''', \qquad [cn.2] = -A'', \qquad [bn.1] = -A'$$

In order, therefore, that the value of $x$ given by the first equation of (74) may become $\frac{1}{p_x}$, we have only to substitute $-A'''$, $-A''$, $-A'$, $-1$, respectively, for $[dn.3]$, $[cn.2]$, $[bn.1]$, $[an]$.

In the same manner, the weight of $y$ being found by putting $[bn] = -1$ and $[an] = [cn] = [dn] = 0$, we have to put

$$[dn.3] = -B''', \qquad [cn.2] = -B'', \qquad [bn.1] = -1$$

in the second equation of (74), in order that we may put $\frac{1}{p_y}$ for $y$.

For the weight of $z$ we have to put

$$[dn.3] = -C''', \qquad [cn.2] = -1$$

in the third equation of (74), and $\frac{1}{p_z}$ for $z$.

For the weight of $w$, we have to put

$$[dn.3] = -1$$

in the last equation of (74), and change $w$ to $\frac{1}{p_w}$.

The final formulæ for the weights are, therefore,

$$\left.\begin{aligned}
\frac{1}{p_x} &= \frac{1}{[aa]} + \frac{A'A'}{[bb.1]} + \frac{A''A''}{[cc.2]} + \frac{A'''A'''}{[dd.3]} \\
\frac{1}{p_y} &= \frac{1}{[bb.1]} + \frac{B''B''}{[cc.2]} + \frac{B'''B'''}{[dd.3]} \\
\frac{1}{p_z} &= \frac{1}{[cc.2]} + \frac{C'''C'''}{[dd.3]} \\
\frac{1}{p_w} &= \frac{1}{[dd.3]}
\end{aligned}\right\} \qquad (79)$$

MEAN ERROR OF A LINEAR FUNCTION OF THE QUANTITIES $x$, $y$, $z$, $w$.

50. *To find the mean error of the function*

$$X = fx + gy + hz + iw + l \qquad (80)$$

*when $x$, $y$, $z$, $w$ are dependent upon the same observations.*

The quantities $x$, $y$, $z$, $w$ not being directly observed, their mean errors cannot be treated as independent, as was done in the case of directly observed quantities in Art. 22. We might proceed by the method of Art. 23; but, as we here suppose $x$, $y$, $z$, $w$ to have been determined from the normal equations (66), we can obtain a more convenient method by the aid of the auxiliaries which have been introduced in the general elimination. The quantities $x$, $y$, $z$, $w$ being functions of the directly observed quantities $n'$, $n''$, $n'''$, ... the mean error of $X$ can be readily obtained by the principles of Art. 22, if we first reduce $X$ to a function of these observed quantities. For this purpose, if the values of $x$, $y$, $z$, $w$ deduced from (70) be substituted in $X$, we shall have an expression of the form

$$X = k_0\,[an] + k_1\,[bn.1] + k_2\,[cn.2] + k_3\,[dn.3] + l \qquad (81)$$

in which the coefficients $k_0$, $k_1$, $k_2$, $k_3$ are functions of $[aa]$, $[ab]$, &c. In order to determine these coefficients, let us substitute in this expression the values of $[an]$, $[bn.1]$, &c. given by (70). We find

$$\begin{aligned} X = -[aa]\,k_0x - [ab]\,k_0y - &[ac]\,k_0z - [ad]\,k_0w + l \\ - [bb.1]\,k_1y - &[bc.1]\,k_1z - [bd.1]\,k_1w \\ - &[cc.2]\,k_2z - [cd.2]\,k_2w \\ &- [dd.3]\,k_3w \end{aligned}$$

which becomes identical with (80) by assuming

$$\left.\begin{aligned} &[aa]\,k_0 = -f \\ &[ab]\,k_0 + [bb.1]\,k_1 = -g \\ &[ac]\,k_0 + [bc.1]\,k_1 + [cc.2]\,k_2 = -h \\ &[ad]\,k_0 + [bd.1]\,k_1 + [cd.2]\,k_2 + [dd.3]\,k_3 = -i \end{aligned}\right\} \qquad (82)$$

These equations fully determine the coefficients. We find $k_0$ directly from the first, and then $k_1$, $k_2$, $k_3$, by successive substitutions in the others.

Now, to find the mean error of $X$ under the form (81), let the mean error of each of the observed quantities $n'$, $n''$, $n'''$ .... be denoted by $\varepsilon$ (these observed quantities being supposed of equal weight, or, rather, the equations of condition being supposed to have been reduced to the same weight), and let the corresponding mean errors of

$$[an], \qquad [bn.1], \qquad [cn.2], \qquad [dn.3], \qquad X,$$

be denoted by

$$E_0, \quad E_1, \quad E_2, \quad E_3, \quad (\varepsilon X).$$

Since we have

$$[an] = a'n' + a''n'' + a'''n''' + \ldots.$$

we have, by Art. 22,

$$E_0^2 = [aa]\,\varepsilon^2$$

Again, we have

$$[bn.1] = [bn] - \frac{[ab]}{[aa]}[an] = \sum\left[\left(b - \frac{[ab]}{[aa]}a\right)n\right]$$

and hence

$$E_1^2 = \varepsilon^2 \sum\left(b - \frac{[ab]}{[aa]}a\right)^2$$

$$= \varepsilon^2\left([bb] - \frac{2\,[ab]}{[aa]}[ab] + \frac{[ab]^2}{[aa]^2}[aa]\right)$$

$$= \varepsilon^2\left([bb] - \frac{[ab]}{[aa]}[ab]\right)$$

$$= [bb.1]\,\varepsilon^2$$

In a similar manner, we have, also,

$$E_2^2 = [cc.2]\,\varepsilon^2, \qquad E_3^2 = [dd.3]\,\varepsilon^2$$

The quantities $x$, $y$, $z$, $w$, being determined from the equations (70), their mean errors involve those of the quantities $[an]$, $[bn.1]$, $[cn.2]$, $[dn.3]$, precisely as if the latter had been independently observed quantities affected by the mean errors just determined. Hence also in (81) we regard $[an]$, $[bn.1]$, &c. as independent; and it then follows directly from the principles of Art. 22 that

$$(\varepsilon X)^2 = k_0^2 E_0^2 + k_1^2 E_1^2 + k_2^2 E_2^2 + k_3^2 E_3^2$$

or

$$(\varepsilon X)^2 = \left(k_0^2 [aa] + k_1^2 [bb.1] + k_2^2 [cc.2] + k_3^2 [dd.3]\right)\varepsilon^2 \quad (83)$$

51. From the preceding article we may easily find the formulæ (74) and (79). The function $X$ becomes $x$ when we assume $f = 1$, $g = h = i = l = 0$; and then (81) gives $x$ while (83) gives $\varepsilon_x^2$, and hence the weight $= \frac{\varepsilon^2}{\varepsilon_x^2}$. This hypothesis gives in (82) $[aa]\,k_0 = -1$; and the remaining equations of (82) are identical with the first three of (73) if we put $[bb.1]\,k_1 = -A'$, $[cc.2]\,k_2 = -A''$, $[dd.3]\,k_3 = -A'''$; and then (81) becomes identical with the first of (74), and (83) with the first of (79). In a similar manner we may deduce the remaining equations of (74) and (79).

EXAMPLE.—In order to exhibit the numerical operations which the preceding method requires, in their proper order and within the limits of the page, I select an example involving but three unknown quantities. The following equations of condition were proposed by GAUSS (*Theoria Motus Corp. Coel.*, Art. 184) to illustrate his method:

$$
\begin{array}{ll}
(1) & x - y + 2z = 3 \\
(2) & 3x + 2y - 5z = 5 \\
(3) & 4x + y + 4z = 21 \\
(4) & -2x + 6y + 6z = 28
\end{array}
$$

of which the first three are supposed to have the weight unity, while the last has the weight $\frac{1}{4}$. Multiplying the last by $\sqrt{\frac{1}{4}} = \frac{1}{2}$ (Art. 41), the equations of condition, reduced to the same weight, are—

$$
\begin{array}{ll}
(1) & x - y + 2z - 3 = 0 \\
(2) & 3x + 2y - 5z - 5 = 0 \\
(3) & 4x + y + 4z - 21 = 0 \\
(4) & - x + 3y + 3z - 14 = 0
\end{array}
$$

The next step is to form the coefficients [*aa*], [*ab*], &c., of the normal equations. In the present example this can be done very easily without the aid of logarithms; but, in order to exhibit the work usually required in practice, I shall give the forms for logarithmic computation. The sums of the coefficients of the unknown quantities will be employed as checks, according to Art. 30. Their logarithms, together with those of *a*, *b*, *c*, *n*, are given in the following table:

| | log *a* | log *b* | log *c* | log *s* | log *n* |
|---|---|---|---|---|---|
| (1) | 0.00000 | *n*0.00000 | 0.30103 | 0.30103 | *n*0.47712 |
| (2) | 0.47712 | 0.30103 | *n*0.69897 | $-\infty$ | *n*0.69897 |
| (3) | 0.60206 | 0.00000 | 0.60206 | 0.95424 | *n*1.32222 |
| (4) | *n*0.00000 | 0.47712 | 0.47712 | 0.69897 | *n*1.14613 |

It is important, where many operations are to be performed, to write down no more figures than are necessary for the clear prosecution of the work. Hence, in combining the preceding logarithms it will be found expedient to proceed as follows. Write each log *a* upon the lower edge of a slip of paper; then, placing this slip so that log *a* shall stand over log *a*, log *b*, log *c*, &c., of the same horizontal line, in succession, add together the two logarithms *mentally*, and, with the sum *in the head*, take from the logarithmic table the corresponding natural number (*aa*, *ab*, *ac*, *as*, or *an*), which place in a column appropriated for the purpose. Then write log *b* in the same manner, and form *bb*, *bc*, *bs*, *bn*, and so proceed to form all the coefficients of the normal equations, as in the following table:

| | [*aa*] | [*ab*] | | [*ac*] | | [*as*] | | [*an*] | | [*bb*] | [*bc*] | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | + | + | − | + | − | + | − | + | − | + | + | − |
| (1) | 1.0 | | 1.0 | 2.0 | | 2.0 | | | 3.0 | 1.0 | | 2.0 |
| (2) | 9.0 | 6.0 | | | 15.0 | | 0.0 | | 15.0 | 4.0 | | 10.0 |
| (3) | 16.0 | 4.0 | | 16.0 | | 36.0 | | | 84.0 | 1.0 | 4.0 | |
| (4) | 1.0 | | 3.0 | | 3.0 | | 5.0 | 14.0 | | 9.0 | 9.0 | |
| | | 10.0 | 4.0 | 18.0 | 18.0 | 38.0 | 5.0 | 14.0 | 102.0 | | 13.0 | 12.0 |
| | + 27.0 | + 6.0 | | 0.0 | | + 33.0 | | − 88.0 | | + 15.0 | + 1.0 | |

| | [*bs*] | | [*bn*] | | [*cc*] | [*cs*] | | [*cn*] | | [*sn*] | | [*nn*] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | + | − | + | − | + | + | − | + | − | + | − | + |
| (1) | | 2.0 | 3.0 | | 4.0 | 4.0 | | | 6.0 | | 6.0 | 9.0 |
| (2) | 0.0 | | | 10.0 | 25.0 | | 0.0 | 25.0 | | 0.0 | | 25.0 |
| (3) | 9.0 | | | 21.0 | 16.0 | 36.0 | | | 84.0 | | 189.0 | 441.0 |
| (4) | 15.0 | | | 42.0 | 9.0 | 15.0 | | | 42.0 | | 70.0 | 196.0 |
| | 24.0 | 2.0 | 3.0 | 73.0 | | 55.0 | 0.0 | 25.0 | 132.0 | 0.0 | 265.0 | |
| | + 22.0 | | − 70.0 | | + 54.0 | + 55.0 | | − 107.0 | | − 265.0 | | + 671.0 |

Having ascertained that the results satisfy the test equations (48), we can write out the normal equations as follows:

$$
\begin{aligned}
27x + 6y \qquad\quad - 88 &= 0 \\
6x + 15y + \quad z - 70 &= 0 \\
y + 54z - 107 &= 0
\end{aligned}
$$

We proceed to determine the values of $x$, $y$, $z$, according to our general formulæ, still carrying out the work with logarithms for the sake of illustration. Here, again, system and conciseness are indispensable. The whole computation is given below nearly in the form proposed by Encke. This form corresponds to the group of equations (70). It is divided into three principal compartments, corresponding, respectively, to the first three equations of (70), each beginning one column farther to the right. In the first compartment the first line of numbers contains the values of [*aa*], [*ab*], &c., the second line their logarithms, and the third line the logarithms of the coefficients of the first equation. The logarithms in this third line are formed by subtracting the first log. in the second line from each of the subsequent ones, for this purpose writing the first logarithm upon the lower edge of a slip of paper.

In the second compartment, the first line contains the values of [*bb*], [*bc*], &c.; the second line, the quantities subtractive from these, according to the formulæ in Art. 42. To form these subtractive quantities, write the logarithm of $\frac{[ab]}{[aa]}$ (which is here 9.34679) upon the lower edge of a slip of paper, and hold it successively over log [*ab*] and each of the subsequent logarithms in the same line; add the two logarithms mentally in each case, take the corresponding natural number from the logarithmic table, and write it in its place below. Subtracting these numbers, we have the values of [*bb*.1], [*bc*.1], &c. The fourth line contains the logarithms of these quantities; the fifth, the logarithms of the coefficients of our second equation, formed by subtracting the first logarithm of the preceding line from each of the subsequent ones in that line.

In the third compartment we have—first, the values of [*cc*], &c.; secondly, the values of the subtractive quantities formed from the last line of the first compartment as before; thirdly, the remainders which are the values of [*cc*.1], &c. The fourth line contains the values of the quantities which are subtractive from the preceding and are formed from the last line of the second compartment by adding the first logarithm of that line to the logarithm immediately above it and to each of the subsequent logarithms in the same line; the fifth line contains the remainders which are the values of [*cc*.2], &c.; the sixth line, the logarithms of these; and the last line, the logarithms of the coefficients of our third equation.

For control, we carry through the operations upon [*as*], [*bs*], &c., precisely as upon the other quantities; and then, according to the arrangement of the scheme, we should have, if we have computed correctly, each sum containing *s* equal to the sum of the quantities on its left in the same line, together with those of the same order in a vertical column over the first number in this line. Thus, we must have, in the present case,

$$[bs.1] = [bb.1] + [bc.1] \qquad [sn.1] = [bn.1] + [cn.1]$$
$$[cs.1] = [cc.1] + [bc.1] \qquad [sn.2] = [cn.2]$$
$$[cs.2] = [cc.2]$$

relations easily proved by means of the formulæ of Art. 42 combined with (48).

The columns [*sn*] and [*nn*] are added to the third compartment in order to form the quantity [*nn*.3], from which the mean error of observation is to be deduced, as will be shown hereafter.

| [aa] | [ab] | [ac] | [as] | [an] |
|---|---|---|---|---|
| + 27.000 | + 6.000 | 0.000 | + 33.000 | − 88.000 |
| 1.43136 | 0.77815 | − ∞ | 1.51851 | n1.94448 |
| | 9.34679 | − ∞ | 0.08715 | n0.51312 |

| | [bb] | [bc] | [bs] | [bn] |
|---|---|---|---|---|
| − 88.000 | | | | |
| 0.000 | + 15.000 | + 1.000 | + 22.000 | − 70.000 |
| + 21.305 | + 1.333 | 0.000 | + 7.333 | − 19.556 |
| − 66.695 | + 13.667 | + 1.000 | + 14.667 | − 50.444 |
| n1.82409 | 1.13566 | 0.00000 | 1.16633 | n1.70281 |
| log x = 0.39273 | | 8.86434 | 0.03067 | n0.56715 |

| | [cc] | [cs] | [cn] | [sn] | [nn] |
|---|---|---|---|---|---|
| − 50.444 | + 54.000 | − 55.000 | − 107.000 | − 265.000 | + 671.000 |
| + 1.916 | 0.000 | 0.000 | 0.000 | − 107.555 | + 286.813 |
| − 48.528 | + 54.000 | − 55.000 | − 107.000 | − 157.445 | + 384.187 |
| n1.68599 | − 0.073 | + 1.073 | − 3.691 | − 54.135 | + 186.191 |
| log y = 0.55033 | + 53.927 | + 53.927 | − 103.309 | − 103.310 | + 197.996 |
| | 1.73181 | | n2.01414 | | + 197.909 |
| | | log (− z) = | n0.28233 | [nn.3] = | + 0.087 |

After $z$ has been found, its value is substituted in the second equation of (70), and $y$ is deduced. Then, the values of $y$ and $z$ being substituted in the first equation, we find $x$. The numerical computations are given above in the margin.

Then, for the weights, by Art. 47, we have first to find the additional auxiliary

$$[cc.1]_a = [cc] - \frac{[bc]}{[bb]}[bc]$$

and by the formulæ of that article we have—

| [bb] | [bc] | | | | |
|---|---|---|---|---|---|
| + 15.000 | + 1.000 | log [bb.1] | 1.13566 | log [cc.2] | 1.73181 |
| 1.17609 | 0.00000 | log [bb] | 1.17609 | log [cc.1] | 1.73239 |
| | 8.82391 | | | log [cc.1] a | 1.73185 |
| | [cc] | | 1.43136 | 1.13566 | 1.73181 |
| | + 54.000 | | 9.95957 | 9.99942 | log $p_z$ |
| | + 0.067 | | 9.99996 | 1.13508 | |
| [cc.1a] = | + 53.933 | | 1.39089 | log $p_y$ | |
| | | | log $p_x$ | | |

The final result is then

$$\begin{aligned} x &= +\,2.4702 \text{ with the weight } 24.597 \\ y &= +\,3.5508 \quad \text{“} \quad\quad \text{“} \quad 13.648 \\ z &= +\,1.9157 \quad \text{“} \quad\quad \text{“} \quad 53.927 \end{aligned}$$

It only remains to substitute the values of $x$, $y$, and $z$ in the original equations of condition, to form the residuals $v$, and from these to determine the mean error of observation. Since here there are but three unknown quantities, we have, by (71),

$$[vv] = [nn.3]$$

and hence the mean error of an observation of the weight unity is, by (61), $m$ being the number of equations of condition,

$$\varepsilon = \sqrt{\left(\frac{[nn.3]}{m-3}\right)} = 0.295$$

The direct computation of the residuals is, therefore, not necessary for determining $\varepsilon$: nevertheless, it is desirable in most cases to resort to the direct substitution also, not only for a final verification, but in order to examine the several observations, and to obtain the data for rejecting any doubtful one by the use of PEIRCE'S Criterion, to be given hereafter. This direct substitution has already been carried out for this example on p. 525, where we have found $[vv] = 0.0804$, which agrees with the above value of $[nn.3]$ as nearly as can be expected with the use of five-decimal logarithms.

52. It not unfrequently happens that one of the unknown quantities is such that the given observations cannot determine it with accuracy. For example, in the reduction of a number of observations of an eclipse, one of the unknown quantities is a correction of the moon's parallax; but, unless the places of observation be remote from each other, the correction will be very uncertain, and this uncertainty will affect all the other quantities which enter into the equations of condition. In such a case, this unknown quantity will come out with a small coefficient, which of itself will reveal the existence of the uncertainty when it is not otherwise anticipated. In order that this uncertainty may not affect those quantities which are well defined by the observations, it is expedient to determine all the latter as functions of the uncertain quantity, which for that purpose must be made the

last in the elimination. Thus, with four unknown quantities $x, y, z, w$, we proceed only as far as the auxiliaries denoted by the numeral 2; then, having found the factors $A'$, $A''$, $A'''$, $B''$, $B'''$, $C'''$, by (73) or (75), if we put

$$\left.\begin{aligned} -x' &= \frac{[an]}{[aa]} + \frac{[bn.1]}{[bb.1]}A' + \frac{[cn.2]}{[cc.2]}A'' \\ -y' &= \frac{[bn.1]}{[bb.1]} + \frac{[cn.2]}{[cc.2]}B'' \\ -z' &= \frac{[cn.2]}{[cc.2]} \end{aligned}\right\} \qquad (84)$$

these will give the values of the unknown quantities which we should obtain from the first three normal equations if the last unknown quantity were disregarded or put $= 0$. Then, by (74), the final values of $x, y, z$, as functions of the uncertain quantity $w$, will be

$$\left.\begin{aligned} x &= x' + A'''w \\ y &= y' + B'''w \\ z &= z' + C'''w \end{aligned}\right\} \qquad (85)$$

The values of $x', y', z'$, will thus be well determined, and a subsequent independent determination of $w$ will enable us to find the final values of $x, y, z$.*

Having found the weights of $x', y', z'$ (which is done as if they were the only quantities under consideration), and their mean errors $\varepsilon_x'$, $\varepsilon_y'$, $\varepsilon_z'$, then, when the quantity $w$ is afterwards found, the mean errors of the final values will be

$$\left.\begin{aligned} \varepsilon^2_x &= \varepsilon_x'^2 + (A'''\varepsilon_w)^2 \\ \varepsilon^2_y &= \varepsilon_y'^2 + (B'''\varepsilon_w)^2 \\ \varepsilon^2_z &= \varepsilon_z'^2 + (C'''\varepsilon_w)^2 \end{aligned}\right\} \qquad (86)$$

as we find from the equations (79), or by Art. 20.

### CONDITIONED OBSERVATIONS.

53. In all that precedes, we have supposed that the several quantities to be found by observation, either directly or indirectly, were independent of each other. Although they were required to satisfy certain equations of condition as nearly as possible, yet they were so far independent that no contradiction was involved in supposing the values of one or more of them to be varied without

* For an example in which three unknown quantities are thus determined as functions of *two* uncertain quantities, see Vol. I. p. 540.

varying the others. By such variations we should obtain systems of values *more or less probable*, but all *possible*.

There is a second class of problems, in which, besides the equations of condition which the unknown quantities are to satisfy approximately, there are also equations of condition which they must satisfy exactly: so that of all the systems of values which may be selected as approximately satisfying the first kind of equations, only those can be admitted as possible which satisfy exactly the equations of the second kind. The number of these rigorous equations of condition must be less than the number of unknown quantities; otherwise they would determine these quantities independently of all observations. These rigorous equations, then, may be satisfied by various possible systems of values, and we can therefore express the problem here to be considered as follows: *Of all the possible systems of values which exactly satisfy the rigorous equations of condition, to find the most probable, or that system which best satisfies the approximate equations of condition.*

The following are simple examples of conditioned observations. The sum of the three angles of a plane triangle must be 180°: so that if we observe each angle directly, and the sum of the observed values differs from 180°, these values must be corrected so as to satisfy this condition. The sum of the angles of a spherical triangle must be 180° + spherical excess. The sum of all the angles around a point, or the sum of all the differences of azimuth observed at a station upon a round of objects in the horizon, must be 360°.

The approximate conditions in these cases are expressed by the observations themselves; for the final values adopted must correspond as nearly as possible to the observed values. The corrections to be applied to the observed values are to be regarded as residual errors with their signs changed; and the solution of our problem is involved in the following statement: *Of all the systems of corrections which satisfy the rigorous equations, that system is to be received as the most probable in which the sum of the squares of the residuals in the approximate equations is a minimum.*

54. The general problem as above stated may be reduced to that of unconditioned observations, already considered. For let us suppose there are $m'$ rigorous equations of condition, and $m$ unknown quantities. From these $m'$ equations let the values of $m'$ unknown quantities be obtained in terms of the remaining

$m - m'$ quantities, and let these values be substituted in all the approximate equations of condition; then there will be left in the latter only $m - m'$ quantities, which may be treated as independent, so that, the approximate equations being now solved by the method of least squares, we have the values of the $m - m'$ quantities, with which we then find the values of the first $m'$ quantities. This is a general solution of the problem; but it is not always the simplest in practice. I shall illustrate it by a simple example, before giving a method applicable to more complicated cases.

EXAMPLE.—At Pine Mount, a station of the U. S. Coast Survey, the angles between the surrounding stations 1, 2, 3, 4 were observed as follows:

| | | | weight |
|---|---|---|---|
| 1.2 | Joscelyne—Deepwater......... | 65° 11′ 52″.500 | 3 |
| 2.3 | Deepwater—Deakyne.......... | 66 24 15 .553 | 3 |
| 3.4 | Deakyne—Burden ............. | 87 2 24 .703 | 3 |
| 4.1 | Burden—Joscelyne............ | 141 21 21 .757 | 1 |

There are here four unknown quantities subjected to the single rigorous condition that their sum must be 360°. But, instead of taking the angles themselves as the unknown quantities, we shall assume approximate values of them, and regard the corrections which they require as the unknown quantities.

We assume

| | | |
|---|---|---|
| 1.2 | Joscelyne—Deepwater, | $65^\circ\ 11'\ 52''.5 + w$ |
| 2.3 | Deepwater—Deakyne, | $66\ \ 24\ \ 15\ .5 + x$ |
| 3.4 | Deakyne—Burden, | $87\ \ \ 2\ \ 24\ .7 + y$ |
| 4.1 | Burden—Joscelyne, | $141\ \ 21\ \ 21\ .8 + z$ |

the sum of which must satisfy the condition

$$359^\circ\ 59'\ 54''.5 + w + x + y + z = 360^\circ$$

or

$$w + x + y + z - 5''.5 = 0$$

The difference between the assumed value and the observed value in each case gives us a residual; and the approximate equations of condition are, therefore,

$$\begin{aligned} w - 0 \quad\ \ &= 0 \\ x - 0.053 &= 0 \\ y - 0.003 &= 0 \\ z + 0.043 &= 0 \end{aligned}$$

We have here but one rigorous condition (or $m' = 1$), and to eliminate this we have only to find from it the value of one unknown quantity in terms of the others, and substitute it in the approximate equations of condition: thus, substituting the value

$$w = -x - y - z + 5''.5$$

our equations of condition, containing now three independent unknown quantities, are

| | weight. |
|---|---|
| $-x - y - z + 5''.5 = 0$ | 3 |
| $x \quad - 0\ .053 = 0$ | 3 |
| $y \quad - 0\ .003 = 0$ | 3 |
| $z - 0\ .043 = 0$ | 1 |

The normal equations, applying the weights, are then

$$\begin{aligned} 6x + 3y + 3z - 16.659 &= 0 \\ 3x + 6y + 3z - 16.509 &= 0 \\ 3x + 3y + 4z - 16.457 &= 0 \end{aligned}$$

which, being solved, give

$$\begin{aligned} x &= +\ 0''.9675 \\ y &= +\ 0\ .9175 \\ z &= +\ 2\ .7005 \end{aligned}$$

whence also

$$w = +\ 0.9145$$

and the corrected values of the angles are

| | | | | | |
|---|---|---|---|---|---|
| 1.2 | Joscelyne—Deepwater.......... | 65° | 11′ | 53″ | .4145 |
| 2.3 | Deepwater—Deakyne........... | 66 | 24 | 16 | .4675 |
| 3.4 | Deakyne—Burden............... | 87 | 2 | 25 | .6175 |
| 4.1 | Burden—Joscelyne............. | 141 | 21 | 24 | .5005 |
| | | 360 | 0 | 0 | .0000 |

55. When the number of unknown quantities is great, or when there are several rigorous conditions to be satisfied, the preceding method would lead to very tedious computations, since we are required to perform two eliminations, the first from our $m'$ rigorous equations to find the first $m'$ quantities in terms of the others, and the second from our normal equations involving all the remaining quantities. In order to obtain the general form

for a more condensed process, let the most probable values of a number ($m$) of directly observed quantities be

$$V', V'', V''', \text{\&c.} \ldots V^{(m)}$$

Let the observed values be

$$M', M'', M''', \text{\&c.} \ldots M^{(m)}$$

Let these observations have the weights

$$p', \ p'', \ p''', \ \text{\&c.} \ldots p^{(m)}$$

Let the equations which the most probable values are required to satisfy rigorously be expressed by

$$\left.\begin{array}{l} \varphi' = f' \ (V', V'', V''', \ldots) = 0 \\ \varphi'' = f'' \ (V', V'', V''', \ldots) = 0 \\ \varphi''' = f''' \ (V', V'', V''', \ldots) = 0 \\ \qquad \text{\&c.} \end{array}\right\} \quad (87)$$

and let

$$m' = \text{the number of these conditions.}$$

Let the most probable corrections of the observed values be

$$v', v'', v''', \text{\&c.} \ldots v^{(m)}$$

so that

$$V' = M' + v', \qquad V'' = M'' + v'', \qquad V''' = M''' + v''', \text{\&c.}$$

Let the values of $\varphi', \varphi'', \varphi''' \ldots$ when the observed values are actually substituted be $n', n'', n''' \ldots$ or

$$\left.\begin{array}{l} f' \ (M', M'', M''', \ldots) = n' \\ f'' \ (M', M'', M''', \ldots) = n'' \\ f''' \ (M', M'', M''', \ldots) = n''' \\ \qquad \text{\&c.} \end{array}\right\} \quad (88)$$

Let the differential coefficients $\frac{d\varphi'}{dV'}, \frac{d\varphi'}{dV''}$, &c., $\frac{d\varphi''}{dV'}, \frac{d\varphi''}{dV''}$, &c. be formed; substitute in them the values $M', M'', M''' \ldots$ for $V'$, $V''$, $V'''$, and denote the resulting values by $a'$, $a''$, &c., $b'$, $b''$, &c.; that is, put

$$\frac{d\varphi'}{dV'} = a', \qquad \frac{d\varphi'}{dV''} = a'', \qquad \frac{d\varphi'}{dV'''} = a''', \text{\&c.}$$

$$\frac{d\varphi''}{dV'} = b', \qquad \frac{d\varphi''}{dV''} = b'', \qquad \frac{d\varphi''}{dV'''} = b''', \text{\&c.}$$

$$\frac{d\varphi'''}{dV'} = c', \qquad \frac{d\varphi'''}{dV''} = c'', \qquad \frac{d\varphi'''}{dV'''} = c''', \text{\&c.}$$

These values of the differential coefficients will generally be sufficiently exact; but if $M'$, $M''$, $M'''$ ... are found very greatly in error, a repetition of the computation might be necessary, in which the more exact values found by the first computation would be used.

The values of $M'$, $M''$, $M'''$ ... being assumed to be so nearly correct that the second and higher powers of the corrections $v'$, $v''$, $v'''$ ... may be neglected, we have at once, by TAYLOR'S Theorem, as in the similar case of Art. 40,

$$\left.\begin{array}{l} \varphi' = n' + a'v' + a''v'' + a'''v''' + \dots + a^{(m)}v^{(m)} = 0 \\ \varphi'' = n'' + b'v' + b''v'' + b'''v''' + \dots + b^{(m)}v^{(m)} = 0 \\ \varphi''' = n''' + c'v' + c''v'' + c'''v''' + \dots + c^{(m)}v^{(m)} = 0 \\ \qquad\qquad \&c. \qquad\qquad \&c. \end{array}\right\} \quad (89)$$

which $m'$ equations must be rigorously satisfied by the values of $v'$, $v''$, $v'''$ ....

The equations

$$V' - M' = 0, \qquad V'' - M'' = 0, \qquad V''' - M''' = 0, \&c.$$

are the approximate equations of condition; or, more strictly,

$$V' - M' = v', \qquad V'' - M'' = v'', \qquad V''' - M''' = v''', \&c.$$

are the equations of condition which are to be satisfied by the most probable system of residuals $v'$, $v''$, $v'''$ .... These, reduced to the unit of weight by Art. 41, become

$$(V' - M')\sqrt{p'} = v'\sqrt{p'}, \qquad (V'' - M'')\sqrt{p''} = v''\sqrt{p''}, \&c. \qquad (90)$$

and the most probable residuals $v'\sqrt{p'}$, $v''\sqrt{p''}$ are those the sum of whose squares is a minimum, or we must have

$$p'v'^2 + p''v''^2 + p'''v'''^2 + \&c. = \text{a minimum.}$$

Putting, then, the differential of this quantity equal to zero, we have

$$p'v'dv' + p''v''dv'' + p'''v'''dv''' + \&c. = 0 \qquad (91)$$

If $v'$, $v''$, $v'''$ ... were independent of each other, each coefficient of this equation would necessarily be zero (as in Art. 28), and then the most probable values of $V'$, $V''$, $V'''$ ... would be the directly observed values $M'$, $M''$, $M'''$ ... But this minimum is here conditioned by the equations (89). If, then, we differentiate (89), the equations

$$\left.\begin{array}{l} a'dv' + a''dv'' + a'''dv''' + \ldots . = 0 \\ b'dv' + b''dv'' + b'''dv''' + \ldots . = 0 \\ c'dv' + c''dv'' + c'''dv''' + \ldots . = 0 \\ \qquad\qquad \&c. \end{array}\right\} \quad (92)$$

must coexist with (91).

The number of the equations (92) is $m'$, while the number of differentials is $m$: and since, by the nature of the case, we must have $m > m'$, we can, by elimination, find from (92) the values of $m'$ differentials in terms of the remaining $m - m'$ differentials. Let us suppose this elimination to be performed, and that the values of the first $m'$ differentials, found in terms of the others, are then substituted in (91); we shall thus have an equation in which the remaining $m - m'$ unknown quantities can be regarded as independent, and the coefficients of these $m - m'$ quantities in this final equation will then *severally* be equal to zero. We can arrive directly at the result of such an elimination and substitution as follows. Multiply the first equation of (92) by $A$, the second by $B$, the third by $C$, &c., and also the equation (91) by $-1$, and form the sum of all these products. Then, if $A$, $B$, $C \ldots .$ are determined so that $m'$ differentials shall disappear from the sum (and they can be so determined, since it only requires $m'$ conditions to determine $m'$ quantities), the final equation obtained will contain only the $m - m'$ remaining differentials. But, the latter being independent, their coefficients must also be severally equal to zero: and hence we have, in all, the following $m$ conditional equations:

$$\left.\begin{array}{l} a'A + b'B + c'C + \ldots . - p'v' = 0 \\ a''A + b''B + c''C + \ldots . - p''v'' = 0 \\ a'''A + b'''B + c'''C + \ldots . - p'''v''' = 0 \\ \qquad \&c. \qquad\qquad \&c. \end{array}\right\} \quad (93)$$

If we multiply the first of these by $\frac{a'}{p'}$, the second by $\frac{a''}{p''}$, &c., and add the products, we have, by comparison with the first equation of (89),

$$\left[\frac{aa}{p}\right]A + \left[\frac{ab}{p}\right]B + \left[\frac{ac}{p}\right]C + \ldots . + n' = 0$$

in which the usual notation for sums is followed. In this way we can form $m'$ normal equations containing $m'$ quantities, namely,

$$\left.\begin{aligned}
&\left[\frac{aa}{p}\right]A + \left[\frac{ab}{p}\right]B + \left[\frac{ac}{p}\right]C + \ldots\ldots + n' = 0 \\
&\left[\frac{ab}{p}\right]A + \left[\frac{bb}{p}\right]B + \left[\frac{bc}{p}\right]C + \ldots\ldots + n'' = 0 \\
&\left[\frac{ac}{p}\right]A + \left[\frac{bc}{p}\right]B + \left[\frac{cc}{p}\right]C + \ldots\ldots + n''' = 0 \\
&\qquad\qquad\qquad\qquad \&c.
\end{aligned}\right\} \quad (94)$$

If the observations are of equal weight, we have only to put $p = 1$, or, in other words, omit $p$.

The factors $A$, $B$, $C$... are called by Gauss the *correlatives* of the equations of condition.

The equations (94) being resolved by the usual method of elimination (Art. 42), the values of the correlatives found are then to be substituted in (93), whence we obtain directly the required corrections,

$$\left.\begin{aligned}
v' &= \frac{1}{p'}\,(a'A + b'B + c'C + \ldots\ldots) \\
v'' &= \frac{1}{p''}\,(a''A + b''B + c''C + \ldots\ldots) \\
v''' &= \frac{1}{p'''}\,(a'''A + b'''B + c'''C + \ldots\ldots) \\
&\quad \&c. \qquad\qquad \&c.
\end{aligned}\right\} \quad (95)$$

and hence, finally, the most probable values of the observed quantities, $V' = M' + v'$, $V'' = M'' + v''$, &c.

The comparative simplicity of this process will best be shown by applying it to the example of the preceding article. We there have given, by observation,

$$\begin{aligned}
M' &= 65^\circ\ 11'\ 52''.500, & p' &= 3 \\
M'' &= 66\ \ 24\ \ 15\ .553, & p'' &= 3 \\
M''' &= 87\ \ \ 2\ \ 24\ .703, & p''' &= 3 \\
M^{\text{iv}} &= 141\ \ 21\ \ 21\ .757, & p^{\text{iv}} &= 1
\end{aligned}$$

with the condition

$$V' + V'' + V''' + V^{\text{iv}} - 360^\circ = 0$$

We have, first,

$$a' = a'' = a''' = a^{\text{iv}} = 1$$

and when $M'$, $M''$, &c. are put for $V'$, $V''$, &c., we have (88)

$$n' = -5''.487$$

As we have but one condition, we have also but one correlative $A$; the equation of condition is, by (89),

$$-5''.487 + v' + v'' + v''' + v^{\text{iv}} = 0$$

and the single normal equation may be constructed according to the following form:

| $p$ | $a$ | $\frac{aa}{p}$ |
|---|---|---|
| 3 | 1 | $\frac{1}{3}$ |
| 3 | 1 | $\frac{1}{3}$ |
| 3 | 1 | $\frac{1}{3}$ |
| 1 | 1 | 1 |

$$\left[\frac{aa}{p}\right] = 2$$

$$2A - 5''.487 = 0$$
$$A = +2''.7435$$

and hence, by (95),

| | | Corrected values. |
|---|---|---|
| $v' = +0.9145$ | | $V' = 65° \; 11' \; 53''.4145$ |
| $v'' = +0.9145$ | | $V'' = 66 \; 24 \; 16\,.4675$ |
| $v''' = +0.9145$ | | $V''' = 87 \; 2 \; 25\,.6175$ |
| $v^{\text{iv}} = +2.7435$ | | $V^{\text{iv}} = 141 \; 21 \; 24\,.5005$ |
| | | $360 \; 0 \; 0$ |

agreeing with the result found by the much longer process of the preceding article.

56. The further prosecution of this branch of the subject belongs more especially to works on Geodesy. For more extended examples, see the special report of Mr. C. A. SCHOTT in the Report of the Superintendent of the U. S. Coast Survey for 1854, from which the above example has been drawn. Consult also BESSEL'S *Gradmessung in Ostpreussen in* 1838; ROSENBERGER, in the *Astronomische Nachrichten*, Nos. 121 and 122; BESSEL, *ibid.* No. 438; T. GALLOWAY, Application of the Method to a Portion

of the Survey of England, in the *Memoirs of the Royal Astronomical Society*, Vol. XV.; J. J. BÆYER'S *Küstenvermessung;* FISCHER'S *Geodæsie;* GERLING'S *Ausgleichungs Rechnungen;* DIENGER'S *Ausgleichung der Beobachtungsfehler;* LIAGRE, *Calcul des Probabilités;* and GAUSS, *Supplementum theoriæ combinationis*, &c.

CRITERION FOR THE REJECTION OF DOUBTFUL OBSERVATIONS.

57. It has been already remarked (p. 490) that the number of large errors occurring in practice usually exceeds that given by theory, and that this discrepancy, instead of invalidating the theory of purely "accidental" errors, rather indicates a source or sources of error of an abnormal character, and calls for a criterion by which such abnormal observations may be excluded. The criterion proposed by Prof. PEIRCE* will be given here with the investigation nearly in the words of its author, and with only some slight changes of notation.

58. "In almost every true series of observations, some are found which differ so much from the others as to indicate some abnormal source of error not contemplated in the theoretical discussions, and the introduction of which into the investigations can only serve, in the present state of science, to perplex and mislead the inquirer. Geometers have, therefore, been in the habit of rejecting those observations which appeared to them liable to unusual defects, although no exact criterion has been proposed to test and authorize such a procedure, and this delicate subject has been left to the arbitrary discretion of individual computers. The object of the present investigation is to produce an exact rule for the rejection of observations, which shall be legitimately derived from the principles of the Calculus of Probabilities.

"*It is proposed to determine in a series of m observations the limit of error, beyond which all observations involving so great an error may be rejected, provided there are as many as n such observations.*

"The principle upon which it is proposed to solve this problem is, *that the proposed observations should be rejected when the probability of the system of errors obtained by retaining them is less than that of the system of errors obtained by their rejection multiplied by the probability of making so many, and no more, abnormal observations.*

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* *Astronomical Journal* (Cambridge, Mass.), Vol. II. p. 161.

"In determining the probability of these two systems of errors, it must be carefully observed that, because observations are rejected in the second system, the corresponding observations of the first system must be regarded, not as being limited to their actual values, but only as surpassing the limit of rejection."

Let

$\mu$ = the number of unknown quantities,
$m$ = the whole number of observations,
$n$ = the number of observations proposed to be rejected,
$n' = m - n$, the number to be retained,
$\Delta, \Delta', \Delta'', \ldots \Delta^{(m)}$ = the system of errors when no observation is rejected,
$\Delta_1, \Delta_1', \Delta_1'', \ldots \Delta_1^{(n')}$ = the system of errors when $n$ observations are rejected,
$\varepsilon, \varepsilon_1$ = the mean errors of the first and second system, respectively,
$y$ = the probability, supposed unknown, of such an abnormal observation that it is rejected on account of its magnitude,
$y' = 1 - y$ = the probability that an observation is not of the abnormal character which involves its rejection,
$\varkappa$ = the ratio of the required limit of error for the rejection of $n$ observations to the mean error $\varepsilon$, so that $\varkappa\varepsilon$ is the limiting error.

The probability of an error $\Delta$ in the first system will be, by (14) and (21),

$$\varphi\Delta = \frac{1}{\varepsilon\sqrt{2\pi}} e^{-\frac{\Delta^2}{2\varepsilon^2}}$$

and the same form will be used for the second system.

The probability of an error which exceeds the limit $\varkappa\varepsilon$ will be expressed by the integral (Arts. 8 and 12)

$$2\int_{\Delta=\kappa\varepsilon}^{\Delta=\infty} \varphi\Delta\, d\Delta$$

or, denoting this by $\psi\varkappa$,

$$\psi\varkappa = \frac{2}{\varepsilon\sqrt{2\pi}} \int_{\Delta=\kappa\varepsilon}^{\Delta=\infty} e^{-\frac{\Delta^2}{2\varepsilon^2}}\, d\Delta$$

which, by putting $t = \frac{\Delta}{\varepsilon\sqrt{2}}$, becomes

$$\psi\varkappa = \frac{2}{\sqrt{\pi}} \int_{t=\frac{\kappa}{\sqrt{2}}}^{\infty} e^{-tt}\, dt$$

and this may be found directly from Table IX. by subtracting the tabular number corresponding to $t = \frac{\varkappa}{\sqrt{2}}$ from unity.

The probability of the first system of errors, embodying the condition that $n$ observations exceed the limit $\varkappa\varepsilon$, is

$$P = \varphi\Delta \,.\, \varphi\Delta' \,.\, \varphi\Delta'' \,\ldots\ldots \left(\frac{\psi\varkappa}{\varphi(\varkappa\varepsilon)}\right)^n$$

$$= \frac{1}{\varepsilon^{n'}(2\pi)^{\frac{1}{2}n'}}\, e^{-\frac{\Sigma\Delta^2 - n\kappa^2\varepsilon^2}{2\varepsilon^2}} (\psi\varkappa)^n$$

in which $\Sigma\Delta^2 = \Delta^2 + \Delta'^2 + \ldots . (\Delta^{(n)})^2$; and by (61) we have $\Sigma\Delta^2 = (m - \mu)\,\varepsilon^2$, whence

$$P = \frac{1}{\varepsilon^{n'}(2\pi)^{\frac{1}{2}n'}}\, e^{\frac{1}{2}(-m + \mu + n\kappa^2)} (\psi\varkappa)^n$$

The probability of the second system of errors is

$$P_1 = y^n y'^{n'} \,.\, \varphi\Delta_1 \,.\, \varphi\Delta_1' \,.\, \varphi\Delta_1'' \ldots = \frac{y^n y'^{n'}}{\varepsilon_1^{n'}(2\pi)^{\frac{1}{2}n'}}\, e^{-\frac{\Sigma\Delta_1^2}{2\varepsilon_1^2}}$$

$$= \frac{y^n y'^{n'}}{\varepsilon_1^{n'}(2\pi)^{\frac{1}{2}n'}}\, e^{\frac{1}{2}(-n' + \mu)}$$

To authorize the proposed rejection of $n$ observations, we must have

$$P < P_1$$

which gives at once

$$\left(\frac{\varepsilon_1}{\varepsilon}\right)^{n'} e^{\frac{1}{2}n(\kappa^2 - 1)} (\psi\varkappa)^n < y^n y'^{n'}$$

The value of $y$ must be determined by the condition that $P_1$ is a maximum, and therefore $y^n y'^{n'} = y^n\,(1 - y)^{n'}$ is a maximum. Taking the logarithm of this quantity, and putting its differential equal to zero, we obtain for the maximum

$$\frac{y}{n} = \frac{y'}{n'} = \frac{1 - y}{n'}$$

whence

$$y = \frac{n}{m} \qquad y' = \frac{n'}{m}$$

Putting then

$$\left.\begin{aligned} T^n &= y^n y'^{n'} = \frac{n^n n'^{n'}}{m^m} \\ R &= e^{\frac{1}{2}(\varkappa^2 - 1)} (\psi\varkappa) \end{aligned}\right\} \quad (96)$$

the limiting value of $\varkappa$, according to the above inequality, must be that which satisfies the equation

$$\left(\frac{\varepsilon_1}{\varepsilon}\right)^{n'} R^n = T^n$$

which gives the required criterion.

The relation of $\varepsilon_1$ to $\varepsilon$ must depend on the nature of the equations which correspond to the rejected observations; but it will give a sufficient approximation to assume that the excess of $\Sigma \Delta^2$ over $\Sigma \Delta_1^2$ is only equal to the sum of the squares of the errors of the rejected observations, which gives the equation

$$(m - \mu)\, \varepsilon^2 - n\varkappa^2 \varepsilon^2 = (m - \mu - n)\, \varepsilon_1^2$$

whence

$$\left(\frac{\varepsilon_1}{\varepsilon}\right)^2 = \frac{m - \mu - n\varkappa^2}{m - \mu - n}$$

which combined with the above equation gives

$$\frac{m - \mu - n\varkappa^2}{m - \mu - n} = \left(\frac{T}{R}\right)^{\frac{2n}{m-n}}$$

Putting, for brevity,

$$\lambda^2 = \left(\frac{T}{R}\right)^{\frac{2n}{m-n}} \quad (97)$$

we find

$$\varkappa^2 - 1 = \frac{m - \mu - n}{n} (1 - \lambda^2) \quad (98)$$

Table X.A gives the logarithms of $T$ and $R$, computed by (96) with the aid of Table IX. We can, therefore, by successive approximations, find the value of $\varkappa$ which satisfies the equations (97) and (98). Since $R$ involves $\varkappa$, we must first assume an approximate value of $\varkappa$ (which the observed residuals will suggest), with which $\lambda^2$ will be computed by (97), and hence $\varkappa$ by (98).

With this first approximate value of $\varkappa$, a new value of $\log R$ will be taken from the table, with which a second approximation to $\varkappa$ will be found. Two or three approximations will usually be found sufficient.

In the application of this criterion, it is to be remembered that it must not be used to reject $n$ observations unless it has previously rejected $n - 1$ observations. Hence we must first determine the limiting value of $\varkappa$ for the hypothesis of one doubtful observation, or $n = 1$, and if this rejects one or more observations, we can pass to the next hypothesis, $n = 2$, or $n = 3$, &c.; and so on until we arrive at the limit which excludes no more observations.

The above arrangement of the tables is nearly the same as that given by Dr. B. A. Gould,* who was the first to prepare such tables and thus render the criterion available to practical computers. The only difference is in my table of Log. $T$, which I have found in practice to be more convenient than the corresponding one of Dr. Gould.

Example.—"To determine the limit of rejection of one or two observations in the case of fifteen observations of the vertical semidiameters of *Venus*, made by Lieut. Herndon, with the meridian circle at Washington, in the year 1846." In the reduction of these observations, Prof. Peirce assumed two unknown quantities, and found the following residuals ($v$):

| | | | |
|---|---|---|---|
| — 0″.30 | — 0″.24 | — 1″.40 | + 0″.18 |
| — 0 .44 | + 0 .06 | — 0 .22 | + 0 .39 |
| + 1 .01 | + 0 .63 | — 0 .05 | + 0 .10 |
| + 0 .48 | — 0 .13 | + 0 .20 | |

We have here $m = 15$, $\mu = 2$, $[vv] = 4.2545$, whence

$$\varepsilon^2 = \frac{4.2545}{13} = 0.3273, \qquad \varepsilon = 0''.572$$

We first try the hypothesis of *one* doubtful observation, or $n = 1$. Assuming $\varkappa = 2$, the successive approximations may be made as follows:

* Report of the Superintendent of the U. S. Coast Survey for 1854, Appendix, p. 131*; also *Astron. Journal*, Vol. IV. p. 81.

| | | 1st Approx. | 2d Approx. |
|---|---|---|---|
| | Table X.A. $\log T$ | 8.404 | 8.4044 |
| | " " $\log R$ | 9.309 | 9.3062 |
| | $\log \frac{T}{R}$ | 9.095 | 9.0982 |
| $\frac{2n}{m-n} = \frac{1}{7}$ | $\log \lambda^2$ | 9.871 | 9.8712 |
| | $\log (1 - \lambda^2)$ | 9.410 | 9.4093 |
| $\frac{m-\mu-n}{n} = 12$ | $\log 12$ | 1.079 | 1.0792 |
| | $\log (\varkappa^2 - 1)$ | 0.489 | 0.4885 |
| | $\log \varkappa^2$ | 0.610 | 0.6106 |
| | $\varkappa$ | 2.02 | 2.020 |

Hence $\varkappa\varepsilon = 1''.16$, which excludes the residual $1''.40$.

We may now try the hypothesis $n = 2$. Commencing again with the assumption $\varkappa = 2$, we have—

| | | 1st Approx. | 2d Approx. | 3d Approx. | 4th Approx. |
|---|---|---|---|---|---|
| | $\log T$ | 8.7210 | 8.7210 | 8.7210 | 8.7210 |
| | $\log R$ | 9.309 | 9.3622 | 9.3544 | 9.3553 |
| | $\log \frac{T}{R}$ | 9.412 | 9.3588 | 9.3666 | 9.3657 |
| $\frac{2n}{m-n} = \frac{4}{13}$ | $\log \lambda^2$ | 9.819 | 9.8027 | 9.8051 | 9.8048 |
| | $\log (1 - \lambda^2)$ | 9.531 | 9.5624 | 9.5582 | 9.5587 |
| $\frac{m-\mu-n}{n} = \frac{11}{2}$ | $\log \frac{11}{2}$ | 0.740 | 0.7404 | 0.7404 | 0.7404 |
| | $\log (\varkappa^2 - 1)$ | 0.271 | 0.3028 | 0.2986 | 0.2991 |
| | $\log \varkappa^2$ | 0.457 | 0.4783 | 0.4755 | 0.4758 |
| | $\varkappa$ | 1.69 | 1.734 | 1.729 | 1.7295 |

Hence $\varkappa\varepsilon = 0''.989$, which excludes the residuals $1''.40$ and $1''.01$.

If we now try the hypothesis $n = 3$, we shall find, in the same manner, $\varkappa\varepsilon = 0''.887$, which does not exclude the residual $0''.63$: so that the residuals $1''.40$ and $1''.01$ are in this case the only abnormal ones. Rejecting these residuals, we shall now find $\varepsilon_1 = 0''.339$.*

59. In order to facilitate the application of PEIRCE'S Criterion

* For another example, in which there were four unknown quantities, and in which the criterion was very useful, see p. 207 of this volume.

in the cases most commonly occurring in practice, Table X. (first given by Dr. GOULD) has been computed by the aid of the log $T$ and log $R$, according to the preceding method.

The first page of this table is to be used when there is but one unknown quantity ($\mu = 1$), or for direct observations. It gives, by simple inspection, the value of $x^2$ for any number of observations from 3 to 60, and for any number of doubtful observations from 1 to 9.

The second page is used in the same manner when there are two unknown quantities ($\mu = 2$).

EXAMPLE.—Same as in the preceding article.—Having found, as above, $\varepsilon^2 = 0.3273$, we first take from Table X. for $\mu = 2$ the value of $x^2$ corresponding to $m = 15$ and $n = 1$, and find

$$x^2 = 4.080, \text{ whence } x^2\varepsilon^2 = 1.3354, \qquad x\varepsilon = 1''.16$$

which rejects the residual 1''.40.

Then, with $m = 15$, $n = 2$, we find, from the same page,

$$x^2 = 2.991, \qquad x^2\varepsilon^2 = 0.9790, \qquad x\varepsilon = 0''.989$$

which rejects the two residuals 1''.40 and 1''.01.

Passing, then, to the hypothesis $n = 3$, we find

$$x^2 = 2.403, \qquad x^2\varepsilon^2 = 0.7865, \qquad x\varepsilon = 0''.887$$

which does not exclude any more residuals.

60. The above investigation of the criterion involves some principles, derived from the theory of probabilities, which may seem obscure to those not familiar with that branch of science. Indeed, the possibility of establishing any criterion whatever for the rejection of doubtful observations, by the aid of the calculus of probabilities, has been questioned even by so distinguished an astronomer as AIRY.* It is easy, however, to derive an approximate criterion *for the rejection of one doubtful observation*, directly from the fundamental formula upon which the whole theory of the method of least squares is based.

We have seen that the function

* Remarks upon PEIRCE's Criterion, *Astronomical Journal* (Cambridge), Vol. IV. p. 137. Professor WINLOCK's reply to the objections of the Astronomer Royal will be found in the same journal, Vol. IV. p. 145.

$$\Theta(\rho t') = \frac{2}{\sqrt{\pi}} \int_0^{\rho t'} e^{-tt}\, dt$$

(the value of which is given in Table IX.A) represents, in general, the number of errors less than $a = rt'$ which may be expected to occur in any extended series of observations when the whole number of observations is taken as unity, $r$ being the probable error of an observation. If this be multiplied by the number of observations $= m$, we shall have the actual number of errors less than $rt'$; and hence the quantity

$$m - m \,.\, \Theta(\rho t') = m\,[1 - \Theta(\rho t')]$$

expresses the number of errors to be expected *greater* than the limit $rt'$. But if this quantity is less than $\frac{1}{2}$, it will follow that an error of the magnitude $rt'$ will have a greater probability against it than for it, and may therefore be rejected. The limit of rejection *of a single doubtful observation*, according to this simple rule, is, therefore, obtained from the equation

$$\tfrac{1}{2} = m\,[1 - \Theta(\rho t')]$$

or

$$\Theta(\rho t') = \frac{2m - 1}{2m} \tag{99}$$

If we express the limiting error under the form $\varkappa\varepsilon$, $\varepsilon$ being the mean error of an observation, we shall have

$$\varkappa = \frac{rt'}{\varepsilon} = 0.6745\,t' \tag{100}$$

With the value of $\Theta(\rho t')$ given by (99), we can find $t'$ from Table IX.A, and hence $\varkappa$ by (100).

EXAMPLE.—To find the limit of rejection of *one* of the observations given on p. 562. We there have $m = 15$, $\varepsilon = 0''.572$; and hence, by (99), $\Theta(\rho t') = 0.96667$, which in Table IX.A corresponds to $t' = 3.155$, whence, by (100), $\varkappa = 2.128$, $\varkappa\varepsilon = 1''.22$, which agrees very nearly with the limit found by PEIRCE'S Criterion.

By the successive application of this rule (with the necessary modifications), it may be used for the rejection of two or more doubtful observations, and I have, by means of it, prepared a table which agrees so nearly with Table X. that, for practical purposes, it may be regarded as identical with that table. For the general case, however, when there are several unknown quantities and several doubtful observations, the modifications which the rule requires render it more troublesome than PEIRCE's formula, and I shall, therefore, not develop it further in this place. What I have given may serve the purpose of giving the reader greater confidence in the correctness and value of PEIRCE's Criterion.

# TABLE IX. Probability of Errors.

(Method of Least Squares.)

$$\Theta(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt$$

| t | Θ(t) | Diff. | t | Θ(t) | Diff. | t | Θ(t) | Diff. | t | Θ(t) | Diff. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **0.00** | 0.00000 | 1128 | **0.50** | 0.52050 | 874 | **1.00** | 0.84270 | 411 | **1.50** | 0.96611 | 117 |
| **0.01** | .01128 | 1128 | **0.51** | .52924 | 866 | **1.01** | .84681 | 403 | **1.51** | .96728 | 113 |
| **0.02** | .02256 | 1128 | **0.52** | .53790 | 856 | **1.02** | .85084 | 394 | **1.52** | .96841 | 111 |
| **0.03** | .03384 | 1127 | **0.53** | .54646 | 848 | **1.03** | .85478 | 387 | **1.53** | .96952 | 107 |
| **0.04** | .04511 | 1126 | **0.54** | .55494 | 838 | **1.04** | .85865 | 379 | **1.54** | .97059 | 103 |
| **0.05** | 0.05637 | 1125 | **0.55** | 0.56332 | 830 | **1.05** | 0.86244 | 370 | **1.55** | 0.97162 | 101 |
| **0.06** | .06762 | 1124 | **0.56** | .57162 | 820 | **1.06** | .86614 | 363 | **1.56** | .97263 | 97 |
| **0.07** | .07886 | 1122 | **0.57** | .57982 | 810 | **1.07** | .86977 | 356 | **1.57** | .97360 | 95 |
| **0.08** | .09008 | 1120 | **0.58** | .58792 | 802 | **1.08** | .87333 | 347 | **1.58** | .97455 | 91 |
| **0.09** | .10128 | 1118 | **0.59** | .59594 | 792 | **1.09** | .87680 | 341 | **1.59** | .97546 | 89 |
| **0.10** | 0.11246 | 1116 | **0.60** | 0.60386 | 782 | **1.10** | 0.88021 | 332 | **1.60** | 0.97635 | 86 |
| **0.11** | .12362 | 1114 | **0.61** | .61168 | 773 | **1.11** | .88353 | 326 | **1.61** | .97721 | 83 |
| **0.12** | .13476 | 1111 | **0.62** | .61941 | 764 | **1.12** | .88679 | 318 | **1.62** | .97804 | 80 |
| **0.13** | .14587 | 1108 | **0.63** | .62705 | 754 | **1.13** | .88997 | 311 | **1.63** | .97884 | 78 |
| **0.14** | .15695 | 1105 | **0.64** | .63459 | 744 | **1.14** | .89308 | 304 | **1.64** | .97962 | 76 |
| **0.15** | 0.16800 | 1101 | **0.65** | 0.64203 | 735 | **1.15** | 0.89612 | 298 | **1.65** | 0.98038 | 72 |
| **0.16** | .17901 | 1098 | **0.66** | .64938 | 725 | **1.16** | .89910 | 290 | **1.66** | .98110 | 71 |
| **0.17** | .18999 | 1095 | **0.67** | .65663 | 715 | **1.17** | .90200 | 284 | **1.67** | .98181 | 68 |
| **0.18** | .20094 | 1090 | **0.68** | .66378 | 706 | **1.18** | .90484 | 277 | **1.68** | .98249 | 66 |
| **0.19** | .21184 | 1086 | **0.69** | .67084 | 696 | **1.19** | .90761 | 270 | **1.69** | .98315 | 64 |
| **0.20** | 0.22270 | 1082 | **0.70** | 0.67780 | 687 | **1.20** | 0.91031 | 265 | **1.70** | 0.98379 | 62 |
| **0.21** | .23352 | 1078 | **0.71** | .68467 | 676 | **1.21** | .91296 | 257 | **1.71** | .98441 | 59 |
| **0.22** | .24430 | 1072 | **0.72** | .69143 | 667 | **1.22** | .91553 | 252 | **1.72** | .98500 | 58 |
| **0.23** | .25502 | 1068 | **0.73** | .69810 | 658 | **1.23** | .91805 | 246 | **1.73** | .98558 | 55 |
| **0.24** | .26570 | 1063 | **0.74** | .70468 | 648 | **1.24** | .92051 | 239 | **1.74** | .98613 | 54 |
| **0.25** | 0.27633 | 1057 | **0.75** | 0.71116 | 638 | **1.25** | 0.92290 | 234 | **1.75** | 0.98667 | 52 |
| **0.26** | .28690 | 1052 | **0.76** | .71754 | 628 | **1.26** | .92524 | 227 | **1.76** | .98719 | 50 |
| **0.27** | .29742 | 1046 | **0.77** | .72382 | 619 | **1.27** | .92751 | 222 | **1.77** | .98769 | 48 |
| **0.28** | .30788 | 1040 | **0.78** | .73001 | 609 | **1.28** | .92973 | 217 | **1.78** | .98817 | 47 |
| **0.29** | .31828 | 1035 | **0.79** | .73610 | 600 | **1.29** | .93190 | 211 | **1.79** | .98864 | 45 |
| **0.30** | 0.32863 | 1028 | **0.80** | 0.74210 | 590 | **1.30** | 0.93401 | 205 | **1.80** | 0.98909 | 43 |
| **0.31** | .33891 | 1022 | **0.81** | .74800 | 581 | **1.31** | .93606 | 201 | **1.81** | .98952 | 42 |
| **0.32** | .34913 | 1015 | **0.82** | .75381 | 571 | **1.32** | .93807 | 195 | **1.82** | .98994 | 41 |
| **0.33** | .35928 | 1008 | **0.83** | .75952 | 562 | **1.33** | .94002 | 189 | **1.83** | .99035 | 39 |
| **0.34** | .36936 | 1002 | **0.84** | .76514 | 553 | **1.34** | .94191 | 185 | **1.84** | .99074 | 37 |
| **0.35** | 0.37938 | 995 | **0.85** | 0.77067 | 543 | **1.35** | 0.94376 | 180 | **1.85** | 0.99111 | 36 |
| **0.36** | .38933 | 988 | **0.86** | .77610 | 534 | **1.36** | .94556 | 175 | **1.86** | .99147 | 35 |
| **0.37** | .39921 | 980 | **0.87** | .78144 | 525 | **1.37** | .94731 | 171 | **1.87** | .99182 | 34 |
| **0.38** | .40901 | 973 | **0.88** | .78669 | 515 | **1.38** | .94902 | 165 | **1.88** | .99216 | 32 |
| **0.39** | .41874 | 965 | **0.89** | .79184 | 507 | **1.39** | .95067 | 162 | **1.89** | .99248 | 31 |
| **0.40** | 0.42839 | 958 | **0.90** | 0.79691 | 497 | **1.40** | 0.95229 | 156 | **1.90** | 0.99279 | 30 |
| **0.41** | .43797 | 950 | **0.91** | .80188 | 489 | **1.41** | .95385 | 153 | **1.91** | .99309 | 29 |
| **0.42** | .44747 | 942 | **0.92** | .80677 | 479 | **1.42** | .95538 | 148 | **1.92** | .99338 | 28 |
| **0.43** | .45689 | 934 | **0.93** | .81156 | 471 | **1.43** | .95686 | 144 | **1.93** | .99366 | 26 |
| **0.44** | .46623 | 925 | **0.94** | .81627 | 462 | **1.44** | .95830 | 140 | **1.94** | .99392 | 26 |
| **0.45** | 0.47548 | 918 | **0.95** | 0.82089 | 453 | **1.45** | 0.95970 | 135 | **1.95** | 0.99418 | 25 |
| **0.46** | .48466 | 909 | **0.96** | .82542 | 445 | **1.46** | .96105 | 132 | **1.96** | .99443 | 23 |
| **0.47** | .49375 | 900 | **0.97** | .82987 | 436 | **1.47** | .96237 | 128 | **1.97** | .99466 | 23 |
| **0.48** | .50275 | 892 | **0.98** | .83423 | 428 | **1.48** | .96365 | 125 | **1.98** | .99489 | 22 |
| **0.49** | .51167 | 883 | **0.99** | .83851 | 419 | **1.49** | .96490 | 121 | **1.99** | .99511 | 21 |
| **0.50** | 0.52050 | | **1.00** | 0.84270 | | **1.50** | 0.96611 | | **2.00** | 0.99532 | |

# TABLE IX.A. Probability of Errors.

(Method of Least Squares.)

$$\Theta(\rho t') = \frac{2}{\sqrt{\pi}} \int_0^{\rho t'} e^{-u}\, dt \qquad t' = \frac{a}{r}$$

| $t'$ | $\Theta(\rho t')$ | Diff. | $t'$ | $\Theta(\rho t')$ | Diff. | $t'$ | $\Theta(\rho t')$ | Diff. | $t'$ | $\Theta(\rho t')$ | Diff. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **0.00** | 0.00000 | 538 | **0.50** | 0.26407 | 508 | **1.00** | 0.50000 | 428 | **1.50** | 0.68833 | 322 |
| **0.01** | .00538 | 538 | **0.51** | .26915 | 506 | **1.01** | .50428 | 425 | **1.51** | .69155 | 319 |
| **0.02** | .01076 | 538 | **0.52** | .27421 | 506 | **1.02** | .50853 | 424 | **1.52** | .69474 | 317 |
| **0.03** | .01614 | 538 | **0.53** | .27927 | 504 | **1.03** | .51277 | 422 | **1.53** | .69791 | 315 |
| **0.04** | .02152 | 538 | **0.54** | .28431 | 503 | **1.04** | .51699 | 420 | **1.54** | .70106 | 313 |
| **0.05** | 0.02690 | 538 | **0.55** | 0.28934 | 502 | **1.05** | 0.52119 | 418 | **1.55** | 0.70419 | 310 |
| **0.06** | .03228 | 538 | **0.56** | .29436 | 500 | **1.06** | .52537 | 415 | **1.56** | .70729 | 309 |
| **0.07** | .03766 | 537 | **0.57** | .29936 | 499 | **1.07** | .52952 | 414 | **1.57** | .71038 | 306 |
| **0.08** | .04303 | 537 | **0.58** | .30435 | 498 | **1.08** | .53366 | 412 | **1.58** | .71344 | 304 |
| **0.09** | .04840 | 538 | **0.59** | .30933 | 497 | **1.09** | .53778 | 410 | **1.59** | .71648 | 301 |
| **0.10** | 0.05378 | 536 | **0.60** | 0.31430 | 495 | **1.10** | 0.54188 | 407 | **1.60** | 0.71949 | 300 |
| **0.11** | .05914 | 537 | **0.61** | .31925 | 494 | **1.11** | .54595 | 406 | **1.61** | .72249 | 297 |
| **0.12** | .06451 | 536 | **0.62** | .32419 | 492 | **1.12** | .55001 | 403 | **1.62** | .72546 | 295 |
| **0.13** | .06987 | 536 | **0.63** | .32911 | 491 | **1.13** | .55404 | 402 | **1.63** | .72841 | 293 |
| **0.14** | .07523 | 536 | **0.64** | .33402 | 490 | **1.14** | .55806 | 399 | **1.64** | .73134 | 291 |
| **0.15** | 0.08059 | 535 | **0.65** | 0.33892 | 488 | **1.15** | 0.56205 | 397 | **1.65** | 0.73425 | 289 |
| **0.16** | .08594 | 535 | **0.66** | .34380 | 486 | **1.16** | .56602 | 396 | **1.66** | .73714 | 286 |
| **0.17** | .09129 | 534 | **0.67** | .34866 | 486 | **1.17** | .56998 | 393 | **1.67** | .74000 | 285 |
| **0.18** | .09663 | 534 | **0.68** | .35352 | 483 | **1.18** | .57391 | 391 | **1.68** | .74285 | 282 |
| **0.19** | .10197 | 534 | **0.69** | .35835 | 482 | **1.19** | .57782 | 389 | **1.69** | .74567 | 280 |
| **0.20** | 0.10731 | 533 | **0.70** | 0.36317 | 481 | **1.20** | 0.58171 | 387 | **1.70** | 0.74847 | 277 |
| **0.21** | .11264 | 532 | **0.71** | .36798 | 479 | **1.21** | .58558 | 384 | **1.71** | .75124 | 276 |
| **0.22** | .11796 | 532 | **0.72** | .37277 | 478 | **1.22** | .58942 | 383 | **1.72** | .75400 | 274 |
| **0.23** | .12328 | 532 | **0.73** | .37755 | 476 | **1.23** | .59325 | 380 | **1.73** | .75674 | 271 |
| **0.24** | .12860 | 531 | **0.74** | .38231 | 474 | **1.24** | .59705 | 378 | **1.74** | .75945 | 269 |
| **0.25** | 0.13391 | 530 | **0.75** | 0.38705 | 473 | **1.25** | 0.60083 | 376 | **1.75** | 0.76214 | 267 |
| **0.26** | .13921 | 530 | **0.76** | .39178 | 471 | **1.26** | .60459 | 374 | **1.76** | 76481 | 265 |
| **0.27** | .14451 | 529 | **0.77** | .39649 | 469 | **1.27** | .60833 | 372 | **1.77** | .76746 | 263 |
| **0.28** | .14980 | 528 | **0.78** | .40118 | 468 | **1.28** | .61205 | 370 | **1.78** | .77009 | 261 |
| **0.29** | .15508 | 527 | **0.79** | .40586 | 466 | **1.29** | .61575 | 367 | **1.79** | .77270 | 258 |
| **0.30** | 0.16035 | 527 | **0.80** | 0.41052 | 465 | **1.30** | 0.61942 | 366 | **1.80** | 0.77528 | 257 |
| **0.31** | .16562 | 526 | **0.81** | .41517 | 462 | **1.31** | .62308 | 363 | **1.81** | .77785 | 254 |
| **0.32** | .17088 | 526 | **0.82** | .41979 | 461 | **1.32** | .62671 | 361 | **1.82** | .78039 | 252 |
| **0.33** | .17614 | 524 | **0.83** | .42440 | 459 | **1.33** | .63032 | 359 | **1.83** | .78291 | 251 |
| **0.34** | .18138 | 524 | **0.84** | .42899 | 458 | **1.34** | .63391 | 356 | **1.84** | .78542 | 248 |
| **0.35** | 0.18662 | 523 | **0.85** | 0.43357 | 456 | **1.35** | 0.63747 | 355 | **1.85** | 0.78790 | 246 |
| **0.36** | .19185 | 522 | **0.86** | .43813 | 454 | **1.36** | .64102 | 352 | **1.86** | .79036 | 244 |
| **0.37** | .19707 | 522 | **0.87** | .44267 | 452 | **1.37** | .64454 | 350 | **1.87** | .79280 | 242 |
| **0.38** | .20229 | 520 | **0.88** | .44719 | 450 | **1.38** | .64804 | 348 | **1.88** | .79522 | 239 |
| **0.39** | .20749 | 519 | **0.89** | .45169 | 449 | **1.39** | .65152 | 346 | **1.89** | .79761 | 238 |
| **0.40** | 0.21268 | 519 | **0.90** | 0.45618 | 446 | **1.40** | 0.65498 | 343 | **1.90** | 0.79999 | 236 |
| **0.41** | .21787 | 517 | **0.91** | .46064 | 445 | **1.41** | .65841 | 341 | **1.91** | .80235 | 234 |
| **0.42** | .22304 | 517 | **0.92** | .46509 | 443 | **1.42** | .66182 | 339 | **1.92** | .80469 | 231 |
| **0.43** | .22821 | 515 | **0.93** | .46952 | 441 | **1.43** | .66521 | 337 | **1.93** | .80700 | 230 |
| **0.44** | .23336 | 515 | **0.94** | .47393 | 439 | **1.44** | .66858 | 335 | **1.94** | .80930 | 228 |
| **0.45** | 0.23851 | 513 | **0.95** | 0.47832 | 438 | **1.45** | 0.67193 | 333 | **1.95** | 0.81158 | 225 |
| **0.46** | .24364 | 512 | **0.96** | .48270 | 435 | **1.46** | .67526 | 330 | **1.96** | .81383 | 224 |
| **0.47** | .24876 | 512 | **0.97** | .48705 | 434 | **1.47** | .67856 | 328 | **1.97** | .81607 | 221 |
| **0.48** | .25388 | 510 | **0.98** | .49139 | 431 | **1.48** | .68184 | 326 | **1.98** | .81828 | 220 |
| **0.49** | .25898 | 509 | **0.99** | .49570 | 430 | **1.49** | .68510 | 323 | **1.99** | .82048 | 218 |
| **0.50** | 0.26407 | | **1.00** | 0.50000 | | **1.50** | 0.68833 | | **2.00** | 0.82266 | |

# TABLE IX.A. Probability of Errors.

(Method of Least Squares.)

$$\Theta(\rho t') = \frac{2}{\sqrt{\pi}} \int_0^{\rho t'} e^{-t^2}\, dt \qquad t' = \frac{a}{r}$$

| $t'$ | $\Theta(\rho t')$ | Diff. | $t'$ | $\Theta(\rho t')$ | Diff. | $t'$ | $\Theta(\rho t')$ | Diff. | $t'$ | $\Theta(\rho t')$ | Diff. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **2.00** | 0.82266 | 215 | **2.50** | 0.90825 | 129 | **3.00** | 0.95698 | 69 | **3.50** | 0.98176 | 306 |
| **2.01** | .82481 | 214 | **2.51** | .90954 | 128 | **3.01** | .95767 | 68 | **3.60** | .98482 | 261 |
| **2.02** | .82695 | 212 | **2.52** | .91082 | 126 | **3.02** | .95835 | 67 | **3.70** | .98743 | 219 |
| **2.03** | .82907 | 210 | **2.53** | .91208 | 124 | **3.03** | .95902 | 66 | **3.80** | .98962 | 185 |
| **2.04** | .83117 | 207 | **2.54** | .91332 | 124 | **3.04** | .95968 | 65 | **3.90** | .99147 | 155 |
| **2.05** | 0.83324 | 206 | **2.55** | 0.91456 | 122 | **3.05** | 0.96033 | 65 | **4.00** | 0.99302 | 129 |
| **2.06** | .83530 | 204 | **2.56** | .91578 | 120 | **3.06** | .96098 | 63 | **4.10** | .99431 | 108 |
| **2.07** | .83734 | 202 | **2.57** | .91698 | 119 | **3.07** | .96161 | 63 | **4.20** | .99539 | 88 |
| **2.08** | .83936 | 201 | **2.58** | .91817 | 118 | **3.08** | .96224 | 62 | **4.30** | .99627 | 73 |
| **2.09** | .84137 | 198 | **2.59** | .91935 | 116 | **3.09** | .96286 | 60 | **4.40** | .99700 | 60 |
| **2.10** | 0.84335 | 196 | **2.60** | 0.92051 | 115 | **3.10** | 0.96346 | 60 | **4.50** | 0.99760 | 48 |
| **2.11** | .84531 | 195 | **2.61** | .92166 | 114 | **3.11** | .96406 | 60 | **4.60** | .99808 | 40 |
| **2.12** | .84726 | 193 | **2.62** | .92280 | 112 | **3.12** | .96466 | 58 | **4.70** | .99848 | 31 |
| **2.13** | .84919 | 190 | **2.63** | .92392 | 111 | **3.13** | .96524 | 58 | **4.80** | .99879 | 26 |
| **2.14** | .85109 | 189 | **2.64** | .92503 | 110 | **3.14** | .96582 | 56 | **4.90** | .99905 | 21 |
| **2.15** | 0.85298 | 188 | **2.65** | 0.92613 | 108 | **3.15** | 0.96638 | 56 | **5.00** | 0.99926 | |
| **2.16** | .85486 | 185 | **2.66** | .92721 | 107 | **3.16** | .96694 | 55 | $\infty$ | 1.00000 | |
| **2.17** | .85671 | 183 | **2.67** | .92828 | 106 | **3.17** | .96749 | 55 | | | |
| **2.18** | .85854 | 182 | **2.68** | .92934 | 104 | **3.18** | .96804 | 53 | | | |
| **2.19** | .86036 | 180 | **2.69** | .93038 | 103 | **3.19** | .96857 | 53 | | | |
| **2.20** | 0.86216 | 178 | **2.70** | 0.93141 | 102 | **3.20** | 0.96910 | 52 | | | |
| **2.21** | .86394 | 176 | **2.71** | .93243 | 101 | **3.21** | .96962 | 51 | | | |
| **2.22** | .86570 | 175 | **2.72** | .93344 | 99 | **3.22** | .97013 | 51 | | | |
| **2.23** | .86745 | 172 | **2.73** | .93443 | 98 | **3.23** | .97064 | 50 | | | |
| **2.24** | .86917 | 171 | **2.74** | .93541 | 97 | **3.24** | .97114 | 49 | | | |
| **2.25** | 0.87088 | 170 | **2.75** | 0.93638 | 96 | **3.25** | 0.97163 | 48 | | | |
| **2.26** | .87258 | 167 | **2.76** | .93734 | 94 | **3.26** | .97211 | 48 | | | |
| **2.27** | .87425 | 166 | **2.77** | .93828 | 94 | **3.27** | .97259 | 47 | | | |
| **2.28** | .87591 | 164 | **2.78** | .93922 | 92 | **3.28** | .97306 | 46 | | | |
| **2.29** | .87755 | 163 | **2.79** | .94014 | 91 | **3.29** | .97352 | 45 | | | |
| **2.30** | 0.87918 | 160 | **2.80** | 0.94105 | 90 | **3.30** | 0.97397 | 45 | | | |
| **2.31** | .88078 | 159 | **2.81** | .94195 | 89 | **3.31** | .97442 | 44 | | | |
| **2.32** | .88237 | 158 | **2.82** | .94284 | 87 | **3.32** | .97486 | 44 | | | |
| **2.33** | .88395 | 155 | **2.83** | .94371 | 87 | **3.33** | .97530 | 43 | | | |
| **2.34** | .88550 | 155 | **2.84** | .94458 | 85 | **3.34** | .97573 | 42 | | | |
| **2.35** | 0.88705 | 152 | **2.85** | 0.94543 | 84 | **3.35** | 0.97615 | 42 | | | |
| **2.36** | .88857 | 151 | **2.86** | .94627 | 84 | **3.36** | .97657 | 41 | | | |
| **2.37** | .89008 | 149 | **2.87** | .94711 | 82 | **3.37** | .97698 | 40 | | | |
| **2.38** | .89157 | 147 | **2.88** | .94793 | 81 | **3.38** | .97738 | 40 | | | |
| **2.39** | .89304 | 146 | **2.89** | .94874 | 80 | **3.39** | .97778 | 39 | | | |
| **2.40** | 0.89450 | 145 | **2.90** | 0.94954 | 79 | **3.40** | 0.97817 | 38 | | | |
| **2.41** | .89595 | 143 | **2.91** | .95033 | 78 | **3.41** | .97855 | 38 | | | |
| **2.42** | .89738 | 141 | **2.92** | .95111 | 76 | **3.42** | .97893 | 37 | | | |
| **2.43** | .89879 | 140 | **2.93** | .95187 | 76 | **3.43** | .97930 | 37 | | | |
| **2.44** | .90019 | 138 | **2.94** | .95263 | 75 | **3.44** | .97967 | 36 | | | |
| **2.45** | 0.90157 | 136 | **2.95** | 0.95338 | 74 | **3.45** | 0.98003 | 36 | | | |
| **2.46** | .90293 | 135 | **2.96** | .95412 | 73 | **3.46** | .98039 | 35 | | | |
| **2.47** | .90428 | 134 | **2.97** | .95485 | 72 | **3.47** | .98074 | 35 | | | |
| **2.48** | .90562 | 132 | **2.98** | .95557 | 71 | **3.48** | .98109 | 34 | | | |
| **2.49** | .90694 | 131 | **2.99** | .95628 | 70 | **3.49** | .98143 | 33 | | | |
| **2.50** | 0.90825 | | **3.00** | 0.95698 | | **3.50** | 0.98176 | | | | |

# TABLE X. Peirce's Criterion.

Values of $x^2$ for $\mu = 1$.

| m | n = 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 1.480 | ...... | ...... | ...... | ...... | ...... | ...... | ...... | ...... |
| 4 | 1.912 | 1.163 | ...... | ...... | ...... | ...... | ...... | ...... | ...... |
| 5 | 2.278 | 1.439 | ...... | ...... | ...... | ...... | ...... | ...... | ...... |
| 6 | 2.592 | 1.687 | 1.208 | ...... | ...... | ...... | ...... | ...... | ...... |
| 7 | 2.866 | 1.910 | 1.409 | 1.045 | ...... | ...... | ...... | ...... | ...... |
| 8 | 3.109 | 2.112 | 1.589 | 1.229 | ...... | ...... | ...... | ...... | ...... |
| 9 | 3.327 | 2.295 | 1.753 | 1.388 | 1.091 | ...... | ...... | ...... | ...... |
| 10 | 3.526 | 2.464 | 1.904 | 1.531 | 1.242 | ...... | ...... | ...... | ...... |
| 11 | 3.707 | 2.621 | 2.045 | 1.662 | 1.373 | 1.122 | ...... | ...... | ...... |
| 12 | 3.875 | 2.766 | 2.176 | 1.785 | 1.492 | 1.249 | 1.018 | ...... | ...... |
| 13 | 4.029 | 2.902 | 2.299 | 1.901 | 1.604 | 1.362 | 1.145 | ...... | ...... |
| 14 | 4.173 | 3.030 | 2.416 | 2.009 | 1.709 | 1.465 | 1.255 | 1.053 | ...... |
| 15 | 4.309 | 3.151 | 2.526 | 2.111 | 1.807 | 1.561 | 1.354 | 1.163 | ...... |
| 16 | 4.436 | 3.264 | 2.630 | 2.207 | 1.898 | 1.651 | 1.445 | 1.259 | 1.080 |
| 17 | 4.555 | 3.371 | 2.729 | 2.300 | 1.985 | 1.736 | 1.529 | 1.347 | 1.176 |
| 18 | 4.668 | 3.475 | 2.824 | 2.389 | 2.069 | 1.817 | 1.609 | 1.428 | 1.261 |
| 19 | 4.776 | 3.571 | 2.914 | 2.474 | 2.150 | 1.895 | 1.685 | 1.504 | 1.341 |
| 20 | 4.878 | 3.664 | 3.001 | 2.556 | 2.227 | 1.970 | 1.757 | 1.576 | 1.415 |
| 21 | 4.975 | 3.755 | 3.084 | 2.634 | 2.301 | 2.041 | 1.827 | 1.644 | 1.483 |
| 22 | 5.068 | 3.840 | 3.164 | 2.709 | 2.373 | 2.109 | 1.893 | 1.710 | 1.549 |
| 23 | 5.157 | 3.923 | 3.240 | 2.782 | 2.442 | 2.176 | 1.957 | 1.773 | 1.612 |
| 24 | 5.242 | 4.002 | 3.315 | 2.852 | 2.509 | 2.240 | 2.019 | 1.833 | 1.671 |
| 25 | 5.324 | 4.078 | 3.387 | 2.920 | 2.573 | 2.302 | 2.079 | 1.892 | 1.729 |
| 26 | 5.403 | 4.151 | 3.456 | 2.986 | 2.636 | 2.362 | 2.137 | 1.948 | 1.784 |
| 27 | 5.479 | 4.222 | 3.523 | 3.049 | 2.697 | 2.420 | 2.194 | 2.003 | 1.838 |
| 28 | 5.552 | 4.291 | 3.588 | 3.111 | 2.756 | 2.477 | 2.249 | 2.056 | 1.891 |
| 29 | 5.622 | 4.358 | 3.651 | 3.171 | 2.813 | 2.532 | 2.302 | 2.108 | 1.941 |
| 30 | 5.690 | 4.422 | 3.712 | 3.229 | 2.869 | 2.586 | 2.354 | 2.158 | 1.990 |
| 31 | 5.756 | 4.484 | 3.772 | 3.285 | 2.923 | 2.638 | 2.404 | 2.207 | 2.038 |
| 32 | 5.820 | 4.545 | 3.829 | 3.340 | 2.976 | 2.689 | 2.454 | 2.255 | 2.085 |
| 33 | 5.882 | 4.604 | 3.884 | 3.394 | 3.028 | 2.738 | 2.502 | 2.302 | 2.130 |
| 34 | 5.942 | 4.661 | 3.939 | 3.446 | 3.078 | 2.787 | 2.549 | 2.347 | 2.175 |
| 35 | 6.001 | 4.717 | 3.992 | 3.497 | 3.127 | 2.834 | 2.594 | 2.392 | 2.218 |
| 36 | 6.058 | 4.771 | 4.044 | 3.547 | 3.174 | 2.880 | 2.639 | 2.436 | 2.261 |
| 37 | 6.113 | 4.823 | 4.095 | 3.595 | 3.221 | 2.926 | 2.683 | 2.478 | 2.302 |
| 38 | 6.167 | 4.874 | 4.144 | 3.643 | 3.267 | 2.970 | 2.726 | 2.520 | 2.343 |
| 39 | 6.219 | 4.925 | 4.192 | 3.689 | 3.312 | 3.013 | 2.768 | 2.561 | 2.383 |
| 40 | 6.270 | 4.974 | 4.239 | 3.734 | 3.356 | 3.055 | 2.809 | 2.601 | 2.422 |
| 41 | 6.320 | 5.022 | 4.285 | 3.779 | 3.398 | 3.097 | 2.849 | 2.640 | 2.460 |
| 42 | 6.369 | 5.069 | 4.331 | 3.822 | 3.440 | 3.138 | 2.888 | 2.678 | 2.497 |
| 43 | 6.416 | 5.114 | 4.375 | 3.865 | 3.481 | 3.178 | 2.927 | 2.716 | 2.534 |
| 44 | 6.463 | 5.159 | 4.418 | 3.906 | 3.521 | 3.217 | 2.965 | 2.753 | 2.570 |
| 45 | 6.508 | 5.202 | 4.460 | 3.947 | 3.561 | 3.255 | 3.002 | 2.789 | 2.606 |
| 46 | 6.552 | 5.245 | 4.501 | 3.987 | 3.600 | 3.293 | 3.039 | 2.825 | 2.641 |
| 47 | 6.596 | 5.287 | 4.542 | 4.026 | 3.638 | 3.330 | 3.075 | 2.860 | 2.675 |
| 48 | 6.639 | 5.328 | 4.581 | 4.065 | 3.675 | 3.366 | 3.110 | 2.894 | 2.708 |
| 49 | 6.681 | 5.368 | 4.620 | 4.103 | 3.712 | 3.401 | 3.145 | 2.928 | 2.741 |
| 50 | 6.720 | 5.408 | 4.657 | 4.140 | 3.748 | 3.436 | 3.179 | 2.962 | 2.774 |
| 51 | 6.761 | 5.447 | 4.695 | 4.176 | 3.784 | 3.471 | 3.213 | 2.994 | 2.806 |
| 52 | 6.800 | 5.484 | 4.732 | 4.212 | 3.819 | 3.505 | 3.246 | 3.027 | 2.838 |
| 53 | 6.838 | 5.522 | 4.768 | 4.247 | 3.853 | 3.538 | 3.279 | 3.059 | 2.869 |
| 54 | 6.876 | 5.559 | 4.804 | 4.282 | 3.887 | 3.571 | 3.311 | 3.090 | 2.899 |
| 55 | 6.913 | 5.595 | 4.839 | 4.316 | 3.920 | 3.603 | 3.342 | 3.121 | 2.929 |
| 56 | 6.950 | 5.630 | 4.873 | 4.349 | 3.952 | 3.635 | 3.373 | 3.151 | 2.959 |
| 57 | 6.986 | 5.665 | 4.907 | 4.382 | 3.984 | 3.666 | 3.404 | 3.181 | 2.988 |
| 58 | 7.021 | 5.699 | 4.941 | 4.415 | 4.016 | 3.697 | 3.434 | 3.210 | 3.017 |
| 59 | 7.056 | 5.733 | 4.974 | 4.447 | 4.047 | 3.728 | 3.463 | 3.239 | 3.046 |
| 60 | 7.090 | 5.766 | 5.006 | 4.478 | 4.078 | 3.758 | 3.492 | 3.268 | 3.074 |

# TABLE X. Peirce's Criterion.

VALUES OF $x^2$ FOR $\mu = 2$.

| m | n: 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 1.484 | ...... | ...... | ...... | ...... | ...... | ...... | ...... | ...... |
| 5 | 1.887 | 1.235 | ...... | ...... | ...... | ...... | ...... | ...... | ...... |
| 6 | 2.230 | 1.479 | 1.114 | ...... | ...... | ...... | ...... | ...... | ...... |
| 7 | 2.528 | 1.705 | 1.288 | 1.025 | ...... | ...... | ...... | ...... | ...... |
| 8 | 2.793 | 1.913 | 1.459 | 1.163 | ...... | ...... | ...... | ...... | ...... |
| 9 | 3.029 | 2.102 | 1.620 | 1.304 | 1.066 | ...... | ...... | ...... | ...... |
| 10 | 3.242 | 2.277 | 1.771 | 1.439 | 1.191 | ...... | ...... | ...... | ...... |
| 11 | 3.437 | 2.440 | 1.913 | 1.566 | 1.310 | 1.098 | ...... | ...... | ...... |
| 12 | 3.616 | 2.592 | 2.046 | 1.687 | 1.423 | 1.208 | 1.015 | ...... | ...... |
| 13 | 3.782 | 2.734 | 2.171 | 1.802 | 1.529 | 1.310 | 1.122 | ...... | ...... |
| 14 | 3.936 | 2.867 | 2.290 | 1.910 | 1.631 | 1.409 | 1.220 | 1.045 | ...... |
| 15 | 4.080 | 2.991 | 2.403 | 2.014 | 1.727 | 1.501 | 1.312 | 1.141 | ...... |
| 16 | 4.215 | 3.109 | 2.510 | 2.112 | 1.819 | 1.589 | 1.398 | 1.229 | 1.070 |
| 17 | 4.342 | 3.221 | 2.611 | 2.206 | 1.907 | 1.673 | 1.480 | 1.311 | 1.157 |
| 18 | 4.462 | 3.328 | 2.708 | 2.295 | 1.991 | 1.753 | 1.557 | 1.388 | 1.236 |
| 19 | 4.576 | 3.429 | 2.801 | 2.382 | 2.072 | 1.830 | 1.631 | 1.461 | 1.310 |
| 20 | 4.684 | 3.526 | 2.890 | 2.465 | 2.150 | 1.904 | 1.703 | 1.531 | 1.380 |
| 21 | 4.787 | 3.619 | 2.975 | 2.544 | 2.225 | 1.976 | 1.772 | 1.598 | 1.447 |
| 22 | 4.885 | 3.707 | 3.057 | 2.621 | 2.298 | 2.045 | 1.838 | 1.663 | 1.511 |
| 23 | 4.979 | 3.792 | 3.136 | 2.695 | 2.368 | 2.112 | 1.902 | 1.725 | 1.572 |
| 24 | 5.069 | 3.874 | 3.212 | 2.766 | 2.435 | 2.176 | 1.964 | 1.785 | 1.631 |
| 25 | 5.155 | 3.953 | 3.286 | 2.835 | 2.501 | 2.239 | 2.024 | 1.843 | 1.688 |
| 26 | 5.238 | 4.029 | 3.357 | 2.902 | 2.565 | 2.299 | 2.082 | 1.900 | 1.743 |
| 27 | 5.317 | 4.103 | 3.426 | 2.967 | 2.626 | 2.358 | 2.139 | 1.955 | 1.796 |
| 28 | 5.394 | 4.174 | 3.492 | 3.030 | 2.686 | 2.415 | 2.194 | 2.008 | 1.848 |
| 29 | 5.468 | 4.242 | 3.556 | 3.091 | 2.744 | 2.471 | 2.248 | 2.060 | 1.898 |
| 30 | 5.539 | 4.309 | 3.619 | 3.150 | 2.801 | 2.525 | 2.300 | 2.111 | 1.948 |
| 31 | 5.608 | 4.373 | 3.680 | 3.208 | 2.856 | 2.578 | 2.351 | 2.160 | 1.996 |
| 32 | 5.675 | 4.435 | 3.739 | 3.264 | 2.909 | 2.630 | 2.401 | 2.208 | 2.042 |
| 33 | 5.740 | 4.496 | 3.796 | 3.319 | 2.961 | 2.680 | 2.449 | 2.255 | 2.088 |
| 34 | 5.803 | 4.555 | 3.852 | 3.372 | 3.012 | 2.729 | 2.496 | 2.301 | 2.132 |
| 35 | 5.864 | 4.613 | 3.906 | 3.424 | 3.062 | 2.777 | 2.543 | 2.345 | 2.176 |
| 36 | 5.924 | 4.669 | 3.959 | 3.474 | 3.111 | 2.824 | 2.588 | 2.389 | 2.219 |
| 37 | 5.981 | 4.723 | 4.011 | 3.523 | 3.158 | 2.870 | 2.632 | 2.432 | 2.260 |
| 38 | 6.037 | 4.776 | 4.061 | 3.572 | 3.205 | 2.914 | 2.675 | 2.474 | 2.301 |
| 39 | 6.092 | 4.827 | 4.111 | 3.619 | 3.250 | 2.958 | 2.717 | 2.515 | 2.341 |
| 40 | 6.145 | 4.878 | 4.159 | 3.665 | 3.294 | 3.001 | 2.759 | 2.555 | 2.380 |
| 41 | 6.197 | 4.927 | 4.206 | 3.710 | 3.338 | 3.043 | 2.800 | 2.595 | 2.419 |
| 42 | 6.247 | 4.975 | 4.252 | 3.755 | 3.381 | 3.084 | 2.840 | 2.634 | 2.457 |
| 43 | 6.297 | 5.022 | 4.297 | 3.798 | 3.422 | 3.124 | 2.879 | 2.672 | 2.494 |
| 44 | 6.345 | 5.068 | 4.341 | 3.840 | 3.463 | 3.164 | 2.917 | 2.709 | 2.530 |
| 45 | 6.392 | 5.113 | 4.384 | 3.882 | 3.503 | 3.203 | 2.955 | 2.746 | 2.566 |
| 46 | 6.438 | 5.157 | 4.426 | 3.923 | 3.543 | 3.241 | 2.992 | 2.782 | 2.601 |
| 47 | 6.483 | 5.200 | 4.468 | 3.963 | 3.581 | 3.278 | 3.029 | 2.817 | 2.635 |
| 48 | 6.527 | 5.242 | 4.508 | 4.002 | 3.619 | 3.315 | 3.064 | 2.852 | 2.669 |
| 49 | 6.570 | 5.283 | 4.548 | 4.040 | 3.656 | 3.351 | 3.099 | 2.886 | 2.703 |
| 50 | 6.612 | 5.323 | 4.587 | 4.078 | 3.693 | 3.386 | 3.134 | 2.920 | 2.736 |
| 51 | 6.653 | 5.362 | 4.626 | 4.115 | 3.728 | 3.421 | 3.168 | 2.953 | 2.768 |
| 52 | 6.694 | 5.401 | 4.663 | 4.151 | 3.764 | 3.456 | 3.201 | 2.986 | 2.800 |
| 53 | 6.734 | 5.440 | 4.700 | 4.187 | 3.798 | 3.489 | 3.234 | 3.018 | 2.831 |
| 54 | 6.773 | 5.478 | 4.736 | 4.222 | 3.833 | 3.523 | 3.266 | 3.049 | 2.862 |
| 55 | 6.811 | 5.515 | 4.772 | 4.257 | 3.867 | 3.555 | 3.298 | 3.080 | 2.892 |
| 56 | 6.848 | 5.551 | 4.807 | 4.291 | 3.900 | 3.588 | 3.329 | 3.111 | 2.922 |
| 57 | 6.885 | 5.587 | 4.842 | 4.325 | 3.932 | 3.619 | 3.360 | 3.141 | 2.951 |
| 58 | 6.921 | 5.622 | 4.876 | 4.357 | 3.964 | 3.650 | 3.390 | 3.171 | 2.980 |
| 59 | 6.957 | 5.656 | 4.909 | 4.390 | 3.996 | 3.681 | 3.419 | 3.200 | 3.009 |
| 60 | 6.993 | 5.690 | 4.942 | 4.421 | 4.027 | 3.711 | 3.448 | 3.229 | 3.037 |

# TABLE X. A. Peirce's Criterion.

Log *T*.

| $m$ | $n$ = 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 9·3979 | ........ | ........ | ........ | ........ | ........ | ........ | ........ | ........ |
| 3 | 9·1707 | 9·5853 | ........ | ........ | ........ | ........ | ........ | ........ | ........ |
| 4 | 9·0231 | 9·3979 | 9·6744 | ........ | ........ | ........ | ........ | ........ | ........ |
| 5 | 8·9134 | 9·2693 | 9·5129 | 9·7283 | ........ | ........ | ........ | ........ | ........ |
| 6 | 8·8259 | 9·1707 | 9·3979 | 9·5853 | 9·7652 | ........ | ........ | ........ | ........ |
| 7 | 8·7532 | 9·0906 | 9·3080 | 9·4810 | 9·6362 | 9·7922 | ........ | ........ | ........ |
| 8 | 8·6910 | 9·0231 | 9·2338 | 9·3979 | 9·5403 | 9·6744 | 9·8130 | ........ | ........ |
| 9 | 8·6365 | 8·9648 | 9·1707 | 9·3287 | 9·4630 | 9·5853 | 9·7042 | 9·8296 | ........ |
| 10 | 8·5882 | 8·9134 | 9·1157 | 9·2693 | 9·3979 | 9·5129 | 9·6210 | 9·7253 | 9·8431 |
| 11 | 8·5447 | 8·8675 | 9·0669 | 9·2172 | 9·3417 | 9·4514 | 9·5527 | 9·6501 | 9·7483 |
| 12 | 8·5051 | 8·8259 | 9·0231 | 9·1707 | 9·2921 | 9·3979 | 9·4943 | 9·5853 | 9·6744 |
| 13 | 8·4689 | 8·7881 | 8·9834 | 9·1288 | 9·2477 | 9·3506 | 9·4433 | 9·5298 | 9·6128 |
| 14 | 8·4355 | 8·7532 | 8·9470 | 9·0906 | 9·2074 | 9·3080 | 9·3979 | 9·4810 | 9·5597 |
| 15 | 8·4044 | 8·7210 | 8·9134 | 9·0555 | 9·1707 | 9·2693 | 9·3570 | 9·4374 | 9·5129 |
| 16 | 8·3754 | 8·6910 | 8·8822 | 9·0231 | 9·1368 | 9·2338 | 9·3197 | 9·3979 | 9·4710 |
| 17 | 8·3483 | 8·6629 | 8·8532 | 8·9930 | 9·1055 | 9·2011 | 9·2854 | 9·3619 | 9·4328 |
| 18 | 8·3227 | 8·6365 | 8·8259 | 8·9648 | 9·0762 | 9·1707 | 9·2537 | 9·3287 | 9·3979 |
| 19 | 8·2986 | 8·6117 | 8·8003 | 8·9383 | 9·0489 | 9·1423 | 9·2242 | 9·2980 | 9·3658 |
| 20 | 8·2757 | 8·5882 | 8·7761 | 8·9134 | 9·0231 | 9·1157 | 9·1966 | 9·2693 | 9·3359 |
| 21 | 8·2540 | 8·5659 | 8·7532 | 8·8898 | 8·9988 | 9·0906 | 9·1707 | 9·2424 | 9·3080 |
| 22 | 8·2333 | 8·5447 | 8·7315 | 8·8675 | 8·9758 | 9·0669 | 9·1463 | 9·2172 | 9·2818 |
| 23 | 8·2136 | 8·5245 | 8·7107 | 8·8462 | 8·9540 | 9·0445 | 9·1231 | 9·1933 | 9·2571 |
| 24 | 8·1947 | 8·5051 | 8·6910 | 8·8259 | 8·9332 | 9·0231 | 9·1012 | 9·1707 | 9·2338 |
| 25 | 8·1766 | 8·4867 | 8·6721 | 8·8066 | 8·9134 | 9·0028 | 9·0803 | 9·1492 | 9·2117 |
| 26 | 8·1592 | 8·4689 | 8·6539 | 8·7881 | 8·8944 | 8·9834 | 9·0604 | 9·1288 | 9·1907 |
| 27 | 8·1425 | 8·4519 | 8·6365 | 8·7703 | 8·8763 | 8·9648 | 9·0414 | 9·1093 | 9·1707 |
| 28 | 8·1264 | 8·4354 | 8·6198 | 8·7532 | 8·8588 | 8·9470 | 9·0231 | 9·0906 | 9·1516 |
| 29 | 8·1109 | 8·4197 | 8·6037 | 8·7368 | 8·8421 | 8·9299 | 9·0056 | 9·0727 | 9·1332 |
| 30 | 8·0959 | 8·4044 | 8·5882 | 8·7210 | 8·8259 | 8·9134 | 8·9888 | 9·0555 | 9·1157 |
| 31 | 8·0814 | 8·3897 | 8·5732 | 8·7057 | 8·8104 | 8·8975 | 8·9726 | 9·0390 | 9·0988 |
| 32 | 8·0674 | 8·3754 | 8·5587 | 8·6910 | 8·7954 | 8·8822 | 8·9571 | 9·0231 | 9·0826 |
| 33 | 8·0538 | 8·3617 | 8·5447 | 8·6767 | 8·7809 | 8·8675 | 8·9420 | 9·0078 | 9·0669 |
| 34 | 8·0407 | 8·3483 | 8·5311 | 8·6629 | 8·7668 | 8·8532 | 8·9275 | 8·9930 | 9·0518 |
| 35 | 8·0279 | 8·3353 | 8·5179 | 8·6495 | 8·7532 | 8·8393 | 8·9134 | 8·9786 | 9·0372 |
| 36 | 8·0155 | 8·3227 | 8·5051 | 8·6365 | 8·7400 | 8·8259 | 8·8998 | 8·9648 | 9·0231 |
| 37 | 8·0034 | 8·3105 | 8·4927 | 8·6239 | 8·7272 | 8·8129 | 8·8865 | 8·9513 | 9·0095 |
| 38 | 7·9917 | 8·2986 | 8·4807 | 8·6117 | 8·7148 | 8·8003 | 8·8737 | 8·9383 | 8·9962 |
| 39 | 7·9803 | 8·2870 | 8·4689 | 8·5998 | 8·7027 | 8·7881 | 8·8613 | 8·9257 | 8·9834 |
| 40 | 7·9691 | 8·2757 | 8·4575 | 8·5882 | 8·6910 | 8·7761 | 8·8492 | 8·9134 | 8·9709 |
| 41 | 7·9583 | 8·2647 | 8·4463 | 8·5769 | 8·6795 | 8·7645 | 8·8374 | 8·9014 | 8·9588 |
| 42 | 7·9477 | 8·2540 | 8·4355 | 8·5659 | 8·6684 | 8·7532 | 8·8259 | 8·8898 | 8·9470 |
| 43 | 7·9373 | 8·2435 | 8·4249 | 8·5552 | 8·6575 | 8·7422 | 8·8148 | 8·8785 | 8·9355 |
| 44 | 7·9272 | 8·2333 | 8·4145 | 8·5447 | 8·6469 | 8·7315 | 8·8039 | 8·8675 | 8·9243 |
| 45 | 7·9174 | 8·2233 | 8·4044 | 8·5345 | 8·6365 | 8·7210 | 8·7933 | 8·8567 | 8·9134 |
| 46 | 7·9077 | 8·2136 | 8·3945 | 8·5245 | 8·6264 | 8·7107 | 8·7829 | 8·8462 | 8·9028 |
| 47 | 7·8983 | 8·2040 | 8·3849 | 8·5147 | 8·6165 | 8·7007 | 8·7728 | 8·8360 | 8·8924 |
| 48 | 7·8890 | 8·1947 | 8·3754 | 8·5051 | 8·6069 | 8·6910 | 8·7629 | 8·8259 | 8·8822 |
| 49 | 7·8800 | 8·1855 | 8·3662 | 8·4958 | 8·5974 | 8·6814 | 8·7532 | 8·8162 | 8·8723 |
| 50 | 7·8711 | 8·1766 | 8·3572 | 8·4867 | 8·5882 | 8·6721 | 8·7438 | 8·8066 | 8·8626 |
| 51 | 7·8624 | 8·1678 | 8·3483 | 8·4777 | 8·5791 | 8·6629 | 8·7345 | 8·7972 | 8·8532 |
| 52 | 7·8539 | 8·1592 | 8·3396 | 8·4689 | 8·5703 | 8·6539 | 8·7254 | 8·7881 | 8·8439 |
| 53 | 7·8456 | 8·1508 | 8·3311 | 8·4603 | 8·5616 | 8·6451 | 8·7166 | 8·7791 | 8·8348 |
| 54 | 7·8374 | 8·1425 | 8·3227 | 8·4519 | 8·5530 | 8·6365 | 8·7079 | 8·7703 | 8·8259 |
| 55 | 7·8293 | 8·1344 | 8·3145 | 8·4436 | 8·5447 | 8·6281 | 8·6993 | 8·7617 | 8·8172 |
| 56 | 7·8214 | 8·1264 | 8·3065 | 8·4355 | 8·5365 | 8·6198 | 8·6910 | 8·7532 | 8·8087 |
| 57 | 7·8137 | 8·1186 | 8·2986 | 8·4275 | 8·5284 | 8·6117 | 8·6828 | 8·7449 | 8·8003 |
| 58 | 7·8060 | 8·1109 | 8·2908 | 8·4197 | 8·5205 | 8·6037 | 8·6747 | 8·7368 | 8·7921 |
| 59 | 7·7986 | 8·1033 | 8·2832 | 8·4120 | 8·5128 | 8·5959 | 8·6668 | 8·7288 | 8·7840 |
| 60 | 7·7912 | 8·0959 | 8·2757 | 8·4044 | 8·5051 | 8·5882 | 8·6590 | 8·7210 | 8·7761 |

# TABLE X. A. Peirce's Criterion.

Log $T$.

| m | n = 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 61 | 7.7840 | 8.0886 | 8.2684 | 8.3970 | 8.4977 | 8.5806 | 8.6514 | 8.7133 | 8.7684 |
| 62 | 7.7768 | 8.0814 | 8.2611 | 8.3897 | 8.4903 | 8.5732 | 8.6439 | 8.7057 | 8.7607 |
| 63 | 7.7698 | 8.0744 | 8.2540 | 8.3825 | 8.4830 | 8.5659 | 8.6365 | 8.6983 | 8.7532 |
| 64 | 7.7629 | 8.0674 | 8.2470 | 8.3754 | 8.4759 | 8.5587 | 8.6293 | 8.6910 | 8.7458 |
| 65 | 7.7562 | 8.0606 | 8.2401 | 8.3685 | 8.4689 | 8.5516 | 8.6222 | 8.6838 | 8.7386 |
| 66 | 7.7495 | 8.0538 | 8.2333 | 8.3617 | 8.4620 | 8.5447 | 8.6152 | 8.6767 | 8.7315 |
| 67 | 7.7429 | 8.0472 | 8.2266 | 8.3549 | 8.4552 | 8.5378 | 8.6082 | 8.6697 | 8.7244 |
| 68 | 7.7364 | 8.0407 | 8.2200 | 8.3483 | 8.4485 | 8.5311 | 8.6015 | 8.6629 | 8.7175 |
| 69 | 7.7300 | 8.0342 | 8.2136 | 8.3418 | 8.4420 | 8.5245 | 8.5948 | 8.6562 | 8.7107 |
| 70 | 7.7237 | 8.0279 | 8.2072 | 8.3353 | 8.4355 | 8.5179 | 8.5882 | 8.6495 | 8.7040 |
| 71 | 7.7175 | 8.0217 | 8.2009 | 8.3290 | 8.4291 | 8.5115 | 8.5817 | 8.6430 | 8.6975 |
| 72 | 7.7114 | 8.0155 | 8.1947 | 8.3227 | 8.4228 | 8.5051 | 8.5753 | 8.6365 | 8.6910 |
| 73 | 7.7054 | 8.0094 | 8.1886 | 8.3166 | 8.4166 | 8.4989 | 8.5690 | 8.6302 | 8.6846 |
| 74 | 7.6994 | 8.0034 | 8.1825 | 8.3106 | 8.4105 | 8.4927 | 8.5628 | 8.6239 | 8.6783 |
| 75 | 7.6936 | 7.9975 | 8.1766 | 8.3045 | 8.4044 | 8.4867 | 8.556[illegible] | 8.6178 | 8.6721 |
| 76 | 7.6878 | 7.9917 | 8.1707 | 8.2986 | 8.3985 | 8.4807 | 8.5506 | 8.6117 | 8.6659 |
| 77 | 7.6820 | 7.9859 | 8.1649 | 8.2928 | 8.3926 | 8.4747 | 8.5447 | 8.6057 | 8.6599 |
| 78 | 7.6764 | 7.9803 | 8.1592 | 8.2870 | 8.3868 | 8.4689 | 8.5388 | 8.5998 | 8.6539 |
| 79 | 7.6708 | 7.9747 | 8.1536 | 8.2813 | 8.3811 | 8.4632 | 8.5330 | 8.5939 | 8.6481 |
| 80 | 7.6653 | 7.9691 | 8.1480 | 8.2757 | 8.3754 | 8.4575 | 8.5273 | 8.5882 | 8.6423 |
| 81 | 7.6599 | 7.9637 | 8.1425 | 8.2702 | 8.3699 | 8.4519 | 8.5216 | 8.5825 | 8.6365 |
| 82 | 7.6546 | 7.9583 | 8.1371 | 8.2647 | 8.3644 | 8.4463 | 8.5161 | 8.5769 | 8.6309 |
| 83 | 7.6493 | 7.9529 | 8.1317 | 8.2593 | 8.3589 | 8.4409 | 8.5106 | 8.5714 | 8.6253 |
| 84 | 7.6440 | 7.9477 | 8.1264 | 8.2540 | 8.3536 | 8.4355 | 8.5051 | 8.5659 | 8.6198 |
| 85 | 7.6389 | 7.9425 | 8.1212 | 8.2487 | 8.3483 | 8.4301 | 8.4998 | 8.5605 | 8.6144 |
| 86 | 7.6337 | 7.9373 | 8.1160 | 8.2435 | 8.3431 | 8.4249 | 8.4945 | 8.5552 | 8.6090 |
| 87 | 7.6287 | 7.9322 | 8.1109 | 8.2384 | 8.3379 | 8.4197 | 8.4892 | 8.5499 | 8.6037 |
| 88 | 7.6237 | 7.9272 | 8.1058 | 8.2333 | 8.3328 | 8.4145 | 8.4841 | 8.5447 | 8.5985 |
| 99 | 7.6187 | 7.9223 | 8.1008 | 8.2283 | 8.3277 | 8.4094 | 8.4790 | 8.5395 | 8.5933 |
| 90 | 7.6139 | 7.9174 | 8.0959 | 8.2233 | 8.3227 | 8.4044 | 8.4739 | 8.5345 | 8.5882 |

Log $R$.

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | 9.5015 | 9.4992 | 9.4969 | 9.4947 | 9.4924 | 9.4902 | 9.4880 | 9.4857 | 9.4835 | 9.4813 |
| 1.1 | 9.4791 | 9.4769 | 9.4747 | 9.4725 | 9.4704 | 9.4682 | 9.4661 | 9.4639 | 9.4618 | 9.4597 |
| 1.2 | 9.4575 | 9.4554 | 9.4533 | 9.4512 | 9.4491 | 9.4470 | 9.4450 | 9.4429 | 9.4408 | 9.4388 |
| 1.3 | 9 4367 | 9.4347 | 9.4327 | 9.4306 | 9.4286 | 9.4266 | 9.4246 | 9.4226 | 9.4206 | 9.4186 |
| 1.4 | 9.4167 | 9.4147 | 9.4127 | 9.4108 | 9.4088 | 9.4069 | 9.4050 | 9.4030 | 9.4011 | 9.3992 |
| 1.5 | 9.3973 | 9.3954 | 9.3935 | 9.3916 | 9.3897 | 9.3878 | 9.3860 | 9.3841 | 9.3823 | 9.3804 |
| 1.6 | 9.3786 | 9.3767 | 9.3749 | 9.3731 | 9.3712 | 9.3694 | 9.3676 | 9.3658 | 9.3640 | 9.3622 |
| 1.7 | 9.3604 | 9.3587 | 9.3569 | 9.3551 | 9.3534 | 9.3516 | 9.3498 | 9.3481 | 9.3464 | 9.3446 |
| 1.8 | 9.3429 | 9.3412 | 9.3395 | 9.3377 | 9.3360 | 9.3343 | 9.3326 | 9.3310 | 9.3293 | 9.3276 |
| 1.9 | 9.3259 | 9.3242 | 9.3226 | 9.3209 | 9.3193 | 9.3176 | 9.3160 | 9.3143 | 9.3127 | 9.3111 |
| 2.0 | 9.3095 | 9.3078 | 9.3062 | 9.3046 | 9.3030 | 9.3014 | 9.2998 | 9.2982 | 9.2966 | 9.2951 |
| 2.1 | 9.2935 | 9.2919 | 9.2904 | 9.2888 | 9.2872 | 9.2857 | 9.2841 | 9.2826 | 9.2811 | 9.2795 |
| 2.2 | 9.2780 | 9.2765 | 9.2750 | 9.2734 | 9.2719 | 9.2704 | 9.2689 | 9.2674 | 9.2659 | 9.2644 |
| 2.3 | 9.2630 | 9.2615 | 9.2600 | 9.2585 | 9.2571 | 9.2556 | 9.2541 | 9.2527 | 9.2512 | 9.2498 |
| 2.4 | 9.2483 | 9.2469 | 9.2455 | 9.2440 | 9.2426 | 9.2412 | 9.2398 | 9.2383 | 9.2369 | 9.2355 |
| 2.5 | 9.2341 | 9.2327 | 9.2313 | 9.2299 | 9.2285 | 9.2272 | 9.2258 | 9.2244 | 9.2230 | 9.2217 |
| 2.7 | 9.2203 | 9.2189 | 9.2176 | 9.2162 | 9.2149 | 9.2135 | 9.2122 | 9.2108 | 9.2095 | 9.2082 |
| 2.7 | 9.2068 | 9.2055 | 9.2042 | 9.2029 | 9.2016 | 9.2002 | 9.1989 | 9.1976 | 9.1963 | 9.1950 |
| 2.8 | 9.1937 | 9.1924 | 9.1912 | 9.1899 | 9.1886 | 9.1873 | 9.1860 | 9.1848 | 9.1835 | 9.1823 |
| 2.9 | 9.1810 | 9.1797 | 9.1785 | 9.1773 | 9.1760 | 9.1748 | 9.1735 | 9.1723 | 9.1711 | 9.1698 |
| 3.0 | 9.1686 | | | | | | | | | |