CREDENCE GOODS MARKET, PRICE DISTRIBUTION UNDER UNCERTAINTY AND ONLINE RATING

by

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Abstract

The thesis contains three chapters on the information transmission in a market place.

The first chapter, joint with Edi Karni, is a study of the nature and prevalence of persistent fraud in a market with two suppliers for credence-quality goods. We model the market as a stochastic game of incomplete information in which the players are customers and suppliers and analyze their equilibrium behavior. Customers characteristics, idiosyncratic search cost and discount rate, are private information. Customers do not possess the expertise necessary to assess the service they need either ex ante or ex post. We show that there exists no fraud-free equilibrium in the markets for credence-quality goods and that fraud is a prevalent and persistent equilibrium phenomenon.

The second chapter shows competition is a double-edged sword in markets with uncertain production costs and consumer search. Even though a market with sufficiently many suppliers does make suppliers to cut unnecessary costs, it also incentivizes suppliers to increase prices. As a result, consumers with search costs will be less willing to search among suppliers, which further enhances suppliers’ market power. When the number of suppliers approaches infinity, the equilibrium in the limit is characterized by a market where consumers accepts whatever is offered, monopolistic price prevails in the market and suppliers provide consumers products with just necessary costs. The implications in the study is robust to assumptions on consumers’ search cost and utility functions.

The third chapter proposes a mechanism to elicit consumer’s subjective opinion toward a product in an online environment where consumer’s purchase decision is known to a market platform, and consumer’s ratings are published by the platform. Truthful equilibrium is shown to be the unique informative equilibrium when consumer’s opinion is binary. In the mechanism, the platform does not need any common knowledge between consumers (e.g., how likely consumer perceives the product is good or bad) and whether truthful equilibrium is played is observable. However, this mechanism is not without limitations: it cannot be free from babbling equilibria, where the consumers rate the product regardless of their true opinion, and consumers simply ignore ratings in their decision-making process. I demonstrate several reasons to argue that babbling equilibria should not prevail in practice. Extension for non-binary opinions is possible with
more assumptions on information structure.

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Chapter 1

Competitive Equilibrium Fraud in Markets for Credence-Goods (with Edi Karni)

1.1 Introduction

Customers seeking to purchase services that require specialized knowledge are susceptible to fraud by suppliers who prescribe unnecessary services. Examples include, medical tests and treatments, auto repairs, equipment maintenance, and taxi cab service. In these markets the service suppliers make diagnostic determinations of the service required and offer to provide it, and the customers must decide whether to purchase the prescribed service or to seek, at a cost, a second service offer. Typically in these situations, the customer can judge, ex post, whether or not the service provided was sufficient to solve the problem, but is unable to assess whether the prescribed service was also necessary.

Darby and Karni (1973) were the first to identify the fundamental ingredients of the problem underlying the provision of what they dubbed credence-quality goods. First, information asymmetry between the customer who lacks the expertise necessary to assess the service needed and service provider who possesses the required expertise and, second, the cost saving of the joint provision of diagnosis and services.\(^1\) They proceeded to discuss and analyze the economic implications of transactions involving this type of asymmetric information. Specifically, Darby and Karni argued that in competitive market equilibrium for

\(^1\)This bundling of information and service is crucial. See Wolinsky (1993) for an analysis of the implication of separation of diagnosis and service.
credence-quality goods there is persistent tendency of suppliers to over-prescribe services (that is, to prescribe services that are sufficient but are unnecessary to solve the problem at hand).

The nature and extent of fraudulent practices depend on the specific characteristics of the credence-good market. For example, the demand for auto repair at a given service station depends on the waiting time (that is, the length of the queue of customers waiting to be served) which is not an issue when it comes to taxi cab service. It also depends on the information the customer may acquire before choosing the service provider and the cost of seeking a second opinion. For instance, in medical diagnosis that requires an invasive procedure the cost of obtaining a second opinion is prohibitively high. It is obvious, therefore, that modeling of credence-goods markets, while incorporating the fundamental ingredients of the problem – information asymmetry and the bundling of diagnosis and service – must be based on the specifics of the market under consideration. In this paper we focus on markets for the provision of services, such as auto-repair services, in which the capacity limitations may result in waiting for service. We underscore this point to avoid the impression that this is a general model of credence-good markets. We believe, however, that the game-theoretic approach invoked here is not specific to the analysis of the model we study in this paper, rather it is a natural framework for the analysis of credence-good markets in general.

Since the publication of Darby and Karni (1973), numerous studies confirm the prevalence of fraudulent behavior in the markets for credence-quality goods.\textsuperscript{2} For medical services, especially physician’s services, over treatment, a phenomenon known in medical literature as supplier induced demand, is widely documented (see McGuire (2000), Currie, W. Lin, and Zhang (2011), and Dranove (1988) found that in Swiss canton of Ticino on average the population has one third more operations than medical doctors and their relatives, suggesting that greater information symmetry tends to reduce overprescription of surgical procedures. The same type of conclusion was reached by Balafoutas et al. (2013). They report the results of a natural field experiment on taxi rides in Athens, Greece, designed to measure different types of fraud and to examine the influence of passengers’ presumed information on the extent of fraud. Their findings indicate that passengers with inferior information about optimal routes are taken on significantly longer detours. Iizuka (2007) finds physicians drugs prescriptions are influenced by markup. Schneider (2012) reports the results of a field experiment designed to assess the accuracy of service provision in the auto repair market. He finds evidence for over prescription of services as well as under prescription. Beck et al. (2014) reports that in experimental setting, car mechanics are significantly more prone to supplying unnecessary services than student subjects.

The work of Darby and Karni, while calling attention to a neglected aspect of economic interactions that results in market failure, lacks the formal structure necessary to derive more subtle implications of the concept they introduced. In this work we take a step towards a more formal analysis of markets for

\textsuperscript{2}Dulleck and Kerschbamer (2006) includes a survey of the literature and provides numerous references.
credence-quality services with some specific characteristics. Specifically, taking a game-theoretic approach we analyze the equilibrium behavior in a market in which two suppliers operating service stations are engaged in Bertrand competition. The suppliers are assumed to be ex ante identical in every respect. The sole asymmetry between the suppliers, which arises endogenously, is the lengths of their queues (i.e., the waiting time for service). The critical aspect of the model is the information asymmetry regarding the service that is required to address the problem at hand. The suppliers are supposed to possess the expertise necessary to assess the required service while the customers do not.

Customers heterogeneity is the consequence of idiosyncratic costs of seeking a second prescription and of waiting for service. We assume that these costs are the customers’ private information. The customers are assumed to discover the lengths of the suppliers queues (that is, the waiting time) only when they visit the supplier’s service outlet.

We study the market in a stationary symmetric equilibrium in which normal profits discourage entry or exit. In other words, the idle time at the service stations is short enough so that no supplier loses money but is sufficiently long so as to discourage new entries or installing additional service capacity. In addition to proving its existence, we show that there exists no fraud-free equilibrium in this market, that the level of fraud committed by the two suppliers depends on the lengths of their queues, and that the short-queue supplier is more likely to overprescribe service than the long-queue supplier. These conclusions highlight the message of this work, namely, that the fraud committed in credence good markets depends on the market’s specific characteristics, suggesting that the study of these markets, while maintaining the unifying characteristics, information asymmetry and the bundling of diagnosis and service, should proceed on a case by case basis.

In the next section we describe the credence good market. The equilibrium analysis appears in section 3. Some economic implications of our analysis are discussed in section 4. Section 5 includes a discussion of related literature and some concluding remarks. To allow for uninterrupted reading we collected the proofs in section 6.

1.2 The Credence Good Market

1.2.1 Overview

Consider a market for credence-quality service populated by infinite number of customers and two suppliers, A and B. The information asymmetry in this market is two sided. The customers’ private information consists of their idiosyncratic search cost and discount rate. The suppliers possess expertise that the customers do not.

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3We confine our analysis to symmetric equilibria. The analysis of possible non-symmetric equilibria is beyond the scope of the this paper.
not have, which allows them to observe the actual state of disrepair and assess the service required to fix the problem. Let \( \tilde{\omega} \) denote a discrete random variable representing the true state of disrepair expressed as the necessary and sufficient number of service hours required to address the problem. We normalize \( \tilde{\omega} \) to take values in \( \Omega := \{\omega_1, \ldots, \omega_n\} \), where \( 0 < \omega_1 < \ldots < \omega_n < 1 \). Denote the distribution of \( \tilde{\omega} \) by \( \mu \in \Delta(\Omega) \), where \( \Delta(\Omega) \) denotes the simplex in \( \mathbb{R}^n \), and assume that \( \mu \) is exogenous and commonly known.

Like the states of disrepair, the prescribed service, denoted by \( q \), is specified in discrete quantities and, to simplify the exposition, we suppose that the prescribed service levels correspond to the states.\(^4\) Moreover, we assume that the prescribed service must fix the problem (e.g., malfunction) or the customer refuses payment. Formally, if the state is \( \omega_i \) then \( q \in \{\omega_i, \ldots, \omega_n\} \). The two suppliers are identical in every respect except the lengths of their queues, which are expressed in terms of service hours committed to serving waiting customers. We assume that the suppliers observe each other’s queue and that customers only discover the lengths of their queues, which are expressed in terms of service hours committed to serving waiting customers. We assume that the suppliers observe each other’s queue and that customers only discover the lengths of a supplier’s queue when they show up at the supplier’s service station.\(^5\) Let \( Q^A(t) \) and \( Q^B(t) \) denote the lengths of the suppliers queues at time \( t \). Formally, \( (Q^A(t), Q^B(t)) \in I := [0, \infty]^2 \), for all \( t \in \mathbb{R}_+ \).

Let \( I \) be endowed with the Borel \( \sigma \)-algebra \( \mathcal{B}(I) \). A customer’s arrival on the market in a state of disrepair, \( \omega \in \Omega \), sets up a stage game \( \Gamma(\omega, Q^A(t), Q^B(t)) \) parameterized by the triplet \( (\omega, Q^A(t), Q^B(t)) \in \Omega \times I \). Let \( \Omega \) be endowed with the discrete topology and \( I \) with the metric topology. Let \( (\Omega \times I, \mathcal{B}(\Omega \times I)) \) be a measurable spaces, where \( \mathcal{B}(\Omega \times I) \) denote the Borel \( \sigma \)-algebra on \( \Omega \times I \).

We assume that the customer’s arrival process is stationary and is depicted by a CDF, \( F \), that is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \) and has full support in \( \mathbb{R}_{++} \). Denote by \( \mathcal{V} \) the set of probability measures on \((I, \mathcal{B}(I))\) and assume the customers’ arrival rate is such that probability of new arrival during a fixed time interval is sufficiently small so that the measures in \( \mathcal{V} \) are tight. Under this assumption, by Prokhorov’s theorem, \( \mathcal{V} \) is sequentially compact. A customer’s type, \((\theta, \beta)\), consists of idiosyncratic search cost, \( \theta \), and discount rate, \( \beta \), both taking values in \([0, 1]\). Thus, the set of customers’ types is \( T = [0, 1]^2 \). Let \( \mathcal{B}(T) \) be the Borel \( \sigma \)-algebra on \( T \) and denote by \( \xi \) a continuous probability measure on the measurable type space \((T, \mathcal{B}(T))\).

When a new customer shows up at a service station, the supplier observes the state of disrepair \( \omega \) and, consequently, the state \((\omega, Q^A(t), Q^B(t))\). The suppliers do not observe the customer’s type. Customers know their types but not the state \( \omega \), and they discover the length of a supplier’s queue upon visiting a service station and receiving a diagnosis. In other words, a customer may discover the lengths of the suppliers queues sequentially, during the search of service process. Insofar as the customers are concerned, what matters are

\(^4\)In view of the common practice of informing the customers what are the parts that need to be fixed or replaced before the actual work begins, this assumption is realistic.

\(^5\)The assumption that the suppliers observe each other’s queue expresses the presumption that survival in competitive markets requires the players to keep tab of their rivals positions and actions. Relaxing this assumption would require a modification of the suppliers strategies described below, and will complicate the analysis without yielding new insights.
the lengths of the queues and not the identity of the suppliers. This assumption rules out suppliers’ identity or reputation as a possible factor.  

Assume that the installed capacity of the two suppliers is the same, that the hourly service price is the same for the two suppliers regardless of the lengths of their queues, and is known to the customers. Assume further that the price is normalized so that the profit generated by servicing customers for a fraction, \( x \), of an hour is \( x \)

We model the credence service market as a stochastic game of incomplete information and analyze it using the concept of stationary Markovian symmetric equilibria. The players in this game are the two suppliers and a finite set of potential customers. We assume that the suppliers earn normal profits, so that there is no incentive for either new suppliers to enter the market or for a current supplier to exit the market or change the level of his service capacity. A customer’s arrival on the market at time \( t \) in a state of disrepair \( \omega \) when the suppliers queues are \( Q^A(t) \) and \( Q^B(t) \) initiates a dynamic stage game, \( \Gamma(\omega, Q^A(t), Q^B(t)) \), depicting the interaction among the customer and the two suppliers A and B. At a state \( s(t, \omega) := (\omega, Q^A(t), Q^B(t)) \) the suppliers and the customer make their decisions, after which the game proceeds to the next state as follows. If the next customer arrives at time \( t' \) in a state \( \omega' \) and accepts the prescription \( q_A \) of supplier A then the new state is

\[
s_A(t', \omega') := (\omega', \max\{Q^A(t) - \Delta t' + q_A, 0 + q_A\}, \max\{Q^B(t) - \Delta t', 0\})
\]

where \( \Delta t' := t' - t \), and if she accepts the prescription \( q_B \) of supplier B then the new state is

\[
s_B(t', \omega') := (\omega', \max\{Q^A(t) - \Delta t', 0\}, \max\{Q^B(t) - \Delta t' + q_B, 0 + q_B\})
\]

The transition probability from the state \( s(t, \omega) \) to the state \( s_j(t', \omega') \) is the product of the probability that the next customer arrives at time \( t' \), the probability that the state of disrepair is \( \omega' \), the probability that supplier \( j \) prescribes \( q_j \), and the probability that the newly arrived customer accepts the prescription \( q_j \).

The rest of the elements of the stochastic game, the players strategies and payoffs are described next.

1.2.1.1 The customers

Upon identifying an equipment malfunction, the customer engages in sequential search for repair service. Diagnosis of the problem and determination of the service needed to solve it requires expert knowledge,

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6 We revisit the issue of reputation in the discussion section.
7 One interpretation is that the price is regulated (e.g., metered cab fare)
8 A detailed exposition of these probabilities and the stochastic evolution of the queues appears in Section 3.1.3 below
which the customer does not have.

The customers’ strategies: Since the posted service prices are the same, the customer chooses one of the two service outlets at random with equal probabilities. Upon visiting a service outlet the customer obtains a service prescription and the length of the supplier’s queue, both expressed in terms of service-hours. The customer must then choose between accepting the prescribed service and waiting in the queue, and rejecting it in favor of seeking a second prescription. If she chooses the latter, the customer visits the second supplier, receives a second prescription and observes the length of the second supplier’s queue. The customer must then decide between accepting the second prescription and waiting to be served and returning to the first supplier. We assume that the search is with full and costless recall. Hence, if the customer decides to seek a second prescription and then return to the first supplier, she maintains her place in the queue and is entitled to obtain the service prescribed by the first supplier. Formally, a customer’s search strategy is a mapping \( \sigma : T \to \Sigma_1 \times \Sigma_2 \), where \( \Sigma_1 := \{ \sigma_1 : \Omega \times [0, \infty] \to \{0, 1\} \} \), \( \Sigma_2 := \{ \sigma_2 : \Omega^2 \times I \to \{0, 1\} \} \). In other words, the strategy assigns to a customer of type \((\theta, \beta)\) two acts depicted by the functions \( \sigma_1^{(\theta, \beta)} : \Omega \times [0, \infty] \to \{0, 1\} \) \( \sigma_2^{(\theta, \beta)} : \Omega^2 \times I \to \{0, 1\} \), where \( \sigma_1^{(\theta, \beta)} (q_1, Q_1) = 1 \) means that the customer accepts the prescription of the first supplier she visits and terminates the search, and \( \sigma_1^{(\theta, \beta)} (q_1, Q_1) = 0 \) means that she seeks a second prescription. Similarly, \( \sigma_2^{(\theta, \beta)} (q_1, q_2, Q_1, Q_2) = 1 \) means that the customer accepts the second supplier’s prescription and \( \sigma_2^{(\theta, \beta)} (q_1, q_2, Q_1, Q_2) = 0 \) means that she rejects the second supplier’s prescription and returns to the first supplier. We denote by \( \Sigma \) the set of customers’ strategies. Let \( \Sigma \) denote the set of customers strategies.

The customers’ beliefs: Since the customers do not observe the suppliers queues, at the outset the customer’s information set is \( \Omega \times I \) and her prior beliefs are captured by \( \mu \in \Delta (\Omega) \) and the \( v \in V \). Upon observing the length of the first supplier’s queue, \( Q_1 \), and obtaining a prescription, \( q_1 \in \Omega \), the customer updates her beliefs about the state \( \omega \) and the waiting time at the second service station. In doing so, the customer applies Bayes’ rule. The updated belief regarding the state \( \omega \) and the second supplier’s queue conditional on the first supplier’s prescription, \( q_1 \), and queue length, \( Q_1 \), is represented by the conditional distribution \( m (\omega, Q_2 \mid q_1, Q_1) \) on \( \Omega \times [0, \infty] \).

The customers’ payoffs: Accepting a prescribed service \( q \) on her first visit from a supplier whose queue

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9 This assumption does not rule out customers loyalty to suppliers or that each customer visits first the supplier whose location is closer provided that the loyalty or proximity are equally divided between the suppliers.

10 The assumption of full recall is intended to simplify the exposition. It is natural to suppose that a customer who decides to return to the first supplier may find out that, with positive probability, her place in the queue is taken. This would call for formulating the customer decision as search with uncertain recall (Karni and Schwartz 1977). This would complicate the exposition without adding insight or change the results. We comment on this alternative when we define the customers’ equilibrium strategies below.

11 This is the sense in which the search involves learning.

12 We examine the updated beliefs in further details below.
length is $Q$, the utility of a customer of type $(\theta, \beta)$ is: $u^\beta(q, Q) = (1 - q) e^{-\beta Q}$. Continuing the search entails a customer-specific additive search cost, $\theta \in [0, 1]$. Thus, the utility of accepting the prescription $q'$ when the queue of the second supplier is $Q'$ is $u^{(\theta, \beta)}(q', Q') = (1 - q') e^{-\beta Q'} - \theta$. Returning to the first supplier after visiting the second supplier, the customer’s payoff is $(1 - q) e^{-\beta Q} - \theta$. If $1 - q < 0$, then the customer is better off not fixing the problem. Under our assumptions $\Omega \subset [0, 1]$, implicitly, this presumes that $\omega > 1$ are states of disrepair that are not worth fixing and, consequently, are not included in $\Omega$.

1.2.1.2 The suppliers

At every point in time each supplier has a queue representing hours committed to serving customers that have already accepted the supplier’s prescriptions. The lengths of the queues are determined by the history of customers arrival, their service prescriptions, and their acceptance decisions. In other words, the lengths of the queues are determined by the realization of an exogenous stochastic process (that is, the arrival rate and the random state $\omega$) and the endogenous decisions of the suppliers and customers.

**The suppliers’ strategies:** The suppliers’ mixed prescription strategies are mappings $G : \Omega \times I \to \mathcal{G}$, where $\mathcal{G}$ denotes the set of CDF on $\Omega$. Formally, for each $q_h \in \Omega$ and $(Q^A, Q^B) \in I$, $G(\omega, Q^A, Q^B)(q_h) := \sum_{i=1}^h g(\omega, Q^A, Q^B)(q_i) \delta_{q_i}$, where $j \in \{A, B\}$ and $\delta_{q_i}$ denotes the distribution function that assigns the unit probability mass to $q_i$. Because the asymmetry between the suppliers is due solely to the lengths of their queues, the suppliers prescriptions are distinct only as a result of the difference of their queues and, in the case of mixed strategies, the randomly selected prescription.

**The suppliers’ payoffs:** Before the start of a stage game, $\Gamma(\omega, Q^A, Q^B)$, at time, say $t = 0$, supplier $j$ anticipates that either his or his rival’s prescription be accepted and, as a result, the state of the queues transitions from the current state $(Q^A, Q^B)$ to the state $(\hat{Q}^A, \hat{Q}^B)$. Following that there is a random waiting time $t' > 0$, before the next customer arrives and initiates the stage game $\Gamma(\omega, \hat{Q}^A - t', \hat{Q}^B - t')$. The transition probabilities from $(Q^A, Q^B)$ to $(\hat{Q}^A, \hat{Q}^B)$, is determined by supplier’s strategies $G(\omega, Q^A, Q^B)$, $j \in \{A, B\}$, and customer’s acceptance rule $\sigma$.

Let $V : \Omega \times I \to \mathbb{R}_+$ be a bounded measurable function representing the suppliers’ anticipated expected discounted value before the start of a stage game. We show next that the value function $V$ exists and is unique.

Just before the start of the stage game $\Gamma$, supplier $j$ expects to receive a cash flow from servicing the

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13 The particular functional form is chosen to simplify the exposition. The critical feature of the customers’ payoff for our analysis that are captured by this functional form are: The utility is monotonic decreasing in the recommended repairs and in the length of the queue.

14 Additive search cost is a standard assumption in the literature on optimal stopping rules.

15 This is the sense in which the recall is costless.

16 We are restricting consideration to history-independent, or Markovian, symmetric strategies.
customers in his queue while waiting the arrival of the next customer, yielding a discounted value

\[
\int_0^{\min(Q^j, t')} e^{-r\tau} d\tau,
\]

and the anticipated discounted expected value from the stage game that follows given by, \(e^{-rt'} \sum_{\omega' \in \Omega} V(\omega', \hat{Q}^j - t', \hat{Q}^{j-} - t') \mu(\omega')\). Thus, the total anticipated payoff \(\hat{V}(\omega, Q^j, Q^{j-})\) from a stage game \(\Gamma(\omega, Q^A, Q^B)\) that is about to be played is:

\[
\hat{V}(\omega, Q^j, Q^{j-}) = \int_0^\infty \int_I \int_0^{\min(Q^j, t')} e^{-r\tau} d\tau \\
+ e^{-rt'} \sum_{\omega' \in \Omega} V(\omega', \hat{Q}^j - t', \hat{Q}^{j-} - t') \mu(\omega') v(\hat{Q}^A, \hat{Q}^B|Q^A, Q^B) d\hat{Q}^A d\hat{Q}^B dF(t')
\]

(1.1)

where the conditional transition probability \(v(\hat{Q}^A, \hat{Q}^B|Q^A, Q^B)\) is a shorthand for the expression \(v(\hat{Q}^A, \hat{Q}^B|Q^A, Q^B, \omega, G, \sigma)\) that makes explicit the fact that the probability of the state \((\hat{Q}^A, \hat{Q}^B)\) of the queues conditional on the state \((Q^A, Q^B)\) depends on the state of disrepair and the suppliers and customers strategies. If the players strategies \((G, \sigma)\) and the anticipated value function \(V\) are all bounded and measurable, then the payoff function \(\hat{V}\) is bounded measurable.

If supplier \(j\) anticipates correctly the strategies of his rival and the customers and chooses his best response, then the payoff function \(\hat{V}\) coincides with the hypothesized anticipated value function \(V\). Formally, for \(j \in \{A, B\}\),

\[
V(\omega, Q^j, Q^{j-}) = \max_{G \in \mathcal{G}} \int_0^\infty \int_I \int_0^{\min(Q^j, t')} e^{-r\tau} d\tau \\
+ e^{-rt'} E_{\Omega} V(\omega', \hat{Q}^j - t', \hat{Q}^{j-} - t') v(\hat{Q}^A, \hat{Q}^B|Q^A, Q^B) d \left(\hat{Q}^A, \hat{Q}^B\right) dF(t')
\]

where \(E_{\Omega} V(\omega', \hat{Q}^j - t', \hat{Q}^{j-} - t') := \sum_{\omega' \in \Omega} V(\omega', \hat{Q}^j - t', \hat{Q}^{j-} - t') \mu(\omega')\).

### 1.2.2 Equilibrium Analysis

#### 1.2.2.1 The players’ behavior and the evolution of the queues

We analyze the credence service market as Markovian sequential equilibrium of a stochastic game of incomplete information. Given \(\omega \in \Omega\), a strategy \(G_k\) is completely mixed with modulus \(k\) if \(g_k(q) \geq k^{-1}, k \geq n\), for all \(q \in \Omega_\omega\). To start with, we study the equilibria of the stage game \(\Gamma(\omega, Q^A, Q^B)\) in completely mixed strategies with modulus \(k\), beginning with the behavior of the customers and the suppliers.

**The customers** **The customers system of beliefs:** At the start nature assigns the customers their types, which is the customers’ private information. When a customer detects a problem and seeks remedial
service, she does not know which particular stage game, $\Gamma (\omega, Q^A, Q^B)$, she initiates. Her prior beliefs are depicted by the distributions $\mu \in \Delta (\Omega)$ and $v \in \mathcal{V}$. In view of the ex-ante symmetry of the suppliers, insofar as the customers are concerned, $v$ is symmetric.\(^{17}\)

Consider the state $(\omega, Q^A, Q^B)$ and let $G_k (\omega, Q^j, Q^{-j})$, $j \in \{A, B\}$, be the suppliers completely mixed strategies with modulus $k$. The customers are supposed to know the strategies of the suppliers as functions of the states but not the current state $(\omega, Q^A, Q^B)$. In particular, the customers do not know which is the short-queue supplier and which is the long-queue supplier. Let $(q_1, Q_1)$ and $(q_2, Q_2)$ denote the prescriptions obtained and queues observed by a customer in her first and second visits, respectively.

Following her visit to the first supplier and having observed $Q_1$, regardless of whether it is $A$ or $B$, the customer updates her beliefs about the state of disrepair, $\omega$, and the length of the queue of the second supplier by applying Bayes’ rule as follows: For all $\omega_i \leq q_1$ and $(Q_1, Q_2) \in \interior I$,

$$m_k (\omega_i, Q_2 \mid q_1, Q_1) = \frac{g_{1,k} (\omega_i, Q_1, Q_2) (q_1) \mu (\omega_i) v (Q_2, Q_1)}{\int_0^\infty [\sum_{\omega_i \leq \omega_j \leq q_1, g_{1,k} (\omega_i, Q_1, Q_2') (q_1) \mu (\omega_i)] v (Q_2', Q_1) dQ_2'], \quad (1.2)$$

where $g_{1,k} (\omega_i, Q_1, Q_2)$ denotes the mixed strategy of the first supplier.\(^{18}\)

The customers expected payoff and best response strategies: Given the suppliers’ completely mixed strategies, $G_k$, with modulus $k$, we explore next the optimal behavior of the customer in the subgame following her visit to the first supplier and the evolution of her beliefs. Having obtained the prescription $q_1$ and observing the length of the queue, $Q_1$, a customer of type $(\theta, \beta)$ can accept the prescription and stop the search or seek a second prescription. In the latter case the customer accepts the second supplier’s prescription if $(1 - q_2) e^{-\beta Q_2} \geq (1 - q_1) e^{-\beta Q_1}$. Otherwise the customer exercises the recall option and returns to the first supplier to obtain the payoff $(1 - q_1) e^{-\beta Q_1} - \theta$.

Because the customer is going to accept or reject the second offer according to whether $u^\beta (q_2, Q_2)$ is greater or smaller than $u^\beta (q_1, Q_1)$, given $q_1$ and $Q_1$ the reservation utility of a customer of type $(\theta, \beta)$, $u^{(\theta, \beta)}_{r,k} (q_1, Q_1)$, is given by

$$u^{(\theta, \beta)}_{r,k} (q_1, Q_1) = \sum_{\omega_i \leq \omega_h \leq q_1} \sum_{\omega_h \leq q_2 \leq \omega_n} \int_0^\infty \max \{u^\beta (q_2, Q_2), u^\beta (q_1, Q_1)\} g_k (\omega_h, Q_2, Q_1) (q_2) m_k (\omega_h, Q_2 \mid q_1, Q_1) dQ_2 - \theta. \quad (1.3)$$

Given her type, $(\theta, \beta)$, and the suppliers’ strategy, $G_k$, the customer’s expected payoff upon observing $(q_1, Q_1)$ given the reservation utility strategy $u^{(\theta, \beta)}_{r,k} (\cdot, \cdot)$ in (1.3), is:

$$\bar{U} (\sigma_k (\theta, \beta), G_k) = \sigma_{1,k}^{(\theta, \beta)} (q_1, Q_1) u^\beta (q_1, Q_1) + \left(1 - \sigma_{1,k}^{(\theta, \beta)} (q_1, Q_1)\right) u^{(\theta, \beta)}_{r,k} (q_1, Q_1). \quad (1.4)$$

\(^{17}\)Because $v$ is part of the equilibrium, it will be shown later that this assumption is validated.

\(^{18}\)For more details see section 3.1.3.
Hence, the customer accepts the first supplier’s offer (that is, set \( \sigma^{(\theta,\beta)}_{1,k}(q_1, Q_1) = 1 \) if \( u^\beta(q_1, Q_1) \geq u^\beta_{r,k}(q_1, Q_1) \)). Otherwise, the customer continues the search (that is, set \( \sigma^{(\theta,\beta)}_{1,k}(q_1, Q_1) = 0 \)). She accepts the second supplier’s offer (that is, set \( \sigma^{(\theta,\beta)}_{2,k}(q_1, q_2, Q_1, Q_2) = 1 \)) if \( u^\beta(q_2, Q_2) > u^\beta(q_1, Q_1) \). Otherwise, she exercises the recall option (that is, set \( \sigma^{(\theta,\beta)}_{2,k}(q_1, q_2, Q_1, Q_2) = 0 \)). With this in mind we make the following definition:

**Definition 1.1.** A reservation-utility search strategy \( \sigma_k : T \to \Sigma_1 \times \Sigma_2 \) consists of two mappings \( \sigma_k^{(\theta,\beta)} : \Omega \times [0, \infty) \to \{0, 1\} \) and \( \sigma^{(\theta,\beta)}_{2,k} : \Omega^2 \to \{0, 1\} \), and a function \( u_k^{(\theta,\beta)} : \Omega \times [0, \infty] \to [0, 1] \) such that:

(a) \( \sigma_k^{(\theta,\beta)}(q, Q) = 1 \) if \( u^\beta(q, Q) \geq u_k^{(\theta,\beta)}(q, Q) \) and \( \sigma_k^{(\theta,\beta)}(q, Q) = 0 \), otherwise.

(b) \( \sigma_{2,k}^{(\theta,\beta)}(q_2, q_1, Q_2, Q_1) = 1 \) if \( \sigma_{1,k}^{(\theta,\beta)}(q_1, Q_1) = 0 \) and \( u^\beta(q_2, Q_2) > u^\beta(q_1, Q_1) \) and \( \sigma_{2,k}^{(\theta,\beta)}(q_2, q_1, Q_2, Q_1) = 0 \), otherwise.

We summarize the above discussion in the following:

**Proposition 1.1.** A reservation-utility strategy is the customers’ unique best response to the suppliers’ strategy profile \( (G_k(\omega, Q^j, Q^{-j}))_{j \in \{A, B\}} \) for all \( (\omega, Q^j, Q^{-j}) \in \Omega \times I \).

The customer’s expected payoff under the reservation-utility strategy is continuous in the suppliers strategies. Formally,

**Lemma 1.1.** For each type \((\theta, \beta) \in T \) and all \((q_1, Q_1) \in \Omega \times [0, \infty] \) the customer’s expected payoff, \( \bar{U}(\sigma_k(\theta, \beta), G_k) \), of the reservation-utility strategy is continuous.

The continuity of \( \bar{U} \) is an immediate implication of its linearity in the strategies and the fact that \( g_k(\omega, Q^j, Q^{-j})(q) > 0, j \in \{A, B\} \) for all \( q \in \Omega_\omega \).

**The suppliers** Because the customers types are private information, the suppliers choose their strategies as best responses against the acceptance probabilities induced by the distribution of customers’ types. We examine next the acceptance probabilities induced by the customers’ reservation utility strategies. Supplier \( j \)’s prescription is accepted in the following cases: (1) \( j \) is the customer’s first call and the customer accepts the prescription \( q \) immediately, (2) \( j \) is the customer’s first call, the customer seeks a second prescription and returns to \( j \) for the service, (3) \( j \) is the customer’s second call and she accepts his prescription. We calculate the probabilities of these events.

---

\( ^{19} \)If the full recall formulation is replaced by search with uncertain recall, the waiting time becomes a random variable, \( \tilde{Q}_1 \), taking values in \([Q_1, \infty]\), whose distribution is determined by the arrival rates. Because the customer is going to accept or reject the second offer according to whether \( u^\beta(q_2, Q_2) \) is greater or smaller than \( u^\beta(q_1, Q_1) \), we use \( u^\beta(q_1, Q_1) \) instead of \( u^\beta(q_2, Q_2) \).
The first-call suppliers face a distribution of acceptance rules induced by the distribution, $\xi$, on the set of types. Thus, for all $(q_1, Q_1) \in \Omega \times [0, \infty]$, the subset of the first callers who do not seek a second prescription when faced with the prescription $q_1$ and queue $Q_1$ is given by the subset of types $A_{1,k} (q_1, Q_1) := \{ (\theta, \beta) \in T \mid u^{(\theta, \beta)} (q_1, Q_1) \geq u_{r,k}^{(\theta, \beta)} \} \in B(T)$. Consequently, the average acceptance rate of first callers who, given the queue length $Q_1$, accepts the prescription $q_1$ immediately is:

$$\sigma_{1,k} (q_1, Q_1) = \int_T \sigma_{1,k}^{(\theta, \beta)} (q_1, Q_1) d\xi(\theta, \beta) = \xi(A_{1,k} (q_1, Q_1)).$$

This may be interpreted as the probabilistic demand function of first callers.

Given the first supplier’s prescription, $q_1$, and the length, $Q_1$, of his queue, the acceptance rate of a second prescription, $q_2$, when the length of the queue of the second supplier is $Q_2$, is:

$$\sigma_{2,k} (q_2, Q_2 \mid q_1, Q_1) = \int_T \sigma_{2,k}^{(\theta, \beta)} (q_2, Q_2; q_1, Q_1) d\xi(\theta, \beta).$$

The second-call supplier does not know that he is the second-call supplier. However, observing $\omega_i$ and $Q_1$, the second supplier can infer that if he is the customer’s second-call then the prescription the customer obtained in her first call is a random variable $\tilde{q}_1$ whose probability distribution is determined by the strategy of the first supplier. Specifically, if the customer first visits supplier $j \in \{A, B\}$ then $q_1$ was determined by the strategy $G_k (\omega_i, Q^j, Q^{-j})$. Moreover, given $Q_1$ and $q_1$, only customers whose type $(\theta, \beta)$ and having obtained the prescription $q$ and observed the queue, $Q$, such that $\sigma_{1,k}^{(\theta, \beta)} (q, Q) = 0$ (that is, customers type for whom $u^{(\theta, \beta)} (q, Q) < u_{r,k}^{(\theta, \beta)} (q, Q)$) seek a second prescription. Consequently, given $(\omega_i, Q^A, Q^B)$, if $j$ is the second supplier the customer calls upon, the probability that his prescribed service is accepted is:

$$\varsigma_{2,k} (q_j, \omega_i, Q^j, Q^{-j}, g_k (\omega_i, Q^{-j}, Q^j)) = \sum_{q \in \Omega_{\omega_i}} \xi((\theta, \beta) \mid \sigma_{1,k}^{(\theta, \beta)} (q-j, Q^{-j}) = 0) \sigma_{2,k}^{(\theta, \beta)} (q_j, Q^j \mid q, Q^{-j}) g_k (\omega_i, Q^{-j}, Q^j) (q_j, Q^j) \mid q, Q^{-j}) g_k (\omega_i, Q^{-j}, Q^j) (q), \ j \in \{A, B\}. $$

Hence, the probability that a newly arrived customer accepts the prescription of supplier $j$ is:

$$\alpha_{j}^k (q_j \mid \sigma_k, G_k (\omega_i, Q^{-j}, Q^j)) :=$$

$$\frac{1}{2} \left[ \sigma_{1,k} (q_j, Q^j) + (1 - \sigma_{1,k} (q_j, Q^j)) \left( 1 - \sum_{q \in \Omega_{\omega_i}} \varsigma_{2,k} (q, Q^{-j}) g_k (\omega_i, Q^{-j}, Q^j) (q) \right) \right] + \sum_{q \in \Omega_{\omega_i}} \left( 1 - \sigma_{1,k} (q, Q^{-j}) \right) \varsigma_{2,k} (q_j, Q^j \mid q, Q^{-j}) g_k (\omega_i, Q^{-j}, Q^j) (q)).$$

Given players strategies $(G, \sigma)$ and current state $(\omega, Q^j, Q^{-j})$, if a customer accepts a prescription $q$ in a
stage game, it must be either with supplier $j$ or supplier $-j$. Moreover, before the start of a stage game, the probability that supplier $j$’s prescription $q_j$ will be accepted against supplier $-j$’s strategy $G_k(\omega, Q^{-j}, Q^j)$ is given by $\alpha^k_j(q_j | \sigma_k, G_k(\omega, Q^{-j}, Q^j))$ in (1.5). Hence, after supplier $j$’s prescription $q_j$, $j \in \{A, B\}$, is accepted, the conditional probability of $(\hat{Q}^j, \hat{Q}^{-j})$ is:

$$v_{k,j}(\hat{Q}^j, \hat{Q}^{-j}|Q^j, Q^{-j}) = \frac{1}{2} \alpha^k_j(q_j | \sigma_k, G_k(\omega, Q^{-j}, Q^j)) g_k(\omega, Q^j, Q^{-j}) (q_j),$$

if $q_j = \hat{Q}^j - Q^j \in \Omega_\omega$ and $\hat{Q}^{-j} = Q^{-j}$, and $v_{k,j}(\hat{Q}^j, \hat{Q}^{-j}|Q^j, Q^{-j}) = 0$, otherwise. Therefore, for any arbitrary state of the queues $(\hat{Q}^A, \hat{Q}^B)$ to occur after a stage game is:

$$v_k(\hat{Q}^A, \hat{Q}^B|Q^A, Q^B) = v_{k,A}(\hat{Q}^A, \hat{Q}^B|Q^A, Q^B) + v_{k,B}(\hat{Q}^A, \hat{Q}^B|Q^A, Q^B)$$

if $\hat{Q}^A - Q^A \in \Omega_\omega$ or $\hat{Q}^B - Q^B \in \Omega_\omega$, and $v((\hat{Q}^A, \hat{Q}^B)|Q^A, Q^B) = 0$, otherwise.

**Lemma 1.2.** For all $\omega, Q^A, Q^B \in \Omega \times I$, the expression (1.1) is a continuous function on the strategy profiles set $\Sigma \times \Delta(\Omega)^2$.

**The evolution of the queues** We show next that there is a stationary distribution of $v_k$ on $I$ perceived by customers. We start with an original distribution $v_k^*$ hypothesized by customers and trace its evolution in the wake of the end of a stage game. Suppose that $v_k^*$ satisfies the following properties (we will show below why we specify these requirements).

1. $v_k^*$ is absolutely continuous (with respect to the Lebesgue measure) except at $(Q^A, 0)$ and $(0, Q^B)$.
2. $v_k^*$ has full-support.
3. $v_k^*$ has marginal probability $v_k^*(Q^A, Q^B)$, for $(Q^A, Q^B) \geq 0$ and $(Q^A, Q^B) \neq (0, 0)$.
4. $v_k^*$ has probability mass $v_k^*(Q^A, Q^B)$, at $(Q^A, Q^B) = (0, 0)$.

The information that the suppliers have and the customer does not have is: (a) How long it has been since the preceding stage game ended (i.e., the waiting time $t$), and (b) The state $(\omega, Q^A, Q^B)$ of the previous stage game. Thus, the customer’s perceived queue distribution is the unconditional expectation of $v_k(\hat{Q}^A, \hat{Q}^B|Q^A, Q^B)$.

For $(\hat{Q}^A, \hat{Q}^B) > 0$, the only possibility to start a stage game with $(\hat{Q}^A, \hat{Q}^B)$ is that the preceding game ended with the state of the queues $(\hat{Q}^A + t, \hat{Q}^B + t)$, $t \in (0, \infty]$. Hence, conditional on the preceding stage game starting with the state $(\omega, Q^A, Q^B)$, the probability of this event is: $\Sigma_{\omega \in \Omega} v_k(\hat{Q}^A + t, \hat{Q}^B +$
The perceived measure $v_k$ on $I$ is absolutely continuous with respect to the Lebesgue measure and has full support.

Starting from the event that both suppliers are idle (i.e., $Q^A = Q^B = 0$) the probability, \( p \), of returning to the same position under the equilibrium strategies is positive. Since the equilibrium is Markovian, this event is encountered infinitely often. Thus, the probability of the event “$Q^A = Q^B = 0$ infinitely often” is: $\lim_{m \to \infty} p^m > 0$. Hence, $p = 1$. In other words, starting from any state of finite queues, $(Q^A, Q^B)$, with probability one the queues will attain the point $Q^A = Q^B = 0$ infinitely often. From this position, the two suppliers are equally likely to become the long-queue supplier. Hence, no supplier enjoys the short-queue advantage persistently. Therefore, the evolution of the queues under the equilibrium strategies requires that the anticipated lengths of the queues be stochastically equal, in the sense that the identity of the short-queue

\[
v_k(\hat{Q}^A, \hat{Q}^B) = \int_0^\infty \int_{\omega \in \Omega} v_k(\hat{Q}^A + t, \hat{Q}^B + t|\omega, Q^A, Q^B) \mu(\omega) \nu_k^*(Q^A, Q^B) d(Q^A \times Q^B) dF(t).
\]

For $(\hat{Q}^A, \hat{Q}^B) \geq (0,0)$ such that $Q^j = 0$, $j \in \{A, B\}$, and $(\hat{Q}^A, \hat{Q}^B) \neq (0,0)$, it can occur if previous game ends with either $(\hat{Q}^A + t, \hat{Q}^B)$ and some $\hat{Q}^B \leq t$, or $(\hat{Q}^A, \hat{Q}^B + t)$ and some $\hat{Q}^A \leq t$. Therefore, the marginal probability of, for example, $\hat{Q}^B = 0$ is the sum of the two unconditional probabilities

\[
v_k(\hat{Q}^A, 0) = \int_0^\infty \int_{\omega \in \Omega} v_k(\hat{Q}^A + t, \hat{Q}^B|\omega, Q^A, Q^B) \mu(\omega) \nu_k^*(Q^A, Q^B) d(Q^A \times Q^B) d\hat{Q}^B dF(t).
\]

For $(\hat{Q}^A, \hat{Q}^B) = (0,0)$, it can be reached by points from the 45 degree line and also points from both axes of the triangle $I$. So the probability at $(0,0)$ is the sum of the three probabilities.

\[
v_k(0,0) = \int_0^\infty \int_{\omega \in \Omega} v_k(t, t|\omega, Q^A, Q^B) \mu(\omega) \nu_k^*(Q^A, Q^B) d(Q^A \times Q^B) d\hat{Q}^B d\hat{Q}^A dF(t).
\]

If the perceived distribution $v_k$ after a stage game coincides with the hypothesized $\nu_k^*$ then, for example of $(\hat{Q}^A, \hat{Q}^B) > (0,0)$

\[
\nu_k^*(\hat{Q}^A, \hat{Q}^B) = \int_0^\infty \int_{\omega \in \Omega} v_k(\hat{Q}^A + t, \hat{Q}^B|\omega, Q^A, Q^B) \mu(\omega) \nu_k^*(Q^A, Q^B) d(Q^A \times Q^B) dF(t).
\]

Note that for $(\hat{Q}^A, \hat{Q}^B)$ with $Q^j = 0$, $j \in \{A, B\}$, $v_k(\hat{Q}^A, \hat{Q}^B)$ is either a marginal probability over one axis, if only one $\hat{Q}^j = 0$, or a probability mass if $\hat{Q}^A = \hat{Q}^B = 0$. 

**Proposition 1.2.** The perceived measure $v_k$ on $I$ is absolutely continuous with respect to the Lebesgue measure and has full support.
supplier is expected to change over time in such a way that the joint distribution of the queues is symmetric around its mean. We summarize this in the following: In symmetric stationary equilibrium, successive stage games induce a joint distribution of the lengths of the queues that is stationary, symmetric and the two suppliers commit the same amount of fraud on average.

1.2.2.2 Equilibrium: Definition and existence

A customer’s system of beliefs $\eta := (\mu, v, m(\omega, Q_2 | q_1, Q_1))$ consists of the prior belief about the stage game being played, which is determined by the prior beliefs $\mu \in \Delta(\Omega)$, $v \in V$, and the updated beliefs $m(\omega, Q_2 | q_1, Q_1)$ on $\Omega \times [0, \infty]$. A strategy profile $(\sigma, G(\omega, Q^j, Q^{-j}))$, $j \in \{A, B\}$, is sequentially rational if, for all $(\omega, Q^A, Q^B) \in \Omega \times I$, given the suppliers objective functions, $G(\omega, Q^j, Q^{-j})$ is best response against $(\sigma, G(\omega, Q^{-j}, Q^j))$, $j \in \{A, B\}$, and, given the customer objective function, $\sigma$ is best response against $(G(\omega, Q^A, Q^B), G(\omega, Q^B, Q^A))$.

Definition 1.2. The strategy profile $(\hat{\sigma}, \hat{G}(\omega, Q^j, Q^{-j}))$, $j \in \{A, B\}$, and a system of beliefs $\eta^* = (\mu, v^*, m^*(\omega, Q_2 | q_1, Q_1))$ constitute a symmetric Markovian sequential equilibrium of the stochastic game induced by the credence good market if:

(i) The strategy profile $(\hat{\sigma}, \hat{G}(\omega, Q^j, Q^{-j}))$, $j \in \{A, B\}$, is sequentially rational given the belief system $\eta^* = (\mu, v^*, m^*)$, where, for $j \in \{A, B\}$, $\hat{G}(\omega, Q^j, Q^{-j})$ is in the arg max$_{G \in G}$ of

$$f_0^\infty \int [f_0^{\min(\hat{Q}^j, t')} e^{-rs} ds + e^{-rt} \Sigma_{\omega' \in \Omega} V(\omega', \hat{Q}^j - t', \hat{Q}^{-j} - t' \mu(\omega')) | v^*(\hat{Q}^A, \hat{Q}^B | Q^A, Q^B) d(\hat{Q}^A, \hat{Q}^B) dF(t')$$

and, for every $(\beta, \theta) \in T$ and $(q_1, Q_1) \in \Omega \times [0, \infty]$,

$$\hat{\sigma}^{(\beta, \theta)}_1(q_1, Q_1) = \arg \max_{\sigma \in \{f: (q, Q) \rightarrow [0, 1]\}} [\sigma_1 u^{(\beta, \theta)}(q_1, Q_1) + (1 - \sigma_1) u^{(\beta, \theta)}(q_1, Q_1)],$$

and $\hat{\sigma}^{(\beta, \theta)}_2(q_1, Q_1, q_2, Q_2) = 1$ if $u^{(\beta, \theta)}(q_1, Q_1) \leq u^{(\beta, \theta)}(q_2, Q_2)$ and $\hat{\sigma}^{(\beta, \theta)}_2(q_1, Q_1, q_2, Q_2) = 0$, otherwise.

(ii) There exists a sequence of completely mixed $G_k$ strategies with modulus $k$ and strategies profiles $(\sigma_k, G_k)$ that is sequentially rational given a belief system $\eta_k := (\mu, v_k, m_k(\omega, Q_2 | q_1, Q_1))$ with $(\hat{\sigma}, \hat{G}) = \lim_{k \rightarrow \infty} (\sigma_k, G_k)$, $\eta^* = \lim_{k \rightarrow \infty} \eta_k$, $v^* = \lim_{k \rightarrow \infty} v_k$, and $m_k(q_2, Q_2 | q_1, Q_1)$ are derived from the $\mu$ and $v_k^*$ and strategy profile $(\sigma_k, G_k)$ using Bayes’ rule.

Theorem 1.1. There exists a symmetric Markovian sequential equilibrium of the stochastic game induced by the credence good market.
In equilibrium, the payoff function \( \hat{V} \) coincides with the anticipated value function \( V \) when every player is adopting its optimal strategy with the transition probability \( v^*(\cdot, |Q^A, Q^B) = v(\cdot, |Q^A, Q^B, G^*, \sigma^*) \) based on equilibrium strategies \( G^*(\omega, Q^i, Q^j), j \in \{A, B\} \), and \( \sigma^* \). Thus, the value function of supplier \( j \in \{A, B\} \) is:

\[
V(\omega, Q^i, Q^{-j}) = \max_{G \in \mathcal{G}} \int_0^\infty \int_0^{r_{\min}(Q^i, t')} e^{-rt} \, dt + e^{-r t'} \sum_{\omega' \in \Omega} V(\omega', \hat{Q}^i - t', \hat{Q}^{-j} - t') \mu(\omega') v^*(\hat{Q}^A, \hat{Q}^B | Q^A, Q^B) d \left( \hat{Q}^A, \hat{Q}^B \right) dF(t')
\]

### 1.2.2.3 The Short-Queue Advantage and Examples

**The short-queue advantage and fraudulent behavior** If \( Q^A \neq Q^B \) the supplier with the shorter queue enjoys a strategic advantage in the sense that, if the two suppliers prescribe the same service, the short-queue supplier is more likely to retain a new customer. However, because of the generality the primitives of the model and the fact that higher prescription entails a trade-off between the benefit of selling extra services and cost represented by lower probability of making the sale, there is no guarantee that the suppliers value functions are necessarily monotonic increasing in the suppliers’ own queues. Consequently, it is not necessary that the short-queue supplier exploits his advantage by prescribing more service than the long-queue supplier. More precisely, suppose that \( Q^A < Q^B \) and the long-queue supplier prescribes \( \omega_i \). If accepted, the marginal value to \( A \) of prescribing \( \omega_{i+1} \) instead of \( \omega_i \) is:

\[
mv := e^{-rQ^A} \int_{\omega_i}^{\omega_{i+1}} e^{-r \tau} \, d\tau + \sum_{\omega \in \Omega} \left[ V(\omega, Q^A + \omega_{i+1}, Q^B) - V(\omega, Q^A + \omega_i, Q^B) \right] \mu(\omega).
\]

If the supplier is interested in selling extra services which, presumably, is the case then this expression must be positive.

The cost to \( A \) of prescribing \( \omega_{i+1} \) instead of \( \omega_i \) is due to the reduction in the probability that the prescription be accepted. Formally, let \( \alpha_A(\omega | Q^A, Q^B) \) denote the probability of supplier \( A \)’s prescription \( \omega \) is accepted, then the marginal cost is:

\[
mc = \left( \alpha_A(\omega_{i+1} | Q^A, Q^B) - \alpha_A(\omega_i | Q^A, Q^B) \right) \left[ e^{-rQ^A} \int_{\omega}^{\omega_{i+1}} e^{-r \tau} \, d\tau + \sum_{\omega \in \Omega} \tilde{V}(\omega, Q^A, Q^B) \mu(\omega) \right],
\]

where

\[
\tilde{V}(\omega, Q^A, Q^B) = \alpha_A(\omega_i | Q^A, Q^B) V(\omega, Q^A + \omega_i, Q^B) + (1 - \alpha_A(\omega_i | Q^A, Q^B)) V(\omega, Q^A, Q^B + \omega_i).
\]

Consider next the difference \( \alpha_A(\omega_{i+1} | Q^A, Q^B) - \alpha_A(\omega_i | Q^A, Q^B) \). The only customers that supplier \( A \) loses by prescribing \( \omega_{i+1} \) instead of \( \omega_i \) are the customers that tried both suppliers and decided to accepts
supplier $B$’s prescription. These are the customers whose type is in the set
\[ \mathcal{I} := \{ (\beta, \theta) \in T | (\beta, \theta) \in T | (1 - \omega_{i+1}) e^{-\beta Q^A} < (1 - \omega_i) e^{-\beta Q^B}, (1 - \omega_i) e^{-\beta Q^B} - \theta < u_r^{(\theta, \beta)}(q_B, Q_B), (1 - \omega_i) e^{-\beta Q^A} - \theta < u_r^{(\theta, \beta)}(q_A, Q_A) \}. \]

Then
\[ (\alpha_A (\omega_{i+1} | Q^A, Q^B) - \alpha_A (\omega_i | Q^A, Q^B)) = -\xi \mathcal{I}. \]

Clearly, this expression decreases monotonically with $Q^B$ and increases monotonically with $Q^A$. Hence, if the spread $Q^B - Q^A$ increases as a result of an increase in $Q^B$ and decrease in $Q^A$ then $-\xi \mathcal{I}$ decreases. But, if $V(\omega, Q^A + \omega, Q^B) - V(\omega, Q^A, Q^B + \omega) > 0$, then $V(\omega, Q^A, Q^B)$ increases with the spread $Q^B - Q^A$ in the same way. Hence, the total effect on $mc$ is ambiguous. However, since $V(\omega, Q^A + \omega, Q^B) - V(\omega, Q^A, Q^B + \omega)$ is bounded, if $Q^B - Q^A$ is sufficient large then the $mc$ tends to zero and the marginal value outweighs the marginal cost. At that point, the short-queue supplier exploits his advantage be prescribing higher level of service then the short-queue supplier.

An equilibrium is said to be fraud-free if the equilibrium strategies are $\hat{G} (\omega, Q^j, Q^{-j}) = \delta_{\omega_i}$, $j \in \{A, B\}$, for all $(\omega, Q^j, Q^{-j}) \in \Omega \times I$. In view of the preceding discussion, the next theorem asserts that fraudulent prescriptions of service is a persistent feature of competitive equilibrium in the credence good market under consideration.

**Theorem 1.2.** There exists no fraud-free equilibrium in the market for credence quality services.

One measure of the short-queue advantage is the difference in the expected change of the lengths of the queues induced by equilibrium strategies. Formally, given a stage game $\Gamma (\omega, Q^A, Q^B)$, if $Q^A < Q^B$ then the measure of the short-queue advantage is:

\[ \Psi \left( \omega_i, Q^A, Q^B \mid \hat{\sigma}, \hat{G} (\omega_i, Q^A, Q^B), \hat{G} (\omega_i, Q^B, Q^A) \right) := \Sigma_{q \in \Omega_{\omega_i}} \left[ \alpha_A \left( q \mid \hat{\sigma}, \hat{G} (\omega_i, Q^A, Q^B) \right) q g (\omega_i, Q^A, Q^B) (q) - \alpha_B \left( q \mid \hat{\sigma}, \hat{G} (\omega_i, Q^B, Q^A) \right) q g (\omega_i, Q^A, Q^B) (q) \right]. \]

The discussion above implies that an increase in the length of the queue of the short-queue supplier reduces its short-queue advantage. Formally, if $A$ is the short-queue supplier then

\[ d\Psi \left( \omega_i, Q^A, Q^B \mid \hat{\sigma}, \hat{G} (\omega_i, Q^A, Q^B), \hat{G} (\omega_i, Q^B, Q^A) \right) / dQ^A < 0. \]

However, because $A$’s objective function is not necessarily monotonic increasing in $Q^A$, the short-queue
advantage does not yield clear cut conclusions concerning its effect on the suppliers’ equilibrium strategies. It is useful, therefore, to consider some simple situations whose analysis would allow us to develop further insights as to the possible nature of fraudulent behavior.

**Simple examples** Suppose that \( \Omega = \{ \omega_L, \omega_H \} \), where \( \omega_H > \omega_L \). Clearly, if the true state is \( \omega_H \) then the only equilibrium is for both suppliers to prescribe the true state. The interesting situation arises when the true state is \( \omega_L \). We consider this case below.

The payoff matrix corresponding to the stage game \( \Gamma (\omega, Q^A, Q^B) \) in which \( A \) is the columns player and \( B \) is the rows player as follows:

\[
\begin{array}{c|cc}
\downarrow B \setminus A \rightarrow & x : \omega_H & (1 - x) : \omega_L \\
y : \omega_H & U^B_{HH}, U^A_{HH} & U^B_{HL}, U^A_{HL} \\
(1 - y) : \omega_L & U^B_{LH}, U^A_{LH} & U^B_{LL}, U^A_{LL}
\end{array}
\]

where

\[
U^j_{ii} = \alpha_j (\omega_i, \omega_i) \sum_{\omega \in \Omega} V (\omega, Q^j + \omega_i, Q^{-j}) \mu (\omega) + (1 - \alpha_j (\omega_i, \omega_i)) \sum_{\omega \in \Omega} V (\omega, Q^j, Q^{-j} + \omega_i) \mu (\omega),
\]

for \( j \in \{A, B\}, \ i \in \{H, L\} \), and

\[
U^j_{HL} = \alpha_A (\omega_H, \omega_L) \sum_{\omega \in \Omega} V (\omega, Q^A + \omega_H, Q^B) \mu (\omega) + (1 - \alpha_A (\omega_H, \omega_L)) \sum_{\omega \in \Omega} V (\omega, Q^A, Q^B + \omega_L) \mu (\omega),
\]

\[
U^j_{LH} = \alpha_A (\omega_L, \omega_H) \sum_{\omega \in \Omega} V (\omega, Q^A + \omega_L, Q^B) \mu (\omega) + (1 - \alpha_A (\omega_L, \omega_H)) \sum_{\omega \in \Omega} V (\omega, Q^A, Q^B + \omega_H) \mu (\omega),
\]

for \( j \in \{A, B\} \). Allowing for mixed strategies, \( x \) and \( y \) denote the probabilities that players \( A \) and \( B \) prescribe \( \omega_H \), respectively. Then

\[
\frac{y}{1 - y} = \frac{U^A_{LL} - U^A_{HL}}{U^B_{HH} - U^B_{HL}} \quad \text{and} \quad \frac{x}{1 - x} = \frac{U^B_{LL} - U^B_{HL}}{U^B_{HH} - U^B_{HL}}.
\]

The primitives of the model, namely, the prior distribution on the customers’ type space, \( T \), the distribution on the possible states of disrepair, \( \Omega \), the stochastic process depicting the arrival of new customers, are quite general. This allows for wide range of values of the suppliers payoffs of the stage games which depend on the states of the queues. Consequently, the model admits a variety of equilibria, including pure strategy and mixed strategy equilibria. In the Appendix we analyze the two stage games \( \Gamma (\omega_L, Q^A, Q^B) \). The first deals with the symmetric case in which the suppliers queues are of equal lengths and the second with the asymmetric case in which the suppliers’ queues are of different lengths. The general conclusions that emerge
are as follows:

If the suppliers queues are equal then, depending on the configurations of the signs of these expressions we may have (a) pure strategy equilibria in which either both suppliers prescribe truthfully or both commit fraud; (b) Two pure strategy equilibria in which one supplier prescribe truthfully and the other overprescribes; (c) A symmetric mixed strategy equilibrium in which each supplier overprescribes service with probability 0.5.

If the suppliers queues are of different lengths then, in mixed strategy equilibrium the short-queue supplier is more likely to commit fraud than the long-queue supplier. In other words, if the true state is $w_L$, and $Q_A < Q_B$ then the equilibrium mixed strategy of supplier $A$ first-order stochastically dominates that of supplier $B$ in the sense that $\Pr_A\{\omega_H\} = x > y = \Pr_B\{\omega_H\}$. Moreover, $\Pr_A\{\omega_H\} > 0.5 > \Pr_B\{\omega_H\}$.

1.3 Related Literature and Concluding Remarks

*Related literature* Shapley (1953) was the first to formulate and prove the existence of equilibrium in two-player, zero-sum, stochastic games. Extensions and review of stochastic games in more general setting are provided in Duggan (2012) and Jaskiewicz and Nowak (2018). While sharing many features of equilibrium analysis that appear in the literature on stochastic games, our model presents an important variation – no player in our model possesses perfect information. In particular, the customer in the model has only partial information about the state that parameterize the stage game and the suppliers are ignorant of the customer’s type. This variation is a contribution to the literature on stochastic games with incomplete information.

Despite evidence regarding the prevalence of fraud in the market for credence goods and the distinguishing features of these markets, the literature dealing with the modeling and analysis of these markets is rather scant. The works that are closest to ours in terms of the questions asked, are Emons (1997), Wolinsky (1995), and Dulleck and Kerschbamer (2006). Despite the shared interest in studying the prevalence of fraud in equilibrium, these works model markets that have distinct structures. Focusing on markets exhibiting features that are different from those of this paper, they reach different conclusions regarding the equilibrium characteristics.

Wolinsky (1995) modeled a market in which the customers bargain with suppliers by offering a price for the repair, and showed that, in interior equilibrium, suppliers commit fraud by employing a strategy that assigns positive probability of rejecting price offers when the state diagnosed requires low service. This strategy reflects the suppliers’ belief that, to avoid the search cost, the customer may offer a higher price rather than seek a second opinion. Wolinsky’s model is different from ours in several important respects. In addition to assuming that the price of service is fixed (no bargaining), a central feature of our model is the
lengths of the suppliers’ queues and the characterization of customers by their idiosyncratic search costs and discount rates. These aspects of our work are absent from Wolinsky’s model. These differences in modeling mandate different equilibrium notions and analysis.

Emons (1997) depicts a credence good market with identical customers and in which the suppliers must decide whether to enter the market. If a supplier enters the market he is endowed with a fixed capacity that can be allocated between diagnosis and repair services. These two functions are priced differently. Suppliers who lack of sufficient capacity, can announce a wrong diagnosis to avoid providing the needed repair. Emons studies conditions under which fraud free equilibrium exists. In addition to its focus on the entry decision, Emons model is different from ours in the specification of the information structure, the characterization of the customers and their behavior, the pricing mechanism, and the suppliers strategies.

Dulleck and Kerschbamer (2006) model a market for credence services in which the customers may experience a need for a high or low level of service. Invoking a game theoretic approach to study conditions under which competition will eliminate fraud.

Hu and J. Lin (2018), Y.-f. Fong and Liu (2018), and Y.-F. Fong et al. (2020) study the efficiency loss due to the asymmetric information in monopolistic credence good market in which the supplier faces uninformed customers. More specifically, Hu and J. Lin (2018) modeled repeated interaction between a customer in occasional need of maintenance service of a durable good and a monopoly supplier. They show that there exists no equilibrium that supports truthful diagnosis. Y.-f. Fong and Liu (2018) investigated the effect of liability on the seller’s incentive to maintain good reputation and its impact on market efficiency. Y.-F. Fong et al. (2020) focus on the use of customer service to build trust between the monopoly supplier and its customers so as to mitigate the efficiency loss.

Heinzel et al. (2019) studied the equilibrium of a price-regulated market in which physicians characterized by heterogeneous cost compete for servicing uninformed patients. Heinzel models the interaction among physicians and patients as a game in which patients may employ mixed strategies in seeking “second opinion” when diagnosed as having a serious problem and physicians may defraud their patients by overtreating them for minor problems. Unlike in the model we present here, the distinct physicians’ types is exogenous and the customer behavior is not derived from optimal search strategy.

**Concluding remarks**  We model a credence service market featuring two identical suppliers engaged in Bertrand competition. The customers care about the prescribed services and the waiting time. Our analysis shows that competition cannot be relayed upon to sustain fraud-free equilibrium and that fraud is a persistent and prevalent phenomenon. The analysis highlights the role of the evolution of the customer’s beliefs in the wake of her visit to the first supplier and the optimal stopping rule that characterizes her best response
strategy, and the suppliers prescription strategies. These aspects of our model and analysis are not specific to the two suppliers case and would show up, in a more complex form, if the number of the suppliers is larger.

The analysis underscores the short-queue supplier’s advantage, its implications for the overprescription of service and the consequent evolution of the queues. It is worth noting that if the waiting time is not an issue (that is, the suppliers have no capacity constraints) so that each customer can be served immediately, then the analysis would change considerably. In this instance, the customers’ utilities depend only on the prescribed service, and their discount rates is no longer a factor. Suppose that \( \theta \in (0, 1] \) then it is easy to verify that the suppliers strategies \( q^i(\omega) = \omega_n \), for all \( \omega \in \Omega \) and \( j \in \{A, B\} \), is an equilibrium. In other words, knowing that the equilibrium prescriptions of the two suppliers are the same, no customer is inclined to search and, consequently, the suppliers have no incentive to try and undercut each other’s prescription. Maximal fraud also characterizes the cab service provided to tourists in an unfamiliar city since the prescription (that is, the route taken) coincides with the service provided, leaving the customer no opportunity for seeking a second prescription. The route taken is only restricted by a tourist’s conception of the reasonable length of the ride.\(^{20}\)

One may think of variations on the model presented here. For instance, there are situations in which, to obtain a diagnosis, one has to schedule an appointment (e.g., a plumber service or medical examination). In these instances, the waiting time is ahead of obtaining the diagnosis and the customer may obtain information about the waiting time at different suppliers prior to deciding which supplier to visit first. This would change the information structure and, consequently, the strategies and equilibrium of the model. The analysis of such variations is left for future research.

An important aspect of the credence good market, discussed in Darby and Karni (1973) but not touched upon in this work, is the possibility of developing a reputation for honest diagnosis and its effect on the commission of fraud. Including reputation in our model would require admitting repeated interactions in which the customers display loyalty (e.g., they visit “their” supplier first) and the suppliers recognize their loyal clients. Under these conditions, the suppliers may establish what Darby and Karni dubbed client relationship. The loss of future business of, and being bad-mouthed by, a dissatisfied customer would increase the cost to the suppliers of “losing” customers, which should serve as a deterrence and, consequently, mitigate the problem of fraud.

\(^{20}\)See also, Stahl (1996) for a discussion of a related issue.
Proofs

Proof of Proposition 1.2 To start with, observe that: (a) Since \( \theta \in [0,1] \), there is a set of positive measures of customer types who accept the prescription \( q = \omega_n \). Thus, every prescription, \( q \in \Omega \) has positive probability of being accepted in a stage game of modulus \( k \) given any queue distribution \( v_k \) (b) Because \( F(t) > 0 \) for all \( t > 0 \), the probability of the event \( (Q^A, Q^B) = (0,0) \) strictly positive. Hence, the state of queues \( (Q^A, Q^B) = (0,0) \) occurs with positive probability after any stage game.

Claim 1.1. Given any initial probability distribution, \( v^0_k \) on \( I \), for every state of the queues, \( (Q^A, Q^B) \), there is a finite \( i \in \mathbb{N} \) such that the customer’s perceived queue distribution \( v^i_k \) in the \( i \)-th stage game has positive probability density on \( (Q_A, Q_B) \).

Proof: Since every point \( (Q^A, Q^B) \) can always be reached by waiting time \( t \) from a stage game that ends with \( (Q^A + t, Q^B + t) \), our question is reduced to whether every point in the set \( I \) can be reached from a finite sequence of stage games. Because \( (0,0) \) has positive probability in the queue distribution after any stage game, the question is further reduced to whether \( (Q^A, Q^B) \) can be reached from origin point \( (0,0) \) with positive probability density. If the answer is yes, then we are done.

To begin with, note that in a stage game of modulus \( k \), the probability that both suppliers prescribe \( \omega \in \Omega \) is at least \( k^{-2} \) and it is necessary that both suppliers have positive probability to be accepted regardless whose queue is longer or shorter. Because the customer waiting time distribution \( F(t) \) is absolutely continuous (with respect to the Lebesgue measure) with full support, \( t \in (0,\infty] \), the points \( (Q^A,0) \) or \( (0,Q^B) \) can be reached with positive marginal probability. For points in \( I \) we can make use the full-support of \( F(t) \) to first choose a proper point \( (Q^A + \Delta,0) \) and follow the sequence of \( m \) stage games, in which supplier \( B \)'s prescriptions gets accepted consecutively to reach the desired \( Q^B \). Let each consecutive stage game starts after a short time interval, \( \delta \), so that the total time during which the \( m \) stage games are played is \( \Delta = m\delta \), so that not only \( Q^B \) is reached as desired, but also \( Q^A + \Delta \) will decrease gradually to \( Q^A \) along the sequence of events. That such \( \Delta \) and such a sequence of events can be constructed is implied by the fact that, for any \( Q^B \), there is a finite set \( \Omega^* = \{ q | q \in \Omega \} \) such that \( Q^B + 1 > \sum_{q \in \Omega^*} q - Q^B \). Define \( \Delta = \sum_{q \in \Omega^*} q - Q^B \) and \( \delta = \frac{\Delta}{|\Omega^*|} \). This completes the proof of the claim.

Claim 1.2. If the distribution \( F \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \) then the equilibrium queue distribution, \( v^*_k \) on \( I \), is absolutely continuous with respect to the Lebesgue measure except in \( (Q^A,0) \cup (0,Q^B) \).

Proof. Given \( (Q^A, Q^B) \in I \), that is the realization of a random variable whose distribution, \( v_k \), is arbitrary, the queue at the start of the next stage game is \( \left((\hat{Q}^A - t), (\hat{Q}^B - t)\right) \in I \). The transition probability from \( (Q^A, Q^B) \) to \( \left((\hat{Q}^A - t), (\hat{Q}^B - t)\right) \) is determined by:
F elements of and 

(a) The customer’s acceptance decision that takes \((Q^A, Q^B)\) to \((\hat{Q}^A, \hat{Q}^B)\).

(b) The waiting time for the arrival of the next customer at time \(t > 0\), that takes \((\hat{Q}^A, \hat{Q}^B)\) to \(((\hat{Q}^A - t), (\hat{Q}^B - t))\)

Step (a) is fully determined by the initial distribution, \(v_k\) on \(I\), the distribution \(\mu\) on \(\Omega\), and the suppliers and customers strategies, \(G\) and \(\sigma\). Thus, without further restrictions on the distributions, because \(v_k\) is arbitrary, the resulting random variable \((\hat{Q}^A, \hat{Q}^B)\) is arbitrarily distributed according to a probability measure \(v_k'\).

Define \(\zeta(t) = (-t, -t), t > 0\) and observe that the sum of the random variables \(\hat{Q} := (\hat{Q}^A, \hat{Q}^B)\) and \((-t, -t)\) is distributed according to \(v_k''\) which is the convolution of \(v_k'\), the measure of \((\hat{Q}^A, \hat{Q}^B)\) and \(F(\zeta^{-1}(-t, -t))\). Thus, by Fubini-Tonelli theorem, \(v_k''\) can be written as:

\[
v_k''(\mathcal{B}) = v_k' * (1 - F(\zeta^{-1}(-t, -t))) (\mathcal{B}) = \int_0^\infty \left[ \int_I 1_{\mathcal{B}} (\hat{Q} + \zeta(t)) v_k' (\hat{Q}) d\hat{Q} \right] dF(t),
\]

for all \(\mathcal{B} \in \mathcal{B}(I)\).

Since \(F\) is absolutely continuous with respect to the Lebesgue measure on \(\mathbb{R}\), the measure of \((\hat{Q}^A - t, \hat{Q}^B - t)\) at \((\hat{Q}^A - t) = 0\) or \((\hat{Q}^B - t) = 0\), must be absolutely continuous with respect to the Lebesgue measure on \(\mathbb{R}\). By the same argument, the measure \(v_k''\) is non-atomic, except at \((0, 0)\).

Suppose that \(\lambda(\mathcal{B}) = 0\) and \(v_k''(\mathcal{B}) > 0\) where \(\lambda\) is Lebesgue measure. Let

\[
\mathcal{B}^0 = \{(Q^A, Q^B) \in I | ((Q^A, Q^B) = (Q^A - t, Q^B - t)), (Q^A, Q^B) \in \mathcal{B}\}.
\]

Then, \(v_k''(\mathcal{B}^0) > 0\). Define \(\mathcal{B}^0_t = \{Q^A + t, Q^B + t \in I | (Q^A, Q^B) \in \mathcal{B}^0\}, t \in (0, \infty)\). Let \(P = \{P_z\}_{z \in \mathbb{N}}\) be a partition of \(\mathcal{B}^0\) such that \(P_z = \{\mathcal{B}^0_t | v_k'(\mathcal{B}^0_t) \in [(z + 1)^{-1}, z^{-1}]\}\). Then, at least one cell of the partition is uncountable. Let this cell be \(P_{z_0}\), then with \(\mathcal{B}^0_t\) in \(P_{z_0}\), \(v_k'(\mathcal{B}^0_t) > z_0^{-1} > 0\). Pick a countable number of elements of \(P_{z_0}\), \(\{\mathcal{B}^0_{t\ell} | \ell \in \mathbb{N}\}\). Then,

\[
v_k'(\bigcup_{\ell \in \mathbb{N}} \mathcal{B}^0_{t\ell}) = \Sigma_{\ell \in \mathbb{N}} v_k'(\mathcal{B}^0_{t\ell}) \geq \Sigma_{\ell \in \mathbb{N}} z_0^{-1} = \infty.
\]

But \(v_k'\) is bounded. A contradiction. Hence, \(v_k''\) is absolutely continuous with respect to the Lebesgue measure on \(I\).

Proof of Lemma 1.2 The customer’s strategy affects \(V_k\) through the probability \(\alpha^k_j\) in (1.5). Since \(V_k\) is continuous in \(\alpha^k_j\) which is continuous in \(\sigma_k\), \(V_k\) is continuous in \(\sigma_k\). To show that \(V_k\) is continuous in
$G_k \left( \omega, Q^j, Q^j \right)$, it suffices to show that
\[
\int_0^\infty \left[ \int_0^{\min(\hat{Q}^j, t')} e^{-r \tau} d\tau + e^{-r t'} \sum_{\omega \in \Omega} V_k \left( \omega, Q^j(t'), Q^{-j}(t') \right) \mu(\omega) \right] dF(t')
\] is continuous in $G_k \left( \omega, Q^{-j}, Q^j \right)$. By equation (1.1), the expression in (1.9) depends on $G_k \left( \omega, Q^{-j}, Q^j \right)$ through $\sum_{q \in \Omega_i} V_k \left( \omega, Q^j, Q^{-j} + q \right) g_k \left( \omega, Q^{-j}, Q^j \right)(q)$. Since the last expression is linear in the probabilities $(g_k \left( \omega, Q^{-j}, Q^j \right)(q))_{q \in \Omega}$, it is continuous in $G_k \left( \omega, Q^j, Q^{-j} \right)$. That $V$ is continuous in $G_k \left( \omega, Q^{-j}, Q^j \right)$ follows from its linearity in the probabilities $(g_k \left( \omega, Q^j, Q^{-j} \right)(q))_{q \in \Omega_i}$.

**Proof of Theorem 1.1** To prove the existence of a symmetric Markovian equilibrium for the stage game $\Gamma(\omega, Q^A, Q^B)$ we first invoke Kakutani-Fan-Glicksberg fixed point theorem to prove the existence of equilibrium in strategies that are totally mixed with modulus $k$. Following that, invoking sequential compactness and letting $k \to \infty$ and we establish the existence of a convergent subsequence of fixed points whose limit is our symmetric Markovian equilibrium.

To start with, we construct a correspondence that maps the sets of suppliers’ value functions, players’ strategies and the distributions on the queues into themselves. Let $M \subset \mathbb{R}^{\Omega \times I}$ whose elements are bounded $\mathcal{B}(\Omega \times I)$–measurable functions and suppose that the suppliers’ continuation functions $V_k$ belong to $M$. Assume that the supplier’s strategies are in:

$$G_k := \{ G : \Omega \times I \to \mathcal{G}|g(\omega, Q^A, Q^B)(q) \geq \frac{1}{k}, \forall q \geq \omega \},$$

the set of totally mixed, $\mathcal{B}(\Omega \times I)$–measurable, CDF with supports $\Omega_\omega$, $\omega \in \Omega$. Trivially, $G_k$ is non-empty, closed and convex set. The set of customers strategies is: $\Sigma_k := (\Sigma_1 \times \Sigma_2)^T$.

Define a correspondence $\Upsilon_k$ from $M \times G_k \times \Sigma_k \times \mathcal{V}$ to itself

$$\Upsilon_k (V_k, G_k, \sigma_k, v_k) \mapsto (\bar{V}_k, \bar{G}_k, \bar{\sigma}_k, \bar{v}_k),$$

such that $\bar{V}_k$ is the suppliers maximized value function given $(G_k, \sigma_k, v_k)$, $\bar{G}_k$ is the maximizer and measurable selection (with respect to $(\omega, Q^A, Q^B)$), $\bar{\sigma}_k$ is the customer’s best response function, and $\bar{v}_k$ is the next stage distribution on $I$.

**Claim 1.3.** The sets $M$, $G_k$, $\Sigma_k$ and $\mathcal{V}$ in the domain of $\Upsilon_k$ are all compact subsets of locally convex Hausdorff spaces.

---

Proof. Consider the set $M$. Since the point-wise limit of measurable functions is measurable, we have that $M$ is bounded and closed in $\mathbb{R}_+^{\Omega \times T}$. Hence, by Tychonoff Theorem, $M$ is compact in the product topology and is a subset of $\mathbb{R}_+^{\Omega \times I}$, which is locally convex Hausdorff space. To show that $G_k$ are compact subsets of locally convex Hausdorff space suffices it to observe that $G_k$ is a compact subset of $\mathbb{R}_+^{n \times \Omega \times I}$, which is locally convex Hausdorff space. Consider next the set $\Sigma_k$. Since $\Sigma_i$, $i = 1, 2$ are compact sets, so is the product, $\Sigma_1 \times \Sigma_2$. Hence, by Tychonoff Theorem, $\Sigma_k$ is compact subset of $\mathbb{R}^{T \times \Omega^2 \times T}$. Also, by Prokhorov’s theorem, $V \subset \mathbb{R}^{2T}$ is compact (in the topology of weak convergence). Hence, both $\Sigma_k$ and $V$ are compact subsets of locally convex Hausdorff spaces. \hfill \Box

We show next that the set $\Sigma_k$ of the customers’ strategies has the required measurability properties.

Claim 1.4. Given $G_k \in G_k$ and $v \in V$, $\sigma_{1,k}^{(\theta, \beta)}(q, Q)$ is measurable with respect to $B(T) \times B(\Omega \times [0, \infty))$, and $\sigma_{2,k}^{(\theta, \beta)}(q_1, Q_1) : \Omega \times [0, \infty) \to \{0, 1\}$ is measurable with respect to $B(T) \times B(\Omega \times [0, \infty))$.\footnote{$B(\Omega \times [0, \infty))$ is the restriction of $B(\Omega \times I)$ to $\Omega \times [0, \infty]$.} Moreover, the mapping $\sigma_{1,k} : \Omega \times [0, \infty) \to [0, 1]$ given by

$$\sigma_{1,k}(q, Q) = \int_{T} \sigma_{1,k}(q, Q) d\xi(\beta, \theta)$$

and $\sigma_{2,k}(q_1, Q_1) : \Omega \times [0, \infty) \to [0, 1]$

$$\sigma_{2,k}(q_1, Q_1)(\cdot, \cdot) = \int_{T} \sigma_{2,k}^{(\beta, \theta)}(q, Q) d\xi(\beta, \theta)$$

are well-defined, $B(\Omega \times [0, \infty])$ –measurable, functions.

Proof. By Definition 2 and proposition 2, the customer’s best responses to $(q_1, Q_1)$ in the first visit, and to $(q_1, Q_1, q_2, Q_2)$ in the second visit, are single valued functions. By Berge Maximum Theorem\footnote{See Aliprantis and Border (2006) Theorem 17.31.} the best response functions $\sigma_{1,k}^{(\theta, \beta)}(\cdot, \cdot)$ and $\sigma_{2,k}^{(\theta, \beta)}(q_1, Q_1)(\cdot, \cdot)$ are continuous on $T$. Hence, they are $B(T)$ –measurable. Thus, $\sigma_k$ is well-defined, $B(\Omega \times I)$ –measurable, vector-valued function.

Next we show that $\sigma_k$ is $B(\Omega \times [0, \infty))$ –measurable function. Since $\sigma_{k}^{(\beta, \theta)}(q, Q)$ is defined over the product space $T \times \Omega \times [0, \infty]$, for every given $(q, Q) \in \Omega \times [0, \infty]$, there exists a sequence $\{\sigma_{k,n}^{(\beta, \theta)}(q, Q)\}$ of simple functions

$$\sigma_{k,n}^{(\beta, \theta)}(q, Q) := \sum_{i=1}^{n} \alpha_i(q, Q) 1_{A_i}(\theta, \beta),$$

that, by Fubini-Tonelli theorem can be chosen to be $B(\Omega \times [0, \infty))$ –measurable, such that

$$\lim_{n \to \infty} \sigma_{k,n}^{(\beta, \theta)}(q, Q) = \sigma_{k}^{(\beta, \theta)}(q, Q).$$
where \((A_i)_{i=1}^n\) is a partition of the customers’ type space \(T\), \(1_{A_i}\) are the indicator functions, and \(\alpha_i(q, Q)\) are the coefficients of the simple functions.

Since \(T\) may be equipped with a finite measure and, for each \((q, Q) \in \Omega \times [0, \infty]\), \(\sigma^{(\beta, \theta)}_k(q, Q)\) is \(\mathcal{B}(T)\) measurable function, we have a well-defined function

\[
\sigma_{1,k}(q, Q) = \int_T \sigma^{(\beta, \theta)}_k(q, Q) d\xi(\beta, \theta) = \lim_{n \to \infty} \sum_{i=1}^n \alpha_i(q, Q) \xi(A_i).
\]

But the simple functions were chosen so that the respective measures \(\sum_{i=1}^n \alpha_i(q, Q) \xi(A_i)\) are \(\mathcal{B}(\Omega \times [0, \infty])\) measurable. Since the sum and pointwise limit of measurable functions are measurable, we get that \(\sigma_{1,k}\) being the pointwise limit of sums of \(\mathcal{B}(\Omega \times [0, \infty])\) measurable functions, is \(\mathcal{B}(\Omega \times [0, \infty])\) measurable. By the same argument \(\sigma_{2,k}(q, Q)\) is \(\mathcal{B}(\Omega \times [0, \infty])\) measurable, for all \((q, Q) \in \Omega \times [0, \infty]\).

**Claim 1.5.** Given any \(\mathcal{B}(\Omega \times [0, \infty])\) measurable \(\sigma_k\), the rival supplier’s \(\mathcal{B}(\Omega \times I)\) measurable strategy \(G'_k\), a probability space \((I, \mathcal{B}(I), \nu)\), and \(\mathcal{B}(\Omega \times I)\) measurable continuation value functions \(V\), a supplier best response correspondence \(\varphi : \Omega \times I \rightrightarrows \mathcal{G}\) is \(\mathcal{B}(\Omega \times I)\) measurable that admits a measurable selector.

**Proof.** Without loss of generality, consider supplier \(A\)’s optimization problem. Since supplier \(A\)’s objective function \((1.1)\) is linear (hence continuous) in his own mixed strategy \(G_k\) the rival’s strategy \(G'_k\), the customer’s \(\sigma_k\), and \(v_k\) and is \(\mathcal{B}(\Omega \times I)\) measurable, it is a Carathéodory function.

Note that \(\mathcal{G}_k\) is a separable metrizable space. Define the constant correspondence \(\zeta : \Omega \times I \rightrightarrows \mathcal{G}\) by \(\zeta(\omega, Q^A, Q^B) = \mathcal{G}_k\), for all \((\omega, Q^A, Q^B) \in \Omega \times I\). Then \(\zeta\) is weakly measurable correspondence with non-empty compact and convex value. Denote by \(V_k^* (\omega, Q^i, Q^{-i})\) the solution to \((1.1)\) for supplier \(j \in \{A, B\}\). Then, by the Measurable Maximum Theorem\(^{24}\) the argmax correspondence \(\varphi : \Omega \times I \rightrightarrows \mathcal{G}_k\) defined by

\[
\varphi(\omega, Q^A, Q^B) = \{G_k \in \mathcal{G}_k \mid V_k(\omega, Q^I, Q^{-i}) = V_k^* (\omega, Q^I, Q^{-i})\}
\]

is weakly measurable correspondence with non-empty compact values. Hence, supplier \(A\)’s point-wise maximization problem has a solution, the maximized value function \(V_k\) is \(\mathcal{B}(\Omega \times I)\) measurable, and the correspondence \(\varphi\) maximizer admits a measurable selector.

**Claim 1.6.** The correspondence \(\Upsilon_k\) is non-empty, compact and convex valued.

**Proof.** Because \(\Delta(\Omega)\) and \([0, 1]\) are both compact sets (in the \(\mathbb{R}^n\) topology), by Tychonoff Theorem, the spaces \(\Delta(\Omega)^{\Omega \times I}\) and \([0, 1]^{\Omega \times [0, \infty]}\) are compact in the product topology. Moreover, because \(\mathcal{G}_k\) is closed subset of \(\Delta(\Omega)^{\Omega \times I}\) it is compact in product topology.

The suppliers objective function, \( V_k \), is linear and, hence, continuous, in \( G_k, \sigma_k \) and \( \nu_k \), and \( \nu_k \) is linear and, hence, continuous, in \( G_k \) and \( \sigma_k \). Moreover, because the domain of \( \Upsilon_k \) is a product of measurable spaces, the objective function is \( \mathcal{B}(\Omega \times I) \)–measurable, and \( \nu_k \) is \( \mathcal{B}(I) \)–measurable, they are Carathéodory functions. Hence, by the Measurable Maximum Theorem, for every given \((\omega, Q^A, Q^B) \in \Omega \times I\), the suppliers’ maximization problems (1.1) has a solution in \( G_k \) and the correspondence \( \nu : G_k \times G_k \times \Sigma_k \Rightarrow M \times \Delta(I) \) has non-empty single values. Given \((q, Q) \in \Omega \times [0, \infty]\), the customers’ optimization problem has a solution in \( \sigma_k(q, Q) \in [0,1] \) and, since the suppliers strategies are in \( G_k \), there is no prescription with probability zero. Thus, the existence of \( \bar{V}_k, \bar{G}_k \) and \( \bar{\sigma}_k \) is implied by the Measurable Maximum Theorem. Moreover, by the same theorem, \( \bar{V}_k \) is \( \mathcal{B}(\Omega \times I) \)–measurable, \( \bar{\sigma}_k \) is \( \mathcal{B}(\Omega \times [0, \infty]) \)–measurable, and we can always select measurable maximizers \( \bar{G}_k \).

Since \( \bar{V}_k, \bar{\sigma}_k \) and \( \hat{\nu}_k \) are single-valued, they are trivially convex and compact. The convexity of \( \bar{G}_k \), is an immediate implication of the linearity of the suppliers’ and customers’ objective function in the maximizers distributions which implies that any convex combination of the maximizers is a maximizer. Furthermore, since the measurable selectors are the intersection of the set of maximizers which, by the Measurable Maximum Theorem are compact valued, and the space \( G_k^{\Omega \times I} \) of measurable functions, \( \bar{G}_k \) is measurable and compact and convex valued.

**Claim 1.7.** The mapping \( \Upsilon_k \) has a closed graph.

**Proof.** Since \( G_k \) is a compact set and the set \( \bar{G}_k = G_k \cap \Lambda_k \), where

\[
\Lambda_k := \text{arg} \max_{(G \in G) \mid g(\omega, Q^A, Q^B) \geq \frac{1}{k}} \int_0^\infty \int_0^{\min(Q^I, t')} e^{-rt} d\tau + e^{-rt'} \left( \sum_{\omega \in \Omega} V_k(Q^A(t'), Q^B(t'), \omega) \right) dF(t'),
\]

is the intersection of all the suppliers maximizers. By the Measurable Maximum Theorem \( \Lambda_k \) is compact and forms a closed graph. In addition, \( G_k \) also forms a closed graph. But the graph of the projection of the correspondence \( \Upsilon_k \) on \( G_k \) constitutes of the intersection of the closed graphs formed by \( \Lambda_k \) and \( G_k \). Moreover, since the projections of \( \Upsilon_k \) on \( M \times \Sigma_k \times \Delta(I^2) \) are continuous functions, their graph is closed. Thus, \( \Upsilon_k \) has a closed graph.

**Proof of Theorem 1.1.** By Claims 1-3 and the Kakutani-Fan-Glicksberg fixed point theorem, the set of fixed points of \( \Upsilon_k \) is non-empty and compact. Since the product set of compact sets is compact, we can construct a sequence \( \left( \hat{V}_k, \hat{G}_k, \hat{\sigma}_k, \hat{\nu}_k \right) \) that has a convergent subsequence. Denote by \( \left( \hat{V}, \hat{G}, \hat{\sigma}, \hat{\nu} \right) \) the sub-sequential limit point. Since it is the limit point of measurable functions, it is measurable.

Let \( \{k_n \mid n = 1, 2, \ldots\} \) be a convergent subsequence and consider supplier \( j \). Given \( \left( \hat{V}_{k_n}, \hat{G}_{k_n}, \hat{\sigma}_{k_n} \right) \), for all
Proof of Theorem 1.2  We need to show that, for some stage game $\Gamma (\omega, Q^A, Q^B), \hat{G} (\omega, Q^j, Q^-j) = \delta_\omega$ is not a best response to $\hat{G} (\omega, Q^-, Q^j) = \delta_\omega$, for some $j \in \{A, B\}$.

Suppose that $Q^A = 0 < Q^B$. In fraud-free equilibrium the customers believe that both suppliers prescribe the necessary service truthfully. Hence, the only reason to obtain a second prescription is the expectation that the second supplier has a sufficiently shorter queue that would justify bearing the cost of obtaining a second prescription. Let the state be $\omega_i$ and suppose that the long-queue supplier prescribes truthfully, (that is, $q_B = \omega_i$). We show that if $Q^B$ is sufficiently large then prescribing $\omega_i$ is not a best response of the short-queue supplier.

To begin with, observe that if the customer visits the short-queue supplier first then, because supplier $B$’s queue cannot possibly be shorter than $Q^A = 0$, the customer never seeks a second prescription.

The probability of a new customer accepting the prescription $\omega_i$ of the long-queue supplier is as follows: If the long-queue supplier (that is, supplier $B$) is the customer’s first call then the probability of acceptance
queue supplier if and only if
\[ \hat{\varrho} \in \mathbb{R} \]
Suppose further that the short-queue supplier prescribes \( q \) and if he prescribes \( A \) implies that \( R \). Hence, \( q \) if he prescribes \( q \).

Suppose that the customer visits the long-queue supplier first and decides to seek a second prescription. Suppose further that the short-queue supplier prescribes \( q_A \in \Omega_{\omega_i} \). The customer will return to the long-queue supplier if and only if

\[ p_2 (q_A, \omega_i, Q^B) := \Pr \{ \beta \in [0, 1] | 1 - q_A < (1 - \omega_i) e^{-\beta Q^B} \}. \]

Define \( p_B (q_A, \omega_i, Q^B) = p_1 (Q^B) + p_2 (q_A, \omega_i, Q^B) \). Since \( 1 - \omega_i > (1 - \omega_i) e^{-\beta Q^B} \) for all \( \beta \in (0, 1) \), \( q_A = \omega_i \) implies that \( p_2 (q_A, \omega_i, Q^B) = 0 \). Thus, \( p_B (q_A, \omega_i, Q^B) = p_1 (Q^B) \). Hence, the short-queue supplier’s payoff if he prescribes \( q_A = \omega_i \) is:

\[ R(\omega_i) = (1 - p_1 (Q^B)) V(\omega_i, 0, Q^B) + p_1 (Q^B) V(\omega_i, 0, Q^B + \omega_i) \]

and if he prescribes \( q_A = \omega_{i+1} \) the short-queue supplier’s payoff is:

\[ R(\omega_{i+1}) = (1 - p_B (\omega_{i+1}, \omega_i, Q^B)) V(\omega_{i+1}, 0, Q^B) + p_B (\omega_{i+1}, \omega_i, Q^B) V(\omega_i, 0, Q^B + \omega_i) \].

Hence,

\[ R(\omega_{i+1}) - R(\omega_i) = \]

\[ (1 - p_1 (Q^B)) (V(\omega_{i+1}, 0, Q^B) - V(\omega_i, 0, Q^B)) - p_2 (\omega_{i+1}, \omega_i, Q^B) (V(\omega_{i+1}, 0, Q^B) - V(0, 0, Q^B + \omega_i)) \]

\[ = (1 - p_1 (Q^B)) (1 - F(\omega_{i+1})) \int_{\omega_i}^{\omega_{i+1}} e^{-rt} dt \]

\[ + \int_{0}^{\omega_{i+1}} e^{-rt'} \sum_{\omega' \in \Omega} [V(\omega', \omega_{i+1} - t', Q^B - t') - V(\omega', \omega_i - t', Q^B - t')] \mu(\omega') dF(t') \]

\[ - p_2 (\omega_{i+1}, \omega_i, Q^B) (V(\omega_{i+1}, 0, Q^B) - V(\omega_i, 0, Q^B + \omega_i)) \].

But \( \lim_{Q^B \to \infty} p_2 (\omega_{i+1}, \omega_i, Q^B) = 0 \) and \( \lim_{Q^B \to \infty} p_1 (Q^B) = 0 \). Hence, in the limit as \( Q^B \to \infty \), \( R(\omega_{i+1}) - R(\omega_i) > 0 \). Thus, there is \( N \) such that, for all \( Q^B > N \), \( \hat{G}(\omega_i, 0, Q^B) = \delta_{\omega_i} \) is not best response to \( \hat{G}(\omega_i, Q^B, 0) = \delta_{\omega_i} \).
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Chapter 2

Strategic Fraud and Competition

2.1 Introduction

Imagine arriving in the airport in a foreign city midnight. You were too busy to do your homework - you do not know how far the hotel is from the airport. You just remember the hotel name. Your smart phone breaks down right when you want to check Google map. Outside the quiet airport waits for you a line of taxis. It seems an advantage to you because economic theory predicts competition in a market of homogeneous goods will get consumers the best deal. However, you just do not feel it is that simple. After all, Uber and Lyft should not have emerged if Bertrand competition is in place as consumer’s wish.

In this study, we are going to show such a feeling does come for a reason. As a passenger, you do not know how costly the service should be (e.g. the route and labor hours, etc.) Ex-ante, you lack the knowledge to judge if the provided service is necessary. Ex-post, even if you use maps to prove your taxi driver took the longer route, the driver can claim traffic at that time is bad and he has taken the best route to drive you to hotel based on his experience. It is also difficult to appeal to third parties such as a judge even if you have doubt in mind. In this case, an uninformed consumer has to search for suppliers not only for better prices but also to learn how costly the service really is because market prices can vary depending on production costs. A high taxi tariff and tips can be acceptable if taking long routes is unavoidable to arrive in the destination and it is not so if the destination is just 5 minutes away from airport. However, the taxi driver may claim it is the former and take unnecessarily long routes in order to make more profits even though the actual situation is the latter. Many industries including cosmetic products, car repairs, and medical services can also have this kind of issue. Incentives to convince consumers of purchasing more costly and unnecessary products does not only create waste but also make consumers engaged in searching, which is another social
inefficiency. It seems reasonable to expect that if competition among suppliers is strong enough, the outcome should favor consumers and the above-mentioned problems will be gone. A straightforward way to enhance competition is to have more suppliers in a market. Naturally one might think that more suppliers will induce lower prices and unnecessary services will be cut. This study shows such an hypothesis is correct only for the second half: whenever the number of suppliers is large enough, suppliers do always provide service that is just necessary. However, the market prices will be as high as if it is set by a monopolist. It resembles the result in Diamond 1971 that search cost leads to monopoly pricing in a market.

To understand the intuition, consider an extreme example. Let’s assume $\omega^* > 0$ is the necessary and sufficient marginal cost of serving the consumer’s need. The number of suppliers is infinite and there are two possible types of consumers. Consumers of the first type are called shoppers and they visit all suppliers in the market with no extra costs. Consumers of the other type are called non-shoppers and they sequentially search for experts with a high search cost. When a supplier is quoting price $p$ for a service of cost $\omega$ (here we implicitly assume $\omega \geq \omega^*$, which means the service $\omega$ is sufficient to address consumer’s need), she needs to consider how likely it will be accepted by consumers. Offering a service with high cost $\omega$, holding prices the same, will squeeze the profits but it may lead the consumer to think the cost is high, which induces consumers infer that other suppliers may also offer higher prices\(^1\). However, due to the large number of suppliers, the supplier will rationally anticipate that the her offer is unlikely to be the best deal in the market. The supplier then infers that, upon encountering a consumer, the service is more likely to be accepted by a non-shopper because shoppers will probably end up with cheaper prices from some other suppliers. With weak incentives to compete for shoppers, suppliers tend to increase the prices offered to consumers and cut unnecessary service to maximize profits. On the other hand, such a pricing policy encourages non-shoppers to not search because even though lower prices may exist, they are costly and unlikely to be found. The non-shoppers’ searching decision in turn further enhances suppliers’ tendency to not compete. A non-competitive equilibrium therefore emerges in the market due to “more competition”. This is similar to the search model depicted in Diamond (1971). With a single price prevailing in market, consumers lose incentive to search, and suppliers will have no incentive to compete because it will only decrease profits.

What is interesting is that the previous intuition applies even when the number of suppliers is finite. This study shows that, when the number of suppliers is higher than a threshold, each supplier’s incentive to compete for consumers will be weak enough so that suppliers will price the service like a monopolist, which further makes consumers with search cost to accept offers immediately. Moreover, it leads suppliers to offer necessary services to save costs because consumers cannot get better prices anywhere and there is no need to

\(^1\text{It is similar to the use of burned money in a signaling game such as education degree in a labor market (Austen-Smith and Banks 2000).}\)
persuade consumers by manipulating costs. In other words, if we define \( p - \omega \) as supplier's profits and \( \omega - \omega^* \) as the degree of inefficiency, the result shows that when a market has enough suppliers, the equilibrium is characterized by maximum efficiency, \( \omega = \omega^* \), and maximum supplier's profits, \( p >> \omega = \omega^* \).

We follow the approach in D. O. Stahl (1989) to model the search process with perfect recall. First, all the qualitative results in this study can be extended to the case of no recall. It is thus without loss of generality to study the problem with perfect recall. Second, it is easier to make direct comparison between our result and others papers because it is a widely used approach in the literature. Finally, perfect recall makes the environment more favorable to consumers because they can get back to suppliers without incurring further costs, which should induce more competition between suppliers. If competition under perfect recall cannot deliver a result in favor of consumers, we cannot expect more from an environment with no recall. One important feature that distinguishes this work from D. O. Stahl (1989) is that he has to assume symmetric equilibrium and also a fixed, publicly known cost to simplify the information structure. Also, assuming symmetric supplier's strategies makes consumer decision problem stationary: there is no learning in the search. As Rothschild (1974) points out, search with learning can lead to non-trivial searching behavior. Consumers can behave in a way significantly different in the case when supplier’s strategies are known in the first place and consumers have a fixed reservation price throughout the search. By relaxing these two assumptions, this study shows the effect of competition is robust to cost uncertainty and learning.

2.2 Literature Review

Price distributions in markets with frictions under competition has been explored widely. As Bertrand competition in a market with no frictions will necessarily lead to a single price equal to supplier’s marginal cost, it may be tempting to conclude that in order to induce a price distribution in markets, it is sufficient to introduce frictions, especially search friction. However, as Diamond (1971) points out, search friction alone is not enough because it still results in a single market price, that is monopolist prices, instead of a price distribution. A seminar paper Varian (1980) addresses this issue by developing a model that combines the insight from Salop and J. Stiglitz (1977) and Shilony (1977) to assume two types of consumers. One type can conduct search with no cost while the other type will make a purchase as long as the offered price is lower than a reservation level. In this paper, only the the former type consumer search for lower prices among suppliers and the latter just randomly shows up in each shop to decide whether to make a purchase. It successfully shows that a distribution of prices emerges endogenously in the model. D. O. Stahl (1989) further takes sequential search into account following Varian (1980). In its model, both types of consumers can search for suppliers strategically. It turns out consumer’s search strategy in equilibrium is
Indeed characterized by a reservation price as Varian (1980) assumed, that is endogenously determined by the equilibrium price distribution.

Sequential search plays an important role in this study since consumers need to search among suppliers both for prices and costs information. This study can be considered in line with studies of oligopolistic competition where sellers face the same marginal cost when deciding price and consumers search to find the best offer. One crucial difference between past studies of oligopolistic markets and ours is that there is no hidden information about seller’s type in oligopolistic models whereas in our study, consumers do not know how costly the necessary service is. M. Janssen, Pichler, and Weidenholzer (2011) analyses a market with hidden cost information and derives conditions under which a reservation equilibrium can exist. A reservation equilibrium is defined as an equilibrium in which a non-shopper will stop search once a supplier quotes prices below a pre-determined level. They show that the existence of reservation equilibrium requires high uncertainty of marginal cost and large search cost. M. C. Janssen, Alexei Parakhonyak, and Anastasia Parakhonyak (2014), however, argues that the reservation equilibrium does not have out-of-equilibrium strategies that satisfy the equilibrium refinement proposed by Cho and Kreps to reduce the multiplicity of Perfect Bayesian equilibria by dominance arguments. They further show there are equilibria that satisfy Cho and Kreps refinement. Though the characterization of equilibria is limited to cases of only two states, multiplicity of equilibria arises as in other signaling game.

This work also shares features in credence good markets. Darby and Karni (1973) first coined “credence good” using auto repair as an example. Credence goods are services that consumers cannot know their necessary level of which is needed before purchase and after purchase. Another common example is an operation of appendix removal. Unless a patient has sufficient medical knowledge, he cannot determine whether such an operation is needed before and after the operation. A important feature of credence goods is information and service are provided jointly instead of separately. This feature implies some fraud will be costly to detect since consumers will have to look for different sources for diagnosis. It also distinguishes credence goods from traditional lemons markets. In lemons markets (such as used cars), consumers sooner or later realize the quality or content of the goods after purchase and can judge if the purchased goods are worthy so contingent contracts may be used to solve the information asymmetry. However, the difficulty in credence goods markets is that those mechanism do not apply due to lack of ex-post information. Wolinsky (1993) proposes an equilibrium where some experts specialize in diagnosis and the other specialize in treatment. In this equilibrium, diagnosis experts will be willing to reveal the true state as they do not get profits from the treatment they propose. However, such an equilibrium is not sub-game perfect since a consumer and expert will find it is a pareto improvement to have treatment right after truthful diagnosis is made so consumer does not have to move.
Past studies (Pitchik and Schotter (1987) and Pitchik and Schotter (1993)) of credence good markets usually model the treatment uncertainty to be represented by two states, high and low. This limits the complexity of learning in the strategic interaction and makes analysis easier. However, it also imposes strong restrictions on the consumer’s search behavior because the consumer always stops whenever low treatment is suggested.

This study is also related with literature on cheap talks. Pitchik and Schotter (1987) is the first study to point out fraud in credence markets can be modeled as a cheap talk game. It studies an environment in which consumers face a monopolistic supplier and have an outside option. In this setting, although there is no competition for the supplier, supplier still plays mixed strategies in equilibrium. Starting with Crawford and Sobel (1982), most studies focus on the cases where sender’s preference is state-dependent and is sender’s private information. Closest to our setting is Lipnowski and Ravid (2020). They study a cheap talk game where sender’s preference is transparent, which is aligned with our setting where supplier’s utility is simply just the profits from services provided and it is independent on the state (e.g. the level of service needed).

Regarding the anti-competitive result due to more suppliers in a market, D. O. Stahl (1989), Rosenthal (1980), J. E. Stiglitz (1987), Schulz and K. Stahl (1996), and Satterthwaite (1979) have found that more suppliers will increase the market price. However, these papers establish the result by assuming symmetric equilibrium. Whether it is also true in asymmetric equilibria is unknown. In addition, the models does not have uncertain costs and choices for suppliers to reveal information to consumers that may complicate the analysis because of learning. The model in this study take these practical consideration into account and prove the result is robust.

2.3 Oligopoly Model

The market consists of $n$ suppliers, identified by an identity index $i \in I = \{1, 2, ..., n\}$, and a consumer arriving with a need for some service or product that requires costs to produce. The cost of least necessary and sufficient service to satisfy consumer’s need is defined as a state of the world $\omega^*$ belonging to an interval of positive real numbers $\Omega := [\omega_0, \omega_1] \subseteq \mathbb{R}^+$. $\omega^*$ is distributed ex ante according to a common prior distribution $G$ of full support in $\Omega$. Consumers do not know $\omega^*$ but suppliers do. Consumers search randomly and sequentially (with perfect recall) for services and prices. Upon visiting a supplier, the supplier learns the state $\omega^*$ and offer a pair $q$ of price $p$ and service cost $\omega$, $q = (p, \omega)$. The offer must address consumer’s need and the consumer will tell whether the need is satisfied. Otherwise the consumer refuses payment. Hence, the offered service should be at least the minimal service required to solve the problem. Formally, $\omega \geq \omega^*$ and we call it “liability constraint” (Dulleck and Kerschbamer (2006)). Upon receiving
an offer from a specific supplier, the consumer must choose between searching for other experts to obtains a second offer, and stop searching accepting the present one. If she chooses to keep searching, she will randomly visit one of the remaining suppliers that have not been visited and repeat the same process in the second, third, fourth,..., and \( n \)-th visit. Continuing the search entails a consumer-specific cost which is consumer’s private information. Let \( 0 < \lambda < 1 \) denotes the probability of the consumer being a shopper, people who has zero search cost. \( 1 - \lambda \) is the probability that the consumer is a non-shopper (i.e. have non-zero search cost \( c > 0 \)).

If she chooses to stop searching because either she has visited all suppliers in the market or she wants or she has made her mind early, the game ends.

### Consumer

Let \( q^m = ([1], q[1]), ([2], q[2]), ..., ([m], q[m]) \) denote the pairs of supplier’s identities and offers that the consumer has collected in \( m \) visits \( (m \leq n) \), where \( [i] \in I = \{1, 2, 3, ..., n\} \) stands for the supplier’s identity index in \( i \)-th visit. Since the consumer only cares about how much she pays for getting the need satisfied, given the assumption that all offers must be sufficient, if the consumer stops searching, the consumer recalls the offer with the lowest price and receive utility payoff \( u(W - \min(p[1], p[2], ..., p[m])) \) where \( p[i] \) denotes the price from the supplier in \( i \)-th visit, \( W \) denotes consumer’s wealth, \( u \) is a real-valued, increasing and continuous function. There are two exceptions to note.

First, this study assumes that the consumer has an outside option denoted by \( p_m > 0 \), a maximum of willingness to pay. If the consumer decides to stop searching and none of the offers is better than \( p_m \), the consumer leaves the market and no supplier receives revenue. To make the model meaningful, it is further assumed \( p_m > \omega_1 \). Second, if there is more than one offer with the lowest price, the consumer chooses one supplier among suppliers whose offer the lowest price with equal probability. If the lowest price she has collected in the market is equal to \( p_m \), the consumer will omit the outside option and choose one supplier in the market according to the rule specified before. \( p_m \) is public information and the same for all consumers.

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2 Readers may wonder why consumer’s choice set is limited to binary decision instead of bargaining with suppliers. Our answer to it is it depends. When the fee schedule is fixed, a passenger may be able bargain with a taxi driver for the time needed to arrive but it is not easy for a patient to argue with a physician whether some material used in a operation is necessary. If a consumer pick up a wrong target to negotiate (e.g. a material that is actually necessary), bargaining may reveal the ignorance of consumers and expose consumers in the risk of being exploited further. We conjecture binary decision instead of bargaining is in favor of consumers who possess very little information.

3 To simplify the exposition we assume that all consumers are identical (that is, have the same wealth and discount factor) and that the only difference among them is the search cost.

4 Although D. O. Stahl (1989) justifies the existence of shoppers by arguing some people receive utility from comparing prices, this argument does not seem reasonable in our setting. If a car breaks down and fails to run, probably there is no one who will enjoy moving the car around to different repair shops. Despite this fact, we keep this assumption because it is useful to contrast our results to models that is commonly used in search theory and see how strategic transmission in information contributes to the difference. In this study, most results extends to settings where a general distribution of search cost is allowed as D. O. Stahl (1996) extending D. O. Stahl (1989).
Suppliers

When a consumer stop searching and decide to accept a supplier’s offer \( q = (p, \omega) \), the supplier receives the payment \( p \) and the profit from the service is

\[ p - \omega \]

whereas it receives nothing if the consumer accepts an offer of other suppliers.

2.4 Equilibrium Analysis

Let \( \mu(\omega|q^m) \) denotes the posterior probability measure of state \( \omega \) conditional on the offers \( q^m \).

Definition 2.1. The consumer’s (mixed) stopping strategy when \( q^m, \forall m \in \{1, 2, 3, ..., n\} \), has been collected is defined as a function that maps \( q^m \) to a real number in \([0, 1]\):

\[ \sigma : \cup_{m \in \{1, 2, ..., n\}} (I \times \Omega \times R^+)^m \to [0, 1] \]

\( \sigma(q^m) = 1 \) means the consumer stops searching and picks one of the offers from the \( m \) suppliers she has visited or the outside option, \( \sigma(q^m) = 0 \) means she keeps searching for next offer. If \( 0 < \sigma(q^m) < 1 \), it means the consumer will stop with probability \( \sigma(q^m) \). Note in the definition we assume that the consumer knows and take into consideration the supplier’s identity index in decision making.

Since the consumer may end up with multiple best offers when she decides to stop searching, in order to break the tie and decide which supplier will be accepted by the consumer, it is assumed that the consumer will randomly choose one of the best offer from suppliers with equal probability. The probability of a offer \( q_i \) from a supplier \( i \) being accepted when a consumer has collected offers \( q^m \) from \( m \) suppliers, including \( (i, q_i) \), and decides to stop searching is defined by a tie breaking rule

\[ \phi(q^m, q_i) = \frac{1}{\#(p|p \in q^m, p = \min(q^m))} 1_{p_i \leq \min(p_m, q^m)} \]

where \( 1_{p_i \leq \min(p_m, q^m)} \) is an indicator function that returns 1 if \( p_i \) is the lowest price in \( q^m \) and no greater than \( p_m \), or 0 otherwise, and \( \#(S) \) is the counting function of a set \( S \) that returns the number of elements in \( S \). The set \( \{p \in q^m, p = \min(q^m)\} \) contains the lowest prices offered by suppliers in \( q^m \).

Definition 2.2. The suppliers (mixed) strategies are defined as distributions of offers \( q \in \Omega \times R^+ \) that depends on the real damage state \( \omega^* \). The cumulative probability function of a offer \( q = (p, \omega) \) from a
supplier $i \in I$ when the true state is $\omega^*$ is defined as

$$F_i(\bullet, \bullet | \omega^*): \mathbb{R}^+ \times [\omega^*, \omega_1] \rightarrow [0, 1]$$

In addition, in order to simplify notations, we will drop $\omega^*$ and write $F_i(p, \omega)$ or $F_i(q)$ interchangeably instead of $F_i(p, \omega|\omega^*)$ when there is no need to specify other states. In some special cases when suppliers’ offer of $\omega$ is deterministic, we will further denote $F_i(p, \omega)$ as $F_i(p)$.

Putting all these elements together, we can define and calculate the probability that a supplier’s offer is accepted by the consumer when the other suppliers’ strategies are fixed and the tie-breaking rule is given as before. Especially, to understand the main result in this study, we will decompose the probability into two parts. One is the probability of acceptance by non-shoppers. The other is by shoppers. Given a vector of probability distributions of supplier’s offer $\{F_i\}_{i \in I}$, the probability of a supplier $i$’s offer being accepted by a non-shopper is

$$\Phi_{ns}(q_i) = \left[ \frac{1}{n} \sigma(q_i) \phi(q^1, q_i) + \frac{1}{n} [1 - \sigma(q_i)] \sum_{j \neq i} \frac{1}{n-1} \int_{\text{supp}(F_j)} \sigma(q_i, q_j) \phi(q^1, q_j) dF_j + \frac{1}{n} [1 - \sigma(q_i)] \right] \int_{\text{supp}(F_k)} \sum_{j \neq i, k} \frac{1}{n-2} \sigma(q_i, q_j, q_k) \phi(q^2, q_k) d(F_j \times F_k) + ...$$

$$= \sum_{m=1}^{n-1} \frac{1}{\prod_{\bar{m}=0}^{m-1} (n-\bar{m})} \int_{(\Omega \times \mathbb{R})^m} \left[ \prod_{\bar{m}=0}^{m-1} \sum_{[\bar{m}]\neq i,[1],[2],...,[\bar{m}-1]} \right] \left[ \phi(q^m) \right] d\Pi_{\bar{m}=0}^{m-1} F_{\bar{m}}$$

where $\text{supp}(F)$ denotes the support of distribution $F$.

The expected utility of supplier $i$ from choosing a cumulative probability distribution of offer $F_i : \Omega \times \mathbb{R}^+ \rightarrow [0, 1]$ for given a vector of supplier’s probability distribution $F_j$ for all suppliers $j \neq i$ and consumer’s choice $\sigma$ is

$$U(\{F_j\}_{j \neq i}, \sigma) = \int_{\text{supp}(F_i)} (p - \omega) \{\lambda \Phi_s(q_i) + (1 - \lambda) \Phi_{ns}(q_i)\} dF_i(q_i)$$

subject to

$$\omega \geq \omega^*$$

where the probability of $i$’s offer being accepted by a shopper is

$$\Phi_s(q_i) = \Pi_{j \neq i} [1 - F_j(q_i)] \phi(q^0, q_i)$$

The non-shopper’s expected utility from choosing a stopping strategy $\sigma$, given supplier’s distribution
Because the tie-breaking rule \( \phi \) is

\[
V_{ns}(q^i, \{F_i\}_{i \in I}) = \sum_{[1] \in I} \frac{1}{n} \int_{\text{supp}(F_{[1]})} \{\sigma(q^i)u(W - \min(p_m, p_{[1]}))dF_{[1]}(q_{[1]}) + (1 - \sigma(q^i)) \sum_{[2] \in I/\{[1]\}} \frac{1}{m-1} \int_{\text{supp}(F_{[2]})} \sigma(q^j)u(W - \min(p_m, p_{[2]}))dF_{[2]}(q_{[2]}))\ldots\}
\]

\[
= \prod_{1 \leq j \leq n} \sum_{[i] \in I/\{[1], [2], \ldots, [j]\}} \int_{\text{supp}(F_{[i]})} \{\prod_{0 \leq i \leq j-1} (1 - \sigma(q^i))\}
\]

\[
\sigma(q^i)u(W - \min(p_m, \{p_{[k]}\}_{k=1}^{j-1}, p_{[j]}) - ic)dF_{[1]}(q_{[1]}))...dF_{[j]}(q_{[j]})
\]

Note that to assume away the need to take care of its decision problem, we explicitly assume the shopper visits all suppliers and pick the best offer so its expected utility can be written as

\[
V_s(q^i, \{F_i\}_{i \in I}) = \int_{\text{supp}(F_n)} \ldots \int_{\text{supp}(F_1)} u(W - \min(p_m, \{p_i\}_{i \in I}))dF_1(q_1)dF_2(q_2)\ldots dF_i(q_i)...dF_n(q_n)
\]

### 2.4.1 Equilibrium

Since suppliers may adopt different strategies \( F_i|_{i \in I} \) under different \( \omega^* \), the consumer needs to infer the state \( \omega^* \) from the suppliers’ offer to decide whether to stop searching. In addition, the game consists of subgames in which the consumer makes stopping decision sequentially and hence perfect Bayesian equilibrium serves as the most appropriate solution concept in this study.

**Definition 2.3.** A (mixed-strategy) perfect Bayesian equilibrium in the game is defined as a pair of strategies \( \{F_i\}_{i \in I}, \sigma \) and a consumer belief system \( \mu \) which satisfy the following conditions

1. Given \( \{F_j\}_{j \in I/\{i\}}, \sigma \) and the state \( \omega^* \in \Omega \), supplier \( i \)'s strategy \( F_i \) maximizes its utility. That is,

\[
F_i(\bullet, \bullet|\omega^*) \in \arg \max_{F \in \Delta(K^+_\times \Omega)} U(\{F_j\}_{j \in I/\{i\}}, \sigma)
\]

2. Given \( \{F_i\}_{i \in I} \), the non-shopper consumer’s strategy \( \sigma \) maximizes its utility \( \sigma(q^i) \in \arg \max_{\sigma \in [0,1]} V_{ns}(q^i, \{F_i\}_{i \in I})\forall j \in I
\]

3. Consumer’s posterior belief \( \mu(\omega|q^i) \) is consistent with \( \{F_i\}_{i \in I} \) according to Bayes’ rule. If an offer in the \( j \) visits, \( q^j \), is probability zero in the equilibrium and the supplier’s strategies \( F_i \) does not have density at \( q^j \), then the posterior belief can be anything. Furthermore, if supplier’s strategies \( F_i \) have density \( f_i \), then the probability density receiving offers \( q^j \) is expressed as

\[
\mu(\omega|q^i) = \frac{P(\omega \cap q^i)}{P(q^i)} = \frac{\prod_{i=1}^{j} f_{[i]}(q^i|\omega)}{\sum_{\omega'} \prod_{i=1}^{j} f_{[i]}(q^i|\omega')}
\]

Because the tie-breaking rule \( \phi \), which causes discontinuity in suppliers' expected utility, the fixed point
theorems do not apply. The strategy for proof is to construct a particular equilibrium by assuming the non-shopper will accept whatever is offered. This assumption largely eases the need to deal with off-equilibrium paths and beliefs. Later, I prove the previously constructed equilibrium strategies are the unique equilibrium strategies when suppliers are sufficiently many.

If the non-shopper will accept any offer for sure, such an offer must not trigger any undercutting among suppliers in equilibrium. In light of this intuition, we can further extend the intuition to every offer if the consumer’s acceptance rule is accepting everything.

**Lemma 2.1.** In any equilibrium characterized by non-shopper consumer’s equilibrium acceptance rule $\sigma(q) = 1$, it is necessary that when fixing a state $\omega^*$

- 1. suppliers’ strategies for offering $\omega$ is deterministic and $\omega = \omega^*$.
- 2. Any offer $q = (p, \omega)$ in any supplier’s equilibrium strategy support with $p < p_m$ must be accepted by shoppers with non-zero probability and $q$ must be in some other supplier’s strategy support.
- 3. For any offer $q = (p, \omega)$ with $p < p_m$ that is offered in the equilibrium, no supplier will offer $q$ with probability mass.
- 4. the equilibrium profits of every supplier are equal to $(p_m - \omega^*) \frac{1 - \lambda}{n}$
- 5. For any supplier $i$ and $j$ that both offer $p$, the cumulative probabilities of supplier $i$ and $j$ at $p$ are symmetric, i.e. $F_i(p|\omega^*) = F_j(p|\omega^*) \forall \omega^*, q$ and

**Proof of Lemma 2.1.** (1) Since the non-shopper will accept whatever is offered and the shopper will visit all suppliers, given any quoted price $p$, it is strictly dominated to offer $\omega > \omega^*$. Therefore the statement is proved.

(2) If there exists a supplier, say supplier A, and a offer $q = (p, \omega)$ such that $p < p_m$ and it is accepted by shoppers with zero probability, then the profits from $q$ is

$$\frac{1}{n}(p - \omega)$$

because non-shoppers accept $q$ with probability $\frac{1}{n}$. However, it is strictly lower than $\frac{1}{n}(p_m - \omega)$ because of $p < p_m$. Thus $q$ should not have been in the supplier’s strategy support. Contradiction.

If $q$ is not in another supplier’s support, then it implies the equilibrium strategies $F_i(q)$ of other suppliers must be constant around $q$, which implies offering $q^+ \equiv (p^+, \omega) = (p + \epsilon, \omega)$ with sufficiently small $\epsilon > 0$,
supplier A will get higher profits because

\[(\lambda \Pi_{i \neq A}(1 - F_i(q^+)) + \frac{1 - \lambda}{n})(p^+ - \omega) = (\lambda \Pi_{i \neq A}(1 - F_i(q)) + \frac{1 - \lambda}{n})(p^+ - \omega) > (\lambda \Pi_{i \neq A}(1 - F_i(q)) + \frac{1 - \lambda}{n})(p - \omega)\]

which contradicts the assumption that \(q\) is an equilibrium offer.

(3) If there exists an offer \(q\) as described and there is at least one supplier offering \(q\) with mass probability, then the other supplier, say supplier A, can profitably deviate to another offer \(q' = (p - \epsilon, \omega)\) with \(\epsilon \simeq 0\) to undercut and attract shoppers with non-zero probability. To see how, denote \(\phi_k\) the probability that there are exactly \(k\) suppliers other than supplier A offering prices equal to \(p\) and all suppliers other than supplier A offering prices no lower than \(p\). Then the profit from supplier A offering \(q\) is

\[(\lambda \Pi_{i \neq A}(1 - F_i(q)) + \sum_{k=1}^{n} \frac{\phi_k}{(k+1)} + \frac{1 - \lambda}{n})(p - \omega)\]

where \(\frac{\phi_k}{(k+1)}\) denotes the probability that supplier A will be accepted by shoppers when there are exactly other \(k\) suppliers offering \(q\) and the other suppliers offering prices higher than \(p\). Offering \(q' = (p - \epsilon, \omega)\), the profits from \(q'\) is greater than

\[(\lambda \Pi_{i \neq A}(1 - F_i(q)) + \sum_{k=1}^{n} \phi_k + \frac{1 - \lambda}{n})(p - \omega - \epsilon)\]

because the probability that all other suppliers offering prices higher than \(p - \epsilon\) must be greater than \(\Pi_{i \neq A}(1 - F_i(q)) + \sum_{k=1}^{n} \phi_k\). As there must be some \(\pi_k > 0\) and \(\epsilon\) is arbitrary, there must exists a \(\epsilon\) such that

\[(\lambda \Pi_{i \neq A}(1 - F_i(q)) + \sum_{k=1}^{n} \phi_k + \frac{1 - \lambda}{n})(p - \epsilon - \omega) > (\lambda \Pi_{i \neq A}(1 - F_i(q)) + \sum_{k=1}^{n} \frac{\phi_k}{(k+1)} + \frac{1 - \lambda}{n})(p - \omega)\]

which contradicts the fact that \(q\) is an equilibrium offer.

(4) Since the supremum of all suppliers’ offer, say \(\bar{p}\), will never be accepted by shoppers, if such a supremum \(\bar{p}\) is less than \(p_m\), then for any \(\omega\), \((\bar{p}, \omega)\) must be dominated by \((p_m, \omega)\) because

\[(\bar{p} - \omega)\{\lambda \Pi_{j \neq i}[1 - F_j(\bar{p})] + (1 - \lambda)\frac{1}{n}\} = (\bar{p} - \omega)(1 - \lambda)\frac{1}{n} < (p_m - \omega)(1 - \lambda)\frac{1}{n}\]

Therefore, there must be a supplier, say A, whose supremum in offers is \(p_m\) and its profit from any equilibrium offer \(q = (p, \omega)\) is

\[(p - \omega)\{\lambda \Pi_{j \neq A}[1 - F_j(p)] + (1 - \lambda)\frac{1}{n}\} = (p_m - \omega)(1 - \lambda)\frac{1}{n}\]
To show every supplier’s profit must be equal, note that if any supplier $B$’s profit is lower than supplier $A$, supplier $B$ can simply offer $(p_m, \omega)$ to make profit as high as supplier $A$. On the contrary, if any supplier $B$’s profit is higher than $A$, supplier $A$ can also offer supplier $B$’s infimum, say $p$ to receive payoffs higher than $B$’s because

$$(p - \omega)\{\lambda \Pi_{j \neq A,B}[1 - F_j(p)] + (1 - \lambda)\frac{1}{n}\} \geq (p - \omega)\{\lambda \Pi_{j \neq B}[1 - F_j(p)] + (1 - \lambda)\frac{1}{n}\}$$

(5) For any $p$ offered by both $i$ and $j$, by the third statement we know

$$\pi_i(p) - \pi_j(p) = (p - \omega)\{\lambda \Pi_{k \neq i,j}[1 - F_k(q)][F_i - F_j](q)\} = 0$$

If $F_i(p) \neq F_j(p)$, then it must be

$$\Pi_{k \neq i,j}[1 - F_k(q)] = 0$$

However, $\Pi_{k \neq i,j}[1 - F_k(q)] = 0$ implies that $p$ is strictly dominated by $p_m$ because

$$\pi_i = \pi_j = (p - \omega)\{\lambda \Pi_{j \neq i}[1 - F_j(q_i)] + (1 - \lambda)\frac{1}{n}\} = (p - \omega)\{(1 - \lambda)\frac{1}{n}\} < (p - \omega)\{(1 - \lambda)\frac{1}{n}\}$$

We conclude that for any commonly offered $p$ by any two suppliers $i, j$, $F_i(p) = F_j(p)$. □

Furthermore, the assumption $\sigma(q) = 1$ reduces the model to the environment in Varian (1980) that has solutions explored by Baye, Kovenock, and De Vries (1992) as follows.

**Lemma 2.2.** (Baye, Kovenock, and De Vries 1992) Any equilibrium with non-shopper consumer’s equilibrium acceptance rule $\sigma(q) = 1$ must be characterized by a vector of prices $(\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n)$ of which $\bar{p}_i$ is the second highest price in supplier $i$’s support. That is for each $\bar{p}_i \forall i \in I$,

$$\bar{p}_i = \sup\{\hat{p} | \lim_{p \to \hat{p}^+} \frac{d}{dp} F(p) = 0, \hat{p} < p_m\} \text{ if it exists}$$

$$p_m \quad \text{otherwise}$$

For any price $p \leq \bar{p}_i$ and any supplier $j$ with $\bar{p}_j \geq p$, the cumulative probability of supplier $i$’s offered price at $p$ is

$$F(p|\omega^*) = 1 - \frac{n - k(p) - 1}{n\Pi_{j < p, j \neq i}}\frac{1 - \lambda}{(1 - F_j(\bar{p}_j))}$$

where number $k(p) \in \{2, 3, 4, \ldots, n\}$ is the number of suppliers with $\bar{p}_i$ lower than $p$.

An immediate result and useful observation is that the above two lemma characterize a set of equilibria in the game:
Corollary 2.1. For any search cost \( c > 0 \), there is a threshold of supplier number \( N \) such that when the number of supplier \( n \) is greater than \( N > 0 \), there are a subset of equilibria, represented by natural number \( k \in \{2, 3, 4, \ldots, n\} \) where

1. Non-shopper accept whatever is offered, i.e., \( \sigma(q) = 1 \forall q \in \Omega \).

2. Suppliers will offer \( \omega \) exactly equal to the true state \( \omega^* \) and there are \( k \) suppliers whose strategy distribution can be characterized as

\[
F(p|\omega^*) = 1 - \frac{1 - \lambda}{\lambda} \left( \frac{p_m - p}{p - \omega^*} \right)^{n-k-1}
\]

and the other \( k \) suppliers offer \( p_m \)

3. Consumer’s posterior belief is that \( \forall q = (p, \omega) \),

\[
\mu(\omega'|q) = \begin{cases} 
1 & \text{if } \omega' = \omega \\
0 & \text{otherwise}
\end{cases}
\]

In this equilibrium, there are two groups of suppliers whose strategies are symmetric within each group. Furthermore, their offer policy is fully honest in offering \( \omega \). Therefore, the consumer will immediately realize the true state in the first visit. Non-shoppers will accept everything in first visit because the price distribution in the market will concentrate on \( p_m \) when \( n \) is large

\[
F(p|\omega^*) = 1 - \frac{1 - \lambda}{\lambda} \left( \frac{p_m - p}{p - \omega^*} \right)^{n-k-1} \rightarrow 0 \forall p < p_m
\] .

One thing to note is that the price distribution is independent of consumer’s search cost \( c \). The search cost only plays a rule to determine how many suppliers to sufficiently weaken suppliers to compete. The implication is that given two markets with the same parameters but different consumer search costs, the price dispersion in equilibrium will be the same when the number of suppliers is large enough. This observation leads to a further striking result is that the equilibria characterised in Lemma 2.2 are the exact set of equilibria when the number of suppliers \( n \) is large enough.

Theorem 2.1. (Uniqueness) There is a threshold of supplier number \( N^* \) such that when the number of supplier \( n \) is greater than \( N^* \), the equilibria characterized in lemma 2.2 are the all possible equilibria.

The strategy to prove the statement is to take the following steps. First, we observe that if an offer \( q = (p, \omega) \) in the first visit, made by supplier \( i \), may be rejected by a non-shopper, then it is necessary
that there is a state \( \omega^{**} \) in which non-shopper should be able to find a more favorable offer with non-trivial probability. Otherwise the non-shopper should not have rejected offer \( q \) in the first place. However, if it is true for a non-shopper, it must also hold true for shoppers. Moreover, it is critical to note that a non-shopper will never want to engage in a number of visits that incurs search costs more than the maximum willingness to pay, \( p_m \), because it is worse than accepting \( q \) in the first visit. In other words, we can be certain that, ex-ante, the willingness of a non-shopper to engage in searching depends on how likely to find a more favorable offer before the search costs accumulated is higher than \( p_m \). If we define \( k^* \) as the number of visits that has accumulated costs just above \( p_m \), then for \( q \) to be rejected, it must imply the probability in not finding a more favorable offer \((p - c) + \epsilon\) with \( \epsilon > 0 \) small enough in the next \( "k^*" \) visits is less than a non-trivial threshold \( 1 > \kappa > 0 \).

\[
\text{Prob}( \min(\hat{p}_{[2]}, \hat{p}_{[3]}, \ldots, \hat{p}_{[k^* + 1]}) > (p - c) + \epsilon | \omega^{**}) < \kappa
\] (2.1)

The expression is telling us that even though there are \( n \) suppliers in the market, the non-shopper is actually confined to the first \( k^* + 1 \) suppliers in visits because of search costs.

The second step is to observe that a shopper will be way more likely to find a more favorable offer than a non-shopper due to the advantage in the number of suppliers to visit (that is all suppliers). A rough estimate of such a probability can be established by noting the probability for non-shoppers is the (arithmetic) average probability for the minimum price of \( k^* \) suppliers drawn randomly among the rest \( n - 1 \) suppliers. Since each sample of \( k^* \) suppliers is equally random for non-shoppers, intuitively speaking the shopper is going to draw \( \frac{n-1}{k^*} \) copies by visiting all \( n - 1 \) suppliers. Therefore, the probability for a shopper to not find a more favorable offer is less than \((1 - \kappa) \frac{n-1}{k^*} \). This intuition is confirmed by exploring the probability distribution function of the minimum among \( k^* \) random variables being (geometric) average of each random variable’s distribution function. By Arithmetic-Geometric inequality and the fact that each supplier appears \( \binom{n-2}{k^*-1} \) times in the geometric average, inequality (2.1) gives the stronger result

\[
\text{Prob}( \min_{j \neq i} \hat{p}_j > p - c + \epsilon | \omega^{**}) = \Pi_{j=1,j\neq i}^{n} \left[ 1 - F_j(p - c + \epsilon) \right] < (1 - \kappa) \frac{n-1}{k^*} \]

**Lemma 2.3.** In an equilibrium with \( n \) suppliers, if non-shoppers does not accept an offer \( q = (p, \omega) \) from a supplier, that is, \( \exists i \in \{1, 2, 3, \ldots, n\} \) such that \( \sigma(q_i) < 1 \) with \( q_i = (p, \omega) \) and \( p \in (c, p_m) \), then there must be a state \( \omega^{**} \) under which the probability of the event in which the minimum price offered among other \( n - 1 \)
suppliers greater than \( p - c + \epsilon \) with \( \epsilon > 0 \) is sufficiently small. Specifically, \( \forall \epsilon > 0 , \exists \kappa > 0 \) such that

\[
\prod_{j=1, j \neq i}^{n} [1 - F_j (p - c + \epsilon)] < (1 - \kappa)^{\frac{n-1}{n}}
\]

and in particular, \( \kappa = \min_{p \in [c, p_m]} \frac{u(W - p) - u(W - p + c - \epsilon)}{u(W - c) - u(W - p + c - \epsilon)} \).

Finally, note that above probability becomes sufficiently small with \( n \) large enough as \( k^* \) is a constant and \( 1 > 1 - \kappa > 0 \). It implies that in state \( \omega^{**} \), a supplier who offers higher than \( p \) should not expect \( p \) will be accepted by shoppers because it is almost certain that some other suppliers will offer prices lower than \( p - c + \epsilon \) in the market. However, we realize \( p \) must not be in \( [c, \frac{3}{2} c] \) because if there is some supplier offering \( p - c + \epsilon \in [c, \frac{1}{2} c + \epsilon] \), the supplier will realize \( p - c + \epsilon \) is dominated by offering \( c \) as \( p - c + \epsilon \) is unlikely to be the lowest price in the market: if the revenue is mostly coming from non-shoppers and non-shoppers will accept immediately any price lower than \( c \), then why to just charge prices less than \( c \)? By continuity of supplier’s distribution in \( [0, c] \), this argument applies to all prices less than \( p - c + \epsilon \) and hence there is a contradiction that shoppers will find prices lower than \( p - c + \epsilon \). Hence, we know all offer with prices in \( [c, \frac{3}{2} c] \) must be accepted with certainty. The last step is just to invoke mathematical induction to all intervals with forms \( [\frac{r+1}{2} c, \frac{r+2}{2} c] \), \( r \in \{1, 2, 3, 4, \ldots \} \) and finish the proof.

### 2.4.2 Consumer Welfare

Based on the equilibrium characterization, we can start to look into the effect of the number of suppliers. In general, it is difficult to derive comparative statics in a model with multiple equilibria, especially one with infinitely many equilibria. However, the following result is showing us that we can find bounds for consumer’s expected utility to see the effect of more suppliers in a market. Moreover, we are going to see that the effects are exactly opposite for shoppers and non-shoppers. For non-shopper, we can find an upper bound for non-shoppers’ expected utility across all equilibria that decreases with the number of suppliers \( n \). Hence the anti-competitive effect of “market competition” is proved. In contrast, we can find that more suppliers increase shopper’s expected utility, which restores the familiar result in a market Bertrand competition.

**Theorem 2.2.** In a market with sufficient many suppliers and in which non-shoppers accept any offer in the first visit, among all possible equilibria, there is an uniform upper bound of non-shopper’s expected utility that is strictly decreasing to \( u(W - p_m) \) in the number of suppliers \( n \).

**Proof.** Since non-shoppers will accept any offer in a market with sufficiently many suppliers, given any
\( \omega^* \in \Omega \), the non-shopper's expected utility is

\[
\frac{1}{n} \sum_{j=1}^{n} u(W - p) dF_j(p)
\]

By integration by part,

\[
\frac{1}{n} \sum_{j=1}^{n} \int_{0}^{p_m} u(W - p) dF_j(p) = \frac{1}{n} \sum_{j=1}^{n} u(W - p) F_j(p) |_{0}^{p_m} + \frac{1}{n} \sum_{j=1}^{n} \int_{0}^{p_m} F_j(p) u'(W - p) dp
\]

re-arranging the expression gives us

\[
u(W - p_m) + \int_{0}^{p_m} \frac{\sum_{j=1}^{n} F_j(p)}{n} u'(W - p) dp
\]

On the other hand, any supplier \( i \)'s profit with an offer \( q = (p, \omega) \) in any equilibrium can be expressed by

\[
(p - \omega)(\lambda \Pi_{j \neq i}(1 - F_j(p)) + (1 - \lambda) \frac{1}{n}) = \frac{1 - \lambda}{n}(p_m - \omega)
\]

Re-arranging the expression will give us

\[
\Pi_{j \neq i}(1 - F_j(p)) = \frac{1 - \lambda}{n\lambda} \left( \frac{p_m - \omega}{p - \omega} - 1 \right)
\]

(2.2)

By arithmetic-geometric inequality, it is straightforward that

\[
\sum_{j=1}^{n} \frac{1 - F_j(p)}{n} \geq [\Pi_{j=1}^{n}(1 - F_j(p))]^{\frac{1}{2}}
\]

and therefore

\[
1 - [\Pi_{j=1}^{n}(1 - F_j(p))]^{\frac{1}{n}} \geq \sum_{j=1}^{n} \frac{F_j(p)}{n}
\]

By plugging equation (2.2) and observing that if \( p < p_m \) and supplier \( i \) offers, \( F_i(p) \) must be lower than 1, we have

\[
\frac{\sum_{j=1}^{n} F_j(p)}{n} \leq 1 - [\frac{1 - \lambda}{n \lambda} \left( \frac{p_m - \omega}{p - \omega} - 1 \right) (1 - F_i(p))]^{\frac{1}{2}} \to 0
\]

By Lemma (2.2), it is immediately that \( 1 - F_i(p) \) is minimized when there are only 2 suppliers not offering
$p_m$ with 100% probability. An upper bound of of a non-shopper expected utility is then

$$u(W - p_m) + \int_0^{p_m} [1 - \sqrt{\frac{1 - \lambda}{n\lambda} \left( \frac{p_m - p}{p - \omega} \right)^2}] u'(W - p) dp$$

which converges to $u(W - p_m)$ monotonely and uniformly.

For shoppers, it is instead that there are upper and lower bound of expected utility that are increasing with the number of suppliers.

**Proposition 2.1.** When the number of suppliers is sufficiently many, there are bounds for shopper’s welfare that are increasing strictly in the number of suppliers $n$ to $u(W - \omega)$ where $\omega$ is the necessary cost of service. The lowest expected utility is attained when the supplier’s strategy is symmetric. The highest expected utility is attained when there are $n - 2$ suppliers offering $p_m$.

**Proof.** Similarly to the proof in Theorem (2.2), the expected utility of a shopper is

$$\int u(W - p) dF_{\min}(p)$$

where $F_{\min}$ is the distribution of the minimum price offered among all suppliers. Specifically,

$$F_{\min}(p) = \text{Prob}(\min p_i \leq p) = 1 - \text{Prob}(\min p_i > p) = 1 - \Pi_{j=1}^n (1 - F_j(p))$$

$\Pi_{j=1}^n (1 - F_j(p))$ attains maximum when all suppliers offer $p$. Hence, shopper’s utility attains the lowest value in the symmetric equilibrium. On the other hand, to see under what condition $\Pi_{j=1}^n (1 - F_j(p))$ attains minimum, let’s decompose it as

$$\Pi_{j=1}^n (1 - F_j(p)) = \Pi_{j \neq i}^n (1 - F_j(p))(1 - F_i(p)) = \left[1 - \frac{\lambda}{n\lambda} \left( \frac{p_m - \omega}{p - \omega} - 1 \right) \right](1 - F_i(p))$$

where $i$ is the supplier that offers $p$. By Lemma (2.2), it is immediately that $1 - F_i(p)$ is minimized when there are only 2 suppliers not offering $p_m$ with 100% probability.

To summarize, we already have the upper bound and the lower bound of $F_{\min}(p)$

$$1 - \left[1 - \frac{\lambda}{n\lambda} \left( \frac{p_m - \omega}{p - \omega} - 1 \right) \right]^2 \geq F_{\min}(p) \geq 1 - \left[1 - \frac{\lambda}{n\lambda} \left( \frac{p_m - \omega}{p - \omega} - 1 \right) \right]^{\frac{n-2}{n}}$$

Both bounds increase with $n$ and converge to 1. Furthermore, note that the infimum of distribution
support in both equilibria is
\[ \omega + (p_m - \omega)/(n\lambda/(1-\lambda) + 1) \to \omega \]
Thus \( F_{\text{min}}(p) \to \delta_{\omega}. \)

Hence, we have found that the number of suppliers have the opposite effects on two types of consumers. For shoppers, it works the same way as in a conventional Bertrand competition model such that shoppers receive services with the lowest price. On the other hand, non-shoppers will in probability pay prices close to the maximum willingness to pay and get low surplus. What is interesting is that this result applies to all equilibria, which implies although competition between suppliers may vary across equilibria, such a difference does not matter for consumer’s surplus.

### 2.5 Conclusion

This study shows that there is always a threshold of supplier number over which the market will be characterized by a price distribution with high probability on monopolistic prices and elimination of unnecessary services. Since the product provided is always necessary, the market is socially efficient although suppliers take most of the surplus. Another point that worth emphasis is that suppliers’ strategies are independent of consumer’s utility function and search cost, which implies that the price distribution is robust to consumer’s characteristics and it gives the model some foundation for empirical tests. But does consumer’s utility and search cost really have no effect on suppliers at all? In fact, consumer’s utility and search cost do matter. They determine the **threshold number of suppliers** above which the paper’s main result starts to have a bite. Why the threshold is determined by consumer’s utility function and search cost can be observed by the proof of theorem 2.1. The intuition is that whether it is better to keep searching depends on consumer’s utility and search cost, and therefore once these two parameters indicate it is not worth searching, then the game among suppliers reduces to just the one described by Varian (1980) in which the solution is characterized by Baye, Kovenock, and De Vries (1992) and it is independent of consumer’s characteristics. The uniformity of price distributions across consumer’s characteristics also eases the difficulty to test the hypothesis empirically as the distribution function is known.

One may ask whether the result in this paper imply price regulations in a market is justified (e.g. meter for taxi required by law, medical insurance with specific rules in reimbursement, etc)? This is beyond this study’s ability to answer. However, this analysis shows that to encourage competition, or increase consumer welfare in a market, it is sometimes, if not always, not just to keep free entry and sit waiting for suppliers to compete. Market characteristics matter. What kind of consumers (shoppers vs non-shoppers) in concern
Proof

Proof of Lemma 2.3. For a non-shopper to reject an offer \( q \) from supplier \( i \), it must imply there is a state \( \omega^{**} \) where the expected utility from searching is no less than \( u(W - p) \) by accepting \( q \).

As the expected utility is not directly tractable, we need to take the estimate. Let’s define \( \min_k p \) as the minimum of prices offered among suppliers in a non-shopper’s first \( k \) visits and obviously it is a random variable. In addition, define \( \bar{k} \) as the natural number that satisfies the following inequality

\[
(k^* - 1)c \leq p_m \leq k^*c
\]

Then before rejecting \( q \), the non-shopper will have a posterior belief on the event that the search will halt before \( k + 1 \) visits and let’s define the posterior probability conditional \( \omega^{**} \) as \( \theta(\omega^{**}) \). Obviously, the conditional expected utility in \( \omega^{**} \) is dominated by \( E[u(W - \min_k p - c)|\omega^{**}] \) because \( \min_k p \) is the lowest price in \( k^* \) visits and the search costs accumulated in \( k^* \) visits must be greater than \( c \). On the other hand, the conditional expected utility from not halting in \( k^* \)-th visit is dominated by \( u(W - (k^* + 1)c) \). Therefore,

\[
u(W - p) \leq \theta(\omega^{**})E(u(W - \min_k p - c)|\omega^{**}) + (1 - \theta(\omega^{**}))u(W - (k + 1)c)
\]

Since

\[u(W - (k^* + 1)c) = u(W - k^*c - c) \leq u(W - p_m - c) < u(W - p)\]

we have

\[u(W - p) < E(u(W - \min_k p - c)|\omega^{**})\]

The expected value \( E(u(W - \min_k p - c)|\omega^{**}) = \int_0^{p_m} u(W - x - c) dF_{\min_k, x}(x) \) can be bounded by

\[
\int_0^{p_m} u(W - x - c) dF_{\min_k, x}(x) = \int_0^{p - \epsilon} u(W - x - c) dF_{\min_k, x}(x) + \int_{p - \epsilon}^{p_m} u(W - x - c) dF_{\min_k, x}(x)
\]

and therefore

\[
\int_0^{p_m} u(W - x - c) dF_{\min_k, x}(x) \leq [F_{\min_k, x}(p - \epsilon)]u(W - c) + [1 - F_{\min_k, x}(p - \epsilon)]u(W - p - \epsilon)
\]

where \( F_{\min_k, x} \) is the distribution function of \( \min_k p \).
Re-arranging the equation, we get

$$F_{\text{min}_k^*} (p - c + \epsilon) > \frac{u(W - p) - u(W - p - \epsilon)}{u(W - c) - u(W - p - \epsilon)}$$

By setting $\epsilon = \frac{1}{2}c$ and $u$ as a continuous function, we can define $\kappa$

$$\kappa \equiv \min_{p \in [c, p_m]} \frac{u(W - p) - u(W - p - \frac{1}{2}c)}{u(W - c) - u(W - p - \frac{1}{2}c)}$$

where $0 < \kappa < 1$. The above result implies

$$F_{\text{min}_k^*} (p - c + \frac{1}{2}c) = F_{\text{min}_k^*} (p - \frac{1}{2}c) > \kappa$$

Furthermore, because a non-shopper will never know which $k^*$ to visit ex-ante, $F_{\text{min}_k^*}$ can be expressed as

$$F_{\text{min}_k^*} (p - \frac{1}{2}c) = 1 - \frac{1}{C_{k^*}^n} \sum_{i=1}^{\lceil \frac{n}{2} \rceil} \{1 - F_j(p - \frac{1}{2}c)\} = \frac{1}{C_{k^*}^n} \sum_{i=1}^{\lceil \frac{n}{2} \rceil} \{1 - \Pi_{j=1}^{k^*} [1 - F_j(p - \frac{1}{2}c)]\}$$

By Arthemetic-Geometric inequality, we have

$$\frac{1}{C_{k^*}^n} \sum_{i=1}^{\lceil \frac{n}{2} \rceil} \{1 - \Pi_{j=1}^{k^*} [1 - F_j(p - \frac{1}{2}c)]\} \geq \frac{1}{C_{k^*}^n} \sum_{i=1}^{\lceil \frac{n}{2} \rceil} \{1 - \Pi_{j=1}^{k^*} [1 - F_j(p - \frac{1}{2}c)]\} \frac{C_{k^*}^n - 1}{C_{k^*}^n}$$

and hence

$$1 - \Pi_{j=1, j \neq i} [1 - F_j(p - \frac{1}{2}c)] \geq 1 - \frac{1}{C_{k^*}^n} \sum_{i=1}^{\lceil \frac{n}{2} \rceil} \{1 - \Pi_{j=1}^{k^*} [1 - F_j(p - \frac{1}{2}c)]\} = F_{\text{min}_k^*} (p - \frac{1}{2}c) > \kappa$$

$$\Pi_{j=1, j \neq i} [1 - F_j(p - \frac{1}{2}c)] < (1 - \kappa)^{\frac{n}{k^*}}$$

Proof of Theorem 2.1

Before we proceed, we need to extend Lemma 2.1 to an arbitrary price interval $[0, \bar{p}]$ so mathematical induction can be invoked easily.

**Lemma 2.4.** Given any equilibrium and fix a state $\omega^*$, if non-shoppers will halt the search and accept any offer with prices in an interval $[0, \bar{p}]$ with $p_m > \bar{p} > 0$, then

1. The equilibrium offer $q = (p, \omega)$ with $p \in [0, \bar{p}]$ must have $\omega = \omega^*$
2. If there is any supplier who offers \( p \in [0, \bar{p}] \), there must be another supplier who offers \( p \) in the distribution support.

3. The equilibrium distributions for each supplier on \([0, \bar{p}]\) must be continuous.

**Proof.** For the first statement, it is obvious that if non-shoppers will stop search and accept when offers are made with prices \( p \in [0, \bar{p}] \) then making offers with \( \omega > \omega^* \) is a dominated strategy because for the consumer to be a shopper, whether the offer is accepted depends only on whether \( p \) is the lowest price in the market.

For the second statement, if it does not hold, then it implies there exists an interval centered at \( p \), \((p - \Delta, p + \Delta)\), with \( \Delta > 0 \) and small enough in which no other suppliers are making offers in between. However, it implies for supplier \( i \), there exists another offer, say \( p + \frac{1}{2} \Delta \), that will dominate \( p \) in profits because

\[
\Pi_j \neq i (1 - F_j(p)|\omega) = \Pi_j \neq i (1 - F_j(p + \frac{1}{2} \Delta|\omega))
\]

and

\[
\pi(p) = (p - \omega) \left\{ \lambda \Pi_{k \neq i}(1 - F_k(p)|\omega) + (1 - \lambda) \left[ \frac{1}{n} + \sum_{k \neq i} \frac{1}{n(n-1)} \int_{c^m} (1 - \sigma(s))dF_k(s|\omega) + \ldots \right] \right\}
\]

and

\[
\pi(p + \frac{1}{2} \Delta) = (p + \frac{1}{2} \Delta - \omega) \left\{ \lambda \Pi_{k \neq i}(1 - F_k(p + \frac{1}{2} \Delta|\omega)) + (1 - \lambda) \left[ \frac{1}{n} + \sum_{k \neq i} \frac{1}{n(n-1)} \int_{c^m} (1 - \sigma(s))dF_k(s|\omega) + \ldots \right] \right\}
\]

\[
> (p + \frac{1}{2} \Delta - \omega) \left\{ \lambda \Pi_{k \neq i}(1 - F_k(p|\omega)) + (1 - \lambda) \left[ \frac{1}{n} + \sum_{k \neq i} \frac{1}{n(n-1)} \int_{c^m} (1 - \sigma(s))dF_k(s|\omega) + \ldots \right] \right\}
\]

Contradiction.

For the third statement, since we know there must be at least two suppliers offering \( p \) in the distribution supports, offering \( p \) with mass probability implies the other supplier can undercut to increase the probability of acceptance from shopper consumer. It contradicts the fact that \( p \) is equilibrium offer in the distribution support.

Now we are ready to show the proof of this paper’s key result.

**Proof of Theorem 2.1.** We prove the statement by showing that for any \( r \in N \), there is a \( N_r > 0 \) such that in a market with \( n > N_r \) suppliers, consumers must accept \( q = (p, \omega) \in [0, (\frac{r+1}{n})c] \times \Omega \) with 100% probability in every equilibrium.

When \( r = 1 \), the offer \( q = (p, \omega) \) must be accepted by consumers as the price \( p \) is in \([0, c]\).
To complete mathematical induction, we will show if the statement is true for $r \in N$, the statement must hold with $r \leftarrow r + 1$. Suppose it is not the case: no matter how large $N$ is, there is a $n > N$ such that with $n$ suppliers, there is an equilibrium in which there is some $i \in I$ and $(p, \omega) \in [(\frac{r+1}{2})c, (\frac{r+2}{2})c] \times \Omega$ such that $\sigma((i, q)) < 1$. Then from Lemma 2.3 we know for any $\epsilon > 0$ there must exist a $N$ such that $\forall n > N$

$$\prod_{j=1, j \neq i}^n[1 - F_j(p - c + \epsilon)] < (1 - \kappa)^{\frac{n-1}{2}}$$

Note that this inequality holds only for supplier $i$, the supplier who makes offers that may be rejected. Let’s decompose the product for any supplier $i' \neq i$ in the market as

$$\prod_{j=1, j \neq i}^n[1 - F_j(p - c + \epsilon)] = [1 - F_{i'}(p - c + \epsilon)] * \prod_{j=1, j \neq i, i'}^n[1 - F_j(p - c + \epsilon)]$$

Then we observe that in order to make the inequality holds, either it is

$$[1 - F_{i'}(p - c + \epsilon)] < (1 - \kappa)^{\frac{1}{2i'}}$$

or

$$\prod_{j=1, j \neq i, i'}^n[1 - F_j(p - c + \epsilon)] < (1 - \kappa)^{\frac{1}{2i'}}$$

We will prove that both cases are leading to contradictions. For following discussions, we will assume $\epsilon < \frac{1}{2c}$.

Suppose the former case is true, given the fact that $\kappa > 0$, we know for any $\epsilon_1 > 0$ there must exist a $N_1$ such that $\forall n > N_1$

$$[1 - F_{i'}(p - c + \epsilon)] < \frac{1 - \lambda}{\lambda} \epsilon_1$$

Thanks to Lemma 2.4, any supplier $j' \neq i'$ who offers $p - c + \epsilon \in [(\frac{r-1}{2})c + \epsilon, (\frac{r}{2})c + \epsilon]$ will have equilibrium profits that can be expressed without worries about tie-breaking

$$\pi(q) = (p - c + \epsilon - \omega)\{\lambda \Pi_{j \neq j'}(1 - F_j(p - c + \epsilon)) + (1 - \lambda)[\frac{1}{n} + \Sigma_{j \neq j'} \frac{1}{n(n-1)} \int_{(\frac{r-1}{2})c}^{(\frac{r}{2})c} (1 - \sigma(s))dF_j(s) + ...]\}$$

or equivalently

$$n\pi(q) = (p - c + \epsilon - \omega)\{\lambda n \Pi_{j \neq j'}(1 - F_j(p - c + \epsilon)) + (1 - \lambda)[1 + \Sigma_{j \neq j'} \frac{1}{(n-1)} \int_{(\frac{r-1}{2})c}^{(\frac{r}{2})c} (1 - \sigma(s))dF_j(s) + ...]\}$$

$$\leq (\frac{r}{2}c + \epsilon - \omega)(1 - \lambda)\{\epsilon_1 + 1 + \Sigma_{j \neq j'} \frac{1}{(n-1)} \int_{(\frac{r-1}{2})c}^{(\frac{r}{2})c} (1 - \sigma(s))dF_j(s) + ...\}$$

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since non-shoppers will accept offers with price in \([0, \frac{r+1}{2}c]\).

The supplier \(j'\) will then find offering \(q' = (\frac{r+1}{2}c, \omega)\) more profitable because

\[
n\pi(q') = (\frac{r+1}{2}c - \omega)\{\lambda n \Pi_{j \neq j'} (1 - F_j(\frac{r+1}{2}c)) + (1 - \lambda)[1 + \Sigma_{j \neq j'} \frac{1}{(n-1)} \int^{p_m}_{\frac{r+1}{2}c}(1 - \sigma(s))dF_j(s) + ...]\}
\[
\geq (\frac{r+1}{2}c - \omega)(1 - \lambda)[1 + \Sigma_{j \neq j'} \frac{1}{(n-1)} \int^{p_m}_{\frac{r+1}{2}c}(1 - \sigma(s))dF_j(s) + ...]
\]

and if \(\epsilon_1 < \frac{\frac{r}{2}c}{2c+\epsilon} < \frac{\frac{r}{2}c}{2c+\epsilon}\)

\[
n\pi(q') - n\pi(q) \geq \frac{r}{2}c(1 - \lambda)[1 + \Sigma_{j \neq j'} \frac{1}{(n-1)} \int^{p_m}_{c}(1 - \sigma(s))dF_j(s) + ...] - (\frac{r}{2}c + \epsilon - \omega)(1 - \lambda)\epsilon_1 > 0
\]

This result implies any supplier \(j' \neq i'\) will not make offers with prices that falls in \([p-c+\epsilon, \frac{r}{2}c+\epsilon]\). However, it also implies supplier \(i'\) will not make offers with prices that falls in \([p-c+\epsilon, \frac{r}{2}c+\epsilon]\) for Lemma 2.4. It contradicts the fact that there is a high probability to find the minimum price offered by suppliers lower than \(p-c+\epsilon\) because the supremum of suppliers’ price support in \([0, p-c+\epsilon]\) is dominated by \(\frac{r+1}{2}c\). We are left with the second case: Suppose the latter case is true, given the fact that \(\kappa > 0\), we know for any \(\epsilon_1 > 0\) there must exist a \(N_1^*\) such that \(\forall n > N_1^*\)

\[
n\Pi_{j=1,j \neq i'}[1 - F_j(p-c+\epsilon)] < \frac{1 - \lambda}{\lambda} \epsilon_1
\]

However, by similar arguments in the previous case, it implies immediately supplier \(i'\) will not make offers with prices that falls in \([p-c+\epsilon, \frac{r}{2}c+\epsilon]\). By repeating the argument for each supplier, we conclude no supplier will make offers with prices that falls in \([p-c+\epsilon, \frac{r}{2}c+\epsilon]\). But it contradicts the fact that there is a high probability to find the minimum price offered by suppliers lower than \(p-c+\epsilon\) because the supremum of suppliers’ price support in \([0, p-c+\epsilon]\) is dominated by \(\frac{r+1}{2}c\).

Therefore, we prove there exists a \(N_1^* = \max(N_1, N_1')\) such that for all \(n > N_1^*\), the market with \(n\) suppliers cannot have equilibrium in which the consumer (non-shopper) does not stop at \(p \in [0, (\frac{r+1}{2})c]\).

By Mathematical Induction and the finiteness of \(p_m\), there must be a \(N^*\) such that with \(n > N^*\) suppliers, the non-shopper’s acceptance rule must accept any \(q' \in (p', \omega') \in [0, p_m] \times \Omega\) in the first visit. With the fact that offering anything of price more than \(p_m\) is a strictly dominated when non-shoppers accept everything of price no greater than \(p_m\). 

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Chapter 3

Online Rating Game with Product Purchase Data

3.1 Introduction

With the recent improvements in technology of individual’s information collection and surging online markets, so-called crowd-wisdom has become topical in various fields including political science, economics, and computer science. However, in order to have reliable data from crowd-wisdom, it is crucial to design a mechanism to incentivize people to report truthfully. For information that has verifiable outcomes, contracts with contingent payoffs can serve as devices to provide incentives. But if the information of interest is purely subjective, it may be difficult to collect crowd-wisdom effectively. If we want to measure how many people in the audience are satisfied with a movie, an intuitive method may be making a poll and ask the audience to answer. But how can we guarantee the audience will provide truthful answers? This issue is pervasive in fields where the crowd’s knowledge is private and unverifiable. There are already many mechanisms proposed in the literature to address how one can elicit answers for unverifiable questions like “How satisfied are you with this product?” (Jurca and Faltings 2008; N. Lambert and Shoham 2008; Radanovic and Faltings 2013; Radanovic and Faltings 2014; Witkowski and Parkes 2012; Miller, Resnick, and Zeckhauser 2005; Baillon 2017). The basic idea underlying these mechanisms is that the mechanism designer can select a participant’s answer as a reference and check whether another participant’s answer agrees with the reference answer. However, most if not all mechanisms proposed in the literature suffer from a common issue: there are multiple equilibria and they are unidentifiable. Truth-telling is indeed a Nash-equilibrium, but there are also non-truthful ones and there is no way for the mechanism designer to detect what equilibrium is played.
For example, since most if not all mechanisms proposed in the literature only checks the correlation between answers, a truthful equilibrium in which satisfied agents report satisfied and unsatisfied agents report unsatisfied will be observationally equivalent to a totally un-truthful equilibrium in which satisfied agents report unsatisfied and unsatisfied agents report satisfied (Miller, Resnick, and Zeckhauser 2005; Kong and Schoenebeck 2016). Some studies also assume that the mechanism designer knows the common prior of information of interest (e.g. the probability distribution of perceived product’s quality) among users, which further restricts the usefulness and robustness of the mechanisms. In some works, the proposed mechanisms may induce equilibrium where the players submit reviews that depend on history or random reviews (Jurca and Faltings 2008). Even worse, the researchers have found equilibria where players submit non-truthful messages may get higher payoffs than truthful equilibrium in which players submit truthful answers (Shnayder et al. 2016). It makes not only the heuristic of equilibrium selection based on Pareto ranking invalid to select the truthful equilibrium, but also implies the possibility for players to coordinate on public signals to play non-truthful equilibria.

In this study, a mechanism is proposed to address these issues in the literature based on two insights borrowed from Bayesian persuasion and auction theory. First, any transmitted information that is decision-relevant will be incorporated in decisions in equilibrium. Even though the user’s satisfaction is not observable and verifiable, a decision-maker will correctly interpret the information in equilibrium. If the mechanism designer is able to observe decisions made after information revelation, the decisions can be a useful proxy for information transmitted in equilibrium. Therefore, the environment we consider has to be platforms such as eBay or Amazon that have both ratings on products and consumers can make a purchase after observing the ratings. Platforms can collect user’s purchase records and possess the power to publish ratings for consumers to observe. To see how non-truthful equilibrium can be eliminated, let’s take an example. Suppose the equilibrium is that reviewers always rate the product as bad when their satisfaction is good and rate the product as good when their satisfaction is bad. If it were an equilibrium, all consumers will interpret the rating correctly because it is common knowledge among users even though the platform will not be able to conclude whether the users are lying or telling the truth if no additional information is provided. However, if the platform observes more purchases when the rating is bad, then it can be inferred that reviewers are lying about their satisfactions. The platform can punish the reviewers based on the future purchase records. At the same time, the platform will reward reviewers if their reviews are aligned with the purchase record. But if we construct a mechanism based on this intuition only, we are still unable to rule out some non-truthful equilibria. If consumers always take reviewer’s reports at face value, reviewers may find it is in their favor to manipulate reports instead of reporting truthfully. The ability to manipulate consumer’s decisions distorts the reviewer’s incentives, a fact that is noted in the literature in cheap-talk game and Bayesian persuasion,
which contradicts the presumption that reviewers are reporting truthfully. In order to further rule out such
an incentive, instead of asking one reviewer at a time to elicit information, the proposed mechanism asks
two agents simultaneously and randomly publishes either one of the two reviews while keeping the other
unpublished. In this setting, the unpublished report is kept as a “reference report” to predict the outcome
after the other report is published and seen by consumers. The platform rewards the reference reviewer
if the prediction is aligned with the outcome and punish it if it is not aligned\(^1\). If the published report
transmits decision-relevant information, the reference report should make a correct prediction because all
players predict player’s reports correctly in an equilibrium. In other words, a reviewer who is satisfied should
predict increasing purchases and a reviewer who is not satisfied should predict decreasing purchases. In this
setting, the reward from a report solely comes from how well a report predicts the outcome of consumer
decisions instead of influencing it. This insight is similar to then central idea of the second-price auction
such that bidder’s incentive is only to maximize winning probability because bidders cannot influence the
winning price.

In the following, proofs are given in the appendix unless they are central to understanding the paper’s
key idea or it is straightforward.

### 3.2 Model

For simplicity, we assume a two-period model\(^2\) where a product is on a platform for purchase and the
consumer will only need the product once in a lifetime so there is no repetitive purchase. There is a public
rating system for the product on the platform. Denote \(t = 0\) as the start period and \(t = 1\) as the ending
period. People surfing the webpage are allowed to see the rating submitted by previous consumers and
chosen by the platform. Here we assume the platform is the monopoly of the product so consumers can
make a purchase nowhere but on the platform. Consumer’s satisfaction after purchase is a binary random
variable \(x \in X = \{0, 1\}\), which means a consumer is either satisfied \((x = 1)\) or not satisfied \((x = 0)\)
with the product. The probability of a consumer being satisfied is fully determined by a state variable
\(\omega\), which we will refer to as “quality”, that belongs to a state space \(\Omega = (\omega_1, \omega_2, ..., \omega_n)\) with a strictly
increasing order: \(\omega_i < \omega_{i+1} \forall i \in \{1, 2, ..., n-1\}\).\(^3\) We can think of the product like a cell phone of which
consumers receive different product samples from a manufacturer. Even though the cellphone prototype
is the same, modeled as the state of the world \(\omega\), the occurrence of malfunctions (e.g., bad signals or
dead battery) still depends on the specific sample a consumer receives. Without loss of generality, we

---

\(^1\)The reward and punishment can be monetary or future purchase discount.

\(^2\)The assumption is to avoid lengthy definitions of dynamic mechanisms without loss of generality.

\(^3\)A technical detail: the state space can also be assumed as a continuum although the underlying prior distribution will have
to be assumed absolutely continuous to apply the Bayesian rule.
assume the probability of being satisfied is increasing with states and the probability is exactly equal to $\omega$: $P(x = 1|\omega) = \omega$ and therefore $\Omega \subseteq [0, 1]$. To simplify the analysis, the quality state $\omega$ is assumed to be a constant throughout all periods. Furthermore, it is assumed that consumer satisfaction is stochastically independent conditional on the state of the world $\omega$. Formally, for arbitrary $m$ consumers and a given quality $\omega$, the probability distribution of satisfactions for consumer 1, 2, 3, ..., and $m$, $(x_1, x_2, ..., x_m)$, can be decomposed as $P(x_1, x_2, ..., x_m|\omega) = P(x_1|\omega)P(x_2|\omega)P(x_3|\omega)...P(x_m|\omega)$. There is a common prior belief on the product’s quality: $P \in \triangle \Omega$, where $\triangle \Omega$ is the set of all probability distribution on $\Omega$, that is known to consumers but may or may not be known to the platform. Note that we will denote $P(\omega)$ as the prior probability of a quality state $\omega$.

We further assume there is a continuum of consumers with outside options $c \in (0, 1)$, endowed with Lebesgue measure$^4$, arriving in the platform in every period $t$. A consumer’s utility derived from realized satisfaction $x$ is

$$
\begin{cases}
    x - c & \text{if the consumer buys the product} \\
    0 & \text{otherwise}
\end{cases}
$$

The outside option $c$ can be thought as an aspiration level for the consumer to be satisfied with the product ex ante. Since $x$ is either 1 or 0, ex post either consumer’s utility is positive ($1 - c$) or negative ($-c$). Furthermore, a consumer’s satisfaction $x$ is independent of its outside option $c$.

Consumers are expected utility maximizer and the probability of $x$’s realization will depend on available information $h$ (e.g. rating history in the platform). Whether the consumer $c$ purchases the product is denoted as

$$
\sigma_{c,t} = \begin{cases}
    1 & \text{if the product is purchased by consumer } c \text{ at time } t \\
    0 & \text{otherwise}
\end{cases} \quad \forall t \in \{0, 1\}
$$

The purchase rate $V_t$ is defined as the Lebesgue integral of $\sigma_{c,t}$ across $c$ $^5$:

$$
V_t = \int_{c \in [0, 1]} \sigma_{c,t} dc \quad \forall t \in \{0, 1\}
$$

$\sigma_{c,t}$ is a function of $c$ when the time variable $t$ is fixed. If $\sigma_{c,t}$ is integrable with respect to $c$, then $V_t$ will be a non-negative real number.

For simplicity, we will suppress subscript $t$ and denote $\sigma_c$ for consumer $c$’s purchase decision for the population in period 1 and $\sigma_{c,0}$ for the population in period 0.

$^4$Lebesgue measure is assumed to simplify the analysis. Other continuous and full-support measures can be assumed without affecting the result.

$^5$The integrability of $\sigma_{c,t}$ is implicitly assumed to avoid technical issues. Readers shall see the assumption’s validity in equilibrium soon.
3.3 Mechanism

With previous definitions, we are ready to show the procedure of the mechanism. In period \( t = 0 \), there is a continuum of consumers that arrive at the platform and make a purchase based on expected satisfaction evaluated according to the common prior belief \( P \) on the state space \( \Omega \). If a consumer makes a purchase, his own satisfaction realizes at the end of period \( t = 0 \). Then the mechanism starts as follows.

1. At the end of period \( t = 0 \), the platform records the purchase rate \( V_0 \).

2. In the beginning of period \( t = 1 \), the platform randomly chooses and sends invitations to two consumers that have made a purchase in period \( t = 0 \). Let’s call the invited reviewers A and B.\(^6\)

3. Reviewers A and B report whether or not they are satisfied in the survey individually. Denote their reports as \( \tilde{x}_A \in \{0, 1\} \) and \( \tilde{x}_B \in \{0, 1\} \).

4. The platform randomly publishes either \( \tilde{x}_A \) or \( \tilde{x}_B \) (say, 50% probability\(^7\)). Here we assume when \( \tilde{x}_i \) is published for some \( i \in \{A, B\} \), the platform announces the reviewer \( i \)’s identity with the review.

5. At the end of period \( t = 1 \), there are other consumers that arrive at the platform. Consumers \( c \in [0, 1] \) choose whether to purchase the product. The platform records the realized purchase rate \( V_1 \) in the end of period \( t = 1 \).

6. \( \forall i \in \{A, B\} \), if \( \tilde{x}_i \) is published, the reward to the reviewer \( i \) is 0 and the reward to the reviewer \( -i \in \{A, B\}\setminus\{i\} \) is

\[
\begin{align*}
V_1 - V_0 & \quad \text{if } x_{-i} = 1 \\
V_0 - V_1 & \quad \text{if } x_{-i} = 0
\end{align*}
\]

To be specific, let’s say A’s review is picked and published. Then A will not get any reward. In this case, reviewer B will receive a security contingent on realized \( V_1 \) and his own review \( x_B \). B will be rewarded the difference of purchase rate \( V_1 - V_0 \) if B’s review is “being satisfied”, \( x_B = 1 \). On the other hand, if B’s review is “not satisfied”, \( x_B = 0 \), its reward will be \( V_0 - V_1 \).

The figure below summarizes the process and the information flows. To put it simply, the reviewer is rewarded/punished if its report’s prediction is aligned/mis-aligned with the direction of the purchase rate change. In order to eliminate the incentive to influence consumer decisions, the mechanism will not reward/punish the reviewer when the reviewer’s report is chosen to be published.

\(^6\)The mechanism can extend to picking arbitrary (even random) number of consumers as long as there are at least two reviewers present. Furthermore, it is assumed that the invitations will be accepted because the equilibrium payoff from participation is no less than zero.

\(^7\)The probability of either report being reported does not change the incentive compatibility as long as the probability is non-zero. In practice, equal probabilities can better balance incentives of both reviewers.
Figure 3.1: A graphical demonstration of information flow

Remark 3.1. There is a critical reason for the mechanism to announce reviewer $i$’s identity. If consumers observe reviewer $i$’s identity, consumers can update posterior belief of $\omega$ solely based on reviewer $i$’s equilibrium strategy. If reviewers are anonymous, consumers then have to take reviewer $-i$’s strategy into consideration, which will in turn alter reviewer $-i$’s incentive to make reports and complicates the analysis. Please refer to the discussion Section 3.7.4 for details.

3.4 Equilibrium

In order to explore possible outcomes of the mechanism, let’s define the induced game. Denote the published rating history up to period $t = 0$ and $t = 1$ as $h_0 = \emptyset$ and $h_1 = \{x\}$ where $x \in \{\tilde{x}_A, \tilde{x}_B\}$ is the chosen reviewer’s report at time $t = 1$. The players in period $t = 1$ are the chosen reviewers $A, B$ and the arriving new continuum of consumers $\{c \mid c \in [0, 1]\}$. We define the pure-strategies of $A$ and $B$ as functions from his own satisfaction $x$ to report:

$$r_i : \{0, 1\} \to \{0, 1\} \forall i \in \{A, B\}$$

and at period $t \in \{0, 1\}$, consumer $c$’s purchase decision as a function from the newly published rating $x$ to $\{0, 1\}$:

$$\sigma_{c,A} : \{0, 1\} \to \{0, 1\}$$

$$\sigma_{c,B} : \{0, 1\} \to \{0, 1\}$$

The payoff function for each player can be defined as follows. For reviewer $i \in \{A, B\}$, the payoff from

---

8Rigorously speaking, $\sigma_c$ also has to be measurable from $c \in [0, 1]$ to $\{\sigma \mid \sigma : \{0, 1\} \to \{0, 1\}\}$.
reporting $r_i$, conditional on realized purchase rates $V_1$ and $V_0$ at periods $t = 0, 1$ is

$$ (V_1 - V_0)r_i + (V_0 - V_1)(1 - r_i) $$

where $V_1 = \int \sigma_c(r_i(x_i))dc$ and for consumer $c$ at period $t$ the utility function is

$$ \sigma_c[t(E_t(x) - c)] \forall c \in [0, 1], t \in \{0, 1\} $$

where the expectation operator $E_1$ is based on the posterior formed by observing published rating $\tilde{x}$, which we will call $\mu_1(\omega|\tilde{x})$, and $E_0$ is expectation operator with respect to prior belief $P$. In this study, we will assume the belief is updated by Bayes’ rule whenever consumers observe reports that are reasonable in equilibrium, which will be introduced later. When Bayes’ rule is not applicable, consumers can make any hypothesis about reviewers’ strategies. Mixed-strategies are defined as distributions on pure-strategies. Since the strategies of the reviewers and consumers are both binary in the range, it is without loss of generality to extend the expression of mixed-strategy in a similar form. Formally, we can define mixed strategies as follows:

**Definition 3.1.** The mixed-strategies of A and B are defined as functions from own satisfaction $x$ to the likelihood of reporting satisfied:

$$ r_i : \{0, 1\} \rightarrow [0, 1] \forall i \in \{A, B\} $$

and the consumer $c$’s purchase decision as a function from the newly published rating $x$ to the likelihood of purchasing the product:

$$ \sigma_{c,0} : \{0, 1\} \rightarrow [0, 1] $$

$$ \sigma_c : \{0, 1\} \rightarrow [0, 1] $$

and it is measurable with respect to $c \in [0, 1]$ with Lebesgue measure.

The equilibrium concept is Perfect Bayesian Equilibrium; henceforth, we briefly call it equilibrium.

**Definition 3.2.** A Perfect Bayesian Equilibrium is a pair of player’s strategies $\{r_A, r_B, \sigma_c\}$ with a belief system $\mu_1$ such that

1. player’s strategies are sequentially rational responses to other’s strategies according to belief system $\mu_0$ and $\mu_1$ and
(2) the belief system $\mu_1$ is consistent with player’s strategies according to Bayes’ rule. That is

$$\mu_1(\omega|\tilde{x}_i) = \frac{P(\omega \cap \tilde{x}_i)}{P(\tilde{x}_i)} = \frac{P(\tilde{x}_i|\omega)P(\omega)}{\sum_{x \in \{0, 1\}} P(\tilde{x}_i|x)P(x = x|\omega)}$$

In particular, consistency of belief system requires the posterior belief on consumer satisfaction $x_i$ to be

$$\mu_1(x_i = 1|\tilde{x}_i = 1) = \frac{r_i(1)P(x_i = 1)}{\sum_{x \in \{0, 1\}} r_i(x)P(x_i = 1)}$$

$$\mu_1(x_i = 1|\tilde{x}_i = 0) = \frac{(1 - r_i(1))P(x_i = 1)}{\sum_{x \in \{0, 1\}} (1 - r_i(x))P(x_i = x)}$$

Sequential rationality requires given $r_{-i}$ and $\sigma_c$, the player $i$ is maximizing utility by choosing the $r_i$

$$r_i(x_i) = \arg \max_{r \in [0, 1]} E[(V_1 - V_0)|x_i]r + E[(V_0 - V_1)|x_i](1 - r)$$

and given $(r_i, r_{-i}, \mu_1)$, the consumer with outside option $c$ decides whether to purchase the product to maximize expected utility

$$\sigma_c(\tilde{x}_i) = \arg \max_{\sigma \in [0, 1]} \sigma [E(x|\tilde{x}_i) - c]$$

It is clear that an equilibrium exists because the action and state space are finite. In addition, the consumer’s best response with sequential rationality is a threshold function that returns 1 (purchase) if given a rating history $h_1 = \{\tilde{x}_i\}$, the expected satisfaction is higher than $c$.

$$\sigma_c(\tilde{x}_i) = \begin{cases} 
1 & E_1(x|\tilde{x}_i) \geq c \\
0 & \text{otherwise}
\end{cases}$$

In addition, measurable maximum theorem\footnote{Theorem 18.19 in Aliprantis & Border 2006} guarantees that $\sigma_c$ is measurable with respect to $c$. This justifies the definition of purchase rates in period $t = 1$ as an integral of $\sigma_c$ across $c$ is well-defined.

Therefore, $V_0 = E_0(x)$ and

$$V_1(\tilde{x}_i) = \int_0^1 \sigma_c(\tilde{x}_i)dc = \int_0^{E_1(x|\tilde{x}_i)} dc = E(x|\tilde{x})$$

Let’s also introduce a notation for expected quality $q_t$ at time $t = 0$ and time $t = 1$ when consumers observe rating $x$ as
\[ q_0 \equiv E(\omega) = \sum \omega P(\omega) \]
\[ q_1(x) \equiv E(\omega|x) = \sum \omega P(\omega|x) \]

**Definition 3.3.** A babbling equilibrium is an equilibrium in which the probability assigned to reporting 1 is independent of reviewer’s satisfaction for both reviewers \(A, B\), that is, \(\forall i \in \{A, B\} \ r_i(1) = r_i(0)\). In addition, we call such a strategy a babbling strategy.

It is trivial to see that in a babbling equilibrium, the consumers will just disregard ratings because it provides no information about the product. On the other hand, as long as one of the reviewers provides a rating that is correlated with his own satisfaction, some information will be revealed from the ratings. This simple observation leads to the following definition:

**Definition 3.4.** An informative equilibrium is an equilibrium in which at least one of reviewer strategies is not babbling: that is \(\exists i \in \{A, B\}\) such that \(r_i(1) \neq r_i(0)\). Furthermore, a truth-telling equilibrium is an informative equilibrium where \(\forall i \in \{A, B\}\) \(r_i(1) = 1, r_i(0) = 0\).

With the above definition, the main result can be stated as follows.

**Theorem 3.1.** There exists only one informative equilibrium and it must be a truth-telling equilibrium. Furthermore, the platform can identify which equilibrium is played by observing \(V_1\). Specifically, it is informative equilibrium if and only if \(V_1 \neq V_0\).

**Remark 3.2.** Readers may find that we have been silent about off-equilibrium paths. It is not because it does not exist but because it is irrelevant in the analysis. We will see why soon in the discussion section.

**Remark 3.3.** It is important to note that this mechanism cannot rule out babbling equilibria where \(r_i(1) = r_i(0) \forall i \in \{A, B\}\). In this case, consumers will make a purchase based on prior belief and the purchase rate will not change, which further justifies the indifference of reviewers to reporting strategies. We will argue how babbling equilibria should not be sustainable in the discussion section.

Before we prove the main result, some preliminary results will be useful.

**Lemma 3.1.** If reviewer \(i\) receives \(x_i = 1\), his posterior belief about the quality will increase for quality state \(\omega > E(\omega)\) and decrease for \(\omega < E(\omega)\). On the other hand, the relation is reversed if reviewer \(i\) receives \(x_i = 0\). In particular,

\[ P(\omega|x_i = 1) = \frac{\omega}{q_0} P(\omega) \]
and
\[ P(\omega|x_i = 0) = \frac{1 - \omega}{1 - q_0} P(\omega) \]

Moreover, the cumulative probability of \( x = 1 \) and \( x = 0 \) can be ordered by first-order stochastic dominance.

\[ P(\omega \leq \omega_j | x_i = 1) < P(\omega \leq \omega_j) < P(\omega \leq \omega_j | x_i = 0) \forall j \in \{1, 2, ..., n\} \]

and therefore the expected quality increases if \( x = 1 \) is received and decreases otherwise. That is

\[
q_1(1) = E(\omega|x = 1) > E(\omega) = q_0 \\
q_1(0) = E(\omega|x = 0) < E(\omega) = q_0
\]

The above lemma tells us that a reviewer’s belief about the other reviewer’s satisfaction will differ based on his own satisfaction: when he is satisfied (\( x = 1 \)), he anticipates the other reviewer is also more likely to be satisfied\(^{10}\). It is the key to drive reviewers with different satisfactions to report differently.

**Lemma 3.2.** Given an informative equilibrium with reviewer \( i \) playing a strategy that is not babbling, it is strictly optimal for the reviewer \( -i \) to report his/her true satisfaction.

**Proof.** Suppose that reviewer \( i \)'s strategy in the equilibrium is informative, that is \( r_i(1) \neq r_i(0) \). Then, when reviewer \( i \)'s report is published, consumer’s posterior belief of quality in equilibrium is

\[
P(\omega|\tilde{x}_i = 1) = \frac{P(\omega \cap \tilde{x}_i = 1)}{P(\tilde{x}_i = 1)} = \frac{\omega r_i(1) + (1 - \omega)r_i(0)}{q_0 r_i(1) + (1 - q_0)r_i(0)} P(\omega)
\]

\[
P(\omega|\tilde{x}_i = 0) = \frac{P(\omega \cap \tilde{x}_i = 0)}{P(\tilde{x}_i = 0)} = \frac{1 - [\omega r_i(1) + (1 - \omega)r_i(0)]}{1 - [q_0 r_i(1) + (1 - q_0)r_i(0)]} P(\omega)
\]

After reviewer \( -i \) receives \( x_{-i} = 1 \), that is reviewer \( -i \) is satisfied, its posterior expectation of \( \omega, q_1 \), will be higher than prior expectation \( q_0 \). Hence, from reviewer \( -i \)'s point of view, the probability of reviewer \( i \) sending \( x = 1 \) can be derived as (by Law of total probability)

\[
P(x_i = 1|x_{-i} = 1)P(\tilde{x}_i = 1|x_i = 1) + P(x_i = 0|x_{-i} = 1)P(\tilde{x}_i = 1|x_i = 0) = q_1 r_i(1) + (1 - q_1)r_i(0)
\]

\(^{10}\)Note that satisfactions between reviewers are independent conditional on a quality state.
and the probability of reviewer \(i\) sending \(x = 0\) is

\[
1 - [P(\tilde{x}_i = 1|x_i = 1)P(x_i = 1|x_{-i} = 1) + P(\tilde{x}_i = 1|x_i = 0)P(x_i = 0|x_{-i} = 1)] \\
= 1 - [q_1r_i(1) + (1 - q_1)r_i(0)]
\]

Given that the consumer’s equilibrium strategy must be sequentially rational conditional on reviewers’ strategies, the expected difference of purchase rate if \(x_i = 1\) is reported by reviewer \(i\) is

\[
E(V_1 - V_0|\tilde{x}_i = 1) = \sum_{\omega \in \Omega} \omega[P(\omega|\tilde{x}_i = 1) - P(\omega)] \\
= \sum_{\omega \in \Omega} \omega \frac{(\omega - q_0)r_i(1) - r_i(0)}{q_0r_i(1) + (1-q_0)r_i(0)} P(\omega) \\
= \sum_{\omega \in \Omega} \omega (\omega - q_0) P(\omega) \frac{r_i(1)-r_i(0)}{q_0r_i(1) + (1-q_0)r_i(0)} Var(\omega)
\]

where \(Var(\omega)\) is the variance of state distribution under prior belief. Similarly, the expected difference of purchase rate if \(x_i = 0\) is reported by reviewer \(i\) is

\[
E(V_1 - V_0|\tilde{x}_i = 0) = \sum_{\omega \in \Omega} \omega[P(\omega|\tilde{x}_i = 0) - P(\omega)] \\
= -\sum_{\omega \in \Omega} \omega \frac{(\omega - q_0)r_i(1) - r_i(0)}{q_0r_i(1) + (1-q_0)r_i(0)} P(\omega) \\
= -\sum_{\omega \in \Omega} \omega (\omega - q_0) P(\omega) \frac{r_i(1)-r_i(0)}{q_0r_i(1) + (1-q_0)r_i(0)} Var(\omega)
\]

From here, we can separate the proof into two cases. One is \(r_i(1) > r_i(0)\) (reviewer \(i\)’s report is positively correlated with its satisfaction) and the other is \(r_i(1) < r_i(0)\) (reviewer \(i\)’s report is negatively correlated with its satisfaction).

**Case \(r_i(1) > r_i(0)\):**

This implies that \(q_1(1)r_i(1) + (1 - q_1(1))r_i(0) > q_0r_i(1) + (1-q_0)r_i(0)\) and also \(E(V_1 - V_0|\tilde{x}_i = 1) > 0\) and \(E(V_1 - V_0|\tilde{x}_i = 0) < 0\). For reviewer \(-i\), the expected utility from sending \(x = 1\) is

\[
E(V_1 - V_0|\tilde{x}_i = 1)P(\tilde{x}_i = 1|x_{-i} = 1) + E(V_1 - V_0|\tilde{x}_i = 0)P(\tilde{x}_i = 0|x_{-i} = 1) \\
= \left\{ \frac{[r_i(1)-r_i(0)]}{q_0r_i(1) + (1-q_0)r_i(0)} \right\} q_1(1) + (1 - q_1(1))r_i(0) \\
+ \left\{ \frac{[r_i(1)-r_i(0)]}{q_0r_i(1) + (1-q_0)r_i(0)} \right\} (1 - q_1(1)r_i(1) + (1 - q_1(1))r_i(0)) \right\} Var(\omega) > 0
\]

On the other hand, the expected utility from sending \(x = 0\) is

\[
E(V_0 - V_1|\tilde{x}_i = 1)P(\tilde{x}_i = 1|x_{-i} = 1) + E(V_0 - V_1|\tilde{x}_i = 0)P(\tilde{x}_i = 0|x_{-i} = 1) \\
= -\left\{ E(V_1 - V_0|\tilde{x}_i = 1)P(\tilde{x}_i = 1|x_{-i} = 1) + E(V_1 - V_0|\tilde{x}_i = 0)P(\tilde{x}_i = 0|x_{-i} = 1) \right\} < 0
\]

Therefore, the utility from sending \(x = 1\) is greater than sending \(x = 0\), which implies \(r_{-i}(1) = 1\).

**Case \(r_i(1) < r_i(0)\):**

This case implies \(q_1(1)r_i(1) + (1 - q_1(1))r_i(0) < q_0r_i(1) + (1-q_0)r_i(0)\) and also \(E(V_1 - V_0|\tilde{x}_i = 1) < 0\)
and $E(V_1 - V_0 | \tilde{x}_i = 0) > 0$. Similarly to the previous analysis, for reviewer $-i$, the expected utility from sending $x = 1$ is

$$E(V_1 - V_0 | \tilde{x}_i = 1) P(\tilde{x}_i = 1 | x_{-i} = 1) + E(V_1 - V_0 | \tilde{x}_i = 0) P(\tilde{x}_i = 0 | x_{-i} = 1)$$

$$= \left\{ \left[ \frac{r_i(1) - r_i(0)}{q_0 r_i(1) + (1 - q_0) r_i(0)} \right] [q_1(1) r_i(1) + (1 - q_1(1)) r_i(0)] \right.$$  

$$+ \left( - \frac{r_i(1) - r_i(0)}{1 - q_0 r_i(1) + (1 - q_0) r_i(0)} \right) (1 - [q_1(1) r_i(1) + (1 - q_1(1)) r_i(0)]) \left\} \text{Var}(\omega) > 0 \right.$$  

and the expected utility from sending $x = 0$ is

$$E(V_0 - V_1 | \tilde{x}_i = 1) P(\tilde{x}_i = 1 | x_{-i} = 1) + E(V_0 - V_1 | \tilde{x}_i = 0) P(\tilde{x}_i = 0 | x_{-i} = 1) < 0$$

Again, we conclude $r_{-i}(1) = 1$ if reviewer $-i$ receives $x_{-i} = 1$ regardless of whether reviewer $i$’s strategy is positively or negatively correlated with true satisfaction. By symmetry, we have proved that $r_{-i}(1) = 1 > 0 = r_{-i}(0)$.

The result of theorem 3.1 is an immediate consequence of lemma 3.2.

**Proof of Theorem 3.1.** By lemma 3.2, reviewers will always report $\tilde{x} = 1$ when $x = 1$ and $\tilde{x} = 0$ when $x = 0$ in an informative equilibrium. Hence there is only truthful equilibrium when informative equilibrium is assumed in play. Furthermore, in truthful equilibrium, $V_1 > V_0$ if $\tilde{x} = 1$ is published and $V_1 < V_0$ if $\tilde{x} = 0$ is published while in a babbling equilibrium $V_1 = V_0$ no matter what $\tilde{x}$ is published. Therefore, the platform can identify what equilibrium is played by observing $V_1$.

**3.5 Multiple Reviewers**

The above mechanism shows the possibility to induce truthful revelation if the platform invites two reviewers from previous consumers. But can we do better to elicit more than two consumers’ satisfaction reports?

One straightforward answer is that the platform can invite multiple reviewers to report satisfactions, say 100 reviewers. The platform then randomly chooses two reviewers to enter the proposed mechanism without letting reviewers know who is chosen into the mechanism. Since reviewers don’t know whether they will be chosen in the mechanism, it is still incentive compatible for them to report truthfully. However, this approach still publishes only one report at a time on the platform. Although this straightforward mechanism induces truthful reports from reviewers, it is obviously far from optimal in terms of information efficiency because there are remaining 99 reports not taken into account when consumers deciding whether to make a purchase. The question we want to address here is: is it possible to utilize almost all the information
revealed by the 100 reports? The section is going to show the answer is affirmative and in fact for N invited reviewers, we can publish $N-1$ reports to consumers without sacrificing incentive compatibility.

**Mechanism**

The proposed mechanism runs as almost exactly like the one with two reviewers. In period $t = 0$, there is a continuum of consumers that arrive at the platform and make a purchase based on expected satisfaction evaluated according to the common prior belief $\mu_0$ on the state space $\Omega$. If a consumer makes a purchase, his own satisfaction realizes at the end of period $t = 0$. The platform sends invitations to reviewers (consumers who have made a purchase) to report their ratings and the platform decides what ratings to publish.

1. At the end of period $t = 0$, the platform records the purchase rate $V_0$.

2. In the beginning of period $t = 1$, the platform randomly chooses and sends invitations to $N$ consumers that have made a purchase in period $t = 0$. Let’s call the invited reviewers $i \in I = \{1, 2, 3, ..., N\}$.

3. Every reviewer reports whether or not they are satisfied in the survey individually. Denote their reports as $\tilde{x}_i \in \{0, 1\}$ for any $i \in I$.

4. The platform randomly chooses $N-1$ reports sent by reviewers to publish. Here we assume when a report $\tilde{x}_i$ is published, the platform announces the reviewer $i$’s identity altogether. Let’s denote the set of chosen reviewers identity index as $I^*$.

5. At the end of period $t = 1$, there is another group of consumers that arrives at the platform. Consumers $c \in [0, 1]$ choose whether to purchase the product. The platform records the realized purchase rate $V_1$ at the end of period $t = 1$.

6. $\forall i \in I^*$ ($\tilde{x}_i$ is published), the reward to the reviewer $i$ is 0 and the reward to the reviewer $-i \in I/I^*$ is

$$
\begin{cases}
V_1 - V_0 & \text{if } \tilde{x}_{-i} = 1 \\
V_0 - V_1 & \text{if } \tilde{x}_{-i} = 0
\end{cases}
$$

To be specific, let’s say $i$’s review is picked and published. Then $i$ will not get any reward. In this case, reviewer $-i$ will receive a security contingent on realized $V_1$ and his own review $x_{-i}$. $-i$ will be rewarded the difference of purchase rate $V_1 - V_0$ if $-i$’s review is “being satisfied”, $\tilde{x}_{-i} = 1$. On the other hand, if $-i$’s review is “not satisfied”, $\tilde{x}_{-i} = 0$, its reward will be $V_0 - V_1$.

In the mechanism described above, we can derive truth-telling as the unique informative equilibrium as previous two-reviewer mechanism.
Theorem 3.2. There is a unique informative equilibrium in the mechanism, in which every reviewer must report truthfully and consumers will take the reports as the face value.

Similarly to two-reviewer mechanism, it is important to note babbling equilibria still exist in the mechanism.

3.6 Non-Binary Opinions

In many scenarios, consumer’s opinion can be non-binary. For example, we can see on Amazon, there are multiple rating choices ranging from 1-star to 5-stars. A reviewer who thinks the product is mediocre can rate it as 3-star instead of necessarily choosing good or bad. At first glance, it may seem trivial to extend the binary model to a non-binary environment. However, note that previously, the binary assumption has given us an advantage to impose first-order stochastic order on posterior distributions and significantly restrict the complexity of type space. Although we can assume that the higher quality state induces a higher distribution in first-order stochastic dominance to restrict our model, this is not what I will do here. I would like to take a more general approach aspired by Bayesian persuasion literature (Gentzkow and Kamenica 2016). Here, we still assume that there is a state space \( \Omega = \{\omega_1, \omega_2, \ldots, \omega_n\} \) but the signal space (the possible satisfaction levels) is a non-binary, finite set \( X = \{x_1, x_2, \ldots x_m\} \subset [0,1] \) where \( x_1 < x_2 < \ldots < x_m \). Similarly, conditional on any state \( \omega \in \Omega \), reviewer’s signals are independent. There is a system of the conditional probability of signals \( P(x|\omega) \) for all \( x \in X \) and \( \omega \in \Omega \) with \( \mu \), a prior distribution of states.

Although the mechanism can be designed in various forms, eventually any message sent to consumers will induce a posterior belief (perceived by consumers) about product’s quality, and the posterior is inferred from the equilibrium strategies and beliefs. Every posterior belief will further induce a purchase rate in equilibrium. Hence, from a reviewer’s perspective, he is facing three sources of randomness:

1. The other reviewer’s satisfaction (type) is random,

2. the other reviewer may play a mixed-strategy in sending messages conditional on own satisfaction, and

3. every message sent to consumers will induce a purchase rate.

However, the third source does not really involve randomness in equilibrium because the consumer’s decision is deterministic and binary to a given posterior. In addition, the second source of randomness can be eliminated by carefully choosing mechanisms. The idea is borrowed from the insights in Karni (2009) and Chambers and N. S. Lambert (2017). Fixing a mechanism (regardless of what message space is defined as), an equilibrium of supplier \( i \)'s strategy (the second source of randomness), and consumer’s purchase decisions
(the third randomness), if reviewer \(-i\) is asked to predict the probability of each possible outcome, there exists a mechanism in which the reviewer \(-i\) will find that it is strictly dominant to report his true belief by first-order stochastic dominance argument. The mechanism can be implemented by randomly generating a cash-settled call option on purchase rate with a strike \(V \in [0, 1]\) and price \(p_c \in [0, 1]\) according to a full-support distribution\(^{11}\). A cash-settled call option, specified by a strike price \(V\) and an underlying price \(\tilde{V}\) that will realize in future, is a contract between two parties, which will be called buyer and seller, under which the buyer can choose whether to request the seller to pay the difference between underlying and strike, that is \(\tilde{V} - V\), when \(\tilde{V}\) realizes. Note that the buyer does not have to exercise such a right (that's why it is called option) therefore the buyer's gross profit from such an option is the higher of \(\tilde{V} - V\) and 0, i.e. \(\max(\tilde{V} - V, 0)\). If the price of such an option is \(p_c\), buyer’s net profit is \(\max(\tilde{V} - V, 0) - p_c\). The seller, on the other hand, collects \(p_c\) from the buyer and makes cash transfer \(\tilde{V} - V\) to the buyer if requested. Hence seller’s net profit is \(-\max(\tilde{V} - V, 0) + p_c\).

After the option is generated and reviewers report their distributions, the mechanism chooses whether to purchase the option \((V, p_c)\) on behalf of the reviewer assuming the reported distribution is the reviewer’s true perceived distribution of purchase rates. Note that reviewers cannot know in advance which option is generated in order to sustain incentive-compatibility. The key driver for reviewer \(-i\) to “reveal truthfully”\(^{12}\) is very similar to the binary setting. Given reviewer \(i\)’s report strategy, since reviewer \(-i\) has no control over purchase rates when reviewer \(i\)’s report is published, in order to incentivize reviewer \(-i\) to report differently across satisfactions, different satisfactions of reviewer \(-i\) have to lead reviewer \(-i\) to perceive different distributions of reviewer \(i\)’s satisfaction, just like in the binary model: satisfied reviewers perceive a distribution of reviewer \(i\)’s satisfaction that stochastically dominates the prior distribution. By symmetry, we can also apply the same argument to reviewer \(i\). This idea leads to the following mechanism.

3.6.1 Mechanism for Non-binary Satisfaction

In order to simplify notations and analysis, we assume that it is still a two-period model, and consumers in period \(t = 0\) have already made purchases so purchase rate \(V_0\) has realized to the platform. Hence the mechanism starts with the following steps at the beginning of period \(t = 1\):

1. The platform invites two reviewers \(A\) and \(B\) who have made a purchase in period \(t = 0\).

2. Reviewers \(A, B\) individually report a distribution of purchase rate \(V\), denoted as \(\tilde{F}_i \in \Delta([0, 1]) \forall i \in \{A, B\}\). For simplicity, it can be an uniform distribution with range \([0, 1]\)\(^2\). But the incentive compatibility is not affected by other distributions as long as it is full-support. For details of the mechanism implementation, see Chambers and N. S. Lambert (2017).

\(^{12}\)Note that the message space is no longer reviewer’s satisfaction so revealing truthfully is not telling reviewer’s own satisfaction.
\{A, B\}. \Delta([0, 1]) is the set of all probability distribution on \([0, 1]\).

3. The platform randomly chooses one of the reviewer’s reports to publish with equal probability. Denote the chosen reviewer as \(i \in \{A, B\}\) and the other \(-i \in \{A, B\}\).\{/i\}.

4. In the meantime, for reviewer \(-i\), the platform will randomly generate a call option with a strike \(\bar{V}\) and price \(p_c\) in the product space of \([0, 1] \times [0, 1]\). The settlement price of the call option is \(V_1\), the realized purchase rate in period 1. The platform will buy or sell the call option on behalf of reviewer \(-i\) depending on the expected value of the call option conditional on the reported distribution \(\tilde{F}_{-i}(V_1)\), which equals to

\[
E(\max(V_1 - \bar{V}, 0) - p_c|\tilde{F}_{-i}) = \int_{V_1 \in [0, 1]} \max(V_1 - \bar{V}, 0)d\tilde{F}_{-i}(V_1) - p_c \tag{3.1}
\]

If the reported distribution \(\tilde{F}_{-i}\) implies the call option’s expected value (3.1) is positive, the mechanism buys the call option on behalf of reviewer \(i \in \{A, B\}\) and pays price \(p_c\). Otherwise, the mechanism sells it on behalf of reviewer \(i \in \{A, B\}\) and receives price \(p_c\).

5. Observing the report published on the platform, consumers make decisions on purchase. The platform records the realized purchase rate \(V_1\) in the end of period 1.

6. Reviewer \(i\) receives nothing. On the other hand, reviewer \(-i\)’s payoff depends on the following. After \(V_1\) realizes, the option generated in step 4 is exercised if the realized purchase rate is higher than the option strike, that is \(V_1 > \bar{V}\). Otherwise the option is left not exercised. Reviewer \(-i\)’s payoff is determined by the option \((\bar{V}, p_c)\) settlement:

\[
\begin{cases} 
\max(V_1 - \bar{V}, 0) - p_c \quad \text{if the option is bought} \\
-(\max(V_1 - \bar{V}, 0) - p_c) \quad \text{if the option is sold}
\end{cases}
\]

3.6.2 Equilibrium for Non-binary Satisfactions

Denote the published rating history up to period \(t = 0\) and \(t = 1\) as \(h_0 = \emptyset\) and \(h_1 = \{F\}\) where \(F \in \{\tilde{F}_A, \tilde{F}_B\}\) is the published reviewer’s "report" at time \(t = 1\). Players in period \(t = 1\) are the chosen reviewers \(A, B\) and the arriving new continuum of consumers \(\{c|c \in [0, 1]\}\). We define pure-strategies of a reviewer \(i \in \{A, B\}\) as functions from own satisfaction \(x\) to report:

\[r_i : X \rightarrow \Delta([0, 1])\]

\(^{13}\)The decision problem has a solution because of the compactness of \(\bar{V}\) and \(p_c\).
and the consumer $c$’s purchase decision as a function from the newly published rating $F$ to $\{0, 1\}$:

$$\sigma_c : h \rightarrow \{0, 1\}$$

The payoff function for each player can be defined as follows. The reviewer whose report is published receives 0 utility, whereas for the reviewer whose report is not published, its utility is

$$\begin{cases} 
\max(V_1 - \bar{V}, 0) - p \quad &\text{if the option is bought} \\
-(\max(V_1 - \bar{V}, 0) - p) \quad &\text{if the option is sold}
\end{cases}$$

where $V_1 = \int \sigma_c(F) dc$ and the utility of a consumer with outside option $c$ is

$$\sigma_c[E(x|F) - c] \forall c \in [0, 1]$$

where the expectation $E(x|F)$ is conditional expectation based on the posterior belief formed after observing published report $F \in \{\tilde{F}_A, \tilde{F}_B\}$.

As was the case in the two-period model, the equilibrium concept is Perfect Bayesian Equilibrium.

**Definition 3.5.** A Perfect Bayesian Equilibrium is a pair of player’s strategies $\{r_A, r_B, \sigma_c\}$ with a belief system $\mu_1$ such that player’s strategies are sequentially rational responses to other strategies according to belief system $\mu_1$ and the belief system is consistent with player’s strategies according to Bayes’ rule. That is

$$\mu_1(\omega|\tilde{F}_i) = \frac{P(\omega \cap \tilde{F}_i)}{P(\tilde{F}_i)} = \frac{\sum_{x \in X} P(\tilde{F}_i|x) P(x|\omega) P(\omega)}{\sum_{\omega \in \Omega, x \in X} P(\tilde{F}_i|x) P(x|\omega) P(\omega)}$$

where $P(\tilde{F}_i|x)$ denotes the probability that $\tilde{F}_i$ is reported by reviewer $i$ if reviewer $i$ receives satisfaction $x$. In particular,

$$P(\tilde{F}_i|x) = \begin{cases} 
1 &\text{if } r_i(x) = \tilde{F}_i \\
0 &\text{otherwise}
\end{cases}$$

Sequential rationality of reviewers requires given $r_{-i}$ and $\sigma_c$, the player $i$ is maximizing utility by choosing the $r_i$

$$r_i(x_i) \in \arg \max_{F \in \Delta([0, 1])} \int_{\bar{V} \times p_c \in [0, 1]^2} s(F, \bar{V}, p_c) \ast E[\max(V - \bar{V}, 0) - p_c | x_i] d(\bar{V} \times p_c) \quad (3.2)$$

where $s(F, \bar{V}, p_c)$ indicates whether the platform buys or sells the randomly generated option on the behalf.
of reviewer $-i$ based on the reported purchase distribution $F$ and option’s parameters $\bar{V}, p_c$, specifically

$$s(F, \bar{V}, p_c) = \begin{cases} 1 & \int_{V \in [0,1]} \max(V - \bar{V}, 0) - p_c \cdot dF(V) > 0 \\ -1 & \text{otherwise} \end{cases}$$

and the expectation $E[\max(V - \bar{V}, 0) - p_c|x_i]$ is the expected value of call option with strike $\bar{V}$ and price $p_c$ conditional on reviewer $i$’s satisfaction $x_i$ with respect to random purchase rate $V$ that is realizing in period $t = 1$.

Furthermore, sequential rationality of consumers requires that given $(r_i, r_{-i}, \mu_1)$, the consumer with outside option $c$ decides whether to purchase the product

$$\sigma_c(\tilde{F}) = \arg\max_{\sigma \in \{0, 1\}} \sigma[E(x|\tilde{F}) - c]$$

Note that an equilibrium exists because the action and state spaces are both compact: the state space is finite and reviewer’s action space is compact in the weak topology. Also, it is without loss of generality to focus on pure-strategy equilibria because (1) the consumer’s best response is a threshold function that returns 1 (purchase) if given rating history, the expected satisfaction is higher than $c$

$$\sigma_c(\tilde{F}) = \begin{cases} 1 & E(x | \tilde{F}) \geq c \\ 0 & \text{otherwise} \end{cases}$$

. Indifference between purchase and no purchase will occur only for the consumer whose outside option is equal to the expected satisfaction: $c = E(x|\tilde{F})$. Since the probability measure for such a consumer is zero, it does not affect reviewers as the purchase rate is determined by the integral of consumer’s purchases, and (2) a reviewer’s best response is always a single-valued function since no matter what distribution of $V_1$ reviewers perceive in equilibrium conditional on its own satisfaction $x$, reviewer’s optimization problem has a unique solution. Claim (2) can be proved as follows.

**Lemma 3.3.** In any equilibrium, for reviewer $i \in \{A, B\}$, there exists a unique distribution of realized purchase rate $V_1$ consistent with reviewer $i$’s satisfaction.

**Lemma 3.4.** The optimization problem in (3.2) has a unique solution and the solution must be reviewer’s belief of realized purchase rate $V_1$ given the other reviewer’ strategies and own satisfaction.

Similarly to the case of binary reports, a strategy is babbling when reviewers adopt the same strategy regardless of their own satisfactions.
Definition 3.6. A babbling equilibrium is an equilibrium in which there exists a pair of distributions $F_A, F_B \in \Delta([0, 1])$ such that reviewers $A$ and $B$ will report $F_A$ and $F_B$ respectively regardless of $A$ and $B$’s satisfactions, that is, $\forall i \in \{A, B\} \ r_i(x) = r_i(x') = F_i \ \forall x, x' \in X$. We call such a strategy a babbling strategy.

In addition, denote a purchase rate $V_x$ as the purchase rate if consumers know the satisfaction of the reviewer whose report is published is $x$, which is equal to

$$V_x = \Sigma_{x' \in X} P(x' | x) = \Sigma_{x' \in X} \frac{P(x' \cap x)}{P(x)} = \Sigma_{x' \in X} \frac{\Sigma_{\omega \in \Omega} P(x' | \omega) P(\omega | x)}{\Sigma_{\omega \in \Omega} P(\omega | x) P(\omega)}$$

Let’s denote the cumulative distribution that concentrates mass probability on a single purchase rate $V$ as

$$\delta_V(v) = \begin{cases} 1 & v \geq V \\ 0 & v < V \end{cases}$$

Definition 3.7. An informative equilibrium is an equilibrium where at least one of the reviewer’s strategy is not babbling: that is $\exists i \in \{A, B\}$ and $\exists x, x' \in X$ such that $r_i(x) \neq r_i(x')$. Furthermore, a truth-telling equilibrium is an informative equilibrium where $\forall i \in \{A, B\}$ and $\forall x, x' \in X$, reviewer $i$ whose satisfaction is $x$ reports a distribution that allocates probability $P(x' | x) = \Sigma_{\omega \in \Omega} P(x' | \omega) P(\omega | x)$ on $V_x$, that is $r_i(x) = \Sigma_{x' \in X} P(x' | x) \delta_{V_x}$.\(^{14}\)

In order to assure truth-telling, we need a particular condition that rules out reviewer’s conditional beliefs to be too “similar”.

Definition 3.8. Reviewer’s posterior belief of the other reviewer’s satisfaction is combinatorially different if conditional on own satisfaction $x$, the posterior belief of the event that other reviewer’s satisfaction belongs to any strict subset $Y$ in $X$, $Y \subset X$, differs for all satisfaction $x \in X$. That is

$$P(Y | x) \neq P(Y | x') \ \forall x, x' \in X, \ \forall Y \subset X$$

Moreover, we assume the environment is non-trivial as there exist at least two satisfactions $x, x' \in X$ such that $V_x \neq V_{x'}$, which means perfectly learning reviewer’s satisfaction leads to different purchase decisions for some consumers. Otherwise, it must be $V_x = V_0 \ \forall x \in X$ and there is no need for consumers to learn from reviewers. With the above two conditions, we can show that there exists a unique informative equilibrium

\(^{14}\)Note that there can be more than two $x', x'' \in X$ such that $V_{x'} = V_{x''}$. In this case, the probability allocated to $V_{x'} = V_{x''}$ will be the $\Sigma_{x'''}: V_{x'''} = V_{x'}, P(x'' | x)$. The express $r_i(x) = \Sigma_{x' \in X} P(x' | x) \delta_{V_{x'}}$ in the definition will derive the same result because $\delta_{V_{x'}} = \delta_{V_{x''}}$. 

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Theorem 3.3. If reviewer’s posterior belief of the other reviewer’s satisfaction is combinatorially different, then in the game induced by the proposed mechanism, the equilibrium is either a babbling equilibrium where reviewers report distribution \( \delta_{V_0}(v) \) and consumers make a purchase ignoring reports or a truth-telling equilibrium in which reviewers of satisfaction \( x \) report \( \sum_{x' \in X} P(x' | x) \delta_{V_{x'}} = \sum_{x' \in X} \sum_{\omega \in \Omega} P(x' | \omega) P(\omega | x) \delta_{V_{x'}} \) and consumers make a purchase (or not) if \( c \leq V_x \) (\( c > V_x \)).

Remark 3.4. Note that the proposed mechanism’s message space for reviewers is not the type space, i.e. \( X \), because reviewers do not report their satisfactions. They report their belief about other’s satisfactions! The reason for constructing an indirect mechanism is that the mechanism designer does not know the player’s common knowledge, including the prior belief of \( \omega \) and consumer’s hypothesis about reviewer’s strategies. If we construct a direct mechanism to implement the social choice function of this indirect mechanism, the mechanism designer will never know how to pay reviewers without knowing the prior belief in order to implement the social choice function.

Remark 3.5. In addition, note that revelation principle does not help much here for this study’s purpose: find a mechanism that reduces possible equilibria to be only babbling ones and truthful one. While revelation principle asserts there is a direct mechanism to implement the social choice function underlying the mechanism, it does not necessarily mean the direct mechanism does not induce a game with other equilibria.

### 3.7 Discussion

#### 3.7.1 Irrelevance of Off-equilibrium Paths

Off equilibrium paths are crucial to every model defined within perfect Bayesian equilibrium. However, throughout the paper, we have not been very clear about what the off-equilibrium paths would be and how consumers’ posterior belief is updated if it is taken. Here we would like to argue that it is not relevant in our mechanism, especially not relevant to key results. To see why it is the case, let’s recall how the mechanism is designed to eliminate the reviewer’s incentive to mis-report own satisfaction. As emphasized in the introduction, the proposed mechanisms isolate prediction and information revelation by randomly choosing a reviewer’s report to publish in platform (revelation) and taking the other to match the outcome of consumer purchase (prediction.) When one’s report is to match the outcome, it is hidden from consumers. On the other hand, if one’s report is published in platform, the reviewer is rewarded nothing. Therefore, a reviewer’s incentive to choose what to report will not take off-equilibrium path into account as the equilibrium definition requires its optimal decision to be made assuming other players in the game
are playing according to equilibrium strategies. Even if a reviewer cannot deny the other reviewer can take off-equilibrium path, by equilibrium definition, it is probability zero. Hence in mechanisms of this study, what can happen on off-equilibrium paths will never enter a reviewer’s objective so it is irrelevant to the key results. This point holds true regardless whether it is the model under binary or non-binary satisfaction environment.

The following discussion will be based on the binary model for its simplicity.

3.7.2 Equilibrium Selection

One thing to note is that the mechanism in this study, although truth-telling equilibrium is the only informative equilibrium, there is a continuum of babbling equilibria.

**Corollary 3.1.** There is a continuum of babbling equilibrium in the game induced by the mechanism, characterized by reviewers playing babbling strategies $r_i(1) = r_{-i}(0) \in [0, 1] \forall i \in \{A, B\}$.

**Proof.** When reviewers play babbling strategies, the consumer’s posterior belief will be equal to the prior belief in equilibrium because

$$P(\omega|x_i = 1) = \frac{P(\omega \cap x_i = 1)}{P(x_i = 1)} = \frac{\omega r_i(1) + (1 - \omega) r_{-i}(0)}{q_0 r_i(1) + (1 - q_0) r_{-i}(0)} P(\omega) = P(\omega)$$

The equality holds as long as $r_i(1) = r_i(0) \in [0, 1] \forall i \in \{A, B\}$. Since the consumer’s posterior belief is equal to prior belief, the purchase rate at period $t = 1$ will be equal to the purchase rate at $t = 0$, i.e. $V_1 = V_0$. Therefore, reviewers are indifferent between reporting truthfully or playing babbling strategies. 

Note that babbling equilibria will hold even if reviewer $i$ reports $x = 1$ with probability different from reviewer $-i$, that is $r_i(1) = r_i(0) \neq r_{-i}(1) = r_{-i}(0)$. With such uncertainty of equilibria, it may seem not so convincing that informative equilibrium necessarily occurs in realistic applications. However, from a reviewer’s perspective, a necessary condition to support a babbling equilibrium is that both reviewers will submit ratings with a probability that is independent of his satisfaction and it is common knowledge between reviewers and consumers. However, since reviewers are randomly chosen and the consumers’ hypothesis about the reviewer’s strategy is out of the reviewers’ control, whether the equilibrium is indeed a babbling equilibrium depends on whether the hypothesis of all players in the game coincides, which should be uncertain in reality. On the other hand, instead of forming a sophisticated hypothesis about strategies of other reviewers and consumers, telling the truth is the only strategy robust to both babbling equilibrium and truth-telling equilibrium: it is no harm to play honestly in a babbling equilibrium, as it leads to zero payoff regardless of what strategy to play, but it is strictly optimal if it is the truth-telling equilibrium.
Another way to think about this issue is based on an observation that babbling equilibria will result in zero rewards for reviewers while the truth-telling equilibrium always results in positive expected reward.

**Corollary 3.2.** The babbling equilibrium yields the lowest reward 0 to reviewers than the informative equilibrium and the informative equilibrium’s expected reward for reviewers is positive, in particular, 
\[ \frac{2q_1(1) - 1 - q_0(1)}{q_0} \text{Var}(\omega) \] for reviewers who are satisfied and 
\[ \frac{-q_1(0)}{q_0} + \frac{1 - q_1(0)}{1 - q_0} \text{Var}(\omega) \] for reviewers who are not satisfied.

**Proof.** Plugging \( r_i(1) = 1, r_i(0) = 0 \) into \( E(V_1 - V_0|\tilde{x}_i = 1)P(\tilde{x}_i = 1|x_{-i} = 1) + E(V_1 - V_0|\tilde{x}_i = 0)P(\tilde{x}_i = 0|x_{-i} = 1) \) and \( E(V_1 - V_0|\tilde{x}_i = 1)P(\tilde{x}_i = 1|x_{-i} = 0) + E(V_1 - V_0|\tilde{x}_i = 0)P(\tilde{x}_i = 0|x_{-i} = 0) \) in lemma 3.2 will give us the desired result. 

If participating and rating involve a small but non-zero cost \( \epsilon \) in forms of either time or cognitive efforts for reviewers, it does not seem plausible the reviewers who expect babbling equilibrium ex ante are still willing to spend the cost \( \epsilon \) to participate in the first place because in babbling equilibria, reviewers payoff will be \( -\epsilon \). Examples of costly rating could be preparing pictures depicting product’s defects and lengthy comments. In other words, when consumers incur a costly rating, it would be more reasonable to believe it is informative rather than babbling. In fact, this line of thinking can lead us to modify the mechanism in a way that the reviewers “signal” that their reports are informative by committing to higher costs, which is an insight highlighted in (Austen-Smith and Banks 2000). The modified mechanism is as follows.

1. The mechanism invites two reviewers to make ratings.
2. The reviewers can decide whether to accept the invitation. If both reviewers accept the invitation, reviewers have to make costly ratings with a cost \( \epsilon \). If either reviewer rejects, the rating game ends.
3. One of the ratings is chosen randomly and published.
4. Consumers see the published rating and make a purchase decision.
5. The reviewer whose review is not published will be rewarded according to the rule in Section 3.3.

It is then clear that if reviewers and consumers expect an informative equilibrium, the small cost will not deter reviewers from accepting the platform’s invitation and there will be no babbling equilibrium.

**Proposition 3.1.** If there is a small fixed cost for participation, either platform’s invitation is rejected and no rating is submitted, or reviewers submit truthful ratings in equilibrium.

Unfortunately, we cannot fully eliminate the possibility of reviewers rejecting to submit ratings. If reviewers and consumers believe that it is a babbling equilibrium, reviewers will reject the invitations. However, if ratings are observed, it must be truthful in the modified mechanism.

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3.7.3 Participation

Another issue is that the reviewer may need to pay the platform if its report prediction is not aligned with the purchase rate change in the mechanism. This can be legally difficult to implement\(^{15}\). A straightforward solution is to linearly transform the payoff with a lump sum transfer so that the lower bound is non-negative. Since the absolute value \(|V_1 - V_0|\) is bounded in \([0, 1]\) by definition, the mechanism designer can impose a linear transformation, say \(\alpha + \beta(V_1 - V_0)\) with \(\beta > 0\), to adjust the expected payoff to the relevant level in practice without changing incentive compatibility.

However, it is important to note that with such a lump sum transfer, reviewer’s incentive may be distorted for participation instead of reporting truthfully. Some people may submit reports even before they receive or use the product for the participation reward \(\alpha\). In contrast, this behavior is discouraged in the original mechanism without a lump sum transfer.

**Proposition 3.2.** For a reviewer who does not experience the product, the expected reward from submitting a review is 0 in the mechanism described in section 3.3.

**Proof.** Since in the informative equilibrium, the reviewer \(i\) will submit the true satisfaction, the probability of the occurrence of \(E(V_1 - V_0|\tilde{x} = 1)\) is \(q_0\) and \(E(V_1 - V_0|\tilde{x} = 0)\) is \(1 - q_0\). By the same argument in proof of Lemma 3.1,

\[
E(V_1 - V_0|\tilde{x} = 1) = \frac{1}{q_0} Var(\omega) \\
E(V_1 - V_0|\tilde{x} = 0) = -\frac{1}{1 - q_0} Var(\omega)
\]

Hence the expected reward from submitting \(x = 1\) for reviewer \(-i\) before knowing \(x_{-i}\) is

\[
E(V_1 - V_0) = E(V_1 - V_0|\tilde{x} = 1)q_0 + E(V_1 - V_0|\tilde{x} = 0)(1 - q_0) = 0
\]

That implies both submitting \(x = 1\) and \(x = 0\) will yield expected reward 0.

\(^{15}\)Legal authority may see the mechanism as a gamble, which could be banned in different jurisdictions.
3.7.4 Anonymous Reviewers

As noted in remark 3.1, to see why anonymity matters, assume that instead of observing $\bar{x}_i$ with being certain about $i = A$ or $i = B$, consumers only observe $\bar{x}_i$’s value without knowing whether it is from $A$ or $B$. In this case, consumers will have to update the posterior belief taking into account that the report can be either from $A$ or from $B$. That is

$$P(\omega|\bar{x}_i = 1) = \frac{\frac{1}{2} P(\omega|\bar{x}_i = 1| i = A) + \frac{1}{2} P(\omega|\bar{x}_i = 1| i = B)}{P(\bar{x}_i = 1| i = A) + \frac{1}{2} P(\bar{x}_i = 1| i = B)}$$

and

$$P(\omega|\bar{x}_i = 0) = \frac{\frac{1}{2} P(\omega|\bar{x}_i = 0| i = A) + \frac{1}{2} P(\omega|\bar{x}_i = 0| i = B)}{P(\bar{x}_i = 0| i = A) + \frac{1}{2} P(\bar{x}_i = 0| i = B)}$$

Denote $\bar{r}(x) = \frac{1}{2}(r_A(x) + r_B(x)) \forall x \in \{0, 1\}$ as the average strategy. The posterior belief can then be re-arranged as

$$P(\omega|\bar{x}_i = 1) = \frac{\omega \bar{r}(1) + (1 - \omega) \bar{r}(0)}{q_0 \bar{r}(1) + (1 - q_0) \bar{r}(0)} P(\omega)$$

Similarly,

$$P(\omega|\bar{x}_i = 0) = \frac{1 - [\omega \bar{r}(1) + (1 - \omega) \bar{r}(0)]}{1 - [q_0 \bar{r}(1) + (1 - q_0) \bar{r}(0)]} P(\omega)$$

If in equilibrium, reviewers $A$ and $B$’s strategies are

$$r_A(x_A) = \begin{cases} 
1 - \epsilon & x_A = 1 \\
\epsilon & x_A = 0 
\end{cases}$$

$$r_B(x_B) = \begin{cases} 
\epsilon & x_B = 1 \\
1 - \epsilon & x_B = 0 
\end{cases}$$

for some $\frac{1}{2} > \epsilon > 0$, then $\bar{r}(1) = \bar{r}(0) = \frac{1}{2}$ and hence consumer’s posterior will be equal to prior belief

$$P(\omega|\bar{x}_i = 1) = P(\omega|\bar{x}_i = 0) = P(\omega)$$

Although both reviewers are not playing babbling strategies as $r_A(1) \neq r_A(0)$ and $r_B(1) \neq r_B(0)$, reviewers are indifferent between reporting satisfied and not satisfied because consumer’s hypothesis will result in $V_1 = V_0$ regardless of what or whose report is published. In addition, since the purchase rate $V_1 = V_0$, reviewers are indifferent to reporting satisfied or not satisfied, which makes the hypothesized strategies
plausible. Our previous result will not hold since an informative equilibrium may still fail to provide decision-relevant information to consumers.

3.7.5 Manipulation

Does implementing the proposed mechanism necessarily imply ratings on the platform are truthful in practice? The answer may not be straightforward. While a platform provides sufficient incentives for eliciting truthful opinions, the credibility of ratings also gives rooms for interested parties to manipulate people’s decisions with false information.

This observation may seem to jeopardize the effectiveness and robustness of the mechanism. However, as I will show in the next result, as long as people are aware of the existence of manipulation and the manipulation is not large in scale, the incentive compatibility of the mechanism is still intact. A simple model is provided to illustrate the result.

It is assumed in addition to reviewers who submit ratings solely based on incentives provided by the platform, there is a group of people who will purchase the product to obtain the eligibility to submit reviews. Let’s call the former non-manipulator and the latter manipulator. Since the invitation is purely random among the consumers, let’s further assume they represent proportion $p + q$ in the pool of possible reviewers, where $p$ represents the manipulators who will always submit satisfied $x = 1$ and $q$ represents the manipulators who will always submit unsatisfied $x = 0$.

The following result demonstrates that their existence does not distort the optimal strategy for non-manipulators.

Corollary 3.3. With a non-trivial proportion of manipulators, the non-manipulator’s optimal response is reporting truthfully in an informative equilibrium.

While the incentives for non-manipulator are not distorted, the posterior belief is nonetheless less accurate for the existence of manipulators. It can be observed from the fact that in the unique informative equilibrium, the rate of difference between posterior and prior belief is

\[
\frac{P(\omega|\hat{x} = 1) - P(\omega)}{P(\omega)} = \frac{q_0 - \omega}{q_0 + \frac{p}{(1-p-q)}}
\]

\[
\frac{P(\omega|\hat{x} = 0) - P(\omega)}{P(\omega)} = \frac{q_0 - \omega}{(1-q_0) + \frac{q}{(1-p-q)}}
\]

This model does not lose generality because the parameter $p$ and $q$ accommodate the possibility that manipulators randomly submit reviews with heterogeneous biases.
Compared to the difference in equilibrium with no manipulator (e.g. \( p + q = 0 \)),

\[
\frac{P(\omega|\hat{x} = 1) - P(\omega)}{P(\omega)} \bigg|_{p+q=0} = \frac{(\omega - q_0)}{q_0}
\]

\[
\frac{P(\omega|\hat{x} = 0) - P(\omega)}{P(\omega)} \bigg|_{p+q=0} = \frac{(q_0 - \omega)}{(1 - q_0)}
\]

the rate of difference in equilibrium with manipulators is lower in magnitude because of the presence of factors \( \frac{p}{(1-p-q)} \) and \( \frac{q}{(1-p-q)} \) in denominator. If the proportion \( p + q \) is large enough, the posterior will, as intuition suggests, approach to prior since

\[
\lim_{(p+q) \to 1} \frac{P(\omega|\hat{x}) - P(\omega)}{P(\omega)} = 0
\]

### 3.7.6 Platform’s Incentive

Although a mechanism that induces unique informative equilibrium exists, it does not necessarily mean the platform will implement it. After all, the platform’s ultimate goal should be to maximize profits instead of providing truthful information. In practice, a platform’s profits are often tied to the sale because either the platform charges commissions from sellers in proportion or the platform itself is the seller. As noted in Bayesian persuasion literature Gentzkow and Kamenica (2016), the platform that serves as the information provider will be able to benefit by distorting the information it receives. For instance, when receiving one positive and one negative rating from receivers, the platform may want to conceal the negative rating and publish the positive one to increase purchases. If the platform has a strong incentive to manipulate ratings, consumers will accordingly adjust their view after reading ratings. The distortion in consumer’s purchase choice will then be taken into consideration when reviewers are submitting ratings. Will the rating received by the platform be distorted then? The answer depends on the degree of the platform’s ability and incentives to deviate from the mechanism. If the platform fakes all ratings to promote the product regardless of what ratings are received from reviewers, the consumers will never trust ratings published, which further discourages reviewer’s incentives.

If we instead assume the platform is committed to a disclosure policy that reveals partial information to consumers, the mechanism can make use of the revealed information to incentivize reviewers sending truthful reviews. However, such a commitment apparently is not that reliable according to recent news about how frequent companies such as YouTube and Google change algorithms for search engines and
content selection\textsuperscript{17}. Moreover, platform may also not want the users to know how the mechanism is changed exactly. The ambiguity of platform algorithms can also influence consumer’s trust.

### 3.8 Conclusion

This study shows that there exists a mechanism to incentivize truthful reports from product reviewers on a platform in an environment in which reviewers have subjective opinions and have no need for future purchase. While the platform cannot verify reviewer’s true opinion by observing whether reviewers make repetitive purchases, the proposed mechanism guarantees that in the induced game, other than babbling equilibria, it must be a truth-telling equilibrium in an environment with binary satisfaction reports. In environments with non-binary reports, a so-called “combinatorial different” condition has to be imposed on reviewer’s posterior belief to guarantee the same result. The key idea in the mechanism is that even though reviewer’s opinion is not directly verifiable, consumers who read the reports will interpret the messages in reviewer’s reports correctly in equilibrium, and their purchase decisions after reading reports will reflect the information from reviewer’s reports. The purchase decision recorded in the platform serves as a lever to design a contingent payoff and incentivize reviewers. Another factor that is key to the result is that reviewer’s satisfactions are correlated. It enables reviewers to infer each other’s strategies and reports.

One question arises in this study: if such a mechanism exists, why is voluntary ratings still the prevailing mechanism in most if not all platforms? Even if platforms may not know the mechanism proposed in this study, it is still not self-evident why mechanisms in the literature are rarely used in practice. Theoretically speaking, voluntary ratings suffer from problems such as under-provision because product reviewers are public goods. Ratings benefit people other than the reviewer who submits the ratings. We should expect product ratings to be insufficiently provided and the quality will also be questionable. Does the prevalence of voluntary ratings mean voluntary reporting, though not theoretically perfect, is useful in practice? Studies show ratings on platforms can be highly inflated and platforms such as eBay are aware and have tried to adjust rating mechanisms (Filippas, Horton, and Golden 2018; Tadelis 2016). While the abovementioned issues have existed for years, leading platforms still have not adopted non-voluntary mechanisms. The reasons behind such persistence of voluntary mechanism remain an open question that needs to be further investigated.

\textsuperscript{17}Jack Nicas/Wall Street Journal/ How YouTube Drives People to the Internet’s Darkest Corners https://www.wsj.com/articles/how-youtube-drives-viewers-to-the-internets-darkest-corners-1518020478
Lauren Feiner/CNBC/Google exercises more direct control over search results than it has admitted, report claims https://www.cnbc.com/2019/11/15/google-tweaks-its-algorithm-to-change-search-results-wsj.html
Proofs

Proof of Lemma 3.1. This first part is derived directly from Bayesian rule,

\[ P(ω|x_i = 1) = \frac{P(ω \cap x_i = 1)}{P(x_i = 1)} = \frac{P(x_i = 1|ω)P(ω)}{\sum_ω P(x_i = 1|ω)P(ω)} = \frac{ωP(ω)}{\sum_ω ωP(ω)} = \frac{ω}{q_0} P(ω) \]

and similarly

\[ P(ω|x_i = 0) = \frac{P(ω \cap x_i = 0)}{P(x_i = 0)} = \frac{P(x_i = 0|ω)P(ω)}{\sum_ω P(x_i = 0|ω)P(ω)} = \frac{(1 - ω)P(ω)}{\sum_ω (1 - ω)P(ω)} = \frac{1 - ω}{1 - q_0} P(ω) \]

Hence if $ω > q_0$, the posterior probability will increase when $x_i = 1$

\[ P(ω|x_i = 1) = \frac{ω}{q_0} P(ω) > P(ω) \]

and decrease when $x_i = 0$

\[ P(ω|x_i = 0) = \frac{1 - ω}{1 - q_0} P(ω) < P(ω) \]

The opposite result holds for $ω < q_0$.

To prove the relation of stochastic dominance, note that by definition of cumulative probability,

\[ P(ω ≤ ω_j|x_i = 1) = \sum_{ω ≤ ω_j} \frac{ω}{q_0} P(ω) = \frac{1}{q_0} E(ω|ω ≤ ω_j)P(ω ≤ ω_j) \]

Since $E(ω|ω ≤ ω_j)$ is a strictly increasing function of $j$ with maximum $q_0$, we can conclude that

\[ P(ω ≤ ω_j|x_i = 1) < P(ω ≤ ω_j) \forall j \in \{1, 2, ..., n\} \]

Similarly,

\[ P(ω ≤ ω_j|x_i = 0) = \sum_{ω ≤ ω_j} \frac{1 - ω}{1 - q_0} P(ω) = \frac{1}{1 - q_0} [1 - E(ω|ω ≤ ω_j)]P(ω ≤ ω_j) \]

this gives us

\[ P(ω ≤ ω_j|x_i = 0) > P(ω ≤ ω_j) \]

It is then evident that the conditional expectation of $ω$ is increasing with $x$ because of stochastic dominance.

Proof of Theorem 3.2. We are going to prove the following by backward induction. Without loss of generality, we assume the proof is done from reviewer1’s perspective and there is some $i ≠ 1$ such that $r_i(1) ≠ r_i(0)$. In
this proof, we will no longer denote reviewer’s report as 1 or 0 but represent them as “+” or “−” instead of the face value according to the following rule:

Specifically, for a reviewer $i$, if its strategy is $r_i(1) > r_i(0)$, then $\tilde{x}_i = 1$ is represented as $\tilde{x}_i = +$ and $\tilde{x}_i = 0$ is represented as $x_i = −$. On the other hand, if its strategy is $r_i(1) < r_i(0)$, then $\tilde{x}_i = 1$ is represented as $\tilde{x}_i = −$ and $\tilde{x}_i = 0$ is represented as $\tilde{x}_i = +$. The rule is summarized as the following table:

<table>
<thead>
<tr>
<th>report \ strategy</th>
<th>$r_i(1) &gt; r_i(0)$</th>
<th>$r_i(1) &lt; r_i(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{x}_i = 1$</td>
<td>$+$</td>
<td>$−$</td>
</tr>
<tr>
<td>$\tilde{x}_i = 0$</td>
<td>$−$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

The reason for transforming the reports is simple: in equilibrium, reviewers should only care about whether a report is going to push the purchase up or down. The nominal value does not really matter. By representing reviewer’s report in this way, we can make the proof more straightforward as in equilibrium reviewers’ strategies are known by players.

Let’s call a set of reports “report record” $H_k$, which contains reports from reviewer 2, 3,... and $k$. That is $H_k = \{\tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_k\}$. Note that given a $k \in I/\{1\}$, $H_k$ may or may not contain reports from reviewers who is playing non-babbling strategy. But by assuming in an informative equilibrium, it is certain that $H_n = \{\tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n\}$ contains at least a report that is non-babbling. The following result tells us that reviewer 1 always perceives higher purchase rates given any reports, which provides the foundation to show the desired result by backward induction.

**Lemma 3.5.** Given any $k \in I/\{1\}$ and any report record $H_k = \{\tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_k\}$ in an informative equilibrium, reviewer 1 will perceive higher purchase rates if his satisfaction is $x_1 = 1$ than $x_1 = 0$. That is

$$E[V_1|x_1 = 1, H_k] \geq E[V_1|x_1 = 0, H_k]$$

with strict inequality when there is at least one non-babbling report in $H_n/H_k = \{\tilde{x}_{k+1}, \tilde{x}_{k+2}, ..., \tilde{x}_n\}$

**Proof.** For $k = n$, it is obviously true because only the published reports $H_n$ influences consumer’s purchases, that is

$$E[V_1|x_1, H_n] = E[V_1|x_1, \tilde{x}_2, ..., \tilde{x}_n] = E[V_1|\tilde{x}_2, ..., \tilde{x}_n] \forall \tilde{x}_i \in \{+, −\}$$

and therefore

$$E[V_1|x_1 = 1, H_n] = E[V_1|x_1 = 0, H_n]$$
For $k = n - 1$

$$E[V_1|x_1, H_{n-1}] = \sum_{x_n \in \{+, -\}} E[V_1|x_1, H_{n-1}, \tilde{x}_n] P(\tilde{x}_n|x_1, H_{n-1}) = \sum_{x_n \in \{+, -\}} E[V_1|x_1, \tilde{x}_n] P(\tilde{x}_n|x_1, H_{n-1})$$

Since

$$E[V_1|H_{n-1}, \tilde{x}_n = +] \geq E[V_1|H_{n-1}, \tilde{x}_n = -]$$

If we can show

$$P(\tilde{x}_n = +|x_1 = 1, H_{n-1}) \geq P(\tilde{x}_n = +|x_1 = 0, H_{n-1})$$

then we are done.

Conditional on $H_k = \{\tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_k\}$, the posterior belief of a given state $\omega$ can be expressed as

$$P(\omega|x_1 = 1, H_k) = \frac{P(\omega \cap (x_1 = 1)|H_k)}{P(x_1 = 1|H_k)} = \frac{\omega}{q_H} P(\omega|H_k)$$

where $q_H = \sum_\omega \omega P(\omega|H_k)$. On the other hand,

$$P(\omega|x_1 = 0, H_k) = \frac{P(\omega \cap (x_1 = 0)|H_k)}{P(x_1 = 0|H_k)} = \frac{1 - \omega}{1 - q_H} P(\omega|H_k)$$

From here, we can already see the result is simply a modification from our previous results in two-reviewer model with $q_0$ replaced by $q_H$ and $P(\omega)$ replaced by $P(\omega|H_k)$. Hence, given $H_{n-1}$, for reviewer 1 whose satisfaction is $x_1 = 1$ will perceive higher probability that $x_N = +$ than $x_1 = 0$.

So, we have

$$E[V_1|x_1 = 1, H_{n-1}] > E[V_1|x_1 = 0, H_{n-1}]$$

Assume for $k = m < n$, the result holds. We can show that for $k = m - 1$

$$E[V_1|x_1, H_{m-1}] = \sum_{\tilde{x}_m \in \{+, -\}} E[V_1|x_1, H_{m-1}, \tilde{x}_m] P(\tilde{x}_m|x_1, H_{m-1})$$

By assumption that the result hold for $k = m$, we have $\forall \tilde{x}_m$ and $\forall H_{m-1}$,

$$E[V_1|x_1 = 1, H_{m-1}, \tilde{x}_m] = E[V_1|x_1 = 1, H_m] > E[V_1|x_1 = 0, H_m] = E[V_1|x_1 = 0, H_{m-1}, \tilde{x}_m]$$

Furthermore, $\forall x_1$ and $\forall H_{m-1}$,

$$E[V_1|x_1, H_{m-1}, \tilde{x}_m = +] > E[V_1|x_1, H_{m-1}, \tilde{x}_m = -]$$
As two-reviewer model, we denote \( q_{1,H_k} \) and \( q_{0,H_k} \) as

\[
q_{1,H_k} = E(\omega|x_1 = 1, H_k)
\]

\[
q_{0,H_k} = E(\omega|x_1 = 0, H_k)
\]

and trivially \( q_{1,H_k} > q_{0,H_k} \). Similarly as the previous step, we only need to show

\[
P(\tilde{x}_m = +|x_1 = 1, H_{m-1}) \geq P(\tilde{x}_m = +|x_1 = 0, H_{m-1})
\]

If \( r_m(1) > r_m(0) \), the probability of reviewer \( m \) reporting + can be similarly expressed as

\[
P(\tilde{x}_m = +|x_1 = 1, H_{m-1}) = q_{1,H_k} r_m(1) + (1 - q_{1,H_k}) r_m(0)
\]

, which is greater than

\[
P(\tilde{x}_m = +|x_1 = 0, H_{m-1}) = q_{0,H_k} r_m(1) + (1 - q_{0,H_k}) r_m(0)
\]

and on the other hand if \( r_m(1) < r_m(0) \),

\[
P(\tilde{x}_m = +|x_1 = 1, H_{m-1}) = q_{1,H_k} (1 - r_m(1)) + (1 - q_{1,H_k})(1 - r_m(0))
\]

which is great than

\[
P(\tilde{x}_m = -|x_1 = 0, H_{m-1}) = q_{0,H_k} (1 - r_m(1)) + (1 - q_{0,H_k})(1 - r_m(0))
\]

Therefore we then have

\[
E[V_1|x_1 = 1, H_{m-1}] \geq E[V_1|x_1 = 0, H_{m-1}]
\]

By induction, we conclude \( \forall k \) such that \( n \geq k \geq 2 \)

\[
E[V_1|x_1 = 1, H_k] \geq E[V_1|x_1 = 0, H_k]
\]

After we prove Lemma 3.5, it is then obvious that for reviewers who receive \( x = 1 \), in an informative equilibrium, they should always submit \( \tilde{x} = 1 \) and for reviewers who receive \( x = 0 \), in an informative equilibrium, they should always submit \( \tilde{x} = 0 \) . \( \square \)
Proof of Lemma 3.3. From reviewer $i$’s perspective, the realized purchase in equilibrium is exactly $E(x|\tilde{F}_{-i})$ because the distribution of $c$ is uniform. In equilibrium, $\tilde{F}_{-i}$ is equal to $r_{-i}(x_{-i})$ and reviewer $i$ perceives $x_{-i}$ as a random variable with a distribution conditional on $x_i$, i.e. $P(x_{-i}|x_i)$. Assuming $r_{-i}$ as a single-valued function, $r_{-i}(x_{-i})$ is a random variable (it is proved in the next proof). Hence $V_1 = E(x|\tilde{F}_{-i}) = E(x \mid r_{-i}(x_{-i}))$ as an conditional expectation of a random variable is a random variable with a probability distribution.

Proof of Lemma 3.4. The proof is a straightforward simplification of the main result in Chambers and N. S. Lambert (2017).

If there are two solutions $F_1$ and $F_2$ to the optimization problem and $F_1 \neq F_2$, then there must exist a $V^*$ such that $\int_0^{V^*} F_1(v)dv \neq \int_0^{V^*} F_2(v)dv$ or equivalently

$$\int_{V^*}^{1} F_1(v)dv \neq \int_{V^*}^{1} F_2(v)dv$$

. However, for an arbitrary call option with strike $\tilde{V}$, the expected value under a distribution $F$ is

$$\int_0^{1} \max(v - \tilde{V}, 0)dF(v) = \int_{\tilde{V}}^{1} (v - \tilde{V})dF(v) = (1 - \tilde{V}) - \int_{\tilde{V}}^{1} F(v)dv$$

Hence if the call option’s strike price is $V^*$, the two distributions $F_1$ and $F_2$ will value the option differently

$$\int_0^{1} \max(v - V^*, 0)dF_1(v) - \int_0^{1} \max(v - V^*, 0)dF_2(v) = \int_{V^*}^{1} F_2(v)dv - \int_{V^*}^{1} F_1(v)dv \neq 0$$

Due to the continuity of $\int_0^{V} F(v)dv$ for all $F \in \Delta([0,1])$ and $V \in [0,1]$, for any pair of $F_1$ and $F_2$, there must exists a open interval of strikes $I_\tilde{V} \subseteq [0,1]$ such that $F_1$ and $F_2$ will value options with strikes $\tilde{V} \in I_\tilde{V}$ differently. Therefore, for any $\tilde{V} \in I_\tilde{V}$, if we define the lower bound $\underline{p}$ and upper bound $\bar{p}$ of a option with strike $\tilde{V}$ with respect to $F_1$ and $F_2$

$$\underline{p} = \min(\int_0^{1} \max(v - \tilde{V}, 0)dF_1(v), \int_0^{1} \max(v - V^*, 0)dF_2(v))$$

and

$$\bar{p} = \max(\int_0^{1} \max(v - \tilde{V}, 0)dF_1(v), \int_0^{1} \max(v - V^*, 0)dF_2(v))$$

, then with $p_c \in I_p = (\underline{p}, \bar{p})$, it must be either reviewers perceiving $F_1$ want to buy the option and reviewers perceiving $F_2$ want to sell the option, or the opposite. To see why, without loss of generality, we can assume $\underline{p} = \int_0^{1} \max(v - \tilde{V}, 0)dF_1(v)$ and $\bar{p} = \int_0^{1} \max(v - V^*, 0)dF_2(v)$. For any $p_c \in I_p = (\underline{p}, \bar{p})$, the following must
be true
\[ \int_0^1 \max(v - \bar{V}, 0) dF_1(v) - p_c < 0 \]
and
\[ \int_0^1 \max(v - \bar{V}^*, 0) dF_2(v) - p_c > 0 \]

In other words, if a reviewer perceiving \( F_1 \) misreports the distribution as \( F_2 \), the platform will at least buy over-valued options or sell under-valued options on the behalf of the reviewer with positive probability because \( I_{\bar{V}} \) and \( I_p \) are both with positive measures. Therefore reviewers will never have incentives to misreport and the reported distribution must be its perceived distribution of purchase rates.

**Proof of Theorem 3.3.** The proof for existence of babbling equilibrium is omitted for its triviality.

In order to prove that the only informative equilibrium is truth-telling, we divide the proof into two parts. First, we show consumers must learn that reviewer’s satisfaction perfectly in any informative equilibrium. Second, we show that since consumers learn that the reviewer’s satisfaction perfectly, the reviewer must report truthfully.

We can assume without loss of generality, in any informative equilibrium, if it exists, there are at least two subsets of \( X \), \( X_1 \) and \( X_2 \), such that for any pair of reviewer \( i \)’s type \( x \in X_1 \) and \( y \in X_2 \), whose reviews \( r_i(x) \) and \( r_i(y) \) are published by platform, they must induce different purchase rates, denoted as \( V(r_i(x)) \) and \( V(r_i(y)) \), such that
\[ V(r_i(x)) \neq V(r_i(y)) \]

If it is not true, reviewer \(-i\) would have preferred to report \( \delta_{\bar{V}_0} \) because once reviewer \( i \)'s report is published, there would have been no change in purchase rate. However, it should have incentivized reviewer \( i \) to report \( \delta_{\bar{V}_0} \) too. Contradiction to the assumption that reviewer \( i \) with satisfaction \( x \) and \( y \) reports different distributions.

Hence, the probability of \( V(r_i(x)) \) to realize is \( \sum_{x \in X_1} P(x) = P(X_1) \) because \( P(X_1) \) is exactly the probability of type \( X_1 \) and similarly, the probability of \( V(r_i(y)) \) to realize is \( \sum_{y \in X_2} P(y) = P(X_2) \). From reviewer \(-i\)'s point of view, when his satisfaction is \( z \), its posterior belief for \( V(r_i(x)) \) to obtain is \( \sum_{x \in X_1} P(x|z) = P(X_1|z) \) and the posterior belief of \( V(r_i(y)) \) is \( \sum_{y \in X_2} P(y|z) = P(X_2|z) \). Since by combinatorial difference, for all \( z, z' \in X \),
\[ P(X_1|z) \neq P(X_1|z') \]
and
\[ P(X_2|z) \neq P(X_2|z') \]
Proof of Corollary 3.3. When the consumer see positive rating, the posterior belief is

\[
P(\omega|\bar{x} = 1) = \frac{p + (1 - p - q)[\omega r_i(1) + (1 - \omega)r_i(0)]}{p + (1 - p - q)[q_0 r_i(1) + (1 - q_0)r_i(0)]} P(\omega)
\]

\[
P(\omega|\bar{x} = 0) = \frac{q + (1 - p - q)[1 - \omega r_i(1) - (1 - \omega)r_i(0)]}{q + (1 - p - q)[1 - q_0 r_i(1) - (1 - q_0)r_i(0)]} P(\omega)
\]

The expected difference of purchase rate for \( \bar{x} = 1 \) is

\[
E(V_1 - V_0|\bar{x} = 1) = \sum \omega [P(\omega|\bar{x} = 1) - P(\omega)] = \sum \omega \{ \frac{\omega r_i(1) + (1 - \omega)r_i(0)}{p + (1 - p - q)[q_0 r_i(1) + (1 - q_0)r_i(0)]} \}
\]

\[
= \frac{1}{p + (1 - p - q)[q_0 r_i(1) + (1 - q_0)r_i(0)]} \left[ (1 - p - q)[r_i(1) - r_i(0)] \right]
\]

Therefore, consumers with outside option \( c \) will make purchase if \( V_\omega > c \) or no purchase otherwise. This completes the proof.
Similarly to the proof in Lemma 3.2, if we assume \( r_i(1) > r_i(0) \), since the probability of observing \( \tilde{x} = 1 \) from a satisfied reviewer will be

\[
p + (1 - p - q)[q_1(1)r_i(1) + (1 - q_1(1))r_i(0)]
\]

which is greater than \( p + (1 - p - q)[q_0r_i(1) + (1 - q_0)r_i(0)] \), the expected reward from submitting \( x = 1 \) is

\[
E(V_1 - V_0|\tilde{x} = 1)p = E(V_1 - V_0|\tilde{x} = 1)\{p + (1 - p - q)[q_1(1)r_i(1) + (1 - q_1(1))r_i(0)]\} > 0
\]

. Immediately, the expected reward from submitting \( x = 0 \) is negative and hence a satisfied reviewer will prefer submitting \( x = 1 \). By symmetry, an unsatisfied reviewer will find \( x = 0 \) dominates \( x = 1 \) and the other case for \( r_i(1) < r_i(0) \) can be done similarly. Therefore, we conclude that the existence of manipulation does not distort non-manipulator’s rating incentives.


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