WEYlâ FORMULA FOR SCHRÖDINGER OPERATORS WITH CRITICALLY SINGULAR POTENTIALS

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Abstract

We record the joint work done by the author and Christopher Sogge [9] on generalizing the classical Weyl formulae to Schrödinger operators involving singular potentials. In a similar vein, we also record a subsequent work done by the author and Cheng Zhang [10] on local pointwise versions of Weyl formulae for Schrödinger operators with same types of potentials.

For Schrödinger operators $H_V = -\Delta_g + V(x)$ on compact boundaryless Riemannian manifolds, we extend the classical results of Avakumović [1], Levitan [12] and Hörmander [8] by obtaining $O(\lambda^{n-1})$ bounds for the error term in the Weyl formula in the universal case when we assume that $V \in L^1(M)$ with the negative part $V^- = \max\{0, -V\}$ belongs to the Kato class, $\mathcal{K}(M)$, which is the minimal assumption to ensure that $H_V$ is essentially self-adjoint and bounded from below or has favorable heat kernel bounds. In this case, we can also obtain extensions of the Duistermaat-Guillemin [5] theorem yielding $o(\lambda^{n-1})$ bounds for the error term under generic conditions on the geodesic flow, and we can also extend Bérard’s [2] theorem yielding $O(\lambda^{n-1}/\log \lambda)$ error bounds under the assumption that the principal curvatures are non-positive everywhere. On the other hand, to obtain local pointwise versions of Weyl formulae for Schrödinger operators, it is expected that extra conditions on the potential $V$ are needed due to the examples constructed in [6], and as we shall show in the last chapter, the $O(\lambda^{n-1})$ bounds for the error term hold if $V \in L^n(M)$.

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Introduction

1.1 The Kato class and self adjoint-ness of Schrödinger operators

Let \((M, g)\) denote a smooth compact \(n\)-dimensional Riemannian manifold without boundary, consider the Schrödinger operators

\[
H_V = -\Delta g + V(x)
\]

defined on the manifold, where \(\Delta g\) denote the Laplace-Beltrami operator. We shall assume throughout that the potentials \(V\) are real-valued. Moreover, we shall assume that

\[
V \in L^1(M), \quad \text{and} \quad V^- = \max\{0, -V\} \in \mathcal{K}(M),
\]

where \(\mathcal{K}(M)\) denotes the Kato class. Recall that \(\mathcal{K}(M)\) is all \(V\) satisfying

\[
\lim_{\delta \to 0} \left( \sup_{x \in M} \int_{B(x, \delta)} |V(y)| h_n(d_g(x, y)) \, dy \right) = 0,
\]

where \(d_g, dy\) and \(B(x, \delta)\) denote geodesic distance, the volume element and the geodesic ball of radius \(\delta\) about \(x\) associated with the metric \(g\) on \(M\), respectively, and

\[
h_n(r) = \begin{cases} 
  r^{2-n}, & n \geq 3 \\
  \log(2 + 1/r), & n = 2.
\end{cases}
\]
Note that our condition on $V$ in (1.1.2) is weaker than $V \in \mathcal{K}(M)$ since we only requires the positive part $V^+ \in L^1$ while $\mathcal{K}(M) \subset L^1(M)$.

As was shown in [3] (see also [15]) the assumption that $V^-$ is in the Kato class is needed to ensure that the eigenfunctions of $H_V$ are bounded. If $H_V$ has unbounded eigenfunctions, then its spectral projection kernels will be unbounded for large enough $\lambda$, and obtaining spectral bounds in this situation seems far-fetched. The assumption that $V^- \in \mathcal{K}(M)$ ensures that this is not the case.

Moreover, if $V$ is as in (1.1.2) then the Schrödinger operator $H_V$ in (1.1.1) is self-adjoint and bounded from below (see e.g. [14],[7]), see also [3] for a simple proof of self adjoint-ness under the stronger assumption that $V \in \mathcal{K}(M)$. Additionally, in this case, since $M$ is compact, by using the compactness of the heat operator $e^{-tH_V}$, it can be shown that the spectrum of $H_V$ is discrete. Also, (see [15]) the associated eigenfunctions are continuous. Assuming, as we may, that $H_V$ is a positive operator, we shall write the spectrum of $\sqrt{H_V}$ as

$$\{\tau_k\}_{k=1}^{\infty}, \tag{1.1.5}$$

where the eigenvalues, $\tau_1 \leq \tau_2 \leq \cdots$, are arranged in increasing order and we account for multiplicity.

For each $\tau_k$ there is an eigenfunction $e_{\tau_k} \in \text{Dom } (H_V)$ (the domain of $H_V$) so that

$$H_V e_{\tau_k} = \tau_k^2 e_{\tau_k}. \tag{1.1.6}$$

We shall always assume that the eigenfunctions are $L^2$-normalized, i.e.,

$$\int_M |e_{\tau_k}(x)|^2 \, dx = 1.$$

After possibly adding a constant to $V$ we may, and shall, assume throughout that $H_V$ is bounded below by one, i.e.,$^1$

$$\|f\|_2^2 \leq \langle H_V f, f \rangle, \quad f \in \text{Dom } (H_V). \tag{1.1.7}$$

Also, to be consistent, we shall let

$$H^0 = -\Delta_g + 1 \tag{1.1.8}$$

be the unperturbed operator also enjoying this lower bound. The corresponding eigenvalues and

$^1$See the remark after Theorem 2.
associated $L^2$-normalized eigenfunctions are denoted by $\{\lambda_j\}_j=1^\infty$ and $\{e^0_j\}_j=1^\infty$, respectively so that

$$H^0 e^0_j = \lambda^2_j e^0_j, \quad \text{and} \quad \int_M |e^0_j(x)|^2 \, dx = 1. \quad (1.1.9)$$

### 1.2 Statement of main results

Let $1_\lambda(P^0)(x,x) = \sum_{\lambda_j \leq \lambda} |e^0_j(x)|^2$, with $e^0_j(x)$ being eigenfunctions of $H^0$ defined above, then one has the following “sharp Weyl formula”

$$1_\lambda(P^0)(x,x) = (2\pi)^{-n} \omega_n \lambda^n + O(\lambda^{n-1}), \quad \text{uniformly in } x \in M, \quad (1.2.1)$$

where $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$. This result is due to Avakumović [1] and Levitan [12], and it was generalized to general self-adjoint elliptic pseudo-differential operators by Hörmander [8]. The bound in (1.2.2) cannot be improved for the standard round sphere, which accounts for the nomenclature “sharp Weyl formula”.

Note that if we integrate both sides of (1.2.1), by (1.1.9), it follows that

$$N^0(\lambda) = (2\pi)^{-n} \omega_n \text{Vol}_g(M) \lambda^n + O(\lambda^{n-1}), \quad N^0(\lambda) = \#\{j : \lambda_j \leq \lambda\}, \quad (1.2.2)$$

where $\text{Vol}_g(M)$ denotes the Riemannian volume of $M$ and $N^0(\lambda)$ denote the eigenvalue counting function for $H^0$.

Our main goal is to show that this sharp Weyl formula also holds for the operators $H_V$ in (1.1.1) involving critically singular potentials $V$ as in (1.1.2). Specifically we have the following.

**Theorem 1** (Theorem 1.1 [9]). Let $V \in L^1(M)$, and $V^- \in \mathcal{K}(M)$, let $H_V$ as above and set

$$N_V(\lambda) = \#\{k : \tau_k \leq \lambda\}. \quad (1.2.3)$$

We then have

$$N_V(\lambda) = (2\pi)^{-n} \omega_n \text{Vol}_g(M) \lambda^n + O(\lambda^{n-1}). \quad (1.2.4)$$

We shall also be able to obtain improved counting estimates under certain geometric assumptions.

The first such result is an extension of the Duistermaat-Guillemin theorem [5]. Recall the assumption in this theorem is that the set $\mathcal{C} \subset S^*M$ of all $(x, \xi)$ lying on a periodic geodesic in $S^*M$ have measure zero (see [18]). Here, $S^*M$ denotes the unit cotangent bundle of $(M, g)$. In this case
Duistermaat and Guillemin [5] showed that one can improve the bounds for the error term in the Weyl law (1.2.2) (assuming that $V = 0$ or $V$ is smooth) to be $o(\lambda^{n-1})$. The proof of this relies on Hörmander’s theory of propagation of singularities for smooth pseudo-differential operators. Even though this theory does not apply to our situation involving very singular potentials, we can extend the theorem of Duistermaat and Guillemin to include the above operators.

**Theorem 2** (Theorem 1.2 [9]). Let $V \in L^1(M)$, and $V^- \in K(M)$, let $H_V$ be as above, and assume that the set $\mathcal{C}$ of directions of periodic geodesics has measure zero in $S^*M$. Then

$$N_V(\lambda) = (2\pi)^{-n}\omega_n \text{Vol}_g(M) \lambda^n + o(\lambda^{n-1}).$$  

(1.2.5)

**Remark.** Note that, under the hypothesis of Theorem 2, the special case where $V \equiv c \geq 0$ implies, for instance, the results where $V \equiv c$. For if $N(\lambda)$ denotes the number of eigenvalues of $\sqrt{-\Delta_g}$ which are $\leq \lambda$ then, under the hypotheses of Theorem 2, by [5, Theorem 3.5], we have $N(\lambda) = (2\pi)^{-n}\omega_n \text{Vol}_g(M) \lambda^n + o(\lambda^{n-1})$. Based on this we see that the results for $V \equiv c$ imply the results where $V \equiv c \geq 0$ since $N_{V\equiv c}(\lambda) = N(\sqrt{\lambda^2 - c})$ and $(\sqrt{\lambda^2 - c})^n = \lambda^n \cdot (1 - c\lambda^{-2})^n = \lambda^n + O(\lambda^{n-2}) = \lambda^n + o(\lambda^{n-1})$. Moreover, if $V \in C^\infty(M)$ then (1.2.5) follows directly from Duistermaat-Guillemin [5, Theorem 3.5] since $\sqrt{-\Delta_g}$ and $\sqrt{H_V}$ have the same principal symbol and zero subprincipal symbol. See [5, p. 41] for the latter. As we shall see in what follows, error terms of order $O(\lambda^{n-2})$ occur recurrently in our arguments.

We also can extend the classical theorem of Bérard [2].

**Theorem 3** (Theorem 1.3 [9]). Assume that the sectional curvatures of $(M, g)$ are non-positive. Then, if $V \in L^1(M)$, and $V^- \in K(M)$,

$$N_V(\lambda) = (2\pi)^{-n}\omega_n \text{Vol}_g(M) \lambda^n + O(\lambda^{n-1}/\log \lambda).$$  

(1.2.6)

We mention that the proof of Theorem 1-3 is based on a perturbative argument which focus on estimating the difference between the counting functions for perturbed and unperturbed case, where the differences only contribute to the error terms in the main theorems. As a result, the arguments in the proof of main theorems should also allow us to obtain improved counting estimates for the Schrödinger operators $H_V$ under more general geometric assumptions, see, e.g., [22], [23], as well as more recent works [11], [4], for more details.

In the other direction, it is natural to ask if the analog of (1.2.1) also holds for Schrödinger
operators. Using a similar perturbative argument, we have the following:

**Theorem 4** (Theorem 2 [10]). Let \( V \in L^n(M) \) and \( e_{\tau_k} \) be eigenfunctions of \( H_V \) defined as in (1.1.6), if we set

\[
\mathbf{I}_\lambda(P_V)(x, x) = \sum_{\tau_k \leq \lambda} |e_{\tau_k}(x)|^2,
\]

(1.2.7)

then

\[
\mathbf{I}_\lambda(P_V)(x, x) = (2\pi)^{-n} \omega_n \lambda^n + O(\lambda^{n-1}), \quad \text{uniformly in } x \in M.
\]

(1.2.8)

The condition \( V \in L^n(M) \) is stronger than the Kato condition since by Hölder’s inequality, \( \mathcal{K}(M) \supset L^p(M) \) for all \( p > n/2 \). In the case \( n = 3 \), Frank and Sabin [6] showed that the pointwise Weyl law can be violated if we merely assume that \( V \in L^p(M) \) for any \( p < n \). It is interesting and still open to us if their examples can be generalized to other dimensions, which would imply the sharpness of above theorem in terms of conditions on the potential.
Preliminaries

2.1 An abstract universal bound and standard reductions

The purpose of this section is to introduce a abstract proposition that will allow us to prove Theorem 1.1-1.3, provided that we have the analogous counting estimates for the unperturbed operators $H_0 = -\Delta_g$.

Throughout this section we shall assume that, if $N_0(\lambda)$ denotes the Weyl counting function for $H_0$, we have

$$N_0(\lambda) = \int_M \sum_{\lambda_j \leq \lambda} |e_j^0(x)|^2 dx = (2\pi)^{-n}\omega_n \text{Vol}_g(M) \lambda^n + O(\epsilon \lambda^{n-1}),$$

(2.1.1)

where $\epsilon = \epsilon(\lambda)$ is a non-increasing function in $\lambda$ which satisfies $\epsilon(2\lambda) \geq \frac{1}{2}\epsilon(\lambda)$ and $0 < \epsilon(\lambda) \leq 1$, $\forall \lambda \geq 1$. The assumption on $\epsilon(\lambda)$ is a very mild one, which, for instance is satisfied whenever $\epsilon(\lambda) = \lambda^{-\sigma}$ with $0 \leq \sigma \leq 1$. We make this assumption, since, as we mentioned before, error terms of the form $O(\lambda^{n-2})$ arise repeatedly.

The abstract proposition we shall need is the following

**Proposition 1.** Let $V \in L^1(M)$, and $V^- \in K(M)$, let $H_V$ as above, if $N_0(\lambda)$ satisfies (2.1.1), then for the same $\epsilon = \epsilon(\lambda)$ appearing in (2.1.1), we have

$$N_V(\lambda) = (2\pi)^{-n}\omega_n \text{Vol}_g(M) \lambda^n + O(\epsilon \lambda^{n-1} + \epsilon^{-1}\lambda^{n-\frac{3}{2}}).$$

(2.1.2)

Note that as a consequence of (1.2.2), (2.1.1) holds with $\epsilon \equiv 1$, thus Theorem 1 follows directly from (2.1.2). Similarly, under the hypothesis of Theorem 2, for any fixed constant $T \gg 1$, (2.1.1)
holds with $\varepsilon = 1/T$ if $\lambda \geq \Lambda(T)$. Since $T\lambda^{n-\frac{3}{2}}$ is bounded by $1/T\lambda^{n-1}$ for sufficiently large $\lambda$, (2.1.2) also implies (1.2.5). Additionally, under the hypothesis of Theorem 3, by the classical theorem of Béard [2], (2.1.1) holds with $\varepsilon = 1/\log \lambda$, and since $\lambda^{n-\frac{3}{2}}\log \lambda$ is bounded by $\lambda^{n-1}(\log \lambda)^{-1}$ for sufficiently large $\lambda$, Theorem 3 also follows from the above proposition.

To prove (2.1.2), we first recall that, if as above, $\{e_{\tau_k}\}$ is an orthonormal basis of eigenfunctions of $H_V$ then

$$N_V(\lambda) = \# \{k : \tau_k \leq \lambda\} = \int_M \sum_{\tau_k \leq \lambda} |e_{\tau_k}(x)|^2 \, dx. \quad (2.1.3)$$

Thus, $N_V(\lambda)$ is the trace of the spectral function

$$E^V_\lambda(x, y) = \sum_{\tau_k \leq \lambda} e_{\tau_k}(x)e_{\tau_k}(y). \quad (2.1.4)$$

Here, we are assuming, as we may, that all the eigenfunctions of $H_V$ in our orthonormal basis are real-valued. To simplify the notation, as we may, we shall assume the same for those of $H^0$, i.e., the $\{e^0_j\}$.

We shall need the following lemma which is key to the proof of (2.1.2).

**Lemma 1.** Let $\chi_{\lambda,\varepsilon}^V(x, y)$ be the kernels of the spectral projection operators

$$\chi_{\lambda,\varepsilon}^V(x, y) = \sum_{\tau_k \in [\lambda, \lambda+\varepsilon)} e_{\tau_k}(x)e_{\tau_k}(y) \quad (2.1.5)$$

for $H_V$, with $\varepsilon = \varepsilon(\lambda)$ defined as in (2.1.1). Then given (2.1.1), we have

$$\int_M \chi_{\lambda,\varepsilon}^V(x, x) \, dx = O(\varepsilon\lambda^{n-1} + \varepsilon^{-1}\lambda^{n-\frac{3}{2}}), \; \lambda \geq 1. \quad (2.1.6)$$

The left side of (2.1.6) is essentially equal to the number of eigenvalues for the operator $\sqrt{H_V}$ inside the interval $[\lambda, \lambda + \varepsilon]$. Note that when $V \equiv 1$, as a consequence of (2.1.1), if

$$\chi_{\lambda,\varepsilon}^0(x, y) = \sum_{\lambda_j \in [\lambda, \lambda+\varepsilon)} e^0_j(x)e^0_j(y),$$

denotes the spectral projection operator onto the interval $[\lambda, \lambda + \varepsilon]$, we have

$$\int_M \chi_{\lambda,\varepsilon}^0(x, x) \, dx = O(\varepsilon\lambda^{n-1}), \; \lambda \geq 1. \quad (2.1.7)$$

We shall postpone the proof of Lemma 1 to the end of next chapter, and first see how we can apply
it to the proof of (2.1.2).

To make use of the above lemma, we shall follow the classical approach of rewriting the traces using the wave equation. To this end, let \( P^0 = \sqrt{H^0} \) and \( P_V = \sqrt{H_V} \) be the square roots of the two Hamiltonians. Then since the Fourier transform of the indicator function \( \mathbf{1}_\lambda(\tau) \) is \( 2\sin \lambda t \), we have for \( \lambda \) not in the spectrum of \( P^0 \)

\[
N^0(\lambda) = \frac{1}{\pi} \int_M \int_{-\infty}^{\infty} \frac{\sin t\lambda}{t} (\cos tP^0)(x, x) \, dt \, dx, \tag{2.1.8}
\]

if

\[
(\cos(tP^0))(x, y) = \sum_j \cos t\lambda_j e^0_j(x)e^0_j(y) \tag{2.1.9}
\]

is the kernel of the solution operator for \( f \to (\cos tP^0)f = u^0(t, x) \), where \( u^0 \) solves the wave equation

\[
(\partial_t^2 + H^0)u^0(x, t) = 0, \quad (x, t) \in M \times \mathbb{R}, \quad u^0|_{t=0} = f, \quad \partial_t u^0|_{t=0} = 0. \tag{2.1.10}
\]

Note that (2.1.9) is the kernel of a bounded operator on \( L^2(M) \), and when we check that (2.1.10) is valid, it suffices to do so when \( f \) is a finite linear combination of the \( \{e^0_j\} \) since such functions are dense in \( L^2(M) \). We shall use similar facts in what follows. See [18] for more details.

Similarly, for \( \lambda \) not in the spectrum of \( P_V \)

\[
N_V(\lambda) = \frac{1}{\pi} \int_M \int_{-\infty}^{\infty} \frac{\sin t\lambda}{t} (\cos(tP_V))(x, x) \, dt \, dx, \tag{2.1.11}
\]

if

\[
(\cos(tP_V))(x, y) = \sum_k \cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y) \tag{2.1.12}
\]

is the kernel of \( f \to \cos(tP_V)f = u_V(x, t) \), where \( u_V \) solve the wave equation

\[
(\partial_t^2 + H_V)u_V(x, t) = 0, \quad (x, t) \in M \times \mathbb{R}, \quad u_V|_{t=0} = f, \quad \partial_t u_V|_{t=0} = 0. \tag{2.1.13}
\]

To exploit (2.1.1) and prove its more general version (1.2.3), in view of (2.1.8)–(2.1.13), it will be useful to relate the kernels in (2.1.9) and (2.1.12). To do so we shall make use of the following simple calculus lemma.
Lemma 2. If $\mu \neq \tau$ we have

$$\int_0^t \sin(t-s)\mu \cos s\tau \, ds = \frac{\cos t\tau - \cos t\mu}{\mu^2 - \tau^2}.$$  \hspace{1cm} (2.1.14)

Similarly,

$$\int_0^t \sin(t-s)\tau \cos s\tau \, ds = \frac{t\sin t\tau}{2\tau}.$$  \hspace{1cm} (2.1.15)

Proof. To prove (2.1.14) we make use of the identity

$$\sin(s(\tau - \mu) + t\mu) = \sin((t - s)\mu + s\tau) = \sin((t - s)\mu) \cos s\tau + \cos((t - s)\mu) \sin s\tau,$$

and, similarly,

$$-\sin((\tau + \mu)s - t\mu) = \sin((t - s)\mu - s\tau) = \sin((t - s)\mu) \cos s\tau + \cos((t - s)\mu) \sin s\tau.$$

Thus,

$$\sin(s(\tau - \mu) + t\mu) - \sin((\tau + \mu)s - t\mu) = 2 \sin((t - s)\mu) \cos s\tau.$$

Consequently, the left side of (2.1.14) equals

$$\frac{1}{2\mu} \left[ \frac{\cos(s(\tau - \mu) + t\mu)}{\mu - \tau} + \frac{\cos(s(\tau + \mu) - t\mu)}{\mu + \tau} \right]_0^t$$

$$= \frac{1}{2\mu} \left[ \cos t\tau \cdot \left( \frac{1}{\mu - \tau} + \frac{1}{\mu + \tau} \right) - \cos t\mu \cdot \left( \frac{1}{\mu - \tau} + \frac{1}{\mu + \tau} \right) \right]$$

$$= \frac{1}{2\mu} \cdot \left( \frac{2\mu \cos t\tau}{\mu^2 - \tau^2} - \frac{2\mu \cos t\mu}{\mu^2 - \tau^2} \right) = \frac{\cos t\tau - \cos t\mu}{\mu^2 - \tau^2},$$

as desired.

The proof of (2.1.15) is similar.

Let us now describe how we shall use (2.1.1) and Lemma 2 to prove the Weyl formula (2.1.2).

If, as above, $1_{\lambda}(\tau)$ is the indicator function of $[-\lambda, \lambda]$, by (2.1.3), proving this amounts to showing that the trace of $1_{\lambda}(P_V)$ satisfies the bounds in (1.2.4). As is the custom (cf. [18]), we shall do this indirectly by showing that an $\varepsilon = \varepsilon(\lambda)$-dependent approximation $\hat{1}_{\lambda}(P_V)$ also enjoys these bounds, and, separately showing that the difference between the trace of $1_{\lambda}(P_V)$ and $\hat{1}_{\lambda}(P_V)$ is $O(\varepsilon\lambda^{n-1} + \varepsilon^{-1}\lambda^{n-\frac{5}{2}})$.
To this end, fix an even real-valued function \( \rho \in C^\infty(\mathbb{R}) \) satisfying

\[
\rho(t) = 1 \text{ on } [-1/2, 1/2] \text{ and supp } \rho \subset (-1, 1). \tag{2.1.16}
\]

We then define

\[
\tilde{I}_\lambda(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \rho(\varepsilon t) \frac{\sin \lambda t}{t} \cos \tau dt. \tag{2.1.17}
\]

Then since the Fourier transform of \( I_\lambda(\tau) \) is \( 2 \frac{\sin \lambda t}{t} \), it is not difficult to see that for \( \tau > 0 \) and large \( \lambda \) we have

\[
I_\lambda(\tau) - \tilde{I}_\lambda(\tau) = O\left((1 + \varepsilon^{-1}|\lambda - \tau|)^{-N}\right) \forall N. \tag{2.1.18}
\]

Also, for later use, for \( \tau > 0 \) we have

\[
\left( \frac{d}{d\tau} \right)^j \tilde{I}_\lambda(\tau) = O(\varepsilon^{-j}(1 + \varepsilon^{-1}|\lambda - \tau|)^{-N}) \forall N, \text{ if } j = 1, 2, 3, \ldots \tag{2.1.19}
\]

If we use Lemma 1, we can estimate the difference between the trace of \( I_\lambda(P_V) - \tilde{I}_\lambda(P_V) \). Indeed, by (2.1.18) we have

\[
\left| \int_M (I_\lambda(P_V)(x, x) - \tilde{I}_\lambda(P_V)(x, x)) \right| = \left| \int_M \sum_k (I_\lambda(\tau_k) - \tilde{I}_\lambda(\tau_k)) |e_{\tau_k}(x)|^2 dx \right|
\]

\[
\lesssim \sum_k \int_M (1 + \varepsilon^{-1}|\lambda - \tau_k|)^{-2n} |e_{\tau_k}(x)|^2 dx \lesssim \varepsilon^{\lambda n-1} + \varepsilon^{-1}\lambda^{n-\frac{3}{2}}, \tag{2.1.20}
\]

using (2.1.6) as well as the condition on \( \varepsilon = \varepsilon(\lambda) \) in (2.1.1) in the last inequality. Here, and in what follows, we are using the notation that \( A \lesssim B \) means that \( A \) is less than or equal to a constant times \( B \) where the constant may change at each occurrence.

Similarly, by (2.1.7), we have

\[
\left| \int_M (I_\lambda(P^0)(x, x) - \tilde{I}_\lambda(P^0)(x, x)) \right| = \left| \int_M \sum_j (I_\lambda(\lambda_j) - \tilde{I}_\lambda(\lambda_j)) |e_{\lambda_j}(x)|^2 dx \right|
\]

\[
\lesssim \sum_j \int_M (1 + \varepsilon^{-1}|\lambda - \lambda_j|)^{-2n} |e_{\lambda_j}(x)|^2 dx \lesssim \varepsilon^{\lambda n-1}. \tag{2.1.21}
\]

Thus, in view of (2.1.20), (2.1.21), and (2.1.1), in order to prove Proposition 1, it suffices to prove our main estimate

\[
\int_M \left( I_\lambda(P_V)(x, x) - \tilde{I}_\lambda(P^0)(x, x) \right) dx = O(\varepsilon^{-1}\lambda^{n-\frac{3}{2}}). \tag{2.1.22}
\]
The implicit constants here of course depend on our $V$ as in Proposition 1.

To prove this, we shall use the fact that, by (2.1.12) and (2.1.17) the kernel of $\tilde{1}_\lambda(P_V)$ is

$$
\tilde{1}_\lambda(P_V)(x, y) = \frac{1}{\pi} \int_{t=-\infty}^{\infty} \rho(\varepsilon t) \sin \frac{\lambda t}{t} \sum_k \cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y) \, dt.
$$

(2.1.23)

To use this formula, we note that, by (2.1.13) if $f$ is a finite combination of the $\{e_{\tau_k}\}$, then

$$
(\partial_t^2 + H^0) \int_M \sum_k \cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y) f(y) \, dy
$$

$$
= -V(x) \cdot \int_M \sum_k \cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y) f(y) \, dy = -V(x) \cdot (\cos tP_V)(f)(x).
$$

Also, since

$$
\left. \left( \frac{d}{dt} \right)^j \left( \int_M \sum_k \cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y) f(y) \, dy - \int_M \sum_j \cos t\lambda_j e_{\lambda_j}^0(x)e_{\lambda_j}^0(y) f(y) \, dy \right) \right|_{t=0} = 0, \ j = 0, 1,
$$

by Duhamel’s principle we have

$$
\int_M \sum_k \cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y) f(y) \, dy - \int_M \sum_j \cos t\lambda_j e_{\lambda_j}^0(x)e_{\lambda_j}^0(y) f(y) \, dy
$$

$$
= - \int_0^t \int_M \sum_j \sin(t-s)\lambda_j e_{\lambda_j}^0(x)e_{\lambda_j}^0(z) V(z) \sum_k \cos s\tau_k e_{\tau_k}(z)e_{\tau_k}(y) f(y) \, dz \, dy \, ds.
$$

By (2.1.17) or (2.1.23) if we integrate this against $\pi^{-1}\rho(\varepsilon t) \sin \frac{\lambda t}{t}$ we obtain $\tilde{1}_\lambda(P_V)f(x) - \tilde{1}_\lambda(P^0)f(x)$. Therefore, by Lemma 2 the kernel of $\tilde{1}_\lambda(P_V) - \tilde{1}_\lambda(P^0)$ is

$$
(\tilde{1}_\lambda(P_V) - \tilde{1}_\lambda(P^0))(x, y) = \frac{1}{\pi} \sum_{j,k} \int_M \int_{-\infty}^{\infty} \rho(\varepsilon t) \sin \frac{\lambda t}{t} \sum_k \cos t\tau_k e_{\tau_k}(x)e_{\tau_k}(y) \, dz \, dt,
$$

(2.1.24)

where

$$
m(\tau, \mu) = \begin{cases} 
\frac{\cos \tau - \cos \mu}{\tau - \mu^2}, & \text{if } \tau \neq \mu \\
\frac{-t \sin \tau}{2\tau}, & \text{if } \tau = \mu.
\end{cases}
$$

(2.1.25)
Thus, by (2.1.23)–(2.1.24) we have
\[
(\tilde{1}_\lambda(P_v) - \tilde{1}_\lambda(P^0))(x, y) = \sum_{j,k} \int_M \frac{\tilde{1}_\lambda(\tau_k) - \tilde{1}_\lambda(\lambda_j)}{\tau_k - \lambda_j^2} e_j^0(x)e_j^0(y)V(z)e_{\tau_k}(z)e_{\tau_k}(y) dz, \tag{2.1.26}
\]
if, by the second part of (2.1.25) we interpret
\[
\tilde{1}_\lambda(\tau) - \tilde{1}_\lambda(\mu) = \frac{\tilde{1}_\lambda'(\tau) - \tilde{1}_\lambda'(\mu)\tau}{\tau^2 - \mu^2}, \quad \text{if} \quad \tau = \mu. \tag{2.1.27}
\]

Thus, we would have (2.1.22) and consequently Proposition 2.1.1 if we could prove the following:

**Proposition 2.** Let \( V \in L^1(M) \) with \( V^- \in \mathcal{K}(M) \), and \( \tilde{1}_\lambda(\tau) \) be defined as in (2.1.17). Then we have
\[
\left\| \sum_{j,k} \int_M \frac{\tilde{1}_\lambda(\lambda_j) - \tilde{1}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y) dx dy \right\| \leq C_V \varepsilon^{-1} \lambda^{n-2}, \tag{2.2.1}
\]
for some constant \( C_V \) depending on \( V \).

As we shall see in the proof of Proposition 2, the constant \( C_V \) depends on \( \|V\|_{L^1(M)} \). The proof will also use heat kernel estimates involving \( H_V \). Steps like this will contribute to the constant \( C_V \) in (2.2.1).

## 2.2 Spectral projection bounds and heat kernel estimates

In this section we shall present several lemmas that are key to the proof the Proposition 2. First of all, note that the kernel in (2.1.26) involves an amalgamation of the kernels of \( \tilde{1}_\lambda(P^0) \), \( \tilde{1}_\lambda(P_v) \) and the resolvent kernels \((H_V - \mu^2)^{-1}\) and \((H^0 - \mu^2)^{-1}\). To prove (2.2.1) we shall attempt to separate the contributions of the various components by using the following simple lemma.

**Lemma 3.** Let \( I \subset \mathbb{R}_+ \) and for eigenvalues \( \tau_k \in I \) assume that \( \delta_{\tau_k} \in [0, \delta] \). Then if \( m \in C^1(\mathbb{R}_+ \times M) \)
\[
\int_M \left| \sum_{\tau_k \in I} m(\delta_{\tau_k}, x) a_k e_{\tau_k}(x) \right| dx \leq \left( \|m(0, \cdot)\|_{L^2(M)} + \int_0^\delta \| \frac{\partial}{\partial s} m(s, \cdot) \|_{L^2(M)} ds \right) \times \left( \sum_{\tau_k \in I} |a_k|^2 \right)^{1/2}. \tag{2.2.1}
\]

**Proof.** We shall use the fact that \( m(\delta_{\tau_k}, x) = m(0, x) + \int_0^{\delta} I_{[0, \delta_{\tau_k}]}(s) \frac{\partial}{\partial s} m(s, x) ds \), where \( I_{[0, \delta_{\tau_k}]}(s) \) is the indicator function of the the interval \([0, \delta_{\tau_k}] \subset [0, \delta] \). Therefore, by Minkowski’s inequality, the
left side of (2.2.1) is dominated by

\[ \int_M |m(0, x) \cdot \sum_{\tau_k \in I} a_k e_{\tau_k}(x) | \, dx + \int_M | \sum_{\tau_k \in I} \int_0^\delta \mathbb{1}_{[0, \delta_{\tau_k}]}(s) \frac{\partial}{\partial s} m(s, x) a_k e_{\tau_k}(x) \, ds \, dx \]

\[ \leq \int_M |m(0, x) \cdot \sum_{\tau_k \in I} a_k e_{\tau_k}(x) | \, dx + \int_0^\delta \left( \int_M \left| \frac{\partial}{\partial s} m(s, x) \right| \, ds \right) \leq \left( \sum_{\tau_k \in I} |a_k|^2 \right)^{1/2}, \]

as desired. \( \square \)

Next, recall that we mentioned that the kernel in (2.1.28) is a juxtaposition of the kernels of \( \tilde{\mathcal{L}}_\lambda(P^0) \) as well as resolvent-type kernels. To handle the former, we shall appeal to the following straightforward result.

**Lemma 4.** Let \( \tilde{\mathcal{L}}_\lambda(P^0) \) be defined by (2.1.17) and the analog of (2.1.23) involving \( P^0 \). Then the kernel of \( (P^0)^\mu \tilde{\mathcal{L}}_\lambda(P^0) \), \( \mu = 0, 1, 2, \ldots \) satisfies

\[ ((P^0)^\mu \tilde{\mathcal{L}}_\lambda(P^0))(x, y) = \sum_j \lambda_j^\mu \tilde{\mathcal{L}}_\lambda(\lambda_j) e_j^0(x) e_j^0(y) = O(\lambda^{n+\mu}), \]  

(2.2.2)

and, moreover,

\[ \|((P^0)^\mu \tilde{\mathcal{L}}_\lambda(P^0))(\cdot, y)\|_{L^2(M)} = O(\lambda^{n/2+\mu}), \]  

(2.2.3)

where the implicit constants are independent of \( \varepsilon \).

The proof of the lemma is very simple. First, by the pointwise Weyl formula of Avakumović [1], Levitan [12] and Hörmander [8] (see also [18]),

\[ \sum_{\lambda_j \in \ell \in \ell + 1} |e_j^0(x) e_j^0(y)| = O(\ell^{n-1}), \quad \ell \in \mathbb{N}. \]  

(2.2.4)

If we use this and (2.1.18), we obtain (2.2.2). To prove the other inequality, (2.2.3), we note that, by orthogonality

\[ \|((P^0)^\mu \tilde{\mathcal{L}}_\lambda(P^0))(\cdot, y)\|_{L^2(M)}^2 = \sum_j \lambda_j^{2\mu} (\tilde{\mathcal{L}}_\lambda(\lambda_j))^2 |e_j^0(y)|^2 = O(\lambda^{n+2\mu}), \]
by this argument, which is (2.2.3).

To deal with the contributions of resolvent type operators in the mixture (2.1.28) we shall need a couple more results. The first is bounds for cutoff resolvent operators for the free operator $H^0$.

**Lemma 5.** Fix $\eta \in C^\infty(\mathbb{R}_+)$ satisfying $\eta(s) = 0$ on $s \leq 2$ and $\eta(s) = 1$, $s > 4$. Then if we set for $\tau \gg 1$

$$R_\tau(x, y) = \sum_j \frac{\eta(\lambda_j/\tau)}{\lambda_j^2 - \tau^2} e_j^0(x) e_j^0(y).$$ (2.2.5)

we have

$$|R_\tau(x, y)| \leq C_N \tau^{n-2} h_n(\tau d_g(x, y)) (1 + \tau d_g(x, y))^{-N},$$ (2.2.6)

for any $N = 1, 2, 3, \ldots$, if $h_n(r)$ is as in (1.1.4). The constant $C_N$ depends on $N$, $(M, g)$ and finitely many derivatives of $\eta$.

Here we are abusing the notation a bit. In (2.2.6) we mean that the inequality holds near the diagonal (so that $d_g(x, y)$ is well-defined) and that outside of this neighborhood of the diagonal $R_\tau(x, y)$ is $O(\tau^{-N})$ for all $N$. We shall state certain inequalities in this manner in what follows.

To verify (2.2.6), we note that the integral operator $R_\tau$ arising from the kernel $R_\tau(x, y)$ is

$$\tau^{-2} m(P^0/\tau),$$

where

$$m(\mu) = \frac{\eta(|\mu|)}{\mu^2 - 1}.$$  

Thus, $m$ is a symbol of order -2, i.e.,

$$\partial^j_\mu m(\mu) = O((1 + \mu)^{-2-j}),$$  

$j = 0, 1, 2, \ldots$.

As a result, one can use the arguments in [19, §4.3] to see that (2.2.6) is valid. Indeed, modulo lower order terms, $R_\tau(x, y)$ equals

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \tau^{-2} \frac{\eta(|\xi|/\tau)}{(|\xi|/\tau)^2 - 1} e^{i d_g(x, y) \xi_1} d\xi,$$

near the diagonal, which satisfies the bounds in (2.2.6), while outside of a fixed neighborhood of the diagonal $R_\tau(x, y) = O(\tau^{-N})$ for all $N$.

We also need bounds for the kernels of $(H_V)^{-j}$. 

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Lemma 6. Let \((H_V)^{-j}(x, y) = \sum_k \tau_k^{-2j} e_{\tau_k}(x)e_{\tau_k}(y)\) be the kernel of \((H_V)^{-j}\), \(j = 1, 2, \ldots\). Then if \(h_n(r)\) is as in (1.1.4)

\[
(H_V)^{-1}(x, y) \lesssim \begin{cases} 
  h_n(d_g(x, y)), & \text{if } d_g(x, y) \leq \text{Inj } (M)/2, \\
  1, & \text{otherwise}. 
\end{cases} 
\]

(2.2.7)

Furthermore, if \(n \geq 5\) and \(j < n/2\), \(j \in \mathbb{N}\) we have

\[
(H_V)^{-j}(x, y) \lesssim \begin{cases} 
  (d_g(x, y))^{-n+2j}, & \text{if } d_g(x, y) \leq \text{Inj } (M)/2, \\
  1 & \text{otherwise}. 
\end{cases} 
\]

(2.2.8)

To prove (2.2.7) or (2.2.8), we note that

\[
(H_V)^{-j}(x, y) = \int_0^\infty t^{-j} (e^{-tH_V})(x, y) \, dt. 
\]

(2.2.9)

We then use the heat kernel estimates of Li and Yau [13] \((V \in C^\infty)\) and Sturm [21, (4.14) Corollary] \((V \in \mathcal{K}(M))\), which say that for \(0 < t \leq 1\) there is a uniform constant \(c = c_{M,V} > 0\) so that

\[
(e^{-tH_V})(x, y) \lesssim \begin{cases} 
  t^{-n/2}\exp(-c(d_g(x, y))^2/t), & \text{if } d_g(x, y) \leq \text{Inj } (M)/2, \\
  1 & \text{otherwise}. 
\end{cases} 
\]

(2.2.10)

Here we do not need \(V^+ \in \mathcal{K}(M)\), since by Feynman-Kac formula, \(e^{-tH_V}|f|\) is monotone decreasing as \(V^+\) increases, so \(V^+ \in L^1(M)\) will not affect the bound in (2.2.10).

As a consequence of (2.2.10), we have for \(0 < t \leq 1\)

\[
\int_M |(e^{-tH_V})(x, y)|^2 \, dy \lesssim t^{-\frac{n}{2}}. 
\]

By Schwarz’s inequality, we have \(\|e^{-tH_V}\|_{L^2 \rightarrow L^\infty} \lesssim t^{-\frac{n}{4}}\). If we consider the kernels of the dyadic spectral projection operators

\[
\hat{\chi}_\lambda^V(x, y) = \sum_{\tau_k \in [\lambda, 2\lambda)} e_{\tau_k}(x)e_{\tau_k}(y), 
\]

(2.2.11)
for $H_V$, then, by the spectral theorem, we have

$$\| \hat{\chi}_\lambda^V \|_{L^2 \to L^\infty} \lesssim \| e^{-\lambda^2 H_V} \|_{L^2 \to L^\infty} \lesssim \lambda^{\frac{n}{2}},$$

which, along with the Cauchy-Schwarz inequality, implies

$$\sup_{x,y \in M} \left| \sum_{\tau_k \in [\lambda, 2\lambda)} e^{\tau_k(x)} e^{\tau_k(y)} \right| \leq \sup_{x \in M} \sum_{\tau_k \in [\lambda, 2\lambda)} |e^{\tau_k(x)}|^2 = \| \hat{\chi}_\lambda^V \|_{L^2 \to L^\infty}^2 \lesssim \lambda^n. \quad (2.2.12)$$

Since the eigenvalues of $H^V$ are all $\geq 1$, by (2.2.12) we have

$$(e^{-tH_V})(x,y) \lesssim e^{-t/2}, \quad t > 1. \quad (2.2.13)$$

If we use (2.2.10), (2.2.13) along with (2.2.9), we obtain (2.2.7) and (2.2.8).
To prove Proposition 2, which, as noted, implies our main result, Theorem 1, we shall split things into three different cases that require slightly different arguments. Specifically, we shall first handle the contribution of frequencies \( \tau_k \) which are comparable to \( \lambda \), and then those that are relatively small followed by ones that are relatively large.

### 3.1 Handling the contribution of comparable frequencies

In this subsection we shall handle frequencies \( \tau_k \) which are comparable to \( \lambda \), which one would expect to be the main contribution to the Weyl error term in (1.2.4). Specifically, we shall prove the following.

**Proposition 3.** Let \( V \in L^1(M) \) with \( V^- \in K(M) \), and \( \tilde{I}_\lambda(\tau) \) be defined as in (2.1.17) with \( \varepsilon = \varepsilon(\lambda) \) satisfying (2.1.1). Then we have

\[
\left| \sum_j \sum \{k: \tau_k \in [\lambda/2, 10\lambda]\} \int_M \frac{\tilde{I}_\lambda(\lambda_j) - \tilde{I}_\lambda(\tau_k)}{\lambda_j - \tau_k^2} e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) dxdy \right| \leq C_V \varepsilon^{-1} \lambda^{\frac{n-2}{2}},
\]

for some constant \( C_V \) depending on \( V \).
To prove Proposition 3, let us fix a Littlewood-Paley bump function $\beta \in C_0^\infty((1/2, 2))$ satisfying

$$
\sum_{\ell=-\infty}^{\infty} \beta(2^{-\ell}s) = 1, \quad s > 0.
$$

Let $\ell_0 \leq 0$ be the largest integer such that $2^{\ell_0} \leq \epsilon$, and set

$$
\beta_{\ell_0}(s) = \sum_{\ell \leq \ell_0} \beta(2^{-\ell}|s|) \in C_0^\infty((-2, 2)),
$$

and

$$
\hat{\beta}(s) = s^{-1}\beta(|s|) \in C_0^\infty(\{|s| \in (1/2, 2)\}).
$$

We then write for $\lambda/2 \leq \tau \leq 10\lambda$

$$
K_\tau(x, y) = \sum_j \frac{\tilde{1}_\lambda(\lambda_j) - \tilde{1}_\lambda(\tau)}{\lambda_j - \tau^2} \frac{\beta_{\ell_0}(\lambda_j - \tau)}{\lambda_j + \tau} e_j^0(x)e_j^0(y)
$$

(3.1.2)

$$
= \sum_j \frac{\tilde{1}_\lambda(\lambda_j) - \tilde{1}_\lambda(\tau)}{\lambda_j - \tau} \frac{\beta_{\ell_0}(\lambda_j - \tau)}{\lambda_j + \tau} e_j^0(x)e_j^0(y)
$$

$$
+ \sum_{\{t \in \mathbb{Z}: 2^t \leq \lambda/100\}} \left( \sum_j \frac{2^{-t}\hat{\beta}(2^{-t}(\lambda_j - \tau))}{\lambda_j + \tau} (\tilde{1}_\lambda(\lambda_j) - \tilde{1}_\lambda(\tau)) e_j^0(x)e_j^0(y) \right)
$$

$$
+ \sum_j \left( \frac{\sum_{\{t \in \mathbb{Z}: 2^t > \lambda/100\}} \beta(2^{-t}(\lambda_j - \tau))}{\lambda_j^2 - \tau^2} \right) (\tilde{1}_\lambda(\lambda_j) - \tilde{1}_\lambda(\tau)) e_j^0(x)e_j^0(y).
$$

Next, let

$$
K_{\tau, \ell_0}(x, y) = \sum_j \frac{\tilde{1}_\lambda(\lambda_j) - \tilde{1}_\lambda(\tau)}{\lambda_j - \tau} \frac{\beta_{\ell_0}(\lambda_j - \tau)}{\lambda_j + \tau} e_j^0(x)e_j^0(y),
$$

$$
R_{\tau, \ell}(x, y) = \sum_j \frac{2^{-t}\hat{\beta}(2^{-t}(\lambda_j - \tau))}{\lambda_j + \tau} e_j^0(x)e_j^0(y), \quad \text{if } \epsilon < 2^\ell \leq \lambda/100,
$$

and

$$
R_{\tau, \infty}(x, y) = \sum_j \left( \frac{\sum_{\{t \in \mathbb{Z}: 2^t > \lambda/100\}} \beta(2^{-t}(\lambda_j - \tau))}{\lambda_j^2 - \tau^2} \right) e_j^0(x)e_j^0(y).
$$

Also, for $\epsilon < 2^\ell \leq \lambda/100$ let

$$
K_{\tau, \ell}^-(x, y) = \sum_j \frac{2^{-t}\hat{\beta}(2^{-t}(\lambda_j - \tau))}{\lambda_j + \tau} (\tilde{1}_\lambda(\lambda_j) - 1) e_j^0(x)e_j^0(y)
$$

$$
K_{\tau, \ell}^+(x, y) = \sum_j \frac{2^{-t}\hat{\beta}(2^{-t}(\lambda_j - \tau))}{\lambda_j + \tau} \tilde{1}_\lambda(\lambda_j) e_j^0(x)e_j^0(y),
$$

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and, finally,

\[
K_{\tau,\infty}^-(x, y) = \sum_j \left( \sum_{\ell \in \mathbb{Z} : 2^{\ell} > \lambda/100} \frac{\beta(2^{-\ell}(\lambda_j - \tau))}{\lambda_j^2 - \tau^2} \right) \left( \tilde{\lambda}_\lambda(\lambda_j) - 1 \right) e_j^0(x)e_j^0(y)
\]

\[
K_{\tau,\infty}^+(x, y) = \sum_j \left( \sum_{\ell \in \mathbb{Z} : 2^{\ell} > \lambda/100} \frac{\beta(2^{-\ell}(\lambda_j - \tau))}{\lambda_j^2 - \tau^2} \right) \hat{\lambda}_\lambda(\lambda_j) e_j^0(x)e_j^0(y).
\]

If \( K \) is as in (3.1.2), our current task, (3.1.1), is to show that

\[
\left| \sum_{\tau_k \in [\lambda/2, 10\lambda]} \iint K_{\tau_k}(x, y) \epsilon_{\tau_k}(x) \epsilon_{\tau_k}(y) V(y) \, dx \, dy \right| \leq C_V \varepsilon^{-1} \lambda^{n-\frac{3}{2}}. \tag{3.1.3}
\]

To prove this, we note that we can write

\[
K_{\tau}(x, y) = K_{\tau,\ell_0}(x, y) + \sum_{\ell \in \mathbb{Z} : \varepsilon < 2^\ell \leq \lambda/100} K_{\tau,\ell}^-(x, y) + K_{\tau,\infty}^-(x, y)
\]

\[
+ \sum_{\ell \in \mathbb{Z} : \varepsilon < 2^\ell \leq \lambda/100} R_{\tau,\ell}(x, y) \left( 1 - \tilde{\lambda}_\lambda(\tau) \right) + R_{\tau,\infty}(x, y) \left( 1 - \tilde{\lambda}_\lambda(\tau) \right), \tag{3.1.4}
\]

or

\[
K_{\tau}(x, y) = K_{\tau,\ell_0}(x, y) + \sum_{\ell \in \mathbb{Z} : \varepsilon < 2^\ell \leq \lambda/100} K_{\tau,\ell}^+(x, y) + K_{\tau,\infty}^+(x, y)
\]

\[
- \sum_{\ell \in \mathbb{Z} : \varepsilon < 2^\ell \leq \lambda/100} R_{\tau,\ell}(x, y) \tilde{\lambda}_\lambda(\tau) - R_{\tau,\infty}(x, y) \tilde{\lambda}_\lambda(\tau). \tag{3.1.5}
\]

We shall use (3.1.4) to handle the summands in (3.1.3) with \( \tau = \tau_k \in [\lambda/2, \lambda] \) and (3.1.5) to handle those with \( \tau = \tau_k \in (\lambda, 10\lambda) \).

For \( \ell \in \mathbb{Z} \) with \( \varepsilon < 2^\ell \leq \lambda/100 \), let for \( j = 0, 1, 2, \ldots \)

\[
I_{\ell,j}^- = (\lambda - (j + 1)2^\ell, \lambda - j2^\ell] \quad \text{and} \quad I_{\ell,j}^+ = (\lambda + j2^\ell, \lambda + (j + 1)2^\ell]. \tag{3.1.6}
\]

Then to use the \( \delta_\tau \)-Lemma (Lemma 3), we shall use the following result whose proof we momentarily postpone.

**Lemma 7.** If \( \ell \in \mathbb{Z}, \varepsilon < 2^\ell \leq \lambda/100, \) and \( j = 0, 1, 2, \ldots, \) we have for each \( N \in \mathbb{N} \)

\[
\| K_{\tau,\ell}^\pm(\cdot, y) \|_{L^2(M)}, \| 2^\ell \frac{\partial}{\partial \tau} K_{\tau,\ell}^\pm(\cdot, y) \|_{L^2(M)}
\]

\[
\lesssim \varepsilon^{-1/2} \lambda^{\frac{n+1}{2} - 1} 2^{-\ell/2} (1 + j)^{-N}, \quad \tau \in I_{\ell,j}^\pm \cap [\lambda/2, 10\lambda]. \tag{3.1.7}
\]
Also,

\[ \|K_{\tau,\ell_0}(\cdot, y)\|_{L^2(M)}, \|\varepsilon \frac{\partial}{\partial \tau} K_{\tau,\ell_0}(\cdot, y)\|_{L^2(M)} \lesssim \varepsilon^{-1} \lambda^{\frac{n-1}{2}} (1 + j)^{-N}, \quad \tau \in I_{\ell_0,j}^\pm \cap [\lambda/2, 10\lambda], \quad (3.1.8) \]

\[ \|K_{\tau,\infty}^+(\cdot, y)\|_{L^2(M)}, \|\lambda \frac{\partial}{\partial \tau} K_{\tau,\infty}^+(\cdot, y)\|_{L^2(M)} \lesssim \lambda^{\frac{n}{2}-2}, \quad \tau \in [\lambda, 10\lambda], \quad (3.1.9) \]

and we can write

\[ K_{\tau,\infty}^-(x, y) = \tilde{K}_{\tau,\infty}^-(x, y) + H_{\tau,\infty}^-(x, y), \]

where for \( \tau \in [\lambda/2, \lambda] \)

\[ \|\tilde{K}_{\tau,\infty}^-(\cdot, y)\|_{L^2(M)}, \|\lambda \frac{\partial}{\partial \tau} \tilde{K}_{\tau,\infty}^-(\cdot, y)\|_{L^2(M)} \lesssim \lambda^{\frac{n}{2}-2} \]

\[ |H_{\tau,\infty}^-(x, y)| \lesssim \lambda^{n-2} h_n(\lambda d_g(x, y))(1 + \lambda d_g(x, y))^{-N}, \quad (3.1.10) \]

where \( h_n \) is as in (1.1.4). Finally, we also have for \( \varepsilon < 2^\ell \leq \lambda/100 \) and \( \tau \in [\lambda/2, 10\lambda] \)

\[ \|R_{\tau,\ell}(\cdot, y)\|_{L^2(M)}, \|2^\ell \frac{\partial}{\partial \tau} R_{\tau,\ell}(\cdot, y)\|_{L^2(M)} \lesssim \varepsilon^{-1/2} \lambda^{\frac{n-1}{2}-\ell/2}, \quad (3.1.11) \]

and

\[ |R_{\tau,\infty}(x, y)| \lesssim \lambda^{n-2} h_n(\lambda d_g(x, y))(1 + \lambda d_g(x, y))^{-N}. \quad (3.1.12) \]

As before, we are abusing notation a bit. First, in (3.1.7) we mean that if \( K_{\tau,\ell} \) equals \( K_{\tau,\ell}^+ \) or \( K_{\tau,\ell}^- \) then the bounds in (3.1.7) for \( \tau \) in \( I_{\ell,j}^+ \cap [\lambda, 10\lambda] \) or \( I_{\ell,j}^- \cap [\lambda/2, \lambda] \), respectively. Also, in both the second inequality in (3.1.10) and in (3.1.12) we mean that the kernels satisfy the bounds when \( x \) is sufficiently close to \( y \) (so that \( d_g(x, y) \) is well-defined) and that they are \( O(\lambda^{-N}) \) away from the diagonal.

We shall also need the following lemma for the proof of Proposition 3.

**Lemma 8.** Let \( I = [a_0, a_0 + \gamma] \) be an interval of length \( \gamma \leq \lambda \), and assume that for any fixed \( \tau \in I \cap [\lambda/2, 10\lambda], \) \( w_\tau(x, y) \in C^1(\mathbb{R} \times M \times M) \) satisfies

\[ \|w_\tau(\cdot, y)\|_{L^2(M)}, \|\gamma \frac{\partial}{\partial \tau} w_\tau(\cdot, y)\|_{L^2(M)} \leq L, \quad (3.1.13) \]
for some constant $L$. Then if $\beta \in C^\infty(\mathbb{R})$ and $V \in L^1(M)$, we have

$$
\left| \sum_{\lambda \leq \tau_k \leq 10\lambda} \int \int w_{\tau_k}(x, y) \beta(\tau_k) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \, dx \right| \lesssim \sup_{\tau \in I \cap [\lambda/2, 10\lambda]} |\beta(\tau)| \cdot L \lambda^{n/2}, \quad (3.1.14)
$$

Proof. For any fixed $y \in M$, by applying Lemma 3 with $\delta = \gamma$, $m(\tau, x) = w_{\tau+a}(x, y)$ and $a_k = \beta(\tau_k) e_{\tau_k}(y)$, we have

$$
\left| \sum_{\tau_k \in I \cap [\lambda/2, 10\lambda]} \int \int w_{\tau_k}(x, y) \beta(\tau_k) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \, dx \right|
\leq \|V\|_{L^1} \cdot \sup_y \left\| \sum_{\tau_k \in I \cap [\lambda/2, 10\lambda]} w_{\tau_k}(x, y) e_{\tau_k}(x) e_{\tau_k}(y) \right\|_{L^1(\mathbb{R})}
\leq \|V\|_{L^1} \cdot \sup_y \left( \|w_{\lambda, \tau_k}(\cdot, y)\|_{L^2(M)} + \int_0^{\gamma} \|\partial_\tau w_{\lambda+a_0}(\cdot, y)\|_{L^2(M)} \, ds \right)
\times \left( \sum_{\tau_k \in I \cap [\lambda/2, 10\lambda]} |\beta(\tau_k) e_{\tau_k}(y)|^2 \right)^{1/2}
\lesssim \sup_{\tau \in I \cap [\lambda/2, 10\lambda]} |\beta(\tau)| \cdot L \lambda^{n/2}.
$$

In the second to last inequality we used (3.1.13) and the fact that, by (2.2.12),

$$
\sum_{\tau_k \in [\lambda/2, 10\lambda]} |e_{\tau_k}(y)|^2 \lesssim \lambda^n, \quad \lambda \geq 1. \quad (3.1.15)
$$

Proof of Proposition 3. First, if we apply Lemma 8 with $w_{\tau}(x, y) = K_{\tau, \ell}^\pm(x, y)$, $\gamma = 2^\ell$ and $\beta(\tau) \equiv 1$, by (3.1.7)

$$
\left| \sum_{\tau_k \in I \cap [\lambda, 10\lambda]} \int \int K_{\tau_k, \ell}(x, y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \, dx \right| \lesssim \varepsilon^{-1/2} \lambda^{-\frac{3}{2}} 2^{-\ell/2} \cdot (1 + j)^{-N} \quad (3.1.16)
$$

If we sum over $j = 0, 1, 2, \ldots$, we see that for $\varepsilon < 2^\ell \leq \lambda/100$, (3.1.16) yields

$$
\left| \sum_{\lambda/2 \leq \tau_k \leq \lambda} \int \int K_{\tau_k, \ell}^+(x, y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dx \, dy \right| + \left| \sum_{\lambda/2 \leq \tau_k \leq \lambda} \int \int K_{\tau_k, \ell}^-(x, y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dx \, dy \right| \lesssim \varepsilon^{-1/2} \lambda^{-\frac{3}{2}} 2^{-\ell/2}. \quad (3.1.17)
$$
If we take $w_{\tau}(x,y) = K_{\tau,\ell_0}(x,y)$, $\gamma = \varepsilon$ and $\beta(\tau) \equiv 1$ in Lemma 8, this argument also gives

$$
\left| \sum_{\lambda/2 \leq \tau_0 \leq \lambda} \left\langle \int K_{\tau_0,\ell_0}(x,y) e_{\tau_0}(x) e_{\tau_0}(y) V(y) \, dx \right\rangle \right|
+ \left| \sum_{\lambda < \tau_0 \leq 10\lambda} \left\langle \int K_{\tau_0,\ell_0}(x,y) e_{\tau_0}(x) e_{\tau_0}(y) V(y) \, dx \right\rangle \right| \lesssim \varepsilon^{-1} \lambda^{n-2}. \quad (3.1.18)
$$

Similarly, if we use Lemma 8 along with (3.1.9) we find that

$$
\left| \sum_{\lambda < \tau_0 \leq 10\lambda} \left\langle \int K_{\tau_0,\ell_0}^+(x,y) e_{\tau_0}(x) e_{\tau_0}(y) V(y) \, dx \right\rangle \right|
\lesssim \lambda^{\frac{3}{2}-2} \|V\|_{L^1} \left( \sum_{\tau_k \in [\lambda/2, 10\lambda]} |e_{\tau_k}(y)|^2 \right)^{1/2} \lesssim \lambda^{n-2}, \quad (3.1.19)
$$

using (3.1.15) for the last inequality.

Next, since $R_{\tau,\ell}$ enjoys the bounds in (3.1.11), we can use Lemma 8 with $w_{\tau}(x,y) = R_{\tau,\ell}(x,y)$, $\gamma = 2^\ell$ and $\beta(\tau) = \tilde{I}_\lambda(\tau)$ to see that for $\varepsilon < 2^\ell \leq \lambda/100$ we have

$$
\left| \sum_{\tau_k \in I_{\ell,j}^r \cap (\lambda, 10\lambda)} \left\langle \int R_{\tau_0,\ell}(x,y) \tilde{I}_\lambda(\tau_k) e_{\tau_0}(x) e_{\tau_0}(y) V(y) \, dx \right\rangle \right|
\lesssim \|V\|_{L^1} \cdot 2^{-\ell/2} \lambda^{\frac{n+1}{2}-1} \sup_{y} \left( \sum_{\tau_k \in I_{\ell,j}^r \cap (\lambda, 10\lambda)} |\tilde{I}_\lambda(\tau_k) e_{\tau_k}(y)|^2 \right)^{1/2}
\lesssim \varepsilon^{-1/2} \lambda^{n-2} 2^{-\ell/2} \|V\|_{L^1} \cdot (1+j)^{-N},
$$
since $\tilde{I}_\lambda(\tau_k) = O((1+j)^{-N})$ if $\tau_k \in I_{\ell,j}^r$. Summing over this bound over $j$ of course yields

$$
\left| \sum_{\lambda < \tau_0 \leq 10\lambda} \left\langle \int R_{\tau_0,\ell}(x,y) \tilde{I}_\lambda(\tau_k) e_{\tau_0}(x) e_{\tau_0}(y) V(y) \, dx \right\rangle \right| \lesssim \varepsilon^{-1/2} \lambda^{n-2} 2^{-\ell/2}. \quad (3.1.20)
$$

The same argument gives

$$
\left| \sum_{\lambda/2 \leq \tau \leq \lambda} \left\langle \int R_{\tau,\ell}(x,y) (1 - \tilde{I}_\lambda(\tau_k)) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dx \right\rangle \right| \lesssim \varepsilon^{-1/2} \lambda^{n-2} 2^{-\ell/2}. \quad (3.1.21)
$$

Also, by (3.1.12) we have

$$
\sup_{y} \int \sup_{\lambda/2 \leq \tau \leq 10\lambda} |R_{\tau,\ell}(x,y)| \, dx \lesssim \lambda^{-2},
$$
\( \sum_{\tau_k \in (\lambda, 10\lambda)} e_{\tau_k}(x) e_{\tau_k}(y) \leq \lambda^n \), we have

\[
\left| \sum_{\tau_k \in (\lambda, 10\lambda)} \int R_{\tau_k, \infty}(x, y) \tilde{f}_\lambda(x) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dx \, dy \right| \leq \lambda^{n-2} \| V \|_{L^1},
\]

(3.1.22)

and since (3.1.15) yields \( \sum_{\tau_k \leq 10\lambda} |e_{\tau_k}(x) e_{\tau_k}(y)| \lesssim \lambda^n \), we have

\[
\left| \sum_{\tau_k \in [\lambda/2, \lambda]} \int R_{\tau_k, \infty}(x, y) (1 - \tilde{f}_\lambda(x)) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dx \, dy \right| \leq \lambda^{n-2} \| V \|_{L^1},
\]

(3.1.23)

If \( H_{-\infty}^{-} \) is as in (3.1.10) this argument also gives us

\[
\left| \sum_{\lambda/2 \leq \tau_k \leq \lambda} \int H_{-\infty}^{-}(x, y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dx \, dy \right| \lesssim \lambda^{n-2} \| V \|_{L^1},
\]

while the proof of (3.1.19) along with the first part of (3.1.10) yields

\[
\left| \sum_{\lambda/2 \leq \tau_k \leq \lambda} \int K_{-\infty}^{-}(x, y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dx \, dy \right| \lesssim \lambda^{n-2} \| V \|_{L^1}.
\]

Since \( K_{-\infty}^{-} = \tilde{K}_{-\infty}^{-} + H_{-\infty}^{-} \), we deduce

\[
\left| \sum_{\lambda/2 \leq \tau_k \leq \lambda} \int K_{-\infty}^{-}(x, y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dx \, dy \right| \lesssim \lambda^{n-2} \| V \|_{L^1}.
\]

We now have assembled all the ingredients for the proof of (3.1.3). If we use (3.1.17), (3.1.18), (3.1.21), (3.1.22) and (3.1.23) along with (3.1.4), we conclude that the analog of (3.1.3) must be valid where the sum is taken over \( \tau_k \in [\lambda/2, \lambda] \). We similarly obtain the analog of (3.1.3) where the sum is taken over \( \tau_k \in (\lambda, 10\lambda] \) from (3.1.5) along with (3.1.17), (3.1.18), (3.1.19), (3.1.20) and (3.1.22).

From this, we deduce that (3.1.3) must be valid, which finishes the proof of Proposition 3.

To finish the present task we need to prove Lemma 3.1.2.

**Proof of Lemma 7.** Since, as defined in (1.1.9), \( \{e_j^0\}_{j=1}^\infty \) are orthonormal bases for \( L^2(M) \). By \( L^2 \) orthogonality, we have

\[
\left( \int_M \left| \sum_i a(\lambda_i, \tau) e_i^0(x) e_i^0(y) \right|^2 \, dx \right)^{1/2} = \left( \sum_i |a(\lambda_i, \tau) e_i^0(y)|^2 \right)^{1/2}, \forall y \in M.
\]

(3.1.24)

To prove the first inequality we note that if \( \tau \in I_{\ell_j}^+ \cap [\lambda/2, 10\lambda] \) with \( \beta(2^{-\ell}(\lambda_i - \tau)) \neq 0 \), then \( |\lambda_i - \tau| \leq 2^{\ell+1}, \lambda_i, \tau \approx \lambda \), and, in this case, we also have \( \tilde{f}_\lambda(\lambda_i) - 1 = \mathcal{O}(1 + |j|^{-N}) \) if \( \tau \in I_{\ell_j}^- \) and

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\( \tilde{\lambda}(\lambda_i) = O((1 + |j|)^{-N}) \) if \( \tau \in I_{t_o,j}^\pm \). Therefore, by (3.1.24) and (2.2.4), we have for \( \varepsilon < 2^\ell \leq \lambda/100 \)

\[
\| K^\pm_{\tau, t}(\cdot, y) \|_{L^2(M)} \lesssim (1 + |j|)^{-N} 2^{-\ell} \lambda^{-1} \left( \sum_{\{\nu: |\lambda_i - \tau| \leq 2^{\ell+1}\}} |e^0_\nu(y)|^2 \right)^{1/2}
\]

\[
\lesssim (1 + |j|)^{-N} 2^{-\ell} \lambda^{-1} \left( \sum_{\{\mu: |\mu - \tau| \leq 2^{\ell+1}\}} \mu^{-N} \right)^{1/2}
\]

\[
\leq \begin{cases} 
(1 + |j|)^{-N} 2^{-\ell} \lambda^{\frac{n-1}{2}} - 1, & \text{if } 2^\ell < 1, \\
(1 + |j|)^{-N} 2^{-\ell/2} \lambda^{\frac{n-1}{2}} - 1, & \text{if } 2^\ell \geq 1,
\end{cases}
\]

which gives us the first part of (3.1.7) since \( \max\{2^{-\ell}, 2^{-\ell/2}\} \leq \varepsilon^{-1/2} 2^{-\ell/2} \). In the second inequality, we used (2.2.4). The other inequality in (3.1.7) follows from this argument since

\[
\frac{\partial}{\partial \tau} \tilde{\beta}(2^{-\ell}(\lambda_i - \tau)) \frac{\lambda_i + \tau}{\lambda_i + \tau} = O(2^{-\ell} \lambda^{-1}),
\]

due to the fact that we are assuming that \( \varepsilon < 2^\ell \leq \lambda/100 \).

This argument in the proof of (3.1.7) also gives us (3.1.8) if we use the fact that \( \tau \to (\tilde{\lambda}(\tau) - \tilde{\lambda}(\mu))/(\tau^2 - \mu^2) \) is smooth if we define it as in (2.1.27) when \( \tau = \mu \) (which is consistent with (2.1.26)) and use the fact that

\[
\partial^k_\tau (\beta_{\varepsilon_0}(\lambda_i - \tau)(\tilde{\lambda}(\lambda_i) - \tilde{\lambda}(\tau))/(\lambda_i - \tau)) = O(\varepsilon^{-1-k}(1 + |j|)^{-N}), \quad k = 0, 1, \quad \tau \in I_{t_o,j}^\pm,
\]

and the fact that, if this expression is nonzero, we must have \( |\lambda_i - \tau| \leq 2\varepsilon \).

To prove (3.1.9) we use the fact that for \( k = 0, 1 \) we have for \( \tau \in (\lambda, 10\lambda) \)

\[
\left| \frac{\partial^k_\tau}{\partial^k_\tau} \left( \sum_{\{\ell \in \mathbb{N}: 2^\ell > \lambda/100\}} \frac{\beta(2^{-\ell}(\lambda_i - \tau))}{\lambda_i^2 - \tau^2} \right) \tilde{\lambda}(\lambda_i) \right| \lesssim \begin{cases} 
\lambda^{-2-k} & \text{if } \lambda_i \leq \lambda, \\
\lambda^{-2-k}(1 + \lambda_i - \lambda)^{-N} & \text{if } \lambda_i > \lambda.
\end{cases}
\]

Thus for \( k = 0, 1 \), by (3.1.24)

\[
\| (\lambda^2 \partial_{\tau})^k K^\pm_{\tau, \infty}(\cdot, y) \|_{L^2(M)} \lesssim \lambda^{-2} \left( \sum_{\lambda_i \leq \lambda} |e^0_\lambda(y)|^2 + \sum_{\lambda_i > \lambda} (1 + \lambda_i - \lambda)^{-N} |e^0_\lambda(y)|^2 \right)^{1/2} \lesssim \lambda^{-2 + \frac{2}{\ell}},
\]

as desired if \( N > 2n \), using (2.2.4) again.

Next we turn to the bounds in (3.1.10) for \( K^\pm_{\tau, \infty} \). To handle this, let \( \eta \) be as in Lemma 5 and
\[
H_{\tau, \infty}(x, y) = -\sum_i \left( \frac{\sum_{\ell \in \mathbb{N}: 2\ell > \lambda/100} \beta(2^{-\ell}(\lambda_i - \tau))}{\lambda_i^2 - \tau^2} \right) \eta(\lambda_i/\tau) e_i^0(x) e_i^0(y)
\]

assuming, as we may, that \(\lambda \gg 1\). The last equality comes from the properties of our Littlewood-Paley bump function, \(\beta\). We then conclude from Lemma 5 that \(H_{\tau, \infty}\) satisfies the bounds in (3.1.10). If we then set

\[
\tilde{K}_{\tau, \infty}(x, y) = \sum_i \left( \frac{\sum_{\ell \in \mathbb{N}: 2\ell > \lambda/100} \beta(2^{-\ell}(\lambda_i - \tau))}{\lambda_i^2 - \tau^2} \right) \eta(\lambda_i/\tau) e_i^0(x) e_i^0(y)
\]

we have \(K_{\tau, \infty} = \tilde{K}_{\tau, \infty} + H_{\tau, \infty}\), and, also, by the proof of (3.1.9), \(\tilde{K}_{\tau, \infty}\) satisfies the bounds in (3.1.10).

It just remains to prove the bounds in (3.1.11) for the \(R_{\tau, \ell}(x, y)\) and that in (3.1.12) for \(R_{\tau, \infty}(x, y)\). The former just follows from the proof of (3.1.7).

To prove the remaining inequality, (3.1.12), we note that if \(\eta\) is as above and we set

\[
\tilde{R}_{\tau, \infty}(x, y) = \sum_i \eta(\lambda_i/\tau) \left( \frac{\sum_{\ell \in \mathbb{N}: 2\ell > \lambda/100} \beta(2^{-\ell}(\lambda_i - \tau))}{\lambda_i^2 - \tau^2} \right) e_i^0(x) e_i^0(y),
\]

then, by Lemma 5, \(\tilde{R}_{\tau, \infty}\) satisfies the bounds in (3.1.12). Also, we have

\[
R_{\tau, \infty}(x, y) = R_{\tau, \infty}^0(x, y) + \tilde{R}_{\tau, \infty}(x, y),
\]

if

\[
R_{\tau, \infty}^0(x, y) = \sum_i (1 - \eta(\lambda_i/\tau)) \left( \frac{\sum_{\ell \in \mathbb{N}: 2\ell > \lambda/100} \beta(2^{-\ell}(\lambda_i - \tau))}{\lambda_i^2 - \tau^2} \right) e_i^0(x) e_i^0(y),
\]

(again using the properties of \(\beta\), and, since the proof of Lemma 5 shows that for \(\tau \in [\lambda/2, 10\lambda]\) we have

\[
|R_{\tau, \infty}^0(x, y)| \lesssim \tau^{n-2} (1 + \tau d_g(x, y))^{-N} \lesssim (1 + \lambda d_g(x, y))^{-N},
\]

we conclude that (3.1.12) must be valid, which completes the proof. \(\square\)
3.2 Handling the contribution of relatively large frequencies of $HV$

In this section we shall handle relatively large frequencies of $HV$ by proving the following.

**Proposition 4.** Let $V \in L^1(M)$ with $V^- \in \mathcal{K}(M)$, and $\tilde{i}_\lambda(\tau)$ be defined as in (2.1.17) with $\varepsilon$ satisfying (2.1.1). Then we have

\[
\left| \sum_j \sum_{\{k : \tau_k > 10\lambda\}} \int_M \int_M \frac{\tilde{i}_\lambda(\lambda_j) - \tilde{i}_\lambda(\tau_k)}{\lambda_j - \tau_k} e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dx \, dy \right| 
\leq C_V \lambda^{n-2} (\log \lambda)^{1/2}, \quad (3.2.1)
\]

for some constant $C_V$ depending on $V$ which is independent of $\varepsilon$.

To prove (3.2.1) fix

\[
\Psi \in C_0^\infty((1/2, 2)), \quad \text{with} \quad \Psi(s) = 1, \quad s \in [3/4, 5/4]. \quad (3.2.2)
\]

To proceed, assume that $\tau_k > 10\lambda$. Since, by the mean value theorem and (2.1.19)

\[
\frac{\tilde{i}_\lambda(\lambda_j) - \tilde{i}_\lambda(\tau_k)}{\lambda_j - \tau_k} = O(\tau_k^{-\sigma}) \quad \forall \sigma, \quad \text{if} \quad \lambda_j \in (\tau_k/2, 2\tau_k), \quad \tau_k > 10\lambda,
\]

by (2.2.4) and (3.1.15), to prove (3.2.1) it suffices to show that

\[
\left| \sum_j \sum_{\{k : \tau_k > 10\lambda\}} \int_M \int_M \frac{\tilde{i}_\lambda(\lambda_j) - \tilde{i}_\lambda(\tau_k)}{\lambda_j - \tau_k} (1 - \Psi(\lambda_j/\tau_k)) e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dx \, dy \right| 
\leq \|V\|_{L^1(M)} \lambda^{n-2} (\log \lambda)^{1/2}, \quad (3.2.3)
\]

since

\[
\left| \sum_j \sum_{\{k : \tau_k > 10\lambda\}} \int_M \int_M \frac{\tilde{i}_\lambda(\lambda_j) - \tilde{i}_\lambda(\tau_k)}{\lambda_j - \tau_k} \Psi(\lambda_j/\tau_k) e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dx \, dy \right| 
\leq \lambda^{-\sigma} \|V\|_{L^1(M)}, \quad \forall \sigma.
\]
As $\mathcal{I}_\lambda(\tau_k) = O(\tau_k^{-\sigma})$ for all $\sigma \in \mathbb{N}$ for $\tau_k > 10\lambda$ and, by Lemma 5,

$$\left| \sum_j \frac{(1 - \Psi(\lambda_j/\tau_k))}{\lambda_j^{\ell} - \tau_k} e_j^0(x) e_j^0(y) \right| \lesssim \begin{cases} \tau_k^{n-2} + (d_g(x,y))^2, & n \geq 3 \\ \log(2 + 1/(\tau_k d_g(x,y))), & n = 2, \end{cases}$$

the analog of (3.2.3) where we replace $(\mathcal{I}_\lambda - \mathcal{I}_\lambda(\tau_k))$ by $\mathcal{I}_\lambda(\tau_k)$ is trivial. Consequently, we would have (3.2.3) and consequently Proposition 4 if we could show that

$$\left| \sum_j \sum_{\{\kappa: \tau_k > 10 \lambda\}} \int \frac{(1 - \Psi(\lambda_j/\tau_k))}{\lambda_j^{\ell} - \tau_k} \mathcal{I}_\lambda(\lambda_j) e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x)e_{\tau_k}(y) dx dy \right|$$

$$\lesssim \|V\|_{L^1(M)} \lambda^{n-2}(\log \lambda)^{1/2}. \quad (3.2.4)$$

If $1 - \Psi(\lambda_j/\tau_k) \neq 0$ we have $\lambda_j \neq \tau_k$, and then can write

$$\frac{1}{\tau_k^{\ell} - \lambda_j^{\ell}} = \tau_k^{-2} + \tau_k^{-2}(\lambda_j/\tau_k)^2 + \cdots + \tau_k^{-2}(\lambda_j/\tau_k)^{2N-2} + (\lambda_j/\tau_k)^{2N} \frac{1}{\tau_k^{\ell} - \lambda_j^{\ell}}.$$ 

As a result, we would have (3.2.4) if we could choose $N \in \mathbb{N}$ so that we have

$$\left| \int \int \sum_j \lambda_j^{2\ell} \mathcal{I}_\lambda(\lambda_j) e_j^0(x) e_j^0(y) \left( \sum_{\tau_k > 10 \lambda} \tau_k^{-2-2\ell}(1 - \Psi(\lambda_j/\tau_k)) e_{\tau_k}(x)e_{\tau_k}(y) \right) V(y) dx dy \right|$$

$$\lesssim \|V\|_{L^1(M)} \lambda^{n-2}(\log \lambda)^{1/2}, \quad \ell = 0, \ldots, N - 1, \quad (3.2.5)$$

as well as

$$\left| \sum_j \sum_{\tau_k > 10 \lambda} \int \frac{(1 - \Psi(\lambda_j/\tau_k))}{\lambda_j^{\ell} - \tau_k} (\lambda_j)^{2N} \mathcal{I}_\lambda(\lambda_j) e_j^0(x) e_j^0(y) V(y) \tau_k^{-2N} e_{\tau_k}(x)e_{\tau_k}(y) dx dy \right|$$

$$\lesssim \lambda^{n-2}\|V\|_{L^1(M)}. \quad (3.2.6)$$

To handle (3.2.5) we start with a trivial reduction. We note that if $\tau_k > 10 \lambda$, then by (2.1.18),
(3.1.15) and (3.2.2)

\[ \left| \tilde{i}_\lambda(\lambda_j) \sum_{\tau_k > 10\lambda} \Psi(\lambda_j/\tau_k) \tau_k^{-2-2\ell} e_{\tau_k}(x)e_{\tau_k}(y) \right| \lesssim \left| \tilde{i}_\lambda(\lambda_j) \sum_{\tau_k \in (\lambda_j/2, 2\lambda_j)} \tau_k^{-2-2\ell} e_{\tau_k}(x)e_{\tau_k}(y) \right| \]

\[ \lesssim \lambda_j^{-\sigma} \sum_{\tau_k \geq \lambda_j} \tau_k^{-\sigma} \tau_k^{-2-2\ell} e_{\tau_k}(x)e_{\tau_k}(y) \]

\[ \lesssim \lambda_j^{n-2\ell-2\sigma}, \]

for any \( \sigma \). If \( \sigma > n \), by (2.2.4) this yields

\[ \left| \int \int \sum_j \lambda_j^{2\ell} \tilde{i}_\lambda(\lambda_j) e_j^0(x)e_j^0(y) \left( \sum_{\tau_k > 10\lambda} \tau_k^{-2-2\ell} \Psi(\lambda_j/\tau_k) e_{\tau_k}(x)e_{\tau_k}(y) \right) V(y) dxdy \right| \]

\[ \lesssim \| V \|_{L^1(M)}, \]

which means that in order to prove (3.2.5) it suffices to show that

\[ \left| \int \int ((P^0)^{2\ell}[\tilde{i}_\lambda(P^0)])(x, y) \left( \sum_{\tau_k > 10\lambda} \tau_k^{-2-2\ell} e_{\tau_k}(x)e_{\tau_k}(y) \right) V(y) dxdy \right| \]

\[ \lesssim \| V \|_{L^1(M)} \lambda^{n-2}(\log \lambda)^{1/2}, \quad \ell = 0, \ldots, N - 1, \quad (3.2.7) \]

since

\[ \sum_j \lambda_j^{2\ell} \tilde{i}_\lambda(\lambda_j) e_j^0(x)e_j^0(y) = ((P^0)^{2\ell}[\tilde{i}_\lambda(P^0)])(x, y). \]

To prove this, in certain cases, we shall rewrite the expression inside the absolute value in the left side of (3.2.7) slightly. Specifically, we can split it into the following two terms

\[ \int \int ((P^0)^{2\ell}[\tilde{i}_\lambda(P^0)])(x, y) \left( \sum_{\tau_k \geq 1} \tau_k^{-2-2\ell} e_{\tau_k}(x)e_{\tau_k}(y) \right) V(y) dxdy \]

\[ - \int \int ((P^0)^{2\ell}[\tilde{i}_\lambda(P^0)])(x, y) \left( \sum_{\tau_k \leq 10\lambda} \tau_k^{-2-2\ell} e_{\tau_k}(x)e_{\tau_k}(y) \right) V(y) dxdy \]

\[ = I + II, \quad \text{if } \ell \leq (n - 4)/4 \text{ and } n \geq 4. \quad (3.2.8) \]

If \( n \leq 3 \) we shall not split things up in this way, and, instead, just deal with the expression in the left side of (3.2.7) directly.
Note that if \( n \geq 5 \) and \( \ell \leq (n-4)/4 \)

\[
|I| = \left| \iint ((P^0)^{2\ell} \mathbb{1}_\lambda(P^0))(x,y) \left( H_V \right)^{-1-\ell}(x,y) V(y) \, dx \, dy \right|
\]

\[
\leq \int_{d_y(x,y) \leq \lambda^{-1}} + \int_{d_y(x,y) \geq \lambda^{-1}} \left( \left| ((P^0)^{2\ell} \mathbb{1}_\lambda(P^0))(x,y) \right| \left| (H_V)^{-1-\ell}(x,y) \right| |V(y)| \right) \, dx \, dy
\]

\[
\lesssim \int_{d_y(x,y) \leq \lambda^{-1}} \lambda^{n+2\ell} (d_y(x,y))^{-n+2+2\ell} |V(y)| \, dx \, dy
\]

\[
+ \|V\|_{L^1} \cdot \sup_y \left( \int_{d_y(x,y) \geq \lambda^{-1}} \left| ((P^0)^{2\ell} \mathbb{1}_\lambda(P^0))(x,y) \right|^2 \, dx \right)^{1/2}
\]

\[
\times \left( \int_{d_y(x,y) \geq \lambda^{-1}} (d_y(x,y))^{-2(n-2-2\ell)} \, dx \right)^{1/2}
\]

\[
\lesssim \|V\|_{L^1} \cdot \lambda^{n+2\ell-(2-2\ell)} + \|V\|_{L^1} \cdot \left( \lambda^{2+2\ell} \cdot \lambda^{2-2-2\ell} \right)
\]

\[
= \lambda^{n-2} \|V\|_{L^1},
\]

which is better than the bounds in (3.2.7). Here we used Lemma 6 to bound \((H_V^{-1-\ell})(x,y)\) (and our momentary assumption \( \ell \leq (n-4)/4 \)). In the second inequality we also used Schwarz’s inequality, while in the second inequality and the second to last step we also used Lemma 4.

If \( n = 4 \) than the requirement in (3.2.8) forces \( \ell = 0 \). In this case, if we repeat the above arguments we obtain slightly worse bounds, i.e.,

\[
|I| \lesssim \lambda^{n-2} (\log \lambda)^{1/2} \|V\|_{L^1},
\]

with the \( \log \lambda \) factor coming from the fact that when \( n = 4 \) we have

\[
\int_{d_y(x,y) \geq \lambda^{-1}} (d_y(x,y))^{-1} \, dx \approx \log \lambda.
\]

On the other hand, this bound is in agreement with the one posited in (3.2.7).

We still need to handle the second term, \( II \), in (3.2.8). To do this we shall again use Lemma 4
and (3.1.15) along with Schwarz’s inequality to deduce that

\[ |II| \leq \|V\|_{L^1} \cdot \sup_y \left( \| (P^0)^{2\ell} \mathbf{1}_\lambda (P^0)(\cdot, y) \|_{L^2} \cdot \left\| \sum_{\tau_k \leq 10\lambda} \tau_k^{-2-2\ell} e_{\tau_k} (\cdot) e_{\tau_k}(y) \|_{L^2} \right) \right) \]

\[ \lesssim \|V\|_{L^1} \cdot \lambda^{\frac{1}{2}} + 2\ell \cdot \left( \sum_{\tau_k \leq 10\lambda} \tau_k^{-4-4\ell}|e_{\tau_k}(y)|^2 \right)^{1/2} \]

\[ \lesssim \|V\|_{L^1} \cdot \lambda^{\frac{1}{2}} + 2\ell \cdot \left( \sum_{\{j \in \mathbb{N}: 2j \leq 10\lambda \}} 2^{-j(4+4\ell)2^{n_j}} \right)^{1/2} \]

\[ \lesssim \|V\|_{L^1} \cdot \lambda^{\frac{1}{2}} + 2\ell \cdot \lambda^{-2-2\ell} + \frac{2}{\lambda} \]

\[ = \|V\|_{L^1} \cdot \lambda^{n-2}, \]

assuming in the last step \( \ell < (n - 4)/4 \). In the remaining case covered in (3.2.8) where \( n \geq 4 \) and \( \ell = (n - 4)/4 \) (forcing \( n \) to be a multiple of 4), as was the case for \( \ell = 0 \) and \( n = 4 \), the bound is somewhat worse and we instead get, in this case,

\[ |II| \lesssim \lambda^{n-2}(\log \lambda)^{1/2} \|V\|_{L^1}, \]

which still better than that of our current goal, (3.2.1).

Since we have obtained favorable estimates for \( I \) and \( II \) in (3.2.8), we have shown that (3.2.7) is valid when \( n \geq 4 \) and \( \ell \leq (n - 4)/4 \). For the remaining cases where \( n = 2, 3 \) and \( 0 \leq \ell \leq N - 1 \) is arbitrary or \( (n - 4)/4 < \ell \leq N - 1 \) for \( n \geq 4 \), we shall just repeat the argument that we used to control \( II \). We have not specified \( N \); however, to get the other inequality, (3.2.6), that is needed to obtain our current goal (3.2.1), \( N \) will have to be chosen to be larger than \( (n - 4)/4 \).

In these remaining cases for (3.2.7) if we argue as above we find that the left side of (3.2.7) is dominated by

\[ \|V\|_{L^1} \cdot \sup_y \left( \| (P^0)^{2\ell} \mathbf{1}_\lambda (P^0)(\cdot, y) \|_{L^2} \cdot \left\| \sum_{\tau_k > 10\lambda} \tau_k^{-2-2\ell} e_{\tau_k} (\cdot) e_{\tau_k}(y) \|_{L^2} \right) \right) \]

\[ \lesssim \|V\|_{L^1} \cdot \lambda^{\frac{1}{2}} + 2\ell \cdot \left( \sum_{\tau_k > 10\lambda} \tau_k^{-4-4\ell}|e_{\tau_k}(y)|^2 \right)^{1/2} \]

\[ \lesssim \|V\|_{L^1} \cdot \lambda^{\frac{1}{2}} + 2\ell \cdot \left( \sum_{\{j \in \mathbb{N}: 2j > 10\lambda \}} 2^{-j(4+4\ell)2^{n_j}} \right)^{1/2} \]

\[ \lesssim \|V\|_{L^1} \cdot \lambda^{\frac{1}{2}} + 2\ell \cdot \lambda^{-2-2\ell} + \frac{2}{\lambda} \]

\[ = \lambda^{n-2} \|V\|_{L^1}, \]
using the fact that our current conditions ensure that $4 + 4\ell > n$. This completes the proof of (3.2.7) and hence (3.2.5).

To finish this subsection we need to show that we can fix $N \in \mathbb{N}$ sufficiently large so that (3.2.6) is valid. As we mentioned before we shall specify our $N > (n - 4)/4$ in a moment.

To prove (3.2.6) for large enough $N$, here too it will be convenient to split matters into two cases. First, let us deal with the sum in (3.2.6) where $\tau_k \geq \lambda^2$. We can handle this case using trivial methods if $N$ is large enough. In fact, by using Schwarz’s inequality, Lemma 4 and orthogonality, we see that for $\tau_k \geq \lambda^2$ we have the uniform bounds

$$
\int \left| \sum_j \frac{1 - \Psi(\lambda_j/\tau_k)}{\lambda_j^2 - \tau_k^2} (\lambda_j)^{2N} \mathbf{I}_\lambda(\lambda_j)e_j^0(x)e_j^0(y) \right| dx 
\lesssim \left\| \sum_j \frac{1 - \Psi(\lambda_j/\tau_k)}{\lambda_j^2 - \tau_k^2} (\lambda_j)^{2N} \mathbf{I}_\lambda(\lambda_j)e_j^0(\cdot)e_j^0(y) \right\|_{L^2} 
\lesssim \left\| (P^0)^{2N} \mathbf{I}_\lambda(P^0)(\cdot, y) \right\|_{L^2} \lesssim \lambda^{2N}.
$$

This does not present a problem since if $N$ is large, since, by (3.1.15) and the Cauchy-Schwartz inequality,

$$
\sum_{\tau_k \geq \lambda^2} \tau_k^{-2N} |e_{\tau_k}(x)e_{\tau_k}(y)| \lesssim \sum_{\{j \in \mathbb{N} : 2^j \geq \lambda^2\}} 2^{-2N} 2^{n_j} \lesssim \lambda^{-4N} \lambda^{2n},
$$

if $N > n$. Using these two inequalities we deduce that

$$
\left| \sum_j \sum_{\tau_k \geq \lambda^2} \int \frac{1 - \Psi(\lambda_j/\tau_k)}{\lambda_j^2 - \tau_k^2} (\lambda_j)^{2N} \mathbf{I}_\lambda(\lambda_j)e_j^0(x)e_j^0(y)V(y)\tau_k^{-2N}e_{\tau_k}(x)e_{\tau_k}(y) dxdy \right| 
\lesssim \lambda^{2N+2} \lambda^{-4N+2n} \|V\|_{L^1(M)} < \|V\|_{L^1},
$$

if we assume, as we may, that $N = 2n$.

Based on this, we would be done with handling relatively large frequencies of $H_V$ if we could show that

$$
\left| \sum_j \sum_{10\lambda < \tau_k < \lambda^2} \int \frac{1 - \Psi(\lambda_j/\tau_k)}{\lambda_j^2 - \tau_k^2} (\lambda_j)^{2N} \mathbf{I}_\lambda(\lambda_j)e_j^0(x)e_j^0(y)V(y)\tau_k^{-2N}e_{\tau_k}(x)e_{\tau_k}(y) dxdy \right| 
\lesssim \lambda^{n-2} \|V\|_{L^1(M)}, \quad \text{if } N = 2n. \quad (3.2.9)
$$

To this end, let $\Phi \in C^\infty(\mathbb{R}_+)$ satisfy $\Phi(s) = 1$, $s \leq 3/2$ and $\Phi(s) = 0$, $s \geq 2$. Then if $\tau_k \geq 10\lambda$,
it follows that
\[ \Phi(\lambda_j/\lambda)(1 - \Psi(\lambda_j/\tau_k)) = \Phi(\lambda_j/\lambda), \]
and also for all \( \sigma \in \mathbb{N} \)
\[ i\lambda(\lambda_j) \left(1 - \Phi(\lambda_j/\lambda)\right)(1 - \Psi(\lambda_j/\tau_k)) = O(\tau_k^{-\sigma}), \quad \text{if} \ 10\lambda \leq \tau_k \leq \lambda^2. \]

Thus, by an earlier argument, modulo \( O(\lambda^{-\sigma}\|V\|_{L^1}) \) \( \forall \sigma \in \mathbb{N} \), the left side of (3.2.9) agrees with the expression where we replace \( (1 - \Psi(\lambda_j/\tau_k)) \) with \( \Phi(\lambda_j/\lambda) \). Therefore since \( (\lambda_j^2 - \tau_k^2)^{-1} = -(1 - (\lambda_j/\tau_k)^2)^{-1} \cdot \tau_k^{-2} \), we would have (3.2.9) if we could show that
\[
\left\| \sum_{j} \sum_{10\lambda < \tau_k < \lambda^2} \int \frac{\Phi(\lambda_j/\lambda)}{1 - (\lambda_j^2/\tau_k)^2} (\lambda_j)^{2N} i\lambda(\lambda_j)e_j^0(x)e_j^0(y)V(y)\tau_k^{-2N-2}e_{\tau_k}(x)e_{\tau_k}(y) \, dx \, dy \right\|
\lesssim \lambda^{n-2}\|V\|_{L^1(M)}, \quad N = 2n. \tag{3.2.10}
\]

Since the left side is dominated by \( \|V\|_{L^1} \) times
\[
\sup_y \left| \int \left( \sum_j \frac{\Phi(\lambda_j/\lambda)}{1 - \tau_k^{-2}\lambda_j^2} (\lambda_j)^{2N} i\lambda(\lambda_j)e_j^0(x)e_j^0(y) \right) \times \left( \sum_{10\lambda < \tau_k < \lambda^2} \tau_k^{-2N-2}e_{\tau_k}(x)e_{\tau_k}(y) \right) \, dx \right|, \tag{3.2.11}
\]
it suffices to show that this expression is \( O(\lambda^{n-2}) \).

To do so, we shall appeal to the \( \delta_r \)-Lemma, Lemma 3. We set for a given \( y \in M \)
\[ m(s, x) = \sum_j \frac{\Phi(\lambda_j/\lambda)}{1 - s^2\lambda_j^2} (\lambda_j)^{2N} i\lambda(\lambda_j)e_j^0(x)e_j^0(y), \quad s \in [0, 1/10\lambda]. \]

Then, since
\[
\left| \frac{\Phi(\lambda_j/\lambda)}{1 - s^2\lambda_j^2} \right| \lesssim 1, \tag{3.2.12}
\]
and
\[
\left| \frac{\partial}{\partial s} \left( \frac{\Phi(\lambda_j/\lambda)}{1 - s^2\lambda_j^2} \right) \right| \lesssim s\lambda^2, \tag{3.2.13}
\]
if \( s \in [0, 1/10\lambda] \), it follows from orthogonality and Lemma 4 that
\[
\|m(0, \cdot)\|_{L^2(M)} + \int_0^{1/10\lambda} \| \frac{\partial}{\partial s} m(s, \cdot) \|_{L^2(M)} \, ds = O(\lambda^{2N+2}). \tag{3.2.14}
\]
Consequently, by Lemma 3, (3.2.11) is dominated by

\[\sum_{10\lambda \tau < \lambda^2} \tau_k^{-2N-2} e_{\tau_k}(\cdot) e_{\tau_k}(y) \|_{L^2} = \lambda^{n+2N} \left( \sum_{10\lambda < \tau < \lambda^2} \tau_k^{-2N-2} e_{\tau_k}(y) \right)^{1/2} \]

\[\lesssim \lambda^{n+2N} \left( \sum_{j \in \mathbb{N}} 2^{-(4N+4)j} \right)^{1/2} \]

\[\lesssim \lambda^{n+2N} \cdot \lambda^{-2N-2} \cdot \lambda^2 = \lambda^n,\]

using (3.1.15) in the second to last step and the fact that \(N = 2n\) in the final one. Thus, the quantity in (3.2.11) is \(O(\lambda^{-2})\), which, by the above, yields (3.2.10) and finishes the proof of Proposition 4.

### 3.3 Handling the contribution of relatively small frequencies of \(H_V\)

In this section we shall handle relatively small frequencies of \(H_V\) and prove the following result.

**Proposition 5.** Let \(V \in L^1(M)\) with \(V^- \in \mathcal{K}(M)\), and \(\tilde{1}_\lambda(\tau)\) be defined as in (2.1.17) with \(\varepsilon\) satisfying (2.1.1). Then we have

\[\left| \sum_j \sum_{\tau_k < \lambda/2} \int_M \int_M \tilde{1}_\lambda(\lambda_j) - \tilde{1}_\lambda(\tau) \frac{e_j(x)e_j(y)}{\lambda_j^2 - \tau_k^2} V(y)e_{\tau_k}(x)e_{\tau_k}(y) \, dx \, dy \right| \leq C_V \lambda^{n-2},\]

(3.3.1)

for some constant \(C_V\) depending on \(V\) which is independent of \(\varepsilon\).

If we combine this with Proposition 3 and Proposition 4 which handle frequencies which are comparable to \(\lambda\) and large compared to \(\lambda\), respectively, we obtain Proposition 2, which by the arguments in §2, yield our main result, Proposition 1.

**Proof of Proposition 5.** As in the earlier cases, we shall first handle a trivial case. To do so, we note that, by (2.1.19) and the mean value theorem,

\[\frac{\tilde{1}_\lambda(\lambda_j) - \tilde{1}_\lambda(\tau)}{\lambda_j^2 - \tau^2} = O(\lambda^{-\sigma}), \quad \forall \sigma \in \mathbb{N}, \quad 1 \leq \tau \leq \lambda/2 \quad \text{and} \quad \lambda_j \leq 7\lambda/8.\]

Also, by (2.2.4) and (3.1.15)

\[\sum_{\lambda_j \leq \lambda} |e_j^0(x)e_j^0(y)|, \quad \sum_{\tau_k < \lambda/2} |e_{\tau_k}(x)e_{\tau_k}(y)| \lesssim \lambda^n.\]

(3.3.2)
To use these and make our first reduction fix $a \in C^\infty(\mathbb{R}_+)$ satisfying

$$a(s) = 0, \quad s \leq \frac{3}{4} \quad \text{and} \quad a(s) = 1, \quad s \geq \frac{7}{8}.$$  

Using the preceding inequalities we see that in order to prove (3.3.1) it suffices to show that

$$\left| \sum_j \sum_{\tau_k < \lambda/2} \int_M \int_M \frac{\mathbf{1}_\lambda(\lambda j) - \mathbf{1}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} a(\lambda_j/\lambda) e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dx \, dy \right|$$

$$\leq C V \lambda^{n-2} \|V\|_{L^1(M)}, \quad (3.3.3)$$

due to the fact that the difference between the quantities inside the absolute values in the left side of (3.3.1) and that of (3.3.3) is $O(\lambda^{-\sigma} \|V\|_{L^1})$ for all $\sigma$.

For the next reduction, note that the proof of Lemma 5 yields

$$\left| \sum_j \frac{a(\lambda_j/\lambda)}{\lambda_j^2 - \tau_k^2} e_j^0(x) e_j^0(y) \right| \lesssim \begin{cases} (d_g(x,y))^2-n, & n \geq 3, \\ \log(2 + (d_g(x,y))^{-1}), & n = 2, \end{cases}$$

if $1 \leq \tau_k \leq \lambda/2$. Based on this and the second part of (3.3.2) and the fact that $1 - \mathbf{1}_\lambda(\tau_k) = O(\lambda^{-\sigma})$ for all $\sigma$ when $1 \leq \tau_k \leq \lambda/2$, we easily see that

$$\left| \sum_j \sum_{\tau_k < \lambda/2} \int_M \int_M \frac{1 - \mathbf{1}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} a(\lambda_j/\lambda) e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dx \, dy \right|$$

$$\leq \lambda^{-\sigma} \|V\|_{L^1(M)}, \quad \forall \sigma \in \mathbb{N}. \quad (3.3.4)$$

Consequently, we would have (3.3.3) if we could show that

$$\left| \sum_j \sum_{\tau_k < \lambda/2} \int_M \int_M \frac{a(\lambda_j/\lambda)}{\lambda_j^2 - \tau_k^2} (\mathbf{1}_\lambda(\lambda j) - 1) e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dx \, dy \right|$$

$$\leq C V \lambda^{n-2} \|V\|_{L^1(M)}. \quad (3.3.4)$$

We need to make one final reduction before we can appeal to the $\delta_r$–Lemma, Lemma 3. For this, let $\eta$ be as in Lemma 5, i.e., $\eta \in C^\infty(\mathbb{R}_+)$ with $\eta(s) = 0$ on $s \leq 2$ and $\eta(s) = 1$, $s > 4$. It then follows that

$$\eta(s) a(s) = \eta(s).$$
Consequently, we can write the quantity inside the absolute value in the left side of (3.3.4) as

\[ \sum_j \sum_{\tau_k < \lambda/2} \int_M \int_M \frac{\eta(\lambda_j/\lambda)}{\lambda_j^2 - \tau_k^2} \left( \hat{I}_\lambda(\lambda_j) - 1 \right) e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y) \, dxdy \]

\[ + \sum_j \sum_{\tau_k < \lambda/2} \int_M \int_M \frac{a(\lambda_j/\lambda)(1 - \eta(\lambda_j/\lambda))}{\lambda_j^2 - \tau_k^2} \left( \hat{I}_\lambda(\lambda_j) - 1 \right) e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y) \, dxdy \]

\[ = I + II. \]

Therefore, in order to prove (3.3.4), it suffices to show that both \(|I|\) and \(|II|\) are dominated by the right side of (3.3.4).

We can easily handle \(I\) without appealing to the \(\delta_r\)-lemma. Indeed since \(\tilde{I}_\lambda(\lambda_j) = O(\lambda_j^{-\sigma})\) for all \(\sigma\) if \(\eta(\lambda_j/\lambda) \neq 0\), we see that (2.2.4) yields

\[ \sum_j \eta(\lambda_j/\lambda) \left( \hat{I}_\lambda(\lambda_j) - 1 \right) e_j^0(x)e_j^0(y) = - \sum_j \eta(\lambda_j/\lambda) e_j^0(x)e_j^0(y) + O(\lambda^{-\sigma}), \quad \forall \sigma. \]

Consequently, by the second part of (3.3.2) we would have the desired bounds for \(I\) if we could show that

\[ \left| \int \int \sum_{\tau_k < \lambda/2} R_{\tau_k}(x,y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dxdy \right| \lesssim \lambda^{n-2}\|V\|_{L^1}, \quad (3.3.5) \]

where

\[ R_{\tau_k}(x,y) = \sum_j \frac{\eta(\lambda_j/\lambda)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y). \]

To use this we note that the proof of Lemma 5 implies that

\[ \sup_{1 \leq \tau_k < \lambda/2} |R_{\tau_k}(x,y)| \leq C_0 \lambda^{n-2}h_\eta(\lambda d_\eta)(x)(1 + \lambda d_\eta)(x,y)^{-\sigma}, \quad \forall \sigma, \]

and, therefore,

\[ \sup_y \int \sup_{1 \leq \tau_k < \lambda/2} |R_{\tau_k}(x,y)| \, dx \lesssim \lambda^{-2}. \]

Since, we always have \(\tau_k \geq 1\) by (1.1.7), by the second part of (3.3.2) we have

\[ \sup_y \left| \int \sum_{\tau_k < \lambda/2} R_{\tau_k}(x,y)e_{\tau_k}(x)e_{\tau_k}(y) \right| \, dx \lesssim \lambda^n \cdot \sup_y \int \sup_{1 \leq \tau_k < \lambda/2} |R_{\tau_k}(x,y)| \, dx \lesssim \lambda^{n-2}, \]

which clearly yields (3.3.5).

Since we have the desired estimate for \(I\) above, it only remains to prove the corresponding
estimate for $\mathcal{I}$. For this, let

$$m(s, x, y) = \sum_j \frac{b(\lambda_j)}{\lambda_j^2 - s^2} (i_\lambda(\lambda_j) - 1) e_j^0(x) e_j^0(y), \quad \text{with } b(s) = a(s)(1 - \eta(s)) \in C^\infty((3/4, 4)).$$

We then can rewrite this desired bound for $\mathcal{I}$ as follows

$$\left| \iint_{\tau_k < \lambda/2} m(\tau_k, x, y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dx \, dy \right| \lesssim \lambda^{n-2} \|V\|_{L^1}.$$

Just as the last step in the proof of Proposition 4 was to establish (3.2.11), the final step here would be to show that

$$\sup_y \int \left| \sum_{\tau_k < \lambda/2} m(\tau_k, x, y) e_{\tau_k}(x) e_{\tau_k}(y) \right| \, dx \lesssim \lambda^{n-2}. \quad (3.3.6)$$

To prove this we shall argue as in the very end of the last subsection and appeal to the $\delta$–Lemma 3 with the $\delta$ there equal to $\lambda/2$. We first note that by (2.2.4) and the fact that $b(\lambda_j/\lambda) \neq 0$ implies $\lambda_j/\lambda \in [3/4, 4]$. Consequently,

$$\left| \left( \frac{\partial}{\partial s} \right)^\ell \left( \frac{b(\lambda_j)}{\lambda_j^2 - s^2} \right) \right| \leq C\lambda^{-2}(s\lambda^{-2})^\ell, \quad \ell = 0, 1, \quad \text{if } 1 \leq s \leq \lambda/2.$$

Using this and the support properties $b$ we can easily see that by the proof of Lemma 4 that (2.2.4) and orthogonality yields for $\ell = 0, 1$

$$\left\| \left( \frac{\partial}{\partial s} \right)^\ell m(s, \cdot, y) \right\|_{L^2(M)} \leq C_0 \lambda^{\frac{n}{2} - 2}(s\lambda^{-2})^\ell, \quad \text{if } y \in M, 0 \leq s \leq \lambda/2.$$

Consequently,

$$\sup_y \left( \|m(1, \cdot, y)\|_{L^2(M)} + \int_1^{\lambda/2} \left\| \left( \frac{\partial}{\partial s} \right)m(s, \cdot, y) \right\|_{L^2(M)} \right) \lesssim \lambda^{\frac{n}{2} - 2}.$$

By Lemma 3 and the second part of (3.3.2) we deduce from this that the left side of (3.3.6) is dominated by

$$\lambda^{\frac{n}{2} - 2} \sup_y \left( \sum_{\tau_k < \lambda/2} \|e_{\tau_k}(y)\|^2 \right)^{1/2} \lesssim \lambda^{n-2},$$

which completes the proof. □
3.4 Proof of Lemma 1

To prove (2.1.6), let us fix a non-negative function \( \chi \in \mathcal{S}(\mathbb{R}) \) satisfying:

\[
\chi(\tau) > 1, \ |\tau| \leq 1 \quad \text{and} \quad \hat{\chi}(t) = 0, \ |t| \geq 1/2.
\]  

(3.4.1)

Then it suffices to show that

\[
\int_{M} \sum_{k=1}^{\infty} \chi(\varepsilon^{-1}(\lambda - \tau_k))|e_{\tau_k}(x)|^2 dx \leq C_{V}(\varepsilon \lambda^{n-1} + \varepsilon^{-1} \lambda^{-\frac{n}{2}}) \]  

(3.4.2)

By Euler's formula we can rewrite the left side of (3.4.2) as

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{M} \varepsilon \hat{\chi}(\varepsilon t) e^{it\lambda} \sum_{k=1}^{\infty} \cos t\tau_k |e_{\tau_k}(x)|^2 dxdt \]  

(3.4.3)

minus

\[
\int_{M} \sum_{k=1}^{\infty} \chi(\varepsilon^{-1}(\lambda + \tau_k))|e_{\tau_k}(x)|^2 dx. \]

Since \( \chi \in \mathcal{S}(\mathbb{R}) \), the last term is rapidly decreasing in \( \lambda \) with bounds independent of \( \varepsilon \leq 1 \). Thus we just need to show that the expression in (3.4.3) is bounded by the right side of (3.4.2).

On the other hand, by (2.1.7), it is straightforward to see that

\[
\int_{M} \sum_{j=1}^{\infty} \chi(\varepsilon^{-1}(\lambda - \lambda_j))|e_j^0(x)|^2 dx \leq C_{\varepsilon} \lambda^{n-1}, \]  

(3.4.4)

as well as

\[
\int_{M} \sum_{j=1}^{\infty} \chi(\varepsilon^{-1}(\lambda + \lambda_j))|e_j^0(x)|^2 dx \leq C_N \lambda^{-N}, \ \forall \ N. \]  

(3.4.5)

By using Euler's formula again, we have

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{M} \varepsilon \hat{\chi}(\varepsilon t) e^{it\lambda} \sum_{j=1}^{\infty} \cos t\lambda_j |e_j^0(x)|^2 dx dt \leq C_{\varepsilon} \lambda^{n-1}. \]  

(3.4.6)

By repeating the previous arguments using Lemma 2 and Duhamel’s principle, we can rewrite
the difference of (3.4.3) and (3.4.6) as

\[
\sum_{j,k} \int_M \int_M \frac{\tilde{\chi}_\lambda(\lambda_j) - \tilde{\chi}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y) \, dx \, dy, \tag{3.4.7}
\]

where

\[
\tilde{\chi}_\lambda(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \varepsilon \hat{\chi}(\varepsilon t) e^{it\lambda} \cos \tau \, dt = \chi(\varepsilon^{-1}(\lambda - \tau)) + \chi(\varepsilon^{-1}(\lambda + \tau)) \tag{3.4.8}
\]

and similarly, we interpret

\[
\frac{\tilde{\chi}_\lambda(\tau) - \tilde{\chi}_\lambda(\mu)}{\tau^2 - \mu^2} = \frac{\chi_\lambda'(\tau)}{2\tau}, \quad \text{if} \quad \tau = \mu \tag{3.4.9}
\]

Since \( \chi \in \mathcal{S}(\mathbb{R}) \), we have

\[
(\frac{d}{d\tau})^j \chi_\lambda(\tau) = O(\varepsilon^{-j}(1 + \varepsilon^{-1}|\lambda - \tau|)^{-N}) \quad \forall N, \quad \text{if} \quad j = 0, 1, 2, 3, \ldots. \tag{3.4.10}
\]

Thus, the proof of Lemma 1 would be complete if we could prove the following

\[
\left| \sum_{j,k} \int_M \int_M \frac{\tilde{\chi}_\lambda(\lambda_j) - \tilde{\chi}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y) \, dx \, dy \right| \leq C \varepsilon^{-1} \lambda^{n-2}, \tag{3.4.11}
\]

The proof of (3.4.11) is completely analogous to the proof of Proposition 2, since the properties of the function \( \tilde{\chi}_\lambda(\tau) \) we used in the proof of (2.1.28) are all satisfied by the function \( \tilde{\chi}_\lambda(\tau) \) as well, e.g., (2.1.18), (2.1.19) and Lemma 4. We skip the proof of (3.4.11) here for the sake of brevity.
4

Sharp pointwise weyl laws
involving singular potentials

In this chapter we shall give the proof of Theorem 4. To proceed, we shall need the following spectral projection bounds:

**Lemma 9** (Spectral projection bounds for $H_V$, [3]). Let $n \geq 2$, if $V \in \mathcal{K}(M) \cap L^{n/2}(M)$, then for $\lambda \geq 1$, we have

$$
\| \Pi_{[\lambda, \lambda+1)}(PV) \|_{L^2 \to L^p} \lesssim \lambda^{\sigma(n,p)}, \ 2 \leq p \leq \infty,
$$

where

$$
\sigma(n,p) = \max\{\frac{n-1}{2} \left(\frac{1}{2} - \frac{1}{p}\right), \frac{n-1}{2} - \frac{n}{p}\}.
$$

Here $1_{[\lambda, \lambda+1)}(PV)$ denotes the spectral projection operator onto the unit band $[\lambda, \lambda+1)$ for the operator $PV = \sqrt{H_V}$. If $V \equiv 0$, Sogge [17] proved that (4.0.1) hold for all $n \geq 2$ by obtaining quasimode estimates for the unperturbed Laplacian, and it was later generalized by Blair, Sire and Sogge [3] to include Schrödinger operators with critically singular potentials. Note that since $\mathcal{K}(M) \supset L^n(M)$, the above lemma holds for potentials $V \in L^n(M)$.

As before, define

$$
\tilde{I}_\lambda(\tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \rho(t) \frac{\sin \lambda t}{t} \cos \tau t \, dt,
$$

where $\rho \in C^\infty(\mathbb{R})$ is an even real-valued function satisfying (2.1.16). Then since the Fourier transform of $I_\lambda(\tau)$ is $2 \frac{\sin \lambda t}{t}$ it is not difficult to see that for $\tau > 0$ and large $\lambda$ we have

$$
I_\lambda(\tau) - \tilde{I}_\lambda(\tau) = O((1 + |\lambda - \tau|)^{-N}) \ \forall \ N.
$$
Also, for later use, for \( \tau > 0 \) we have

\[
(\frac{d}{d\tau})^j \tilde{\lambda}_\tau(x) = O((1 + |\lambda - \tau|)^{-N}) \quad \forall N, \quad \text{if } j = 1, 2, 3, \ldots
\] (4.0.4)

If we use Lemma 9 for the exponent \( p = \infty \), we can estimate the difference between \( \tilde{1}_\lambda(P_V) - \tilde{\lambda}_\tau(P_V) \). Indeed, by (4.0.3) we have

\[
\left\| \tilde{1}_\lambda(P_V)(x, x) - \tilde{\lambda}_\tau(P_V)(x, x) \right\| = \left\| \sum_k \left( \tilde{1}_\lambda(\tau_k) - \tilde{\lambda}_\tau(\tau_k) \right) |e_{\tau_k}(x)|^2 \right\|
\leq \sum_k (1 + |\lambda - \tau_k|)^{-2n} |e_{\tau_k}(x)|^2 dx \lesssim \lambda^{n-1} \quad (4.0.5)
\]

using (4.0.1) in the last inequality.

Similarly, when \( V \equiv 0 \), by applying Lemma 9 again, we have

\[
\left\| \tilde{1}_\lambda(P^0)(x, x) - \tilde{\lambda}_\tau(P^0)(x, x) \right\| = \left\| \sum_j \left( \tilde{1}_\lambda(\lambda_j) - \tilde{\lambda}_\tau(\lambda_j) \right) |e_{\lambda_j}(x)|^2 \right\|
\leq \sum_j (1 + |\lambda - \lambda_j|)^{-2n} |e_{\lambda_j}(x)|^2 \lesssim \lambda^{n-1} \quad (4.0.6)
\]

Thus, in view of (4.0.5), (4.0.6) and (1.2.1), in order to prove Theorem 4, it suffices to prove our main estimate

\[
\hat{1}_\lambda(P_V)(x, x) - \hat{1}_\lambda(P^0)(x, x) = O(\lambda^{n-1}),
\] (4.0.7)

where the implicit constants here of course depend on our \( V \) as before. And by (2.1.26), we would have (4.0.7) if we could prove the following

**Proposition 6.** Let \( V \in L^n(M) \), and \( \hat{\lambda}_\tau(\lambda) \) be defined as in (4.0.2). Then we have

\[
\left| \sum_{j,k} \int_M \frac{\hat{1}_\lambda(\lambda_j) - \hat{1}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y) dy \right| \leq C_V \lambda^{n-1},
\] (4.0.8)

for some constant \( C_V \) depending on \( V \).

To prove (4.0.8), we shall need the following lemma which is a variant of Lemma 3.

**Lemma 10.** Let \( I \subset \mathbb{R}_+ \) and for eigenvalues \( \tau_k \in I \) assume that \( \delta_{\tau_k} \in [0, \delta] \). Then if \( V \in L^n(M) \)
and \( m \in C^1(\mathbb{R}_+ \times M) \), we have

\[
\int_M \left| \sum_{\tau_k \in I} m(\delta_{\tau_k}, y) a_k V(y) e_{\tau_k}(y) \right| dy \\
\leq \|V\|_{L^n(M)} \cdot \left( \|m(0, \cdot)\|_{L_n^\infty(M)} + \int_0^\delta \left\| \frac{\partial}{\partial s} m(s, \cdot) \right\|_{L_n^\infty(M)} ds \right) \times \left( \sum_{\tau_k \in I} |a_k|^2 \right)^{\frac{1}{2}}. \quad (4.0.9)
\]

**Proof of Lemma 10.** We shall use the fact that

\[ m(\delta_{\tau_k}, y) = m(0, y) + \int_0^\delta 1_{[0, \delta_{\tau_k}]}(s) \frac{\partial}{\partial s} m(s, y) ds, \]

where \( 1_{[0, \delta_{\tau_k}]}(s) \) is the indicator function of the the interval \([0, \delta_{\tau_k}] \subset [0, \delta]\). Therefore, by Hölder's inequality and Minkowski's inequality, the left side of (4.0.9) is dominated by \( \|V\|_{L^n(M)} \) times

\[
\left( \int_M |m(0, y) \cdot \sum_{\tau_k \in I} a_k e_{\tau_k}(y)| \frac{n}{n-1} \ dy \right)^{\frac{n-1}{n}} \\
+ \left( \int_M \sum_{\tau_k \in I} \int_0^\delta 1_{[0, \delta_{\tau_k}]}(s) \frac{\partial}{\partial s} m(s, y) a_k e_{\tau_k}(y) ds \right| \frac{n}{n-1} \ dy \right)^{\frac{n-1}{n}}
\]

\[
\leq \|m(0, \cdot)\|_{L_n^2} \| \sum_{\tau_k \in I} a_k e_{\tau_k} \|_2 + \int_0^\delta \left( \| \frac{\partial}{\partial s} m(s, \cdot) \|_{L_n^2} \cdot \| \sum_{\tau_k \in I} 1_{[0, \delta_{\tau_k}]}(s) a_k e_{\tau_k} \|_2 \right) ds
\]

\[
\leq \|m(0, \cdot)\|_{L_n^2} \| \sum_{\tau_k \in I} a_k e_{\tau_k} \|_2 + \int_0^\delta \left( \| \frac{\partial}{\partial s} m(s, \cdot) \|_{L_n^2} ds \cdot \sup_{s \in [0, \delta]} \| \sum_{\tau_k \in I} 1_{[0, \delta_{\tau_k}]}(s) a_k e_{\tau_k} \|_2 \right)
\]

\[
\leq \left( \|m(0, \cdot)\|_{L_n^2} + \int_0^\delta \left\| \frac{\partial}{\partial s} m(s, \cdot) \right\|_{L_n^2} ds \right) \times \left( \sum_{\tau_k \in I} |a_k|^2 \right)^{\frac{1}{2}}
\]

as desired.

As before, we shall split things into three different parts that require slightly different arguments. Specifically, we shall first handle the contribution of frequencies \( \tau_k \) which are comparable to \( \lambda \), and then those that are relatively small followed by ones that are relatively large.

**Handling the contribution of comparable frequencies**

In this section we shall handle frequencies \( \tau_k \) which are comparable to \( \lambda \). Specifically, we shall prove the following:
Proposition 7. Let $V \in L^n(M)$, and $\tilde{1}_a(\tau)$ be defined as in (4.0.2). Then we have

$$\left| \sum_j \sum_{\tau_k \in [\lambda/2,10\lambda]} \int_M \tilde{1}_a(\lambda_j) - \tilde{1}_a(\tau_k) \frac{c_j^0(x)c_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)}{\lambda_j - \tau_k} dy \right| \leq C_V \lambda^n - 1, \quad (4.0.10)$$

for some constant $C_V$ depending on $V$.

Let $K_\tau$ be defined as in (3.1.2), to prove (4.0.10), it suffices to show that

$$\left| \sum_{\tau_k \in [\lambda/2,10\lambda]} \int_K \tau_k(x,y)e_{\tau_k}(x)e_{\tau_k}(y)V(y) dy \right| \leq C_V \lambda^n - 1. \quad (4.0.11)$$

Similar to the proof of Proposition 2, we shall use (3.1.4) to handle the summands in (4.0.11) with $\tau = \tau_k \in [\lambda/2,\lambda]$ and (3.1.5) to handle those with $\tau = \tau_k \in (\lambda, 10\lambda]$.

For $\ell \in \mathbb{Z}^+$ with $2^\ell \leq \lambda/100$, let for $j = 0, 1, 2, \ldots$

$$I_{\ell,j}^- = (\lambda - (j + 1)2^\ell, \lambda - j2^\ell] \quad \text{and} \quad I_{\ell,j}^+ = (\lambda + j2^\ell, \lambda + (j + 1)2^\ell]. \quad (4.0.12)$$

Then to use the $\delta_\tau$–Lemma (Lemma 10), we shall use the following result whose proof we momentarily postpone.

Lemma 11. If $\ell \in \mathbb{Z}^+$, $2^\ell \leq \lambda/100$, and $j = 0, 1, 2, \ldots$, we have for each $N \in \mathbb{N}$

$$\|K_{\tau,j}^\pm(x, \cdot)\|_{L^{\frac{n+2}{\lambda^2}}(M)}, \quad \|2^j \frac{\partial}{\partial \tau} K_{\tau,j}^\pm(x, \cdot)\|_{L^{\frac{n+2}{\lambda^2}}(M)} \lesssim \lambda^\frac{1}{2}(1 + j)^{-N}, \quad \tau \in I_{\ell,j}^\pm \cap [\lambda/2, 10\lambda]. \quad (4.0.13)$$

Also,

$$\|K_{\tau,0}(x, \cdot)\|_{L^{\frac{n+2}{\lambda^2}}(M)}, \quad \|\frac{\partial}{\partial \tau} K_{\tau,0}(x, \cdot)\|_{L^{\frac{n}{\lambda^2}}(M)} \lesssim \lambda^\frac{1}{2}(1 + j)^{-N}, \quad \tau \in I_{\ell,j}^\pm \cap [\lambda/2, 10\lambda], \quad (4.0.14)$$

$$\|K_{\tau,\infty}^+(x, \cdot)\|_{L^{\frac{n+2}{\lambda^2}}(M)}, \quad \|\lambda \frac{\partial}{\partial \tau} K_{\tau,\infty}^+(x, \cdot)\|_{L^{\frac{n+2}{\lambda^2}}(M)} \lesssim \lambda^\frac{1}{2} \quad (4.0.15)$$

and we can write

$$K_{\tau,\infty}^-(x,y) = \tilde{K}_{\tau,\infty}^-(x,y) + H_{\tau,\infty}^-(x,y),$$

where for $\tau \in [\lambda/2, \lambda]$

$$\|\tilde{K}_{\tau,\infty}^-(x, \cdot)\|_{L^{\frac{n+2}{\lambda^2}}(M)}, \quad \|\lambda \frac{\partial}{\partial \tau} \tilde{K}_{\tau,\infty}^-(x, \cdot)\|_{L^{\frac{n+2}{\lambda^2}}(M)} \lesssim \lambda^\frac{1}{2} \quad (4.0.16)$$

$$|H_{\tau,\infty}^-(x,y)| \lesssim \lambda^{n-2} h_\tau(\lambda d_\tau(x,y))(1 + \lambda d_\tau(x,y))^{-N}. \quad (4.0.17)$$
Finally, we have for \( \tau \in (\lambda, 10\lambda) \)

\[
\|R_{\tau, \ell}(x, \cdot)\|_{L^{2n/2}(M)}, \quad \|2^{\ell} \frac{\partial}{\partial \tau} R_{\tau, \ell}(x, \cdot)\|_{L^{2n/2}(M)} \lesssim \lambda^{2\ell - 1}. \tag{4.0.18}
\]

and

\[
|R_{\tau, \infty}(x, y)| \lesssim \lambda^{n-2} h_n(\lambda d_y(x, y))(1 + \lambda d_y(x, y))^{-N} \tag{4.0.19}
\]

By Lemma 10 with \( \delta = 2^\ell \) along with the Lemma 11 and Lemma 9, we have

\[
\left| \sum_{\tau_k \in \mathcal{I}_{\lambda}^{\tau} \cap [\lambda/2,10\lambda]} \int K_{\tau_k, \ell}^+(x, y) e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \tag{4.0.20}
\]

\[
\lesssim \|V\|_{L^n(M)} \cdot \sup_x \left( \|K_{\lambda, \ell}^+(x, \cdot)\|_{L^{2n/2}(M)} + \int_{\mathcal{I}_{\lambda}^{\tau,j}} \|\frac{\partial}{\partial s} K_{\lambda, \ell}^+(x, \cdot)\|_{L^{2n/2}(M)} \, ds \right) \times \left( \sum_{\tau_k \in \mathcal{I}_{\lambda}^{\tau} \cap [\lambda/2,10\lambda]} |e_{\tau_k}(x)|^2 \right)^{1/2} \]

\[
\lesssim \lambda^{2\ell - 1}(1 + j)^{-N} \cdot \lambda^{\frac{n+2\ell}{2}} 2^{\ell/2} \cdot \|V\|_{L^n(M)} \]

\[
\lesssim \lambda^{n-2} 2^{\ell/2}(1 + j)^{-N} \cdot \|V\|_{L^n(M)}. \]

If we sum over \( j = 0, 1, 2, \ldots, \), we see that (4.0.20) yields

\[
\left| \sum_{\lambda < \tau_k \leq 10\lambda} \int K_{\tau_k, \ell}^+(x, y) e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| + \left| \sum_{\lambda/2 \leq \tau_k \leq \lambda} \int K_{\tau_k, \ell}^+(x, y) e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \lesssim \|V\|_{L^n(M)} \lambda^{n-\frac{3}{2}} 2^{\ell/2}. \tag{4.0.21}
\]

If we take \( \delta = 1 \) in Lemma 10, this argument also gives

\[
\left| \sum_{\lambda < \tau_k \leq 10\lambda} \int K_{\tau_k, 0}(x, y) e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| + \left| \sum_{\lambda/2 \leq \tau_k \leq \lambda} \int K_{\tau_k, 0}(x, y) e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \lesssim \|V\|_{L^n(M)} \lambda^{n-\frac{3}{2}}. \tag{4.0.22}
\]

If we take \( \delta = \lambda \) in Lemma 10, this argument gives

\[
\left| \sum_{\lambda < \tau_k \leq 10\lambda} \int K_{\tau_k, \infty}^+(x, y) e_{\tau_k}(x)e_{\tau_k}(y)V(y) \, dy \right| \lesssim \|V\|_{L^n(M)} \lambda^{n-1}. \tag{4.0.23}
\]
Similarly,
\[ | \sum_{\lambda/2 < \tau_k \leq \lambda} \int \tilde{K}_{\tau_k,\infty}(x,y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy | \lesssim \| V \|_{L^n(M)} \lambda^{n-1}. \tag{4.0.24} \]

By (4.0.18), if we repeat the argument above, we have
\[ | \sum_{\lambda < \tau_k \leq 10\lambda} \int R_{\tau_k,\ell}(x,y) h(\tau_k) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy | \]
\[ + \left| \sum_{\lambda/2 < \tau_k \leq \lambda} \int R_{\tau_k,\ell}(x,y) (1 - h(\tau_k)) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \right| \lesssim \| V \|_{L^n(M)} \lambda^{n-\frac{3}{2}} 2^{\ell/2}. \tag{4.0.25} \]

Moreover, by using (4.0.17), we have \( \| H_{\tau,\infty}(x,\cdot) \|_{L^\infty(M)} \lesssim \lambda^{-1} \) for \( \tau \in [\lambda/2, 10\lambda] \), and then
\[ | \sum_{\lambda/2 < \tau_k \leq \lambda} \int H_{\tau_k,\infty}(x,y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy | \]
\[ \lesssim \| V \|_{L^n(M)} \cdot \lambda^{-1} \sum_{\tau_k \in [\lambda/2, 10\lambda]} | e_{\tau_k}(x) e_{\tau_k}(\cdot) |_{\infty} \]
\[ \lesssim \| V \|_{L^n(M)} \lambda^{n-1}. \]

Similarly,
\[ | \sum_{\lambda < \tau_k \leq 10\lambda} \int R_{\tau_k,\infty}(x,y) h(\tau_k) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy | \]
\[ + \left| \sum_{\lambda/2 < \tau_k \leq \lambda} \int R_{\tau_k,\infty}(x,y) (1 - h(\tau_k)) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \right| \lesssim \| V \|_{L^n(M)} \lambda^{n-1}. \tag{4.0.26} \]

Hence, using the estimates above and summing over \( \ell \), we obtain (4.0.10).

**Proof of Lemma 11.** To prove Lemma 11, we shall need the fact that, by Lemma 9, for any fixed \( \ell \) with \( 1 \leq 2^{\ell} \leq \lambda/100 \), we have the following spectral projection estimates (see also [17])
\[ \| \mathbb{1}_{[\lambda, \lambda+2^\ell]}(P^0) \|_{L^2 \rightarrow L^{\frac{2n}{2n-2}}} \lesssim 2^{\ell/2} \lambda^{1/2}. \tag{4.0.27} \]

To prove the first inequality we note that if \( \tau \in I^\pm_{1,j} \cap [1, 10\lambda] \) then \( |\lambda_i - \tau| \leq 2^\ell+1 \) if \( \beta(2^{-\ell}(\lambda_i - \tau)) \neq 0 \), and, in this case, we also have \( h(\lambda_i) - 1 = O((1 + j)^{-N}) \) if \( \tau \in I^-_{1,j} \) and \( h(\lambda_i) = O((1 + j)^{-N}) \) if
\( \tau \in I_{\ell,j}^+ \), Therefore, we have

\[
K_{\tau,\ell}^\pm (\cdot, y) \|_{L^2(M)} \lesssim 2^{\ell/2} \lambda^{1/2} \| K_{\tau,\ell}^\pm (\cdot, y) \|_{L^2(M)} \\
\lesssim 2^{\ell/2} \lambda^{1/2} (1 + j)^{-N} 2^{-\ell} \lambda^{-1} \left( \sum_{\{i: |\lambda_i - \tau| \leq 2^{\ell+1} \}} |e^{0}_i(y)|^2 \right)^{1/2} \\
\lesssim (1 + j)^{-N} 2^{-\ell/2} \lambda^{-1/2} \left( \sum_{\mu \in \{N: |\mu - \tau| \leq 2^{\ell+1} \}} \mu^{n-1} \right)^{1/2} \\
\leq (1 + j)^{-N} \lambda^{2} \lambda^{-1}.
\]

In particular, if \( n = 2 \), the same argument implies that

\[
\| K_{\tau,\ell}^\pm (\cdot, y) \|_{L^\infty(M)} \leq (1 + j)^{-N},
\]

which proves the first part of (4.0.13). The other inequality in (4.0.13) follows from this argument since

\[
\frac{\partial}{\partial \tau} \beta(|\lambda_i - \tau| \approx 2^{\ell}) \frac{\lambda^2}{\lambda_i^2 - \tau^2} = O(2^{-2\ell} \lambda^{-1}),
\]

due to the fact that we are assuming that \( 2^{\ell} \leq \lambda/100 \).

This argument also gives us (4.0.14) if we use the fact that \( \tau \to (\mathbf{1}_\lambda(\tau) - \mathbf{1}_\lambda(\mu))/(\tau^2 - \mu^2) \) is smooth and if we define it as in (2.1.27) when \( \tau = \mu \) and use the fact that

\[
\frac{\partial^k}{\partial \tau^k} \beta(|\lambda_i - \tau| \lesssim 1)(h(\lambda_i) - h(\tau))/(\lambda_i - \tau)) = O((1 + j)^{-N}), \; k = 0, 1, \; \tau \in I_{0,j}^\pm.
\]

To prove (4.0.15) we use the fact that for \( k = 0, 1 \) we have for \( \tau \in (\lambda, 10\lambda] \)

\[
\left| \left( \frac{\partial}{\partial \tau} \right)^k \left( \beta(|\lambda_i - \tau| \geq \lambda) \frac{\lambda^2}{\lambda_i^2 - \tau^2} \right) h(\lambda_i) \right| \lesssim \begin{cases} 
\lambda^{-2-k} & \text{if } \lambda_i \leq \lambda \\
\lambda^{-2-k}(1 + |\lambda_i - \lambda|)^{-N} & \text{if } \lambda_i > \lambda.
\end{cases}
\]

Thus for \( k = 0, 1 \), by (4.0.27)

\[
\|(\lambda \partial \tau)^k K_{\tau,\infty}^+ (\cdot, y) \|_{L^2(M)} \lesssim \| (\lambda \partial \tau)^k K_{\tau,\infty}^+ (\cdot, y) \|_{L^2(M)} \\
\lesssim \lambda^{-1} (\sum_{\lambda_i \leq \lambda} |e^{0}_i(y)|^2 + \sum_{\lambda_i > \lambda} (1 + |\lambda_i - \lambda|)^{-N} |e^{0}_i(y)|^2)^{1/2} \\
\lesssim \lambda^{-1 + \frac{1}{2}},
\]

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as desired if $N > 2n$. Similarly, $\tilde{K}_{\tau,\infty}$ satisfies the bounds in (4.0.16).

Moreover, we can conclude from Lemma 7 that $H_{\tau,\infty}$ satisfies the bounds in (4.0.17) and $R_{\tau,\infty}(x, y)$ satisfies the bounds in (4.0.19). It just remains to prove the bounds in (4.0.18) for the $R_{\tau,\ell}(x, y)$, which follows from the same argument as in the proof of (4.0.13). \hfill \square

**Handling the contribution of relatively large frequencies of $H_V$**

In this section we shall handle relatively large frequencies of $H_V$ by proving the following.

**Proposition 8.** Let $V \in L^n(M)$, and $\tilde{1}_\lambda(\tau)$ be defined as in (4.0.2). Then we have

$$\left| \sum_j \sum_{\{k: \tau_k > 10\lambda\}} \int_{M} \frac{\tilde{1}_\lambda(\lambda_j) - \tilde{1}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dy \right| \leq C_V \lambda^{n-1}, \quad (4.0.28)$$

for some constant $C_V$ depending on $V$.

To prove (4.0.28) fix

$$\Psi \in C_0^\infty((1/2, 2)), \quad \text{with } \Psi(s) = 1, \ s \in [3/4, 5/4]. \quad (4.0.29)$$

To proceed, assume that $\tau_k > 10\lambda$. Since, by the mean value theorem and (4.0.4)

$$\frac{\tilde{1}_\lambda(\lambda_j) - \tilde{1}_\lambda(\tau_k)}{\lambda_j - \tau_k} = O(\tau_k^-\sigma) \quad \forall \sigma, \ \text{if } \lambda_j \in (\tau_k/2, 2\tau_k), \ \tau_k > 10\lambda,$$

by (2.2.4) and (3.1.15), to prove (4.0.28) it suffices to show that

$$\left| \sum_j \sum_{\{k: \tau_k > 10\lambda\}} \int_{M} \frac{\tilde{1}_\lambda(\lambda_j) - \tilde{1}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} (1 - \Psi(\lambda_j/\tau_k)) e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dy \right|$$

$$\lesssim \|V\|_{L^n(M)} \lambda^{n-1}, \quad (4.0.30)$$

since

$$\left| \sum_j \sum_{\{k: \tau_k > 10\lambda\}} \int_{M} \frac{\tilde{1}_\lambda(\lambda_j) - \tilde{1}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} \Psi(\lambda_j/\tau_k) e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dy \right|$$

$$\lesssim \lambda^{-\sigma}\|V\|_{L^1(M)}, \ \forall \sigma.$$
As $\hat{I}_\lambda(\tau_k) = O(\tau_k^{-\sigma})$ for all $\sigma \in \mathbb{N}$ for $\tau_k > 10\lambda$ and, by Lemma 5,

$$\left| \sum_j (1 - \Psi(\lambda_j/\tau_k)) e^0_j(x)e^0_j(y) \right| \lesssim \begin{cases} \tau_k^{-n-2} + (d_g(x,y))^2, & n \geq 3 \\ \log(2 + (\tau_k d_g(x,y))), & n = 2, \end{cases}$$

the analog of (4.0.30) where we replace $(\hat{I}_\lambda(\lambda_j) - \hat{I}_\lambda(\tau_k))$ by $\hat{I}_\lambda(\tau_k)$ is trivial, since by Hölder’s inequality

$$\int_M h_n(d_g(x,y))V(y)dy \leq C\|V\|_{L^n(M)}$$

Consequently, we would have (4.0.30) and consequently Proposition 4 if we could show that

$$\left| \sum_j \sum_{k: r_k > 10\lambda} \int (1 - \Psi(\lambda_j/\tau_k)) \frac{\hat{I}_\lambda(\lambda_j)}{\lambda_j^2 - \tau_k^2} e^0_j(x)e^0_j(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y) dy \right| \lesssim \|V\|_{L^n(M)} \lambda^{n-1}. \quad (4.0.31)$$

We shall first focus on the terms in the sum where $\lambda_j \lesssim \lambda$. If $1 - \Psi(\lambda_j/\tau_k) \neq 0$ we have $\lambda_j \neq \tau_k$, and then can write

$$\begin{align*}
- \left( \frac{1 - \Psi(\lambda_j/\tau_k)}{\lambda_j^2 - \tau_k^2} \right) \hat{I}_\lambda(\lambda_j) &= \int_0^\infty (1 - \Psi(\lambda_j/\tau_k)) \hat{I}_\lambda(\lambda_j)e^{t(\lambda_j^2 - \tau_k^2)} dt \\
&= \int_0^{\lambda^{-2}} (1 - \Psi(\lambda_j/\tau_k)) \hat{I}_\lambda(\lambda_j)e^{t(\lambda_j^2 - \tau_k^2)} dt \\
&\quad - (1 - \Psi(\lambda_j/\tau_k)) \frac{\hat{I}_\lambda(\lambda_j)e^{\lambda^{-2}(\lambda_j^2 - \tau_k^2)}}{\lambda_j^2 - \tau_k^2} \\
&= M_1(\tau_k, \lambda_j) + M_2(\tau_k, \lambda_j)
\end{align*} \quad (4.0.32)$$

To estimate the first term, note that by the heat kernel estimates in Lemma 6, for $\lambda_j \lesssim \lambda$, we have

$$\left| \int_0^{\lambda^{-2}} e^{t\lambda_j^2} \sum_{\tau_k} e^{-t\tau_k^2} e_{\tau_k}(x)e_{\tau_k}(y) dt \right| \lesssim \int_0^{\lambda^{-2}} | \sum_{\tau_k} e^{-t\tau_k^2} e_{\tau_k}(x)e_{\tau_k}(y) | dt \lesssim \int_0^{\lambda^{-2}} t^{-\frac{1}{2}} e^{-cd_g(x,y)^2} dt \quad (4.0.33)$$

$$\lesssim \begin{cases} \log(2 + (\lambda d_g(x,y))^{-1})(1 + \lambda d_g(x,y))^{-N}, & n = 2 \\ d_g(x,y)^{2-n}(1 + \lambda d_g(x,y))^{-N}, & n \geq 3 \end{cases}$$

$$\lesssim \lambda^{n-2} h_n(\lambda d_g(x,y))(1 + \lambda d_g(x,y))^{-N}, \; \forall N.$$
Also by Lemma 2.2.4
\[
\sum_{\lambda_j \leq \lambda} |\tilde{u}_\lambda(\lambda_j)||e_j^0(x)e_j^0(y)| \lesssim \lambda^n. \quad (4.0.34)
\]

Thus by using Hölder inequality along with the estimates (4.0.33) and (4.0.34) we get
\[
\left| \int_M \sum_{\lambda_j \leq \lambda} \sum_{\tau_k} \int_0^{\lambda-2} (1 - \Psi(\lambda_j/\tau_k)) \tilde{u}_\lambda(\lambda_j)e^{t(\lambda_j^2 - \tau_k^2)}e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dtdy \right|
\lesssim \lambda^n \cdot \|V\|_{L^n(M)} \cdot \left| \sum_{\tau_k} e^{-\tau_k}e_{\tau_k}^*(x) \right| dt \|_{L^{\frac{n}{n-1}}(M)}
\lesssim \|V\|_{L^n(M)} \lambda^{n-1},
\]
where in the last inequality we used (4.0.33) and the fact that
\[
\|\lambda^{n-2}h_n(\lambda\eta, \cdot)(1 + \lambda\eta, \cdot)^{-N}\|_{L^{\frac{n}{n-1}}(M)} \lesssim \lambda^{-1}, \text{ if } N > n. \quad (4.0.36)
\]

On the other hand, by applying Lemma 10 with \(\delta = 10\lambda\), and using Sobolev estimates, Lemma 4 as well as (2.2.12), we have
\[
\left| \int_M \sum_{\lambda_j \leq \lambda} \sum_{\tau_k \in [1,10\lambda]} \int_0^{\lambda-2} (1 - \Psi(\lambda_j/\tau_k)) \tilde{u}_\lambda(\lambda_j)e^{t(\lambda_j^2 - \tau_k^2)}e_j^0(x)e_j^0(y)e_{\tau_k}(x)e_{\tau_k}(y)V(y)dtdy \right|
\lesssim \|V\|_{L^n(M)} \cdot \lambda^{-2}\lambda^{n/2} \cdot \lambda \cdot \left( \sum_{\tau_k \in [1,10\lambda]} |e_{\tau_k}^*(x)|^2 \right)^{\frac{1}{2}}
\lesssim \|V\|_{L^n(M)} \cdot \lambda^{-2}\lambda^{n/2} \cdot \lambda \cdot \lambda^{n/2}
\lesssim \|V\|_{L^n(M)} \lambda^{n-1}. \quad (4.0.37)
\]
A combination of (4.0.35) and (4.0.37) implies that
\[
\left| \sum_{\lambda_j \leq \lambda} \sum_{\lambda_j \geq \lambda} \int M_1(\tau_k, \lambda_j)e_j^0(x)e_j^0(y)V(y)e_{\tau_k}(x)e_{\tau_k}(y)dy \right| \lesssim \|V\|_{L^n(M)} \lambda^{n-1}. \quad (4.0.38)
\]

To estimate the second term involving \(M_2(\tau_k, \lambda_j)\), we shall decompose \((10\lambda, \infty) = \bigcup_{\ell \geq 0} I_\ell\), where
\[
I_\ell = (10 \cdot 2^\ell \lambda, 10 \cdot 2^\ell+1 \lambda].
\]
Then by classical Sobolev estimates

\[
\| \sum_{\lambda_j \leq \lambda} (1 - \Psi(\lambda_j/\tau_k)) \frac{\hat{h}_k(\lambda_j)}{\lambda_j^2 - (10 \cdot 2^\ell \lambda)^2} \frac{\partial \| e_j^0(x) e_j^0(\cdot) \|_{L^2(M)}}{L^2(M)} \lesssim \lambda \left| \sum_{\lambda_j \leq \lambda} \frac{\hat{h}_k(\lambda_j)}{\lambda_j^2 - (10 \cdot 2^\ell \lambda)^2} e_j^0(x) e_j^0(\cdot) \right|_{L^2(M)} \lesssim (2^\ell \lambda)^{-2}\lambda_0^{n/2},
\]

and similarly, for \( s \in I_\ell \)

\[
\| \frac{\partial}{\partial s} \sum_{\lambda_j \leq \lambda} (1 - \Psi(\lambda_j/s)) \frac{\hat{h}_k(\lambda_j)}{\lambda_j^2 - s^2} e_j^0(x) e_j^0(\cdot) \|_{L^2(M)} \lesssim (2^\ell \lambda)^{-3}\lambda_0^{n/2+1}.
\]

By Lemma 10 with \( \delta = 2^\ell \lambda \), and using Lemma 2.2.4 as well as (2.2.12), we have

\[
| \int_M \sum_{\lambda_j \leq \lambda} \sum_{\tau_k \in I_\ell} (1 - \Psi(\lambda_j/\tau_k)) \frac{\hat{h}_k(\lambda_j)}{\lambda_j^2 - \tau_k^2} \frac{e_j^0(x) e_j^0(y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) dy}{L^n(M)} | \lesssim \| V \|_{L^n(M)} \cdot (2^\ell \lambda)^{-2}\lambda_0^{n/2} \cdot \lambda \cdot ( \sum_{\tau_k \in I_\ell} | e^{-\lambda_0^2 \tau_k} e_{\tau_k}(x) \|^2 )^{1/2}
\]

\[
\lesssim \| V \|_{L^n(M)} \cdot (2^\ell \lambda)^{-2}\lambda_0^{n/2} \cdot \lambda \cdot e^{-2^\ell \lambda} \cdot (2^\ell \lambda)^{n/2}
\]

\[
\lesssim \| V \|_{L^n(M)} \lambda^{n-1} \cdot e^{-2^\ell \lambda} 2^{(n/2-2)\ell}.
\]

Summing over \( \ell \), we get

\[
| \int_M \sum_{\lambda_j \leq \lambda} \sum_{\tau_k > 10^\lambda} M^2(\tau_k, \lambda_j) e_j^0(x) e_j^0(y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) dy | \lesssim \| V \|_{L^n(M)} \lambda^{n-1}.
\]

Thus the proof of (4.0.31) is complete when \( \lambda_j \leq \lambda \). For general \( \lambda_j \approx 2^\ell \lambda \) with \( 2^\ell \geq 1 \), write

\[
- \frac{(1 - \Psi(\lambda_j/\tau_k))}{\lambda_j^2 - \tau_k^2} \tilde{h}_k(\lambda_j) = \int_0^{(2^\ell \lambda)^{-2}} \frac{(1 - \Psi(\lambda_j/\tau_k))}{\lambda_j^2 - \tau_k^2} \tilde{h}_k(\lambda_j) e^{t(\lambda_j^2 - \tau_k^2)} dt
\]

\[
= \int_0^{(2^\ell \lambda)^{-2}} (1 - \Psi(\lambda_j/\tau_k)) \tilde{h}_k(\lambda_j) e^{t(\lambda_j^2 - \tau_k^2)} dt
\]

\[
- (1 - \Psi(\lambda_j/\tau_k)) \frac{\tilde{h}_k(\lambda_j)}{\lambda_j^2 - \tau_k^2} e^{(2^\ell \lambda)^{-2}(\lambda_j^2 - \tau_k^2)}
\]

\[
= M_1(\tau_k, \lambda_j) + M_2(\tau_k, \lambda_j).
\]
If we repeat the argument above, it is not hard to see that
\[
\left| \sum_{\{j: \lambda_j \approx 2^\ell \lambda\}} \sum_{\{k: \tau_k > 10\lambda\}} \int \frac{(1 - \Psi(\lambda_j/\tau_k))}{\lambda_j^2 - \tau_k^2} \tilde{h}_\lambda(\lambda_j) e_j^0(x) e\tau_k^0(y)V(y) e\tau_k(x) e\tau_k(y) \, dy \right| \\
\lesssim \|V\|_{L^n(M)} \lambda^{n-1} 2^{-\ell N}, \quad \forall N > 0, \quad (4.0.42)
\]
where the $2^{-\ell N}$ factor comes from the fact that $\tilde{h}_\lambda(\lambda_j)$ is rapidly decreasing when $\lambda_j \geq 2\lambda$. After summing over $\ell \geq 1$, (4.0.42) along with (4.0.40) give us the desired inequality (4.0.31), thus the proof of Proposition 8 is complete.

**Handling the contribution of relatively small frequencies of $HV$**

In this section we shall handle relatively small frequencies of $HV$ and prove the following result.

**Proposition 9.** Let $V \in L^n(M)$, and $\tilde{h}_\lambda(\tau)$ be defined as in (4.0.2). Then we have
\[
\left| \sum_j \sum_{\tau_k < \lambda/2} \int_M \frac{\tilde{h}_\lambda(\lambda_j) - \tilde{h}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} e_j^0(x) e\tau_k^0(y) V(y) e\tau_k(x) e\tau_k(y) \, dy \right| \leq C_V \lambda^{n-1}, \quad (4.0.43)
\]
for some constant $C_V$ depending on $V$.

If we combine this with Proposition 7 and Proposition 8 which handle frequencies which are comparable to $\lambda$ and large compared to $\lambda$, respectively, we obtain Proposition 6, which by the arguments in §2, yield Theorem 4.

**Proof of Proposition 9.** As in the earlier cases, we shall first handle a trivial case. To do so, we note that, by (4.0.4) and the mean value theorem,
\[
\frac{\tilde{h}_\lambda(\lambda_j) - \tilde{h}_\lambda(\tau)}{\lambda_j^2 - \tau^2} = O(\lambda^{-\sigma}), \quad \forall \sigma \in \mathbb{N}, \quad \text{if} \quad 1 \leq \tau \leq \lambda/2 \quad \text{and} \quad \lambda_j \leq 7\lambda/8.
\]
Also, by (2.2.4) and (3.1.15)
\[
\sum_{\lambda_j \leq \lambda} |e_j^0(x) e\tau_k^0(y)|, \quad \sum_{\tau_k < \lambda/2} |e\tau_k(x) e\tau_k(y)| \lesssim \lambda^n. \quad (4.0.44)
\]
To use these and make our first reduction fix $a \in C^\infty(\mathbb{R}_+) \text{ satisfying} \quad a(s) = 0, \quad s \leq 3/4 \text{ and } a(s) = 1, \quad s \geq 7/8.$

Using the preceding inequalities we see that in order to prove (4.0.43) it suffices to show that

$$
| \sum_j \sum_{\tau_k < \lambda/2} \int_M \frac{\mathbb{I}_\lambda(\lambda_j) - \mathbb{I}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} a(\lambda_j/\lambda) e_j^0(x)e_j^0(y) V(y)e_{\tau_k}(x)e_{\tau_k}(y) \, dy |
\leq C_V \lambda^{n-1} \| V \|_{L^n(M)}, \quad (4.0.45)
$$

due to the fact that the difference between the quantities inside the absolute values in the left side of (4.0.43) and that of (4.0.45) is $O(\lambda^{-\sigma} \| V \|_{L^1})$ for all $\sigma$.

For the next reduction, note that the proof of Lemma 5 yields

$$
| \sum_j \frac{a(\lambda_j/\lambda) e_j^0(x)e_j^0(y)}{\lambda_j^2 - \tau_k^2} e_j^0(x)e_j^0(y) \left( \begin{array}{l} (d_g(x,y))^2 - n, \quad n \geq 3, \\ \log(2 + (d_g(x,y))^{-1}), \quad n = 2, \end{array} \right) \right| \leq C_V \lambda^{n-1} \| V \|_{L^n(M)}, \quad \forall \sigma \in \mathbb{N},
$$

if $1 \leq \tau_k \leq \lambda/2$. Based on this and the second part of (4.0.44) and the fact that $1 - \mathbb{I}_\lambda(\tau_k) = O(\lambda^{-\sigma})$ for all $\sigma$ when $1 \leq \tau_k \leq \lambda/2$, we easily see that

$$
| \sum_j \sum_{\tau_k < \lambda/2} \int_M \frac{1 - \mathbb{I}_\lambda(\tau_k)}{\lambda_j^2 - \tau_k^2} a(\lambda_j/\lambda) e_j^0(x)e_j^0(y) V(y)e_{\tau_k}(x)e_{\tau_k}(y) \, dy |
\leq \lambda^{-\sigma} \| V \|_{L^n(M)}, \quad \forall \sigma \in \mathbb{N},
$$

since, by H"older's inequality

$$
\int_M h_n(d_g(x,y)) V(y) \, dy \leq C \| V \|_{L^n(M)}.
$$

Consequently, we would have (4.0.45) if we could show that

$$
| \sum_j \sum_{\tau_k < \lambda/2} \int_M \frac{a(\lambda_j/\lambda)}{\lambda_j^2 - \tau_k^2} (\mathbb{I}_\lambda(\lambda_j) - 1) e_j^0(x)e_j^0(y) V(y)e_{\tau_k}(x)e_{\tau_k}(y) \, dy |
\leq C_V \lambda^{n-1} \| V \|_{L^n(M)}. \quad (4.0.46)
$$

We need to make one final reduction before we can appeal to the $\delta_r$–Lemma, Lemma 10. For this, let $\eta$ be as in Lemma 5, i.e., $\eta \in C^\infty(\mathbb{R}_+) \text{ with } \eta(s) = 0 \text{ on } s \leq 2 \text{ and } \eta(s) = 1, \quad s > 4.$ It then
follows that
\[ \eta(s) a(s) = \eta(s). \]

Consequently, we can write the quantity inside the absolute value in the left side of (4.0.46) as
\[
\sum_j \sum_{\tau_k < \lambda/2} \int_M \frac{\eta(\lambda_j/\lambda)}{\lambda_j^2 - \tau_k^2} (\tilde{I}_\lambda(\lambda_j) - 1) e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dy 
+ \sum_j \sum_{\tau_k < \lambda/2} \int_M \frac{a(\lambda_j/\lambda)(1 - \eta(\lambda_j/\lambda))}{\lambda_j^2 - \tau_k^2} (\tilde{I}_\lambda(\lambda_j) - 1) e_j^0(x) e_j^0(y) V(y) e_{\tau_k}(x) e_{\tau_k}(y) \, dy
\]
\[ = I + II. \]

Therefore, in order to prove (4.0.46), it suffices to show that both \(|I|\) and \(|II|\) are dominated by the right side of (4.0.46).

We can easily handle \(I\) without appealing to the \(\delta_\tau\)-lemma. Indeed since \(\tilde{I}_\lambda(\lambda_j) = O(\lambda_j^{-\sigma})\) for all \(\sigma\) if \(\eta(\lambda_j/\lambda) \neq 0\), we see that (2.2.4) yields
\[
\sum_j \frac{\eta(\lambda_j/\lambda)}{\lambda_j^2 - \tau_k^2} (\tilde{I}_\lambda(\lambda_j) - 1) e_j^0(x) e_j^0(y) = - \sum_j \frac{\eta(\lambda_j/\lambda)}{\lambda_j^2 - \tau_k^2} e_j^0(x) e_j^0(y) + O(\lambda^{-\sigma}), \quad \forall \sigma.
\]

Consequently, by the second part of (3.3.2) we would have the desired bounds for \(I\) if we could show that
\[
\left| \int \sum_{\tau_k < \lambda/2} R_{\tau_k}(x, y) e_{\tau_k}(x) e_{\tau_k}(y) V(y) \, dy \right| \lesssim \lambda^{n-1} \| V \|_{L^n}, \quad (4.0.47)
\]
where
\[ R_{\tau_k}(x, y) = \sum_j \frac{\eta(\lambda_j/\lambda)}{\lambda_j^2 - \tau_k^2} e_j^0(x) e_j^0(y). \]

To use this we note that the proof of Lemma 5 implies that
\[
\sup_{1 \leq \tau_k < \lambda/2} |R_{\tau_k}(x, y)| \leq C_0 \lambda^{n-2} h_\alpha(\lambda d_g(x, y))(1 + \lambda d_g(x, y))^{-\sigma}, \quad \forall \sigma,
\]
and, therefore,
\[
\sup_x \left( \int \sup_{1 \leq \tau_k < \lambda/2} |R_{\tau_k}(x, y)|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \lesssim \lambda^{-1}.
\]

Since, we always have \(\tau_k \geq 1\) by (1.1.7), by the second part of (3.3.2) we have
\[
\sup_x \left| \int \sum_{1 \leq \tau_k < \lambda/2} R_{\tau_k}(x, y) e_{\tau_k}(x) e_{\tau_k}(y) \, dy \right| \lesssim \lambda^{n-1} \| V \|_{L^n(M)},
\]
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which clearly yields (4.0.47).

Since we have the desired estimate for $I$ above, it only remains to prove the corresponding estimate for $II$. For this, let

$$m(s, x, y) = \sum_j \frac{b(\lambda_j/\lambda)}{\lambda_j^2 - s^2} \left( \mathbf{1}_k(\lambda_j) - 1 \right) e_j^0(x)e_j^0(y), \quad \text{with } b(s) = a(s)(1 - \eta(s)) \in C^\infty((3/4, 4)).$$

We then can rewrite this desired bound for $II$ as follows

$$\left| \int \sum_{\tau_k < \lambda/2} m(\tau_k, x, y) e_{\tau_k}(x) V(y) \, dy \right| \lesssim \lambda^{n-1} \|V\|_{L^n}. \tag{4.0.48}$$

To prove this we first note that by (2.2.4) and the fact that $b(\lambda_j/\lambda) \neq 0$ implies $\lambda_j/\lambda \in [3/4, 4]$. Consequently,

$$\left| \left( \frac{\partial}{\partial s} \right)^\ell \left( \frac{b(\lambda_j/\lambda)}{\lambda_j^2 - s^2} \right) \right| \leq C \lambda^{-2}(s\lambda^{-2})^\ell, \quad \ell = 0, 1, \quad \text{if } 1 \leq s \leq \lambda/2.$$

Using this and the support properties $b$ we can easily see that by the proof of Lemma 4 that (2.2.4) and orthogonality yields for $\ell = 0, 1$

$$\|\left( \frac{\partial}{\partial s} \right)^\ell m(s, x, \cdot)\|_{L^2(M)} \leq \lambda \|\left( \frac{\partial}{\partial s} \right)^\ell m(s, x, \cdot)\|_{L^2(M)} \leq C_0 \lambda^{\frac{n}{2} - 1}(s\lambda^{-2})^\ell, \quad \text{if } x \in M, \ 0 \leq s \leq \lambda/2,$$

where in the first inequality we used classical Sobolev estimates.

Consequently,

$$\sup_x \left( \|m(1, x, \cdot)\|_{L^\infty(M)} + \int_1^{\lambda/2} \|\left( \frac{\partial}{\partial s} \right)^\ell m(s, x, \cdot)\|_{L^\infty(M)} \right) \lesssim \lambda^{\frac{n}{2} - 1}.$$

By Lemma 10 and the second part of (4.0.44) we deduce from this that the left side of (4.0.48) is dominated by

$$\lambda^{\frac{n}{2} - 1} \sup_y \left( \sum_{\tau_k < \lambda/2} |e_{\tau_k}(y)|^2 \right)^{1/2} \lesssim \lambda^{n-1},$$

which completes the proof. \qed

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Bibliography


Curriculum Vitae

Xiaoqi Huang was born on December 21, 1992 in Jinhua, Zhejiang Province, China. He received his B.S. degree in Pure and Applied Mathematics from the Zhejiang University in 2015. He started his Ph.D. program in the Mathematics Department at Johns Hopkins University in 2016. He received an M.A. degree in Mathematics from Johns Hopkins University in 2020. His dissertation was completed under the guidance of Professor Christopher D. Sogge and defended on May 26th, 2021.