ON MEAN CURVATURE FLOWS COMING OUT OF CONES

by

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Abstract

In this thesis, we study several aspects of mean curvature flows (MCFs) coming out of cones. As geometric heat flows, MCFs are expected to immediately smooth out singular initial data, so it is natural to consider this phenomenon in the model case of cones. These flows also serve as models of possible continuations of MCFs after conical singularities. The topics discussed include the existence, uniqueness and symmetry of the evolution of cones.

In the first part, we study the existence and uniqueness properties of such flows. The most natural family are called self-expanders, which are extensively studied by Bernstein–Wang in a series of work starting from [11]. Building on the theory of self-expanders, we first construct a family of non-self-similar MCFs coming out of a given cone, provided there is an unstable self-expander asymptotic to the cone. Using spectral analysis, we will then classify solutions coming out of a generic cone with low entropy.

In the second part, we show rigidity when we impose additional symmetry on the cone structure. In particular, we prove that for a rotationally symmetric double cone, any initially smooth MCF coming out of the cone remains rotationally symmetric at all future times, even past singularities. Note that the flows we consider are not only smooth self-expanders, but also the examples constructed in the first part. This generalizes earlier results of Fong–McGrath [42] and Bernstein–Wang [9].
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Contents

Abstract ......................................................... ii

Acknowledgements ........................................ iv

Contents ....................................................... vi

List of Figures ................................................ viii

Chapter 1 Introduction ...................................... 1

Chapter 2 Notations and Preliminaries ..................... 11
  2.1 Notations .................................................. 11
  2.2 Basic geometric measure theory ....................... 12
  2.3 Integral Brakke flows .................................. 13
  2.4 Function spaces .......................................... 16
  2.5 Self-expanders .......................................... 17

Chapter 3 Existence of Mean Curvature Flows Coming out of Cones 23
  3.1 Properties of flows coming out of cones .......... 23
  3.2 Existence of tame ancient flows .................... 28
  3.3 More on weak flows .................................... 36
  3.4 The smooth one-sided flow ........................... 40
  3.5 Construction of the matching motion ............... 43
List of Figures

Figure 5-1 A typical picture of the moving plane ............................. 77
Figure 5-2 Boundary touching ...................................................... 81
Figure 5-3 Interior touching ......................................................... 81
Chapter 1

Introduction

In this thesis, we study several related problems arising in the study of mean curvature flow. A family of properly embedded hypersurfaces \( \{ \Sigma_t \}_{t \in I} \subset \mathbb{R}^{n+1} \) is a mean curvature flow if it satisfies the equation:

\[
\frac{\partial x}{\partial t} \perp = H_{\Sigma_t}.
\]  

(1.1)

Here \( x \) is the position vector in \( \mathbb{R}^{n+1} \), \( H_{\Sigma_t} \) is the mean curvature vector of the hypersurface \( \Sigma_t \) given by the trace of the second fundamental form \( A \), i.e.

\[
H_{\Sigma_t} = \text{tr} A = \sum_{i=1}^{n} A(E_i, E_i).
\]

(1.2)

The equation (1.1) can be thought as the (negative) gradient flow of the area function. Each point of the hypersurface moves in the direction in which the area decreases the fastest.

Mean curvature flow was perhaps first introduced by Mullins in the 1950s as a model for grain boundary growths in material science. The mathematical study of the flow began with the foundational work of Brakke [15], who studied a weaker formulation of the flow using notions from geometric measure theory. The study of equation (1.1) using PDE methods was initiated by Huisken [52] and Ecker–Huisken [38]. We remark that when \( n = 1 \), equation (1.1) is known as the curve shortening flow, which has much better properties and also has an almost complete theory — see
Gage–Hamilton [43] and Grayson [45]. It is also worth noting that the static solutions
to the mean curvature flow, i.e. hypersurfaces with $H = 0$, known as the *minimal
hypersurfaces*, has been a central object of study in geometry since the times of Euler
and Lagrange.

Standard examples of mean curvature flows in $\mathbb{R}^{n+1}$ include the shrinking spheres
$\sqrt{-2nt}S^n$ and the shrinking (generalized) cylinders $\sqrt{-2(n - k)t}S^{n-k} \times \mathbb{R}^k$, for $k = 1, \ldots, n - 1$. These solutions are examples of *self-shrinkers*. Self-shrinkers are of
particular importance because they model the singularities of the flow; see eg. [52,
55, 32]. Solutions such as the grim reaper in $\mathbb{R}^2$, given by the curve $y = t + \log \sec x$,
are *translators* of the mean curvature flow. The terminology is self-explanatory as the
curve moves in the positive $y$-direction with constant speed under the flow. There is
also a rich literature in the study of translators, and we refer to [84, 50] and references
therein for more information. Both self-shrinkers and translators are examples of
*soliton* solutions of the flow. Solutions defined on the interval $(-\infty, 0)$, like the self-
expanders, are known as the *ancient solutions*, while solutions defined on $(-\infty, \infty)$,
like the translators, are known as the *eternal solutions*.

The recent study of mean curvature flows has centered around their singularity
formations. By the parabolic maximum principle, any closed hypersurface must develop
a singularity in finite time under the flow. It is a central problem to understand the
type of singularities that a mean curvature flow can encounter. For example, when
$n = 1$, every closed curve will disappear at a round point under the flow, by the
celebrated theorem of Grayson [45]. Unfortunately, examples such as the neck-pinch
singularity show that the higher-dimensional analogues do not hold, and the situation
there is much more complicated. Although there have been recent significant progress
for $n = 2$ — see [16, 28, 8] among others — the general picture remains unclear.

Our study has two main motivations. The first is the following basic question: is it
possible to continue the flow after a singularity? Inevitably, the mean curvature flow
equation (1.1) becomes undefined after the flow encounters a singularity. The first approach, also the one taken in this thesis, is to embrace the singularity and try to define a flow that makes sense even with singularities. This was first achieved, as we alluded to before, by Brakke [15] using notions from geometric measure theory (see Chapter 2 for the precise definition). Later on, it was realized that the mean curvature flow also has a natural set-theoretic formulation. This is the so-called level set flow, first introduced by Osher–Sethian [70] and then extensively studied by Evans–Spruck [40] in the sense of viscosity solutions (cf. Chen–Giga–Goto [23]). These two approaches were united by the seminal works of Ilmanen [55], who proved a very general existence and regularity theorem for weak mean curvature flows combining both ideas. These weak formulations of (1.1) will be the starting point of our investigation.

Another notable approach is the surgery method. Roughly speaking, it is done by cutting the hypersurface before the singular time, attaching handles to it and flowing the topologically modified hypersurface in hope of avoiding the singularity. This method has found great success in the Ricci flow in 3-manifolds, famously employed by Perelman in his resolution of the Poincaré conjecture [72]. In the mean curvature flow setting, there has been partial theory of surgery, for singularities modeled on a shrinking cylinder, developed by Huisken–Sinestrari [54], Brendle–Huisken [18] and Haslhofer–Kleiner [47]. However, all the above results require very strong convexity assumptions on the flow, which limits its applicability.

Secondly, equation (1.1) can be thought of as a geometric version of the heat equation. One important feature of the heat equation is its strong smoothing property, namely the solution of the heat equation has the tendency to resolve rough initial data immediately. We study the same property in the geometric setting, namely to determine whether a MCF starting from a rough initial data has the same smoothing phenomenon. The simplest rough initial data (that makes sense geometrically) to study are the smooth cones, which has its only singular point at the origin. Here by a
cone we mean a hypersurface in $\mathbb{R}^{n+1}$ that is invariant under dilations.

In this thesis, we focus primarily on singularities modeled on cones. Existence of such singularity models is obtained by Kapouleas–Kleene–Møller [59] (see also Nguyen [69]). A mean curvature flow before encountering a conical singularity, under suitable rescaling, can be roughly thought of as flows into the cone. Therefore, to continue the flow past a singularity, one may seek a way of flowing out of the cone. The natural candidates of such flows are modeled on flows that achieve the cone as the initial data. In other words, we seek solutions to (1.1) on $I = (0, T)$ such that

$$\lim_{t \to 0} \Sigma_t = C \text{ in } C^\infty_{\text{loc}}(\mathbb{R}^{n+1} \setminus \{0\}),$$

where $C$ is the (asymptotic cone of the) given conical singularity model. We note importantly that a non-flat cone is necessarily singular at the origin, so the initial data is not achieved in the classical sense. This singular nature of the cone is a primary source of complexity in our analysis, which we shall see below.

**(Existence of mean curvature flows coming out of cones)**

Standard parabolic theory implies the short-time existence and uniqueness of a mean curvature flow if the initial hypersurface is smooth and closed (see eg. [37] or [2, Chapter 6]). However, as a cone is neither smooth nor closed, the standard existence and uniqueness theory does not apply, and we must work it out by hand. First, we discuss the existence problem.

As cones are invariant under dilations, it is natural to consider homothetically self-similar solutions of the flow.

**Definition 1.1.** A *self-expander* is a hypersurface $\Sigma$ such that

$$H_\Sigma = \frac{x^\perp}{2}.$$  \hspace{1cm} (1.4)

We say a self-expander is *asymptotically conical* if there exists a cone $C$ smooth away
from the origin such that

\[ \lim_{\rho \to 0^+} \rho \Sigma = \mathcal{C} \text{ in } C^\infty_{\text{loc}}(\mathbb{R}^{n+1} \setminus \{0\}). \]

It is easy to check that \( \Sigma \) is a smooth self-expander asymptotic to \( \mathcal{C} \) if and only if \( \{\sqrt{t} \Sigma\}_{t \in (0, \infty)} \) is a solution to (1.1) with (1.3). Hence the existence problem will be solved if one can find a smooth self-expander asymptotic to the given cone. This reduces the parabolic problem to an elliptic one (as in (1.4)), and constructions for different classes of cones, using various methods, are available in the literature, see for example Ding [35], Ecker–Huisken [38] and Bernstein–Wang [12].

Having dealt with existence, it is natural to ask if the solutions given by self-expanders above are unique. It turns out that uniqueness does not hold in general. It was first observed by Angenent–Ilmanen–Chopp [4] that uniqueness does not necessarily hold for a mean curvature flow coming out of a cone. To elaborate, they showed numerically that there are at least two possible evolutions of the cone

\[ \mathcal{C}_m = \{x_1^2 = m^2(x_2^2 + x_3^2)\} \subset \mathbb{R}^3, \]

provided \( m \) is smaller than a critical value related to the cone angle. Helmensdorfer [48] proved rigorously that for these values of \( m \) there are at least three distinct smooth self-expanders asymptotic to \( \mathcal{C}_m \). Our aim in the first part of this thesis is to provide a more refined understanding of the nonuniqueness phenomenon in general. First of all, we will show that as long as the cone has an unstable asymptotic self-expanders in the sense of Morse (see Chapter 2 for the precise definitions), then there exists a solution to (1.1) with (1.3) that is not self-similar.

**Theorem 1.2** (Theorem 1.1 of [20]). Let \( \mathcal{C} \) be a smooth hypercone in \( \mathbb{R}^{n+1} \) and suppose \( \Sigma \) is an unstable self-expander asymptotic to \( \mathcal{C} \) of index \( I \), then there exists an \( I \)-parameter family of distinct non-self-similar solutions of the mean curvature flow satisfying (1.3).
According to Theorem 1.2, when $I \geq 2$, there exists an infinite number of possible solutions coming out of a given cone. When $I = 1$, the family produced by Theorem 1.2 is parametrized by time translation, and it turns out that the solution is unique up to time translation (see Proposition 4.12).

The proof of Theorem 1.2 is mostly PDE-based, and is based the work of Choi–Mantoulidis [30], who constructed the ancient flows for suitable elliptic functionals. The main difference is that we are dealing with non-compact hypersurfaces, and so slightly more sophisticated estimates are needed. Indeed, under the change of variables

$$s = \log t \text{ and } \tilde{\Sigma}_s = t^{-1/2} \Sigma_t,$$  

the new family of hypersurfaces $\{\tilde{\Sigma}_s\}_{s \in (-\infty, S)}$ satisfies the following rescaled mean curvature flow equation

$$\left(\frac{\partial \mathbf{x}}{\partial s}\right)_{\mathbf{\perp}} = H_{\tilde{\Sigma}_s} - \frac{\mathbf{x}_{\mathbf{\perp}}}{2},$$  

whose static solutions are precisely the self-expanders (by (1.4)). To construct the desired solutions, we consider solutions that can be written as entire graphs over the given self-expanders. It turns out that equation (1.6) can be reduced to the following second-order nonlinear PDE:

$$\frac{\partial u}{\partial s} = L_\Sigma u + Q(u),$$  

for some linear elliptic operator $L_\Sigma$ (called the stability operator) and a nonlinear error term $Q$ which has a very specific asymptotic expansion. By solving the above PDE with prescribed data coming from the unstable directions at time $s = 0$, we arrive at the desired solutions.

In the particular case of $I = 1$, the (unique) mean curvature flow constructed by Theorem 1.2 actually lies on one side of $\Sigma$. Furthermore it has a long-time regularity, and the forward limit of the flow, under the rescaling (1.5), is a stable self-expander. We are able to give a more geometric interpretation of this flow. For expository reasons,
we will provide a short proof of these statements based on the constructions in [22],
with the extra assumption that the cone is geometrically simple. A full statement
using a different construction can be found in [6].

We need to use the notion of entropy, which is a geometric quantity that is
monotone under the mean curvature flow, introduced by Colding–Minicozzi in their
seminal work [32]. Roughly speaking, entropy measures the geometric complexity of a
hypersurface (cf. [31, 7, 94]). Given a hypersurface $\Sigma$, the entropy $\lambda[\Sigma]$ of $\Sigma$ is defined
to be the supremum of the Gaussian area under all translation and dilations; i.e.

$$\lambda[\Sigma] = \sup_{x_0 \in \mathbb{R}^{n+1}, t > 0} (4\pi t)^{-n/2} \int_{\Sigma} e^{\frac{|x-x_0|^2}{4t}} d\mathcal{H}^n.$$ 

Here the quantity is normalized so that the hyperplane in $\mathbb{R}^{n+1}$ has entropy 1. Therefore
$\lambda[\Sigma] \geq 1$ for any smooth hypersurface, since the entropy of the tangent plane is 1.

**Theorem 1.3** (Theorem 1.5 of [22]). Let $C$ be a smooth hypercone in $\mathbb{R}^{n+1}$ with
$\lambda[C] < 2$, and suppose $\Sigma$ is an unstable self-expander asymptotic to $C$, then there exists
a non-self-similar mean curvature flow satisfying (1.3). Moreover, this flow lies on
one side of $\Sigma$.

It is a natural to ask the converse question; namely, if the solutions constructed
in Theorem 1.2 are indeed all the possible flows coming out of the cone. While the
general structure of the space of such flows remains unclear, in $\mathbb{R}^3$ we can give an
affirmative answer to a class of cones that are not too complicated geometrically.

**Theorem 1.4** (Corollary 5.6 of [20]). Let $C \subset \mathbb{R}^3$ be a generic cone, in the sense of
Baire categories, with $\lambda[C] < 2$. Then any mean curvature flow with (1.3) is either a
smooth self-expander, or one of the non-self-similar flows starting from an unstable
self-expander, as constructed in Theorem 1.2.

The proof of Theorem 1.4 relies on the ODE lemma of Merle–Zaag type [68],
which has found great success in recent advances for classification problems in ancient
geometric flows – see [3, 28, 29, 36, 27] for example. First of all, without too much work we can show that any mean curvature flow satisfying (1.3), under the change of variables (1.5), will converge backwards subsequentially to a self-expander \( \Sigma \) asymptotic to \( \mathcal{C} \). To upgrade the subsequential convergence to full convergence, the usual tool is the Łojasiewicz–Simon inequality [77] (cf. [26, 75, 33] for applications in mean curvature flow), which we have if we know \( \Sigma \) is smooth. However, this is not always true as singular self-expanders do exist. This is why the dimension and the entropy restrictions are imposed. Secondly, the solutions constructed in Theorem 1.2 only accounts for the negative eigenspace, and without any further assumption it could be the case that there are more solutions coming from the zero eigenspace. Thanks to a theorem of Bernstein and Wang [9], the set of cones whose asymptotic self-expanders have non-trivial zero eigenspace is meager, so we are able to complete the classification for a reasonable set of cones.

**Rotational symmetry of solutions**

In the second part of the thesis, we investigate how symmetries of the cone affects the symmetry of the self-expanders asymptotic to the cone.

Consider a rotationally symmetric hypercone \( \mathcal{C} \subset \mathbb{R}^{n+1} \). If \( \mathcal{C} \) is a single cone (i.e. \( \mathcal{C} \cap S^n \) has one connected component, then a classical theorem of Ecker–Huisken [38] and an application of the maximum principle shows that there is a unique forward evolution. By solving equation (1.4) after imposing the rotational symmetry, one easily sees that there exists a rotationally symmetric self-expander that is an entire graph over \( \mathcal{C} \). Together with the uniqueness this completes the picture for single cones.

We are interested in the next non-trivial case, namely when \( \mathcal{C} \) is a double cone. Here by a double cone we mean a cone \( \mathcal{C} \) such that \( \mathcal{C} \cap S^n \) has two connected components lying on opposite hemispheres. The simplest examples include the cones \( \mathcal{C}_\alpha \) introduced earlier, which has \( O(2) \times O(1) \) symmetry. Clearly, the forward evolution of such cones
\( C \) needs not be unique by \([4]\).

**Theorem 1.5** (Theorem 1.1 of [21]). *Any smooth self-expander asymptotic to a rotationally symmetric double cone \( C \) is also rotationally symmetric, with the same axis of symmetry.*

Theorem 1.5 generalizes earlier results of Fong–McGrath [42], who assume mean-convexity, i.e. \( H_{\Sigma} > 0 \) and Bernstein–Wang [14], who assume weak stability of \( \Sigma \). The proof relies on the celebrated moving plane method, first used by Alexandrov to prove that compact embedded CMC hypersurfaces are round spheres. The method later found applications in solutions to certain elliptic PDEs - see works of Serrin [76] and Gidas–Ni–Nirenberg [44]. In a more geometric context, Schoen [74] used the method to prove a uniqueness of catenoid result for minimal surfaces, and Martin–Savas-Halilaj–Smoczyk [66] proved a rotational symmetry result for translating paraboloids of mean curvature flows. All the above methods deal with elliptic equations, and although the self-expander equation (1.4) is also elliptic, it is not preserved under reflections (in other words, the reflected hypersurface does not satisfy the same equation). This causes difficulty in the application of maximum principle. To circumvent this, we have to utilize a parabolic version of the method similar to [28], where the entire flow will be reflected. Rotational symmetry results in other geometric flows have been widely studied and we refer to [25, 17, 93] for some recent advancements and other approaches.

Upon inspection, the same proof as in Theorem 1.5 will work for any forward evolution of the cone \( C \), as long as the flow remains smooth. However, generally speaking, singularities of a general flow coming out of the cone are not avoidable (see Proposition 5.17 for an explicit example). In these cases, by employing a novel moving plane method of Choi–Haslhofer–Hershkovits–White [29] (see also [46]), we are able to prove that the rotational symmetry is preserved even after singularities.
Theorem 1.6 (Corollary 1.2 of [22]). Suppose $C$ is a rotationally symmetric double cone and $\lambda[C] < 2$. Then any unit-regular, cyclic integral Brakke flow with (1.3) remains rotationally symmetric with the same axis of symmetry. Moreover, the possible singularities of the flow are all on the axis of symmetry.

Here we need to use the notion of Brakke flow in order to make sense of the flow with singularities. In particular, this covers all the flows we constructed in Theorem 1.2. The entropy condition $\lambda[C] < 2$ is technical and likely redundant, as every rotationally symmetric double cone is conjectured to have entropy at most 2 (see Section 5.3 for further discussions). The striking part about the conclusion of Theorem 1.6 is that the regularity of the flow can also be upgraded concurrently with the moving plane.

It is worth mentioning that if $C \cap S^n$ has three or more components, then the rotational symmetry is not expected to hold. It is likely that one can use the desingularization method to glue together a plane and a stable one-sheeted self-expander (such as the examples from [4]) to construct a self-expander asymptotic to a rotationally symmetric double cone union a plane. This self-expander should have high genus and is not rotationally symmetric. We refer to [51, 58–60, 69] for various related constructions including the desingularization method.

Outline of the thesis

The thesis is organized as follows. In Chapter 2 we present our notations and list some preliminary facts of self-expanders. In Chapter 3 we prove Theorem 1.2 using PDE method and then Theorem 1.3 using a construction from matching motions. In Chapter 4 we prove the spectral uniqueness theorem and the Łojasiewicz inequality to prove Theorem 1.4. In Chapter 5 we study the rotational symmetry of self-expanders and prove Theorem 1.5 and Theorem 1.6. We also present a proof of the existence of a singular flow line Proposition 5.17 and a singular self-expander Theorem 5.20 to illustrate the necessity of some assumptions made in Theorem 1.5.
Chapter 2

Notations and Preliminaries

2.1 Notations

Throughout the thesis, lower case letters such as $x$ denote points in $\mathbb{R}^{n+1}$, while upper case letters such as $X$ denote points in the spacetime $\mathbb{R}^{n+1} \times \mathbb{R}$. $B_r(x)$ denotes the Euclidean ball of radius $r$ centered at $x$, and $P_r(X)$ denotes the (backwards) parabolic cylinder centered at $X = (x, t)$ of radius $r$, i.e.

$$P_r(X) = B_r(x) \times (t - r^2, t].$$

$T_r(A)$ denotes the tubular neighborhood of $A \subset \mathbb{R}^{n+1}$ of radius $r$. Finally, for $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}$, let

$$B^n_r(x) = \{(y', y_{n+1}) \in \mathbb{R}^{n+1} \mid |x' - y'| < r, y_{n+1} = x_{n+1}\},$$

and let $C_r(x)$ be the open cylinder of height $r$ over $B^n_r(x)$, i.e.

$$C_r(x) = \{(y', y_{n+1}) \in \mathbb{R}^{n+1} \mid |x' - y'| < r, |x_{n+1} - y_{n+1}| < r\}.$$ 

By a $C^{k,\alpha}$-hypersurface in $\mathbb{R}^{n+1}$ we mean a properly embedded $n$-dimensional $C^{k,\alpha}$-submanifold in $R^{n+1}$. We will distinguish a point $p \in \Sigma$ and its position vector $x(p)$. Intrinsic balls on a hypersurface $\Sigma$ will be denoted by $B^\Sigma_r(p)$, and the normal vector field will be denoted by $\mathbf{n}_\Sigma$. Given a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ and any function
v : Σ → ℝ, let Σ_v be the normal graph of v over Σ, i.e. Σ_v = f_v(Σ) where

\[ f_v(p) = x(p) + v(p)n_Σ(p), \quad p ∈ Σ. \]

By a hypercone we mean a set \( C ⊂ \mathbb{R}^{n+1} \) that is invariant under dilations, i.e. \( ρC = C \) for all \( ρ > 0 \). We will use the word cone to refer to a hypercone for the rest of the thesis. The link of \( C \) is \( \mathcal{L}(C) = C \cap S^n \). We say \( C \) is of class \( C^{k,α} \) if \( \mathcal{L}(C) \) is a \( C^{k,α} \)-hypersurface of \( S^n \). A hypersurface \( Σ \) is \( C^{k,α} \)-asymptotically conical to \( C \) if

\[ \lim_{ρ→0^+} ρΣ = C \text{ in } C^{k,α}_{\text{loc}}(\mathbb{R}^{n+1} \setminus \{0\}). \]

### 2.2 Basic geometric measure theory

In this section recall some basic facts from geometric measure theory that we will use throughout the presentation. The basic references are [78] and [55]. The main objects we study are rectifiable integral varifolds, which is a generalization of differentiable manifolds.

**Definition 2.1.** A \( n \)-dimensional varifold on a domain \( U ⊂ \mathbb{R}^{n+1} \) is a Radon measure on \( U \times G(n+1, n) \), where \( G(n+1, n) \) denotes the Grassmann bundle of all \( n \)-dimensional linear subspace of \( \mathbb{R}^{n+1} \).

Notice that an \( n \)-dimensional varifold gives rise to a Radon measure on \( Ω \) via the projection \( U × G(n+1, n) → U \). A Radon measure \( μ \) is integer \( n \)-countably rectifiable if there exists a \( \mathcal{H}^n \)-measurable set \( X ⊂ \mathbb{R}^{n+1} \) and a integer-valued locally integrable density function \( θ ≥ 0 \) such that \( X = \{θ > 0\} \) \( \mathcal{H}^n \)-a.e. and \( μ = \mathcal{H}^n \cap θ \). Equivalently, this means that \( μ \) has an approximate tangent space of integer multiplicity \( \mathcal{H}^n \)-a.e.

Given an integer \( n \)-rectifiable Radon measure \( μ \), define a varifold \( V(μ) \) by

\[ V(μ)(φ) = \int φ(x, T_xμ)dμ(x), \]

12
for \( \phi \in C^0_c(U \times G(n+1,n), \mathbb{R}) \). Here \( T_x \mu \) denotes the approximate tangent space of \( \mu \) at \( x \). Such a varifold \( V \) is called an integer rectifiable \( n \)-varifold. For our purposes, we will refer to such an object simply as an integral varifold (as the codimension is always 1).

Let \( H \) be the generalized mean curvature vector of \( V(\mu) \) given by the first variation formula: for any compactly supported \( C^1 \) vector field \( X \),

\[
\int \text{div}_{V(\mu)} X d\mu = - \int H \cdot X d\mu.
\]

Notice that in the case \( V(\mu) \) is a classical smooth manifold, \( H \) agrees with the standard mean curvature vector.

### 2.3 Integral Brakke flows

Here we define the most important weak formulation of mean curvature flow in our study based on the seminal work of Brakke [15] (cf. Ilmanen [55], Tonegawa [82] and Kasai–Tonegawa [61]). Note however that the definition below makes no use of the deep orthogonality theorem therein. Roughly speaking, we replace the smooth manifold by varifold and flow by the generalized mean curvature instead of the usual mean curvature vector.

**Definition 2.2.** Given an open set \( U \subset \mathbb{R}^{n+1} \), by an integral \( n \)-Brakke flow in \( U \) we mean a family of integral \( n \)-rectifiable Radon measures \( \mathcal{M} = \{ \mu_t \}_{t \in I} \) such that:

(a) For a.e. \( t \in I \), \( V(\mu_t) \) has locally bounded first variation and its generalized mean curvature vector \( H \) is orthogonal to the approximating tangent space of \( V(\mu_t) \) \( x \)-a.e.

(b) For any bounded interval \( [a, b] \subset I \) and compact set \( K \subset U \),

\[
\int_a^b \int_K (1 + |H|^2) d\mu_t dt < \infty.
\]
(c) For \([a, b] \subset I\) and every \(\phi \in C^1_c(U \times [a, b], \mathbb{R}^+)\),
\[
\int \phi d\mu_b - \int \phi d\mu_a \leq \int_a^b \int \left( -\phi |H|^2 + H \cdot \nabla \phi + \frac{\partial \phi}{\partial t} \right) d\mu_t dt.
\]

Notice that in condition (c) above, an inequality is present instead of an equality. This is to account for the potential sudden loss of mass that may occur during the flow. As Brakke noted in [15], sometimes this loss of mass is unavoidable. However, we want to exclude certain artificial examples, constructed by replacing a Brakke flow with empty sets after a certain time. Given a space time point \((x_0, t_0)\) and a sequence \(\lambda_i \to \infty\), consider the parabolic rescaling of the flow \(M\):
\[
M_i^t = \lambda_i(M_{\lambda_i^{-2}t+t_0} - p_0),
\]
defined on \(I_i = \{\lambda_i^2(t - t_0) \mid t \in I \cap (-\infty, t_0)\}\). By compactness theorem for Brakke flows (see eg. [55, 7.1]), up to passing to a subsequence, the rescaled flows \(M^i\) converges to an integral Brakke flow \(M^\infty\) defined on \((-\infty, 0)\). We call any such limit a tangent flow to \(M\) at \((x_0, t_0)\). Note importantly that the tangent flow needs not be unique (cf. [75, 26, 33]). Near a smooth point of the Brakke flow, all tangent flows will be a multiplicity one hyperplane. Tangent flows obtained this way turn out to be self-similar shrinking, a fact which forms the basis of the singularity analysis of Brakke flows. However we will rarely discuss this aspect of the tangent flow in this thesis.

**Definition 2.3.** An integral Brakke flow \(M\) is unit-regular if the following holds: for each \(t_0 \in I\) and \(x_0 \in \text{supp} \mu_{t_0}\), if one of the tangent flows at \((x_0, t_0)\) is given by a multiplicity 1 hyperplane, then \(M\) is smooth in a spacetime neighborhood of \((x_0, t_0)\).

Unit-regularity prevents the sudden loss of mass when the flow is near a multiplicity 1 hyperplane. In particular the flow given by the replacement example is not unit-regular. This does not prevent all sudden vanishing as such behavior could happen near a higher multiplicity hyperplane.
We will also need the notion of cyclicity. An integral Brakke flow is said to be cyclic if the associated mod 2 flat chain \([V(\mu_t)]\) is boundaryless for a.e. \(t \in I\) (see works of White for a quick overview of flat chains [89] or [87]). This condition is imposed to ensure that our flow does not encounter a singularity modeled on a triple junction, i.e. three hyperplanes meeting at an equal angle. This configuration is troublesome in some application of the Hopf lemma (and indeed exists - see Theorem 5.20).

Henceforth, we will assume that all Brakke flows in our consideration are integral, unit-regular and cyclic, and refer to such a flow simply as a Brakke flow. Notice that a smooth mean curvature flow, treated as a Brakke flow, will have these properties. Moreover, any Brakke flow produced by the elliptic regularization method [55] will also automatically have these properties.

We recall the monotonicity formula of Huisken [53]. Note that the formula was originally obtained for smooth MCFs, but the identical proof works for Brakke flows.

**Theorem 2.4** (Monotonicity formula, [53]). Let \(\{\mu_t\}_{t \geq 0}\) be a Brakke flow on \(\mathbb{R}^{n+1}\) with bounded area ratio, i.e.

\[
\sup_{x \in \mathbb{R}^{n+1}} \sup_{r > 0} \frac{\mu_0(B_r(x))}{\omega_n r^n} \leq D < \infty,
\]

then, for any \(x_0 \in \mathbb{R}^{n+1}\) and \(t_0 > 0\), we have for \(0 \leq t_1 < t_2 < t_0\),

\[
\int \rho_{(x_0,t_0)}(\cdot,t_1)d\mu_t - \int \rho_{(x_0,t_0)}(\cdot,t_2)d\mu_t \geq \int_{t_1}^{t_2} \int \left| H + \frac{(x - x_0)^1}{2(t_0 - t)} \right|^2 \rho_{x_0,t_0}d\mu_t dt \geq 0,
\]

where

\[
\rho_{(x_0,t_0)}(x,t) = (4\pi(t_0 - t))^{-n/2}e^{-|x - x_0|^2/(4(t_0 - t))}
\]

is the backward heat kernel centered at \((x_0, t_0)\).

Motivated by the above theorem, Colding and Minicozzi defined the notion of
entropy of a hypersurface $\Sigma$ in their seminal work [32]:

$$
\lambda[\Sigma] = \sup_{x_0 \in \mathbb{R}^{n+1}, t > 0} (4\pi t)^{-\frac{n}{2}} \int_\Sigma e^{\frac{|x(p)-x_0|^2}{4t}} \, d\mathcal{H}^n.
$$

In other words, the entropy is the maximum of the Gaussian area of $\Sigma$ under all possible dilations and translations. It roughly measures the complexity of the hypersurface. The hyperplane has entropy 1, while the spheres $\mathbb{S}^n$ has slightly larger entropy $\lambda[\mathbb{S}^n] > \sqrt{2}$, by a computation of Stone [81] (The round spheres, in fact, have the lowest entropy among closed hypersurfaces – see [7] and [94]). The important observation is that, by Theorem 2.4, the entropy is nonincreasing along a MCF (or more generally a Brakke flow). Hence it makes sense define the entropy of a MCF to be the entropy of its initial data.

Finally, under the rescaling (1.5), a Brakke flow becomes a rescaled Brakke flow, which satisfies the same conditions as above, except that condition (c) is replaced by

(c') For $[a, b] \subset I$ and every $\phi \in C^1_c(U \times [a, b], \mathbb{R}^+)$,

$$
\int_a^b \int (-\phi \mathbf{H} \cdot \left(-\frac{\mathbf{x}^\perp}{2} + \mathbf{H}\right) + \left(-\frac{\mathbf{x}^\perp}{2} + \mathbf{H}\right) \cdot \nabla \phi + \frac{\partial \phi}{\partial t}) \, d\mu_t \, dt.
$$

Of course, a smooth rescaled Brakke flow satisfies the RMCF equation (1.6).

### 2.4 Function spaces

In this section we define the relevant function spaces on manifolds which we will use later. Let $\Sigma$ be a smooth hypersurface and $\nabla_\Sigma$ be the induced Levi-Civita connection on $\Sigma$. For $\alpha \in (0, 1)$ we define the Hölder seminorm

$$
[f]_{\alpha; \Sigma} = \sup_{\substack{p \neq q \in \Sigma \atop q \in B_{\delta}(p)}} \frac{|f(p) - f(q)|}{d_\Sigma(p, q)^\alpha},
$$

where $\delta$ is such that $B_\delta^\Sigma(p)$ is strictly geodesically convex. For $k$ an integer, define $C^{k, \alpha}(\Sigma)$ to be the Hölder space with norm

$$
\|f\|_{C^{k, \alpha}(\Sigma)} = \sum_{i=0}^k \sup_{\Sigma} \left| \nabla_\Sigma^i f \right| + \left| [\nabla_\Sigma^k f]_{\alpha; \Sigma} \right|.
$$
The parabolic Hölder seminorm is defined similarly: for an interval $I$ let

$$[f]_{\alpha;\Sigma \times I} = \sup_{(p,t_1)\neq(q,t_2)\in\Sigma \times I \atop q \in B^\Sigma_{\delta}(p)} \frac{|f(p) - f(q)|}{d\Sigma(p,q)^\alpha + |t_1 - t_2|^{\alpha/2}}.$$ 

The corresponding parabolic Hölder space $C^{k,\alpha}_P(\Sigma \times I)$ consists of functions $f$ such that

$$\|f\|_{C^{k,\alpha}_P(\Sigma \times I)} = \sum_{i+2j \leq k} \sup_{\Sigma \times I} \left| \nabla^i_{\Sigma} \nabla^j_t f \right| + \sum_{i+2j = k} \left[ \nabla^i_{\Sigma} \nabla^j_t f \right]_{\alpha;\Sigma \times I}.$$ 

We shall write $C^{k,\alpha} = C^{k,\alpha}(\Sigma)$ and $C^{k,\alpha}_P = C^{k,\alpha}_P(\Sigma \times (-\infty,0])$ when $\Sigma$ is clear from context.

Finally we define the weighted Sobolev space. For an integer $k$ we say $f \in W^{k,\frac{1}{4}}(\Sigma)$ if

$$\|f\|_{W^{k,\frac{1}{4}}(\Sigma)} = \left( \int_{\Sigma} \sum_{i=0}^{k} \left| \nabla^i_{\Sigma} f \right|^2 e^{\frac{|x|^2}{4}} d\mathcal{H}^n \right)^{1/2} < \infty.$$ 

It is clear that $W^{k,\frac{1}{4}}$ are Banach spaces. Moreover, we say a map $f(x,s) \in W^{k,\frac{1}{4}}(\Sigma \times (-\infty,0])$ if $f(\cdot, t) \in W^{k,\frac{1}{4}}(\Sigma)$ and $f(x, \cdot) \in L^2((-\infty,0))$. Throughout the thesis, the only relevant weight is $e^{\frac{|x|^2}{4}}$, so we will write $W^k = W^k_{\frac{1}{4}}$ and further write $W^0 = W$.

The square bracket $\langle \cdot, \cdot \rangle$ will denote the standard inner product in $W$, i.e.

$$\langle f, g \rangle = \int_{\Sigma} f g e^{\frac{|x|^2}{4}},$$ 

whereas $\cdot$ will denote the standard inner product in $\mathbb{R}^{n+1}$.

### 2.5 Self-expanders

In this section we recall the basics of the geometry of self-expanders, which form the building blocks of our construction of the flows. Our main references are the foundational works of Bernstein [5] and Bernstein–Wang [11, 9, 14, 13, 12]. Recall that we defined a self-expander to be a solution of the following geometric PDE (1.4):

$$H_{\Sigma} = \frac{x_1}{2}.$$
We will start by giving a variational structure to self-expanders. Define the *expander functional*  
\[
E[\Sigma] = \int_{\Sigma} e^{\frac{|x|^2}{4}} d\mathcal{H}^n.  
\]

(2.1)

Due to the large weight, the functional is not finite in general, and so it can be only formally defined. Nevertheless, we can compute its first and second variations.

**Proposition 2.5** (First variation formula of \(E, [11]\)). Let \(\Sigma \subset \mathbb{R}^{n+1}\) be a \(C^2\)-hypersurface and \(\{\Psi_s\}_{s \in (-\varepsilon, \varepsilon)}\) be a compactly supported variation of \(x|_\Sigma\) with \(\frac{\partial \Psi_s}{\partial s}|_{s=0} = V\), then

\[
\frac{d}{ds}\bigg|_{s=0} E[\Psi_s(\Sigma)] = -\int_{\Sigma} V \cdot \left( H_\Sigma - \frac{x^\perp}{2} \right) e^{\frac{|x|^2}{4}} d\mathcal{H}^n.  
\]

From this, we have the following equivalent characterizations of self-expanders.

**Proposition 2.6.** Let \(\Sigma \subset \mathbb{R}^{n+1}\) be a smooth hypersurface. The following are equivalent.

(1) \(\Sigma\) satisfies the self-expander equation (1.4).

(2) \(\Sigma\) is a critical point of the expander functional, \(E\).

(3) \(\{\sqrt{t}\Sigma\}_{t \in (0, \infty)}\) is a smooth mean curvature flow.

**Proof.** (1) and (2) are equivalent due to Proposition 2.5. Indeed, equation (1.4) is simply the Euler–Lagrange equation of the expander functional.

Assume (1) holds. Then we can compute for any \(p \in \Sigma\),

\[
\left( \frac{\partial \sqrt{t}x}{\partial t} \right)^\perp = \frac{1}{2\sqrt{t}} x^\perp = \frac{1}{\sqrt{t}} H_\Sigma(p) = H_{\sqrt{t}\Sigma}(\sqrt{t}p),
\]

where \(x = x(p)\). The converse can be deduced similarly. 

The following proposition provides us with the basic curvature decay of a asymptotically conical self-expander.
**Proposition 2.7.** Let $\mathcal{C}$ be a smooth cone. Suppose $\Sigma$ is a self-expander $C^{2,\alpha}$-asymptotic to $\mathcal{C}$. There exists $C > 0$ such that $|A_{\Sigma}(p)| \leq C |x(p)|^{-1}$. Moreover, there is $N > 0$ such that $\Sigma \setminus B_{NR}(0) \subset \mathcal{T}_{R^{-1}}(\mathcal{C})$ for $R > 1$.

**Proof.** Since $\Sigma$ is smooth and $\rho\Sigma \to \mathcal{C}$ in $C^{2,\alpha}_{loc}(\mathbb{R}^n \setminus \{0\})$, for sufficiently small $\rho$ we have on $\rho\Sigma \cap (B_1(0) \setminus \{0\})$

$$|A_{\rho\Sigma}| \leq C.$$ 

Hence for $p \in \Sigma$ with $|x(p)|$ sufficiently large depending on the above,

$$|A_{\Sigma}(p)| = \rho |A_{\rho\Sigma}(\rho p)| \leq C |x(p)|^{-1}$$

if we pick $\rho = \frac{1}{2} |x(p)|^{-1}$ so that $\rho p \in B_1(0)$. This proves the bound on the second fundamental form.

Together with the self-expander equation, these imply that there is $C > 0$ with

$$\text{dist}(p, \mathcal{C}) < C |x(p)|^{-1}$$

for $p \in \Sigma \setminus B_1(0)$. Finally by scaling it follows that there is $N$ such that for $R \geq 1$.

$$\Sigma \setminus B_{NR}(0) \subset \mathcal{T}_{R^{-1}}(\mathcal{C}).$$

**Remark.** Note that similar to the above we can also estimate the derivatives of $A_{\Sigma}$:

$$|\nabla^m_{\Sigma} A_{\Sigma}(p)| \leq C |x(p)|^{-m-1},$$

provided the cone is sufficiently regular.

Next we consider the second variation of the expander functional. Note here we are only interest in a one-parameter family of (compactly supported) diffeomorphisms.

**Proposition 2.8** (Second variation formula, Proposition 4.2 of [11]). Given a self-expander $\Sigma$ and a compactly supported variation $\{\Psi_s\}_{s \in (-\varepsilon, \varepsilon)}$ such that $\frac{d}{ds}|_{s=0} \Psi_s(\Sigma) = V$, then

$$\left. \frac{d^2}{ds^2} \right|_{s=0} E[\Psi_s(\Sigma)] = - \int_{\Sigma} (V \cdot n_{\Sigma}) L_{\Sigma}(V \cdot n_{\Sigma}) e^{-\frac{|x|^2}{4}} d\mathcal{H}^n,$$
where \( L_\Sigma \) is the stability operator of \( \Sigma \) given by

\[
L_\Sigma = \Delta_\Sigma + \frac{1}{2} \mathbf{x} \cdot \nabla_\Sigma + |A_\Sigma|^2 - \frac{1}{2}.
\]

This allows us to discuss the stability of a self-expander in the sense of Morse. For a connected self-expander \( \Sigma \), the Morse index of \( \Sigma \), \( \text{ind}(\Sigma) \), is the biggest dimension of a linear subspace \( V \subset C^2_0(\Sigma) \) such that

\[
- \int_\Sigma v L_\Sigma ve \left| x \right|^2 4 d\mathcal{H}^n < 0 \text{ for all } v \in V \setminus \{0\}.
\]

Notice that the choice of \( V \) needs not be unique. \( \Sigma \) is a stable self-expander if it has Morse index 0. A number \( \mu \in \mathbb{R} \) is an eigenvalue of \( -L_\Sigma \) if there exists a function \( f \in W^2 \) such that \( -L_\Sigma f = \mu f \). According to Bernstein–Wang [9, Proposition 4.1], when \( \Sigma \) is smooth, \( -L_\Sigma \) has a discrete spectrum and we can therefore order the eigenvalues of \( -L_\Sigma \) as \( \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \) (counting multiplicities). By considering the Rayleigh quotient

\[
\frac{\int_\Sigma |\nabla_\Sigma v|^2 e^{|x|^2}}{\int_\Sigma |v|^2 e^{|x|^2}}
\]

for \( v \in C^{2,\alpha} \cap W^2(\Sigma) \setminus \{0\} \), it is easy to see that the first eigenvalue is indeed achieved by some eigenfunction. Moreover, standard spectral theory implies that the first eigenvalue has multiplicity 1. It follows that the index of \( \Sigma \) is equal to the number of negative eigenvalues of \( -L_\Sigma \) with multiplicity, and a self-expander is stable if and only if \( \lambda_1 \geq 0 \). We further say the self-expander \( \Sigma \) is strictly stable if \( \lambda_1 > 0 \), weakly stable if \( \lambda_1 = 0 \) and unstable otherwise. Additionally, we call \( f \in W^2 \) a Jacobi field if \( f \in \ker L_\Sigma \). Bernstein–Wang showed in [9, Lemma 7.1] (cf. earlier work of White [85]) that the set of cones that has an asymptotic self-expander with nontrivial Jacobi field is of first Baire category (i.e. a union of countably nowhere dense sets), so that for a generic cone, all smooth self-expanders asymptotic to the cone have no nontrivial Jacobi field.
One of the features of self-expanders is that the eigenfunctions have very fast exponential decay to offset the large exponential weight in (2.1). This is in contrast with the self-shrinker case where one has to be very careful about the decay rate.

**Proposition 2.9** (Proposition 2.1 of [9]). Suppose \( f \in W^2(\Sigma \setminus B_R(0)) \) for some \( R > 0 \) depending on \( \Sigma \) and satisfies \(-L \Sigma f = \mu f\), then

\[
\left\| e^{\frac{\beta}{2}|x|^2} f \right\|_{C^{k,\alpha}(\Sigma \setminus B_{4R}(0))} < \infty \tag{2.2}
\]

for any \( \beta < \frac{1}{2} \).

In fact, the precise \( C^0\)-asymptote of the first eigenfunction is also available:

**Theorem 2.10** (Proposition 3.2 of [12]). Suppose \( \Sigma \) is an asymptotically conical self-expander. If \( f \in W^2 \) satisfy \(-L \Sigma f = \lambda_1 f \) with \( \|f\|_W = 1 \), then there is \( C = C(\Sigma, \mu) \) such that

\[
C^{-1}(1 + |x(p)|^2)^{-\frac{1}{2}(n+1-2\lambda_1)} e^{-\frac{|x(p)|^2}{4}} \leq f(p) \leq C(1 + |x(p)|^2)^{-\frac{1}{2}(n+1-2\lambda_1)} e^{-\frac{|x(p)|^2}{4}}
\]

for any \( p \in \Sigma \).

Finally we record some computations for a normal graph over an asymptotically conical self-expander. Suppose \( v \in C^2(\Sigma) \) has \( \|v\|_{C^2} < \varepsilon \) sufficiently small so that \( f_v = x|_{\Sigma} + v n|_{\Sigma} \) is an embedding. It follows from a computation of Bernstein and Wang [13, Proposition A.1] that

\[
H_{\Sigma_v} + \frac{x}{2} \cdot n_{\Sigma_v} = -L_{\Sigma} v + Q(v, x \cdot \nabla_{\Sigma} v, \nabla_{\Sigma} v, \nabla_{\Sigma}^2 v), \tag{2.3}
\]

where \( H \) is the scalar mean curvature and \( Q \) is a homogeneous quadratic polynomial with bounded coefficients. Furthermore, \( Q \) can be written as

\[
Q(s, \rho, d, T) = a(s, \rho, d, T) \cdot d + b(s, d, T)s, \tag{2.4}
\]
where \( a, b \) are homogeneous polynomial of degree 1 with bounded coefficients. \( a \) and \( b \) satisfy the estimates

\[
\begin{align*}
|\mathbf{x}|^{2+j-\ell} \left| \nabla_{\Sigma}^{i} \nabla_{s}^{j} \nabla_{d}^{k} \nabla_{T}^{\ell} b(s, d, T) \right| & \leq C(|\mathbf{x}|^{-1} |s| + |d| + |\mathbf{x}| |T|), \\
|\mathbf{x}|^{1+j-\ell} \left| \nabla_{\Sigma}^{i} \nabla_{s}^{j} \nabla_{d}^{k} \nabla_{T}^{\ell} a(s, \rho, d, T) \right| & \leq C(|\mathbf{x}|^{-1} |s| + |d| + |\mathbf{x}| |T|).
\end{align*}
\] (2.5)

This follows, essentially, from Proposition 2.7. See [24, Lemma 3.6] for a proof in the self-shrinker case, which carries almost identically to our case (cf. [34, Theorem 2.1]).
Chapter 3

Existence of Mean Curvature Flows Coming out of Cones

The goal of this chapter is to establish Theorem 1.2 and Theorem 1.3. Throughout the chapter, the time variable $t$ is used for a mean curvature flow (MCF), and the variable $s$ is used for a rescaled mean curvature flow (RMCF) via (1.5).

3.1 Properties of flows coming out of cones

In this section we consider general MCFs coming out of cones. First we record a distance estimate along MCFs, using small balls as barriers.

Proposition 3.1 (cf. Lemma 4.3 of [8]). Suppose $\Sigma$ is an hypersurface $C^{2,\alpha}$-asymptotic to $\mathcal{C}$, and there is $N > 0$ such that $\Sigma \setminus B_{NR}(0) \subset \mathcal{T}_{R^{-1}}(\mathcal{C})$ for $R > 1$. If $\{\Sigma_t\}_{t \in [1,T]}$ is an integral Brakke flow starting from $\Sigma$ in $\mathbb{R}^{n+1}$, then there is a constant $N' > 0$ such that

$$\Sigma_t \setminus B_{N'R\sqrt{t}}(0) \subset \mathcal{T}_{R^{-1}\sqrt{t}}(\mathcal{C})$$

for $R > 1$.

Proof. For any $x \in \mathbb{R}^{n+1} \setminus (B_{NR}(0) \cup \mathcal{T}_{R^{-1}}(\mathcal{C}))$ let

$$\rho(x) = \inf \{ \rho' \geq 0 \mid B_{\rho'}(x) \cap (B_{NR}(0) \cup \mathcal{T}_{R^{-1}}(\mathcal{C})) \neq \emptyset \}$$

23
and let
\[
\rho_t(x) = \begin{cases} 
\sqrt{\rho(x)^2 - 2n(t-1)} & \rho(x)^2 \geq 2n(t-1) \\
0 & \rho(x)^2 < 2n(t-1)
\end{cases}
\]
be the corresponding MCF starting from \( B_\rho(x) \). Let
\[
U_t = \bigcup_{\rho_t(x) > 0} B_{\rho_t(x)}(x).
\]
be the time \( t \) slice of the above MCF. Since initially \( \Sigma \cap U_1 = \emptyset \), maximum principle implies that \( U_t \cap \Sigma_t = \emptyset \) for all \( t \in [1,T) \). This proves that
\[
\Sigma_t \setminus B_{NR + \sqrt{2n(t-1)}}(0) \subset T_{R^{-1} + \sqrt{2n(t-1)}}(C)
\]
(3.1)
since \( \mathbb{R}^{n+1} \setminus (B_{NR + \sqrt{2n(t-1)}}(0) \cup T_{R^{-1} + \sqrt{2n(t-1)}}(C)) \subset U_t \).

Next we consider the map \( \Phi : C \setminus \{0\} \times \mathbb{R} \to \mathbb{R}^{n+1} \) given by
\[
\Phi(x, \lambda) = x + \lambda n(x).
\]
Choose \( \lambda_0 < 1/2 \) depending on \( C \) small enough so that \( \Phi|_{C \times (-2\lambda_0,2\lambda_0)} \) is a diffeomorphism onto its image. It follows, since \( C \) is a cone, that \( \Phi \) is a diffeomorphism on the set \( \{(x, \lambda) \mid |\lambda| < 2\lambda_0 |x(x)|\} \). Consider, for some \( N' \) to be chosen later,
\[
V_t = \mathbb{R}^{n+1} \setminus (U_t \cup B_{N'\sqrt{t}}(0)).
\]
We first claim that \( N' \) can be chosen so that \( y \in V_t \) can be written as \( x + \lambda' |x| n(x) \) for some \( |\lambda'| < \lambda_0 \). Indeed, since \( y \notin U_t \) we have that \( \text{dist}(y,C) \leq R^{-1} + \sqrt{2n(t-1)} \) provided we choose \( N' \) large enough so that
\[
N'\sqrt{t} > NR + \sqrt{2n(t-1)}.
\]
Let \( x \) be the nearest point projection of \( y \) onto \( C \), then using \( R > 1 \) we have that
\[
\frac{|y - x|}{|x|} \leq \frac{1 + \sqrt{2nt}}{N'\sqrt{t} - 1 - \sqrt{2nt}} = \frac{(\sqrt{t})^{-1} + \sqrt{2n}}{N' - (\sqrt{t})^{-1} - \sqrt{2n}} < \frac{1 + \sqrt{2n}}{N' - 1 - \sqrt{2n}} < \lambda_0
\]
provided we choose \( N' \) large depending on \( n \) only. Hence the claim holds.
Let $y \in \Sigma_i \setminus B_{N'R/\sqrt{t}}(0) \subset V_i$. We claim that, up to further increasing $N'$, $\text{dist}(y, C) < R^{-1}\sqrt{t}$ and this will finish the proof. To this end consider $y_0 = x + \lambda_0 |x| n_C$. We have that

$$|y_0| = |x + \lambda_0 |x| n_C| = \sqrt{1 + \lambda_0^2} |x| > NR + \lambda_0 |x| - \frac{1}{3} R^{-1},$$

if $N'$ is chosen large enough so that

$$|x| \geq (N'R - \sqrt{2n}) \sqrt{t} - R^{-1} > 4NR - R^{-1},$$

where we used the fact that $\sqrt{1 + \lambda_0^2} - \lambda_0 > \frac{1}{3}$. This implies that $B_{\lambda_0 |x| - R^{-1}}(y_0) \cap B_{NR}(0) = \emptyset$. Since $\text{dist}(y_0, C) = \lambda_0 |x|$ we conclude that

$$\rho(y_0) = \lambda_0 |x| - R^{-1}.$$

Increasing $N'$ if necessary we can also ensure that

$$\lambda_0 |x| - R^{-1} > \sqrt{2n(t-1)}.$$

Since $y \not\in B_{\rho_t(y_0)}(y_0) \subset U_i$ we can estimate

$$\text{dist}(y, C) \leq \text{dist}(y_0, C) - \rho_t(y_0) = \lambda_0 |x| - \sqrt{(\lambda_0 |x| - R^{-1})^2 - 2n(t-1)}.$$

To finish the proof we compute, for large $N'$,

$$(\lambda_0 |x| - R^{-1}\sqrt{t})^2 - ((\lambda_0 |x| - R^{-1})^2 - 2n(t-1))$$

$$= 2R^{-1} \lambda_0 |x| (1 - \sqrt{t}) + R^{-2} (t-1) + 2n(t-1)$$

$$= R^{-1}(\sqrt{t}-1)(-2\lambda_0 |x| + (R^{-1} + 2nR)(\sqrt{t}+1)) \leq 0.$$

which is equivalent to

$$\lambda_0 |x| - R^{-1}\sqrt{t} \leq \sqrt{(\lambda_0 |x| - R^{-1})^2 - 2n(t-1)} \implies \text{dist}(y, C) \leq R^{-1}\sqrt{t}$$

provided $N'$ is chosen large enough so that

$$\lambda_0 |x| \geq (N'R - \sqrt{2n}) \sqrt{t} - R^{-1} \geq (R^{-1} + 2nR)\sqrt{t}.$$

□
Next we recall the pseudolocality theorem for MCFs (see [56, Theorem 1.5] or [19]). The theorem asserts that, locally, graphicality propagates with suitable scale along MCF. In particular, this theorem guarantees that, if the flow is graphical over a neighborhood of \( x \in \Sigma_t \), there is a spacetime neighborhood of \((x,t)\) such that the flow remains graphical.

**Theorem 3.2.** Let \( \{\Sigma_t\}_{t \in [0,T)} \) be a mean curvature flow in \( \mathbb{R}^{n+1} \). Given any \( \eta > 0 \), there is \( \delta, \varepsilon > 0 \) such that: if \( x \in \Sigma_0 \) and \( \Sigma_0 \cap C_1(x) \) is a graph over \( B_1^n(x) \) with Lipschitz constant bounded by \( \varepsilon \), then \( \Sigma_t \cap C_\delta(x) \) can be written as a graph over \( B_\delta^n(x) \) with Lipschitz bounded by \( \eta \) for any \( t \in [0,\delta^2) \cap [0,T) \).

Using the above, we can show that a MCF coming out of a cone is fully smooth away from a large compact set, a key regularity property repeatedly used in this thesis. The proof is standard (see eg. [14, Lemma 5.3]), as once we know that the flow remains graphical in a spacetime neighborhood, the MCF equation can be written as a parabolic PDE, where standard estimates will be available.

**Proposition 3.3.** Suppose \( \Sigma \) is a hypersurface \( C^{2,\alpha} \)-asymptotic to \( C \) and let \( \{\Sigma_t\}_{t \in [1,T)} \) be a MCF starting from \( \Sigma \). If there is \( N > 0 \) such that \( \Sigma_1 \setminus B_{NR}(0) \subset T_{R^{-1}}(C) \) for all \( R > 1 \), then there is \( N' > 0 \) such that \( \Sigma_t \setminus B_{N'\sqrt{t}}(0) \) can be written as a (smooth) normal graph over \( C \setminus B_{N'\sqrt{t}}(0) \). In particular we have the uniform curvature bound

\[
\sup_{t \in [1,T)} \sup_{B_{N'\sqrt{t}}(0)} |A_{\Sigma_t}| < \infty.
\]

**Proof.** Fix \( t \in [1,T) \) and let \( \delta > 0 \). Since \( \Sigma \) is asymptotically conical, by Proposition 2.7 there is \( N_1 \) and \( \varepsilon > 0 \) depending on \( \delta \) such that for any \( x_0 \in C \setminus B_{N_1}(0) \), \( \Sigma_1 \cap C_\eta(x_0) \) can be written as a graph of \( \tilde{f}_{x_0}(x) \) over some neighborhood of \( x_0 \) in \( T_{x_0}C \) containing \( B_\eta^n(x_0) \). Here \( \eta = \varepsilon |x| \). Moreover up to increasing \( N_1 \) we can ensure Proposition 3.1 holds, and that \( \tilde{f}_{x_0} \) satisfies the estimates

\[
\sum_{i=0}^{2} \eta^{-1+i} \sup_{B_\eta^n(x_0)} |\nabla^i \tilde{f}_{x_0}| + r^{1+\alpha} |\nabla^2 \tilde{f}_{x_0}| \alpha < \delta \quad (3.2)
\]
By Theorem 3.2, given $\varepsilon > 0$, there is $N_1 > 0$ such that for every $x_0 \in \Sigma_1 \setminus B_{N_1}(0)$ and $\tau \in [1, t]$, $\Sigma_\tau \cap C_{\eta/2}(x_0)$ can be written as a normal graph over $\Sigma_1 \cap B_{\eta/2}^n(x_0)$. Combining the above two facts, we see that for sufficiently small $\delta$ and $\varepsilon$, $\Sigma_\tau \cap C_{\eta/2}(x_0)$ can be written as the graph of a function $f_{x_0}(\tau, x)$ over some neighborhood of $T_{x_0}C$ for all $\tau \in [1, t]$ and $x_0 \in C \setminus B_{N_1}(0)$. Moreover, $f_{x_0}$ satisfies the pointwise estimates

$$(\eta/2)^{-1} \sup_{B_{\eta/4}^n(x_0)} |f_{x_0}(\tau, \cdot)| + \sup_{B_{\eta/2}^n(x_0)} |\nabla_x f_{x_0}(s, \cdot)| < 1.$$ 

For the rest of the proof we fix an $x_0$ and put $f = f_{x_0}$. Since $\{\Sigma_s\}_{\tau \in [1, t]}$ is a graphical MCF near $x_0$, $f$ satisfies the evolution equation:

$$\frac{\partial f}{\partial \tau} = \sqrt{1 + |\nabla_x f|^2} \text{div} \left( \frac{\nabla_x f}{\sqrt{1 + |\nabla_x f|^2}} \right).$$

This is a quasilinear parabolic equation, so we may use Hölder estimates (e.g. Theorem 1.1 in Chapter 4 of [63]) to get that

$$\sup_{\tau \in [1, t]} \left[ \nabla_x f(\tau, \cdot) \right]_{\alpha; B_{\eta/4}(x_0)} + \sup_{B_{\eta/4}^n(x_0)} \left[ \nabla_x f(\cdot, x) \right]_{\alpha/2; [1, t]} \leq C(\eta/4)^{-\alpha},$$

for any $\alpha \in (0, 1)$. Standard Schauder estimates (see e.g. Chapter 5 of [64] or Theorem 5.1 in Chapter 4 of [63]) yield higher order estimates of the form

$$\sum_{i=0}^2 (\eta/8)^{-1} \sup_{B_{\eta/8}^n(x_0)} \left| \nabla^i f(\tau, \cdot) \right| + (\eta/8)^{1+\alpha} \left[ \nabla^2 f(\tau, \cdot) \right]_{\alpha; B_{\eta/8}(x_0)} \leq C$$

for $s \in [1, t]$ and

$$\sup_{x \in B_{\eta/8}^n(x_0)} \left[ \nabla_x f_{x_0}(\tau, x) \right]_{\frac{3}{2}; [1, t]} \leq C(\eta/8)^{-1}.$$ 

From the above we may estimate

$$|f_{x_0}(\tau, x) - f_{x_0}(1, x_0)| \leq C(\tau - 1)(\eta/8)^{-1} + \delta |x - x_0| + C(\eta/8)^{-1} |x - x_0|^2$$

where we also used the evolution equation and the fact that $|\nabla_x f(1, x_0)| < \delta$ from (3.2). These implies that, for $\rho < 1/8$, a fixed $\tau \in [1, t]$ and $x_0 \in C \setminus B_{\sqrt{\eta/8}}(0)$,

$$(\rho\eta)^{-1} \sup_{x \in B_{\rho\eta}^n(x_0)} |f(\tau, x)| \leq (\rho\eta)^{-1} N_1 |x_0|^{-1} + C(\tau - 1)\rho^{-1} \eta^{-2} + \delta + C\rho \leq \frac{(\rho\varepsilon)^{-1} N_1}{\tilde{N}^2 \tau} + \frac{C(\tau - 1)\rho^{-1}}{\tilde{N}^2 \tau} + \delta + C\rho$$

27
where we used that \(|f(1, x_0)| < N_1 |x_0|^{-1}\) by Proposition 3.1. The right hand side of the above equation can be made arbitrarily small provided we choose \(\delta\) and \(\rho\) small enough and \(\tilde{N}\) large enough. Similarly we can estimate the derivative
\[
|\nabla_{x_0} f(x_0)(\tau, x) - f(x_0(1, x_0))| \leq C(\eta/8)^{-1}|x - x_0| + C(\eta/8)^{-1}\sqrt{\tau - 1}
\]
and
\[
\sup_{x \in B^n_{\rho}(x_0)} |\nabla_x f(\tau, x)| \leq \delta + C\rho + C\frac{\sqrt{\tau - 1}}{N\sqrt{\tau}}
\]
which can be made arbitrarily small as well. These two decay estimates together with Schauder estimates above with \(\eta/8\) replaced by \(\rho\eta\) give
\[
\sum_{i=0}^2 (\eta/8)^{i-1} \sup_{B^n_{\eta/8}(x_0)} |\nabla^i f(\tau, \cdot)| + (\eta/8)^{1+\alpha}|\nabla^2 f(\tau, \cdot)|_{\alpha; B^n_{\eta/8}(x_0)} \leq \frac{1}{2} + C(\rho + \rho^{1+\alpha})
\]
which can be made to be less than 1 provided \(\rho\) is chosen small enough. This proves that \(\Sigma_t \cap C_{\rho\eta}(x_0)\) is a graph over (a neighborhood of) \(T_{x_0}C\) for \(x_0 \in C \setminus B_{\tilde{N}\sqrt{\tau}}(0)\) with derivative bounds up to the second order. The curvature bounds follow easily, and higher order bounds follow similarly using Schauder estimates.

As \(C\) is trivially asymptotic to \(C\), the above yields:

**Corollary 3.4.** Suppose \(C\) is a smooth cone and \(M = \{\mu_t\}_{t \in (0, \infty)}\) is a Brakke flow satisfying (1.3). If the flow is initially smooth, then there exists a constant \(R > 1\) depending only on \(\Sigma\) such that \(M_t \setminus B_{R\sqrt{\tau}}(0)\) is a smooth MCF for all \(t\).

## 3.2 Existence of tame ancient flows

In this section we prove Theorem 1.2 by showing the existence of rescaled mean curvature flows coming out of a given unstable self-expander. To simplify some terminologies, we call an ancient RMCF (or Brakke flow) tame if there exists a smooth self-expander \(\Sigma\) such that
\[
\lim_{s \to -\infty} \tilde{\Sigma}_s = \Sigma \text{ in } C_{\text{loc}}^\infty(\mathbb{R}^{n+1}).
\]
Let $C \subset \mathbb{R}^{n+1}$ be a smooth cone and fix a smooth self-expander $\Sigma$ asymptotic to $C$. Let $\tilde{\mathcal{M}} = \{\tilde{\mu}_s\}_{s < 0}$ be a tame ancient rescaled Brakke flow coming out of $\Sigma$. In view of the rescaling (1.5), Corollary 3.4 immediately yields:

**Proposition 3.5.** Let $\tilde{\mathcal{M}}$ be as above. There is $R = R(\Sigma) > 0$ such that $\tilde{\mathcal{M}}_s \setminus B_R(0)$ is a smooth RMCF. Moreover, $\tilde{\mathcal{M}}_s \setminus B_{2R}(0)$ can be written as a normal graph of $v$ over an open set $V \subset \Sigma \setminus B_R(0)$ such that

$$|v(p, s)| < C |x(p)|^{-1} \quad \text{for } p \in \Sigma \setminus B_R(0).$$

**Proof.** The distance estimate follows from Proposition 3.1 and the regularity follows from Corollary 3.4. □

The key fact we need to use in the subsequent analysis is that all such RMCF, up to a time translation, can be written as an entire normal graph over $\Sigma$.

**Proposition 3.6.** Given $\varepsilon > 0$, there is $s_\varepsilon$ such that $\tilde{\mathcal{M}}$ can be written as a normal graph of a smooth function $v$ over $\Sigma$ on $(-\infty, s_\varepsilon)$ such that

$$\|v\|_{C^{2,\alpha}_p} < \varepsilon \quad \text{for all } s < s_\varepsilon.$$  

**Proof.** Since the varifold convergence (3.3) has multiplicity 1, we may apply the Brakke regularity theorem [15] (see also [88]) to conclude that the convergence is in fact smooth on compact sets. Moreover, Proposition 3.5 shows that there exists $R = R(\Sigma)$ such that $\operatorname{supp} \tilde{\mu}_s \setminus B_{2R}(0)$ can be written as a normal graph $v$ over $\Sigma \setminus B_R(0)$ such that

$$\|v(p, s)\|_{C^{2,\alpha}(\Sigma \setminus B_R(0))} \leq C |x(p)|^{-1}.$$  

Now given $\varepsilon$ we choose $R$ sufficiently large so that $CR^{-1} < \varepsilon$ and then $s_\varepsilon$ sufficiently negative such that $\operatorname{supp} \tilde{\mu}_s \cap B_R(0)$ is a normal graph over $\Sigma \cap B_R(0)$ with $C^{2,\alpha}$ norm bounded by $\varepsilon$ as well. A similar argument applied to $\frac{\partial v}{\partial s}$ using the fact that $v$ satisfies the graphical RMCF equation gives the full parabolic estimate (up to decreasing $s_\varepsilon$). □
Remark. Arguing as in [24, Lemma 7.18], it is possible to obtain a weighted estimate as well, but in the self-expander case, we do not need to work with weighted Hölder space in the following due to the rapid decay of eigenfunctions.

Now suppose further that $\Sigma$ is unstable. We establish, using PDE method similar to [30], the existence of an $I$-parameter family of ancient solutions to the RMCF starting from $\Sigma$. Each one of these solutions will correspond to a MCF coming out of $\mathcal{C}$ that is not self-similar.

We work with the RMCF equation (1.6). The tame ancient flows will be constructed as limits of suitable perturbations of the initial hypersurface $\Sigma$ by its eigenfunctions. To this end, we fix an orthonormal basis $\{\phi_i\}_{i=1}^\infty \subset W$ consisting of eigenfunctions of $-L_\Sigma$, where $\phi_1, \ldots, \phi_I$ correspond to the negative eigenvalues, $\phi_{I+1}, \ldots, \phi_{I+K}$ correspond to the 0 eigenvalue and $\phi_{I+K+1}, \ldots$ correspond to positive eigenvalues. Define the map $\tau_- : \mathbb{R}^I \to W(\Sigma \times [-\infty, 0))$ by

$$\tau_-(a_1, \ldots, a_I) = \sum_{i=1}^I a_i e^{-\lambda_i s} \phi_i(x).$$

Let $\Pi_- : W \to W$ be the projection onto the negative eigenspace, i.e.

$$\Pi_-(v) = \sum_{i=1}^I \langle v, \phi_i \rangle \phi_i(x).$$

Using (2.3), the RMCF equation (1.6) reduces to the following nonlinear problem:

$$\frac{\partial}{\partial s} v = L_\Sigma v + Q(v, x \cdot \nabla_\Sigma v, \nabla_\Sigma v, \nabla^2_\Sigma v). \quad (3.4)$$

To solve this nonlinear PDE, we first consider the inhomogeneous linear PDE:

$$\frac{\partial}{\partial s} v = L_\Sigma v + h(p, s), \quad (p, s) \in \Sigma \times (-\infty, 0), \quad (3.5)$$

where $h(p, s)$ is a smooth function. We will assume in addition that $h(p, s)$ is exponentially decaying in time, i.e. there exists some $\delta > 0$ such that

$$\int_{-\infty}^0 |e^{-\delta s} \| h(\cdot, s) \|_W^2 | ds < \infty. \quad (3.6)$$
Using separation of variables, we can write a solution to (3.5) as
\[
v(p, s) = \sum_{i=1}^{\infty} v_i(p) \phi_i(s),
\]
where \(v_i\) formally solves the ODE
\[
v_i'(s) = -\lambda_i v_i(s) + \sum_{j=1}^{\infty} h_j(s) \phi_j(p),
\]
where \(h_j(s) = \langle h(\cdot, s), \phi_j \rangle\). This allows us to establish the existence of the \(L^2\)-solutions to (3.5).

**Proposition 3.7** (cf. Lemma 3.1 of [30]). Let \((a_1, \ldots, a_I) \in \mathbb{R}^I\) and suppose \(h\) satisfies (3.6). There exists a unique solution \(v\) to (3.5) such that \(\Pi_{-}(v(\cdot, 0)) = \sum_{i=1}^{I} a_i \phi_i\) and
\[
\int_{-\infty}^{0} \left| e^{-\delta' s} \|v(\cdot, s)\|_{W^1} \right|^2 ds < \infty,
\]
for some \(0 < \delta' < \min\{\delta, -\mu_I\}\). \(v\) can be written as
\[
v(p, s) = \sum_{i=1}^{\infty} v_i(s) \phi_i(p),
\]
where
\[
v_i(s) = \begin{cases} a_i e^{-\lambda_i s} - \int_{s}^{0} e^{\lambda_i (\sigma-s)} h_i(\sigma) d\sigma & i = 1, \ldots, I, \\ \int_{-\infty}^{s} e^{\lambda_i (\sigma-s)} h_i(\sigma) d\sigma & i = I+1, I+2, \ldots. \end{cases}
\]
Here \(h_i(s) = \langle h(p, \cdot), \phi_i(x) \rangle\). Additionally, \(v\) satisfies the estimate
\[
e^{-\delta' s} \|v(\cdot, s) - \tau_{-}(a_1, \ldots, a_I)\|_{W} \leq C \left( \int_{-\infty}^{0} \left| e^{-\delta \sigma} \|h(\cdot, \sigma)\|_{W} \right|^2 d\sigma \right)^{\frac{1}{2}}
\]
for all \(s < 0\).

**Proof.** Let us check that \(v\) solves (3.5). Indeed, differentiating under the integral sign gives
\[
\frac{\partial}{\partial s} v(s) = \sum_{i=1}^{I} \left[ -\lambda_i a_i e^{-\lambda_i s} + h_i(s) + \lambda_i \int_{s}^{0} e^{\lambda_i (\sigma-s)} h_i(\sigma) d\sigma \right] \phi_i
\]
\[
+ \sum_{i=I+1}^{\infty} \left[ h_i(s) - \lambda_i \int_{-\infty}^{s} e^{\lambda_i (\sigma-s)} h_i(\sigma) d\sigma \right] \phi_i
\]
Using $h(p, s) = \sum_{i=1}^{\infty} h_i(s) \phi_i(p)$ and $\lambda_i \phi_i = -L_{\Sigma} \phi_i$ we get that

\[
\frac{\partial}{\partial s} v(s) = \sum_{i=1}^{I} L_{\Sigma} \phi_i \left( a_i e^{-\lambda_i s} - \int_{s}^{0} e^{\lambda_i (\sigma - s)} h_i(\sigma) d\sigma \right) + \sum_{i=I+1}^{\infty} L_{\Sigma} \phi_i \int_{-\infty}^{s} e^{\lambda_i (\sigma - s)} h_i(\sigma) d\sigma + h(\cdot, s).
\]

Evidently this is (3.5). The fact that $\Pi_{-}(v(\cdot, 0)) = \sum_{i=1}^{I} a_i \phi_i$ is a straightforward computation. The uniqueness of the solution follows from the uniqueness of the linear parabolic equation $\frac{\partial}{\partial s} v = L_{\Sigma} v$ (with 0 initial data). (3.7) follows from the assumption on $h$ and the fact that, if $\lambda_i < 0$,

\[
\int_{-\infty}^{0} e^{-2\delta s - 2\lambda_i s} ds < \infty
\]
as long as $\delta' < -\lambda_i$. Finally the last estimates follow from Hölder’s inequality as the subtraction kills the $a_i e^{-\lambda_i s}$ terms.

We now follow the ideas of [11] to establish higher regularity of the solutions obtained above. First notice that for a given initial data $a = (a_1, \ldots, a_I)$, $\tau_-(a_1, \ldots, a_I)$ solves the linear homogeneous equation $\frac{\partial}{\partial s} v = L_{\Sigma} v$, hence by replacing $v$ by $v - \tau_-(a_1, \ldots, a_I)$ we can, without loss of generality, assume that $v$ is a solution to (3.5) with

\[
\Pi_{-}(v(\cdot, 0)) = 0.
\]

We will prove a Hölder estimate of the following type:

**Proposition 3.8.** Suppose that $h \in C_p^{0, \alpha}(\Sigma \times (-\infty, 0])$. Let $v$ be the solution to (3.5) with (3.8) constructed in Proposition 3.7, then for $s < 0$,

\[
[\nabla^2_{\Sigma} v]_{\alpha; \Sigma \times [s-1, s]} + \sup_{\Sigma \times [s-1, s]} \left( |\nabla_{\Sigma} v| + |\nabla^2_{\Sigma} v| \right) \leq C \| h \|_{C_p^{0, \alpha}(\Sigma \times (-\infty, 0])}.
\]

**Remark.** We emphasize that, in the self-expander case, all eigenfunctions are decreasing rapidly in the spatial variables by Proposition 2.9, and therefore we do not need to
invoke any weighted Schauder estimates. This is in contrast to the self-shrinker case in [24] as the eigenfunctions of the stability operator of self-shrinkers can potentially have polynomial growth in space, which is much larger than the weight $e^{-|x|^2/4}$.

**Proof.** WLOG we assume $s = 0$. In this proof it is often convenient to work with the original MCF. To this end let $\Sigma_t = \sqrt{t} \Sigma$ denote the MCF associated to the self-expander $\Sigma$. Since $\Sigma$ is asymptotically conical, we can find $R > 0$ such that, for any $p \in \Sigma \setminus B_R(0)$ and $t \in [e^{-1}, 1]$, $\Sigma_t \cap B_2(p)$ can be parametrized as a normal graph over some open subset of $T_p \Sigma$ with small $C^3$ norm, i.e. there exists $\psi_p : \Omega \times [e^{-1}, 1] \subset T_p \Sigma \times [0, 1] \to \mathbb{R}$ such that $\psi_p(0, 1) = 0$,

$$\sum_{i=0}^{3} \left| \nabla_{\mathbb{R}^n}^i \psi_p \right| + \sum_{i=0}^{1} \left| \partial_t \nabla_{\mathbb{R}^n}^i \psi_p \right| \leq \varepsilon, \quad (3.9)$$

and

$$\Psi_p(x, t) = x(p) + x(x) + \psi_p(x, t)n_\Sigma(p)$$

parametrizes $\Sigma_t \cap B_2(p), t \in [e^{-1}, 1]$.

We now fix $p \in \Sigma \setminus B_R(0)$ and consider, on $\Omega \times [e^{-1}, 1]$,

$$w(x, t) = t^{\frac{1}{2}}v(t^{-\frac{1}{2}}\Psi_p(x, t), \log t).$$

By (3.5) and the chain rule we see that $w$ satisfies the equation

$$\frac{\partial w}{\partial t} - \Delta_{\Sigma_t} w - |A_{\Sigma_t}|^2 w = t^{-\frac{1}{2}}h(t^{-\frac{1}{2}}\Psi_p(x, t), \log t),$$

Since $h \in C^0_{\mathcal{P}, \alpha}$, $|\Psi_p(x, t)| < 2|x(p)|$, and $t \in [e^{-1}, 1]$ which is compact, it follows that the right hand side $g_p = t^{-\frac{1}{2}}h(t^{-\frac{1}{2}}\Psi_p(x, t), \log t)$ is also Hölder continuous in spacetime with

$$|g_p(x, t_1) - g_p(y, t_2)| \leq C \|h\|_{C^0_{\mathcal{P}, \alpha}} (|x - y| + |t_1 - t_2|^{\frac{\alpha}{2}}).$$
We may now invoke interior Schauder estimates (see eg. [63]) to conclude that (note that \( |A_{\Sigma_{t}}| \leq C |x|^{-2} \) on \([e^{-1}, 1]\) by Proposition 2.7)

\[
\sup_{\Omega \times [e^{-1}, 1]} \left( |\nabla_{R^{n}} w| + \left| \nabla_{R^{n}}^{2} w \right| + \left[ \nabla_{R^{n}}^{2} w \right]_{\alpha} \right) \leq C \|h\|_{C^{0, \alpha}_{p}(\Sigma \times (-\infty, 0])}.
\]

Rescaling back and using (3.9), we see that

\[
\sup_{s \in [-1, 0]} \left( |\nabla_{\Sigma} v(p, s)| + \left| \nabla_{\Sigma}^{2} v(p, s) \right| + \left[ \nabla_{\Sigma}^{2} v \right]_{\alpha} \right) \leq C \|h\|_{C^{0, \alpha}_{p}}.
\]

As \( p \in \Sigma \setminus B_{R}(0) \) is arbitrary, we see the desired estimates hold on \((\Sigma \setminus B_{R}(0)) \times [-1, 0]\).

The estimates on \( \Sigma \cap B_{R}(0) \) follows from standard parabolic Schauder estimates for compact domains.

It is standard to bootstrap Theorem 3.8 to obtain higher order Schauder estimates, and a simple modification of the proof gives the corresponding estimates for \( \partial s/\partial v \). These lead to the following exponentially weighted variant, which follows easily from Theorem 3.8.

**Corollary 3.9.** Suppose that there is \( \delta > 0 \) such that

\[
\sup_{s < 0} e^{-\delta s} \|h\|_{C^{0, \alpha}_{p}(\Sigma \times [s-1, s])} < \infty.
\]

Let \( v \) be the solution to (3.5) with (3.8) constructed in Proposition 3.7. Then for every \( 0 < \delta' < \min \{ \delta, -\lambda_{I} \} \) we have, for \( s < 0 \)

\[
e^{-\delta' s} \|v\|_{C^{2, \alpha}_{p}(\Sigma \times [s-1, s])} \leq C \sup_{\sigma < 0} e^{-\delta \sigma} \|h\|_{C^{0, \alpha}_{p}(\Sigma \times [-\sigma, \sigma])}.
\]

We are now in position to prove the existence and uniqueness theorem for (3.4). Theorem 1.2 follows easily from the following.

**Theorem 3.10** (cf. Theorem 3.3 of [30]). Let \( \delta_{0} \in (0, -\lambda_{I}) \). For every sufficiently large \( \beta \), there exists \( \varepsilon = \varepsilon(\Sigma, \delta_{0}, \beta) \) such that the following holds: for any \((a_{1}, \ldots, a_{I}) \in \)
there exists a unique ancient solution \( v \) to (3.4) satisfying \( \Pi_- v(\cdot, 0) = \sum_{i=1}^I a_i \phi_i \) and

\[
e^{-\delta_0 s} \left\| v - \tau_-(a_1, \ldots, a_I) \right\|_{C^2_P(\Sigma \times [s-1, s])} \leq \beta \sum_{i=1}^I a_i^2
\]

for all \( s \leq 0 \).

**Proof.** In this proof we will also treat the nonlinear term \( Q \) as a function of \( s \) and \( p \in \Sigma \), and will write \( Q(p, s) \) whenever we do so. Let us fix \( a = (a_1, \ldots, a_I) \in B^I_\varepsilon(0) \).

Consider the space

\[
C^* = \left\{ v : \Sigma \times (-\infty, 0] \to \mathbb{R} \mid e^{-\delta_0 s} \left\| v \right\|_{C^2_P(\Sigma \times [s-1, s])} < \infty \text{ for all } s \leq 0 \text{, and } \Pi_- (v(\cdot, 0)) = \sum_{i=1}^I a_i \phi_i \right\}.
\]

Evidently \( C^* \) is a Banach space equipped with the norm

\[
\left\| v \right\|_{C^*} = \sup_{s < 0} e^{-\delta_0 s} \left\| v \right\|_{C^2_P(\Sigma \times [s-1, s])}.
\]

Given \( v \in C^* \cap C^\infty(\Sigma \times (-\infty, 0]) \), we let \( \Psi(v, a) \) be a solution to the linear problem

\[
\left( \frac{\partial}{\partial t} - L_\Sigma \right) \Psi(v, a) = Q(v, x \cdot \nabla_\Sigma v, \nabla_\Sigma v, \nabla_\Sigma^2 v).
\]

Using the estimates on the nonlinear term \( Q \), (2.5), we can find \( \eta \) sufficiently small depending on \( \Sigma \) and \( \alpha \) such that

\[
\left\| v \right\|_{C^*} < \eta \implies \int_{-\infty}^0 \left| e^{-\delta_0 s} \left\| Q(\cdot, s) \right\|_{W} \right|^2 ds < \infty.
\]

This allows us to apply (3.7) to conclude that \( \Psi(v, a) \) is well-defined (i.e. the solution exists and is unique). The function

\[
\tilde{\Psi}(v, a) = \Psi(v, a) - \sum a_i e^{-\lambda_i s} \phi_i
\]

then (uniquely) solves the problem

\[
\begin{align*}
\left( \frac{\partial}{\partial s} - L_\Sigma \right) w &= Q(v, x \cdot \nabla_\Sigma v, \nabla_\Sigma v, \nabla^2_\Sigma v) \text{ on } \Sigma \times (-\infty, 0], \\
\Pi_- w(\cdot, 0) &= 0.
\end{align*}
\]
Using the parabolic Schauder estimates Corollary 3.9, we get that for all \( s < 0 \)

\[
e^{-\delta_0 s} \| \Psi(v, a) \|_{C^2_P(\Sigma \times [s-1, s])} \\
\leq C \sup_{\sigma < 0} e^{-\delta_0 \sigma} \| Q(v, \nabla_{\Sigma} v, \mathbf{x} \cdot \nabla_{\Sigma} v, \nabla^2_{\Sigma} v) \|_{C^{2,0}_P(\Sigma \times [\sigma-1, \sigma])}.
\]

Using (2.5) again, we see that, upon taking supremum on the left hand side

\[
\| \Psi(v, a) \|_{C^*} \leq C \sup_{\sigma \leq 0} e^{-\delta_0 \sigma} \| v \|_{C^2_P(\Sigma \times [\sigma-1, \sigma])}.
\]

This shows that \( \Psi(v, a) \) is a well-defined map from \( C^* \) to itself provided \( \| v \|_{C^*} \) is sufficiently small.

Repeating the above argument for another function \( w \) with \( \| w \|_{C^*} \) sufficiently small shows that

\[
\| \Psi(v, a) - \Psi(w, a) \|_{C^*} \leq C(\| v \|_{C^*} + \| w \|_{C^*}) \| v - w \|_{C^*}.
\]

Thus \( \Psi(v, a) \) is a continuous contraction mapping on \( C^* \) provided \( \eta \) is chosen small enough. Hence by the contraction mapping theorem, there exists a unique fixed point \( \Psi(a) \) of the map \( \Psi(\cdot, a) \) in \( C^* \). This \( \Psi(a) \) is the solution we seek for the nonlinear problem (3.4) with \( \Pi_\perp \Psi(a) = \sum_{i=1}^I a_i \phi_i \). The regularity of \( \Psi(a) \) follows from standard parabolic regularity theory (with Corollary 3.9).

3.3 More on weak flows

In the next few sections, we establish, using a completely different method, Theorem 1.3. The proof relies heavily on tools from geometric measure theory (in addition to materials from Chapter 2), which we will briefly introduce.

The first tool we need is a generalization of the Brakke flow developed by [90]. The main issue is that, in general, there is no good existence theory even for Brakke flows, so we have to use an exhaustion approach on the initial data. Brakke flow with boundary ensures that we can get a reasonable flow for each truncated initial data.
For simplicity we will only work in the ambient manifold $\mathbb{B}_R(0)$. Given a hypersurface $\Sigma$ with boundary $\Gamma \subset \partial \mathbb{B}_R(0)$ in an open set $U \subset \mathbb{B}_R(0)$ and an integral $n$-rectifiable Radon measure $\mu$, the first variation formula with boundary asserts that

$$\int \text{div}_{V(\mu)} X d\mu = - \int H \cdot X d\mu + \int \nu_\mu \cdot X d(\mathcal{H}^{n-1} \lfloor \Gamma)$$

for any compactly supported $C^1$ vector field $X$, where $H$ is the generalized mean curvature vector and $\nu$ the approximating normal vector to $\Gamma$. By an integral $n$-Brakke flow with boundary $\Gamma$ in $U$ we mean a family of integral $n$-rectifiable Radon measures $\mathcal{M} = \{\mu_t\}_{t \in I}$ satisfying the items (a), (b) and (c) in Definition 2.2 with the extra condition:

(d) For a.e. $t \in I$ the normal vector satisfies $|\nu_{\mu_t}| \leq 1 \mathcal{H}^{n-1}$-a.e on $\Gamma$.

For simplicity we will refer to the above as Brakke flow with boundary $\Gamma$. This is unambiguous since, by item (d), the boundary $\Gamma$ stays unchanged under the Brakke flow.

As before, a Brakke flow with boundary $\Gamma$ is unit-regular if, for a spacetime point $X = (x, t)$, $\mathcal{M}$ is smooth and has no sudden mass loss if a tangent flow at $X$ is a multiplicity one plane or half-plane. $\mathcal{M}$ is cyclic if the associated mod-2 flat chain $[V(\mu_t)]$ has boundary equal to $\Gamma$. By works of White [90], Brakke flows with boundary $\Gamma$ produced by elliptic regularization are unit-regular and cyclic.

**Theorem 3.11** (Theorem 1.1, Theorem 14.1 of [90]). Let $\Sigma \subset \overline{\mathbb{B}_R(0)}$ be a hypersurface with boundary $\Gamma \subset \partial \mathbb{B}_R(0)$. There exists a unit-regular and cyclic Brakke flow with boundary $\Gamma$, $\mathcal{M} = \{\mu_t\}_{t \in [0, \infty)}$ with $\mu_0 = \mathcal{H}^n \lfloor \Sigma$.

Similarly, Brakke flow with boundary needs not be unique, but White’s theorem says that a unit-regular and cyclic one always exists. White proved in addition a strong boundary regularity theorem (Theorem 17.1 of [90]) in the codimension one
case, ruling out a scenario where interior singularities could accumulate to a boundary
singularity. Hence the boundary $\Gamma$ truly remains unchanged in the classical sense.

To ensure regularity of the limit, we will couple the above Brakke flow with
boundary with a set-theoretic flow, called the level set flow. The definition we use
below is from Evans–Spruck [40], who define the level set flow as viscosity solutions to
certain parabolic PDEs. Alternatively, it has been observed that the level set flow can
be characterized as the 'biggest flow' of a closed set satisfying the avoidance principle.
There is a rich literature on this more geometrically intuitive way of handling set flows
and we refer to [86], [49] and [6] for more information on this approach.

Given a closed set $\Gamma_0 \subset \mathbb{R}^{n+1}$, we choose any uniformly continuous function $u_0$
such that $\Gamma_0 = \{ x \in \mathbb{R}^{n+1} \mid u_0(x) = 0 \}$. It was shown in [40] that there exists a unique
$u \in C(\mathbb{R}^{n+1} \times [0, \infty))$ which is a viscosity solution to the problem
\[
\begin{cases}
  u_t = \sum_{i,j=1}^{n+1} \left( \delta_{ij} - \frac{u_{x_i}u_{x_j}}{|\nabla u|^2} \right) u_{x_i}x_j & \text{on } \mathbb{R}^{n+1} \times [0, \infty) \\
  u(x, 0) = u_0(x) & \text{on } \mathbb{R}^{n+1} \times \{0\}.
\end{cases}
\]
Let $\Gamma_t = \{ x \in \mathbb{R}^{n+1} \mid u(x, t) = 0 \}$. We call $K = \bigcup_{t \in [0, \infty)} \Gamma_t \times \{t\}$ the level set flow of
$\Gamma_0$.

Since the viscosity solution is unique, level set flow is also unique with given initial
data. However, level set flows might fatten, i.e. $K$ might develop a non-empty interior
(for example the figure eight fattens immediately). Formally, the level set flow of $\Gamma_0$
fattens if $\mathcal{H}^{n+1}(\Gamma_t) > 0$ for some $t > 0$. A theorem of Ilmanen [55, 11.3] shows that
fattening phenomenon is not generic and can therefore be perturbed away.

The fattening phenomenon of the level set flow is related to the nonuniqueness of
Brakke flow. In particular, the level set of a cone fattens if there is more than one
self-expander asymptotic to the cone (in the case where there is a unique self-expander,
the level set flow agrees with the usual smooth MCF associated to the expander).
Ilmanen [55] combined ideas from Brakke flows and level set flows and introduced
the notion of a matching motion, which turns out to be the suitable notion for our
purposes.

Recall that an n-dimensional current on $\mathbb{R}^{n+1}$ is an element of the dual space of compactly supported n-forms in $\mathbb{R}^{n+1}$. Given an integral Radon measure $\mu$ on $U \subset \mathbb{R}^{n+1}$, one can associate a current $T$ by letting

$$T(\omega) = \int_U \omega \cdot \xi d\mu$$

for a compactly supported n-form $\omega$ and a choice of orientation $\xi$ of the approximate tangent space of $T_x\mu$. We call $T$ an integer rectifiable current. If additionally the boundary of $T$ has locally finite mass, we say $T$ is an integral current. Conversely, given an integral current $T$ on $\mathbb{R}^{n+1}$, we can define its mass measure by

$$\mu_T(U) = \sup \{ T(\omega) \mid \omega \in \Lambda^k(M), \supp \omega \subset U, |\omega| \leq 1 \}.$$ 

Let $I_n(U)$ be the set of n-dimensional integral current in $U$.

**Definition 3.12** (8.1, 9.1 of Ilmanen [55]). Let $K \in I_{n+1}(\mathbb{R}^{n+1} \times \mathbb{R}^+), M = \{\mu_t\}_{t \in [0,\infty)}$ be a Brakke flow and $\Gamma_0 \in I_n(\mathbb{R}^{n+1})$ with finite mass and empty boundary. The pair $(K, M)$ is an enhanced motion with initial data $\Gamma_0$ if

(a) $\partial K = \Gamma_0$ and $K_t \in I_n(\mathbb{R}^{n+1})$ for a.e. $t \geq 0$;

(b) $\partial K_t = 0$ and $t \to K_t$ is continuous in the flat topology for $t \geq 0$;

(c) $\mu_0 = \mu_{\Gamma_0}, M[\mu_t] \leq M[\mu_0]$ and $\mu_{K_t} \leq \mu_t$ for a.e. $t \geq 0$.

If the pair $(K, M)$ further satisfies

(d) $\mu_t = \mu_{K_t} = \mu_{V(\mu_t)}$ for $t \geq 0$,

then we call it a matching motion.

In our applications $K$ is going to be the level set flow from $\Gamma_0$. A fundamental result of Ilmanen ([55, Chapter 12]) shows that a nonfattening level set flow is a matching motion, which justifies our abuse of notation here.
We will use the following result of S. Wang [83] which asserts that limit of low entropy matching motions is a matching motion. Recall that a sequence of Brakke flow $\mathcal{M}_i = \{\mu^i_t\}_{t \in [0, \infty)}$ converges to $\mathcal{M} = \{\mu_t\}_{t \in [0, \infty)}$ if $\mu^i_t \to \mu_t$ as Radon measures and, after possibly passing to a subsequence, $V(\mu^i_t) \to V(\mu_t)$ as varifolds for a.e $t \in [0, \infty)$.

**Theorem 3.13** (Theorem 3.5 of [83]). Let $(\mathcal{K}_i, \mathcal{M}_i)$ be a sequence of matching motions converging to an enhanced motion $(\mathcal{K}, \mathcal{M})$ with $\lambda[\mathcal{M}] < 2$, then $(\mathcal{K}, \mathcal{M})$ is a matching motion.

Remark. The theorem fails without the entropy assumption as the set-theoretic limit of a sequence of grim reapers (which has entropy 2) is two lines but the limit in the sense of currents is empty (as the two lines cancel each other).

### 3.4 The smooth one-sided flow

Let us now summarize the idea of the proof of Theorem 1.3. Given a (connected) unstable self-expander $\Sigma$, let $f = \phi_1$ be its first eigenfunction of norm 1. We perturb $\Sigma$ by $\varepsilon f$ and call the resulting hypersurface $\Sigma^\varepsilon$. We construct a classical MCF starting from $\Sigma^\varepsilon$, which, after the rescaling (1.5), will be defined on $s \in (0, S)$ where $S$ is its first singular time. To obtain an ancient flow, we then construct a matching motion starting from $\Sigma^\varepsilon$. Matching property tells us that the weak flow is smooth and agrees with the classical flow on $(0, s)$. We then take $\varepsilon \to 0$ after time translation to obtain the desired ancient flow. The limit-taking procedure has to be done carefully, as one wants to avoid getting back the original self-expander.

In this section we give an overview of the first part of the construction, namely the construction of the smooth flow, which was done essentially by Bernstein–Wang in [14] (here the only difference is that [14] has an even stronger entropy condition which ensures the long-time regularity of the flow, but in the present case we will have singularities). The presentation of this section largely follows [14, Section 5].

40
Given a smooth MCF $\mathcal{M} = \{\Sigma_t\}_{t \in I}$, the expander mean curvature of $\Sigma_t$ is

$$E_{\Sigma_t}(p) = 2t H_{\Sigma_t} + x(p) \cdot n_{\Sigma_t}.$$  

We say $\{\Sigma_t\}$ is expander mean convex if the $E_{\Sigma_t}(x) > 0$ along the flow. For a fixed time $t$ and a hypersurface $\Sigma$, the relative expander mean curvature of $\Sigma$ is

$$E^t_\Sigma(p) = 2t H_\Sigma + x(p)^\perp.$$  

For $\beta > 0$ define the auxiliary function $g_\beta : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$g_\beta(r) = r^{-\beta} e^{-\beta r}.$$  

We have the (rather standard) existence theorem for expander mean convex hypersurfaces.

**Theorem 3.14** (Existence Theorem, cf. Proposition 5.1 of [14]). Let $\Sigma$ be a hypersurface $C^{2,\alpha}$-asymptotic to $\mathcal{C}$ with no closed components. Suppose that there is $N$ such that $\Sigma \setminus B_{NR}(0) \subset T_{R^{-1}}(\mathcal{C})$ and that there is $c, \beta > 0$ such that

$$E_\Sigma(p) \geq cg_\beta(1 + |x(p)|^2) > 0, \ p \in \Sigma.$$  

Then there exists a unique smooth MCF, $\{\Sigma_t\}_{t \in [1,T]}$, with $\Sigma_1 = \Sigma$, where $T$ is the first singular time (possibly $\infty$). Moreover the MCF satisfies

1. $\Sigma_t$ is $C^{2,\alpha}$-asymptotic to $\mathcal{C}$ for all $t \in [1,T)$.

2. $E_{\Sigma_t}(p) > cg_\beta(1 + |x(p)|^2 + 2n(t-1))$ for all $t \in [1,T)$ and $p \in \Sigma_t$.

3. If $T < \infty$, we have

$$\lim_{t \to T} \sup_{\Sigma_t \cap B_{N/\sqrt{T}}} |A_{\Sigma_t}| = \infty.$$  

**Proof.** Consider the map $\Phi : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$ given by

$$\Phi(p, \lambda) = x(p) + \lambda n_\Sigma.$$  

41
Since $\Sigma$ is asymptotically conical, we can choose $\varepsilon$ sufficiently small so that the above map is a diffeomorphism onto its image for every $\lambda \in (-\varepsilon, \varepsilon)$. Using this parametrization we can invoke standard existence theorem for MCF to conclude that there exists a unique MCF starting from $\Sigma_1 = \Sigma$. Proposition 3.3 immediately implies Item (1) and (3) of the properties. For Item (2), see [14, Lemma 5.4].

Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a smooth cone. Given an unstable self-expander $\Sigma$ asymptotic to $\mathcal{C}$, let $f$ be the unique positive first eigenfunction of $-L_\Sigma$ with $\|f\|_W = 1$. We wish to apply Theorem 3.14 to the perturbed hypersurface $\Sigma^\varepsilon$ defined by

$$
\Sigma^\varepsilon = f^\varepsilon(\Sigma) \text{ where } f(p) = x(p) + \varepsilon f(p)n_\Sigma.
$$

It remains to check that $\Sigma^\varepsilon$ satisfies the assumption of Theorem 3.14. Theorem 2.10 and Proposition 2.7 imply that there exists $N > 0$ such that $\Sigma^\varepsilon \setminus B_{NR}(0) \subset \mathcal{T}_{R^{-1}}(\mathcal{C})$ for $R > 1$, so the first condition to apply Theorem 3.14 is satisfied. It remains to show that $\Sigma^\varepsilon$ is expander mean convex.

**Lemma 3.15.** There exists $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$ there is $\beta = \beta(\varepsilon)$ and $c > 0$ such that

$$
E_{\Sigma^\varepsilon}(p) \geq c\beta(1 + |x(p)|^2).
$$

**Proof.** By the expander mean curvature formula (2.3), and the fact that $f$ is the first eigenfunction of $-L_\Sigma$, we get that

$$
E_{\Sigma^\varepsilon} = \varepsilon\lambda_1 f + \varepsilon^2 Q(f, x \cdot \nabla_\Sigma f, \nabla_\Sigma f, \nabla^2_\Sigma f).
$$

When $\varepsilon > 0$, it follows from Theorem 2.10, that, up to further shrinking $\varepsilon$,

$$
E_{\Sigma^\varepsilon}(p) \geq \varepsilon\lambda_1 C^{-1}(1 + |x(p)|^2)^{-\frac{1}{2}(n+1-2\mu_1)} \varepsilon^{-\frac{1}{4} + |x(p)|^2} \geq c\beta(1 + |x(p)|^2)
$$

where $\beta = \frac{1}{2}(n + 1 - 2\mu_1) > 0$. The case $\varepsilon < 0$ can be handled similarly. \qed
Lemma 3.15 shows that $\Sigma^\varepsilon$ satisfies the second condition of Theorem 3.14 for sufficiently small $\varepsilon$. As such, Theorem 3.14 can be applied to conclude the short-time existence of an expander mean-convex MCF starting from $\Sigma^\varepsilon$. Applying Theorem 3.14 to $\Sigma^\varepsilon$, we get, for sufficiently small $\varepsilon$, a unique MCF $\mathcal{M}^\varepsilon = \{\Sigma^\varepsilon_t\}_{t \in [0,T^\varepsilon)}$ with $\Sigma^\varepsilon_1 = \Sigma^\varepsilon$. Moreover, by Theorem 2.10 and Proposition 2.7, $\Sigma^\varepsilon$ has uniformly bounded curvature, so the interior estimates of Ecker–Huisken [39] implies that the interval of existence of is independent of $\varepsilon$. Moreover, at the first singular time $T^\varepsilon$,

$$\lim_{t \to T^\varepsilon} \sup_{\Sigma^\varepsilon_t \cap B_{N', \sqrt{t}}} |A_{\Sigma^\varepsilon_t}| = \infty$$

for some constant $N' > 0$.

### 3.5 Construction of the matching motion

We now turn to the construction of the weak flow starting from $\Sigma^\varepsilon$. It is possible to achieve this by capping off the hypersurfaces $\Sigma^\varepsilon \cap \overline{B_R(0)}$, rounding the corners, and taking a sequence of weak flows starting from these capped-off hypersurfaces. This is the construction used in [6] (see also [24, Section 7] where they produced a one-sided MCF starting from an asymptotically conical self-shrinker). For simplicity, however, in this section we will use an alternative method based on Brakke flow with boundary. This method works well with the entropy bound and is much simpler to state and prove. For a full construction of the one-sided flow without any entropy bound we refer again to [6] for details (note that the construction therein is not compactible with the entropy bound as the caps might increase the entropy). We should note importantly that the capping method has the advantage that, by a suitable choice of the caps, expander mean-convexity is preserved through the cap and hence in the limits (here one has to interpret the mean-convexity in the weak sense as well). This allows us in [6] to deduce further regularity of the one-sided flow even after singularities, after a careful analysis of the construction. The method we use here only produce a flow that
agrees with the flow from [6] until the first singular time, and after that we have no information on the regularity of the flow. In particular, we do not know if the flow stays one-sided after singularity.

We consider $\Sigma^\varepsilon$ from the previous section and WLOG assume $\varepsilon > 0$, as the construction for $\varepsilon < 0$ is identical. The following proposition is the weak flow analogy of the smooth flow produced in Theorem 3.14.

**Proposition 3.16.** There exists $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$, there exists an immortal matching motion $(K^\varepsilon, M^\varepsilon)$ where $K^\varepsilon = \{\Gamma^\varepsilon_t\}_{t \in [1, \infty)}$ and $M^\varepsilon = \{\mu^\varepsilon_t\}_{t \in [1, \infty)}$ such that $\Gamma^\varepsilon_1 = \Sigma^\varepsilon$ and $\mu^\varepsilon_1 = \mathcal{H}^{n-1} \Sigma^\varepsilon$. Moreover the flow $(K^\varepsilon, M^\varepsilon)$ agrees with the smooth flow starting from $\Sigma^\varepsilon$ for $t \in [1, T^\varepsilon)$.

**Proof.** Let $\Sigma^\varepsilon, R = \Sigma^\varepsilon \cap \overline{B_R(0)}$ be the hypersurface in $\overline{B_R(0)}$ with boundary $\Sigma^\varepsilon \cap \partial B_R(0)$. By Theorem 3.11, there exists an unit-regular and cyclic Brakke flow with boundary $M^{\varepsilon, R} = \{\mu^{\varepsilon, R}_t\}_{t \in [0, \infty)}$ starting from $\Sigma^{\varepsilon, R}$. The flow $M^{\varepsilon, R}, B_{R/2}(0)$ is therefore a (usual) Brakke flow inside $B_{R/2}(0)$. Since nonfattening is generic, we may choose a sequence $R_i \to \infty$ such that the associated level set flow of $M^{\varepsilon, R_i} B_{R_i/2}(0)$ is nonfattening. This produces a sequence of matching motions

$$(K^{\varepsilon, R_i}, M^{\varepsilon, R_i} B_{R_i/2}(0)).$$

By compactness of Brakke flow we may now pass to a subsequence $R_i \to \infty$ to obtain a limiting enhanced motion $(K^\varepsilon, M^\varepsilon)$ in $\mathbb{R}^{n+1}$ starting from $\Sigma^\varepsilon$.

By [10, Lemma 3.5], $\lambda[\Sigma] = \lambda[C] < 2$, and by [14, Lemma 6.2], for every $\delta > 0$ there exists $\varepsilon_0$ such that $|\lambda[\Sigma^\varepsilon] - \lambda[\Sigma]| < \delta$ for $\varepsilon < \varepsilon_0$. Choosing $\delta$ small enough so that $\lambda[C] + \delta < 2$ and $\varepsilon_0$ small according to $\delta$ ensures that $\lambda[M^\varepsilon] = \lambda[\Sigma^\varepsilon] < 2$, and so, in view of Theorem 3.13, $(K^\varepsilon, M^\varepsilon)$ is also a matching motion.

Finally, using the argument in Proposition 3.3 with pseudolocality of MCF replaced by that of Brakke flow there exist $\delta > 0$ and $N' > 0$ such that $\text{supp} \mu^\varepsilon_t \setminus B_{N' \sqrt{t}}(0) =$
Since \(\mathcal{K}_\varepsilon, \mathcal{M}_\varepsilon\) is matching, \(\Gamma^\varepsilon_t \setminus B_{N^\varepsilon \sqrt{t}}(0) = \Sigma^\varepsilon_t \setminus B_{N^\varepsilon \sqrt{t}}(0)\) as well. It follows from uniqueness of level set flow that \(\Gamma^\varepsilon_t\) agrees with \(\Sigma^\varepsilon_t\). Using the matching property again we infer that \(\text{supp } \mu^\varepsilon_t = \Sigma^\varepsilon_t\). It is easy to see that these flow agree up to the first singular time of \(\Sigma^\varepsilon_t\) (i.e. \(T^\varepsilon\)).

We can once again take a limit as \(\varepsilon \to 0^+\) to obtain a limiting enhanced motion \((\mathcal{K}, \mathcal{M})\) where \(\mathcal{K} = \{\Gamma_t\}_{t \in [1, \infty)}\) and \(\mathcal{M} = \{\mu_t\}_{t \in [1, \infty)}\) such that \(\Gamma_0 = \Sigma\) and \(\mu_0 = \mathcal{H}^n \Sigma\). However, this limit is not enough to prove Theorem 1.3, as we will only recover the flow of the self-expander \(\Sigma\) (as in the case of the smooth flow). We must analyze the flow in the rescaled variables in order to obtain a non-trivial limit.

Let \((\tilde{\mathcal{K}}, \tilde{\mathcal{M}})\) denote the rescaled matching motion under (1.5), and suppose \(S^\varepsilon = \log T^\varepsilon\) is the first singular time of \(\tilde{\mathcal{M}}^\varepsilon\). We first establish the long-time regularity of the one-sided flow.

**Proposition 3.17.** If \(S^\varepsilon = \infty\) for some \(\varepsilon > 0\) sufficiently small, then there exists a stable self-expander \(\Gamma\) asymptotic to \(\mathcal{C}\) such that

\[
\lim_{s \to \infty} \mathcal{H}^n \Sigma^\varepsilon_s = \mathcal{H}^n \Gamma.
\]

Moreover, \(\Gamma\) is smooth away from a set of Hausdorff dimension at most \((n - 7)\). In addition, there is a \(\beta_1 = \beta_1(\Sigma) > 0\), so that

\[
\Gamma \setminus \mathcal{T}_{\beta_1}(\Sigma) \neq \emptyset.
\]

**Proof.** The first claim follows in exactly the same way as in the proof of [14, Proposition 5.1]. We sketch a proof here. Fix a sufficiently small \(\varepsilon\) and write for simplicity \(\Sigma^\varepsilon_s = \text{supp } \tilde{\mathcal{M}}^\varepsilon_s\). As the flow \(\tilde{\mathcal{M}}^\varepsilon\) is one-sided when smooth, we can let \(\Omega^\varepsilon_s\) be the closure of the region bounded by \(\Sigma^\varepsilon_s\) that is inside \(\Sigma\). Consider the translations \(\Sigma^\varepsilon_{s+\sigma} = \Sigma^\varepsilon_s + \sigma\). By compactness of rescaled Brakke flows, for each \(s\) there is a sequence \(\sigma_i \to \infty\) such that \(\mathcal{H}^n \Sigma^\varepsilon_s \to \tilde{\mathcal{M}}_s\), and that \(\{\tilde{\mathcal{M}}(s)\}_{s \in \mathbb{R}}\) is an expander Brakke flow.
with \( \text{supp} \tilde{M}_s = \partial \Omega \) where

\[
\Omega = \bigcap_{s>0} \Omega_s.
\]

Hence, \( \tilde{M} \) is a static solution of the rescaled Brakke flow, and so the varifold associated to \( \Gamma = \partial \Omega \) is \( E \)-stationary. Moreover, by the expander mean-convexity of \( \partial \Omega \), \( \{ \tilde{\Sigma}_s \}_{s \geq s_0} \) forms a foliation of a neighborhood of \( \partial K \) in \( \mathbb{R}^{n+1} \setminus \text{int}(K) \), and so the varifold associated to \( \Gamma \) is locally \( E \)-minimizing on one-side. It follows from the regularity theory of one-sided \( E \)-minimizing varifolds [73], that \( \Gamma \) is smooth away from a singular set of Hausdorff dimension at most \( (n - 7) \).

We now show the second claim. Suppose for a contradiction that there is a sequence of stable self-expanders \( \Gamma^i \) asymptotic to \( C \) such that \( \Gamma^i \subseteq T_{i-1}(\Sigma) \). It follows from [10, Lemmas 3.5 and 3.6] that \( \lambda[\Gamma^i] = \lambda[C] \) is constant, and so there are uniform area bounds for \( \Gamma^i \). Hence, up to passing to a subsequence the \( \Gamma^i \)'s converge weakly to an integral varifold \( V \). The hypotheses on \( \Gamma^i \) ensure that \( \text{supp} \ V \subseteq \Sigma \). By compactness properties of asymptotically conical self-expanders, [10, Corollary 3.4], up to passing to a further subsequence, there is a self-expanding end, \( \Sigma' \), and a radius \( R' = R(C) > 0 \) such that \( \Gamma^i \rightarrow \Sigma' \) in \( C^\infty_{\text{loc}}(\mathbb{R}^{n+1} \setminus \overline{B}_{R'}(0)) \) with multiplicity one. Since \( \Gamma^i \subseteq T^{i-1}(\Sigma) \), we have \( \Sigma' \) agrees with \( \Sigma \) outside of \( B_{R'}(0) \). These two convergences together give that \( \text{supp} \ V = \Sigma \) in \( B_{2R'}(0) \) and \( V = \mathcal{H}^{n-1} \Sigma' \) in \( \mathbb{R}^{n+1} \setminus B_{R'}(0) \). Hence, in fact, the convergence is smooth on \( B_{2R'}(0) \setminus B_{R'}(0) \). The constancy theorem [78, Theorem 41.1] and unique continuation imply that \( V = \mathcal{H}^{n-1} \Sigma \) (see the proof of [10, Item (3) of Theorem 1.1] for details, where we also use the fact that there is no closed self-expander). Hence, by Allard’s regularity theorem [1], \( \Gamma^i \rightarrow \Sigma \) in \( C^\infty_{\text{loc}}(\mathbb{R}^{n+1}) \) with multiplicity one. As the convergence is locally smooth and of multiplicity one, \( L_{\Gamma^i}u \rightarrow L_{\Sigma}u \) for all test functions \( u \). It follows that \( \Sigma \) is a stable self-expander, a contradiction.

Using the above, we will show that the rescaled flows must leave a tubular neigh-
Proposition 3.18. There exists a $\beta = \beta(\Sigma) > 0$ so that for every $0 < \varepsilon < \frac{1}{2}\beta$ there exists an $s^\varepsilon \in (0, S^\varepsilon)$ so that, for $s \in [0, s^\varepsilon)$, $\tilde{\Sigma}_s^\varepsilon \subseteq T_\beta(\Sigma)$, while, for $s \in (s^\varepsilon, S^\varepsilon)$, $\tilde{\Sigma}_s^\varepsilon \setminus T_\beta(\Sigma) \neq \emptyset$. Moreover, $s^\varepsilon \to \infty$ as $\varepsilon \to 0$.

Proof. Let $\beta_0$ be another parameter to be specified below and take $\beta = \min\{\beta_0, \beta_1\}$, where $\beta_1 = \beta_1(\Sigma)$ is from Proposition 3.17. Denote $\tilde{\Sigma}_s^\varepsilon = \text{supp} \, \tilde{\mathcal{M}}_s^\varepsilon$ for $s \leq S^\varepsilon$. For $\varepsilon < \beta$, set
$$s^\varepsilon = \sup\{s \in [0, S^\varepsilon) : \tilde{\Sigma}_s^\varepsilon \subseteq T_\beta(\Sigma)\}.$$ 
First observe that by the choice of $\varepsilon$, $s^\varepsilon > 0$ is well defined. It also follows from the expander mean convexity of the construction that $\tilde{\Sigma}_s^\varepsilon$ is moving away from $\Sigma$ and so for $s \in (s^\varepsilon, S^\varepsilon)$, $\tilde{\Sigma}_s^\varepsilon \setminus T_\beta(\Sigma) \neq \emptyset$.

It remains to show that $s^\varepsilon < S^\varepsilon$. Recall that $S^\varepsilon$ is uniformly bounded below by $\delta_0$ for $\varepsilon$ sufficiently small. Thus $\tilde{\Sigma}_s^\varepsilon$ is a graph over $\Sigma$ with $C^0$-norm bounded by $\beta_0$ for $s \in [0, \delta_0]$. Thus pseudolocality implies that, for $s_0 < s^\varepsilon$, $\tilde{\Sigma}_s^\varepsilon$ is the graph of a function $g_s$ over $\Sigma$ that satisfies $\|g_s\|_{C^2} \leq \alpha_0$ for $s \in [s, s + \delta_0]$. (see [6, Proposition 3.1]) for a detailed argument). For $\beta_0$ sufficiently small, this implies that $s^\varepsilon + \delta_0 \leq S^\varepsilon$, as the existence time only depends on the curvature estimate [38]. This proves the claim when $S^\varepsilon < \infty$. When $S^\varepsilon = \infty$, the claim instead follows from Proposition 3.17.

Finally, the claim that $s^\varepsilon \to \infty$ as $\varepsilon \to 0$ follows from the Arzela-Ascoli theorem and the uniqueness result of [19].

We are now in position to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\tilde{\mathcal{M}}_{s^\varepsilon}$ be the rescaled MCF obtained from time translating $\tilde{\mathcal{M}}_s$ by $-s^\varepsilon$, where $s^\varepsilon$ is from Proposition 3.18. Let $\tilde{\Sigma}_s^{\varepsilon, s^\varepsilon} = \text{supp} \, \tilde{\mathcal{M}}_{s^\varepsilon}^{\varepsilon, s^\varepsilon}$. Then, by definition, for $s \in (-s^\varepsilon, 0)$,
$$\tilde{\Sigma}_s^{\varepsilon, s^\varepsilon} \subseteq T_\beta(\Sigma).$$
while for $s \in (0, S^\varepsilon - s^\varepsilon)$,
\[
\tilde{\Sigma}_{s} \setminus \mathcal{T}_\beta(\Sigma) \neq \emptyset.
\]

We may now take a subsequential limit as $\varepsilon \to 0^+$ to obtain a RMCF $\mathcal{M}$ such that for $s \in (0, \delta_0)$, $\tilde{\Sigma}_s$ exists as a smooth hypersurface and
\[
\tilde{\Sigma}_s \setminus \mathcal{T}_\beta(\Sigma) \neq \emptyset.
\]

This guarantees that the flow $\mathcal{M}$ is not that of the self-expander $\Sigma$. Next, note that
\[
\lim_{s \to -\infty} \tilde{\Sigma}_s = \Sigma,
\]
so $\mathcal{M}$ is an ancient flow from $\Sigma$. Moreover, since each $\mathcal{M}^\varepsilon$ is strictly expander mean convex, and $\mathcal{M}$ is not the static flow of $\Sigma$, the flow $\mathcal{M}$ is strictly expander mean convex by the strong maximum principle; e.g., [80, Section 2].  

$\square$
Chapter 4

Generic Uniqueness of Tame Ancient Flows

In this chapter we prove Theorem 1.4. To do so, we first work with the rescaled flow \( (1.6) \) and show that for a generic cone \( C \), all tame ancient flows are either self-expanders or given by the ones constructed in Theorem 1.2. This part uses spectral analysis. Roughly speaking we decompose the solution in terms of eigenfunctions and find the dominate mode to show that the solution satisfies the decay rate in Theorem 1.2, which enables us to invoke the uniqueness aspect of the theorem. We then study a subtle question of when a MCF corresponds to a tame ancient RMCF, which will involve the entropy bound and the Łojasiewicz inequality.

We continue to use the convention that \( t \) denotes the time variable in MCF, and \( s \) the time variable in RMCF.

4.1 The relative entropy

Recall from Section 2.5 that self-expanders are only formally critical points of the \( E \)-functional. This is problematic as the RMCF does not have a well-defined gradient flow structure (in that RMCFs flow from infinity to infinity, so there is no definite drop of energy) coming from the \( E \)-functional, even though self-expanders are static solutions of the flow. To fix this issue, in this section we define a version of the relative
expander entropy which will serve as a substitute quantity for the $E$-functional. The notion of relative entropy was first studied by Bernstein–Wang [13] which was then used to prove a mountain-pass theorem for self-expanders (cf. Deruelle–Schulze [34], and, for a similar notion for minimal hypersurfaces in hyperbolic spaces, Yao [92, 91]). However, for technical reasons, we find it more suitable to define a version of the relative entropy adapted to normal graphs (as we will see later, the two notions agree in low dimensions).

Throughout the section, fix a smooth cone $C \subset \mathbb{R}^{n+1}$ and a self-expander $\Sigma$ asymptotic to $C$. Let $\chi_R : \mathbb{R}^{n+1} \to \mathbb{R}$ be smooth cutoff functions supported on $B_{R+2}(0)$ and identically 1 on $B_R(0)$. For any function $v \in C^{2,\alpha}_0(\Sigma)$, define the relative expander entropy of $\Sigma_v$ to be

$$E^*_{\text{rel}}[\Sigma_v, \Sigma] = \int_{\Sigma_v} e^{\frac{|x|^2}{4}} d\mathcal{H}^n - \int_{\Sigma} e^{\frac{|x|^2}{4}} d\mathcal{H}^n.$$  

For a general function $v \in W^1 \cap C^{2,\alpha}(\Sigma)$, define $E^*_{\text{rel}}[\Sigma_v, \Sigma]$ by

$$E^*_{\text{rel}}[\Sigma_v, \Sigma] = \lim_{R \to \infty} E^*_{\text{rel}}[\Sigma_{\chi_R v}, \Sigma]$$

whenever the limit exists.

We now comment on the difference between the above definition and the usual definition of the relative entropy from [13] and [34]. We will denote by $E_{\text{rel}}$ the quantity defined by Bernstein–Wang; $E_{\text{rel}}$ is given by the formula

$$E_{\text{rel}}[\Sigma_1, \Sigma_2] = \lim_{R \to \infty} \left( \int_{\Sigma_1 \cap B_R(0)} e^{\frac{|x|^2}{4}} d\mathcal{H}^n - \int_{\Sigma_2 \cap B_R(0)} e^{\frac{|x|^2}{4}} d\mathcal{H}^n \right),$$

for two hypersurfaces $\Sigma_1$ and $\Sigma_2$, whenever the limit is defined (possibly $\infty$). In particular, they showed in [13] that when $\Sigma_1$ is a hypersurface trapped between two self-expanders asymptotic to the same cone $C$, then $E_{\text{rel}}[\Sigma_1, \Gamma]$ is well-defined (possibly $\infty$, but not $-\infty$) for any self-expander $\Gamma$ asymptotic to $C$. Because of this, $E_{\text{rel}}$ is the natural and more suitable quantity to study in the trapped case (and, in fact,
\( E_{\text{rel}} = E_{\text{rel}}^* \) in the trapped case - see Proposition 4.13). Unfortunately, in order for a graph \( \Sigma_v \) to be trapped, \( v \) needs to have a very good spatial decay near infinity:

\[
v(p) = O(|x(p)|^{-n-1} e^{-|x(p)|^2/4}).
\]

As we are working with normal graphs that a priori do not have the sharp decay (rather only an energy bound), we will have to use \( E_{\text{rel}}^* \) instead of \( E_{\text{rel}} \).

Fortunately \( E_{\text{rel}}^* \) is enough for our purposes. We show that \( E_{\text{rel}}^* \) is well-defined if the function \( v \) has small \( C^{2,\alpha} \) norm.

**Proposition 4.1.** Suppose \( v \in C^{2,\alpha}_0(\Sigma) \). There exists \( \varepsilon = \varepsilon(\Sigma) \) sufficiently small such that the following inequality holds:

\[
\left| E_{\text{rel}}^*[\Sigma_v, \Sigma] - \frac{1}{2} \int_\Sigma (|\nabla v|^2 + (1/2 - |A_\Sigma|^2)v^2) e^{\frac{|x|^2}{4}} \right| \leq C \varepsilon \|v\|_{W^1}^2 \tag{4.1}
\]

whenever \( \|v\|_{C^{2,\alpha}} < \varepsilon \). Here \( C = C(\Sigma) \).

**Remark.** Integrating by parts (which is justified as \( v \) is compactly supported) gives that the second order term in (4.1) is exactly equal to

\[
-\frac{1}{2} \int_\Sigma v L_\Sigma v e^{\frac{|x|^2}{4}}. \tag{4.2}
\]

This is not surprising as \( \Sigma_v \) can be thought of as a perturbation of \( v \) when \( \Sigma_v \) is sufficiently close to \( \Sigma \). Since \( \Sigma \) is \( E \)-stationary, the relative entropy should pick up the second-order information, which is precisely the stability operator.

**Proof.** We will proceed by explicit computation. By the area formula we can write

\[
\int_{\Sigma_v} e^{\frac{|x|^2}{4}} dH^n = \int_\Sigma \sqrt{\det((Dv)^T(Dv))} e^{\frac{|x|^2}{4}} e^{-vH_\Sigma}, \tag{4.3}
\]

where we used the self-expander equation (1.4). Here

\[
Dv = I_{(n+1)\times n} + \nabla_\Sigma v \otimes n_\Sigma + v \nabla_\Sigma n_\Sigma,
\]
where $I_{(n+1)\times n}$ denotes the $(n+1)$-by-$n$ matrix which is the identity in the top $n$ rows and 0 in the last row. Given $p \in \Sigma$, the matrix $Dv$ can be written in normal coordinates centered at $p$ as

$$Dv = \begin{pmatrix}
1 + v\kappa_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 + v\kappa_n \\
\partial_1 v & \cdots & \partial_n v
\end{pmatrix},$$

where $\kappa_1, \ldots, \kappa_n$ are the principle curvatures of $\Sigma$. Hence the Jacobian matrix $(Dv)^T(Dv)$ takes the form

$$
\begin{pmatrix}
(1 + v\kappa_1)^2 + (\partial_1 v)^2 & \partial_1 v\partial_2 v & \cdots & \partial_1 v\partial_n v \\
\partial_1 v\partial_2 v & (1 + v\kappa_2)^2 + (\partial_2 v)^2 & \cdots & \partial_2 v\partial_n v \\
\vdots & \ddots & \vdots \\
\partial_1 v\partial_n v & \partial_2 v\partial_n v & \cdots & (1 + v\kappa_n)^2 + (\partial_n v)^2
\end{pmatrix}
$$

Equivalently, the above can be written as $I_n + A$, where $A$ has entries

$$A_{ij} = \partial_i v\partial_j v + (v^2\kappa_i^2 + 2v\kappa_i)\delta_{ij}$$

We now expand the determinant using its series expansions, whose validity is justified by the fact that $\|v\|_{C^{2,\alpha}} < \varepsilon$:

$$\det(I + \varepsilon A) = 1 + \text{tr}(A) + \frac{1}{2}\varepsilon^2(\text{tr}^2(A) - \text{tr}(A^2)) + O(\varepsilon^3).$$

In the following we will use $M(v, \nabla\Sigma v)$ to denote a polynomial of degree at least 3 in $v$ and $\nabla\Sigma v$ coming from the remainder form of the Taylor expansion. The coefficients of $M$ depend only on $\Sigma$, but the exact form of $M$ may change from line to line. We compute that

$$\text{tr}(A) = |\nabla\Sigma v|^2 + |A\Sigma|^2 v^2 + 2vH_{\Sigma}$$

$$\text{tr}(A^2) = \sum_{i,j} |\partial_i v\partial_j v|^2 + \sum_{i=1}^n (2|\partial_i v|^2(v^2\kappa_i^2 + 2v\kappa_i) + v^4\kappa_i^4 + 4v^3\kappa_i^3) + 4v^2 |A\Sigma|^2.$$
Thus
\[ \text{tr}^2(A) - \text{tr}(A^2) = 4(|H\Sigma|^2 - |A\Sigma|^2)v^2 + M(v, \nabla\Sigma v). \]

Putting the above computations together gives
\[ \det(I + A) = 1 + |\nabla\Sigma v|^2 + 2H\Sigma v + (2|H\Sigma|^2 - |A\Sigma|^2)v^2 + M(v, \nabla\Sigma v). \quad (4.4) \]

Finally, using the Taylor expansion
\[ \sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3), \]
we get that
\[ \sqrt{\det(I + A)} = 1 + \frac{1}{2}|\nabla\Sigma v|^2 + H\Sigma v + \frac{1}{2}(|H\Sigma|^2 - |A\Sigma|^2)v^2 + M(v, \nabla\Sigma v). \]

Using the above in (4.3) together with a Taylor expansion on the exponential terms, we obtain (where we wrote \( \varepsilon = \|v\|_{C^2,\alpha} \))
\[
\int_{\Sigma} e^{\frac{|v|^2}{4}} dH^n = \int_{\Sigma} \left(1 + \frac{1}{2}|\nabla\Sigma v|^2 + H\Sigma v + \frac{1}{2}(|H\Sigma|^2 - |A\Sigma|^2)v^2 + M(v, \nabla\Sigma v)\right) e^{\frac{|v|^2}{4}}
\]
\[ = \int_{\Sigma} \left(1 + \frac{1}{2}|\nabla\Sigma v|^2 + \frac{1}{2}v^2 - |A\Sigma|^2v^2 + M(v, \nabla\Sigma v)\right) e^{\frac{|v|^2}{4}} \quad (4.5) \]

Subtracting the above from \( \int_{\Sigma} e^{\frac{|v|^2}{4}} \) and using that
\[ \int_{\Sigma} M(v, \nabla\Sigma v)e^{\frac{|v|^2}{4}} \leq C\|v\|_{C^2,\alpha}\|v\|^2_{W^1} \leq C\varepsilon\|v\|^2_{W^1} \]
gives the desired formula.

**Corollary 4.2.** There exists \( \varepsilon > 0 \) such that if \( v \in W^1 \cap C^2,\alpha(\Sigma) \) satisfies \( \|v\|_{C^2,\alpha} < \varepsilon \), then \(-\infty < E_{\text{rel}}[\Sigma v, \Sigma] < \infty\).
Proof. For a general \( v \), inserting \( \chi_R v \) in place of \( v \) in (4.5) yields
\[
E_{\text{rel}}^*[\Sigma_{\chi_R v}, \Sigma] = \frac{1}{2} \int_{\Sigma} \left( |\nabla_{\Sigma} (\chi_R v)|^2 + \left( \frac{1}{2} - |A_{\Sigma}|^2 \right) v^2 \chi_R^2 + M(\chi_R v, \nabla_{\Sigma} \chi_R v) \right) e^{\frac{|x|^2}{4}}
\]
\[
= \frac{1}{2} \int_{\Sigma} \left( |\nabla_{\Sigma} \chi_R|^2 + \frac{1}{2} - |A_{\Sigma}|^2 \right) v^2 + \chi_R^2 |\nabla_{\Sigma} v|^2
+ 2\chi_R v \nabla_{\Sigma} \chi_R \cdot \nabla_{\Sigma} v + M(\chi_R v, \nabla_{\Sigma} \chi_R v) e^{\frac{|x|^2}{4}}
\] (4.6)
Thus for \( R_1 > R_2 \), we get from (4.6)
\[
E_{\text{rel}}^*[\Sigma_{\chi_{R_1} v}, \Sigma] - E_{\text{rel}}^*[\Sigma_{\chi_{R_2} v}, \Sigma]
= \frac{1}{2} \int_{\Sigma} \left( (|\nabla_{\Sigma} \chi_{R_1}|^2 - |\nabla_{\Sigma} \chi_{R_2}|^2) v^2 + (\chi_{R_1}^2 - \chi_{R_2}^2) |\nabla_{\Sigma} v|^2
+ 2\chi_{R_1} v \nabla_{\Sigma} \chi_{R_1} \cdot \nabla_{\Sigma} v - 2\chi_{R_2} v \nabla_{\Sigma} \chi_{R_2} \cdot \nabla_{\Sigma} v
+ M(\chi_{R_1} v, \nabla_{\Sigma} \chi_{R_1} v) - M(\chi_{R_2} v, \nabla_{\Sigma} \chi_{R_2} v) \right) e^{\frac{|x|^2}{4}}.
\]
As \( v \in W^2 \), given \( \eta \), there is \( R_0 \) such that for \( R_1, R_2 > R_0 \),
\[
\int_{B_{R_1}(0) \setminus B_{R_2}(0)} (|\nabla_{\Sigma} v|^2 + v^2) e^{\frac{|x|^2}{4}} < \eta.
\]
As \( \nabla_{\Sigma} \chi_{R_1} - \nabla_{\Sigma} \chi_{R_2} \) is supported on \( \mathbb{R}^{n+1} \setminus B_{R_2}(0) \) and \( \chi_{R_1} - \chi_{R_2} \) is supported on \( B_{R_1+2}(0) \setminus B_{R_2+2}(0) \), there is \( R_0 \) such that \( R_1 > R_2 > R_0 \) implies
\[
\int_{\Sigma} \left( (|\nabla_{\Sigma} \chi_{R_1}|^2 - |\nabla_{\Sigma} \chi_{R_2}|^2) v^2 + (\chi_{R_1}^2 - \chi_{R_2}^2) |\nabla_{\Sigma} v|^2 \right) e^{\frac{|x|^2}{4}} < \frac{\eta}{3},
\]
\[
\int_{\Sigma} \left( 2\chi_{R_1} v \nabla_{\Sigma} \chi_{R_1} \cdot \nabla_{\Sigma} v - 2\chi_{R_2} v \nabla_{\Sigma} \chi_{R_2} \cdot \nabla_{\Sigma} v \right) e^{\frac{|x|^2}{4}} < \frac{\eta}{3}.
\]
For the error term \( M \), we also have, as a sixth-degree polynomial in \( v \) and \( \nabla_{\Sigma} v \), for \( R_1 \) sufficiently large,
\[
\int_{\Sigma} |M(\chi_{R_1} v, \nabla_{\Sigma} \chi_{R_1} v) - M(\chi_{R_2} v, \nabla_{\Sigma} \chi_{R_2} v)| e^{\frac{|x|^2}{4}} < \frac{\eta}{3}.
\]
Hence \( E_{\text{rel}}^*[\Sigma_{\chi_R v}, \Sigma] \) is a Cauchy sequence, and it follows that
\[
-\infty < E_{\text{rel}}^*[\Sigma_{\chi_R v}, \Sigma] = \lim_{R \to \infty} E_{\text{rel}}^*[\Sigma_{\chi_R v}, \Sigma] < \infty.
\]

Another immediate consequence of Proposition 4.1 is the reverse Poincaré inequality for \( E_{\text{rel}}^* \).
Corollary 4.3. If \( v \in W^1 \cap C^{2,\alpha}(\Sigma) \) satisfies \( \|v\|_{C^{2,\alpha}} < \varepsilon \), then
\[
E_{rel}^*[\Sigma_v, \Sigma] \geq C_1 \int_{\Sigma} |\nabla \Sigma v|^2 e^{\frac{|x|^2}{4}} - C_2 \int_{\Sigma} v^2 e^{\frac{|x|^2}{4}}.
\]
for \( C_1, C_2 \) depending only on \( \Sigma \). Specifically, if \( E_{rel}^*[\Sigma_v, \Sigma] \leq 0 \),
\[
\int_{\Sigma} v^2 e^{\frac{|x|^2}{4}} \geq C \int_{\Sigma} |\nabla \Sigma v|^2 e^{\frac{|x|^2}{4}}.
\]

Proof. This follows from the expansion (4.5), using the fact that \( |A_\Sigma| \) is bounded on \( \Sigma \) and the bound
\[
\int_{\Sigma} M(v, \nabla \Sigma v) e^{\frac{|x|^2}{4}} \geq -C\varepsilon \|v\|_{W^2}^2 - C\varepsilon \|
abla \Sigma v\|_{W^2}^2.
\]

4.2 Spectral uniqueness

In this section we deduce uniqueness of tame ancient RMCFs. The main theorem is:

Theorem 4.4. Suppose \( C \subset \mathbb{R}^{n+1} \) is a generic smooth cone. Suppose \( \tilde{M} = \{\tilde{\Sigma}_s\}_{s \in (-\infty,0)} \) is a smooth RMCF such that
\[
\lim_{s \to -\infty} \tilde{\Sigma}_s = \Sigma \text{ in } C^\infty_{loc}(\mathbb{R}^{n+1})
\]
for some smooth self-expander \( \Sigma \) asymptotic to \( C \). Then \( \tilde{M} \) is either the static flow of \( \Sigma \) or \( \Sigma \) is unstable and \( \tilde{M} \) coincides with one of the tame ancient RMCFs constructed from Theorem 1.2.

To prove Theorem 4.4, we will adapt the spectral analysis technique of Choi–Mantoulidis [30] (cf. [3, 28, 36, 24] and references therein) to the self-expander setting. Since \( \Sigma \) is not compact, we will use \( E_{rel}^* \) to give the equation (1.6) a gradient flow structure.

Let \( \Sigma \) be an unstable self-expander asymptotic to some smooth cone \( C \). Let \( I = \text{ind}(\Sigma) \) and again fix an orthonormal basis \( \{\phi_i\}_{i=1}^\infty \subset W^2(\Sigma) \) as before. Let \( v(p, s) : \Sigma \times (-\infty, 0] \to \mathbb{R} \) be a solution to the nonlinear problem (3.4). Theorem 4.4
is a consequence of the following spectral uniqueness theorem. Recall that we are using the notation $\Pi_-$ as in Section 3.2.

**Theorem 4.5.** There exists $\varepsilon = \varepsilon(\Sigma) > 0$ such that the following hold: suppose $v$ solves (3.4) and further satisfies

$$
\|v\|_{C^1_{P,\alpha}} \leq 1, \quad \|v(\cdot, s)\|_{C^2_{P,\alpha}} < \varepsilon \text{ for all } s < 0,
$$

and

$$
\|v(\cdot, s)\|_W \leq C \|\Pi_- v(\cdot, s)\|_W
$$

then, up to a time translation, $v$ agrees with $\Phi(a)$ for some $a = (a_1, \ldots, a_I)$ from Theorem 3.10.

**Remark.** The condition (4.9) is needed to rule out the possibility of neutral mode being dominant. If the neutral mode is dominant, there might exist extra solutions that shuffle the neutral modes around. (4.9) is in general a hard condition to check, and is precisely the reason why we need the genericity of $C$ in Theorem 4.4.

**Proof of Theorem 4.4.** Given an RMCF $\mathcal{M}$ satisfying (4.7), we use (4.7) to get that, up to time translation, there exists a function $v$ satisfying (1.7) such that the $\tilde{\Sigma}_s$ can be written as a normal graph over $\Sigma$ with $\|v\|_{C^2_{P,\alpha}} < \varepsilon$. This gives (4.8) of Theorem 4.5. Genericity of $C$ implies that (4.9) is satisfied. Hence the theorem follows from Theorem 4.5.

Thus we focus on the proof of Theorem 4.5 in the following. Given a solution $v$ to (3.4), we also introduce the following extra notations for projections throughout the section: Given $\mu \in \mathbb{R}$,

$$
\Pi_{\sim\mu} v = \sum_{\lambda_i \sim \mu} \langle v, \phi_i \rangle \phi_i \text{ and } V_{\sim\mu}(s) = \|\Pi_{\sim\mu} v(\cdot, s)\|_W.
$$
Here $\sim$ refers to any of $>,$ $<,$ or $=.$ When $\mu = 0,$ we will instead use $V_+, V_-$ and $V_0$ instead of $V_{>0}, V_{<0}$ and $V_{=0}.$ Let also

$$V(s) = \|v(\cdot, s)\|_W \text{ and } \delta(s) = \|v(\cdot, s)\|_{C^{2,\alpha}}.$$

From now on we assume that $v$ satisfies the assumptions of Theorem 4.5, i.e. (4.8) and (4.9). Using the asymptotic expansion of $Q,$ (2.4), we see that, up to a time translation, on $(-\infty, 0),$

$$\left\|(\frac{\partial}{\partial s} - L_{\Sigma})v(\cdot, s)\right\|_W \leq C\delta(s)\|v(\cdot, s)\|_W.$$  \hfill (4.10)

The gradient flow structure of the relative expander entropy implies that $E_{rel}^*[\Sigma, \Sigma] \leq 0.$ Moreover, by the reverse Poincaré inequality, Corollary 4.3,

$$0 \geq E_{rel}^*[\Sigma, \Sigma] \geq C_1 \int_{\Sigma} |\nabla_{\Sigma} v|^2 e^{|x|^2} - C_2 \int_{\Sigma} |v|^2 e^{|x|^2},$$

so that (4.10) implies

$$\left\|(\frac{\partial}{\partial s} - L_{\Sigma})v(\cdot, s)\right\|_W \leq C\delta(s)\|v(\cdot, s)\|_W.$$  

Knowing this, we can apply the projection operator $\Pi_{\sim \mu}$ to (3.4) and obtain, following [3], for each $\mu \in \{\lambda_1, \ldots, \lambda_I\} \cup \{0\},$ the system:

$$\frac{d}{ds} V_{\geq \mu} + \bar{\mu} V_{\geq \mu} \leq \delta(s) V,$$ \hfill (4.11)

$$\left|\frac{d}{ds} V_{= \mu} + \mu V_{= \mu}\right| \leq C\delta(s) V,$$ \hfill (4.12)

$$\frac{d}{ds} V_{\leq \mu} + \underline{\mu} V_{\leq \mu} \geq -\delta(s) V.$$ \hfill (4.13)

Here $\bar{\mu}$ is the smallest eigenvalue above $\mu$ and $\underline{\mu}$ is the largest eigenvalue below $\mu.$

Recall the key improved Merle–Zaag type ODE lemma:

**Lemma 4.6** (Lemma B.1 in [30]). Suppose $x, y, z : (-\infty, 0] \to [0, \infty)$ are absolutely continuous functions such that $x + y + z > 0$ and

$$\liminf_{s \to -\infty} y(s) = 0.$$  

57
If there is $\epsilon > 0$ such that $x, y, z$ satisfy the following system of differential inequalities

\begin{align*}
|x'| &\leq \epsilon(x + y + z), \\
y' + y &\leq \epsilon(x + z), \\
z' - z &\geq -\epsilon(x + y).
\end{align*}

Then there exists $\epsilon_0 > 0$ such that if $\epsilon \leq \epsilon_0$, $y \leq 2\epsilon(x + z)$. Moreover, either there exists $-\infty < s_* \leq 0$ such that $z \leq 8\epsilon x$ on $(-\infty, s_*)$ or $x \leq c\epsilon z$ on $(-\infty, 0]$ for some $c$ depending on $\epsilon_0$.

When $\epsilon$ is sufficiently small and $v$ not the trivial solution, we may apply Lemma 4.6 after multiplying $e^{\mu s}$ to the system to obtain that

\[ V_{>\mu}(s) \leq C\delta(s)(V_{=\mu}(s) + V_{<\mu}(s)) \text{ for } s \in (-\infty, 0], \]

and that either there exists $s_0 \in (-\infty, 0)$ such that $V_{<\mu}(s) \leq C\delta(s)V_{=\mu}(s)$ on $(-\infty, s_0]$, or $V_{=\mu}(s) \leq C\delta(s)V_{<\mu}(s)$ on $(-\infty, 0]$. (4.9) immediately implies that that the second case happens when $\mu = 0$.

We claim that $\delta(s) \leq Ce^{-\lambda_I s}$. To see this, by (4.13), for every $\epsilon > 0$ there exists $s_\epsilon < 0$ such that

\[ \frac{d}{ds}\log V_- \geq -\lambda_I - C\delta(s) \geq -\lambda_I - \epsilon \text{ on } (-\infty, s_\epsilon) \]

as $\delta(s) \to 0$ as $s \to -\infty$. Integrating this gives the pointwise bound

\[ V_-(s) \leq V_-(s_\epsilon)e^{(-\lambda_I - \epsilon)(s-s_\epsilon)}, \]

for $s < s_\epsilon$, which, together with the interior Schauder estimates [30, Theorem C.2], implies the pointwise decay (note that here it suffices to use the usual estimate for $L^2(\Sigma)$, as $W(\Sigma) \subset L^2(\Sigma)$)

\[ \delta(s) \leq C_\epsilon e^{(-\lambda_I - \epsilon)s}. \]
Going back to (4.13) and multiplying both sides by $e^{\lambda_1 s}$ implies that
\[
\frac{d}{ds} \log(e^{\lambda_1 s}V_-(s)) \geq -C \delta(s) \geq C e^{(-\lambda_1 - \varepsilon)s}.
\]
Integrating the above from $s$ to 0 gives that
\[
\log(e^{\lambda_1 s}V_-(s)) \leq C + V_-(0) \implies V_-(s) \leq C e^{-\lambda_1 s}.
\]
Finally, interior Schauder estimates give that
\[
\delta(s) \leq C e^{-\lambda_1 s}
\]
as desired. We now refine this a priori decay to obtain the sharp asymptotic decay of the solution in the negative mode.

**Proposition 4.7.** Suppose $\mu \in \{\lambda_1, \ldots, \lambda_I\}$ is such that
\[
\lambda_i \geq \mu \implies \lim_{s \to -\infty} e^{\lambda_1 s} \langle v(\cdot, s), \phi_i \rangle \phi_i = 0. \quad (4.14)
\]
Then $\mu \neq \lambda_1$ and $V_{\geq \mu}(s) \leq C \delta(s) V_{< \mu}(s)$ for all $s \leq 0$.

**Proof.** We proceed by induction. Suppose $\mu = \lambda_I$, and suppose for a contradiction that there is $s_0$ such that $V_{< \lambda_I}(s) \leq C \delta(s) V_{= \lambda_I}(s)$ on $(-\infty, s_0]$. Multiplying (4.12) by $e^{\lambda_1 s}$ gives that
\[
\left| \frac{d}{ds} (e^{\lambda_1 s} V_{\leq \lambda_I}) \right| \leq C e^{\lambda_1 s} \delta(s) V(s) \leq C e^{\lambda_1 s} \delta(s) V_-(s) \leq C e^{\lambda_1 s} \delta(s) V_{= \lambda_I}(s),
\]
where we used (4.9). This implies that
\[
\left| \frac{d}{ds} \log(e^{\lambda_1 s} V_{= \lambda_I}(s)) \right| \leq C \delta(s) \text{ on } (-\infty, s_0]
\]
As $\delta(s) \leq C e^{-\lambda_1 s}$, we see that this gives a contradiction upon integrating. Thus the alternative must hold, i.e.
\[
V_{= \lambda_I}(s) \leq C \delta(s) V_{< \lambda_I}(s) \text{ on } (-\infty, 0].
\]
In particular this means that $\mu \neq \lambda_1$. In general, suppose the proposition is true for $\mu = \lambda_I$, then for $\mu = \lambda_{J'}$ the largest eigenvalue below $\lambda_J$, we can repeat the above argument using the fact that
\[
V_-^2 = V_{0 > \mu > \lambda_J}^2 + V_{= \lambda_J}^2 + V_{< \lambda_J}^2,
\]
and the proof follows verbatim until \( \lambda_j^* = \lambda_1. \)

Let \( I^* \) be the largest index for which (4.14) fails, then it follows, from (4.13) applied to the smallest eigenvalue above \( \lambda_{I^*}, \) that

\[
\frac{d}{ds} V_{\leq \lambda_{I^*}} + \lambda_{I^*} V_{\leq \lambda_{I^*}} \geq -C \delta(s) V_{\leq \lambda_{I^*}}.
\] (4.15)

Using (4.15) in place of (4.13), a similar argument as above shows that \( \delta(s) \leq C e^{-\lambda_{I^*} s}, \) which is the sharp asymptotic decay in time for the solution.

To finish the proof, we seek to apply the uniqueness aspect of Theorem 3.10. It suffices to establish (3.10). For \( \sigma > 0, \) let \( v^{(\sigma)}(p, s) = v(p, s - \sigma) \) be the translated solution. Then by definition of \( I^* \) we have

\[
\limsup_{\sigma \to \infty} e^{-\lambda_{I^*} \sigma} \| \Pi_{-} v^{(\sigma)}(\cdot, 0) \|_W > 0.
\]

**Proposition 4.8.** Given \( \sigma \geq 0, \) then it holds for \( s \leq 0, \)

\[
e^{-2\lambda_{I^*} \sigma} \| v^{(\sigma)}(\cdot, s) \| - \sum_{i \leq I} e^{-\lambda_i(s-\sigma)} \langle v(\cdot, s - \sigma), \phi_i \rangle \phi_i \|_W \leq C e^{-\lambda_I s}.
\]

**Proof.** As \( \delta(s) \leq C e^{-\lambda_{I^*} s} \) and the negative mode is dominant, we have

\[
e^{-2\lambda_{I^*} \sigma} (V_0 + V_+)(\cdot, \cdot) - \sum_{i \leq I} e^{-\lambda_i(s-\sigma)} \langle v(\cdot, s - \sigma), \phi_i \rangle \phi_i \|_W \leq C e^{-\lambda_I s}.
\]

On the other hand, (4.13) with \( \mu = 0 \) implies that for every \( 1 \leq i \leq I, \)

\[
\| \frac{d}{ds} u_i + \lambda_i u_i \|_W \leq C \delta(s) V_{\leq I^*},
\]

where \( u_i = e^{-\lambda_i(s-\sigma)} \langle v(\cdot, s - \sigma), \phi_i \rangle \phi_i. \) Multiplying this equation by \( e^{\lambda_i s} \) and integrating from \( s - \sigma \) to \( -\sigma \) yields,

\[
\| e^{\lambda_i(s-\sigma)} u_i(\cdot, s - \sigma) - e^{-\lambda_i s} u_i(-\sigma) \|_W \leq C \int_{s-\sigma}^{\sigma} e^{\lambda_i \rho} e^{-2\lambda_{I^*} \rho} d\rho.
\]

Thus

\[
e^{-2\lambda_{I^*} \sigma} \| u_i(\cdot, s - \sigma) - e^{-\lambda_i s} u_i(-\sigma) \|_W \leq C \int_{s-\sigma}^{\sigma} e^{-\lambda_i(s-\sigma-\rho)} e^{-2\lambda_{I^*} \rho} d\rho
\]

\[
\leq C |s| e^{-2\lambda_{I^*} s} \leq C e^{-\lambda_I s},
\]

60
provided \( \sigma \) is sufficiently large.

Now let \( 0 < \delta < -\lambda_I \). Using interior Schauder estimates \([30, \text{Theorem C.2}]\) and Corollary 3.9 together with Proposition 4.8, we see that, for every \( s \leq 0 \),

\[
e^{(\lambda_I + \delta)s} \left\| v^{(\sigma)}(\cdot, s) \right\| - \sum_{i \leq I} e^{-\lambda_i(s-\sigma)} \left\langle v(\cdot, s - \sigma), \phi_i \right\rangle \phi_i \left\| \right\|_{C^{2,\alpha}_p(\Sigma \times [s-1, s])} \leq C e^{2\lambda_I^* s}.
\]

Theorem 4.5 now follows from Theorem 3.10 after choosing \( \beta \) sufficiently large depending on \( C \) and then \( \sigma \) sufficiently large so that \( |\left\langle v(\cdot, -\sigma), \phi_i \right\rangle| < \eta \), where \( \eta = \eta(\beta) \).

Before we prove Theorem 1.4, we briefly mention two other related results that fit into the spectral analysis frame. First, In \( \mathbb{R}^3 \), using the classification of low entropy self-shrinkers of Bernstein and Wang [8], we have the following strong converse in terms of Morse flow lines to Theorem 3.10 in the low entropy setting. Roughly speaking, the constructed flows, in fact, connect two self-expanders as in the classical Morse theory.

**Definition 4.9.** A rescaled Brakke flow \( \tilde{\mathcal{M}} \) is a *Morse flow line* if there exists (distinct) self-expanders \( \Sigma, \Gamma \) asymptotic to the same cone \( C \) such that

\[
\lim_{s \to -\infty} \tilde{\mu}_s = \mathcal{H}^n \llcorner \Sigma \text{ and } \lim_{s \to \infty} \tilde{\mu}_s = \mathcal{H}^n \llcorner \Gamma.
\]

For more discussions on the importance of such flow lines, we refer to [6].

**Corollary 4.10.** Suppose \( C \subset \mathbb{R}^3 \) is a cone with \( \lambda[C] < \lambda[\mathbb{S} \times \mathbb{R}] \). Then any tame ancient RMCF is a Morse flow line. In particular, there exist an \( I \)-parameter family of Morse flow lines coming out of an index \( I \) self-expander asymptotic to \( C \).

**Proof.** By Huisken’s monotonicity formula Theorem 2.4, any singularity of the flow must have entropy less than \( \lambda[\mathbb{S} \times \mathbb{R}] \). By [8, Corollary 1.2], it must be a round sphere \( \mathbb{S}^2 \). However, as any tame ancient RMCF is asymptotically conical (as \( \Sigma \) is asymptotically conical), it cannot encounter a compact singularity at the first singular time. Thus, any such flow must remain smooth for all time. The second conclusion follows in view of Theorem 1.2. \(\square\)
Remark. We suspect that the entropy bound can be relaxed, with a suitable surgery procedure, to $\lambda[C] < \lambda[S \times \mathbb{R}] + \delta$.

Secondly, if we know that the flow is expander mean convex at some time, it is also possible to extend an ancient RMCF to a Morse flow line. In view of (3.4), a graphical RMCF over a self-expander $\Sigma$ is expander mean convex if and only if $v > 0$; that is, the RMCF lies on one side of $\Sigma$. An expander mean convex RMCF stays expander mean convex for all future time as long as it is smooth, by Theorem 3.14. In [6], we extended the notion of expander mean convexity past singularities. In particular, we showed that a smooth expander mean convex RMCF can be extended in some appropriate weak sense to stay expander mean convex in all future time, regardless of singularities, and that the extended flow is a rescaled Brakke flow. Moreover, the forward limit of such a flow is always a stable self-expander. This limit is unique due to expander mean convexity, and is smooth in low dimensions.

Using a slight modification of the argument used in [6], we have the following partial converse to Theorem 4.5.

**Proposition 4.11.** Let $2 \leq n \leq 6$. If $v$ is an ancient solution constructed in Theorem 3.10 and there is $s_0 < 0$ such that $v(\cdot, s_0) > 0$, then $v$ can be extended to a Morse flow line.

**Proof.** Since $v(\cdot, s_0) > 0$, by the strong maximum principle (see eg. [80]), we have $v > 0$ on $(s_0, 0]$. We can then follow the construction in [6, Section 3] to extend the flow in an expander mean convex way past the singularity (the results there are stated for perturbations of the first eigenfunction $\phi_1$, but the proof only uses the fact that $\phi_1$ has a sign). Since we are in low dimensions, the limiting self-expander $\Gamma$ must be smooth, and so $v$ gives rise to a Morse flow line between $\Sigma$ and $\Gamma$. $\square$

The prototypical example of an expander mean convex RMCF is the one-sided RMCF constructed in Theorem 1.3, which is expander mean convex for all time.
However, the above proposition is not so effective in general, as the condition \( v(\cdot, s) > 0 \) is hard to check. Indeed, the first eigenfunction \( \phi_1 \) is the only eigenfunction that has a sign, but at the same time it also has the best asymptotic decay among all eigenfunctions (cf. Theorem 2.10). In fact, an ambitious conjecture would be \( v(\cdot, s_0) > 0 \) for some \( s_0 \in (-\infty, 0) \) implies the same for all \( s_0 < 0 \). One can in fact show that the only tame ancient RMCF that is expander mean convex for all time is the one constructed in Theorem 1.3.

**Proposition 4.12.** Suppose \( \mathcal{C} \) is generic, then, up to time translation, there is a unique solution \( v \) to (3.4) such that \( v > 0 \) on \( (-\infty, 0) \).

**Proof.** Let \( v > 0 \) be a positive solution to (3.4) on \( (-\infty, 0) \). As \( \mathcal{C} \) is generic, we can follow the spectral analysis above to get the sharp asymptotic decay

\[
\delta(s) \leq Ce^{-\lambda_{I_s}s}.
\]  

(4.16)

We claim that \( \lambda_{I_s} = \lambda_1 \). This follows from the fact that

\[
V_{\neq \lambda_{I_s}}(s) \leq C\delta(s)V_{= \lambda_{I_s}}
\]

when \( s \leq s_0 \) for some \( s_0 < 0 \) (indeed if the above were true, it will violate the sharpness of the estimate (4.16)). To see this, note that \( v > 0 \) implies that

\[
-\min\{0, \Pi_{= \lambda_{I_s}}v\} \leq \left\| \Pi_{\neq \lambda_{I_s}}v \right\| \implies \left\| -\min\{0, \Pi_{= \lambda_{I_s}}v\} \right\| \leq C\delta(s)V_{= \lambda_{I_s}}
\]

Now let \( h^s = V_{= \lambda_{I_s}}^{-1}(s)\Pi_{= \lambda_{I_s}}v(\cdot, s) \), then as \( s \to -\infty \), \( h^s \) converges to an \( \lambda_{I_s} \)-eigenfunction \( h \) of norm 1. Since \( \delta(s) \to 0 \) as \( s \to -\infty \), it follows that

\[
\left\| -\min\{0, h\} \right\|_W = 0,
\]

and so \( h \geq 0 \). By standard spectral theory the only eigenfunction that does not change sign corresponds to the lowest eigenvalue. Hence \( \lambda_{I_s} = \lambda_1 \). In particular, for any positive solution \( v \), there exists a constant \( \alpha_1 \neq 0 \) such that

\[
\lim_{s \to -\infty} e^{s\lambda_1}v(\cdot, s) = \alpha_1 \phi_1.
\]
Since $C$ is generic, we can apply the strong uniqueness theorem Theorem 4.5 to conclude that $v = \Phi(a)$ for some $a = (a_1, \ldots, a_I) \in \mathbb{R}^I$. By (3.10), we have that, for $s \leq 0$,

$$
\left\| v - \sum_{i=1}^{I} a_i e^{-\lambda_i s} \phi_i \right\|_{C^{2,\alpha}_{p,s}(\Sigma \times [s-1,s])} \leq e^{\delta_0 s} \beta \sum_{i=1}^{I} a_i^2
$$

where $0 < \delta_0 < -\lambda_I$ and $\beta > 0$. Multiplying by $e^{\lambda_1 s}$ on both sides yields

$$
\left\| e^{\lambda_1 s} v - \sum_{i=1}^{I} a_i e^{(-\lambda_i + \lambda_1) s} \phi_i \right\|_{C^{2,\alpha}_{p,s}(\Sigma \times [s-1,s])} \leq e^{\delta_0 + \lambda_1 s} \beta \sum_{i=1}^{I} a_i^2
$$

Since $v(\cdot, s) \leq C e^{-\lambda_1 s}$ and $\delta_0 + \lambda_1 < \lambda_1 - \lambda_I \leq \lambda_1 - \lambda_i$ for all $i \geq 1$, the above can only hold if $a_i = 0$ for all $2 \leq i \leq I$. This proves that there is a one-parameter family of positive solutions to (3.4), which corresponds precisely to time translations. \qed

Remark. By modifying a beautiful iteration argument of Chodosh–Choi–Mantoulidis–Schulze [24, Corollary 5.2], we expect that the uniqueness continues to hold without the genericity assumption (essentially, the nonlinear term $Q$ in the expander case satisfies the same estimates as in the shrinker case). As our article mostly concerns with generic cones, we have chosen to state the simpler version of the uniqueness result.

4.3 Łojasiewicz inequality and uniqueness of tangents

In this section, we prove Theorem 1.4. We study mean curvature flows coming out of cones and try to understand when such a flow will correspond to a tame ancient RMCF under (1.5). First of all let us impose the trapping hypothesis, which we did not utilize in our previous discussions. We say an asymptotically conical hypersurface $\Gamma$ is trapped if there exist two self-expanders $\Sigma_1$ and $\Sigma_2$ asymptotic to $C$ such that there is some radius $R_0 > 0$ such that

$$
\Gamma \setminus B_{R_0}(0) \subset \Omega_1^+ \cap \Omega_2^-.
$$
where \( \Omega^+_1, \Omega^+_2 \) denote the connected components of \( \mathbb{R}^{n+1} \setminus \Sigma_1 \) and \( \mathbb{R}^{n+1} \setminus \Sigma_2 \) respectively, oriented in a way such that \( \Omega^+_1 \subset \Omega^+_2 \). In the special case \( \Gamma = \Sigma_v \), using the asymptotic structure of self-expanders at infinity [5], we deduce that \( v \) satisfies the sharp decay estimates
\[
|v(p)| \leq C |x(p)|^{-n-1} e^{-\frac{|x(p)|^2}{4}} \quad \text{when} \quad |x(p)| > R_0
\] (4.17)

We will work in low dimensions, i.e. \( 2 \leq n \leq 6 \), where the structure theory of self-expanders is best known. First of all, under the trapping assumption, we prove that the relative expander entropy \( E^*_\text{rel}[\Sigma_v, \Sigma] \) indeed coincides with the usual notion \( E_{\text{rel}} \) from [13]. This seemingly innocent fact has to do with the failure of the normal graph map \( f_v \) being bijective in the annulus \( B_{R+2}(0) \setminus B_R(0) \). Due to the large weight \( e^{\frac{|x|^2}{4}} \), this difference cannot be killed unless the function \( v \) has a very good decay.

**Proposition 4.13.** Suppose \( v : \Sigma \to \mathbb{R} \) is a \( W^1 \) function such that \( E^*_\text{rel}[\Sigma_v, \Sigma] < \infty \). If \( \Sigma_v \) is trapped between two self-expanders \( \Gamma_1, \Gamma_2 \) asymptotic to \( C \) then
\[
E_{\text{rel}}[\Sigma_v, \Sigma] = E^*_\text{rel}[\Sigma_v, \Sigma] < \infty.
\]

**Proof.** We fix \( R \) sufficiently large and write \( \chi_R = \chi \) for simplicity. Let \( A_{r,R} = B_R(0) \setminus \bar{B}_r(0) \). Since \( f_{\chi v} \) is an embedding, it is a diffeomorphism between \( \Sigma_{\chi v} \cap B_{R+2}(0) \) and \( \Sigma \cap B_{R+2}(0) \). Let \( Y_R = f^{-1}_{\chi v}(\Sigma_{\chi v} \cap B_R(0)) \subset \Sigma \cap B_{R+2}(0) \) and \( Z_R = (\Sigma \cap B_{R+2}(0) \setminus Y_R) \). By the triangle inequality
\[
|f_{\chi v}(p)| \leq |x(p)| + |v(p)|,
\]
consequently the sets \( Y_R \) satisfy
\[
Y_R \Delta (\Sigma \cap B_R(0)) \subset \Sigma \cap A_{R-\bar{v}_R,R+\bar{v}_R}
\] (4.18)
where \( \bar{v}_R = \sup_{\Sigma \cap A_{R,R+2}} |v| \), and \( \Delta \) denotes the symmetric difference of two sets. Since \( \Sigma \) is trapped, it follows from (4.17) that
\[
|Y_R \Delta (\Sigma \cap B_R(0))| = O(R^{-2} e^{-\frac{R^2}{4}}).
\]
Now write
\[
\int \chi v \sum e^{\frac{|x|^2}{4}} - \int \sum e^{\frac{|x|^2}{4}} = \left( \int \chi v \cap B_R(0) e^{\frac{|x|^2}{4}} - \int_{Y_R} e^{\frac{|x|^2}{4}} \right) \\
+ \left( \int \chi v \cap A_{R, R+2} e^{\frac{|x|^2}{4}} - \int_{Z_R} e^{\frac{|x|^2}{4}} \right).
\]

We can estimate
\[
\left| \int_{Y_R} e^{\frac{|x|^2}{4}} - \int \chi v \cap B_R(0) e^{\frac{|x|^2}{4}} \right| \leq \int_{Y_R} \Delta (\chi v \cap B_R(0)) e^{\frac{|x|^2}{4}} = O(R^{-2}),
\]
where we used the fact that, when \( R \) is sufficiently large
\[
e^{\frac{|x|^2}{4}} \leq e^{\frac{(R + v_R)^2}{4}} \leq e^{\frac{R^2}{4}} \left(1 + v_R R + 2v_R^2\right) \leq e^{\frac{R^2}{4}} + CR^{-n} \leq e^{\frac{R^2}{4}} + 1.
\]

Thus
\[
\lim_{R \to \infty} \int_{Y_R} e^{\frac{|x|^2}{4}} - \int \chi v \cap B_R(0) e^{\frac{|x|^2}{4}} = 0. \tag{4.19}
\]

Moreover, using (4.5), we get that
\[
\int e^{\frac{|x|^2}{4}} - \int \chi v \cap A_{R, R+2} e^{\frac{|x|^2}{4}} \\
\leq \frac{1}{2} \int_{Z_R} \left( |\nabla \chi|^2 + |\nabla \chi|^2 \right) \chi (|\nabla \chi|^2 + v^2) + \left( \frac{3}{2} - |A\chi|^2 \right) v^2 + C\varepsilon (|\nabla \chi|^2 + v^2) e^{\frac{|x|^2}{4}}.
\]

As \( v \in W^1 \),
\[
\lim_{R \to \infty} \int_{Z_R} (|\nabla \chi|^2 + v^2) e^{\frac{|x|^2}{4}} = 0.
\]

Since \( |A\chi| \) is bounded, the above implies that
\[
\lim_{R \to \infty} \int e^{\frac{|x|^2}{4}} - \int \chi v \cap A_{R, R+2} e^{\frac{|x|^2}{4}} = 0 \tag{4.20}
\]

(4.19) and (4.20) imply the desired equality.

The above equivalence means that we have all the tools from [13] in our disposal, and from now on we will write unambiguously \( E_{\text{rel}} \) for the relative entropy.
Now let \( \mathcal{M} = \{ \mu_t \}_{t \in (0,T]} \) be an integral Brakke flow coming out of \( C \) in the sense that
\[
\lim_{t \to 0} \mu_t = \mathcal{H}^n \res C.
\]
\( \mathcal{M} \) is contained in the level set flow of \( C \) (which necessarily fattens as long as there are more than one self-expanders asymptotic to \( C \)). The key fact is that, when \( 2 \leq n \leq 6 \), by [24, Theorem 8.21], the two outermost flows of the cone \( C \) corresponding to the boundary of the level set flow are given by two stable self-expanders (which are smooth when \( 2 \leq n \leq 6 \)). Hence \( \mathcal{M} \) is, in fact, trapped between two asymptotically conical self-expanders. In particular the following forward monotonicity formula holds (here we have taken \( f = 1 \)).

**Proposition 4.14** (Proposition 6.5 of [13]). Let \( \mathcal{M} \) be as above, and let \( \tilde{\mathcal{M}} \) denote the corresponding rescaled Brakke flow defined on \( (-\infty, S) \). Then for any \( -\infty < s_1 \leq s_2 \leq S \), we have
\[
E_{\text{rel}}[\tilde{\mu}_{s_1}, \Sigma] \geq E_{\text{rel}}[\tilde{\mu}_{s_2}, \Sigma] + \int_{s_1}^{s_2} \left| \mathbf{H}_{\tilde{\mu}_s} - \frac{|x|}{2} \right|^2 e^{-|x|^2} d\tilde{\mu}_s ds.
\]

The following backward convergence to a self-expander is the starting point of our analysis.

**Proposition 4.15.** Let \( \mathcal{M} \) be as above. Up to passing to a subsequence \( s_i \to -\infty \), there exists a self-expander \( \Sigma \) such that
\[
\lim_{i \to \infty} \tilde{\mu}_{s_i} = \mathcal{H}^n \res \Sigma.
\]

**Proof.** Fix a reference self-expander \( \Sigma' \). Then \( E_{\text{rel}}[\tilde{\mu}_s, \Sigma'] \) is finite by appealing to the trapping and [13]. Consider the translated flow \( \tilde{\mu}_s^{s_0} = \tilde{\mu}_{s+s_0} \) defined on \( s \in (-\infty, S-s_0) \). By compactness of Brakke flows, up to passing to a subsequence \( \{s_i\} \), as \( s_i \to -\infty \), the sequence of translated flows converges to a rescaled Brakke flow \( \tilde{\mu} \) defined on \( (-\infty, \infty) \). Moreover,
\[
E_{\text{rel}}[\tilde{\mu}_s, \Sigma'] = \lim_{i \to \infty} E_{\text{rel}}[\tilde{\mu}_s^{s_i}, \Sigma'] < \infty.
\]
In view of the forward monotonicity formula Proposition 4.14, we have

\[ \int_{s_1}^{s_2} \int \left| H_{\bar{\mu}_s} - \frac{x^\perp}{2} \right|^2 d\bar{\mu}_s ds = 0 \]

for all \(-\infty < s_1 < s_2 < \infty\). Hence \(\bar{\mu}\) is \(E\)-stationary, and therefore a self-expander. Moreover as \(\tilde{M}\) is asymptotic to \(C\), so is \(\bar{\mu}\). \[\square\]

In general, however, \(\Sigma\) is only an \(E\)-stationary varifold and can have very large singular sets even in low dimensions. This is, in some sense, the key obstruction in the correspondence between tame ancient RMCFs and MCFs coming out of \(C\). One way to resolve this issue is to impose a low entropy condition, which forces the expanders to be smooth. This is the approach taken in this thesis.

The advantage of \(\Sigma\) being smooth is the uniqueness of tangent flows, proved using the Łojasiewicz-Simon inequality. Let \(N_\Sigma\) be the Euler-Lagrange operator associated with \(E_{\text{rel}}[\cdot, \Sigma]\) given by

\[ \frac{d}{ds} \bigg|_{s=0} E_{\text{rel}}[\Sigma_{v+sw}, \Sigma] = \langle N_\Sigma v, w \rangle. \]

As 0 is a critical point for \(E_{\text{rel}}\), the operator \(N_\Sigma\) takes the form

\[ N_\Sigma v = L_\Sigma v + Q(v), \]

which, in fact, agrees with the expander mean curvature of the hypersurface \(\Sigma_v\). Here we record a version of the Łojasiewicz inequality for generic cones, which is good enough for our applications. Since we do not use the trapping assumption in the following proof, the same argument will work for \(E^*_{\text{rel}}\) if we do not know a priori that \(E_{\text{rel}}\) and \(E^*_{\text{rel}}\) agree. However, in the proof of the uniqueness of tangent flows, Corollary 4.17, trapping is necessary.

**Theorem 4.16.** Let \(\Sigma\) be a self-expanders asymptotic to a generic cone \(C\). There is \(\varepsilon = \varepsilon(\Sigma)\) such that the following holds: suppose \(v \in C^{2,\alpha} \cap W^2(\Sigma)\) satisfies \(\|v\|_{C^{2,\alpha}} < \varepsilon\), then

\[ C \|N_\Sigma(v)\|_W \geq |E_{\text{rel}}[\Sigma_v, \Sigma]|^{1/2}. \] (4.21)
Proof. Let $L_\Sigma$ denote the operator

$$L_\Sigma v = \Delta_\Sigma v + \frac{1}{2} x \cdot \nabla_\Sigma v - \frac{1}{2} v = L_\Sigma v - |A_\Sigma|^2 v.$$ 

By [9, Proposition 3.4], $L_\Sigma$ is an isomorphism between $W^2$ and $W$, so there exists a constant $C = C(\Sigma) > 0$ such that

$$\|v\|_{W^2} \leq C \|L_\Sigma v\|_W \leq C \|L_\Sigma v\|_W + C C' \|v\|_W,$$

where $C' = C'(\Sigma) > 0$ depends only on $|A_\Sigma|^2$. As $C$ is generic, we have

$$\|L_\Sigma v\|_W \geq c \|v\|_W$$

for some constant $c > 0$ depending on the spectral gap $\tilde{\lambda} = \min\{|\lambda_i|\} > 0$. Thus

$$\|v\|_{W^2} \leq C \|L_\Sigma v\|_W + c^{-1} C C' \|L_\Sigma v\|_W \implies C_\Sigma \|v\|_{W^2} \leq \|L_\Sigma v\|_W,$$

where the constant $C_\Sigma > 0$ in the last inequality depends only on $\Sigma$. Using the expansion (2.4), there is $\varepsilon > 0$ such that when $\|v\|_{C^{2,\alpha}} < \varepsilon$, we have

$$\|Q(v, \nabla_\Sigma v, x \cdot \nabla_\Sigma v, \nabla^2_\Sigma v)\|_{W} \leq \frac{1}{2} C_\Sigma \|v\|_{W^1} \leq \frac{1}{2} C_\Sigma \|v\|_{W^2}.$$ 

Hence it follows from the triangle inequality that

$$\|N_\Sigma v\|_W = \|L_\Sigma v + Q(v)\|_W \geq \frac{1}{2} C_\Sigma \|v\|_{W^2} \quad (4.22)$$

On the other hand, Proposition 4.1 implies that

$$|E_{\text{rel}}[\Sigma, v, \Sigma]| \leq C_0 \|v\|_{W^1}^2 \leq C_0 \|v\|_{W^2}^2 \quad (4.23)$$

where $C_0 = C_0(\Sigma) > 0$. Combining (4.22) and (4.23), we see that

$$4C_0 C_\Sigma^{-2} \|N_\Sigma v\|_W^2 \geq C_0 \|v\|_{W^2}^2 \geq |E_{\text{rel}}[\Sigma, v, \Sigma]|$$

holds whenever $\|v\|_{C^{2,\alpha}} < \varepsilon$. 

\qed
Remark. Without the genericity assumption, one instead expects a constant $\gamma \in (0, 1)$ such that

$$C \|N_{\Sigma}(v)\|_W \geq |E_{\text{rel}}[\Sigma_v, \Sigma]|^{1-\gamma/2},$$

(4.24)

which is the usual Łojasiewicz inequality for $E_{\text{rel}}$ proved by Park–Wang [71] (note here we use $E_{\text{rel}}$ instead of $E_{\text{rel}}^*$). This is, of course, also enough to deduce the uniqueness of tangent. However, as Theorem 4.4 requires genericity anyways, we have opted to prove the inequality in the generic case separately. We also emphasize that the proof of the theorem makes no use of the classical Łojasiewicz–Simon inequality from [77] and, therefore, also does not rely on the analyticity of the relative expander functional (which is indeed true). See [41] for some related discussion on analyticity vs. Morse-Bott conditions.

Remark. Here we shall explain that the exponent $\gamma = 1$ in (4.24) is the best possible by showing (4.21) implies (4.24) when $\Sigma_v$ is trapped. It is enough to show

$$\|N_{\Sigma}(v)\|_W^{\gamma_2} \leq C^{\gamma_2-\gamma_1} \|N_{\Sigma}(v)\|_W^{\gamma_1},$$

(4.25)

whenever $\gamma_1 \leq \gamma_2$ and $\|v\|_{C^{2,\alpha}} < \varepsilon$ for $\varepsilon$ sufficiently small. We point out that, in the following argument, the trapping assumption is also essential.

As $L_{\Sigma}$ is an isomorphism between $W^2$ and $W$,

$$\|L_{\Sigma} v\|_W^2 \leq C \|v\|_{W^2}^2.$$

Together with the expansion of $Q(v)$, (2.4), we conclude that

$$\|N_{\Sigma}(v)\|_W^2 \leq C \|v\|_{W^2}^2.$$

Given $\delta > 0$, we can choose $R_1 > R_0$ depending on $\delta$ such that

$$\int_{\Sigma \setminus B_{R_1}(0)} |v|^2 e^{\frac{|x|^2}{4}} \leq \frac{\delta}{2},$$

70
where we used the sharp decay rate of $v$ in (4.17). As $\Sigma \cap B_{R_1}(0)$ is compact, we can find $\varepsilon = \varepsilon(\delta, \Sigma)$ such that
\[
\int_{\Sigma} |v|^2 e^{|x|^2} = \int_{\Sigma \cap B_{R_1}(0)} |v|^2 e^{|x|^2} + \int_{\Sigma \cap B_{R_1}(0)} |v|^2 e^{|x|^2} \leq \delta
\]
whenever $\|v\|_{C^{2,\alpha}} < \varepsilon$. A similar argument shows the same for $\|v\|_{W^2}$ (possibly shrinking $\varepsilon$). This means that we can guarantee
\[
\|N_{\Sigma}(v)\|_W^2 \leq C\delta^2
\]
whenever $\|v\|_{C^{2,\alpha}} < \varepsilon$. (4.25) immediately follows (by setting $\delta = 1$, for example).

Using Theorem 4.16 we can upgrade subsequential convergence to full convergence in the generic case. Note that in the following, we only require the limit to be smooth for one subsequence in the following.

**Corollary 4.17.** Let $\mathcal{M}$ and $\Sigma$ be as above. If $\Sigma$ is smooth, then
\[
\lim_{s \to -\infty} \bar{\mu}_s = \mathcal{H}^n|\Sigma.
\]

**Proof.** The proof is standard (see eg. [75] or [79, Chapter 3]), and we present it for the sake of completeness. For any function $v(\cdot, s) \in C^{2,\alpha} \cap W^2$ with $\|v(\cdot, s)\|_{C^{2,\alpha}} < \varepsilon$ sufficiently small, we compute, using the forward monotonicity formula Proposition 4.14:

\[
\frac{d}{ds} E_{\text{rel}}[\Sigma v(\cdot, s), \Sigma] = - \int_{\Sigma v(\cdot, s)} \left| H_{\Sigma v(\cdot, s)} - \frac{x^1}{2} e^{|x|^2} \right|^2_{\Sigma v(\cdot, s)} \, d\mathcal{H}^n \\
\leq - \left( \int_{\Sigma} |N_{\Sigma} v|^2 \right)^{1/2} \left( \int_{\Sigma} \left| \frac{\partial v}{\partial s} \right|^2 e^{|x+s(\cdot)\eta_3|^2} \frac{|x+s(\cdot)\eta_3|^2}{4} \text{Jac}(v) \right)^{1/2} \\
\leq C \|N_{\Sigma} v\|_W \left\| \frac{\partial v}{\partial s} \right\|_W,
\]

where $C = C(\Sigma)$. Hence, by Theorem 4.16,
\[
- \frac{d}{ds} \left| E_{\text{rel}}[\Sigma v(\cdot, s), \Sigma] \right|^{1/2} \geq C \|E_{\text{rel}}[\Sigma v(\cdot, s), \Sigma]\|^{-1/2} \|N_{\Sigma} v\|_W \left\| \frac{\partial v}{\partial s} \right\|_W \geq C \left\| \frac{\partial v}{\partial s} \right\|_W
\]
Integrating the above gives, for $s_2 \geq s_1$

$$\int_{s_1}^{s_2} \left\| \frac{\partial v}{\partial s} \right\|_W ds \leq C_0 \left| E_{\text{rel}}[\Sigma_{v(\cdot, s_1)}, \Sigma] \right|^{1/2},$$

(4.26)

and therefore

$$\|v(\cdot, s_2)\|_W \leq \|v(\cdot, s_1)\|_W + C_0 \left| E_{\text{rel}}[\Sigma_{v(\cdot, s_1)}, \Sigma] \right|^{1/2}.$$

(4.27)

for $C_0 = C_0(\Sigma)$.

Let $\varepsilon_1 > 0$ be such that the nearest point projection onto $\Sigma$ is smooth in a tubular neighborhood of radius $\varepsilon_1$. Let $\delta$ be such that the extension lemma [75, Lemma 2.2] holds with $\beta = \frac{1}{2}$ and $\sigma = \varepsilon_1$ (strictly speaking we need a slightly modified version with $L^2$ norm replaced by the $W$ norm). Choose $\varepsilon_0 = \varepsilon_0(\delta, \Sigma)$ such that $\|v\|_{C^{2,\alpha}} < \varepsilon_0$ implies $\|v\|_{W^2} \leq \delta/(3C_0)$ (as $\mathcal{M}$ is trapped) and

$$C_0 \left| E_{\text{rel}}[\Sigma_{v}, \Sigma] \right|^\frac{1}{2} \leq 2C_0 \left\| v \right\|_{W^2} \leq \frac{2\delta}{3},$$

in view of Proposition 4.1. Now let $s_i \to -\infty$ be a convergent subsequence, i.e.

$$\lim_{i \to \infty} \bar{\mu}_s = \mathcal{H}^n \Sigma.$$

As $\Sigma$ is a smooth self-expander, by Brakke regularity theorem [15], for each $i$ there is $\eta_i$ such that $\bar{\mu}_s$ converges to the static flow of $\Sigma$ on $(s_i - \eta_i, s_i + \eta_i)$. By subsequential convergence, given $\varepsilon > 0$ we may assume $i$ is taken large enough so that the flow can be written as a normal graph $v$ over $\Sigma$ with $\|v(\cdot, s)\|_{C^{2,\alpha}} < \varepsilon_0$ on $(s_i - \eta_i, s_i + \eta_i)$. By the interior estimates of Ecker–Huisken [38] we may assume $\eta_i > 1$ when $i$ is sufficiently large.

Fix an $i$ such that all of the above is satisfied. (4.27) implies that

$$\|v(\cdot, s_i - \log(2))\|_W \leq \|v(\cdot, s_i)\|_W + C_0 \left| E_{\text{rel}}[\Sigma_{v(\cdot, s_i)}, \Sigma] \right|^{1/2} < \delta.$$  

Applying the extension lemma [75, Lemma 2.2], we get that $v$ can be extended to a solution $v$ to (3.4) on $(s_i - 2\log(2), s_i + \log(2))$ with $\|v\|_{C^{2,\alpha}} < \varepsilon_1$. Iterating the above
using the forward monotonicity formula Proposition 4.14 (note that $E_{rel}$ in our case is negative and decreasing from $-\infty$, so $|E_{rel}|$ is increasing from $-\infty$), we get a solution $v$ to (3.4) on $(-\infty, s_i)$ with $\|v\|_{C^{2,\alpha}} < \varepsilon_1$. By (4.26), we have $v \in W(\Sigma \times (-\infty, s_i))$. Hence

$$\lim_{s \to -\infty} \left\| \frac{\partial v}{\partial s} (\cdot, s) \right\|_W = 0.$$ 

Arguing as before using interior Schauder estimates together with Theorem 3.8 shows that after a suitable time translation, $v$ converges backwards to a (smooth) static solution $\Sigma'$ to the RMCF that is graphical over $\Sigma$. As $\mathcal{M}$ is trapped, unique continuation [5, Theorem 1.4] implies that $\Sigma' = \Sigma$, as desired.

**Remark.** As a byproduct of Theorem 4.16, we can determine the rate of decay of the flow to the static solution, similar to the ODE case. Indeed, the rate (in the rescaled setting) is exponential if (4.21) holds and polynomial depending on $\gamma$ if (4.24) holds.

**Proof of Theorem 1.4.** It suffices to establish that any blow up limit of such flow is automatically smooth in view of (4.21). This argument is essentially carried out in [28, Lemma 3.1] (cf. [7, Lemma 4.1]). Let $\Sigma$ be a subsequential limit from Proposition 4.15 and let $p \in \text{sing}(\Sigma)$. By lower-semicontinuity of the entropy, any tangent cone $\nu$ at $p$ is a stationary cone with entropy less than 2. We claim that any such cone must be flat. Take an iterated tangent cone $\nu'$ at a singular point $q \in \text{supp} \nu$ (that is not the vertex). $\nu'$ is then a 2-dimension stationary cone with entropy at most 2 that splits off a line, which we write as $\nu' = \nu'' \times \mu_{\mathbb{R}}$ for some 1-dimensional stationary cone $\nu''$. Any such $\nu''$ is a union of rays. Since the entropy of $\nu''$ is less than 2, $\nu''$ is either a flat line or a triple junction. However, as the MCF is smooth, it is cyclic as a mod 2 flat chain, which cannot encounter any singularity modeled on a triple junction by works of White [89]. Hence $\nu''$ is a flat line, and $\nu'$ is flat. This implies $\nu$ is smooth away from the origin. Since $\nu$ is stationary and has entropy less than 2, $\text{supp} \nu \cap S^2$ is
a closed geodesic of $S^2$, which must be a multiplicity 1 great circle. This implies that $\nu$ is flat and $\Sigma$ is smooth by Allard regularity.

In $\mathbb{R}^4$, we also have a similar result but with a more restrictive entropy bound in order to make the subsequential limit smooth.

**Corollary 4.18.** Suppose $C \subset \mathbb{R}^4$ is a generic cone with $\lambda[C] < \frac{\pi}{2}$, then any smooth MCF coming out of $C$ is either a (smooth) self-expander or a tame ancient RMCF starting from a self-expander $\Sigma$ asymptotic to $C$, as constructed in ??.

**Proof.** The proof is similar except one uses [29, Lemma 4.2] instead of [28, Lemma 3.1]. Essentially, the same argument follows through until the conclusion $\text{supp} \, \nu \cap S^3$ is a closed smooth minimal surface in $S^3$. It follows from the resolution of Willmore conjecture [65] that $\text{supp} \, \nu \cap S^3$ must be an equatorial sphere as any other such minimal surface has (Gaussian) area ratio at least $\frac{2\pi^2}{4\pi} = \frac{\pi}{2}$. Hence $\nu$ is flat and $\Sigma$ is smooth. \qed
Chapter 5

Rotational Symmetry

In this chapter we prove Theorem 1.5 and Theorem 1.6. In contrast to the conventions used in the previous two chapters, the parameter $s$ will not be reserved for the time variable in RMCF (as we do not need to use it). We use the convention that the lower case letters like $x$ denote points in $\mathbb{R}^{n+1}$, where as upper case letters like $X$ denote spacetime points in $\mathbb{R}^{n+1} \times \mathbb{R}$.

Up to an ambient rotation so that the axis of symmetry is $x_1$-axis, we consider double cones of the form

$$x_1^2 = \begin{cases} 
\alpha_1(x_2^2 + x_3^2 + \cdots + x_n^2) & x_1 \geq 0 \\
\alpha_2(x_2^2 + x_3^2 + \cdots + x_n^2) & x_1 < 0 
\end{cases} \quad (5.1)$$

where $\alpha_1, \alpha_2 > 0$ are constants related to the aperture of the cone. When $m_1 = m_2$, the cone has full $O(n) \times O(1)$-symmetry.

5.1 The smooth case

In this section we prove Theorem 1.5 using the celebrated moving plane method. Let us first recall the parabolic maximum principle and Hopf lemma. If $u : P_r(0,0) \to \mathbb{R}$ is a graphical solution of MCF, $u$ satisfies the following parametrized PDE:

$$u_t = \sqrt{1 + |\nabla u|^2} \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).$$
Observe that if \( u_t \equiv 0 \), the equation is nothing but the usual minimal surface equation. The difference of two graphical solutions to MCF satisfies a second-order linear parabolic PDE (provided the gradients are bounded a priori, which is guaranteed by Proposition 3.3), so by standard theory of linear parabolic PDEs we have (cf. Section 6.2 in [28]):

**Lemma 5.1** (Maximum Principle). Suppose \( u, v \) are graphical solutions to the MCF in a parabolic cylinder \( P_r(0,0) \) with \( u(0,0) = v(0,0) \). If \( u \leq v \) in \( P_r(0,0) \), then \( u = v \) in \( P_r(0,0) \).

**Lemma 5.2** (Hopf Lemma). Suppose \( u, v \) are graphical solutions to the MCF in a half parabolic cylinder \( P_r(0,0) \cap \{ x_1 \geq 0 \} \) with \( u(0,0) = v(0,0) \) and \( \frac{\partial u}{\partial x_1}(0,0) = \frac{\partial v}{\partial x_1}(0,0) \). If \( u \leq v \) in \( P_r(0,0) \cap \{ x_1 \geq 0 \} \), then \( u = v \) in \( P_r(0,0) \cap \{ x_1 \geq 0 \} \).

The rough idea of the moving plane method is the following: By pseudolocality properties of MCF Proposition 3.3, we know that the solutions are smooth outside of a large ball \( B_{\sqrt{\sqrt{t}R}}(0) \) for \( R = R(\Sigma) \) and have very good asymptotic properties. Consider a hyperplane \( \Pi_s \) containing the axis of symmetry of the cone of height \( s \). The asymptotic property of the flow will guarantee that the flow has the properties of a reflectionally symmetric solution for \( s \) sufficiently large. By pushing the height \( s \) down, the maximum principle and the Hopf lemma will force the symmetry to be propagated to the interior region. When \( s \) reaches 0 we deduce the reflection symmetry, and the rotational symmetry follows since the direction of the hyperplane was arbitrary.

We now establish the preliminary graphical property of such flows, which is a basic consequence of the pseudolocality theorem, Theorem 3.2. The following lemmas are proved in the context of Brakke flows for uses later in the non-smooth case. Let \( C \) be a rotationally symmetric double cone and consider a Brakke flow \( \mathcal{M} = \{ \mu_t \}_{t \in [0,\infty]} \) that is initially smooth and that

\[
\lim_{t \to 0} \mu_t = \mathcal{H}^n \cup C.
\]
Figure 5-1. A typical picture of the moving plane

Fix an open half space $\mathbb{H} \subset \mathbb{R}^{n+1}$ and write $\Pi = \partial \mathbb{H}$. In the following we also write $\mathcal{M}_t^+ = \mathcal{M}_t \cap \mathbb{H}$.

**Lemma 5.3** (cf. Proposition 3.1). Let $\mathcal{M}$ be as above, then for every $t > 0$ there is $R = R(t)$ and a smooth function $u$ on $\mathcal{C}$ such that

$$\mathcal{M}_t^+ \setminus B_R(0) \subset \{ x(p) + u(p)n_C(p) \mid p \in \mathcal{C} \}.$$  

and $|u(p)| \leq C |x(p)|^{-1}$ for $p \in \mathcal{C}$ for some constant $C = C(t)$.

**Proof.** Fix a time $t_0$. Let us first show that $\mathcal{M}_t^+ \setminus B_R(0)$ can be written as a smooth normal graph over $\mathcal{C}$. By Theorem 3.2 and an argument similar to Proposition 3.3 with $\Sigma$ replaced by $\mathcal{C}$, there exists $R = R(t) > 0$ such that $\mathcal{M}_t \setminus B_R(0)$ is asymptotically conical to $\mathcal{C}$. Moreover, given $\eta > 0$, Theorem 3.2 implies that there exists $t_1$ such that for $0 < t < t_1$ and $x \in \mathcal{C} \setminus B_1(0)$, $\mathcal{M}_t \cap C_{\sqrt{t_1}}(x)$ can be written as a normal graph over $B^n_{\sqrt{t_1}}(x) \cap T_x \mathcal{C}$ with Lipschitz constant bounded by $\eta$. By parabolic rescaling, we see that, for $0 < t < 2t_0$ and $x \in \mathcal{C} \setminus B_{\sqrt{2t_0}}(0)$, $\mathcal{M}_t \cap C_{\sqrt{2t_0}}(x)$ can be written as a normal graph over $B^n_{\sqrt{2t_0}}(x)$ with Lipschitz constant bounded by $\eta$. In particular putting $t = t_0$ gives the desired graphicality. The regularity of $u$ follows from Proposition 3.3.

To see that the function $u$ decays near infinity, by a similar argument as in
Proposition 3.1, there exists $N$ such that for all $R > 1$ we have

$$\mathcal{M}_{t_0} \setminus B_{NR\sqrt{t_0+1}}(0) \subset T_{R^{-1}\sqrt{t_0+1}}(C).$$

Equivalently, for $R > N\sqrt{t_0+1}$,

$$\mathcal{M}_{t_0} \setminus B_R(0) \subset T_{N(t_0+1)(R')^{-1}}(C).$$

Enlarge $R$ if needed so that $\mathcal{M}_{t_0}^+ \setminus B_R(0)$ is a normal graph over $C$. We see that $u$ satisfies $|u(p)| \leq C|x(p)|^{-1}$ for $p \in C$ and $u(p) \in \mathcal{M}_{t_0}^+ \setminus B_{2R}(0)$. □

If $C \cap \mathbb{H}$ is a Lipschitz graph, the above graphicality can be upgraded to over $\Pi$.

**Lemma 5.4.** Suppose $C \cap \mathbb{H}$ is a Lipschitz graph over $\Pi$. Let $\mathcal{M}$ be as above. For every $t > 0$ there is $R = R(t)$ such that $\mathcal{M}_{t}^+ \setminus B_R(0)$ can be written as a graph over $\Pi$; that is, the projection $\pi : \mathcal{M}_{t}^+ \setminus B_R(0) \to \Pi$ is injective.

**Proof.** By Lemma 5.3, for every $\eta > 0$, there is $R = R(t)$ such that $\mathcal{M}_{t_0}^+ \setminus B_R(0)$ is a normal graph over $C$ with Lipschitz constant bounded by $\eta$. Since $C \cap \mathbb{H}$ is a Lipschitz graph over $\Pi$, the unit normal vector $n_\mathbb{H}$ is not contained in any tangent space to $x' \in (C \cap \mathbb{H}) \setminus \{0\}$. Therefore by taking $\eta$ sufficiently small we may make sure that $n_\mathbb{H}$ is also not contained in any tangent space to $x \in \mathcal{M}_{t}^+ \setminus B_R(0)$ (here $R = R(t, \eta)$, but of course $\eta$ in turn depends on $t$). This proves that $\mathcal{M}_{t}^+ \setminus B_R(0)$ is graphical over $\Pi$ as well. □

Let

$$\Pi^s = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = s\} \times [0, \infty) \subset \mathbb{R}^{n+1} \times [0, \infty)$$

be the hyperplane at level $s$ in spacetime. Given a set $A \subset \mathbb{R}^{n+1} \times [0, \infty)$ and $s \in [0, \infty)$ we let

$$A^{s+} = \{(x, x_{n+1}, t) \in A \mid x_{n+1} > s\} \quad \text{and} \quad A^{s-} = \{(x, x_{n+1}, t) \in A \mid x_{n+1} < s\}$$

78
be the parts of $A$ lying above $\Pi_s$ and below $\Pi_s$ respectively. Finally, the set

$$A^s = \{(x, x_{n+1}, t) \mid (x, 2s - x_{n+1}, t) \in A\}$$

is the reflection of $A$ across $\Pi_s$. We say $A > B$ for $A, B \subset \mathbb{R}^{n+1} \times [0, \infty)$ provided for any $(x, s, t) \in A$ and $(x, s', t) \in B$ we have $s > s'$. In contrast, a subscript $t$ will continue to denote the time $t$ slice of a spacetime set.

Theorem 1.5 is a straightforward consequence of the following reflection symmetry theorem.

**Theorem 5.5.** Suppose $C \cap \mathbb{H}$ is a Lipschitz graph and $C$ is symmetric with respect to the reflection across $\Pi = \partial \mathbb{H}$, then any smooth MCF coming out of $C$ inherits the reflection symmetry.

To set up the proof of Theorem 5.5, WLOG we may assume the half space is $\mathbb{H} = \{x_{n+1} > 0\}$. Let $T$ be the first singular time of the flow $\mathcal{M}$. Fix a time $T > T_0 > 0$. We consider $\mathcal{M}$ on $[0, T_0]$ as its spacetime track; namely,

$$\mathcal{M} = \bigcup_{t=0}^{T_0} \mathcal{M}_t \times \{t\} \subset \mathbb{R}^{n+1} \times [0, T_0].$$

Finally, let

$$S = \{s \in (0, \infty) \mid (\mathcal{M}^s)^* > \mathcal{M}^{s^-}, \text{ and } (\mathcal{M}^s)^*_t \text{ is graphical over } (\Pi^s)_t \text{ for } t \in [0, T_0]\}.$$ 

Here graphicality means that the projection $\pi_s : (\mathcal{M}^s)^*_t \to (\Pi^s)_t$ is injective for $t \in [0, T]$. Since each $(\mathcal{M}^s)^*_t$ is countably $n$-rectifiable, graphicality is equivalent to that the unit normal $e_{n+1} = (0, \ldots, 0, 1)$ of $(\Pi^s)_t$ is not contained in the approximate tangent space of $(\mathcal{M}^s)^*_t$ for $t \in [0, T_0]$. Observe that $(\mathcal{M}^s)^*$ is asymptotically conical to the translated cone $(C + 2se_{n+1}) \times [0, \infty)$ (in the sense of Lemma 5.3 — this ensures that a hypersurface cannot be simultaneously asymptotic to two distinct cones).

**Proof of Theorem 5.5.** We must show $S$ is nonempty, $S$ is open, and $S$ is closed.
By Lemma 5.3, for sufficiently large $s$ we can make sure that $\mathcal{M}^{s+}$ is a smooth MCF,

$$(\mathcal{M}^{s+})^* \cap ((\mathcal{M}^{s-}) \cap \{x_{n+1} \geq 0\}) = \emptyset,$$

and that for any $(x, s_1, t) \in (\mathcal{M}^{s+})^*$ and $(x, s_2, t) \in \mathcal{M}^{0-}$ it holds that $s_1 - s_2 \geq 2s - 1$.

These two facts imply that for sufficiently large $s$ the inequality $(\mathcal{M}^{s+})^* > \mathcal{M}^{s-}$ is valid. On the other hand, by Lemma 5.4, there is $R = R(T_0)$ such that $(\mathcal{M}^{0+})_{T_0} \setminus B_R(0)$ is graphical over $(\Pi^0)_{T_0}$. So for $s > R$ we have $(\mathcal{M}^{s+})_t$ is graphical over $(\Pi^s)_t$ for all $t \in [0, T_0)$. This shows $S$ is not empty.

It is clear that $(\mathcal{M}^{s+})^* > \mathcal{M}^{s-}$ is an open condition. To see that the graphicality condition is also an open condition, let $\theta_t(x)$ be the angle between the unit normal to the approximate tangent space at a point $x \in \mathcal{M}_t$ and $e_{n+1}$. Suppose that $s \in S$, then graphicality is equivalent to $\theta_t(x) < \frac{\pi}{2}$ for all $t \in [0, T_0)$ and $x \in (\mathcal{M}^{s+})_t$. Since the flow $\mathcal{M}$ is $C^{2,\alpha}$-asymptotically conical, for given $t$ there exists $\varepsilon > 0$ such that $\theta_t(x) < \pi/2$ for all $x \in (\mathcal{M}^{s'+})_t$ where $|s' - s| < \varepsilon$. Since the time interval is compact, there is a universal $\varepsilon$ such that the above holds for all $t \in [0, T_0]$. This shows openness of $S$.

Finally we show $S$ is closed. Obviously if $s \in S$ then $[s, \infty) \subset S$. So we assume $(s, \infty) \subset S$ and suppose for a contradiction that $s \notin S$. At level $s$, either $(\mathcal{M}^{s+})^* \cap \mathcal{M}^{s-} \neq \emptyset$ or there is some $t_0 \in [0, T_0)$ such that $(\mathcal{M}^{s+})_{t_0}$ fails to be graphical over $(\Pi^s)_{t_0}$.

In the first case, $s$ is necessarily the first level of contact. By choosing $r$ small enough we can ensure $(\mathcal{M}^{s+})^*$ and $\mathcal{M}^{s-}$ are graphical in $P_t(X)$ where $(x_0, t_0) \in (\mathcal{M}^{s+})^* \cap \mathcal{M}^{s-}$. This allows us to apply Lemma 5.1 to see $(\mathcal{M}^{s+})^*$ and $\mathcal{M}^{s-}$ agree in an open neighborhood of $(x_0, t_0)$. The same reasoning applied to any other point in $\mathcal{M}^{s-} \cap (\mathcal{M}^{s+})^*$ shows that a connected component of $\mathcal{M}^{s-}$ agrees with a connected component of $(\mathcal{M}^{s+})^*$. This implies that $(\mathcal{M}^{s+})^*$ is simultaneously asymptotic to
\[ C \times [0, \infty) \text{ and } (C + 2se_{n+1}) \times [0, \infty), \text{ a contradiction.} \]

In the second case, WLOG we may assume \( t_0 \) is the first time the graphicality condition fails. Then there necessarily exists a point \( (x, s, t_0) \in \mathcal{M}_{t_0} \cap \{ x_{n+1} = s \} \) whose tangent space contains the vector \( e_{n+1} \). Thus the tangent planes of \( (\mathcal{M}^{s+})^* \) and \( \mathcal{M}^{s-} \) at the point \( (x, s, t_0) \) must coincide. If we choose \( r \) small enough we can ensure that \( (\mathcal{M}^{s+})^* \) and \( \mathcal{M}^{s-} \) are graphical solutions of MCF in \( P_r(x, s, t_0) \cap \{ x_{n+1} \leq s \} \). Since the tangent planes coincide we can apply the Hopf Lemma, Lemma 5.2, to conclude again that \( (\mathcal{M}^{s_0+})^* \) is simultaneously asymptotic to \( C \times [0, \infty) \) and \( (C + 2se_{n+1}) \times [0, \infty) \), a contradiction.

This shows that \( S = (0, \infty) \). At \( s = 0 \), one sees that the graphicality condition is preserved (alternatively one can run the moving plane method from the other side, i.e. \( s < 0 \)), but the strict inequality \( (\mathcal{M}^{0+})^* > \mathcal{M}^{0-} \) does not hold anymore, which implies that \( (\mathcal{M}^{0+})^* \cap \mathcal{M}^{0-} \neq \emptyset \). Applying the maximum principle Lemma 5.1 once again we conclude \( (\mathcal{M}^{0+})^* = \mathcal{M}^{0-} \), and this is the required reflection symmetry across \( \Pi^0 = \{ x_{n+1} = 0 \} \).

\[ \text{Figure 5-2. Boundary touching} \]

\[ \text{Figure 5-3. Interior touching} \]
5.2 The general case

In this section we prove Theorem 1.6 using a variant of the moving plane method without smoothness. Notice that the standard parabolic maximum principle and Hopf lemma from the previous section does not necessarily apply anymore.

It is easy to generalize the parabolic maximum principle Lemma 5.1 to the Brakke flow setting. Recall that given a Brakke flow $M$ and a spacetime point $X = (x_0, t_0) \in \text{supp} M$, the Gaussian density at $X$ is

$$\Theta_M(X) = \lim_{\rho \to 0} \frac{1}{(4\pi \rho^2)^{n/2}} \int_{M_{t_0-\rho}^t} e^{-\frac{|x-x_0|^2}{4\rho^2}} dH^n.$$  

$\Theta_M(X)$ is well-defined by Huisken’s monotonicity formula Theorem 2.4. Observe that an entropy upper bound automatically gives upper bounds on all Gaussian densities.

Lemma 5.6 (Maximum principle for Brakke flows, Theorem 3.4 of [29]). Let $M$ be a smooth MCF defined in a parabolic ball $P_r(X)$, where $X = (x_0, t_0) \in \text{supp} M$ and $r > 0$ is sufficiently small such that $\text{supp} M$ separates $P_r(X)$ into two open connected components $U$ and $U'$. Let $M'$ be an integral Brakke flow in $P_r(X)$ with $X \in \text{supp} M'$ and Gaussian density $\Theta_X(M') < 2$. If $\text{supp} M' \subset U \cup \text{supp} M$, then $X$ is a smooth point for $M'$, and $M'$ agrees with $M$ in a small parabolic ball.

However, it turns out to be very difficult to generalize the Hopf lemma Lemma 5.2 to the Brakke flow setting. We first need a technical notion that is essential in the breakthrough of Choi–Haslhofer–Hershkovits–White [29].

Definition 5.7 (Definition 3.1 in [29]). For an integral Brakke flow $M$ in $\mathbb{R}^{n+1}$, we say $X \in M$ is a tame point of the flow if the $-1$ time slice of every tangent flow at $X$ is smooth with multiplicity one away from a singular set $S$ with $\mathcal{H}^{n-1}(S) = 0$. We say $M$ is a tame flow if every point $X \in M$ is a tame point.

For instance, a tame flow should not have a singularity modeled on a triple junction nor a multiplicity 2 hyperplane. The generalized Hopf lemma is:
Lemma 5.8 (Hopf lemma for tame Brakke flows, Theorem 3.19 of [29]). Let $\mathcal{M}$ and $\mathcal{M}'$ be two integral Brakke flows defined in a parabolic ball $P_r(X)$ where $X = (x_0, t_0) \in \text{supp} \mathcal{M} \cap \mathcal{M}'$. Suppose $X$ is a tame point (see Theorem 5.7) for both $\mathcal{M}$ and $\mathcal{M}'$ and let $\mathbb{H} \subset \mathbb{R}^{n+1}$ be an open half space with $x_0 \in \partial \mathbb{H}$. If in addition $\partial \mathbb{H}$ is not the tangent flow to either $\mathcal{M}$ or $\mathcal{M}'$, and $\text{reg} \mathcal{M}_t \cap \mathbb{H}$ and $\text{reg} \mathcal{M}'_t \cap \mathbb{H}$ are disjoint for $t \in (t_0 - r^2, t_0)$, then $\mathcal{M}$ and $\mathcal{M}'$ are smooth at $(x_0, t_0)$ with distinct tangents.

The striking part about the above theorem is that one obtains full regularity for the flow in the parabolic cylinder.

Consider a double cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ and a Brakke flow $\mathcal{M}$ coming out of $\mathcal{C}$ that is initially smooth, as before. Fix an open half-space $\mathbb{H}$ with $\Pi = \partial \mathbb{H}$. Theorem 1.6 is a straightforward consequence of the following reflection symmetry theorem.

**Theorem 5.9.** Suppose $\lambda[\mathcal{C}] < 2$ and $\mathcal{C} \cap \mathbb{H}$ is a Lipschitz graph over $\Pi$. If $\mathcal{C}$ is symmetric across $\Pi$, then so is $\mathcal{M}$ for $t \in [0, \infty)$. Moreover, $\mathcal{M}$ is smooth away from $\Pi$.

The basic idea to prove Theorem 5.9 is the same as that of Theorem 5.5, except one substitutes Lemma 5.1 and Lemma 5.2 with Lemma 5.6 and Lemma 5.8 respectively. We first verify that $\mathcal{M}$ is a tame flow in order to apply Lemma 5.8.

**Proposition 5.10.** Let $\mathcal{M}$ be as above. Then $\mathcal{M}$ is a tame flow.

**Proof.** It suffices to check the definition. Let $\mathcal{X} = \{\nu_t\}_{t \in (-\infty, 0]}$ be a tangent flow at $(x_0, t_0)$. Since $\lambda[\mathcal{C}] < 2$, $\mathcal{X}$ has multiplicity 1 (i.e. the Gaussian density is $1 \mathcal{H}^{n-1}$-a.e. on $\mathcal{X}$, t-a.e.).

If $\mathcal{X}$ is static or quasi-static, then $\nu_{-1} = \mathcal{H}^{n} \sqcup \Gamma$ for some stationary cone $\Gamma$. If $\Gamma$ splits off $(n - 2)$-lines, then $\nu_{-1} = \mu_{\mathbb{R}^{n-2}} \times \nu'$ where $\nu'$ is a one-dimensional stationary cone in $\mathbb{R}^2$. Hence $\nu'$ is a union of half-rays. Since $\lambda[\nu'] = \lambda[\nu_{-1}] < 2$, there are at most 3 rays. Since $\mathcal{M}$ is cyclic, $\nu'$ cannot be 3 rays meeting at the origin. This is because
the triple junction is not cyclic, and a cyclic Brakke flow cannot have a singularity modeled on a non-cyclic singularity by [89]. Therefore $\nu'$ consists of 2 lines and in fact $\nu_{-1}$ is smooth. So any singular cone $\nu_{-1}$ can split off at most $(n - 3)$-lines, and consequently the singular part of $\nu_{-1}$ has codimension at least 3.

If $\mathcal{X}$ is a non-flat self-shrinker, then any tangent cone $\nu'$ to $\nu$ is a stationary cone with entropy at most 2 (here we used the fact that self-shrinkers are minimal surfaces with respect to the metric $g_{ij} = e^{-\frac{|x|^2}{2n}}\delta_{ij}$ which is conformal to the Euclidean metric). It follows from the above discussion that $\nu'$ can split off at most $(n - 3)$-lines (if $\nu'$ splits of $(n - 2)$-lines then it is a multiplicity 1 hyperplane, which, by Allard regularity theorem, means that $\mathcal{S}(\mathcal{X}) \subset \{0\}$ and consequently $\mathcal{X}$ is a multiplicity 1 hyperplane), and so $\nu_{-1}$ is smooth away from a set of Hausdorff dimension at most $(n - 2)$. Thus $\mathcal{M}$ is tame.

Remark. With a more restrictive entropy bound it is possible to refine the codimension of the singular set even more. See [7] or [29, Section 4].

Using the notations from the previous section, set

$$\Pi^s = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = s\} \times [0, \infty) \subset \mathbb{R}^{n+1} \times [0, \infty)$$

and

$$S = \{s \in (0, \infty) \mid (\mathcal{M}^{s^+})^* > \mathcal{M}^{s^-}, \text{ and } (\mathcal{M}^{s^+})_t \text{ is graphical over } (\Pi^s)_t \text{ for } t \in [0, T_0]\}.$$

In order to show $S$ is non-empty, we need to prove the smoothness of the top part of the flow similar to [29, Proposition 7.4].

**Lemma 5.11.** Suppose $s > 0$ and $s \in S$. Then $\mathcal{M}^{s^+}$ is a smooth MCF asymptotic to $\mathcal{C}$. Moreover, every point on $\mathcal{M} \cap \{x_{n+1} = s\}$ is a regular point of the flow.

**Proof.** Let

$$I_s = \{s' \geq s \mid \mathcal{M}^{s^s} \text{ is smooth}\}.$$

84
As before, there is $R = R(t)$ such that $\mathcal{M}_t \setminus B_R(0)$ is a smooth MCF asymptotic to $\mathcal{C}$.

So for sufficiently large $s$ depending on $T_0$ we see that $\mathcal{M}^s$ is asymptotic to $\mathcal{C} \times [0, T_0]$. This shows $I_s$ is not empty.

Let $s_0 = \inf I_s$. We first argue that $\mathcal{M} \cap \{x_{n+1} = s_0\}$ consists of regular points. Let $(x_0, t_0) \in \mathcal{M} \cap \{x_{n+1} = s_0\}$ and let $\mathcal{M}^*$ be the flow reflected across $\Pi_s$. We wish to apply the Hopf lemma Lemma 5.8 to $\mathcal{M}$, $\mathcal{M}^*$ and $H = \{x_n < s\}$ to conclude that $(x_0, t_0)$ is a regular point. To this end we must check that the conditions are satisfied. Tameness follows from Proposition 5.10. We may also assume that $\partial H$ is not a tangent flow to either $\mathcal{M}$ or $\mathcal{M}^*$ at $(x_0, t_0)$, because otherwise the entropy bound together with Brakke regularity theorem implies $(x_0, t_0)$ is a regular point. Finally, we claim $\text{reg } \mathcal{M}_t \cap H$ and $\text{reg } \mathcal{M}^*_t \cap H$ are disjoint for $t$ sufficiently close to $t_0$. Suppose not, then there must be a first time of contact:

$$t_1 = \inf \{t \mid (\mathcal{M}_t^{s_0-}) \cap (\mathcal{M}_t^{s_0+})^* \cap H \neq \emptyset\}$$

in $H$. Given any point $(x_1, t_1) \in (\mathcal{M}_t^{s_0-}) \cap (\mathcal{M}_t^{s_0+})^* \subset H$.

By definition of $s_0$ we know $(\mathcal{M}_t^{s_0+})^*_t$ is in fact a smooth MCF, so maximum principle Lemma 5.6 implies that $\mathcal{M}^{s_0-}$ agrees with $\mathcal{M}^{s_0+}$ in some parabolic cylinder around $(x_1, t_1)$. The same reasoning applied to any other point in $\mathcal{M}^{s_0-} \cap (\mathcal{M}^{s_0+})^*$ shows that a connected component of $\mathcal{M}^{s_0-}$ agrees with a connected component of $(\mathcal{M}^{s_0+})^*$. This implies that $(\mathcal{M}^{s_0+})^*$ is simultaneously asymptotic to $\mathcal{C} \times [0, \infty)$ and $(\mathcal{C} + 2s_0e_{n+1}) \times [0, \infty)$, a contradiction. Hence the last condition in order to apply Lemma 5.8 is satisfied and we conclude that $\mathcal{M} \cap \{x_{n+1} = s_0\}$ is regular.

Lastly we show that $s_0 = s$. This is a consequence of the fact that $\mathcal{M} \cap \Pi^{s_0}$ is compact. Using small balls as barriers similar to Proposition 3.1, one sees that there exists some constant $N_1$ such that

$$\mathcal{M}_t \setminus B_{N_1 R^\sqrt{\tau-1}}(0) \subset T_{R^{-1}\sqrt{\tau-1}}(\mathcal{C})$$
for $R > 1$. On the other hand, for a fixed $t$ there is a constant $N_2$ such that

$$\mathcal{M}_t \cap B_{N_1 \sqrt{t+1}}(0) \subset T_{N_2}(C).$$

as the first set is clearly compact. These two facts together imply the existence of a constant $N_3$ such that

$$\mathcal{M}_t \cap \{x_{n+1} = s_0\} \subset T_{N_3}(C) \cap \{x_{n+1} = s_0\}.$$

This shows that $\mathcal{M}_t \cap \{x_{n+1} = s_0\}$ is compact (as $C \cap \{x_{n+1} = s_0\}$ is compact), and since $t \in [0, T_0]$, $\mathcal{M} \cap \Pi^{s_0}$ is compact as well. To finish the proof, note that by the previous paragraph $\mathcal{M} \cap \{x_{n+1} = s_0\}$ consist of regular points only. At each regular point $(x_0, t_0)$ there is some $r = r(x_0, t_0)$ such that $\mathcal{M}$ is smooth in $P((x_0, t_0), r)$. Since $\mathcal{M} \cap \{x_{n+1} = s_0\}$ is compact, $r$ is uniformly bounded below away from 0, and this is a contradiction unless $s_0 = s$.

Proof of Theorem 5.9. The proof is identical to that of Theorem 5.5 except for showing $S$ is closed. In that argument, the first part can be easily adapted by using Lemma 5.6 in place of Lemma 5.1. In the second case, WLOG we may again assume $t_0$ is the first time the graphicality condition fails. Then there necessarily exists a point $X = (x, s, t_0) \in \mathcal{M}_{t_0} \cap \{x_{n+1} = s\}$ whose tangent space contains the vector $e_{n+1}$. We check the condition to apply Lemma 5.8 to $\mathcal{M}_1 = (\mathcal{M}^{s^+})^*, \mathcal{M}_2 = \mathcal{M}^{s^+}$ and $\mathbb{H} = \{x_{n+1} < s\}$ as in the proof of Lemma 5.11. Tameness follows from Proposition 5.10. Since $e_{n+1}$ is normal to the hyperplane $\{x_{n+1} = s\}$ and $X$ is a regular point of $\mathcal{M}$ by Lemma 5.11, we see that $\partial \mathbb{H}$ is not the tangent flow to either $\mathcal{M}_1$ or $\mathcal{M}_2$ (here we used the fact that the tangent flow at a regular point agrees with the static flow of the tangent plane). The disjointness of the regular parts of $\mathcal{M}_1$ and $\mathcal{M}_2$ in $\mathbb{H}$ follows identically as in the proof of Lemma 5.11. Hence, we may apply Lemma 5.8 to conclude that $\mathcal{M}_1$ and $\mathcal{M}_2$ have distinct tangents, which is a contradiction since the tangent spaces agree at $X$. This concludes the proof that $S$ is closed.
We can also apply the same method to cones with more general symmetry groups. For this part we will work, for convenience, in $\mathbb{R}^{n+2}$ instead. Let $O(p)$ denote the symmetry group of $S^{p-1} \subset \mathbb{R}^{p}$. Fix an integer $1 \leq p \leq n - 1$ and suppose $\mathcal{C}$ is a smooth double cone with $\lambda[\mathcal{C}] < 2$ that has symmetry group $O(p + 1) \times O(n - p + 1)$. Typical examples are cones $\mathcal{C}_{n,p}$ over the families of minimal hypersurfaces in $S^{n+1}$ given by

$$S_{n,p} = \sqrt{\frac{p}{n}}S^{p} \times \sqrt{\frac{n-p}{n}}S^{n-p} \subset S^{n+1}.$$ 

The cones $\mathcal{C}_{n,p}$ are known as Simons-type cones. We have the following corollary of Theorem 5.9:

**Corollary 5.12.** Suppose $n \geq 2$ and $1 \leq p \leq n - 1$. Let $\mathcal{C} \subset \mathbb{R}^{n+2}$ be a cone invariant under $O(p + 1) \times O(n - p + 1)$ with $\lambda[\mathcal{C}] < 2$. Let $\mathcal{M} = \{\mu_t\}_{t \in [0, \infty)}$ be an integral, unit-regular and cyclic Brakke flow coming out of $\mathcal{C}$. If there is $T > 0$ such that $\mathcal{M}$ is smooth on $(0, T)$, then $\mathcal{M}$ inherits the $O(p + 1) \times O(n - p + 1)$ symmetry (with the same axes of symmetry).

**Proof.** Write $(x_1, \ldots, x_{p+1}, y_1, \ldots, y_{n-p+1})$ the standard coordinates on $\mathbb{R}^{n+2}$. WLOG we may assume $\mathcal{L}(\mathcal{C}) = \sigma_1 \times \sigma_2$ where $\sigma_1$ is rotationally symmetric across the $x_1$ axis and $\sigma_2$ rotationally symmetric across the $y_1$-axis. Evidently showing the rotational symmetry in $x$-coordinates is enough, as the identical argument works for the $y$-coordinates. It suffices to show that the reflection symmetry is preserved through all hyperplanes of the form

$$\sum_{i=2}^{p+1} c_i x_i = 0,$$

which, up to an ambient rotation in the $x$-coordinates, we may assume to be $\{x_{p+1} = 0\}$. Note that the a double cone with $O(p + 1) \times O(n - p + 1)$ symmetry takes the form (up to relabeling)

$$\sum_{i=1}^{p+1} a_i x_i^2 = \sum_{j=1}^{n-p+1} b_j y_j^2.$$
for suitable choices of coefficients $a_i$ and $b_j$. It is not hard to see that $\mathcal{C} \cap \{x_{p+1} > 0\}$ is a graph over $\{x_{p+1} = 0\}$ via

$$x_{p+1} = a_{p+1}^{-1/2} \left( \sum_{j=1}^{n-p+1} b_jy_j^2 - \sum_{i=1}^{p} a_ix_i^2 \right)^{1/2}.$$

Hence Theorem 5.5 applies and the desired reflection symmetry follows (of course, we have to put

$$\Pi^s = \{(x, x_{p+1}, y) \in \mathbb{R}^{n+2} \mid x_{p+1} = s\} \times [0, \infty) \subset \mathbb{R}^{n+2} \times [0, \infty).$$

and change the dimensions in the proofs accordingly).

An immediate consequence of the above and the fact that $\lambda[\mathcal{C}_{n,p}] < 2$ (see Proposition 5.15) is that:

**Corollary 5.13.** Any smooth self-expander coming out of a Simons-type cone $\mathcal{C}_{n,p}$ inherits the $O(p + 1) \times O(n - p + 1)$-symmetry of $\mathcal{C}_{n,p}$.

**Remark.** Similar results for minimal surfaces have been obtained by Mazet [67], using the elliptic moving plane method. We also note that the existence of such self-expanders is not trivial (and does not follow from standard method such as ??), as the cones $\mathcal{C}_{n,p}$ are often area-minimizing.

### 5.3 Entropy of cones

In this section we discuss some aspects of the technical condition $\lambda[\mathcal{C}] < 2$ in the previous section. As we remarked earlier, for rotationally symmetric double cones, we expect that $\lambda[\mathcal{C}] < 2$ holds automatically so that the assumption in Theorem 1.6 is redundant. More precisely we conjecture that

**Conjecture 5.14.** If $\mathcal{C}$ is of the form (5.1), then $\lambda[\mathcal{C}] < 2$. 

88
When $m_1 = m_2 = m$, observe that when $m \to 0$, $C$ converges to a multiplicity 2 plane which has entropy 2, and that when $m \to \infty$, $C$ converges after suitable translations to a cylinder $\mathbb{R} \times S^{n-1}$ which has entropy strictly less than 2. This also explains why the two connected components of $L(C)$ need to be in different half-spaces, for otherwise when $m$ is small the double cone could be close to a multiplicity two cylinder which has entropy strictly larger than 2.

It is not so hard to calculate the Gaussian area of $C$ at the origin, but unlike self-shrinkers, the maximum in our case needs not to happen at the origin (the argument for self-shrinkers can be found in eg. Section 7 of [32]). In fact when $m$ is large, the entropy is achieved far away from the origin (in a region where the cone looks more like a cylinder). For example in $\mathbb{R}^3$ when $m_1 = m_2$, the bifurcation happens at $m = 1$. The cone $\{x_1^2 = x_2^2 + x_3^2\}$ has entropy $\sqrt{2}$, which is achieved at the origin, and for $m > 1$ the entropy is bigger than $\sqrt{2}$ and is achieved away from the origin. Although we have strong numerical evidence that the conjecture is true, the analysis of the Gaussian area functional on the cone centered away from the axis of symmetry is complicated to handle.

For cones with $O(p) \times O(n + 1 - p)$-symmetry in $\mathbb{R}^{n+2}$ the situation is not as clear. Ilmanen and White [57] give an exact formula for the Gaussian density of $C_{n,p}$ at the origin:

$$\Theta_0(C_{n,p}) = \frac{\sigma_p \sigma_{n-p}}{\sigma_n} \left(\frac{p}{n}\right)^{p/2} \left(\frac{n - p}{n}\right)^{(n-p)/2},$$

where $\sigma_p$ is the volume of the unit sphere in $\mathbb{R}^{p+1}$. Since $C_{n,p}$ are minimal, they are also self-shrinkers. It follows from a theorem of Colding–Minicozzi [32] that the entropy of $C_{n,p}$ is achieved at the origin. Hence, in fact, $\lambda[C_{n,p}] = \Theta_0(C_{n,p})$. For example, the cone $C_{2,1} \subset \mathbb{R}^4$ has entropy $\frac{\pi}{2}$.

**Proposition 5.15.** $\lambda[C_{n,p}] < 2$ for all $n$ and $p$.

**Proof.** Let us fix $n$ and consider the quantity $\Theta_0(C_{n,p})$ as a function of $p = x \in (0, n)$. 89
After clearing out the denominators and throwing out factors only depending on \( n \), we are left with

\[
f(x) = \frac{(x + 1)(n - x + 1)x^{x/2}(n - x)^{(n - x)/2}}{\Gamma((x + 1)/2 + 1)\Gamma((n - x + 1)/2 + 1)}.
\]

Let \( g = \log f \) and take its derivative.

\[
g'(x) = -\frac{1}{2}\psi^{(0)}((x + 1)/2 + 1) + \frac{1}{2}\psi^{(1)}((n - x + 1)/2 + 1) + \frac{1}{2}(\log x - \log(n - x)),
\]

where \( \psi^{(0)}(x) = \frac{\Gamma'(x)}{\Gamma(x)} \). Let \( h(x) = \psi^{(0)}((x + 1)/2 + 1) - \log(x) \), \( g(x) = 0 \) is then equivalent to \( h(x) = h(n - x) \). We claim that \( h(x) \) is monotonically decreasing on \((0, n)\), so that the only solution to \( g(x) = 0 \) is \( x = \frac{n}{2} \). To see that, we have

\[
h'(x) = \frac{1}{2}\psi^{(1)}((x + 1)/2 + 1) - \frac{1}{x},
\]

where \( \psi^{(1)}(x) = (\psi^{(0)})'(x) \). Now making the change of variable \( y = \frac{x + 1}{2} + 1 \), and using the fact that

\[
\frac{1}{y} + \frac{1}{2y^2} \leq \psi^{(1)}(y) \leq \frac{1}{y} + \frac{1}{y^2}
\]

for \( y > 0 \), we see that

\[
\frac{1}{2}\psi^{(1)}(y) - \frac{1}{2y - 3} = -\frac{3y + 3}{2y^2(2y - 3)} < 0.
\]

Hence \( h'(x) < 0 \) for all \( x > 0 \) and the only critical point of \( f \) on happens on \( x = \frac{n}{2} \).

Since \( f(0) = f(n) = 0 \), the function \( f \) increases on \((0, n/2]\) and decreases on \([n/2, n)\).

We can now show the desired bound. Let

\[
f(n) = \frac{\sqrt{\pi} (n/2 + 1)^2}{2^n n + 1} \frac{\Gamma((n + 1)/2 + 1)}{\Gamma((n/2 + 1)/2 + 1)^2}.
\]

We compute \( f(1) = \frac{\sqrt{\pi}}{2\Gamma(3/4)^2} \), \( f(2) = \frac{\sqrt{\pi}}{2} \), \( f(3) = \frac{\sqrt{\pi}}{2\Gamma(5/4)^2} \) and \( f(4) = \frac{3}{2} \), all of which are smaller than 2. Now, using \( \Gamma(n) = n\Gamma(n - 1) \), we have

\[
\frac{f(n + 4)}{f(n)} = \frac{(n + 1)(n/2 + 3)^2(n/2 + 7/2)(n/2 + 5/2)}{4(n + 5)(n/2 + 1)^2(n/4 + 5/2)^2}.
\]
It can be easily checked that the above is less than or equal to 1 for $n \geq 1$. This means that $f(n)$ is, in fact, monotonically decreasing for positive integer values of $n$, and this completes the proof.

In this case we make the following loose conjecture, partly due to Solomon:

**Conjecture 5.16** (cf. Section 4 of [57]). Any double cone with $O(p+1) \times O(n-p+1)$ symmetry has entropy at least that of $C_{n,p}$ and at most 2.

### 5.4 Singularity formation

In this short section we provide an example of a tame ancient flow with a singularity based on Theorem 1.3. This shows that Theorem 1.6 is not void.

**Proposition 5.17.** Suppose $C \subset \mathbb{R}^{n+1}$, $2 \leq n \leq 6$, is a smooth double cone with $\lambda[C] < 2$ and that the two connected components of $L(C)$ are graphs over some (fixed) hyperplane. Suppose that there is a connected self-expander asymptotic to $C$. Then there exists an integral Brakke flow coming out of $C$ that has a singularity in finite time.

**Remark.** We note that the above is consistent with the topological uniqueness result of [14]. Indeed, by Proposition 5.6 of [14], if $\lambda[C] < \lambda[S^{n-1} \times \mathbb{R}]$, the flow produced by Theorem 1.3 is smooth for all time. On the other hand, any such double cone will not have a connected self-expander.

**Proof.** Let $\sigma = L(C)$, and let $W$ be the connected component of $S^n$ lying between the two connected components of $\sigma$. By Corollary 1.2 of [11], the set of generic cones (in the sense that there is no $C^2$-asymptotically self-expander with nontrivial Jacobi fields that fix the infinity) whose link lie in $W$, is dense near $C$. These facts allow us to take a sequence of $C^{2,\alpha}$-hypersurfaces $\sigma_i$ in $S^2$ such that

- $\sigma_i \to \sigma$ in $C^{2,\alpha}(S^n)$ as $i \to \infty$;
• $C_i$ is a generic, smooth double cone for all $i$, where $C_i$ is the cone over $\sigma_i$;

• $\lambda[C_i] < 2$ for sufficiently large $i$, by Lemma 6.2 of [14].

From the above we immediately see that there exists a unique disconnected, stable self-expander $\Gamma_i$, $C^{2,\alpha}$-asymptotic to $C_i$ (by evolution of entire graph [38]). We also see that $C_i \subset \Omega$, where $\Omega$ is the connected component of $\mathbb{R}^{n+1} \setminus C$ that contains $W$. Denote by $\Sigma_0$ the connected self-expander asymptotic to $C$. Using a direct method with $\Sigma_0$ as the barrier, similar to Lemma 8.2 of [9], we can find a connected self-expander asymptotic to $C_i$ in $\Omega'$, where $\Omega'$ is the connected component of $\mathbb{R}^{n+1} \setminus \Sigma_0$ such that the outward unit normal of $C$ points into $\Omega'$.

Since there exists a unique disconnected self-expander $\Gamma_i$ asymptotic to $C_i$, by the partial ordering of self-expanders asymptotic to a fixed cone (Theorem 4.1 of [14]), we can pick an innermost connected self-expander $\Sigma_i$, $C^{2,\alpha}$-asymptotic to $C_i$ (i.e. pick any $\Sigma_i$ such that the only self-expander lying on the inside of $\Sigma_i$ is the disconnected $\Gamma_i$ - note that $\Sigma_i$ might not be unique). We claim that $\Sigma_i$ is unstable. If not, the mountain pass theorem (Corollary 1.2 [12], this requires $2 \leq n \leq 6$) and the genericity of $C_i$ imply the existence of an unstable self-expander $\Sigma'$ lying between $\Sigma_i$ and $\Gamma_i$. $\Sigma'$ must then be connected, but this contradicts the partial ordering.

Since $\Sigma_i$ is unstable and $\lambda[C_i] < 2$, we can produce using Theorem 1.3 an integral Brakke flow $\mathcal{M}^i = \{\mu^i_t\}_{t \in (0, \infty)}$ that moves inwards initially (by expander mean convexity) and satisfies $\lim_{t \to 0} \mu^i_t = \mathcal{H}^n \cap C_i$. Suppose for a contradiction that $\mathcal{M}^i$ is smooth for all $t$, then the flow is expander mean convex (in the classical sense) and moves inwards for all time. Moreover, using an almost identical argument as in Proposition 5.1(3) of [14], the rescaled flow (rescaling as in the proof of Theorem 1.3) $\tilde{\mathcal{M}}^i$ converges as $s \to \infty$ to a smooth, stable self-expander asymptotic to $C_i$, which must lie inside $\Sigma_i$. Since $\Sigma_i$ is an innermost connected self-expander, the stable limit must be $\Gamma_i$ which is disconnected, a contradiction.
Now let $s^i$ denote the first singular time of the rescaled flows $\tilde{M}^i$, and time translate $\tilde{M}^i$ by $-s^i$ to obtain rescaled flows $\tilde{M}^{i,s^i}$ with a singularity at time 0. By compactness of MCF we obtain a rescaled flow $\hat{M}$ such that $\tilde{M}^{i,s^i} \to \hat{M}$ subsequentially. Rescaling back we see that, by upper semicontinuity of the Gaussian density, $\mathcal{M}$ has its first singularity at time $t = 1$. Finally, we claim that $\mathcal{M}$ indeed comes out of the cone $C$.

As $\mathcal{M}$ is smooth on $(0, 1)$, it is enough to show that

$$\lim_{t \to 0} \supp \mu_t \cap S^n = \sigma \text{ in } C^{2,\alpha}(S^n).$$

(5.3)

Since each $\mathcal{M}^i$ attains $C_i$ as initial data and $C_i$ is $C^{2,\alpha}$-regular, we have

$$\lim_{t \to 0} \supp \mu^i_t \cap S^n = \sigma^i \text{ in } C^{2,\alpha}(S^n).$$

As $\sigma^i \to \sigma$ in $C^{2,\alpha}(S^n)$, by a diagonalization argument, we see that (5.3) holds. This completes the proof.

Assuming Conjecture 5.14, the assumptions in Proposition 5.17 are satisfied by cones of in the form (5.1) with $m_1 = m_2 = m$, given that the parameter $m$ is sufficiently small. In fact, numerical computations do confirm that the cones $x_1^2 = m^2(x_2^2 + x_3^2)$ for $m \leq 1$ have entropy less than 2. Moreover, by [4], there exists a connected self-expander for sufficiently small $m$. Moreover, since $C$ is symmetric across the hyperplane $\{x_1 = 0\}$ as well, we can apply Theorem 5.5 to $\mathbb{H} = \{x_1 > 0\}$ to conclude that $\mathcal{M}$ is smooth away from $\{x_1 = 0\}$. Together with Theorem 1.6 we infer that the only possible singularity of $\mathcal{M}$ is at the origin. Moreover, any tangent flow $\mathcal{X}$ at the first singular time must also be rotationally symmetric. By the classification of rotationally symmetric self-shrinkers of Kleene and Møller [62], $\mathcal{X}$ has to be one of the following: a round sphere, a round cylinder $\mathbb{R} \times S^{n-1}$ or a smooth embedded $S^1 \times S^{n-1}$. Since $\mathcal{M}$ is not closed, we conclude that $\mathcal{X}$ has to be a round cylinder. The uniqueness of the cylinder follows from the work of Colding–Minicozzi [33]. In conclusion, we have the much strong conclusion that the singularity is a neck pinch at the origin in this case.
5.5 Self-expanders with triple junctions

In this section we prove a simple ODE result on the existence of two self-expanders with triple junctions for rotationally symmetric double cones with sufficiently large cone angle. The proof roughly follows the setup of Helmensdorfer [48], although we do not need the clearing out lemma for MCF in the following analysis. This provides an example of a singular self-expander, and also illustrates that the cyclicity assumption in Theorem 5.5 is essential as tameness (Proposition 5.10) clearly fails for self-expanders with triple junction singularities. However, the examples constructed below are in fact still rotationally symmetric.

We denote by $C_m$ the cone in the form (5.1) with $m_1 = m_2 = m$. Observe that $C_m$ has an $O(n) \times O(1)$-symmetry. Assume the expander $\Sigma$ has a triple junction singularity at a point $(0, x_0)$. Imposing rotational symmetry on $\Sigma$ across the $x_1$-axis we may assume $x_0 = (a, 0, \ldots, 0)$. Note also at a triple junction singularity, the tangent cone is stationary and is therefore the union of three half-lines meeting at an angle of $2\pi/3$. These observations reduce the problem to finding a function $u : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following ODE (written in spherical coordinates):

$$\frac{u_{rr}}{1+u_r^2} - \frac{n-1}{u} + \frac{1}{2}ru_r - \frac{1}{2}u = 0. \quad (5.4)$$

with initial data $u(0) = a$ and $u'(0) = \sqrt{3}$. The solution is asymptotic to $C_m$ if

$$\lim_{r \to \infty} \frac{u(r)}{r} = \frac{1}{m}. \quad (5.5)$$

Of course, by the usual ODE existence and uniqueness theorem, the solution to the problem (5.4) is unique with a given initial data $a$. First we show that any solution to (5.4) is asymptotically conical.

**Proposition 5.18.** For any $a > 0$, there is a (unique) $m = m(a)$ such that (5.5) holds.
Proof. First we prove \( u > 0 \) for all \( r > 0 \). Suppose for a contradiction that there is \( r_0 \) such that \( u(r_0) < 0 \). Since initially \( u \) and \( u_r \) are both positive, by the mean value theorem there must be a local maximum \( r_1 \in (0, r_0) \) with \( u(r_1) > 0 \), but at such an \( r_1 \) we can use (5.4) to get

\[
\frac{u_{rr}(r_1)}{u(r_1)} = \frac{n-1}{2} u(r_1) > 0,
\]
a contradiction. So \( u \) is indeed positive. By the above calculation, this implies that all critical points of \( u \) are local minima.

To continue, let \( \alpha(r) = \arctan(u/r) \). Differentiating, we obtain

\[
\alpha'(r) = \frac{ru_r - u}{u^2 + r^2} \quad \text{and} \quad \alpha''(r) = \frac{ru_{rr} - (ru_r - u)(2uu_r + 2r)}{(u^2 + r^2)^2}.
\]

At a critical point \( r_0 \) of \( \alpha(r) \), we have \( r_0u_r(r_0) - u(r_0) = 0 \) and (5.4) implies that \( u_{rr}(r_0) = \frac{n-1}{u(r_0)} > 0 \). Hence \( \alpha''(r_0) = \frac{ru_{rr}}{u^2 + r^2} > 0 \). Therefore all critical points of \( \alpha(r) \) are local minima as well. Since \( \alpha(r) \) is bounded and only has local minima, monotone convergence theorem shows that \( \lim_{r \to \infty} \alpha(r) \) exists.

It remains to show that \( \lim_{r \to \infty} \alpha(r) \in (0, \frac{\pi}{2}) \). If the limit is 0, then \( m = 0 \) and the MCF of \( \Sigma \) is contained in the level set flow of the \( x_1 \)-axis, which disappears immediately. This is impossible. If the limit is \( \frac{\pi}{2} \), the MCF of \( \Sigma \) is contained in the level set flow of the hyperplane \( \{x_1 = 0\} \), which is static. This is again impossible by, say, the initial condition \( u_r(0) = \sqrt{3} \).

Knowing the above, our problem becomes essentially a shooting problem: Given \( m \), we wish to find the appropriate initial condition \( a \) so that the solution to (5.4) satisfies (5.5). Let \( u^a(r) \) be the solution to (5.4) with initial condition \( u^a(0) = a \). Consider the asymptotic cone angle parameter \( m \) as a function of \( a \):

\[
m(a) = \lim_{r \to \infty} \frac{u^a(r)}{r} = \lim_{r \to \infty} u^a_r(r) \in (0, \infty).
\]

**Proposition 5.19.** \( m(a) \) is a continuous function on \((0, \infty)\).
Proof. First we record that

$$u_{rr}(0) = \frac{4}{3} \left( \frac{n-1}{a} + \frac{1}{2}a \right) > 0.$$  

From the proof of Theorem 5.18, we see that a critical point of $u$ must be a local minimum, but since $u$ is initially increasing and smooth, there cannot be any critical point at all. So $u$ is strictly increasing and we deduce that $u_r > 0$ for all $r > 0$. On the other hand, l'Hôpital’s rule on (5.5) yields $\lim_{r \to \infty} u_r = m$. Going back to (5.4) and taking the limit as $r \to \infty$ yield also $\lim_{r \to \infty} u_{rr} = 0$.

Fix an $a \in (0, \infty)$. Clearly $u^a \geq a$ and $u^a_r$ is bounded above. Therefore we may fix a constant $N$ with $\frac{1}{N} < a$ such that $|a' - a| < \frac{1}{N}$ implies that

$$\frac{1}{u^{a'}} \leq c$$  \hfill (5.6)

for some constant $c$ depending on $a$ and $N$. We compute

$$\left( \frac{u}{r} \right)_r = \frac{1}{r} \left( u_r - \frac{u}{r} \right) = \frac{2}{r^2} \left( \frac{n-1}{u} - \frac{u_{rr}}{1 + (u_r)^2} \right).$$  \hfill (5.7)

For $|a - a'| < \frac{1}{N}$ there are two cases. If $u^{a'}_{rr}$ is never zero, then it is always positive. This immediately gives the bound:

$$\left( \frac{u^{a'}}{r} \right)_r \leq \frac{c}{r^2}. $$  \hfill (5.8)

If $u^{a'}_{rr}(r_0) = 0$ at some point $r_0$, differentiating (5.4) once we get

$$\frac{1}{1 + u^2_r} \left( u_{rrr} - \frac{2u_r(u_{rr})^2}{1 + u^2_r} \right) + \frac{n-1}{u^2} u_r + \frac{1}{2} ru_{rr} = 0. $$  \hfill (5.9)

From this we immediately see that when $u^{a'}_{rr}(r_0) = 0$,

$$u^{a'}_{rr}(r_0) = -(1 + u^2_r) \frac{n-1}{u^2} < 0,$$

so every critical point of $u^{a'}_r$ is a local maximum, for which there can be at most one of them. Therefore $r_0$ is the only zero of $u^{a'}_r$. In this case, we see from (5.9) that, at any negative local minimum of $u^{a'}_r$, we have

$$\frac{2u^{a'}_{rr}}{(1 + (u^{a'}_r)^2)^2} (u^{a'}_r)^2 \leq \frac{(n-1)u^{a'}_r}{(u^{a'}_r)^2} \implies \frac{(u^{a'}_{rr})^2}{(1 + (u^{a'}_r)^2)^2} \leq \frac{n-1}{2(u^{a'}_r)^2} \leq c,$$
where we used (5.6). Going back to (5.7), this yields the same type of uniform upper bound as (5.8) (up to increase $c$). Therefore we conclude (5.8) holds for all $|a - a'| < \frac{1}{N}$.

Integrating (5.8) from $r$ to $\infty$, we obtain the estimate:

$$m(a') - \frac{u^a(r)}{r} < \frac{c}{r}, \quad |a' - a| < \frac{1}{N}.$$  

Now given $\varepsilon > 0$ we can pick $r_0 > 0$ such that $c/r_0 < \varepsilon/3$ and $\delta$ so small that $|a' - a| < \delta$ implies that $|u^{a'} - u^a| < \varepsilon/3$ on $(0, r_0]$ (this follows from continuous dependence on initial data as we are now in a compact set). Using the triangle inequality we get that

$$|m(a) - m(a')| \leq \left| m(a) - \frac{u^a(r_0)}{r_0} \right| + \left| \frac{u^a(r_0)}{r_0} - \frac{u^{a'}(r_0)}{r_0} \right| + \left| m(a') - \frac{u^{a'}(r_0)}{r_0} \right| < \varepsilon.$$  

This finishes the proof of continuity.  

We are now in the position to prove the existence theorem.

**Theorem 5.20.** There is an $M_0 > 0$ such that for all $M > M_0$, there exists at least two distinct values $a_1, a_2 \in (0, \infty)$ depending on $M$ such that $m(a_1) = m(a_2) = M$.

**Proof.** We will show that $m(a) \to \infty$ both as $a \to 0$ and as $a \to \infty$. In view of Theorem 5.19 this will prove the theorem.

Let us show that $m(a) \to \infty$ as $a \to \infty$. First of all, we show that $u^a_r$ cannot be uniformly bounded above. Indeed if $u^a_r$ is uniformly bounded above by some constant $C$, then $u - ru_r \geq a - C$ for $r \in [0, 1]$ and so $u_{rr} \geq \frac{1}{2}(a - C)$ on $[0, 1]$ by (5.4). But then

$$u^a_r(1) = \frac{\sqrt{3}}{3} + \int_0^1 u^a_{rr}(r)dr \geq \frac{\sqrt{3}}{3} + \frac{1}{2}(a - C) \to \infty$$  

as $a \to \infty$, a contradiction.

97
Recall from the proof of Theorem 5.19 that \( u_{rr}^a \) can have at most one zero. If there is a sequence \( a_i \rightarrow \infty \) such that \( u_{rr}^{a_i} \) has one zero \( r_i \), then (5.7) immediately implies that

\[
\left( \frac{u^{a_i}}{r} \right)_r > 0, \ r > r_i.
\]

Integrating the above from \( r_i \) to \( \infty \) we get

\[
m(a_i) > \frac{u^{a_i}(r_i)}{r_i}.
\]  
(5.10)

On the other hand, at a zero of \( u_{rr} \), (5.4) gives that

\[
u_{rr}^{a_i}(r_i) = u_{rr}^{a_i}(r_i) - \frac{2(n-1)}{u_{a_i}^{a_i}(r_i)r_i}
\]  
(5.11)

Observe that \( u_{rr}^{a_i}(r_i) = \sup_r u_{rr}^{a_i}(r) \) because \( r_i \) is a local maximum of \( u_{rr}^{a_i} \) and \( u_{rr}^{a_i}(r_i) < 0 \) for all \( r > r_i \). Since \( u_{rr}^{a_i} \) is not uniformly bounded, we have that

\[
\lim_{i \rightarrow \infty} \frac{u_{a_i}^{a_i}(r_i)}{r_i} = \lim_{i \rightarrow \infty} u_{rr}^{a_i}(r_i) - \frac{2(n-1)}{u_{a_i}^{a_i}(r_i)r_i} = \infty,
\]

where we also used \( u_{a_i}^{a_i}(r_i) > a_i \). Recalling (5.10), we see that \( \lim_{i \rightarrow \infty} m(a_i) = \infty \).

Otherwise there is \( a_0 > 0 \) such that \( u_{rr}^a \) has no zero for all \( a > a_0 \). This means that \( u_{rr}^a \) is strictly increasing for all \( a > a_0 \). Since \( u_{rr}^a \) is not uniformly bounded we can find a sequence \( a_i \rightarrow \infty \) and \( \{r_i\} \subset [0, \infty) \) such that \( u_{a_i}^{a_i}(r_i) > i \). Of course monotonicity implies that \( m(a_i) \geq u_{a_i}^{a_i}(r_i) > i \). This shows that \( m(a) \rightarrow \infty \) as \( a \rightarrow \infty \).

Next we will show that \( m(a) \rightarrow \infty \) as \( a \rightarrow 0 \) as well. Again we will first argue that \( u_{rr}^a \) cannot be uniformly bounded near 0. Suppose for a contradiction that there is \( a_0 > 0 \) such that \( |u_{rr}^a| \leq C \) for all \( 0 < a < a_0 \), then since \( u_{rr}^a > 0 \) we get

\[
u^a(r) = a + \int_0^r u_{rr}^a(t)dt \leq a + Cr, \ a < a_0.
\]  
(5.12)

On the other hand, (5.4) gives that

\[
u_{rr}^a \geq \frac{n-1}{u^a} + \frac{1}{2}(u^a - ru_{rr}^a) \geq \frac{n-1}{2u^a} + \sqrt{n-1} - Cr, \ a < a_0.
\]
In particular for sufficiently small $a$ we can ensure $\sqrt{n-1} \geq Cr$ and so that $u^a_{rr}(r) \geq \frac{n-1}{2u^a(r)}$ for $r < \sqrt{a}$. Hence, using (5.12), we may estimate

\[
u^a_r(\sqrt{a}) \geq \frac{\sqrt{3}}{3} + \int_0^{\sqrt{a}} \frac{n-1}{2u^a(r)}dr
\]

\[
\geq \frac{\sqrt{3}}{3} + \frac{n-1}{2} \int_0^{\sqrt{a}} \frac{1}{r + Cr}dr = \frac{\sqrt{3}}{3} + \frac{n-1}{2C} \log(1 + Ca^{-\frac{1}{2}}) \to \infty
\]
as $a \to 0$, a contradiction. This shows that $u^a_i$ is not uniformly bounded near 0.

Suppose there is a sequence $a_i \to 0$ such that $u^a_{rr}$ has one zero $r_i$. In view of (5.11), if $r_i$ is bounded away from 0, then taking $i \to \infty$ will give

\[
\lim_{i \to \infty} \frac{u^a_i(r_i)}{r_i} = \lim_{i \to \infty} \frac{u^a_i(r_i) - 2(n-1)}{u^a_i(r_i) - 2} = \infty,
\]

where we used the fact that $u^a_i(r_i) \geq a_i + \frac{\sqrt{3}}{3} r_i$ and that $u^a_i$ is not uniformly bounded near 0. By (5.10), we can conclude $m(a_i) \to \infty$ as $i \to \infty$ as before. So we henceforth assume that $r_i \to 0$ and assume for a contradiction that there is a constant $C$ such that $r^{-1}u^a_i(r) \leq C$ uniformly for $r \geq r_i$. Since $u_r$ is decreasing on $[r_i, \infty)$ we get

\[
u^a_i(2r_i) \geq \int_{r_i}^{2r_i} u^a_i(t)dt \geq r_i u^a_i(2r_i) \implies u^a_i(2r_i) \leq \frac{u^a_i(2r_i)}{r_i} \leq 2C.
\]

Since $u^a_{rr}(2r_i) < 0$, (5.4) gives that

\[
0 < \frac{u^a_i(2r_i)}{2r_i} \leq \frac{-2(n-1)}{2r_i u^a_i(2r_i)} + u^a_{rr}(2r_i).
\]

Rearranging, we deduce that $r_i u^a_i(2r_i) \geq (n-1) u^a_{rr}(2r_i) \geq \frac{1}{2}(n-1)C^{-1}$, and so

\[
2Cr_i^2 \geq r_i u^a_i(2r_i) \geq \frac{1}{2}(n-1)C^{-1} \implies r_i^2 \geq \frac{1}{4}(n-1)C^{-2}
\]
a contradiction as $r_i \to 0$. Hence $r^{-1}u^a_i(r)$ is not uniformly bounded on $(r_i, \infty)$, so for each $i$ we may find $r'_i \in (r_i, \infty)$ such that $(r'_i)^{-1}u^a_i(r'_i) > i$. (5.10) then implies $m(a_i) \to \infty$ as $i \to \infty$.

Otherwise $u^a_{rr}$ has no zero for sufficiently small $a$ and monotonicity as before implies that $m(a) \to \infty$ as $a \to 0$. This completes the proof.
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Curriculum Vitae

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Education

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• BMath in Pure Mathematics & Applied Mathematics
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Preprints


Publications


Invited Talks

• Yale Geometric Analysis Seminar (online), Mar. 2022.
• Johns Hopkins Analysis and PDE Seminar, Sept. 2022.
• MIT Geometric Analysis Seminar, Nov. 2022.
• Topics in Differential Geometry, Brown University, Mar. 2023
• AMSS Geometric PDE Seminar, Apr. 2023

Other Talks

• Johns Hopkins Graduate Student Lunch Seminar, multiple times.
• Junior Colloquium, Johns Hopkins University (online), Mar. 2021.

Teaching Experience

• Graduate Teaching Assistant
  Department of Mathematics, Johns Hopkins University
  Sept. 2018 -
Teaching Assistant/Grader for: Calculus II (Biology), Calculus III, Differential Equations and Applications, Real Analysis I, Honors Complex Analysis.

- **Undergraduate Teaching Assistant**  
  University of Waterloo  
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  Graded assignments for: Advanced Linear Algebra (MATH 146), Advanced Calculus 2 (MATH 148), Differential Equations for Engineers (MATH 218), Calculus 3 for Honours Physics (MATH 227).

**Awards and Scholarships**

- Pino Tenti Memorial Scholarship, University of Waterloo, 2018.
- Faculty of Mathematics Scholarship, University of Waterloo, 2015-2018.
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**Service**

- **Mentor**: Directed Reading Program, Johns Hopkins University (multiple times).
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