STABILITY OF QUERMASSINTEGRAL INEQUALITIES

by

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Abstract

In this thesis, we study stability in the quermassintegral inequalities for nearly spherical sets. Our work studying stability in the quermassintegral inequalities is inspired by results that established a quantitative isoperimetric inequality for nearly spherical sets.

Just as in the classical isoperimetric inequality, we achieve equality in the quermassintegral inequalities precisely when the surface is a sphere. To study stability in the quermassintegral inequalities, we analyze how the deficit in the quermassintegral inequality controls the closeness between the surface to a sphere. In Chapter 3, we establish quantitative quermassintegral inequalities, where we find lower bounds for the deficit in terms of the Fraenkel asymmetry and spherical deviation for nearly spherical sets.

Additionally, we study certain curvature flows to further analyze the stability of the quermassintegral inequalities. In Chapter 5, we study the behavior of quantities related to the quantitative quermassintegral inequality of along the flow. With some symmetry restrictions on the surface, this gives a new method to prove the results from Chapter 3.

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Chapter 1

Introduction

1.1 Quermassintegral inequalities

For a convex body $\Omega \subset \mathbb{R}^{n+1}$, the $k$-th *quermassintegral* of $\Omega$ is the mixed volume

$$W_k(\Omega) := V(\Omega, \ldots, \Omega, B, \ldots, B), \quad (1.1)$$

where $\Omega$ appears in the first $n + 1 - k$ entries and $B$, which is the unit ball in $\mathbb{R}^{n+1}$, appears in the last $k$ entries. The famous Steiner formula states that the volume of $\Omega + tB$ is a polynomial in $t$. In particular,

$$\text{Vol}(\Omega + tB) = \sum_{k=0}^{n+1} \binom{n+1}{k} W_k(\Omega) t^k. \quad (1.2)$$

Next, denoting $\omega_m$ as the volume of the unit $m$-ball, we set

$$V_k(\Omega) := \frac{\omega_k}{\omega_{n+1}} W_{n+1-k}(\Omega). \quad (1.3)$$
Note that $V_{n+1}(A) = \Vol(\Omega)$ and $V_n(\Omega) = \frac{\omega_{n+1}}{(n+1)\omega_n}\Area(\partial\Omega)$. We obtain, as a consequence of the Alexandrov-Fenchel inequalities, the *quermassintegral inequalities*

\[
\left( \frac{V_{k+1}(\Omega)}{V_{k+1}(B)} \right)^{\frac{1}{k+1}} \le \left( \frac{V_k(\Omega)}{V_k(B)} \right)^{\frac{1}{k}}. \tag{1.4}
\]

When $k = n$, the inequality in (1.4) is simply the classical isoperimetric inequality. For smooth, convex domains and $k \ge 1$, quermassintegrals have the useful integral formula

\[
V_{n+1-k} = \frac{(n+1-k)!(k-1)!}{(n+1)!} \frac{\omega_{n+1-k}}{\omega_{n+1}} \int_M \sigma_{k-1}(L)d\mu, \tag{1.5}
\]

where $M := \partial\Omega$, $L$ is the second fundamental form of $M$, and $\sigma_k(L)$ is the $k$-*th mean curvature* of $M$. The $k$-th mean curvature is the $k$-th elementary symmetric polynomial of the principal curvatures. The inequalities in (1.4) equivalently state that for convex domains

\[
\left( \int_M \sigma_{k-1}(L)d\mu \right)^{\frac{1}{n-k+1}} \le C(n, k) \left( \int_M \sigma_k(L)d\mu \right)^{\frac{1}{n-k}}, \tag{1.6}
\]

where $C(n, k)$ is the constant that gives equality in the case where $M$ is a sphere. More generally, for any $-1 \le j < k$, we have the $(k, j)$-*quermassintegral inequality*,

\[
\left( \int_M \sigma_j(L)d\mu \right)^{\frac{1}{n-j}} \le C(n, k, j) \left( \int_M \sigma_k(L)d\mu \right)^{\frac{1}{n-k}}, \tag{1.7}
\]

where again $C(n, k, j)$ gives equality in the case of the sphere.

Much of the previous work to establish (1.7) relies heavily on working with convex domains. There has been work extending (1.7) to a class of non-convex domains known as *k-convex domains*, where $\sigma_j(L) \ge 0$ for $1 \le j \le k$.

In Chapter 4, we introduce some results on curvature flows that have been used to study the quermassintegral inequalities in the non-convex case. We further study these flows in Chapter 5 to look at stability of the inequalities. These flows look at the surfaces $M(t)$,
which at each time $t$ is given by an embedding $X : S^n \to \mathbb{R}^{n+1}$ and satisfies

$$X_t = G\nu,$$  \hspace{1cm} (1.8)

where $G$ is a symmetric function of the principal curvatures and $\nu$ is the outward pointing vector. Notably, to prove the $(k, k-1)$-quermassintegral inequality for $k$-convex starshaped domains, Guan and Li in [15] studied the flow

$$X_t = \frac{\sigma_{k-1}(L)}{\sigma_k(L)}\nu.$$  \hspace{1cm} (1.9)

Urbas in [23] and Gerhardt in [14] show that the solution exists for all $t \geq 0$ with any initial surface $M(0)$ that is smooth, strictly $k$-convex, and starshaped (and these conditions are preserved for $M(t)$ for all $t \geq 0$). Furthermore, they apply a rescaling to obtain the surfaces $\{\tilde{M}(t)\}$, which they showed converges to a sphere. Additionally, Guan and Li proved that $\frac{d}{dt}\int_{\tilde{M}(t)} \sigma_k(\tilde{L})d\mu_t \leq 0$ and $\frac{d}{dt}\int_{\tilde{M}(t)} \sigma_{k-1}(\tilde{L})d\mu_t = 0$. These results, combined with an approximation argument to include non-strictly $k$-convex domains, prove the $(k, k-1)$-quermassintegral inequalities.

Later in Chapter 5 we study stability of the $(k, -1)$-quermassintegral inequality along the volume preserving flow:

$$X_t = (-\sigma_k(L) + h(t))\nu.$$ \hspace{1cm} (1.10)

The results on this flow are a bit more restrictive. In [2], Cabezas-Rivas and Sinestrari found solutions existing for certain convex domains that satisfied a pinching condition. Just like the argument in [15], they prove a monotonicity result of the quermassintegral, which provides a proof of the $(k, -1)$ quermassintegral inequalities for this restricted class of surfaces.

In [5], Chang and Wang were able to show (1.6) without the requirement of a starshaped domain, but with the added assumption of having $(k+1)$-convexity instead of just
$k$-convexity, and the constant $C(n,k)$ is non-optimal. They proved this using optimal transport methods. See also [3], [4], and [24].

1.2 Quantitative isoperimetric inequality

Our work to find a quantitative quermassintegral inequality is largely motivated by work done on the stability in the classical isoperimetric inequality. In particular, we are motivated by analysis done on the isoperimetric deficit $\delta(\Omega)$ of a domain $\Omega \subseteq \mathbb{R}^{n+1}$, defined as

$$\delta(\Omega) := \frac{P(\Omega) - P(B_{\Omega})}{P(B_{\Omega})}. \tag{1.11}$$

Here $|B_{\Omega}|$ is the volume of $B_{\Omega}$, $B_{\Omega}$ is a ball such that $|\Omega| = |B_{\Omega}|$, and $P(\cdot)$ gives the perimeter of a set. The classical isoperimetric inequality is equivalent to $\delta(\Omega) \geq 0$, with equality if and only if $\Omega$ is a ball. There has been a lot of work studying quantitative isoperimetric inequalities inspired by the Bonnesen type inequalities, which was named by Osserman in [21]. This was based off work by Bonnesen, where he studied inequalities in the form

$$L^2 - 4\pi A \geq \lambda(C). \tag{1.12}$$

In this setting, $L$ and $A$ represent the length and area enclosed by a simple closed curve $C$ in $\mathbb{R}^2$. Moreover, $\lambda(C)$ satisfies three conditions:

1. $\lambda(C) \geq 0$.

2. $\lambda(C) = 0$ precisely when $C$ is a circle.

3. $\lambda(C)$ measures geometrically how close $C$ is to a circle.

Fuglede worked to expand these results to higher dimensions in [8] and [9], where they proved a quantitative isoperimetric inequality for nearly spherical sets. They used this result to study the stability for convex domains using the spherical deviation.
**Definition 1.2.1.** For a domain $\Omega \subseteq \mathbb{R}^{n+1}$, set $\tilde{\Omega} := \frac{\text{Vol}(B)}{\text{Vol}(\Omega)}(\Omega - \text{bar}(\Omega))$, where $\text{bar}(\Omega)$ is the barycenter of $\Omega$ and $B$ is the unit ball. The *spherical deviation* of $\Omega$, $d(\Omega)$, is defined as

$$d(\Omega) := d_H(\tilde{\Omega}, B), \quad \text{(1.13)}$$

where $d_H(\cdot, \cdot)$ gives the Hausdorff distance between two sets.

Specifically, for a convex domain $\Omega$, they established an inequality in the form $d(\Omega) \leq f(\delta(\Omega))$, for some function $f$.

To establish a quantitative isoperimetric inequality for more general domains, the *Fraenkel asymmetry*, $\alpha(\Omega)$, is a well-studied quantity used as a lower bound.

**Definition 1.2.2.** Suppose $\Omega \subseteq \mathbb{R}^{n+1}$. The *Fraenkel asymmetry* of $\Omega$ is denoted by $\alpha(\Omega)$, where

$$\alpha(\Omega) := \inf \left\{ \frac{|\Omega \Delta (x + B_\Omega)|}{|B_\Omega|}, x \in \mathbb{R}^{n+1} \right\}. \quad \text{(1.14)}$$

$B_\Omega$ denotes the ball centered at the origin with the same volume as $\Omega$, and $\Delta$ denotes the symmetric difference between two sets.

Using the Fraenkel asymmetry in the study of stability brings us to the *quantitative isoperimetric inequality*, which asks if there is a fixed $C(n) > 0$ such that all Borel sets $\Omega \subseteq \mathbb{R}^{n+1}$ with finite measure satisfy the inequality

$$\delta(\Omega) \geq C(n)\alpha^m(\Omega), \quad \text{(1.15)}$$

for some exponent $m$. The quantitative isoperimetric inequality $\delta(\Omega) \geq C\alpha^2(\Omega)$ was shown for Steiner symmetrical sets in [17] by Hall, Hayman, and Weitsman. Later, by using results on the Steiner symmetrical, Hall showed (1.15) in [16], but with a suboptimal exponent of $m = 4$. 

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Finally, in [10], Fusco, Maggi, and Pratelli showed, by using symmetrizations of $\Omega$, that (1.15) holds with optimal exponent $m = 2$ for Borel sets $\Omega \subseteq \mathbb{R}^{n+1}$ of finite measure. Figalli, Maggi, and Pratelli in [7] proved this optimal result in the more general setting of the anisotropic perimeter.

**Theorem 1.2.1** ([7], [10]). Suppose $n \geq 1$. Then for any Borel set $\Omega \subseteq \mathbb{R}^{n+1}$ of finite measure,

$$\delta(\Omega) \geq C(n)\alpha^2(\Omega),$$

(1.16)

where $C(n) > 0$ depends only on $n$.

This theorem was proved in [10] by reducing the problem to $n$-symmetric sets and then using the method of Steiner symmetry. To prove the result in [7], the authors did not use symmetrization arguments, as done previously. Instead, they applied the Brenier map to employ methods in mass transportation theory. For further reading on the quantitative isoperimetric inequality, see [11] and [20].

Our study of stability in the quermassintegral inequalities is inspired by work done by Fuglede in [9] and by Cicalese and Leonardi in [6] on nearly spherical sets.

**Definition 1.2.3.** Suppose $M = \{(1 + u(x))x : x \in \partial B\}$, where $\partial B$ is the unit sphere in $\mathbb{R}^{n+1}$ and $u : \partial B \to (-1, \infty)$ is a smooth function on the unit sphere. $M$ is referred to as a nearly spherical set when we have suitable, small bounds on $|u|$, $|\nabla u|$, and $|D^2u|$.

**Remark.** Nearly spherical sets in [8] and [6] only require bounds on $|u|$ and $|\nabla u|$. However, since we will be working with curvature terms, we will require small bounds on $|D^2u|$ as well.

In [6], Cicalese and Leonardi introduced a new method to show (1.16) for all Borel sets of finite measure. In this paper, they utilized the results in [8], which they reformulated by assuming $|u|_{W^{1,\infty}} < \epsilon$ and found

$$\delta(\Omega) \geq \frac{1 - \Omega(\epsilon)}{2}||u||_{L^2}^2 + \frac{1}{4}||\nabla u||_{L^2}^2,$$

(1.17)

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where \( u \) is the function from Definition 1.2.3. Note that functions in \( O(\epsilon) \) may obtain either positive or negative values. It quickly follows that for nearly spherical sets,

\[
\delta(\Omega) \geq C(1 + O(\epsilon))\alpha^2(\Omega).
\]  

(1.18)

They proved the Selection Principle, which provided a new proof of the quantitative isoperimetric inequality by reducing the problem to nearly spherical sets converging to the unit ball.

To get (1.18) from (1.17), only the weaker statement \( \delta(\Omega) \geq \frac{1-O(\epsilon)}{2}||u||^2_{L^2} \) is needed. However, in [13], Fusco and Julin showed a stronger result for stability in the isoperimetric problem, where they bound the asymmetry index \( A(\Omega) \), so that \( \delta(\Omega) \geq CA^2(\Omega) \). To do this, they need the full result that \( \delta(\Omega) \geq C||u||^2_{W^{1,2}} \) for nearly spherical sets. Then, as in [6], they are able to use the methods of the Selection Principle to reduce the general problem to the results for nearly spherical sets.

### 1.3 Main results

We aim to prove a version of (1.17) as it applies to the quermassintegral inequalities. First, we define \( I_k(\Omega) \) by integrating the \( k \)-th mean curvature of \( M \) for \( k \geq 0 \), that is

\[
I_k(\Omega) := \int_M \sigma_k(L) \, dA.
\]  

(1.19)

Also, for \( k = -1 \) we define

\[
I_{-1}(\Omega) := \text{Vol}(\Omega).
\]  

(1.20)

We are now able to define a natural generalization of the isoperimetric deficit for quermassintegrals.
**Definition 1.3.1.** For $-1 < k \leq n$ and $-1 \leq m < k$, the $(k,m)$-isoperimetric deficit is denoted by $\delta_{k,m}(\Omega)$, where

$$
\delta_{k,m}(\Omega) := \frac{I_k(\Omega) - I_k(B_{\Omega,m})}{I_k(B_{\Omega,m})}.
$$

Here $B_{\Omega,m}$ is the ball centered at the origin where $I_m(B_{\Omega,m}) = I_m(\Omega)$.

Our first main theorems are contained in Chapter 3. In 3.1, we compute an explicit formula for the $k$-th mean curvature of nearly spherical sets. Then in 3.2, we add in the assumption that $||u||_{W^{2,\infty}} < \epsilon$ to expand out $I_k(\Omega)$. This brings us to our first main theorem.

**Theorem 1.3.1.** Suppose $\Omega = \{(1 + u(\frac{x}{|x|}))x : x \in B\} \subseteq \mathbb{R}^{n+1}$, where $u \in C^3(\partial B)$, $\text{Vol}(\Omega) = \text{Vol}(B)$, and $\text{bar}(\Omega) = 0$. For all $\eta > 0$, there exists $\epsilon > 0$ such that if $||u||_{W^{2,\infty}} < \epsilon$, then

$$
\delta_{k,-1}(\Omega) \geq \left(\frac{(n-k)(k+1)}{2n(n+1)^2} - \eta\right) \alpha^2(\Omega). \tag{1.22}
$$

In 3.3, we prove a similar theorem, but we assume that $I_j(\Omega) = I_j(B)$ for $j \geq 0$, instead of $\text{Vol}(\Omega) = \text{Vol}(B)$.

**Theorem 1.3.2.** Fix $0 \leq j < k$. Suppose $\Omega = \{(1 + u(\frac{x}{|x|}))x : x \in B\} \subseteq \mathbb{R}^{n+1}$, where $u \in C^3(\partial B)$, $I_j(\Omega) = I_j(B)$, and $\text{bar}(\Omega) = 0$. For all $\eta > 0$, there exists $\epsilon > 0$ such that if $||u||_{W^{2,\infty}} < \epsilon$, then

$$
\delta_{k,j}(\Omega) \geq \left(\frac{n(n-k)(k-j)}{4(n+1)^2} - \eta\right) \alpha^2(\Omega). \tag{1.23}
$$

We remark, for sufficiently small $||u||_{W^{2,\infty}}$, that $\Omega$ is a convex domain. Then, we already know from the result of Guan and Li, which assumes $\Omega$ is $k$-convex and starshaped, that $\delta_{k,j}(\Omega) \geq 0$. So, we are establishing a quantitative isoperimetric inequality in this case.

In 3.4, we prove the following theorem.
Theorem 1.3.3. Suppose \( \Omega = \{(1 + u(\frac{x}{|x|})) x : x \in B\} \subseteq \Omega \), where \( u \in C^3(\partial B) \), \( \text{Vol}(\Omega) = \text{Vol}(B) \), and \( \text{bar}(\Omega) = 0 \). There exists an \( \eta > 0 \) so if \( ||u||_{W^{2, \infty}} < \eta \), then

\[
||u||_{L^\infty}^n \leq \begin{cases} 
C\delta_{k,-1}^{1/2}(\Omega) & n = 1 \\
C\delta_{k,-1}(\Omega) \log \frac{A}{\delta_{k,-1}^{1/2}(\Omega)} & n = 2 \\
C\delta_{k,-1}(\Omega) & n \geq 3,
\end{cases}
\]

where \( A, C > 0 \) depend only on \( n \).

Theorem 1.3.3 shows that the \((k, -1)\)-deficit gives a control on \( ||u||_{L^\infty} \), which is equivalent to the spherical deviation \( d(\Omega) \). The proof of this theorem follows closely to Fuglede’s in [9], where they proved this theorem when \( k = 0 \) (for the classical isoperimetric deficit). In [12], Fusco, Gelli, and Pisante expanded this stability result for domains where they impose a uniform cone condition on the boundary. Studying the control on \( ||u||_{L^\infty} \) gives a stronger result than just the Fraenkel asymmetry, although there must be some regularity condition imposed for it to hold. In [9], bounding \( ||u||_{L^\infty} \) by the classical isoperimetric deficit for nearly spherical domains was a key result to establish stability results for convex domains (without the assumption that the domain is nearly spherical). We hope Theorem 1.3.3 will be useful for establishing a stability result with the spherical deviation for less restrictive \( k \)-convex domains.

We now turn our attention specifically to the \((k, k - 1)\)-quermassintegral inequalities. A key proposition we proved to show our theorems from Chapter 3 states that when \( I_{k-1}(\Omega) = I_{k-1}(B) \),

\[
I_k(\Omega) - I_k(B) \geq (1 + O(\epsilon))A(t),
\]

(1.24)
where
\[ A(t) := \binom{n}{k} \left( \frac{n-k}{2n} \right) \left( \|u\|_{L^2}^2 + \frac{1}{2} \|\nabla u\|_{L^2}^2 \right). \] (1.25)

We aim to further study this inequality by studying the flow in (1.9), and we scale the surfaces \( M(t) \) to \( \tilde{M}(t) \) so that
\[ \tilde{X} = e^{-rt} X, \quad r = \binom{n}{k-1} \binom{n}{k}. \] (1.26)

**Theorem 1.3.4.** Suppose \( \tilde{M}(t) \) is the rescaled surface in (1.26) of \( M(t) \) (which is a solution to the flow (1.9)). Additionally, assume at \( t_0 \) that \( M(t_0) \) is nearly spherical with \( \|u(t_0)\|_{W^{2,\infty}} < \epsilon \) and that the barycenter of \( \tilde{M}(t_0) \) satisfies \( \|\text{bar}(\tilde{M}(t_0))\| \leq K \epsilon \|u(t_0)\|_{W^{2,2}}^2 \) for fixed a \( K > 0 \). Then, for any small \( \eta > 0 \),
\[ \frac{d}{dt}(I_k(\tilde{\Omega}(t_0)) - I_k(B)) \leq (1 - \eta) \frac{d}{dt} A(t_0), \] (1.27)
and the choice of a sufficiently small \( \epsilon > 0 \) depends on \( \eta \) and \( K \).

Moreover, along any solution to the flow (5.1) where \( |\text{bar}(\tilde{M}(t))| \leq K \epsilon \|u\|_{W^{2,2}}^2 \) holds for sufficiently large \( t \), we have
\[ \liminf_{t \to \infty} \frac{I_k(\tilde{\Omega}(t)) - I_k(B)}{A(t)} \geq 1. \] (1.28)

With some additional work, we are able to show in Corollary 5.1.5 that this provides an alternative proof to the key inequality in (1.24) in the case where \( M \) is \( n \)-symmetric. Therefore, we view Theorem 1.3.4 as a stronger statement to Theorem 1.3.2 in this case, as it gives us additional information on the behavior of the derivatives of relevant quantities along the flow.

We also look at the flow (1.10) to prove the following theorem.
Theorem 1.3.5. Suppose $M(t)$ is a solution of surfaces to the flow (1.10), and at $t_0$ the surface $M(t_0)$ satisfies, for a fixed $K$, that $|\text{bar}(M(t_0))|^2 \leq K\epsilon||u||_{W^{2,2}}^2$ and $||u(t_0)||_{W^{2,\infty}} < \epsilon$. Then, for any small $\eta > 0$

$$\frac{d}{dt} \left( I_{k-1}(\Omega(t_0)) - I_{k-1}(B) \right) \leq (1 - \eta) \frac{d}{dt} \frac{k(n-k+1)}{2n}||u(t_0)||_{L^2}^2, \quad (1.29)$$

where the choice of a sufficiently small $\epsilon > 0$ depends on $\eta$ and $K$. Additionally, if $|\text{bar}(M(t))|^2 \leq K\epsilon||u||_{W^{2,2}}^2$ holds for sufficiently large $t$, then

$$\liminf_{t \to \infty} \frac{I_{k-1}(\Omega(t)) - I_{k-1}(B)}{||u||_{L^2}^2} \geq \frac{k(n-k+1)}{2n}. \quad (1.30)$$

A few things have prevented us from getting the same results here that we have in Theorem 1.3.4. First, we do not have $\frac{d}{dt}||\nabla u||_{L^2}$ on the right-hand side of (1.29). Also, we were not able to use this result to get a quantitative quermassintegral inequality, as we did in a corollary to Theorem 1.3.4. This is because we miss a key lemma stating that if $M(t_0)$ is nearly spherical, then $M(t)$ remains nearly spherical for all $t \geq t_0$. However, this theorem still provides additional information on the stability in the $(k, -1)$-quermassintegral inequalities along the flow.
Chapter 2

Preliminaries

2.1 The $k$-th mean curvature

For $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$, we denote $\sigma_k(\lambda)$ as the $k$-th elementary symmetric polynomial of $(\lambda_1, ..., \lambda_n)$. That is, for $1 \leq k \leq n$,

$$\sigma_k(\lambda) = \sum_{i_1 < i_2 < \ldots < i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}, \quad (2.1)$$

and

$$\sigma_0(\lambda) = 1. \quad (2.2)$$

This leads to a natural generalization of the mean curvature of a surface.

**Definition 2.1.1.** Suppose $\Omega$ is a smooth, bounded domain in $\mathbb{R}^{n+1}$. For $x \in M := \partial \Omega$, the $k$-th mean curvature of $M$ at $x$ is $\sigma_k(\lambda)$, where $\lambda = (\lambda_1(x), ..., \lambda_n(x))$ are the principal curvatures of $M$ at $x$.

Observe that in this definition, $\sigma_1(\lambda)$ is the mean curvature and $\sigma_n(\lambda)$ is the Gaussian curvature. When $(\lambda_1, ..., \lambda_n)$ are the eigenvalues of a matrix $A = \{A_i^j\}$, we denote $\sigma_k(A) =$
\( \sigma_k(\lambda) \), which can be equivalently calculated as

\[
\sigma_k(A) = \frac{1}{k!} \delta_{i_1\cdots i_k}^{j_1\cdots j_k} A_{j_1}^{i_1} \cdots A_{j_k}^{i_k},
\]

(2.3)

using the Einstein convention to sum over repeated indices.

So, if \( L \) is the second fundamental form of \( M \), we can use this expression for \( \sigma_k(L) \) to compute the \( k \)-th mean curvature of \( M \). Throughout this paper, we will be working with a family of surfaces where, for \( 0 < j \leq k \), \( \sigma_j(L) \geq 0 \) at each point. Such surfaces are called \( k \)-convex.

**Definition 2.1.2.** Let \( \Omega \) be a domain in \( \mathbb{R}^{n+1} \). Then the hypersurface \( M := \partial \Omega \) is said to be strictly \( k \)-convex if the principal curvatures \( \lambda = (\lambda_1, \ldots, \lambda_n) \) lie in the Gårding cone \( \Gamma_k^+ \), which is defined as

\[
\Gamma_k^+ := \{ \lambda \subseteq \mathbb{R}^n : \sigma_j(\lambda) > 0, 1 \leq j \leq k \}. \quad (2.4)
\]

Note that \( n \)-convexity is the same as normal convexity. A useful operator related to \( \sigma_k \) is the Newton transformation tensor \( [T_k]_{li}^j \).

**Definition 2.1.3.** The Newton transformation tensor, \( [T_k]_{li}^j \), of \( n \times n \) matrices \{\( A_1, \ldots, A_k \)\} is defined as

\[
[T_k]_{li}^j(A_1, \ldots, A_k) := \frac{1}{k!} \delta_{ii_1\cdots i_k}^{jj_1\cdots j_k} (A_1)^{i_1}_{j_1} \cdots (A_k)^{i_k}_{j_k}. \quad (2.5)
\]

When \( A_1 = A_2 = \ldots = A_k = A \), we denote \( [T_k]_{li}^j(A) = [T_k]_{li}^j(A, \ldots, A) \).

A related operator is \( \Sigma_k \), which the polarization of \( \sigma_k \).
Definition 2.1.4. Suppose \( \{A_1, ..., A_k\} \) is a collection of \( n \times n \) matrices. We denote

\[
\Sigma_k(A_1, ..., A_k) := (A_1)^{i_1}_{[T_{k-1}]} (A_2, ..., A_k)
\]

\[
= \frac{1}{(k-1)!} \delta_{i_1...i_k}^{j_1...j_k} (A_1)^{i_1}_{j_1} \cdots (A_k)^{i_k}_{j_k}.
\]  

(2.6)

Two useful identities are

\[
\sigma_k(A) = \frac{1}{k} \Sigma_k(A, ..., A) = \frac{1}{k} A^i_j [T_{k-1}]^j_i (A),
\]  

(2.7)

and

\[
\frac{\partial \sigma_k(A)}{\partial A^i_j} = \frac{1}{k} [T_{k-1}]^j_i (A).
\]  

(2.8)

We will also use the identity

\[
A^i_j [T_m]^j_i (A) = \delta^i_s \sigma_{m+1}(A) - [T_{m+1}]^i_s (A).
\]  

(2.9)

2.2 The \((k, m)\)-isoperimetric deficit

In this paper, we look at a more general notion of the isoperimetric deficit as it pertains to the \( k \)-th mean curvature. We consider a bounded domain \( \Omega \subset \mathbb{R}^{n+1} \) where \( M := \partial \Omega \) is a smooth hypersurface. First, we define \( I_k(\Omega) \) by integrating the \( k \)-th mean curvature of \( M \). That is,

\[
I_k(\Omega) := \int_M \sigma_k(L) d\mu.
\]  

(2.10)

The definition extends to \( k = -1 \) so that

\[
I_{-1}(\Omega) := \text{Vol}(\Omega).
\]  

(2.11)
Furthermore, because $\sigma_0(L) = 1$,

$$I_0(\Omega) = \text{Area}(M). \quad (2.12)$$

As seen from the identity (1.5), $I_k(\Omega)$ is equal to a quermassintegral of $\Omega$ up to a constant.

We aim to study the $(k, m)$-deficit, $\delta_{k,m}(\Omega)$, from Definition [1.3.1]. Note that $\delta_{0,-1}(\Omega)$ is the classical isoperimetric deficit from (1.11).

For $k \geq 0$, $I_k(B) = \binom{n}{k}\text{Area}(\partial B)$. If we scale $\Omega$ by a fixed $r > 0$, so $rE = \{rx : x \in \Omega\}$, then $I_k(rE) = r^{n-k}I_k(\Omega)$. It follows that

$$I_m(r\Omega) = I_m(rB_{\Omega,m}). \quad (2.13)$$

Therefore, $\delta_{k,m}(\Omega)$ is invariant under scaling. It is also invariant under translation.

Furthermore, if $r$ is the radius of $B_{\Omega,m}$, then $I_m(\Omega) = \binom{n}{m}r^{n-m}\text{Area}(\partial B)$, giving

$$r = \left(\frac{I_m(\Omega)}{\binom{n}{m}\text{Area}(\partial B)}\right)^{\frac{1}{n-m}}. \quad (2.14)$$

Then,

$$I_k(B_{\Omega,m}) = \left(\frac{I_m(\Omega)}{\binom{n}{m}\text{Area}(\partial B)}\right)^{\frac{n-k}{n-m}}\binom{n}{k}\text{Area}(\partial B) = \left(\frac{n}{m}\right)^{\frac{n-k}{n-m}}\text{Area}(\partial B)\binom{n-m}{n-k}I_m(\Omega)^{\frac{n-k}{n-m}}. \quad (2.15)$$

In particular, when setting $m = k - 1$,

$$\delta_{k,k-1}(\Omega) = \frac{I_k(\Omega)}{\frac{n-k+1}{k}I_{k-1}(B)\frac{1}{n-k+1}I_{k-1}(\Omega)^{n-k+1}} - 1. \quad (2.16)$$

Thus, the inequality $\delta_{k,k-1}(\Omega) \geq 0$ is equivalent to the quermassintegral inequalities in [1.6].

Our goal is to look at the quantitative isoperimetric inequality in the setting of the $k$-th
mean curvature, which we refer to as the quantitative \((k,m)\)-isoperimetric inequality. That is, we aim to compare \(\delta_{k,m}(\Omega)\) to the Fraenkel asymmetry of \(\Omega\), \(\alpha(\Omega)\), which measures how close \(\Omega\) is to a ball (see Definition 1.2.2).

The Fraenkel asymmetry of a set is invariant under scaling and translation. Therefore, when studying the quantitative \((k,m)\)-isoperimetric inequality, i.e. when there is a fixed \(C > 0\) so that

\[
\delta_{k,m}(\Omega) \geq C\alpha^2(\Omega), \tag{2.17}
\]

we only need to consider sets where \(I_m(\Omega) = I_m(B)\) and \(\text{bar}(\Omega) = 0\).

### 2.3 Nearly spherical sets

The focus of this paper is to establish the \((k,m)\)-isoperimetric inequality for nearly spherical sets. Our approach is inspired by Cicalese and Leonardi’s work in the classical quantitative isoperimetric inequality for nearly spherical sets in [6]. That is, we consider a smooth, bounded domain \(\Omega\) that is starshaped with respect to the origin, which is enclosed by \(M := \partial\Omega\). We write \(M = \{(1+u(x))x : x \in \partial B\}\), where \(u : \partial B \rightarrow \mathbb{R}\) is a smooth function. The set \(M\) is referred to as a nearly spherical set when there is a suitable, small bound on \(||u||_{W^{2,\infty}}\).

In this section, we establish some useful formulas for nearly spherical sets.

We write \(\mathbb{R}^{n+1}\) in spherical coordinates with the tangent basis \(\{\frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2}, \ldots, \frac{\partial}{\partial \theta_n}, \frac{\partial}{\partial r}\}\). Denoting \(s_{ij}\) as the metric on the sphere, we have \(<\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial r}> = 0\), \(<\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}> = 1\), and \(<\frac{\partial}{\partial r}, \frac{\partial}{\partial r}> = r^2s_{ij}\). Set \(u_i = \frac{\partial u}{\partial \theta_i}\). Then, \(\{e_i\}\) forms a tangent basis of \(M\) where

\[
e_i = \frac{\partial}{\partial \theta_i} + u_i \frac{\partial}{\partial r}. \tag{2.18}
\]
We find,

\[ N = -\sum_{i=1}^{n} s^{ij} u_i \frac{\partial}{\partial \theta_j} + (1 + u)^2 \frac{\partial}{\partial r}, \tag{2.19} \]

where \( N \) is the outward unit normal on \( M \), and the norm \(|\nabla u|\) is taken with respect to the standard metric on \( \partial B \). We compute the metric \( g_{ij} \) on \( M \) as

\[ g_{ij} = \langle e_i, e_j \rangle = (1 + u)^2 s_{ij} + u_i u_j, \tag{2.20} \]

where \( \langle \cdot, \cdot \rangle \) is the standard Euclidean inner product on \( \mathbb{R}^{n+1} \). Setting \( g^{ij} \) to be the inverse of \( g_{ij} \), we have

\[ g^{ij} = \frac{s^{ij}}{(1 + u)^2} - \frac{u_k u_l s^{ki} s^{lj}}{(1 + u)^2 |\nabla u|^2 + (1 + u)^2}. \tag{2.21} \]

We denote \( h_{ij} \) as the second fundamental form on \( M \). That is, \( h_{ij} = -\langle N, \nabla e_i e_j \rangle \), and we form the shape operator \( h^i_j \) by

\[ h^i_j = g^{ik} h_{kj}. \tag{2.22} \]

We now explicitly calculate \( h^i_j \). First, note

\[ \left( \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta_j} \right)^k = \frac{1}{2} s^{kl} \left( \partial_k (r^2 s_{il}) + \partial_l (r^2 s_{ik}) - \partial_i (r^2 s_{lj}) \right) \tag{2.23} \]

\[ = \frac{1}{2} s^{kl} \left( \partial_i s_{kl} + \partial_j s_{il} - \partial_l s_{ij} \right) \tag{2.24} \]

\[ = \Gamma^k_{ij}, \tag{2.25} \]
where $\Gamma_{ij}^k$ refers to the Christoffel symbol on $S^n$, and

$$\left(\nabla \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_j}\right)^r = -rs_{ij}. \tag{2.26}$$

We thus obtain

- $\nabla \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_j} = \Gamma_{ij}^k \frac{\partial}{\partial \theta_k} - rs_{ij} \frac{\partial}{\partial r}$.

Similarly,

- $\nabla \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial r} = \nabla \frac{\partial}{\partial r} \frac{\partial}{\partial \theta_i} = \frac{1}{r} \frac{\partial}{\partial \theta_i}$,

- $\nabla \frac{\partial}{\partial r} \frac{\partial}{\partial r} = 0$.

Then,

$$\nabla e_i e_j = \nabla \frac{\partial}{\partial \theta_i} + u_i \frac{\partial}{\partial r} \left( \frac{\partial}{\partial \theta_j} + u_j \frac{\partial}{\partial r} \right)$$

$$= \nabla \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_j} + \nabla \frac{\partial}{\partial \theta_i} \left( u_j \frac{\partial}{\partial r} \right) + u_i \nabla \frac{\partial}{\partial \theta_j} \left( u_j \frac{\partial}{\partial r} \right)$$

$$= \nabla \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_j} + u_j \nabla \frac{\partial}{\partial r} \frac{\partial}{\partial \theta_j} + \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} u \right) \frac{\partial}{\partial r} + u_i \nabla \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_j} + u_i u_j \nabla \frac{\partial}{\partial r} \frac{\partial}{\partial \theta_j} + u_i \left( \frac{\partial}{\partial r} \frac{\partial}{\partial r} \right) \frac{\partial}{\partial r}. \tag{2.27}$$

So,

$$h_{ij} = -\left( \frac{-s_{pq} u_{pq} \frac{\partial}{\partial r}}{(1 + u)^2 \sqrt{|\nabla u|^2 + (1 + u)^2}}, \frac{\partial}{\partial \theta_b} \frac{\partial}{\partial \theta_c} \Gamma_{ij}^b \frac{\partial}{\partial \theta_k} - (1 + u)s_{ij} \frac{\partial}{\partial r} + \frac{1}{(1 + u)} \left( u_j \frac{\partial}{\partial \theta_i} + u_i \frac{\partial}{\partial \theta_j} \right) \right)$$

$$+ \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} u \right) \frac{\partial}{\partial r}. \tag{2.29}$$
Thus,

\[ h_{ij} = \frac{1}{\sqrt{|\nabla u|^2 + (1 + u)^2}} \left( (1 + u)u_k \Gamma^k_{ij} + (1 + u)^2 s_{ij} + 2u_i u_j - (1 + u) \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} u \right) \right) \]

\[ = \frac{1}{\sqrt{|\nabla u|^2 + (1 + u)^2}} \left( 2u_i u_j + (1 + u)^2 s_{ij} - (1 + u)u_{ij} \right), \]

where \( u_{ij} \) denotes the Hessian of \( u \) on \( S^n \). Set

\[ D := \sqrt{|\nabla u|^2 + (1 + u)^2}. \]

Then,

\[ h^i_j = g^{ik} h_{kj} \]

\[ = \left( \frac{s^{ik}}{(1 + u)^2} - \frac{1}{(1 + u)^2} \frac{u_m u_l s^{mi} s^{lk}}{D^2} \right) \frac{1}{D} \left( 2u_k u_j + (1 + u)^2 s_{kj} - (1 + u)u_{kj} \right) \]

\[ = \frac{2u^i u_j}{(1 + u)^2 D} + \frac{\delta^i_j}{D} - \frac{u^i_j}{(1 + u) D} - \frac{2u^i u_j |\nabla u|^2}{(1 + u)^2 D^3} - \frac{u^i u_j}{D^3} + \frac{u^i u_j}{(1 + u) D^3}. \]

(2.32)

We observe that

\[ \frac{2u^i u_j}{(1 + u)^2 D} - \frac{2u^i u_j |\nabla u|^2}{(1 + u)^2 D^3} - \frac{u^i u_j}{D^3} = \frac{u^i u_j}{D^3}, \]

(2.33)

which yields

\[ h^i_j = \frac{\delta^i_j}{D} - \frac{u^i_j}{(1 + u) D} + \frac{u^i u_j u^l_j}{(1 + u) D^3} + \frac{u^i u_j}{D^3}. \]

(2.34)

Next, note

\[ \sqrt{\det g_{ij}} = (1 + u)^n \sqrt{\frac{|\nabla u|^2}{(1 + u)^2} + 1}. \]

(2.35)
Therefore,

$$\text{Area}(M) = \int_{\partial B} (1 + u)^n \sqrt{\frac{|\nabla u|^2}{(1 + u)^2} + 1} \, dA. \quad (2.36)$$

We list a few more relevant formulae below:

$$|\Omega| = \frac{1}{n+1} \int_{\partial B} (1 + u)^{n+1} dA, \quad (2.37)$$

$$|\Omega \Delta B| = \sum_{k=1}^{n+1} \int_{\partial B} \frac{1}{n+1} \binom{n+1}{k} |u|^k dA, \quad (2.38)$$

$$\bar{\text{Area}}(\Omega) = \frac{1}{\text{Area}(\partial B)} \int_{\partial B} (1 + u)^{n+2} x dA. \quad (2.39)$$

Finally, we consider how to compute $\nabla_j [T_m]^j_i (D^2 u)$ for nearly spherical sets. This is particularly useful when applying integration by parts to $I_k(\Omega)$ in Section 4. See [5] for a similar computation. We compute,

$$\nabla_j [T_m]^j_i (D^2 u) = \frac{1}{m!} \nabla_j \delta^{ij_1 j_2 ... j_m} u_{i_1, j_1} u_{i_2, j_2} \cdots u_{i_m, j_m}$$

$$= \frac{m}{m!} \delta^{ij_1 j_2 ... j_m} (\nabla_j u_{i_1, j_1} u_{i_2, j_2} \cdots u_{i_m, j_m}). \quad (2.40)$$

Note that

$$\delta^{ij_1 j_2 ... j_m} (\nabla_j u_{i_1, j_1}) u_{i_2, j_2} \cdots u_{i_m, j_m} = -\delta^{ij_1 j_2 ... j_m} (\nabla_j u_{i_1, j_1}) u_{i_2, j_2} \cdots u_{i_m, j_m}. \quad (2.41)$$

We obtain

$$\nabla_j [T_m]^j_i (D^2 u) = \frac{1}{2(m-1)!} \delta^{ij_1 j_2 ... j_m} (\nabla_j u_{i_1, j_1} - \nabla_{j_1} u_{i_1, j_1}) u_{i_2, j_2} \cdots u_{i_m, j_m}$$

$$= \frac{1}{2(m-1)!} \delta^{ij_1 j_2 ... j_m} (u_p R_{s j}^{pl}) u_{i_1, j_1} \cdots u_{i_{m-1}, j_{m-1}}, \quad (2.42)$$
where \( R_{sj}^{pl} \) is the curvature tensor on \( \Omega \). On \( S^n \), we know by the Gauss equation,

\[
R_{sj}^{pl} = h^p_s h_j^l - h^p_l h_j^s = \delta^p_s \delta^l_j - \delta^p_l \delta^s_j.
\] (2.43)

Therefore,

\[
\nabla_j[T_m]_{ij}^2(D^2u) = \frac{1}{(m-1)!} \left( \frac{1}{2} u_p(\delta^p_s \delta^l_j - \delta^p_l \delta^s_j) \delta^{jsj_1\ldots jm-1}_{ih_1\ldots im-1} u_j^{i_1} \ldots u_j^{i_{m-1}} \right)
\]

\[
= \frac{1}{(m-1)!} \left( \frac{1}{2} \left( u_s \delta^{jsj_1\ldots jm-1}_{ij_1\ldots im-1} u_j^{i_1} \ldots u_j^{i_{m-1}} - u_j \delta^{ij_1\ldots jm-1}_{ih_1\ldots im-1} u_j^{i_1} \ldots u_j^{i_{m-1}} \right) \right)
\]

\[
= \frac{-1}{(m-1)!} u_j \delta^{ij_1\ldots jm-1}_{ih_1\ldots im-1} u_j^{i_1} \ldots u_j^{i_{m-1}}
\]

\[
= -(n-m) u_j [T_{m-1}]_i^j(D^2u).
\] (2.44)
Chapter 3

Establishing quantitative quermassintegral inequalities

3.1 Computation of $\sigma_k(L)$ for nearly spherical sets

Our goal is to control a lower bound on the $(k, m)$-isoperimetric deficit $\delta_{k,m}(\Omega)$, where $M := \partial \Omega$ is a nearly spherical set. In this section, we focus on calculating $\sigma_k(h^i_j)$, where

$$
\sigma_k(h^i_j) = \frac{1}{k!} \delta_{i_1 \ldots i_k}^{j_1 \ldots j_k} \left( \frac{-u_{j_1}^{i_1}}{D(1 + u)} + \frac{\delta_{j_1}^{i_1}}{D} + \frac{u_{j_1}^i u_s u_{i_1}}{D^3(1 + u)} + \frac{u_{i_1}^i u_{j_1}}{D^3} \right)
\ldots \left( \frac{-u_{j_k}^{i_k}}{D(1 + u)} + \frac{\delta_{j_k}^{i_k}}{D} + \frac{u_{j_k}^i u_s u_{i_k}}{D^3(1 + u)} + \frac{u_{i_k}^i u_{j_k}}{D^3} \right).
$$

(3.1)

In particular, the mean curvature on $M$ is given by

$$
\sigma_1(h^i_j) = \delta_{i_1}^{j_1} \left( \frac{-u_{j_1}^{i_1}}{D(1 + u)} + \frac{\delta_{j_1}^{i_1}}{D} + \frac{u_{j_1}^i u_s u_{i_1}}{D^3(1 + u)} + \frac{u_{i_1}^i u_{j_1}}{D^3} \right)
= \frac{-\Delta u}{D(1 + u)} + \frac{n}{D} + \frac{u_s^i u_s u^i}{D^3 (1 + u)} + \frac{|\nabla u|^2}{D^3}.
$$

(3.2)
Computing $\sigma_k(h^i_j)$ for any $k > 0$ requires a bit more work, as we show in the following lemma.

**Lemma 3.1.1.** Suppose $\Omega \subseteq \mathbb{R}^{n+1}$ where $M = \{(1 + u(x))x : x \in \partial B\}$ and $u \in C^2(\partial B)$.

Then $\sigma_k(h^i_j)$ equals:

$$
\frac{1}{((1 + u)^2 + \lvert \nabla u \rvert^2)^{\frac{k+2}{2}}} \sum_{m=0}^{k} \frac{(-1)^m \binom{n-m}{k-m}}{(1 + u)^m} \left( (1 + u)^2 \sigma_m(D^2 u) + \frac{n + k - 2m}{n - m} u^i u^j T_{m,i,j}(D^2 u) \right).
$$

(3.3)

**Proof.** In this proof we set

$$
D := \sqrt{(1 + u)^2 + \lvert \nabla u \rvert^2}.
$$

(3.4)

We expand out each term of $\sigma_k(h^i_j)$ in (3.1). Many of the terms in the expansion of this sum turn out to be zero. We compute the terms in the following cases:

1. $m \geq 0$ instances of $\frac{-u^i}{D(1+u)}$, and the rest are in the form $\frac{\delta^i_j}{D}$.

First, consider the sum of all terms where $\frac{-u^i_j}{D(1+u)}$ occurs in the first $m$ terms. This equals

$$
\frac{1}{k!} \delta^i_{j_1 \cdots j_k} \frac{-u^{i_1}_{j_1}}{D(1 + u)} \cdots \frac{-u^{i_m}_{j_m}}{D(1 + u)} \frac{\delta^{i_{m+1}}_{j_{m+1}}}{D} \cdots \frac{\delta^{i_k}_{j_k}}{D} \\
= \frac{1}{k!} \frac{(-1)^m}{D^k(1 + u)^m} \delta^i_{j_1 \cdots j_m} u^{i_1}_{j_1} \cdots u^{i_m}_{j_m} \binom{n - m}{k - m} (k - m)!
$$

(3.5)

However, to account for any permutation of the ordering of the terms above, we multiply (3.5) by $\binom{k}{m}$.
So, the sum of all terms in this case is:

\[
\frac{(-1)^m \binom{n-m}{k-m} \sigma_m(D^2 u)}{D^k(1 + u)^m}.
\]  

(3.6)

2. One instance of \( \frac{u^i u_j}{D^j} \) and \( m \geq 1 \) instances of \( \frac{-u_j}{D(1+u)} \).

The sum of these terms is equal to:

\[
\begin{align*}
\frac{(-1)^m k \cdot \binom{k-1}{m}}{k!} & \delta_{j_1 \cdots j_k} u^{i_1} u_{j_1} \frac{u^{i_2}}{D^3} \frac{u^{i_2}}{D(1 + u)} \cdots \frac{u^{i_{m+1}}}{D(1 + u)} \frac{\delta_{j_m+1}}{D} \cdots \frac{\delta_{j_k}}{D} \\
& = \frac{(-1)^m \binom{k-1}{m}}{(k-1)!} \frac{1}{D^{k+2}(1 + u)^m} \delta_{j_1 \cdots j_{m+1}} u^{i_1} u_{j_1} \frac{u^{i_2}}{D(1 + u)} \cdots \frac{u^{i_{m+1}}}{D(1 + u)} \frac{(n - (m + 1))}{k - (m + 1)}(k - (m + 1))! \\
& = \left( \frac{n - (m + 1)}{k - (m + 1)} \right) \frac{(-1)^m}{D^{k+2}(1 + u)^m} u^{i_1} u_{j_1} \left[ T_{m+1} \right]_i ^j (D^2 u). 
\end{align*}
\]  

(3.7)

3. One instance of \( \frac{u^i u_s u^i}{D^s(1+u)} \) and \( m \geq 1 \) instances of \( \frac{u^i}{D(1+u)} \).

In this case, the sum of all the terms is:

\[
\begin{align*}
\frac{(-1)^m \binom{k}{m} (k - m)}{k!} & \delta_{j_1 \cdots j_k} u^{i_1} u_s u^{i_1} \frac{u^{i_2}}{D^3} \frac{u^{i_2}}{D(1 + u)} \cdots \frac{u^{i_{m+1}}}{D(1 + u)} \frac{\delta_{j_m+1}}{D} \cdots \frac{\delta_{j_k}}{D} \\
& = \frac{(-1)^m}{k!} \frac{\binom{k}{m} (k - m)}{D^{2+k}(1 + u)^{m+1}} \delta_{j_1 \cdots j_{m+1}} u^{i_1} u_s u^{i_1} \frac{u^{i_2}}{D(1 + u)} \cdots \frac{u^{i_{m+1}}}{D(1 + u)} \left( \frac{n - (m + 1)}{k - (m + 1)} \right) ! (k - (m + 1))! \\
& = \left( \frac{n - (m + 1)}{k - (m + 1)} \right) \frac{(-1)^m}{D^{2+k}(1 + u)^{m+1}} u^{i_1} u_s u^{i_1} \left[ T_{m+1} \right]_i ^j (D^2 u). 
\end{align*}
\]  

(3.8)

4. When there are either two instances of \( \frac{u^i u_j}{D^j(1+u)} \), two instances of \( \frac{u_{j_s} u^i u^i}{D^j(1+u)} \), or one instance of \( \frac{u^{i_1} u_{j_s} u^i u^i}{D^j(1+u)} \) and one instance of \( \frac{u^{i_1} u_{j_s} u^i u^i}{D^j(1+u)} \).

In this case, we apply the following Lemma 3.1.2 to conclude the sum of all these terms is zero.

Next, we simplify the expression for \( \sigma_k(h_j^i) \) by noting the identity

\[
\begin{align*}
\frac{u^s}{T_{m+1} \left[ D^2 u \right]} = \delta_i^s \sigma_{m+1}(D^2 u) - \left[ T_{m+1} \right]_i ^s (D^2 u). 
\end{align*}
\]  

(3.9)
We compute

\[
\sigma_k(h^i_j) = \sum_{m=0}^{k} \left( \frac{(n-m)(-1)^m \sigma_m(D^2u)}{D^k(1 + u)^m} + \sum_{m=0}^{k-1} \frac{(n - (m + 1)) (-1)^m u^i u^j [T_{m+1}^j_i](D^2u)}{k - (m + 1)} \frac{(1 + u)^2}{D^{k+2}(1 + u)^{m+1}} \right)
\]

Now we give a quick proof of the lemma that was used in case 4 in Lemma 3.1.1.

**Lemma 3.1.2.** Suppose \(2 \leq k \leq n\) where \(M_1, \ldots, M_{k-2}\) are \(n \times n\) matrices, and \(w_1, w_2, v\) are \(n\)-dimensional vectors. Then,

\[
\Sigma_k(w_1 v^T, w_2 v^T, M_1, M_2, M_3, \ldots, M_{k-2}) = 0.
\]
Proof. We compute,
\[
\Sigma_k(\langle v^T \rangle, w^T, M_1, M_2, ..., M_{k-2}) = \frac{1}{(k-1)!} \delta_{i_1 i_2 ... i_k} \varepsilon_{j_1 j_2 ... j_k} \varepsilon_{j_1 i_2 ... i_k} w_1 v_{j_1} w_2 v_{j_2} M_{1,j_3} \cdots M_{k-2,j_k}
\]
\[
= \frac{1}{(k-1)!} \delta_{i_1 i_2 ... i_k} \varepsilon_{j_1 i_2 ... i_k} w_1 v_{j_2} w_2 v_{j_1} M_{1,j_3} \cdots M_{k-2,j_k}
\]
\[
= -\frac{1}{(k-1)!} \delta_{i_1 i_2 ... i_k} \varepsilon_{j_1 i_2 ... i_k} w_1 v_{j_2} w_2 v_{j_1} M_{1,j_3} \cdots M_{k-2,j_k}
\]
\[
= -\Sigma_k(\langle v^T \rangle, w^T, M_1, M_2, ..., M_{k-2}).
\]
(3.12)

Since \( \Sigma_k(\langle v^T \rangle, w^T, M_1, M_2, ..., M_{k-2}) = -\Sigma_k(\langle v^T \rangle, w^T, M_1, M_2, ..., M_{k-2}) \), we conclude

\[
\Sigma_k(\langle v^T \rangle, w^T, M_1, M_2, ..., M_{k-2}) = 0.
\]
(3.13)

\[\Box\]

3.2 \( I_k(\Omega) \) for nearly spherical sets

We continue to look at \( I_k(\Omega) \) for starshaped domains, but now we add in the additional assumption that \( ||u||_{W^{2,\infty}} < \epsilon \), making \( M := \partial \Omega \) a nearly spherical set as described in Definition 1.2.3. Note,

\[
\int_M \sigma_k(h^i_j) d\mu = \int_{\partial B} \sigma_k(h^i_j) (1 + u)^n \sqrt{1 + \frac{|\nabla u|^2}{(1 + u)^2}} dA.
\]
(3.14)

Using our formula for \( \sigma_k(h^i_j) \) in Lemma 3.1.1, we have the following expression for \( \int_M \sigma_k(h^i_j) d\mu \):

\[
\int_{\partial B} \sum_{m=0}^{k} \frac{(-1)^m (n-m)(1 + u)^{n-m-1}}{(1 + u)^2 + |\nabla u|^2} \left( (1 + u)^2 \sigma_m(D^2u) + \frac{n + k - 2m}{n - m} u^i u_j [T_m]^i_j (D^2u) \right) dA.
\]
(3.15)
Later, in the main theorems, our analysis deals mainly with lower order terms in $O(|u|)$ and $O(|\nabla u|)$. In the following lemma, we expand out $\frac{1}{((1+u)^2+|\nabla u|^{2})^{(k+1)/2}}$ in the integral using its Taylor expansion, and we are able to group all the higher order terms in $O(\epsilon)||u||^2_{L^2} + O(\epsilon)||\nabla u||^2_{L^2}$.

**Lemma 3.2.1.** Suppose $u \in C^1(\partial B)$ and for sufficiently small $\epsilon > 0$ that $||u||_{L^\infty}, ||\nabla u||_{L^\infty} < \epsilon$. Then,

$$\frac{1}{(|\nabla u|^2 + (1 + u)^2)^{\frac{k}{2}}} = 1 - mu + \frac{m(m+1)}{2}u^2 - \frac{m}{2}||\nabla u||^2 + O(\epsilon)u^2 + O(\epsilon)||\nabla u||^2. \quad (3.16)$$

**Proof.** First, we will expand out $(|\nabla u|^2 + (1 + u)^2)^{-1/2}$, which we rewrite as $(1 + 2u + u^2 + |\nabla u|^2)^{-1/2}$. Setting $f(x) = (1 + x)^{-1/2}$, we obtain

$$f^n(0) = \frac{(-1)^n \prod_{m=1}^n (2m-1)}{2^n}. \quad (3.17)$$

In the radius of convergence for its Taylor expansion, $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = \frac{(-1)^n \prod_{m=1}^n (2m-1)}{n!2^n}$ for $n \geq 1$ and $c_0 = 1$. Note that

$$|c_n| = \frac{\prod_{m=1}^n (2m-1)}{n!2^n} = \frac{(2n)!}{(n!)^24^n} \leq 1. \quad (3.18)$$

Then, for small $|u|$ and $|\nabla u|$, and setting $g(u) := 2u + u^2 + |\nabla u|^2$,

$$\frac{1}{(|\nabla u|^2 + (1 + u)^2)^{\frac{k}{2}}} = 1 - \frac{1}{2}(g(u)) + \frac{3}{8}(g(u))^2 + \sum_{n=3}^{\infty} c_n(g(u))^n. \quad (3.19)$$

Furthermore, for $|g(u)| \leq \frac{1}{2}$,

$$|\sum_{n=3}^{\infty} c_n(g(u))^n| \leq \sum_{n=3}^{\infty} |g(u)|^n = \frac{|g(u)|^3}{1 - g(u)} \leq 2|2u + u^2 + |\nabla u|^2|^3 = O(\epsilon)u^2 + O(\epsilon)||\nabla u||^2. \quad (3.20)$$

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Therefore,

$$\frac{1}{(|\nabla u|^2 + (1 + u)^2)^{\frac{1}{2}}} = 1 - \frac{1}{2}(2u + u^2 + |\nabla u|^2) + \frac{3}{8}(2u + u^2 + |\nabla u|^2)^2 + \sum_{n=3}^{\infty} c_n (g(u))^n$$

$$= 1 - u + u^2 - \frac{1}{2}|\nabla u|^2 + O(\epsilon)u^2 + O(\epsilon)|\nabla u|^2. \quad (3.21)$$

We conclude,

$$\frac{1}{(|\nabla u|^2 + (1 + u)^2)^{\frac{1}{2}}} = \left(1 + (-u + u^2 - \frac{1}{2}|\nabla u|^2) + O(\epsilon)u^2 + O(\epsilon)|\nabla u|^2\right)^m$$

$$= \sum_{j=0}^{m} \binom{m}{j} (-u + u^2 - \frac{1}{2}|\nabla u|^2)^j + O(\epsilon)u^2 + O(\epsilon)|\nabla u|^2$$

$$= 1 - mu + \frac{m(m+1)}{2} u^2 - \frac{m}{2} |\nabla u|^2 + O(\epsilon)u^2 + O(\epsilon)|\nabla u|^2. \quad (3.22)$$

Using the expansion in Lemma 3.2.1, we further expand $\int_M \sigma_k(h^i_j) \ d\mu$. We perform integration by parts on many of the terms to convert them to include $|\nabla u|^2$. The new format of the integral will be useful later on when we find a lower bound involving the Fraenkel asymmetry.

**Lemma 3.2.2.** Suppose $\Omega = \{(1+u(x^i))x : x \in B\} \subseteq \mathbb{R}^{n+1}$ and $u \in C^3(\partial B)$. If $||u||_{W^{2,\infty}} < \epsilon$, then

$$\int_M \sigma_k(h^i_j) \ d\mu = \int_{\partial B} \binom{n}{k} + \binom{n}{k} (n-k)u + \binom{n}{k} \frac{(n-k)(n-k-1)}{2} u^2$$

$$+ \sum_{m=0}^{k} (-1)^m \binom{n-m}{k-m} \frac{(n-k)(k+1)}{2(m+1)(n-m)} |\nabla u|^2 \sigma_m(D^2 u) \ dA + O(\epsilon)||u||_{W^{1,2}}^2. \quad (3.23)$$
Proof. Applying the Taylor expansion in Lemma 3.2.1, we find $\int_M \sigma_k(h^i_j) d\mu$ equals:

$$
\sum_{m=0}^{k} (-1)^m \left( \frac{n-m}{k-m} \right) \int_{\partial B} \left( 1 + (n-m-k)u + \frac{(n-m-k)(n-m-k-1)}{2} u^2 \right.
- \frac{k+1}{2} |\nabla u|^2 \left) \sigma_m(D^2 u) + \frac{n+k-2m}{n-m} u^i u_j [T_m]^j_i (D^2 u) dA + O(\epsilon) \|u\|^2_{W^{1,2}}. \right) (3.24)
$$

Next, recall from the preliminaries that

$$
\nabla_j [T_m]^j_i (D^2 u) = -(n-m)u_j [T_{m-1}]^j_i (D^2 u), \quad (3.25)
$$

and

$$
\sigma_m(D^2 u) = \frac{1}{m} u^i_j [T_{m-1}]^j_i (D^2 u). \quad (3.26)
$$

Using these identities, we rewrite many of the terms using integration by parts. First, for $m \geq 1$

$$
\int_{\partial B} |\nabla u|^2 \sigma_m(D^2 u) dA = \frac{1}{m} \int_{\partial B} |\nabla u|^2 u^i_j [T_{m-1}]^j_i (D^2 u) dA
- \frac{1}{m} \int_{\partial B} u^i 2 u^s u^j [T_{m-1}]^j_s [T_{m-1}]^s_i (D^2 u) dA
- \frac{2}{m} \int_{\partial B} u^i u^s u^j [T_{m-1}]^s_j (D^2 u) dA + O(\epsilon) \|\nabla u\|^2_{L^2}
= \frac{2}{m} \int_{\partial B} u^i u_j [T_m]^j_i (D^2 u) - |\nabla u|^2 \sigma_m(D^2 u) dA + O(\epsilon) \|\nabla u\|^2_{L^2}. \quad (3.27)
$$

Therefore,

$$
\int_{\partial B} u^i u_j [T_m]^j_i (D^2 u) dA = \frac{m+2}{2} \int_{\partial B} |\nabla u|^2 \sigma_m(D^2 u) dA + O(\epsilon) \|\nabla u\|^2_{L^2}. \quad (3.28)
$$
Next, we integrate each $\sigma_m(D^2 u)$ term. For $m \geq 2$, we have

$$\int_{\partial B} \sigma_m(D^2 u) dA = \frac{1}{m} \int_{\partial B} u_i [T_{m-1}]^j_i (D^2 u) dA$$

$$= \frac{-1}{m} \int_{\partial B} u^i \nabla_j [T_{m-1}]^j_i (D^2 u) dA$$

$$= \frac{n-m+1}{m} \int_{\partial B} u^i u_j [T_{m-2}]^j_i (D^2 u) dA. \quad (3.29)$$

By (3.28),

$$\int_{\partial B} \sigma_m(D^2 u) dA = \frac{n-m+1}{2} \int_{\partial B} |\nabla u|^2 \sigma_{m-2}(D^2 u) dA + O(\epsilon) ||\nabla u||^2_{L^2}. \quad (3.30)$$

Also, for $m = 0$ and 1,

$$\int_{\partial B} \sigma_0(D^2 u) dA = \int_{\partial B} 1 dA \text{ and } \int_{\partial B} \sigma_1(D^2 u) dA = 0. \quad (3.31)$$

Similarly, for $m \geq 1$,

$$\int_{\partial B} u \sigma_m(D^2 u) dA = \frac{1}{m} \int_{\partial B} u u_i [T_{m-1}]^j_i (D^2 u) dA$$

$$= \frac{-1}{m} \int_{\partial B} u^i u_j [T_{m-1}]^j_i (D^2 u) + uu^i \nabla_j [T_{m-1}]^j_i (D^2 u) dA$$

$$= \frac{-1}{m} \int_{\partial B} u^i u_j [T_{m-2}]^j_i (D^2 u) dA + O(\epsilon) ||\nabla u||^2_{L^2}. \quad (3.32)$$

By (3.28),

$$\int_{\partial B} u \sigma_m(D^2 u) dA = \frac{-(m+1)}{2m} \int_{\partial B} |\nabla u|^2 \sigma_{m-1}(D^2 u) dA + O(\epsilon) ||\nabla u||^2_{L^2}. \quad (3.33)$$

And,

$$\int_{\partial B} u \sigma_0(D^2 u) dA = \int_{\partial B} u dA. \quad (3.34)$$
Lastly, for \( m \geq 1 \),

\[
\int_{\partial B} u^2 \sigma_m (D^2 u) dA = \frac{1}{m} \int_{\partial B} u^2 u_j^j [T_{m-1}]^j_i (D^2 u) dA \\
= \frac{-1}{m} \int_{\partial B} u^i \left( 2 uu_j [T_{m-1}]^j_i (D^2 u) + u^2 \nabla_j [T_{m-1}]^j_i (D^2 u) \right) dA \\
= O(\epsilon) \left\| \nabla u \right\|_{L^2}^2. \tag{3.35}
\]

And,

\[
\int_{\partial B} u^2 \sigma_0 (D^2 u) dA = \int_{\partial B} u^2 dA. \tag{3.36}
\]

All together, we find

\[
\int_M \sigma_k (h^i_j) d\mu = \int_{\partial B} \left( \begin{array}{c} n \\ k \end{array} \right) + \left( \begin{array}{c} n \\ k \end{array} \right) (n-k)u + \left( \begin{array}{c} n \\ k \end{array} \right) \frac{(n-k)(n-k-1)}{2} u^2 \\
+ \sum_{m=0}^{k} (-1)^m \left( \begin{array}{c} n-m \\ k-m \end{array} \right) \frac{(n-k)(k+1)}{2(m+1)(n-m)} |\nabla u|^2 \sigma_m (D^2 u) dA + O(\epsilon) \left\| u \right\|_{W^{1,2}}^2. \tag{3.37}
\]

Next we turn our attention to the \((k, -1)\)-isoperimetric deficit. Recall from Section 2.2,

\[
\delta_{k,-1}(\Omega) = \frac{I_k(\Omega) - I_k(B_{\Omega,-1})}{I_k(B_{\Omega,-1})} \tag{3.38}
\]

where \( B_{\Omega,-1} \) is the ball centered at the origin satisfying \( \text{Vol}(B_{\Omega,-1}) = \text{Vol}(\Omega) \). In particular, if \( \Omega \) is normalized so that \( \text{Vol}(\Omega) = \text{Vol}(B) \), then

\[
\delta_{k,-1}(\Omega) = \frac{I_k(\Omega) - I_k(B)}{I_k(B)}. \tag{3.39}
\]

In the next proposition, with the additional assumption that the barycenter of \( \Omega \) is at the
origin, we are able to bound $I_k(\Omega) - I_k(B)$ below by terms involving $||u||_{W^{1,2}}$. In order to get our main theorem bounding $\delta_{k-1}(\Omega)$ below by $\alpha^2(\Omega)$ (see Section 2.2), we only need the term with $||u||^2_{L^2}$ in the lower bound. However, we form a stronger statement that also includes $||\nabla u||^2_{L^2}$ in the lower bound.

**Proposition 3.2.3.** Suppose $\Omega = \{(1 + u(x/|x|)) : x \in B\} \subseteq \mathbb{R}^{n+1}$ where $u \in C^3(\partial B)$, $\text{Vol}(\Omega) = \text{Vol}(B)$, and bar $\Omega = 0$. Additionally, assume for sufficiently small $\epsilon > 0$ that $||u||_{W^{2,\infty}} < \epsilon$. Then,

$$I_k(\Omega) - I_k(B) \geq \left( \frac{n}{k} \right) \frac{(n-k)(k+1)}{2n} \left( \left( 1 + O(\epsilon) \right) ||u||^2_{L^2} + \left( \frac{1}{2} + O(\epsilon) \right) ||\nabla u||^2_{L^2} \right).$$

(3.40)

**Proof.** From Lemma 3.2.2,

$$I_k(\Omega) - I_k(B) = \int_{\partial B} \left( \frac{n}{k} \right) (n-k)u + \left( \frac{n}{k} \right) \frac{(n-k)(n-k-1)}{2} u^2$$

$$+ \sum_{m=0}^{k} (-1)^m \frac{(n-m)(k+1)}{2(m+1)(n-m)} ||\nabla u||^2 \sigma_m(D^2u) \ dA + O(\epsilon)||u||^2_{W^{1,2}}.$$  

(3.41)

Using the assumption that $\text{Vol}(\Omega) = \text{Vol}(B)$, we have from formula (2.37) for the volume that

$$\int_{\partial B} u \ dA = \int_{\partial B} \frac{-n}{2} u^2 \ dA + O(\epsilon)||u||^2_{L^2}.$$  

(3.42)

Substituting this expression into (3.41) yields

$$I_k(\Omega) - I_k(B) = \int_{\partial B} \left( \frac{n}{k} \right) \frac{(n-k)(k+1)}{2n} ||\nabla u||^2 - \left( \frac{n}{k} \right) \frac{(n-k)(k+1)}{2} u^2$$

$$+ \sum_{m=1}^{k} (-1)^m \frac{(n-m)(k+1)}{2(m+1)(n-m)} ||\nabla u||^2 \sigma_m(D^2u) \ dA + O(\epsilon)||u||^2_{W^{1,2}}.$$  

(3.43)
Then, using the assumptions that $\text{Vol}(\Omega) = \text{Vol}(B)$ and $\text{bar}(\Omega) = 0$, Cicalese and Leonardi showed (see Lemma 4.2 in [6]), by writing $u$ in terms of its spherical harmonics basis, that

$$||\nabla u||^2_{L^2} \geq 2(n+1)||u||^2_{L^2} + O(\epsilon)||u||^2_{L^2}. \quad (3.44)$$

Finally, by applying the inequality (3.44) to (3.43), we find that $I_k(\Omega) - I_k(B)$ is bounded below by the following expression:

$$\left(\binom{n}{k}\frac{(n-k)(k+1)}{2n}\right)\left(\frac{1}{2}||\nabla u||^2_{L^2} + (n+1)||u||^2_{L^2} - n||u||^2_{L^2}\right) + O(\epsilon)||\nabla u||^2_{L^2} + O(\epsilon)||u||^2_{L^2}. \quad (3.45)$$

Now, with the lower bound on $I_k(\Omega) - I_k(B)$ being controlled by $||u||^2_{L^2}$, we are equipped to show one of our main results. Observe, as shown in [6], that for $||u||_{L^\infty} < \epsilon$, Hölder’s inequality yields

$$\frac{\Omega \Delta B}{|B|} = \frac{1}{|B|} \left(||u||_{L^1} + \sum_{k=2}^{n+1} \int_{\partial B} \frac{1}{n+1}\binom{n+1}{k} |u|^k \ dA\right) \leq \frac{1}{|B|} \left(\text{Area}(\partial B)^{1/2}||u||_{L^2} + \sum_{k=2}^{n+1} \int_{\partial B} \frac{1}{n+1}\binom{n+1}{k} |u|^k \ dA\right). \quad (3.46)$$

Therefore, when $\text{Vol}(\Omega) = \text{Vol}(B)$,

$$\alpha^2(\Omega) \leq \frac{|\Omega \Delta B|^2}{|B|^2} \leq \frac{\text{Area}(\partial B)}{|B|^2} ||u||^2_{L^2} + O(\epsilon)||u||^2_{L^2} = \frac{(n+1)^2}{\text{Area}(\partial B)} ||u||^2_{L^2} + O(\epsilon)||u||^2_{L^2}. \quad (3.47)$$

Now we are ready to prove Theorem 1.3.1

**Theorem 1.3.1.** Suppose $\Omega = \{(1 + u(x))/x : x \in B\} \subseteq \mathbb{R}^{n+1}$, where $u \in C^3(\partial B)$, $\text{Vol}(\Omega) = \text{Vol}(B)$, and $\text{bar}(\Omega) = 0$. For all $\eta > 0$, there exists $\epsilon > 0$ such that if $||u||_{W^{2,\infty}} < \epsilon$,
then
\[
\delta_{k,-1}(\Omega) \geq \left( \frac{(n-k)(k+1)}{2n(n+1)^2} - \eta \right) \alpha^2(\Omega).
\] (3.48)

**Proof.** Suppose \( ||u||_{W^{2,\infty}} < \epsilon \). Since \( \text{Vol}(\Omega) = \text{Vol}(B) \), the definition for the \((k,-1)\)-isoperimetric deficit becomes

\[
\delta_{k,-1}(\Omega) = \frac{I_k(\Omega) - I_k(B)}{I_k(B)} = \frac{I_k(\Omega) - I_k(B)}{\binom{n}{k} \text{Area}(\partial B)}.
\] (3.49)

From Proposition 3.2.3, we have

\[
\delta_{k,-1}(\Omega) \geq \frac{(n-k)(k+1)}{2n \text{Area}(\partial B)} ||u||_{L^2}^2 + O(\epsilon)||u||_{L^2}^2.
\] (3.50)

Next, as noted in (3.47),

\[
\alpha^2(\Omega) \leq \frac{|\Omega \Delta B|^2}{|B|^2} \leq \frac{(n+1)^2}{\text{Area}(\partial B)} ||u||_{L^2}^2 + O(\epsilon)||u||_{L^2}^2.
\] (3.51)

It follows that

\[
||u||_{L^2}^2 \geq \frac{\text{Area}(\partial B)}{(n+1)^2} \alpha^2(\Omega) + O(\epsilon)\alpha^2(\Omega).
\] (3.52)

Therefore,

\[
\delta_{k,-1}(\Omega) \geq \frac{(n-k)(k+1)}{2n \text{Area}(\partial B)} (1 + O(\epsilon))||u||_{L^2}^2
\]
\[
\geq \frac{(n-k)(k+1)}{2n(n+1)^2} \alpha^2(\Omega) + O(\epsilon)\alpha^2(\Omega).
\] (3.53)
3.3 Quantitative isoperimetric inequality for $\delta_{k,j}(\Omega)$ when $j \geq 0$

In this section, we extend the result from Theorem 1.3.1 to $\delta_{k,j}(\Omega)$ for $0 \leq j < k$. The proof turns out to be quite similar to the case for $\delta_{k,-1}(\Omega)$ in the previous section. The expression for $I_k(\Omega)$ will contain the quantity $C \int_{\partial B} |\nabla u|^2 - nu^2 dA$ under the assumption $I_j(\Omega) = I_j(B)$, which we bound in the same manner as in Proposition 3.2.3.

We begin with the following proposition.

**Proposition 3.3.1.** Fix $j$ where $0 \leq j < k$. Suppose $\Omega = \{(1 + u(|x|))x : x \in B\} \subseteq \mathbb{R}^{n+1}$, where $u \in C^3(\partial B)$, $I_j(\Omega) = I_j(B)$, and $\text{bar}(\Omega) = 0$. Assume for sufficiently small $\epsilon > 0$ that $\|u\|_{W^{2,\infty}} < \epsilon$. Then,

$$I_k(\Omega) - I_k(B) \geq \binom{n-k}{k} \frac{(n-k)(k-j)}{2n} \left( \left( 1 + O(\epsilon) \right) \|u\|^2_{L^2} + \left( \frac{1}{2} + O(\epsilon) \right) \|\nabla u\|^2_{L^2} \right). \quad (3.54)$$

**Proof.** First, for any $s \geq 0$, we have from Lemma 3.2.2 that

$$I_s(\Omega) - I_s(B) = \int_{\partial B} \binom{n}{s} (n-s)u \binom{n-s}{s} \frac{(n-s)(n-s-1)}{2} u^2$$

$$+ \sum_{m=0}^{s} (-1)^m \binom{n-m}{s-m} \frac{(n-s)(s+1)}{2(m+1)(n-m)} |\nabla u|^2 \sigma_m(D^2 u) dA + O(\epsilon)\|u\|^2_{W^{1,2}}. \quad (3.55)$$

Therefore, if $I_j(\Omega) = I_j(B)$,

$$\int_{\partial B} u dA = - \int_{\partial B} \frac{n-j-1}{2} u^2 + \frac{j+1}{2n} |\nabla u|^2$$

$$+ \left( \frac{1}{k!} \sum_{m=1}^{k} (-1)^m \binom{n-m}{j-m} \frac{j+1}{2(m+1)(n-m)} \sigma_m(D^2 u) \right) dA + O(\epsilon)\|u\|^2_{W^{1,2}}. \quad (3.56)$$
Substituting this expression in (3.55) for $s = k$ yields

$$I_k(\Omega) - I_k(B) = \left(\frac{n}{k}\right) \frac{(n-k)(k-j)}{2n} \int_{\partial B} |\nabla u|^2 - nu^2 + \left(\sum_{m=1}^{k} d_m \sigma_m(D^2 u)\right) |\nabla u|^2 dA$$

$$+ O(\epsilon)||\nabla u||^2_{L^2} + O(\epsilon)||u||^2_{L^2},$$

(3.57)

where each $d_m$ is coefficient for $\sigma_m(D^2 u)|\nabla u|^2$. Using the assumptions that $I_j(\Omega) = I_j(B)$ and $\text{bar}(\Omega) = 0$, we show in the following lemma that

$$||\nabla u||^2_{L^2} \geq 2(n+1)||u||^2_{L^2} + O(\epsilon)||u||^2_{L^2} + O(\epsilon)||\nabla u||^2_{L^2}.$$  

(3.58)

Therefore,

$$I_k(\Omega) - I_k(B) \geq \left(\frac{n}{k}\right) \frac{(n-k)(k-j)}{2n} \left(\frac{1}{2}||\nabla u||^2_{L^2} + (n+1)||u||^2_{L^2} - n||u||^2_{L^2}\right)$$

$$+ \int_{\partial B} \left(\sum_{m=1}^{k} d_m \sigma_m(D^2 u)\right) |\nabla u|^2 dA + O(\epsilon)||\nabla u||^2_{L^2} + O(\epsilon)||u||^2_{L^2}$$

$$\geq \left(\frac{n}{k}\right) \frac{(n-k)(k-j)}{4n} ||\nabla u||^2_{L^2} + \left(\frac{n}{k}\right) \frac{(n-k)(k-j)}{2n} ||u||^2_{L^2} + O(\epsilon)||u||^2_{W^{1,2}}.$$  

(3.59)

We now prove the lower bound for $||\nabla u||^2_{L^2}$ used in the previous lemma. The proof closely resembles that in [6] by Cicalèse and Leonardi in their work with the classical quantitative isoperimetric inequality (see also [8] and [9] by Fuglede).

**Lemma 3.3.2.** Suppose $\Omega = \{(1 + u(|x|))x : x \in B\} \subseteq \mathbb{R}^{n+1}$, with $u \in C^3(\partial B)$, $\text{bar}(\Omega) = 0$, and $I_j(\Omega) = I_j(B)$ for a fixed $j$ where $0 \leq j \leq n$. It holds that

$$||\nabla u||^2_{L^2} \geq 2(n+1)||u||^2_{L^2} + O(\epsilon^2)||u||^2_{L^2} + O(\epsilon^2)||\nabla u||^2_{L^2}.$$  

(3.60)
Proof. We write
\[
  u = \sum_{k=0}^{\infty} a_k Y_k,
\]
where \( \{Y_k\} \) are spherical harmonics that form an orthonormal basis for \( L^2 \). Since \( Y_0 = 1 \) we have
\[
a_0 = \langle u, 1 \rangle_{L^2} = \int_{\partial B} u \, dA .
\]
Additionally, using the assumption \( I_j(\Omega) = I_j(B) \), we have from (3.56) that
\[
\int_{\partial B} u \, dA = -\int_{\partial B} \frac{n - j - 1}{2} u^2 + \frac{j + 1}{2n} |\nabla u|^2 \\
+ \left( \frac{1}{n} \right) \sum_{m=1}^{k} (-1)^m \binom{n-m}{j-m} \frac{j+1}{2(m+1)(n-m)} \sigma_m(D^2 u) \right) \, dA + O(\epsilon ||u||_{W^{2,2}}^2). \tag{3.63}
\]
This further implies that \( \int_{\partial B} u \, dA = O(\epsilon^2) \). Hence,
\[
a_0^2 = O(\epsilon^2) ||u||_{L^2}^2 + O(\epsilon^2) ||\nabla u||_{L^2}^2. \tag{3.64}
\]
As shown in [6], combining \( \text{bar}(\Omega) = 0 \) and \( \int_{\partial B} Y_1 \, dA = 0 \) gives
\[
\int_{\partial B} ((1 + u)^{n+2} - 1)Y_1 \, dA = 0. \tag{3.65}
\]
So,
\[
\int_{\partial B} u Y_1 \, dA = \frac{-1}{n + 2} \sum_{k=2}^{n+2} \binom{n+2}{k} \int_{\partial B} u^k Y_1 \, dA = O(||u||_{L^2}^2). \tag{3.66}
\]
Therefore,

\[ a_1^2 = O(\varepsilon^2)||u||_{L^2}^2. \quad (3.67) \]

Next, we consider the corresponding eigenvalue of the spherical harmonic \( Y_k \), which is explicitly given by \( \lambda_k = -k(k+n-1) \). Noting that \(|\lambda_k| \geq 2(n+1)\) when \( k \geq 2 \), we compute

\[
||\nabla u||_{L^2}^2 = \sum_{k=1}^{\infty} |\lambda_k| a_k^2
\]

\[
= \sum_{k=2}^{\infty} |\lambda_k| a_k^2 + na_1^2
\]

\[
\geq 2(n+1) \sum_{k=2}^{\infty} a_k^2 + na_1^2
\]

\[
= 2(n+1) \sum_{k=0}^{\infty} a_k^2 - 2(n+1)a_0^2 - (n+2)a_1^2
\]

\[
= 2(n+1)||u||_{L^2}^2 + O(\varepsilon^2)||u||_{L^2}^2 + O(\varepsilon)||\nabla u||_{L^2}^2. \quad (3.68)
\]

\[ \Box \]

Next, we aim to use Proposition 3.3.1 to bound the \((k,j)\)-isoperimetric deficit below by the Fraenkel asymmetry \( \alpha(\Omega) \). When \( j = -1 \) (when the volume is preserved), estimating \( \alpha(\Omega) \) was reduced to being bounded above by \( |\Omega \Delta B|^2 \), which was bounded by \( ||u||_{L^2}^2 \) (up to a constant). When \( \text{Vol}(\Omega) \neq \text{Vol}(B) \), estimating this quantity is a bit more difficult, and we show in the next theorem that we can bound it above by \( ||\nabla u||_{L^2}^2 \).

**Lemma 3.3.3.** Suppose \( M := \partial \Omega \) is a nearly spherical set, then

\[
\frac{|\Omega \Delta B_{\Omega}|^2}{|B_{\Omega}|^2} \leq \frac{(n+1)^2}{n^2 \text{Area}(\partial B)} ||\nabla u||_{L^2}^2 + O(\varepsilon) ||\nabla u||_{L^2}^2, \quad (3.69)
\]

where \( ||u||_{W^{1,\infty}} < \varepsilon \).

**Remark.** Note that we do not need to assume \( ||D^2 u||_{L^\infty} < \varepsilon \) in this lemma.
Proof. Recall the formula

\[ |\Omega| = \frac{1}{n+1} \int_{\partial B} (1 + u)^{n+1} dA = |B| + \sum_{k=1}^{n+1} \frac{(n+1)}{n+1} \int_{\partial B} u^k dA. \quad (3.70) \]

And, if \( r \) is the radius of \( B_\Omega \), then \( |\Omega| = |B_\Omega| = r^{n+1} |B| \). Hence, \( r^{n+1} = \frac{|\Omega|}{|B|} \). We compute,

\[ \frac{|\Omega \Delta B_\Omega|}{|B_\Omega|} = \frac{1}{n+1} \frac{1}{|B_\Omega|} \int_{\partial B} \left| (1 + u)^{n+1} - r^{n+1} \right| dA \\
= \frac{1}{n+1} \frac{1}{|B_\Omega|} \int_{\partial B} \left( 1 + u \right)^{n+1} - \frac{|B| + \sum_{k=1}^{n+1} \frac{(n+1)}{n+1} \int_{\partial B} u^k dA}{|B|} dA \\
= \frac{1}{n+1} \frac{1}{|B_\Omega|} \int_{\partial B} \sum_{k=1}^{n+1} \binom{n+1}{k} u^k - \frac{1}{\text{Area}(\partial B)} \binom{n+1}{k} \int_{\partial B} u^k dA dA \\
= \frac{1}{n+1} \frac{1}{|B_\Omega|} \int_{\partial B} \sum_{k=1}^{n+1} \binom{n+1}{k} \left( u^k - \text{Avg}(u^k) \right) dA \\
\leq \frac{1}{n+1} \frac{1}{|B_\Omega|} \sum_{k=1}^{n+1} \binom{n+1}{k} ||u^k - \text{Avg}(u^k)||_{L^1}, \quad (3.71) \]

where \( \text{Avg}(u^k) \) denotes the average value of \( u^k \) on \( \partial B \). Then, by applying Hölder’s inequality and the Poincaré inequality, we continue to bound

\[ \frac{|\Omega \Delta B_\Omega|}{|B_\Omega|} \leq \frac{1}{n+1} \frac{\text{Area}(\partial B)^{1/2}}{|B_\Omega|} \sum_{k=1}^{n+1} \binom{n+1}{k} ||u^k - \text{Avg}(u^k)||_{L^2} \]

\[ \leq \frac{1}{n(n+1)} \frac{\text{Area}(\partial B)^{1/2}}{|B_\Omega|} \sum_{k=1}^{n+1} \binom{n+1}{k} ||\nabla(u^k)||_{L^2} \]

\[ \leq \frac{\text{Area}(\partial B)^{1/2}}{n|B_\Omega|} \sum_{k=1}^{n+1} \frac{n}{k-1} ||u^{k-1}||_{L^\infty} ||\nabla u||_{L^2}. \quad (3.72) \]
Therefore, noting that \( \| u^{k-1} \|_{L^\infty} = O(\epsilon) \) for \( k \geq 2 \),

\[
\frac{|\Omega \Delta B_\Omega|^2}{|B_\Omega|^2} \leq \frac{1}{n^2} \frac{\text{Area}(\partial B)}{|B_\Omega|^2} \| \nabla u \|_{L^2}^2 + O(\epsilon) \| \nabla u \|_{L^2}^2 \\
= \frac{(n+1)^2}{n^2 \text{Area}(\partial B)} |B|^2 \| \nabla u \|_{L^2}^2 + O(\epsilon) \| \nabla u \|_{L^2}^2.
\]

(3.73)

Then, because \( \frac{|B|^2}{|B_\Omega|^2} = 1 + O(\epsilon) \)

\[
\frac{|\Omega \Delta B_\Omega|^2}{|B_\Omega|^2} \leq \frac{(n+1)^2}{n^2 \text{Area}(\partial B)} \| \nabla u \|_{L^2}^2 + O(\epsilon) \| \nabla u \|_{L^2}^2.
\]

(3.74)

We now prove the main theorem of this section, where we obtain a quantitative isoperimetric inequality for the \((k, j)\)-isoperimetric deficit. Recall from Section 2.2 that normalizing \( \Omega \) such that \( I_j(\Omega) = I_j(B) \) yields

\[
\delta_{k,j}(\Omega) = \frac{I_k(\Omega) - I_k(B)}{I_k(B)}.
\]

(3.75)

**Theorem 1.3.2.** Fix \( 0 \leq j < k \). Suppose \( \Omega = \{(1 + u(x/|x|))x : x \in B\} \subseteq \mathbb{R}^{n+1} \), where \( u \in C^3(\partial B) \), \( I_j(\Omega) = I_j(B) \), and bar(\( \Omega \)) = 0. For all \( \eta > 0 \), there exists \( \epsilon > 0 \) such that if \( \| u \|_{W^{2,\infty}} < \epsilon \), then

\[
\delta_{k,j}(\Omega) \geq \left( \frac{n(n-k)(k-j)}{4(n+1)^2} - \eta \right) \alpha^2(\Omega).
\]

(3.76)

**Proof.** Suppose \( \| u \|_{W^{2,\infty}} < \epsilon \). Applying Lemma 3.3.3

\[
\alpha^2(\Omega) \leq \frac{|\Omega \Delta B|^2}{|B|^2} \leq \left( \frac{(n+1)^2}{n^2 \text{Area}(\partial B)} + O(\epsilon) \right) \| \nabla u \|_{L^2}^2.
\]

(3.77)
Thus,
\[ \|\nabla u\|_{L^2}^2 \geq \left( \frac{n^2}{(n+1)^2} \text{Area}(\partial B) + O(\epsilon) \right) \alpha^2(\Omega). \] (3.78)

Additionally, since \( I_j(\Omega) = I_j(B) \),
\[ \delta_{k,j}(\Omega) = \frac{I_k(\Omega) - I_k(B)}{I_k(B)} = \frac{I_k(\Omega) - I_k(B)}{\left(\frac{n}{k}\right) \text{Area}(\partial B)}. \] (3.79)

Therefore, applying Proposition 3.3.1 when \( \|u\|_{W^{2,\infty}} < \epsilon \), we have that
\[
\delta_{k,j}(\Omega) \geq \left( \frac{(n-k)(k-j)}{4n\text{Area}(\partial B)} + O(\epsilon) \right) \|\nabla u\|_{L^2}^2 \\
\geq \left( \frac{(n-k)(k-j)}{4n\text{Area}(\partial B)} + O(\epsilon) \right) \left( \frac{n^2}{(n+1)^2} \text{Area}(\partial B) + O(\epsilon) \right) \alpha^2(\Omega) \\
= \left( \frac{n(n-k)(k-j)}{4(n+1)^2} + O(\epsilon) \right) \alpha^2(\Omega). \] (3.80)

3.4 Bounds on \( \|u\|_{L^\infty} \)

Following the argument of Fuglede in [9] for stability of the classical isoperimetric inequality, we control \( \|u\|_{L^\infty} \) using the \((k, -1)\)-isoperimetric deficit. Because we consider \( \Omega \) when \( \text{Vol}(\Omega) = \text{Vol}(B) \) and \( \text{bar}(\Omega) = 0 \), \( \|u\|_{L^\infty} \) is simply the spherical deviation \( d(\Omega) \) from Definition 1.2.1. First, we state a lemma from [9].

**Lemma 3.4.1.** *(Fuglede [9], Lemma 1.4)* Suppose \( w : \partial B \to \mathbb{R} \) is a Lipschitz function where
\[ \int_{\partial B} w \, dA = 0. \]

Then
\[
\|w\|^n_{L^\infty} \leq \begin{cases} 
\pi \|\nabla w\|_{L^1} \leq \pi \|\nabla w\|_{L^2} & n = 1 \\
4\|\nabla w\|_{L^2}^2 \log \frac{\delta}{\|\nabla w\|_{L^2}} & n = 2 \\
C\|\nabla w\|_{L^2}^2 \|\nabla w\|_{L^\infty}^{n-2} & n \geq 3,
\end{cases}
\]

where \(C > 0\) depends only on \(n\).

This lemma is useful because the assumption that \(\text{Vol}(\Omega) = \text{Vol}(B)\) is equivalently expressed as

\[ \int_{\partial B} (1 + u)^{n+1} - 1 \, dA = 0. \quad (3.81) \]

So, we set \(w = \frac{1}{n+1}((1 + u)^{n+1} - 1)\) and apply Lemma 3.4.1 to \(w\).

**Theorem 1.3.3.** Suppose \(\Omega = \{(1 + u(x/|x|))x : x \in B\} \subset \mathbb{R}^{n+1}\), where \(u \in C^3(\partial B)\), \(\text{Vol}(\Omega) = \text{Vol}(B)\), and \(\text{bar}(\Omega) = 0\). There exists an \(\eta > 0\) so if \(\|u\|_{W^{2, \infty}} < \eta\), then

\[
\|u\|^n_{L^\infty} \leq \begin{cases} 
C\delta_{k-1}^{1/2}(\Omega) & n = 1 \\
C\delta_{k-1}^2(\Omega) \log \frac{A}{\delta_{k-1}(\Omega)} & n = 2 \\
C\delta_{k-1}(\Omega) & n \geq 3
\end{cases}
\]

where \(A, C > 0\) depend only on \(n, k\).

**Proof.** As noted above, setting \(w = \frac{1}{n+1}((1 + u)^{n+1} - 1)\) gives

\[ \int_{\partial B} w \, dA = 0. \quad (3.82) \]

Therefore, Lemma 3.4.1 applies to \(w\). Moreover, as shown in \[9\], there exists an \(\eta > 0\) such
that when $|u|_{W^{2,\infty}} < \beta$, then

\[
(1 - O(\eta))|u| \leq |w| \leq (1 + O(\eta))|u|,
\]
and

\[
(1 - O(\eta))|\nabla u| \leq |\nabla w| \leq (1 + O(\eta))|\nabla u|.
\]

First we suppose $n \geq 3$. We will then prove the theorem for $n = 2$, and $n = 1$ follows similarly. Applying Lemma 3.4.1, there is a constant $C > 0$ (possibly changing from line to line) where

\[
\|\nabla u\|^2_{L^2} \geq C\|\nabla w\|^2_{L^2} \geq \frac{C\|w\|^n_{L^\infty}}{C(n)\|\nabla w\|^{n-2}_{L^\infty}} \geq \frac{C\|u\|^n_{L^\infty}}{C(n)\|\nabla u\|^{n-2}_{L^\infty}} \geq C\|u\|^n_{L^\infty}.
\]

By Proposition 3.2.3, for small enough $\|D^2 u\|_{L^\infty}$,

\[
\delta_{k,-1}(\Omega) \geq C\|\nabla u\|^2_{L^2},
\]
which together with (3.85) gives the statement of the theorem for $n \geq 3$.

Next suppose $n = 2$. Then, there is a $M > 0$ such that

\[
\|\nabla u\|^2_{L^2} \log \frac{M\|\nabla u\|_{L^\infty}}{\|\nabla u\|_{L^2}} \geq C\|\nabla w\|^2_{L^2} \log \frac{8e\|\nabla w\|_{L^\infty}}{\|\nabla w\|_{L^2}} \\
\geq C\|w\|^n_{L^\infty} \\
\geq C\|u\|^n_{L^\infty}.
\]
Furthermore,

\[ ||\nabla u||_{L^2}^2 \log \frac{M||\nabla u||_{L^\infty}^2}{||\nabla u||_{L^2}^2} \leq C_1 \delta_{k,-1}(\Omega) \log \frac{C_2}{||\nabla u||_{L^2}^2} \leq C_1 \delta_{k,-1}(\Omega) \log \frac{C_2}{\delta_{k,-1}(\Omega)}. \tag{3.88} \]

The last line follows from the observation in (3.43), where for sufficiently small \( ||u||_{W^{2,\infty}} \) we have \( \delta_{k,-1}(\Omega) \leq C(n, k)||\nabla u||_{L^2}^2 \) for some positive constant \( C(n, k) > 0 \). Combining (3.87) and (3.88) concludes the statement of the theorem for \( n = 2 \). The proof for \( n = 1 \) follows similarly. \( \square \)
Chapter 4

Flows involving the $k$-th mean curvature

4.1 Inverse curvature flow

In [15], Guan and Li gave a proof of the quermassintegral inequalities for $k$-convex starshaped domains. They used a special case of the flows studied by Urbas in [23] and Gerhardt in [14], which looked at certain flows involving a function $f$ that is symmetric in its inputs $\lambda = (\lambda_1, ..., \lambda_n)$, which are the principal curvatures of a hypersurface in $\mathbb{R}^{n+1}$. We take $\Gamma \subseteq \mathbb{R}^n$ to be any set that is an open, convex cone with its vertex at the origin, which also contains $\Gamma_n = \{(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n : \lambda_i > 0 \text{ for each } 1 \leq i \leq n\}$. The function $f$ is assumed to satisfy the following properties:

\[ f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma}); \]  
\[ \frac{\partial f}{\partial \lambda_i} > 0 \text{ for } \lambda \in \Gamma; \]  
\[ \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \leq 0; \]  
\[ f \equiv 0 \text{ on } \partial \Gamma. \]
We now examine the flow of surfaces $M(t)$, which at each time $t$ is an embedding given by $X: S^n \to \mathbb{R}^{n+1}$, and satisfies the equation

$$X_t = \frac{1}{f} \nu,$$  \hspace{1cm} (4.5)

where $\nu$ is the outer unit normal vector on $M(t)$. The following theorem, proved by Urbas and Gerhardt, discusses the existence of starshaped solutions.

**Theorem 4.1.1** (Urbas, [23] and Gerhardt, [14]). Suppose $M_0$ is a smooth surface that is starshaped with respect to a point $p$ and with principal curvatures in $\Gamma$. There is a unique smooth solution of surfaces $\{M(t)\}$ to the flow (4.5) for $t \in [0, \infty)$, and $M(t)$ remains starshaped with respect to $p$.

Furthermore, when rescaling the surfaces to $\{\tilde{M}(t)\}$ so that $\tilde{X} = e^{-\beta t} X$, where $\beta = f(I)$, $\{\tilde{M}(t)\}$ converges to a sphere exponentially fast.

**Remark.** From here on, we assume the surfaces $\{M(t)\}$ are starshaped with respect to the origin. We denote $1 + u(t)$ as the radius of $\tilde{M}(t)$. As a result of the exponential convergence of $\tilde{M}(t)$ to the sphere, for each $k \geq 0$ there exist $K, \gamma > 0$ where $||D^k u(t)||_\infty < Ke^{-\gamma t}$.

To study the quermassintegral inequalities, Guan and Li in [15] specifically look at the case where $f = \frac{\sigma_k(L)}{\sigma_{k-1}(L)}$. That is,

$$X_t = \frac{\sigma_{k-1}(L)}{\sigma_k(L)} \nu.$$  \hspace{1cm} (4.6)

We state some useful derivatives along the flow (4.5) from [15] in the following proposition.

**Proposition 4.1.2.** (Guan and Li, [15] Proposition 4)

1. $\partial_t g_{ij} = \frac{2}{f} h_{ij}$
2. $\partial_t \nu = -\nabla^1 \frac{1}{f}$
3. $\partial_t d\mu_g = \frac{1}{f} \sigma_1(L) d\mu_g$
4. $\partial_t h_{ij} = -\nabla_i \nabla_j \frac{1}{f} + \frac{(h^2)_{ij}}{f}$

5. $\partial_t h^i_j = -\nabla^i \nabla_j \frac{1}{f} - \frac{(h^2)^i_j}{f}$

6. $\partial_t \sigma_m(L) = -\frac{1}{m} \nabla_j ([T_{m-1}]^i_j \nabla_i \frac{1}{f}) - \frac{1}{m} \Sigma_{m-1,1} (h^i_j, (h^2)^i_j)$,

where $(h^2)_{ij} = g^{kl} h_{ik} h_{lj}$.

In [15], the authors show

$$\frac{d}{dt} \int_{M(t)} \sigma_m(L) d\mu_t = (m + 1) \int_{M(t)} \frac{\sigma_{m+1}(L) \sigma_{m-1}(L)}{\sigma_m(L)} d\mu_t. \quad (4.7)$$

Their argument extends to any curvature flow in the form $X_t = \frac{1}{f} \nu$, and so we have the following proposition.

**Proposition 4.1.3** (Guan and Li [15], Lemma 5). **Along the flow** (4.5), **it holds that**

$$\frac{d}{dt} \int_{M(t)} \sigma_m(L) d\mu_t = (m + 1) \int_{M(t)} \frac{\sigma_{m+1}(L)}{f} d\mu_t. \quad (4.8)$$

**Proof.**

$$\frac{d}{dt} \int_{M(t)} \sigma_m(L) d\mu_t = \int_{M(t)} \partial_t \sigma_m(L) d\mu_t + \int_{M(t)} \frac{1}{f} \sigma_1(L) \sigma_m(L) d\mu_t$$

$$= -\int_{M(t)} \frac{1}{m} \nabla_j ([T_{m-1}]^i_j \nabla_i \frac{1}{f}) + \frac{1}{m} \frac{1}{f} \Sigma_{m-1,1} (h^i_j, (h^2)^i_j) d\mu_t$$

$$+ \int_{M(t)} \frac{1}{f} \sigma_1(L) \sigma_m(L) d\mu_t$$

$$= \int_{M(t)} \frac{1}{f} \left( \frac{-1}{m} \Sigma_{m-1,1} (h^i_j, (h^2)^i_j) + \sigma_1(L) \sigma_m(L) \right) d\mu_t. \quad (4.9)$$

Last, we get the result by using the identity:

$$-\frac{1}{m} \Sigma_{m-1,1} (h^i_j, (h^2)^i_j) + \sigma_1(L) \sigma_m(L) = (m + 1) \sigma_{m+1}(h^i_j). \quad (4.10)$$

□
Guan and Li continue to rescale the surfaces to a new family \( \tilde{M}(t) \) where \( \tilde{X} = e^{-rt}X \).

In this paper, they set
\[
r(t) = \frac{\int_{M(t)} \frac{\sigma_{k+1}(L)\sigma_{k-1}(L)}{\sigma_k(L)} d\mu_t}{C_{n,k+1} \int_{M(t)} \sigma_k(L) d\mu_t},
\]  
where \( C_{n,k+1} = \frac{\binom{n}{k+1}}{\binom{n}{k}} \). Then, for a strictly \( k \)-convex surface \( M(t) \), they show that
\[
\frac{d}{dt} \int_{\tilde{M}(t)} \sigma_k(\tilde{L}) d\mu_t = 0 \quad \text{and} \quad \frac{d}{dt} \int_{\tilde{M}(t)} \sigma_{k-1}(\tilde{L}) d\mu_t \geq 0.
\]  
That is to say, if \( I_k(\Omega_0) = I_k(B) \), then \( I_k(\tilde{\Omega}(t)) = I_k(B) \) through the entire flow. Since \( \tilde{\Omega}(t) \) is converging to \( B \) and \( I_{k-1}(\tilde{\Omega}(t)) \) is decreasing, we conclude that \( I_{k-1}(\Omega(0)) \leq I_{k-1}(B) \). These conclusions together prove the \((k,k-1)\)-quermassintegral inequalities for strictly \( k \)-convex, starshaped surfaces. They continue to extend the result to all \( k \)-convex starshaped domains (instead of only strictly \( k \)-convex) through an approximation argument.

For our computations, we find it convenient to consider a different normalization that fixes \( I_{k-1}(\Omega(t)) = I_{k-1}(B) \). Notice that if \( I_{k-1}(\tilde{\Omega}(t)) \) remains constant and \( I_k(\tilde{\Omega}(t)) \) decreases along the flow, we make the same conclusion as in [15] to prove the \((k,k-1)\)-quermassintegral inequalities. Our choice of normalization in this case is the constant \( r = \frac{\binom{n}{k-1}}{\binom{n}{k}} \). The computation is quite similar. Setting \( \tilde{X} = e^{-\int_0^t r(s) ds}X \), we apply formula (4.7) to compute
\[
\frac{d}{dt} \int_{\tilde{M}} \sigma_m(\tilde{L}) d\mu_t = \frac{d}{dt}\left( e^{-(n-m)\int_0^t r(s) ds} \int_{\tilde{M}} \sigma_m(L) d\mu_t \right) = (m+1)e^{-(n-m)\int_0^t r(s) ds} \left( \int_{\tilde{M}} \frac{\sigma_{m+1}(L)}{f} d\mu_t - \frac{(n-m)}{m+1} r(t) \int_{\tilde{M}} \sigma_m(L) d\mu_t \right) = (m+1) \left( \int_{\tilde{M}} \frac{\sigma_{m+1}(\tilde{L})}{f} d\mu_t - \frac{n}{m+1} r(t) \int_{\tilde{M}} \sigma_m(\tilde{L}) d\mu_t \right) .
\]  
Next, using our choice of \( r(t) = \frac{\binom{n}{k-1}}{\binom{n}{k}} \) for the rescaling and substituting in \( \frac{1}{f} = \frac{\sigma_{k-1}(L)}{\sigma_k(L)} \),
we have

\[
\frac{d}{dt} \int_{\tilde{M}} \sigma_m(\tilde{L})d\mu_t = (m + 1) \left( \int_{\tilde{M}} \frac{\sigma_{m+1}(\tilde{L})\sigma_{k-1}(\tilde{L})}{\sigma_k(L)} d\mu_t - \frac{n}{m} \left( \frac{k-1}{m} \right)^{n} \int_{\tilde{M}} \sigma_m(\tilde{L})d\mu_t \right). \tag{4.14}
\]

Applying the Newton-Maclaurin inequalities (as in [15]), which states \(\sigma_{k+1}(\tilde{L})\sigma_{k-1}(\tilde{L}) \leq \frac{(\binom{n}{k+1})(\binom{n}{k-1})}{(n)_2} \sigma^2_k(\tilde{L})\), we find

\[
\frac{d}{dt} \int_{\tilde{M}} \sigma_k(\tilde{L})d\mu_t = (k + 1) \left( \int_{\tilde{M}} \frac{\sigma_{k+1}(\tilde{L})\sigma_{k-1}(\tilde{L})}{\sigma_k(L)} d\mu_t - \frac{n}{k+1} \left( \frac{k-1}{k} \right)^{n} \int_{\tilde{M}} \sigma_k(\tilde{L})d\mu_t \right) \leq 0,
\tag{4.15}
\]

and,

\[
\frac{d}{dt} \int_{\tilde{M}} \sigma_{k-1}(\tilde{L})d\mu_t = k \left( \int_{\tilde{M}} \frac{\sigma_k(\tilde{L})\sigma_{k-1}(\tilde{L})}{\sigma_k(L)} d\mu_t - \int_{\tilde{M}} \sigma_{k-1}(\tilde{L})d\mu_t \right) = 0. \tag{4.16}
\]

Additionally, we generalize the result to when \(f = \left( \frac{\sigma_k(L)}{\sigma_{k-p}(L)} \right)^{1/p} \) for \(1 \leq p \leq k\). In the general case, we normalize the flow such that \(I_{k-p}(\tilde{\Omega}(t))\) remains fixed and \(I_k(\tilde{\Omega}(t))\) is decreasing, thereby proving the \((k, k - p)\)-quermassintegral inequality for strictly \(k\)-convex starshaped domains. The case \(p = k\) corresponds to fixing the surface area. We have considered every case here except for when the volume is fixed, which we consider in Lemma 4.1.5 and requires a slightly different argument.

**Lemma 4.1.4.** Suppose for some \(1 \leq p \leq k\) that \(f = \left( \frac{\sigma_k(L)}{\sigma_{k-p}(L)} \right)^{1/p} \) in the flow (4.5). If we rescale the solution so that \(\tilde{X} = e^{-\int_0^t r(s)ds}X\) where

\[
r(s) := \frac{\left( \frac{n}{k-p} \right) \int_{\tilde{M}(s)} \sigma_{k-p+1}(\tilde{L}) \left( \frac{\sigma_{k-p}(\tilde{L})}{\sigma_k(L)} \right)^{1/p} d\mu_s}{\left( \frac{n}{k-p+1} \right) \int_{\tilde{M}(s)} \sigma_{k-p}(\tilde{L}) d\mu_s}.
\tag{4.17}
\]

then \(I_{k-p}(\tilde{\Omega}(t))\) remains constant and \(I_k(\tilde{\Omega}(t))\) is decreasing for all \(t \in [0, \infty)\).
Proof. First, substituting our choice of \( r \) into formula (4.13) yields

\[
\frac{d}{dt} \int_{\tilde{M}} \sigma_{k-p}(\tilde{L}) \, d\mu_t = 0. \tag{4.18}
\]

Next,

\[
\frac{1}{k+1} \frac{d}{dt} \int_{\tilde{M}} \sigma_k(\tilde{L}) \, d\mu_t = \int_{\tilde{M}} \sigma_{k+1}(\tilde{L}) \left( \frac{\sigma_{k-p}(\tilde{L})}{\sigma_k(\tilde{L})} \right)^{1/p} \, d\mu_t \tag{4.19}
\]

Next, we apply the inequality (see [19] Lemma 15.13)

\[
\frac{n}{k+1} \left( \frac{n}{k-p} \right)^{1/p} \frac{(\sigma_{k-p}(\tilde{L})}{\sigma_k(\tilde{L})} \right)^{1/p} \geq \left( \frac{n}{k-p+2} \right)^{1/2} \left( \frac{\sigma_{k-p}(\tilde{L})}{\sigma_{k-p+2}(\tilde{L})} \right)^{1/2}, \tag{4.21}
\]

and the Newton-Maclaurin inequality,

\[
\sigma_{k-p+1}^2(\tilde{L}) \geq \frac{1}{n(k-p+2)} \sigma_{k-p}(\tilde{L}) \sigma_{k-p+2}(\tilde{L}), \tag{4.22}
\]

to find

\[
\frac{n}{k+1} \left( \frac{n}{k-p} \right)^{1/p} \int_{\tilde{M}} \sigma_{k-p+1}(\tilde{L}) \left( \frac{\sigma_{k-p}(\tilde{L})}{\sigma_k(\tilde{L})} \right)^{1/p} \, d\mu_t \int_{\tilde{M}} \sigma_k(\tilde{L}) \, d\mu_t
\]

\[
\geq \frac{n}{k-p} \left( \frac{n}{k-p+2} \right)^{1/2} \left( \frac{n}{k} \right)^{1/2} \int_{\tilde{M}} \left( \frac{\sigma_{k-p+1}(\tilde{L}) \sigma_{k-p}(\tilde{L})}{\sigma_{k-p+2}(\tilde{L})} \right)^{1/2} \, d\mu_t \int_{\tilde{M}} \sigma_k(\tilde{L}) \, d\mu_t
\]

\[
\geq \frac{n}{k-p} \left( \frac{n}{k-p+2} \right)^{1/2} \left( \frac{n}{k} \right)^{1/2} \int_{\tilde{M}} \left( \frac{\sigma_{k-p+1}(\tilde{L})^2}{\sigma_{k-p+2}(\tilde{L})^2} \right)^{1/2} \, d\mu_t \int_{\tilde{M}} \sigma_k(\tilde{L}) \, d\mu_t. \tag{4.23}
\]
After simplifying the coefficients, we have

$$\left( \frac{n}{k+1} \right) \left( \frac{n}{k-p+1} \right) \int_{\tilde{M}} \sigma_{k-p+1}(\tilde{L}) \left( \frac{\sigma_{k-p}(\tilde{L})}{\sigma_k(L)} \right)^{1/p} \, d\mu_t \int_{\tilde{M}} \sigma_k(\tilde{L}) \, d\mu_t \geq \left( \frac{n}{k+1} \right)^{1/p} \left( \frac{n}{k} \right)^{1/p} \int_{\tilde{M}} \sigma_k(\tilde{L}) \, d\mu_t. \quad (4.24)$$

Which gives,

$$\frac{1}{k+1} \frac{d}{dt} \int_{\tilde{M}} \sigma_k(\tilde{L}) \, d\mu_t \leq \int_{\tilde{M}} \sigma_{k+1}(\tilde{L}) \left( \frac{\sigma_{k-p}(\tilde{L})}{\sigma_k(\tilde{L})} \right)^{1/p} \, d\mu_t - \frac{n}{k+1} \left( \frac{n}{k} \right)^{1/p} \int_{\tilde{M}} \sigma_k(\tilde{L}) \, d\mu_t. \quad (4.25)$$

Next, observing that

$$\left( \frac{\sigma_{k-p}(\tilde{L})}{\sigma_k(\tilde{L})} \right)^{1/p} \leq \left( \frac{n}{k-p} \right)^{1/p} \left( \frac{n}{k} \right)^{1/p} \int_{\tilde{M}} \sigma_k(\tilde{L}) \, d\mu_t \leq \frac{n}{k+1} \left( \frac{n}{k} \right)^{1/p} \sigma_k(\tilde{L}), \quad (4.26)$$

we find

$$\frac{1}{k+1} \frac{d}{dt} \int_{\tilde{M}} \sigma_k(\tilde{L}) \, d\mu_t \leq \int_{\tilde{M}} \sigma_k(\tilde{L}) \sigma_k(\tilde{L}) \, d\mu_t - \int_{\tilde{M}} \sigma_k(\tilde{L}) \, d\mu_t = 0. \quad (4.27)$$


\[\square\]

In the next lemma, we extend the previous results to the case when the volume of $\tilde{\Omega}(t)$ is fixed. This requires a slightly different choice of $f$ and an additional inequality.

**Lemma 4.1.5.** In the flow (4.5), set $f = \sigma_{k+1}^{1/(k+1)}$. If we rescale the solution so that $\tilde{X} = e^{-\int_0^t r(s) \, ds} X$ where $r(s) = \frac{\int_{M(s)} \sigma_k(\tilde{L}) \, d\mu}{\text{Vol}(\tilde{\Omega}(s))}$, then $\text{Vol}(\tilde{\Omega}(t))$ remains constant and $I_k(\tilde{\Omega}(t))$ is decreasing for $t \in [0, \infty)$. 

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Proof. We find,
\[
\frac{d}{dt} \text{Vol}(\Omega) = e^{-(n+1) \int_0^t r(s)ds} \text{Vol}(\Omega)
\]
\[
= e^{-(n+1) \int_0^t r(s)ds} \left( \frac{d}{dt} \text{Vol}(\Omega) - (n + 1)r(t)\text{Vol}(\Omega) \right)
\]
\[
= e^{-(n+1) \int_0^t r(s)ds} \left( \int_M \frac{1}{L} d\mu_t - (n + 1)r(t)\text{Vol}(\Omega) \right)
\]
\[
= 0.
\] (4.28)

And, applying the Newon-Maclaurin inequalities,
\[
\frac{1}{k+1} \frac{d}{dt} \int_M \sigma_k(\bar{L})d\mu_t = \left( \int_M \sigma_{k+1}(\bar{L})^{k/(k+1)}d\mu_t - \frac{n}{(n+1)} \int_M \sigma_{k+1}^{-1/(k+1)}(\bar{L})d\mu_t \int_M \sigma_k(\bar{L})d\mu_t \right)
\]
\[
\leq \left( \frac{n}{(n+1)} \right)^{k/(k+1)} \int_M \sigma_k(\bar{L})d\mu_t - \frac{n}{(n+1)} \int_M \sigma_{k+1}^{-1/(k+1)}(\bar{L})d\mu_t \int_M \sigma_k(\bar{L})d\mu_t
\]
\[
\leq \left( \frac{n}{(n+1)} \right)^{k/(k+1)} \left( 1 - \frac{n}{(n+1)\text{Vol}(\bar{M})} \right) \int_M \sigma_k(\bar{L})d\mu_t.
\] (4.29)

Finally, Ros’s inequality (Theorem 1 in [22]) yields \(\int_M \frac{n}{\sigma_1(\bar{L})}d\mu_t \geq (n + 1)\text{Vol}(\bar{\Omega})\) for hypersurfaces in \(\mathbb{R}^{n+1}\), from which we conclude \(\frac{d}{dt} \int_M \sigma_k(\bar{L})d\mu_t \leq 0\). \(\square\)

In the next chapter, we will study the curvature flows to prove quantitative quermassintegral inequalities. In doing so, we need to study the quantities \(\frac{d}{dt}||u||^2_{L^2}\) and \(\frac{d}{dt}||\nabla u||^2_{L^2}\). For this, we need to understand how to convert the equation \(X_t = \frac{1}{T}v\) into an equation involving the radius of the surface, instead of \(X\). Both Urbas and Gerhardt derive a formula for differentiation of the radial function \(w\), which we state in the following lemma. We state the formula in a slightly more general form \(X_t = G\nu\) (and \(g\) does need to satisfy the properties of \(f\) stated at the beginning of this chapter). This way, we can apply it to other flows, as well.

**Lemma 4.1.6** (See [23] and [18]). Suppose \(\{M(t)\}\) is a collection of smooth hypersurfaces
in $\mathbb{R}^{n+1}$ such that each $M(t)$ is starshaped with respect to the origin and satisfies

$$X_t = G \nu.$$  \hfill (4.30)

If we write $M(t)$ in spherical coordinates with coefficients $(\theta_1, \ldots, \theta_n, w)$, where $w$ is the radius, we have

$$w_t = G \sqrt{1 + \frac{|\nabla w|^2}{w^2}}.$$  \hfill (4.31)

**Proof.** Note, we are using $w_t$ and $\frac{\partial w}{\partial t}$ interchangeably. The outer unit normal at a point on $M(t)$ in spherical coordinates is given by

$$\nu = -\sum_{i=1}^{n} s^{ij} w_i \frac{\partial}{\partial \theta_j} + \frac{\partial}{\partial r},$$  \hfill (4.32)

where $P = \sqrt{1 + \frac{|\nabla w|^2}{w^2}}$. For a solution to the flow $X_t = g \nu$, we can write $X(\theta, t) = w(\phi(\theta, t), t) \phi(\theta, t)$, where $w$ is the radial function of $M(t)$ at $\phi(\theta, t)$, and $\phi(\cdot, t) : \mathbb{S}^n \rightarrow \mathbb{S}^n$ is a suitable diffeomorphism of $\mathbb{S}^n$. Then,

$$X_t = (w_m \frac{\partial \phi^m}{\partial t} + \frac{\partial w}{\partial t}) \phi(\theta) + w \frac{\partial \phi}{\partial t}.$$  \hfill (4.33)
And,

\[ G = \langle X_t, \nu \rangle \]

\[ = \frac{1}{P} \left< w_m \frac{\partial \phi^m}{\partial t}, \nu \right> + \frac{1}{P} \left< w \frac{\partial \phi}{\partial t}, \nu \right> - \sum_{i=1}^{n} s^{ij} \frac{w_i}{w^2} \frac{\partial}{\partial \theta_j} \frac{\partial}{\partial r} \right> \]

\[ = \frac{1}{P} \left< w_m \frac{\partial \phi^m}{\partial t} \frac{\partial}{\partial r}, \nu \right> + \frac{1}{P} \left< w \frac{\partial \phi}{\partial t} \frac{\partial}{\partial r}, \nu \right> - \sum_{i=1}^{n} s^{ij} \frac{w_i}{w^2} \frac{\partial}{\partial \theta_j} \frac{\partial}{\partial r} \right> \]

\[ = \frac{1}{P} \left( \frac{w_m}{w^2} \frac{\partial \phi^m}{\partial t} + \frac{\partial w}{\partial t} \right) \]

\[ = \frac{1}{P} \left( \frac{w_m}{w^2} \frac{\partial \phi^m}{\partial t} + \frac{\partial w}{\partial t} \right) \]

\[ = \frac{1}{P} w_t. \quad (4.34) \]

\[ \square \]

### 4.2 \( \sigma_k \) curvature flow

In \[2\] Cabezas-Rivas and Sinestrari studied the volume preserving flow

\[ X_t = (-\sigma_k(L)^\alpha + h(t)) \nu, \quad (4.35) \]

where \( \nu \) is the outer unit normal, \( \alpha > \frac{1}{k} \), and \( h(t) \) is the normalization factor

\[ h(t) = \frac{1}{\text{Area}(M(t))} \int_{M(t)} \sigma_k(L)^\alpha d\mu_t. \quad (4.36) \]
This choice of $h$ ensures the volume remains fixed under the flow since

\[
\frac{d}{dt}\text{Vol}(\Omega(t)) = \int_{M(t)} <X_t, \nu > d\mu_t \\
= \int_{M(t)} -\sigma_k(L)^\alpha + h(t)d\mu_t = 0. \tag{4.37}
\]

They assume a pinching condition on $M_0$ and show the condition is preserved along the flow. The pinching condition is a stronger condition than convexity, and their main theorem on the existence of a solution to the flow is summarized in the following theorem.

**Theorem 4.2.1** (Cabezas-Rivas and Sinestrari, [2] Theorem 1.1). Given a fixed $\alpha > 0$ and $1 \leq k \leq n$, there exists a constant $C_p \in (0, \frac{1}{n^k})$ depending only on $k, \alpha,$ and $n$, such that if the initial surface $M_0$ has the property that at every point

\[
\sigma_n(L) > C_p\sigma_1^n(L) > 0, \tag{4.38}
\]

then there exists a unique solution of surfaces $\{M(t)\}$ such that

1. The pinching condition (4.38) holds for all $t > 0$ for which a solution to the flow exists,

2. $M(t)$ exists for all $t \in (0, \infty)$,

3. $\{M(t)\}$ converges exponentially fast to a sphere and the volume is preserved along the flow.

Note that the Newton-Maclaurin inequality gives an opposite inequality of the pinching condition, that is $\frac{1}{n^n}\sigma_1^n(L) \geq \sigma_n(L)$, but we get equality when the surface is a sphere. So, with the established $C_p$, we know that a nearly spherical surface, with $||u|| < \epsilon$ for sufficiently small $\epsilon$, will satisfy the pinching condition. Additionally, any surface satisfying the pinching condition will be convex. In [1], Bertini and Sinestrari were able to remove the pinching condition when $k = 2$, and they only assumed convexity on the initial surface to show the existence of a solution.
Just as with the inverse curvature flow, we are concerned with the derivatives of various quantities along the flow to establish a quantitative quermassintegral inequality. We note that all the derivative equations in Proposition 4.1.2 hold for this flow, but we replace \( \frac{1}{f} \) with \(- (\sigma_k^\alpha(L) - h) \nu\). The next lemma is contained in [1], which states that \( \int_{M(t)} \sigma_k(L) d\mu_t \) is decreasing along the flow. This, along with (4.37), proves the \((k, -1)\)-quermassintegral inequality in the case where \( M \) satisfies the pinching condition.

**Lemma 4.2.2** (Bertini and Sinestrari, [1] Lemma 3.3). For a solution of hypersurfaces \( \{M(t)\} \) along the flow (4.35),

\[
\frac{d}{dt} \int_{M(t)} \sigma_{k-1}(L) = -k \int_{M(t)} \sigma_k(\sigma_k^\alpha(L) - h) d\mu_t = -k \int_{M(t)} (\sigma_k(L) - (\frac{1}{\alpha})^\alpha(\sigma_k^\alpha(L) - h)) d\mu_t \leq 0,
\]

(4.39)

and there is equality only when \( M(t) \) is a sphere.

**Proof.** We find from the results of Proposition 4.1.2 that, setting \( G := -\sigma_k^\alpha(L) + h\),

\[
\frac{d}{dt} \int_{M(t)} \sigma_m(L) d\mu_t = \int_{M(t)} \partial_t(\sigma_m(L)) d\mu_t + \int_{M(t)} \sigma_m(L) \partial_t d\mu_t
\]

\[
= - \int_{M(t)} \frac{1}{m} \nabla_j([T_{m-1}]^i_j \nabla_i G) + \frac{G}{m} \Sigma_{m-1,1} (h_j^1, (h^2_j)) d\mu_t
\]

\[
+ \int_{M(t)} \sigma_{k-1}(L) \sigma_1(L) G d\mu_t.
\]

(4.40)

Then, for the same reasoning as in Proposition 4.1.2

\[
\frac{d}{dt} \int_{M(t)} \sigma_{k-1}(L) d\mu_t = -k \int_{M(t)} \sigma_k(\sigma_k^\alpha(L) - h) d\mu_t.
\]

(4.41)

Observing \(- (\frac{1}{\alpha})^\alpha(\sigma_k^\alpha(L) - h) d\mu_t = 0\), we find

\[
\frac{d}{dt} \int_{M(t)} \sigma_{k-1}(L) d\mu_t = -k \int_{M(t)} (\sigma_k(L) - (\frac{1}{\alpha})^\alpha(\sigma_k^\alpha(L) - h)) d\mu_t.
\]

(4.42)
Since \((\sigma_k(L) - h^{1/\alpha})\) and \((\sigma_k^\alpha(L) - h)\) share the same sign, we find \(\frac{d}{dt} \int_{\mathcal{M}(t)} \sigma_{k-1}(L) d\mu_t \leq 0\). \(\square\)
Chapter 5

Quermassintegral inequalities along curvature flows

5.1 Stability along the inverse curvature flow

In this section, we analyze the stability of the \((k, k - 1)\)– quermassintegral inequalities for \(k\)-convex starshaped surfaces, \(M(t)\), along the flow (4.5) when \(f = \frac{\sigma_k(L)}{\sigma_{k-1}(L)}\). That is the flow

\[ X_t = \frac{\sigma_{k-1}(L)}{\sigma_k(L)} \nu. \]  

(5.1)

Writing \(M(t)\) is written in spherical coordinates where \(w\) is the radial function, we saw Lemma 4.1.6 that

\[ w_t = \frac{\sqrt{1 + \frac{[\nabla w]^2}{w^2}}}{f}. \]  

(5.2)
We scale the surfaces to $\tilde{M}(t)$ so that the new radius satisfies
\[
\tilde{w} = e^{-rt}w, \quad r = \left(\frac{n}{k-1}\right) \left(\frac{n}{k}\right).
\tag{5.3}
\]

In (4.15) and (4.16), we saw that this choice of $r$ ensures $\int_{\tilde{M}(t)} \sigma_k(\tilde{L})dA$ is decreasing and $\int_{\tilde{M}(t)} \sigma_{k-1}(\tilde{L})dA$ is constant along the flow. We set
\[
(1 + u) := \tilde{w}. \tag{5.4}
\]

For the computations below, we take $M(t)$ to be nearly spherical so that $||u||_{C^2} < \epsilon$.

Using the formula in Lemma 3.1.1 and Lemma 3.2.2 and denoting $O(W) := O(u\Delta u) + O(u^2) + O(|\nabla u|^2) + O(|D^2 u|^2)$, we find
\[
\sigma_{k-1}(\tilde{L}) = \frac{\binom{n}{k-1}}{(1+u)^2 + |\nabla u|^2} \left(1 + 2u - \frac{k-1}{n}(1 + u)\Delta u\right) + O(W)
\]
\[
= \binom{n}{k-1}(1 - (k+1)u) \left(1 + 2u - \frac{k-1}{n}\Delta u\right) + O(W)
\]
\[
= \binom{n}{k-1} \left(1 - (k-1)u - \frac{k-1}{n}\Delta u\right) + O(W). \tag{5.5}
\]

And,
\[
\frac{1}{\sigma_k(\tilde{L})} = \frac{1}{\binom{n}{k}} \frac{1}{\frac{1}{1 - (1 - \sigma_k(\tilde{L})/\binom{n}{k})}}
\]
\[
= \frac{1}{\binom{n}{k}} \left(1 + \left(1 - \frac{\sigma_k(\tilde{L})}{\binom{n}{k}}\right) + \sum_{j=2}^{\infty} \left(1 - \frac{\sigma_k(\tilde{L})}{\binom{n}{k}}\right)^j\right)
\]
\[
= \frac{1}{\binom{n}{k}} \left(2 - \frac{\sigma_k(\tilde{L})}{\binom{n}{k}}\right) + O(W)
\]
\[
= \frac{1}{\binom{n}{k}} \left(2 - (1 - (k+2)u)(1 + 2u - \frac{k}{n}\Delta u)\right) + O(W)
\]
\[
= \frac{1}{\binom{n}{k}} \left(1 + ku + \frac{k}{n}\Delta u\right) + O(W). \tag{5.6}
\]

These formulas are utilized in the following lemma.
Lemma 5.1.1. Suppose \{\tilde{M}(t)\} are the rescaled surfaces, as in (5.3), of a solution to the flow (5.1). If at some \(t_0 \geq 0\), \(\tilde{M}(t_0)\) is a nearly spherical surface where \(\|u(t_0)\|_{C^2} < \epsilon\) for sufficiently small \(\epsilon > 0\), it holds that at \(t_0\)

\[
\frac{d}{dt} \|u\|_{L^2}^2 = -\frac{2}{n} \binom{n}{k} \|\nabla u\|_{L^2}^2 + O(\epsilon)\|u\|_{W^{2,2}}^2, \tag{5.7}
\]

and

\[
\frac{d}{dt} \|\nabla u\|_{L^2}^2 = -\frac{2}{n} \binom{n}{k} \|\Delta u\|_{L^2}^2 + O(\epsilon)\|u\|_{W^{2,2}}^2. \tag{5.8}
\]

Proof. Combining the expansions of \(\sigma_{k-1}(\tilde{L})\) and \(\frac{1}{\sigma_k(\tilde{L})}\) from (5.5) and (5.6),

\[
\frac{\sigma_{k-1}(\tilde{L})}{\sigma_k(\tilde{L})} = \binom{n}{k} \left(1 - (k-1)u - \frac{k-1}{n} \Delta u\right) \left(1 + ku + \frac{k}{n} \Delta u\right) + O(W)
= \binom{n}{k} \left(1 + u + \frac{1}{n} \Delta u\right) + O(W). \tag{5.9}
\]

We use this expansion to compute

\[
\frac{d}{dt} \|u\|_{L^2}^2 = 2 \int_{\partial B} uu_t \, dA
= 2 \int_{\partial B} u\tilde{w}_t \, dA
= 2 \int_{\partial B} u \left( -re^{-rt}w + e^{-rt}w_t \right) \, dA
= 2 \int_{\partial B} u \left( - \binom{n}{k} \frac{k-1}{n} (1 + u) + e^{-rt} \sqrt{1 + \frac{\|\nabla u\|^2}{(1 + u)^2} \frac{\sigma_{k-1}(L)}{\sigma_k(L)}} \right) \, dA. \tag{5.10}
\]
After rescaling, we have
\[
\frac{d}{dt} ||u||_{L^2}^2 = 2 \int_{\partial B} u \left( - \frac{n}{k} \right) (1 + u) + \sqrt{1 + \frac{|\nabla u|^2}{(1 + u)^2}} \frac{\sigma_{k-1}(\tilde{L})}{\sigma_k(\tilde{L})} \ dA
\]
\[
= 2 \frac{n}{k} \int_{\partial B} u \left( - (1 + u) + (1 + u + \frac{1}{n} \Delta u) \right) dA + O(\epsilon)||u||_{W^{2,2}}
\]
\[
= \frac{2}{n} \int_{\partial B} u \Delta u dA + O(\epsilon)||u||_{W^{2,2}}
\]
\[
= - \frac{2}{n} \int_{\partial B} |\nabla u|^2 dA + O(\epsilon)||u||_{W^{2,2}}. \quad (5.11)
\]

Next, we find
\[
\int_{\partial B} \frac{d}{dt} |\nabla w|^2 dA = \int_{\partial B} 2w^{j} (w_t)_{j} dA
\]
\[
= - \int_{\partial B} 2w_t \Delta w dA
\]
\[
= - \int_{\partial B} 2G \Delta w \sqrt{1 + \frac{|\nabla w|^2}{w^2}} dA. \quad (5.12)
\]

Therefore,
\[
\int_{\partial B} \frac{d}{dt} |\nabla u|^2 dA = \int_{\partial B} \frac{d}{dt} \left( e^{-2rt} |\nabla w|^2 \right) dA
\]
\[
= \int_{\partial B} -2re^{-2rt} |\nabla w|^2 + e^{-2rt} \frac{d}{dt} |\nabla w|^2 dA
\]
\[
= \int_{\partial B} -\frac{n}{k} |\nabla u|^2 + e^{-2rt} \frac{d}{dt} |\nabla w|^2 dA
\]
\[
= \int_{\partial B} -\frac{n}{k} |\nabla u|^2 - 2e^{-2rt} \frac{\sigma_{k-1}(L)}{\sigma_k(L)} \left( \Delta w \sqrt{1 + \frac{|\nabla w|^2}{w^2}} \right) dA. \quad (5.13)
\]
After rescaling, we find
\[
\int_{\partial B} \frac{d}{dt} |\nabla u|^2 dA = \int_{\partial B} -2\binom{n}{k-1} \frac{\sigma_{k-1}(\tilde{L})}{\sigma_k(\tilde{L})} \left( \Delta u \sqrt{1 + \frac{|\nabla u|^2}{(1 + u)^2}} \right) dA \\
= -2\binom{n}{k-1} \int_{\partial B} |\nabla u|^2 + \Delta u \left( 1 + u + \frac{1}{n} \Delta u \right) dA + O(\epsilon)||u||_{W^{2,2}} \\
= -2\binom{n}{k-1} \int_{\partial B} |\nabla u|^2 + u\Delta u + \frac{1}{n}(\Delta u)^2 dA + O(\epsilon)||u||_{W^{2,2}} \\
= -2\binom{n}{k-1} \int_{\partial B} \frac{1}{n}(\Delta u)^2 dA + O(\epsilon)||u||_{W^{2,2}}. \tag{5.14}
\]

Next, we expand the derivative of \(\int_M \sigma_k(\tilde{L}) dA\) along the flow.

**Lemma 5.1.2.** For the nearly spherical surface \(\tilde{M}(t_0)\) in Lemma 5.1.1 at \(t_0\) it holds that
\[
\frac{d}{dt} \int_M \sigma_k(\tilde{L}) dA \leq (k + 1) \binom{n}{k-1} \binom{n}{k+1} \left( \frac{1}{n} ||\nabla u||_{L^2}^2 - \frac{1}{n^2} ||\Delta u||_{L^2}^2 \right) + O(\epsilon)||u||_{W^{2,2}}^2. \tag{5.15}
\]

**Proof.** From formula (4.15), we find
\[
\frac{d}{dt} \int_M \sigma_k(\tilde{L}) dA = (k + 1) \int_{\partial B} \frac{1}{\sigma_k(\tilde{L})} \left( \sigma_{k-1}(\tilde{L})\sigma_{k+1}(\tilde{L}) - \binom{n}{k-1} \binom{n}{k+1} \sigma_k^2(\tilde{L}) \right) \sqrt{\det g} dA, \tag{5.16}
\]
and we found in (2.35) that
\[
\sqrt{\det g} = (1 + u)^n \sqrt{1 + \frac{|\nabla u|^2}{(1 + u)^2}}. \tag{5.17}
\]
From the inequality \(\sigma_{k-1}(\tilde{L})\sigma_{k+1}(\tilde{L}) - \binom{n}{k-1} \binom{n}{k+1} \sigma_k^2(\tilde{L}) \leq 0\) (the Newton-Maclaurin inequality)
and formula (5.6), we conclude

\[
\frac{d}{dt} \int_M \sigma_k(\tilde{L}) dA \leq \frac{k+1}{n} \int_{\partial B} \sigma_{k-1}(\tilde{L}) \sigma_{k+1}(\tilde{L}) - \frac{n}{k} \left( \frac{k-1}{k+1} \right) \sigma_k^2(\tilde{L}) dA + O(\epsilon) \int_{\partial B} \sigma_{k-1}(\tilde{L}) \sigma_{k+1}(\tilde{L}) - \frac{n}{k} \left( \frac{k-1}{k+1} \right) \sigma_k^2(\tilde{L}) dA.
\]

(5.18)

Now we expand the expression

\[
\sigma_{k-1}(\tilde{L}) \sigma_{k+1}(\tilde{L}) - \frac{n}{k} \left( \frac{k-1}{k+1} \right) \sigma_k^2(\tilde{L})
\]

from the formula in Lemma 3.1.1. We find

\[
\sigma_{k-1}(\tilde{L}) \sigma_{k+1}(\tilde{L}) - \frac{n}{k} \left( \frac{k-1}{k+1} \right) \sigma_k^2(\tilde{L}) \text{ is equal to the product:}
\]

\[
\left( \sum_{m=0}^{k-1} \frac{(-1)^m}{(1+u)_m} \frac{n-m}{n-k-m} \left( (1+u)^2 \sigma_m(D^2u) + \frac{n+k-1-2m}{n-m} u^i u_j[T_m]_i^j(D^2u) \right) \right) \times \left( \sum_{m=0}^{k+1} \frac{(-1)^m}{(1+u)_m} \frac{n-m}{n-k+1-m} \left( (1+u)^2 \sigma_m(D^2u) + \frac{n+k+1-2m}{n-m} u^i u_j[T_m]_i^j(D^2u) \right) \right).
\]

(5.19)

And, \(\sigma_k^2(\tilde{L})((1+u)^2 + |\nabla u|^2)^{k+2}\) equals:

\[
\left( \sum_{m=0}^{k} \frac{(-1)^m}{(1+u)_m} \frac{n-m}{n-k-m} \left( (1+u)^2 \sigma_m(D^2u) + \frac{n+k-2m}{n-m} u^i u_j[T_m]_i^j(D^2u) \right) \right)^2.
\]

(5.20)

We collect the coefficients occurring in front of the lower order terms in the expression

\[
((1+u)^2 + |\nabla u|^2)^{k+2}(\sigma_{k-1}(\tilde{L}) \sigma_{k+1}(\tilde{L}) - \frac{n}{k} \left( \frac{k-1}{k+1} \right) \sigma_k^2(\tilde{L})),
\]

(5.21)

for which we have the following:

- \((1+u)^4\) : 0
- \((1+u)^2|\nabla u|^2\) : 0
- \((1+u)^3 \Delta u\) : 0
- \((1+u)^2 \sigma_2(D^2u)\) : \(\frac{n}{k-1} \left( \frac{n}{k+1} \right) \frac{2}{n-1} \)
- \((1+u)^2(\Delta u)^2\) : \(-\left( \frac{n}{k-1} \right) \left( \frac{n}{k+1} \right) \frac{1}{n^2}\)
The rest of the terms are in \( O(\epsilon)u^2 + O(\epsilon)|\nabla u|^2 + O(\epsilon)|D^2 u|^2 \). Because \( \frac{1}{((1+u)^2+|\nabla u|^2)^{1/2}} \) is in \( 1 + O(\epsilon) \), we obtain \( \sigma_{k-1}(\tilde{L})\sigma_{k+1}(\tilde{L}) = \frac{(k-1)(k+1)}{(n)} \sigma_k^2(\tilde{L}) \) is equal to:

\[
\left( \begin{array}{c}
\frac{n}{k-1} \\
\frac{n}{k+1}
\end{array} \right) \left( \frac{2}{n(n-1)} \sigma_2(D^2 u) - \frac{1}{n^2}(\Delta u)^2 \right) + O(\epsilon)u^2 + O(\epsilon)|\nabla u|^2 + O(\epsilon)|D^2 u|^2.
\]

(5.22)

Hence,

\[
\frac{d}{dt} \int_M \sigma_k(\tilde{L}) dA \leq \frac{n}{2} \frac{(k-1)(k+1)}{(n)} \int_{\partial B} \frac{2}{n(n-1)} \sigma_2(D^2 u) - \frac{1}{n^2}(\Delta u)^2 dA + O(\epsilon)||u||^2_{W^{2,2}}.
\]

(5.23)

In the proof of Lemma 3.2.2, we used integration by parts to find \( \int_{\partial B} \sigma_2(D^2 u) dA = \frac{n-1}{2} \int_{\partial B} |\nabla u|^2 dA + O(\epsilon)||\nabla u||^2_{L^2}. \) Hence,

\[
\frac{d}{dt} \int_M \sigma_k(\tilde{L}) dA \leq \frac{n}{2} \frac{(k-1)(k+1)}{(n)} \left( \frac{1}{n} ||\nabla u||^2_{L^2} - \frac{1}{n^2} ||\Delta u||^2_{L^2} \right) + O(\epsilon)||u||^2_{W^{2,2}}.
\]

(5.24)

\[
\square
\]

In Lemma 3.3.2, we obtained a Poincaré inequality when the barycenter of a surface is at the origin. Following a similar argument to this lemma, we now obtain a similar inequality, but instead we compare \( ||\Delta u||^2_{L^2} \) and \( ||\nabla u||^2_{L^2} \). We also relax the conditions on the barycenter slightly.

**Lemma 5.1.3.** Suppose \( M \) is a hypersurface in \( \mathbb{R}^{n+1} \) which is starshaped with respect to the origin, so that \( M = \{(1 + u(x))x : x \in \partial B\} \) where \( u \in C^2(\partial B) \). Further assume that \( M \) is nearly spherical with \( ||u||_{C^2} < \epsilon \), and for some fixed \( K > 0 \) the barycenter of \( M \) satisfies \( |\text{bar}(M)|^2 \leq K \epsilon ||u||^2_{W^{2,2}} \). Then,

\[
||\Delta u||^2_{L^2} \geq 2(n+1)||\nabla u||^2_{L^2} - K' \epsilon ||u||^2_{W^{2,2}},
\]

(5.25)
where $K' > 0$ depends on the choice of $K$.

**Proof.** Write $u = \sum_{k=0}^{\infty} a_k Y_k$. From the proof of Lemma 3.3.2 observe that if $|\text{bar}(M)|^2 \leq K\epsilon||u||^2_{W^{2,2}}$, then for some $K' > 0$ (depending on $K$) we have $a_1^2 \leq K'\epsilon||u||^2_{W^{2,2}}$. Furthermore, we find that

$$||\Delta u||^2_{L^2} = \sum_{k=1}^{\infty} \lambda_k^2 a_k^2 \geq |\lambda_2| \sum_{k=1}^{\infty} |\lambda_k| a_k^2 + (|\lambda_1| - |\lambda_2|)|\lambda_1| a_1^2$$

$$= 2(n+1)||\nabla u||^2_{L^2} + (|\lambda_1| - |\lambda_2|)|\lambda_1| a_1^2. \quad (5.26)$$

Now we are fully equipped with the formulas needed to prove the next proposition. Under the conditions of Proposition 3.3.1 when $I_{k-1}(\Omega) = I_{k-1}(B)$ we have

$$I_k(\Omega) - I_k(B) \geq (1 + O(\epsilon))A(t), \quad (5.27)$$

where we denote

$$A(t) := \left( \frac{n}{k} \right) \frac{(n-k)}{2n} \left( ||u||^2_{L^2} + \frac{1}{2} ||\nabla u||^2_{L^2} \right). \quad (5.28)$$

The quantitative quermassintegral inequalities that we established for nearly spherical sets quickly followed from this proposition. The rest of this section is devoted to providing a new approach to proving (5.27) by comparing $\frac{d}{dt}(I_k(\tilde{\Omega})(t) - I_k(B))$ and $\frac{d}{dt}A(t)$ along the flow.

**Theorem 1.3.4.** Suppose $\tilde{M}(t)$ is the rescaled surface in (5.3) of $M(t)$ (which is a solution to the flow (5.1)). Additionally, assume at $t_0$ that $M(t_0)$ is nearly spherical with $||u(t_0)||_{W^{2,\infty}} < \epsilon$ and that the barycenter of $\tilde{M}(t_0)$ satisfies $|\text{bar}(\tilde{M}(t_0))| \leq K\epsilon||u(t_0)||^2_{W^{2,2}}$ for fixed a $K > 0$. 65
Then, for any small $\eta > 0$,

$$
\frac{d}{dt}(I_k(\Omega(t_0)) - I_k(B)) \leq (1 - \eta)\frac{d}{dt}A(t_0),
$$

(5.29)

and the choice of a sufficiently small $\epsilon > 0$ depends on $\eta$ and $K$.

Moreover, along any solution to the flow [5.1] where $|\bar{M}(t)| \leq K\epsilon||u||^2_{W^{2,2}}$ holds for sufficiently large $t$, we have

$$
\liminf_{t \to \infty} \frac{I_k(\tilde{\Omega}(t)) - I_k(B)}{A(t)} \geq 1.
$$

(5.30)

**Remark.** If the initial surface $M(0)$ is n-symmetric (symmetric with respect to reflection over each coordinate axis), then $M(t)$ remains n-symmetric throughout the flow. In this example, the barycenter remains at the origin during the entire flow, thus satisfying the conditions on the barycenter for this theorem.

**Proof.** Lemma [5.1.1] immediately yields

$$
\frac{d}{dt}A(t) = -\left(\begin{array}{c} n \\ k \end{array}\right)\frac{n-k+2}{2n} \frac{n}{(k)} \left(||\nabla u||^2_{L^2} + \frac{1}{2}||\Delta u||^2_{L^2}\right) + O(\epsilon)||u||^2_{W^{2,2}}
$$

$$
= -\left(\begin{array}{c} k+1 \\ n \end{array}\right)\frac{n}{(k-1)} \frac{n}{(k)} \left(||\nabla u||^2_{L^2} + \frac{1}{2}||\Delta u||^2_{L^2}\right) + O(\epsilon)||u||^2_{W^{2,2}}.
$$

(5.31)

Combining the inequalities in (5.15) and (5.26) yields

$$
\frac{d}{dt} \int_{\hat{M}} \sigma_k(\hat{\Omega}) dA \leq \left(\begin{array}{c} k+1 \\ n \end{array}\right)\frac{n}{(k-1)} \frac{n}{(k)} \left(n||\nabla u||^2_{L^2} - \frac{1}{2}||\Delta u||^2_{L^2}ight) + O(\epsilon)||u||^2_{W^{2,2}}.
$$

$$
\leq \left(\begin{array}{c} k+1 \\ n \end{array}\right)\frac{n}{(k-1)} \frac{n}{(k)} \left(-||\nabla u||^2_{L^2} + \frac{1}{2}||\Delta u||^2_{L^2}\right) + O(\epsilon)||u||^2_{W^{2,2}}.
$$

(5.32)

At this point, we can almost conclude the first part of the theorem, but the term $O(\epsilon)||u||^2_{W^{2,2}}$ in both inequalities above does not allow for the conclusion that $\frac{d}{dt}(I_k(E) -$
\[ I_k(B) \leq \frac{d}{dt} A(t). \] However, if we multiply the left-hand side of (5.32) by \( \frac{1}{1-\eta} \), we find

\[
\frac{1}{1-\eta} \frac{d}{dt} \int_{\tilde{M}} \sigma_k(\tilde{L}) dA \leq \frac{(k+1)\left(\frac{n}{k-1}\right)\left(\frac{n}{k+1}\right)}{n^2} \left( - ||\nabla u||_{L^2}^2 - \frac{1}{2} ||\Delta u||_{L^2}^2 \right)
- C\|\Delta u\|^2_{L^2} + O(\epsilon)\|u\|_{W^{2,2}}\),
\]

(5.33)

where \( C > 0 \) depends only on the choice of \( \eta \).

Furthermore, noting that \( \|\Delta u\|^2_{L^2} = \|D^2 u\|^2_{L^2} + (n-1)||\nabla u||_{L^2}^2 \) and the inequalities in Lemma 5.1.3 and Lemma 3.3.2 we find that \( C\|\Delta u\|^2_{L^2} \) dominates \( O(\epsilon)\|u\|_{W^{2,2}}\). Thereby, for any choice of \( 0 < \eta < 1 \), there is an \( \epsilon > 0 \) where \( \|u\|_{C^2} < \epsilon \) ensures

\[
\frac{d}{dt}(I_k(\tilde{\Omega}(t)) - I_k(B)) \leq (1-\eta)\frac{d}{dt} A(t).
\]

(5.34)

Both sides of the inequality above are negative quantities, which gives

\[
\frac{\frac{d}{dt}(I_k(\tilde{\Omega}(t)) - I_k(B))}{\frac{d}{dt} A(t)} \geq 1 - \eta.
\]

(5.35)

Next, as a consequence of the exponential convergence of \( M(t) \) to a sphere shown in [23] and [14], we have that \( \|u\|_{C^2} \) converges to 0 as \( t \to \infty \). Hence,

\[
\lim_{t \to \infty} \inf \frac{\frac{d}{dt}(I_k(\tilde{\Omega}(t)) - I_k(B))}{\frac{d}{dt} A(t)} \geq 1.
\]

(5.36)

Because both \( I_k(M(t)) - I_k(B) \) and \( A(t) \) approach 0, we obtain, from a generalized version of L’Hopital’s Rule, that

\[
\lim_{t \to \infty} \inf \frac{\frac{I_k(\tilde{\Omega}(t)) - I_k(B)}{A(t)}} \geq 1.
\]

(5.37)

Before we can use the previous proposition to give a different proof of Proposition 3.3.1,
at least in the case where the barycenter remains close enough to the origin, we need the following lemma. We need to show that if $\tilde{M}(0)$ is close enough to the sphere initially, then $\tilde{M}(0)$ remains nearly spherical, which will allow us to apply the previous lemma for all $\tilde{M}(t)$ along the flow.

**Lemma 5.1.4.** Along the flow \[5.1\], for any $\epsilon > 0$ there is a $\delta > 0$ such that when $||u(0)||_{C^2} < \delta$, it holds that $||u(t)||_{C^2} < \epsilon$ for all $t \geq 0$.

**Proof.** In this proof, we primarily examine some results from [23] to help prove the lemma. In this paper, Urbas normalizes $f$ in the flow so that $f(\delta_j^i) = 1$. This normalization would require that the rescaling to $\tilde{M}(t)$ satisfies $\tilde{w}(t) = e^{-rt}w$ (instead of $e^{-rt}w$). In this proof, we continue using our conventions from this chapter and don’t use their normalization. Note that in our rescaling we have $\tilde{f} = e^{rt}f$, where we recall $f = \frac{\sigma_k(L)}{\sigma_{k-1}(L)}$. We change constants from their proof to appropriately adapt to our notation. In Lemma 3.1, Urbas proves

\[
\min_{\partial B} w(0) \leq w(t)e^{-rt} \leq \max_{\partial B} w(0),
\]

which immediately yields $|u(0)| \leq \max_{\partial B} |u(0)|$. They continue in Lemma 3.2 to show

\[
\frac{|\nabla w(t)|}{w(t)} \leq \max_{\partial B} \frac{|\nabla w(0)|}{w(0)},
\]

which yields

\[
|\nabla u(t)| \leq \frac{1 + u(t)}{\min_{\partial B}(1 + u(0))} \max_{\partial B} |\nabla u(0)|.
\]

At this point, we have concluded that $||u||_{C^1} < \epsilon$ for some chosen $\delta > 0$ on the initial conditions, and next we find bounds on the principal curvatures to obtain the desired bound on $||D^2 u||_{L^\infty}$. At the end of Lemma 3.3, Urbas shows

\[
h_{\text{max}}(t) \leq h_{\text{max}}(0) - 2rt,
\]

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where \( h = \log \left( \kappa \langle \xi, \nu \rangle \right) \) and \( h_{\text{max}}(t) \) is the maximum value taken over all points on the surface \( M(t) \) and the principal curvatures \( \kappa \). When rescaling the inputs, we observe

\[
h = \log \left( \frac{e^{-rt} \tilde{\kappa}}{\langle e^{rt} \tilde{X}, \nu \rangle} \right) = \log \left( \frac{e^{-2rt} \tilde{\kappa}}{\langle \tilde{X}, \nu \rangle} \right).
\] (5.42)

Because of the bound on \( ||u||_{C^1} \), we have that \( \langle \tilde{X}, \nu \rangle \) can be made close to 1 for small \( \delta \). Then, given any \( \beta > 0 \) we can ensure, for small enough \( \delta \), that

\[
e^{-2rt}(\tilde{\kappa}_{\text{max}}(t) - \beta) \leq e^{-2rt}(\tilde{\kappa}_{\text{max}}(0) + \beta),
\] (5.43)

where we have used that the inequality (5.41) can be rewritten as \( e^{h_{\text{max}}(t)} \leq e^{-2rt} e^{h_{\text{max}}(0)} \).

This inequality quickly simplifies to

\[
\tilde{\kappa}_{\text{max}}(t) \leq \tilde{\kappa}_{\text{max}}(0) + 2\beta.
\] (5.44)

For sufficiently small \( ||u(0)||_{C^2} \), we find \( \tilde{\kappa}_{\text{max}}(0) \) is close to 1 so that

\[
\tilde{\kappa}_{\text{max}}(t) \leq 1 + 3\beta.
\] (5.45)

Next we find a similar lower bound on the principal curvatures. In Lemma 3.5, Urbas sets

\[
G := \left( \frac{n}{n-1} \right) \sqrt{1 + \frac{||u||^2}{n^2}},
\]

and they conclude

\[
\min_{\partial B} G(0) \leq G \leq \max_{\partial B} G(0).
\] (5.46)
Using \( \tilde{f} = e^{rt}f \), we rewrite the above inequality as

\[
\left( \frac{n}{k} \right) \left( \frac{n}{k-1} \right) \frac{\tilde{w}}{\sqrt{1 + \frac{|\nabla \tilde{w}|^2}{\tilde{w}^2}}} \frac{1}{\max_{\partial B} G(0)} \leq \tilde{f} \leq \left( \frac{n}{k} \right) \left( \frac{n}{k-1} \right) \frac{\tilde{w}}{\sqrt{1 + \frac{|\nabla \tilde{w}|^2}{\tilde{w}^2}}} \frac{1}{\min_{\partial B} G(0)}. \tag{5.47}
\]

We find for sufficiently small \( \delta \) that

\[
1 - \beta \leq \left( \frac{n}{k} \right) \tilde{f} \leq 1 + \beta. \tag{5.48}
\]

for all \( t \geq 0 \). Now, using the inequality \( \frac{1}{n} \sigma_1(\tilde{L}) \geq \left( \frac{k-1}{k} \right) \tilde{f} \) (see, for example, Lemma 15.13 in [19]), we sum over the principal curvatures at any point on \( M(t) \) to find \( \frac{1}{n} \sum_{i=1}^{n} \tilde{\kappa}_i \geq 1 - \beta \). Therefore,

\[
\tilde{\kappa}_1 \geq n - \sum_{i=2}^{n} \tilde{\kappa}_i - n\beta \\
\geq n - (n - 1)(1 + 3\beta) - n\beta = 1 - (4n - 3)\beta. \tag{5.49}
\]

Finally, we observe that the principal curvatures remaining close to 1, together with \( ||u(t)||_{C^1} \) remaining close to 0, lets us ensure \( ||u||_{C^2} < \epsilon \).

We are now ready to give a proof of the following corollary, which gives a new proof of Proposition 3.3.1 when \( j = k - 1 \) in the case that \( M \) is n-symmetric (that is, \( M \) is preserved under reflection over each coordinate axis). When setting \( M(0) = M \), the symmetry condition will be preserved throughout the entire flow, thus keeping the barycenter of \( M(t) \) at the origin for all \( t \geq 0 \). This allows us to apply Theorem 1.3.4 in the proof of the following corollary.

**Corollary 5.1.5.** Given any \( \eta > 0 \), there is an \( \epsilon > 0 \) such that any smooth n-symmetric, nearly spherical set \( M \) that, where \( M = \partial \Omega \), satisfies the inequality

\[
I_k(\Omega) - I_k(B) \geq (1 - \eta)A, \tag{5.50}
\]

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when $||u||_{C^2} < \epsilon$ and $I_{k-1}(\Omega) = I_{k-1}(B)$.

Proof. Suppose $M$ is the initial surface, $M(0)$, to a solution of the flow in (5.1). Along the rescaled solution to the flow from (5.3), we denote $\tilde{M}(t) = \partial \tilde{\Omega}(t)$ as the rescaled surface. Next, we set

$$S(t) := \frac{I_k(\tilde{\Omega}(t)) - I_k(B)}{A(t)}, \quad (5.51)$$

For any $\eta > 0$ we find $\epsilon > 0$, as in Theorem 1.3.4, so that $\frac{d}{dt} I_k(\tilde{\Omega}(t)) \leq (1 - \eta) \frac{d}{dt} A(t)$ when $||u(t)||_{C^2} < \epsilon$. From Lemma 5.1.4 we know for small enough $||u(0)||_{C^2}$ that $M(t)$ remains nearly spherical throughout the flow to ensure $\frac{d}{dt} I_k(\tilde{\Omega}(t)) \leq (1 - \eta) \frac{d}{dt} A(t)$. We aim to prove the conclusion of this corollary for $M(0)$, which we note is the same as $\tilde{M}(0)$.

We find

$$\frac{d}{dt} S(t) = \frac{A(t) \frac{d}{dt} I_k(\tilde{\Omega}(t)) - (I_k(\tilde{\Omega}(t)) - I_k(B)) \frac{d}{dt} A(t)}{A^2(t)} \leq \frac{\frac{d}{dt} A(t)}{A^2(t)} ((1 - \eta) \cdot A(t) - (I_k(\tilde{\Omega}(t)) - I_k(B))). \quad (5.52)$$

Given any $t_0 \geq 0$, there are two cases:

1. $S(t_0) \geq 1 - \eta$. This is equivalent to $(1 - \eta) \cdot A(t_0) \leq I_k(\tilde{\Omega}(t_0)) - I_k(B)$.

2. $S(t_0) < 1 - \eta$, which implies $S$ is decreasing at $t_0$ because in this case $(1 - \eta) \cdot A(t_0) - (I_k(\tilde{\Omega}(t_0)) - I_k(B)) > 0$, and we always have $\frac{\frac{d}{dt} A(t_0)}{A^2(t_0)} \leq 0$ for small $||u||_{C^2}$.

Suppose when $t_0 = 0$ we have case 1. This means $S(0) \geq 1 - \eta$, which concludes this corollary for $M(0)$.

Now we consider what would happen if case 2 were to occur at $t_0 = 0$, for which we will find a contradiction and conclude that case 1 must occur at $t_0 = 0$. For case 2 to occur, we would have $S(0) < 1 - \eta$ and that $S$ begins decreasing initially. This implies that $S(t) < 1 - \eta$ for all $t > 0$. So, $S(t)$ remains in case 2 for all $t > 0$ and is thereby decreasing throughout the
entire flow. This is a contradiction, however, since that implies \( \liminf_{t \to \infty} S(t) \leq 1 - \eta < 1 \), but Theorem 13.3.4 shows that \( \liminf_{t \to \infty} S(t) \geq 1 \).

\[ \square \]

5.2 Stability along the \( \sigma_k \) curvature flow

In this section, we examine stability of the \((k, -1)\)-quermassintegral inequality of the flow of surfaces \( M(t) \) along \( (4.35) \) with \( \alpha = 1 \), which is the flow

\[ X_t = (-\sigma_k(L) + h(t))\nu, \tag{5.53} \]

where \( h(t) = \frac{1}{\text{Area}(M(t))} \int_{M(t)} \sigma_k(L) d\mu_t \). The volume along the flow is preserved. Therefore, when studying stability for nearly spherical sets, the Fraenkel Asymmetry can be approximated above by \( ||u||^2_{L^2} \), which we observed in (3.47). So, next we differentiate the quantities \( ||u||^2_{L^2} \) and \( \frac{d}{dt} \int_{M(t)} \sigma_{k-1}(L) dA \) along the flow.

**Lemma 5.2.1.** For sufficiently small \( \epsilon > 0 \), if \( ||u(t)||_{C^2} < \epsilon \) along the flow \( (5.53) \), then

\[ \frac{d}{dt} ||u||^2_{L^2} = 2\frac{k}{n} \binom{n}{k} \int_{\partial B} nu^2 - |\nabla u|^2 dA + O(\epsilon)||u||^2_{W^{2,2}}, \tag{5.54} \]

and

\[ \frac{d}{dt} \int_{M(t)} \sigma_{k-1}(L) d\mu_t = k^3 \binom{n}{k}^2 \int_{\partial B} -u^2 - \frac{1}{n^2} (\Delta u)^2 + \frac{2}{n} |\nabla u|^2 dA + O(\epsilon)||u||^2_{W^{2,2}}. \tag{5.55} \]

**Proof.** We begin by estimating the function \( G := -\sigma_k(L) + h(t) \). To compute \( h(t) = \frac{1}{\text{Area}(M(t))} \int_{M(t)} \sigma_k(L) dA \), recall from the proof of Proposition 3.2.3 that when \( V(M(t)) = V(B) \), we have \( \int_{M(t)} \sigma_k(L) d\mu_t \) equals:

\[ \binom{n}{k} \text{Area}(\partial B) + \binom{n}{k} \frac{(n-k)(k+1)}{2n} \left( ||\nabla u||^2_{L^2} + n||u||^2_{L^2} \right) + O(\epsilon)||u||^2_{W^{1,2}}. \tag{5.56} \]
Additionally,

\[
\text{Area}(M(t)) = \int_{\partial B} (1 + u)^{n-1} \sqrt{(1 + u)^2 + |\nabla u|^2} dA
\]
\[
= \int_{\partial B} (1 + (n - 1)u + \frac{(n - 1)(n - 2)}{2}u^2)(1 + u + \frac{1}{2} |\nabla u|^2) dA + O(\|u\|_{W^{1,2}})
\]
\[
= \int_{\partial B} 1 + nu + \frac{n(n - 1)}{2} u^2 + \frac{1}{2} |\nabla u|^2 dA + O(\|u\|_{W^{1,2}}).
\]  

(5.57)  

(5.58)

Using that \(\text{Vol}(\Omega) = \text{Vol}(\partial B)\) we substitute, just as in (3.42), the expression \(\int_{\partial B} u dA = (\frac{-n}{2} + O(\epsilon))\|u\|^2_{L^2}\) to find

\[
\text{Area}(M(t)) = \int_{\partial B} 1 - \frac{n}{2} u^2 + \frac{1}{2} |\nabla u|^2 dA O(\|u\|_{W^{1,2}}).
\]  

(5.59)

Hence,

\[
\frac{1}{\text{Area}(M(t))} = \frac{1}{\text{Area}(\partial B)} \left( 1 - \frac{1}{\text{Area}(\partial B)} \left( \frac{n}{2} \|u\|^2_{L^2} - \frac{1}{2} |\nabla u|^2_{L^2} \right) + O(\|u\|_{W^{1,2}}) \right)
\]
\[
= \frac{1}{\text{Area}(\partial B)} + O(\|u\|^2_{W^{1,2}}).
\]  

(5.60)

Thus,

\[
h(t) = \frac{1}{\text{Area}(M(t))} \left( \binom{n}{k} \text{Area}(M(t)) + O(||u||^2_{W^{1,2}}) \right)
\]
\[
= \binom{n}{k} + O(||u||^2_{W^{1,2}}).
\]  

(5.61)
Using the formulas in Lemma 3.2.1 and Lemma 3.1.1, we have $\sigma_k(L)$ equals:

$$
\frac{1}{(1 + u)^2 + |\nabla u|^2} \sum_{m=0}^{k} \frac{(-1)^m (n-m)^{k-m}}{(1 + u)^m} \left( (1 + u)^2 \sigma_m(D^2 u) + \frac{n + k - 2m}{n - m} u' u_j [T_m]_{ij}^j (D^2 u) \right)
$$

$$
= \binom{n}{k} (1 - (k + 2)u)(1 + 2u - \frac{k}{n} \Delta u) + O(u^2, |\nabla u|^2, |D^2 u|^2, u \Delta u)
$$

$$
= \binom{n}{k} (1 - ku - \frac{k}{n} \Delta u) + O(u^2, |\nabla u|^2, |D^2 u|^2, u \Delta u). \quad (5.62)
$$

Thereby,

$$
G = (-\sigma_k(L) + h) = k \binom{n}{k} (u + \frac{1}{n} \Delta u) + O(u^2, |\nabla u|^2, |D^2 u|^2, u \Delta u). \quad (5.63)
$$

And,

$$
G^2 = k^2 \binom{n}{k}^2 \left( u^2 + \frac{1}{n^2} (\Delta u)^2 + \frac{2}{n} u \Delta u \right) + O(\epsilon)O(u^2, |\nabla u|^2, |D^2 u|^2). \quad (5.64)
$$

Next, as in (5.2), $u_t = G \sqrt{1 + \frac{|\nabla u|^2}{(1+u)^2}}$. Then, since $\sqrt{1 + \frac{|\nabla u|^2}{(1+u)^2}} = 1 + O(u^2, |\nabla u|^2)$, we obtain

$$
\frac{d}{dt} ||u||_{L^2} = 2 \int_{\partial B} u u_t dA
$$

$$
= 2 \binom{n}{k} \frac{k}{n} \int_{\partial B} (nu^2 + u \Delta u) \sqrt{1 + \frac{|\nabla u|^2}{(1+u)^2}} dA + O(\epsilon)||u||_{W^{2,2}}^2
$$

$$
= 2 \binom{n}{k} \frac{k}{n} \int_{\partial B} nu^2 - |\nabla u|^2 dA + O(\epsilon)||u||_{W^{2,2}}^2. \quad (5.65)
$$
Using the formula in Lemma 4.2.2 with \( \alpha = 1 \), we compute

\[
\frac{d}{dt} \int_{M(t)} \sigma_{k-1} \, d\mu_t = -k \int_{\partial B} G^2 (1 + u)^2 \sqrt{1 + \frac{\|u\|^2}{(1 + u)^2}} \, dA
\]

\[
= -k^3 \left( \frac{n}{k} \right)^2 \int_{\partial B} u^2 + \frac{1}{n^2} (\Delta u)^2 + \frac{2}{n} u \Delta u \, dA + O(\epsilon) \|u\|_{W^{2,2}}^2.
\]

\[
= -k^3 \left( \frac{n}{k} \right)^2 \int_{\partial B} u^2 + \frac{1}{n^2} (\Delta u)^2 - \frac{2}{n} |\nabla u|^2 \, dA + O(\epsilon) \|u\|_{W^{2,2}}^2.
\]

(5.66)

Now we revisit Proposition 3.3.1 along the flow, from which we conclude that if \( M \) is a nearly spherical surface, where the barycenter is at the origin and \( V(M) = V(B) \), then

\[
I_{k-1}(\Omega) - I_{k-1}(B) \geq \left( n \frac{k(n-k+1)}{k-1} \right) \frac{\|u\|_{L^2}^2 + O(\epsilon) \|u\|_{W^{1,2}}^2}{2n}.
\]

(5.67)

This is a weaker statement than Proposition 3.3.1 because we have left out the gradient term. However, the gradient term is not needed when using the Fraenkel asymmetry in the quantitative quermassintegral inequality.

In the following theorem, we compare the derivatives of both sides of the inequality (5.67). Importantly, just as in Theorem 1.3.4, we need a condition that ensures the barycenter remains near the origin throughout the flow. We know this condition will be preserved for n-symmetric sets, for example.

**Theorem 1.3.5** Suppose \( M(t) \) is a solution of surfaces to the flow (5.53), and at \( t_0 \) the surface \( M(t_0) \) satisfies, for a fixed \( K \), that \( |\text{bar}(M(t_0))|^2 \leq K \epsilon \|u(t_0)\|_{W^{2,2}}^2 \) and \( \|u(t_0)\|_{W^{2,\infty}} < \epsilon \). Then, for any small \( \eta > 0 \)

\[
\frac{d}{dt} \left( I_{k-1}(\Omega(t_0)) - I_{k-1}(B) \right) \leq (1 - \eta) \frac{d}{dt} \frac{k(n-k+1)}{2n} \|u(t_0)\|_{L^2}^2.
\]

(5.68)

where the choice of a sufficiently small \( \epsilon > 0 \) depends on \( \eta \) and \( K \). Additionally, if \( |\text{bar}(M(t))|^2 \leq \ldots \)
\( K\epsilon||u||_{W^{2,2}}^2 \) holds for sufficiently large \( t \), then

\[
\liminf_{t \to \infty} \frac{I_{k-1}(\Omega(t)) - I_{k-1}(B)}{||u(t)||_{L^2}^2} \geq \frac{k(n-k+1)}{2n}.
\] (5.69)

**Proof.** First, from (5.54),

\[
\frac{d}{dt} \frac{k^2(n-k+1)}{2n} (\begin{pmatrix} n \\ k-1 \end{pmatrix}) ||u||_{L^2}^2 = k^3 \left( \begin{pmatrix} n \\ k \end{pmatrix} \right)^2 \int_{\partial B} nu^2 - |\nabla u|^2 dA + O(\epsilon)||u||_{W^{2,2}}^2.
\]

From Lemma 5.1.3, when \(|\text{bar}(M(t))| \leq K\epsilon||u||_{W^{2,2}}\), we have \( ||\Delta u||_{L^2}^2 \geq 2(n+1)||\nabla u||_{L^2}^2 + O(\epsilon)||u||_{W^{2,2}}^2 \). Thus, we can bound the deficit with formula (5.55),

\[
\frac{d}{dt} (I_{k-1}(M(t)) - I_{k-1}(B)) = k^3 \left( \begin{pmatrix} n \\ k \end{pmatrix} \right)^2 \int_{\partial B} -n^2u^2 - (\Delta u)^2 + 2n|\nabla u|^2 dA + O(\epsilon)||u||_{W^{2,2}}^2
\]

\[
\leq k^3 \left( \begin{pmatrix} n \\ k \end{pmatrix} \right)^2 \int_{\partial B} -n^2u^2 - |\nabla u|^2 dA + O(\epsilon)||u||_{W^{2,2}}^2.
\] (5.71)

We achieve both (5.68) and (5.69) by the same reasoning as in Theorem 1.3.4 which is made possible from the convergence of \( M(t) \) to a sphere as stated in Lemma 4.2.1.

We are not at the moment able to conclude an analogous statement to Corollary 5.1.5 because we have not proven a similar result to Lemma 5.1.4.

\[ \square \]


